CompSci 102 Discrete Mathematics for CS
Spring 2006 Forbes

HW 1 Solutions

1. (2 pts.) Basics
Thanks for your responses

2. (13 pts.) Book problems - Warmup for Recitation

(a) 1.2 Exercise 36:

The statement of the problem is really the solution. Each line of the truth table corresponds to exactly one combination of truth values for the n atomic propositions involved. We can write down a conjunction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false.If we do this for each line of the truth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form.

(b) 1.3 Exercise 26:

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives. Let:
$R(x) = x$ is in the correct place.
$E(x) = x$ is in excellent condition.
$T(x) = x$ is a (your) tool; and let the universe of discourse be all things.

a) There exists something not in the correct place: $\exists x \neg R(x)$

b) If something is a tool, then it is in the correct place and in excellent condition:
$\forall x(T(x) \Rightarrow (R(x) \wedge E(x)))$.

c) $\forall x(R(x) \wedge E(x))$.

d) This is saying that everything fails to satisfy the condition:
$\forall x \neg(R(x) \wedge E(x))$.

e) There exists a tool with this property:
$\exists x(T(x) \wedge \neg R(x) \wedge E(x))$.

(c) 1.4 Exercise 34:

In each case, you need to specify predicates and the universe of discourse (noted in italics).

a) Let: $L(x, y)$ = *person* x has lost y *dollars* playing the lottery
Statement: $\neg\exists x\exists y(y > 1000 \wedge L(x, y)$
Negation: $\exists x\exists y(y > 1000 \wedge L(x, y)$: someone has lost more than \$1000 playing the lottery

b) Let: $C(x, y)$ means that *student in class* x has chatted with *student in class* y
Statement: $\exists x, \exists y(y \neq x \wedge \forall z(z \neq x \Rightarrow (z = y \Leftrightarrow C(x, z))))$
Negation: $\forall x, \forall y(y \neq x \Rightarrow \exists z(z \neq x \wedge \neg(z = y \Leftrightarrow C(x, z))))$: everybody in this class has either chatted with no one else or has chatted with two or more others.

c) Let $E(x, y)$ mean that *student in class* x has sent email to *student in class* y.
Statement: $\neg\exists x\exists y\exists z(y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w(w \neq x \Rightarrow (E(x, w) \Leftrightarrow (w = y \vee w = z))))$
Negation: $\exists x\exists y\exists z(y \neq z \wedge x \neq y \wedge x \neq z \wedge \forall w(w \neq x \Rightarrow (E(x, w) \Leftrightarrow (w = y \vee w = z))))$: some student in this class has sent email to exactly two other students in the class.

d) Let $S(x, y)$ mean that *student in class* x has solved exercise y
Statement: $\exists x\forall y S(x, y)$
Negation: $\forall x\exists y\neg S(x, y)$: For every student in the class, there is some exercise that he or she has not solved

e) Let $S(x, y)$ mean that *student in class* x has solved exercise y and let $B(y, z)$ mean that *exercise* y is in *section* z of the book.
Statement: $\neg\exists x\forall z\exists y(B(y, z) \wedge S(x, y))$
Negation: $\exists x\forall z\exists y(B(y, z) \wedge S(x, y))$: some student has solved at least one exercise in every section of the book

(d) 1.4 Exercise 50:

We just need to translate the words into symbols:

$$\lim_{n\to\infty} a_n = L$$

means

$$\forall\epsilon\exists N\forall n(n > N \Rightarrow |a_n - L| < \epsilon)$$

where n and N range over positive integers and *epsilon* ranges over positive real numbers.

3. (20 pts.) Knights and Knaves
Monique is either a knight or a knave. Knights always tell the truth, and only the truth; knaves always tell falsehoods, and only falsehoods. Someone asks Monique, "Are you a knight?" She replies, "If I am a knight, then I'll stand on my head."

A. yes

B.

$$\begin{aligned} P &\implies (P \implies Q) \\ \neg P &\implies \neg(P \implies Q) \end{aligned}$$

C. Using proof by enumeration, prove that your answer from part (a) follow from the premises you wrote in part (b). (No inference rules allowed)

P	Q	$P \Longrightarrow Q$	$P \Longrightarrow (P \Longrightarrow Q)$	$\neg P \Longrightarrow \neg(P \Longrightarrow Q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	T	F

For both implications to be satisfied, ,P and Q must both be true. Therefore, Monique must stand on her head.

4. (5 pts.) Unicorns!

Given the following, can you prove that the unicorn is mythical? How about magical? Horned?

> If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

P1. $Mythical \Rightarrow Immortal$

P2. $\neg Mythical \Rightarrow \neg Immortal \wedge Mammal$

P3. $Immortal \vee Mammal \Rightarrow Horned$

P4. $Horned \Rightarrow Magical$

1. $\neg Immortal \Rightarrow \neg Mythical$ — P1 contrapositive
2. $\neg Immortal \Rightarrow \neg Immortal \wedge Mammal$ — 1, P2, hypothetical syllogism
3. $Immortal \vee (\neg Immortal \wedge Mammal)$ — 2, implication defn
4. $(Immortal \vee \neg Immortal) \wedge (Immortal \vee Mammal)$ — 3, distributive
5. $Immortal \vee Mammal$ — 4, tautology, identity
6. $Horned$ — 5, P3, modus ponens
7. $Mythical$ — 6, P4, modus ponens

So the unicorn is horned and mythical. However, there is no way to show the unicorn is mythical.

5. (15 pts.) Inference rules

(a) P = "Kangaroos live in Australia."
Q = "Kangaroos are marsupials."
Rule: $\frac{P \wedge Q}{Q}$, *and-elimination.*

(b) P = "It is hotter than 100 degrees today."
Q = "the pollution is dangerous."
Rule: $\frac{P \vee Q, \quad \neg P}{Q}$, *disjunctive syllogism.*

(c) P = "Linda is an excellent swimmer."
Q = "Linda can work as a lifeguard."
Rule: $\frac{P \Rightarrow Q, P}{Q}$, *modus ponens.*

(d) $P =$ "Steve will work at a computer company this summer."
$Q =$ "He will be a beach bum."
Rule: $\frac{P}{P \vee Q}$, *addition.*

(e) $P =$ "I work all night on this homework."
$Q =$ "I will answer all the exercises."
$R =$ "I will understand the material."
Rule: $\frac{P \Rightarrow Q,\ \ Q \Rightarrow R}{P \Rightarrow R}$, *hypothetical syllogism.*

6. (20 pts.) Setting up a minesweeper problem

7. (25 pts.) Truth Tables

Columns `i` and `m` are given:

$$\texttt{i} \ : \ p \,|\, q$$
$$\texttt{m} \ : \ p \,|\, p \ .$$

The same trick we used for `m` also holds for `k`, which is $\neg q$:

$$\texttt{k} : \ q \,|\, q \ ,$$

and combining `i` and `m` yields the "and" of column `h`:

$$\texttt{h} : \ (p \,|\, q) \,|\, (p \,|\, q) \ .$$

We now just need to find half of the solutions, and the negations of these can be derived by this same trick.

Since the "nand" of column `i` has one `T` and three `F` in it, it is easy to obtain the other columns with this property by "switching" rows. For instance, the implication of column `e` resembles the "nand" of column `i` except that its first and second row are switched. If you look at the "input" columns of the truth table (the first two, with headers p and q), you see that this switch can be achieved by replacing q with $\neg q$. This also switches the third and fourth row of the truth table, but this does not matter, because these rows in `e` are equal to each other, and so are the corresponding rows in `i`. Thus, `e` can be implemented as $p \,|\, \neg q$, that is,

$$\texttt{e} : \ p \,|\, (q \,|\, q) \ .$$

Exchanging the role of p and q yields the converse of `e`, that is, `c`:

$$\texttt{c} : \ (p \,|\, p) \,|\, q \ .$$

Column `b` is the truth table for the inclusive "or," and is the same as the "nand" of column `i` read from the bottom up. Therefore, this truth table can be obtained by negating both p and q:

$$\texttt{b} : \ (p \,|\, p) \,|\, (q \,|\, q) \ .$$

Note that this is consistent with De Morgan's rule: to "or" propositions p and q we negate p (this is $p \,|\, p$) and q (this is $q \,|\, q$), "and" the results together, and negate the result of that (the negation of "and" is "nand").

Columns `l`, `n`, and `o` can now be obtained by negating ("nand"-ing with themselves) columns `e`, `c`, and `b`, respectively:

$$\begin{array}{lll} \texttt{l} & : & (p \,|\, (q \,|\, q)) \,|\, (p \,|\, (q \,|\, q)) \\ \texttt{n} & : & ((p \,|\, p) \,|\, q) \,|\, ((p \,|\, p) \,|\, q) \\ \texttt{o} & : & ((p \,|\, p) \,|\, (q \,|\, q)) \,|\, ((p \,|\, p) \,|\, (q \,|\, q)) \end{array}$$

We are left with six columns: tautology (column `a`), contradiction (column `r`), and the four tables that have two `T`s and two `F`s (columns `d`, `f`, `g`, and `j`).

Tautology can be written as $p \vee (\neg p)$, so we can resort to column `b` (for "or") and negation:

$$\texttt{a}: \ (p \,|\, p) \,|\, ((p \,|\, p) \,|\, (p \,|\, p)) \ .$$

Similarly, contradiction can be written as $p \wedge (\neg p)$, which is the "and" (column `h`) of p and $p \,|\, p$:

$$\texttt{r}: \ (p \,|\, (p \,|\, p)) \,|\, (p \,|\, (p \,|\, p)) \ .$$

Of the remaining columns, `d` and `f` are trivially copies of the inputs p and q, respectively:

$$\begin{array}{lll} \texttt{d} & : & p \\ \texttt{f} & : & q \ . \end{array}$$

The remaining two columns are the exclusive "or" (column `g`) and the bi-implication (column `j`). These are negations of each other, so we just need to find one, and derive the other by "nand"-ing the first by itself. Bi-implication differs by the "nand" (column `i`) in its last row, and can therefore be expressed as follows:

$$\begin{array}{cl} \texttt{j} \ : & (p \,|\, q) \wedge \neg(\neg p \wedge \neg q) \\ \equiv & (p \,|\, q) \wedge (\neg p \,|\, \neg q) \\ \equiv & \neg((p \,|\, q) \,|\, (\neg p \,|\, \neg q)) \\ \equiv & \neg((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q))) \\ \equiv & ((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q))) \,|\, ((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q))) \ . \end{array}$$

Finally, by negating `j` (that is, "nand"-ing it with itself) we obtain the exclusive "or:"

$$\texttt{g}: \ (((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q))) \,|\, ((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q)))) \,|\, (((p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q))) \,|\, ((p \,|\, q) \,|\, ((p \,|\, p) \,|\, ($$

A simpler way to obtain the exclusive "or" of column `g` is to take the "nand" of column `i` (the "nand") and column `b` (the "or"):

$$\begin{array}{cl} \texttt{g} \ : & (p \,|\, q) \,|\, (p \vee q) \\ \equiv & (p \,|\, q) \,|\, ((p \,|\, p) \,|\, (q \,|\, q)) \ . \end{array}$$

In summary, here is one version of the complete table:

$$
\begin{array}{lll}
\texttt{a} & : & (p\,|\,p)\,|\,((p\,|\,p)\,|\,(p\,|\,p)) \\
\texttt{b} & : & (p\,|\,p)\,|\,(q\,|\,q) \\
\texttt{c} & : & (p\,|\,p)\,|\,q \\
\texttt{d} & : & p \\
\texttt{e} & : & p\,|\,(q\,|\,q) \\
\texttt{f} & : & q \\
\texttt{g} & : & (p\,|\,q)\,|\,((p\,|\,p)\,|\,(q\,|\,q)) \\
\texttt{h} & : & (p\,|\,q)\,|\,(p\,|\,q) \\
\texttt{i} & : & p\,|\,q \\
\texttt{j} & : & ((p\,|\,q)\,|\,((p\,|\,p)\,|\,(q\,|\,q)))\,|\,((p\,|\,q)\,|\,((p\,|\,p)\,|\,(q\,|\,q))) \\
\texttt{k} & : & q\,|\,q \\
\texttt{l} & : & (p\,|\,(q\,|\,q))\,|\,(p\,|\,(q\,|\,q)) \\
\texttt{m} & : & p\,|\,p \\
\texttt{n} & : & ((p\,|\,p)\,|\,q)\,|\,((p\,|\,p)\,|\,q) \\
\texttt{o} & : & ((p\,|\,p)\,|\,(q\,|\,q))\,|\,((p\,|\,p)\,|\,(q\,|\,q)) \\
\texttt{r} & : & (p\,|\,(p\,|\,p))\,|\,(p\,|\,(p\,|\,p))\ .
\end{array}
$$

Other implementations are possible.