Multiple regression with interaction terms

Luis Castro-de-Araujo[a]

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Virginia Institute for Psychiatric and Behavioral Genetics

[a] Post-doc T32. luis.araujo@vcuhealth.org

Multiple regression recap

Interaction terms

Visualizing interactions

Marginal effects

Multiple regression recap

Multiple Linear Regression

- Extension of the simple linear regression model to two or more independent variables

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p + \epsilon$$

- Expression = Baseline + Age + Tissue + Sex + Error
- Partial Regression Coefficients: effect on the dependent variable when increasing the i^{th} independent variable by 1 unit, **holding all other predictors constant**

Categorical Independent Variables

- Qualitative variables are easily incorporated in regression framework through ***dummy variables***
- Simple example: sex can be coded as 0/1
- What if my categorical variable contains three levels:

$$x_1 = \begin{cases} 0 \text{ if } \mathrm{AA} \\ 1 \text{ if } \mathrm{AG} \\ 2 \text{ if } \mathrm{GG} \end{cases}$$

Categorical Independent Variables

- Previous coding would result in ***colinearity***
- Solution is to set up a series of dummy variable.
- for k levels you need k-1 dummy variables

	x1	x2
AA	1	0
AG	0	1
GG	0	0

$$x_1 = \begin{cases} 1 \text{ if } AA \\ 0 \text{ otherwise} \end{cases}$$

$$x_2 = \begin{cases} 1 \text{ if } AG \\ 0 \text{ otherwise} \end{cases}$$

ASSUMPTIONS

Validity Does the data we're modeling matches the problem we're actually trying to solve?

Representativeness Is the sample data used in the regression model representative of the population to which it will be applied?

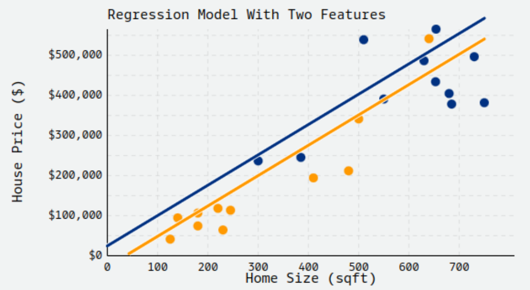
Additivity and Linearity The deterministic component of a regression model is a linear function of the separate predictors: $y = B_0 + B_1x_1 + \ldots + B_px_p$

Independence of Errors The errors from our model are independent.

Homoscedasticity The errors from our model have equal variance.

Normality of Errors The errors from our model are normally distributed.

MULTIVARIATE REGRESSION

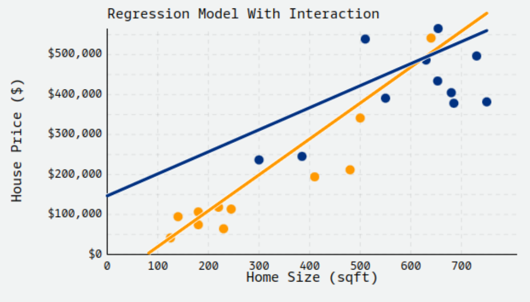


$$houseprice = -27154 + 757 * sqft + 51867 * pool$$

- In our example, we model home prices as a function of both the size of the house (sqft) and whether or not it has a pool

- intercept: -$27,154, the predicted average housing price for houses with all $x_i = 0$. Or the cost of houses with no pools and a square-footage of zero.
- coefficient of pool: $51,867, average expected price difference in houses of the same size (in sqft) if they do or do not have a pool. In other words, we expect, on average, houses of the same size to cost $51,867 more if they have a pool than if they do not.
- coefficient of sqft: $757, average expected price difference in housing price for houses that have the same value of pool but differ in size by one square-foot.
- We assume the same slope for sqft.Hence, two lines. This isn't always a valid assumption to make.

Back to our housing example, now with interactions



$$houseprice = -70296 + 899 * sqft + 217111 * pool - 347 * (sqft : pool)$$

- If we believe that the slope for sqft should differ between houses that do have pools and houses that do not, we can add an interaction term to our model, (sqft:pool).

- interaction term: -$347, represents the difference in the slope for sqft, comparing houses that do and do not have pools. Visually, this represents the difference between the slopes of the two lines.
- intercept: -$70,296, represents the predicted housing price for houses with no pools and a square-footage of zero.
- coefficient of pool: $217,111, represents the average expected difference in houses of the same size (0 sqft) that differed in whether or not they had a pool. (It's not super useful since we don't have houses with 0 square-feet).
- coefficient of sqft: $899, represents the average expected difference in housing price for houses that do not have a pool (pool= 0) but differ in size by one square-foot.

Interaction terms

What is an Interaction?

- An interaction is a predictor that is some combination of the other predictors.

Constructing an Interaction

- Interactions are often the product of two or more predictors.
- Can be written as,

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + \epsilon$$

Conditional vs. Marginal Effects

- Conditional effects: the effect of a predictor on the response, holding all other predictors constant.
- Marginal effects: the effect of a predictor on the response, averaged over all values of the other predictors.

- If the conditional effects of X1 on Y at different levels of X2 are all the same then there is no interaction.

Interpreting Interaction Effects

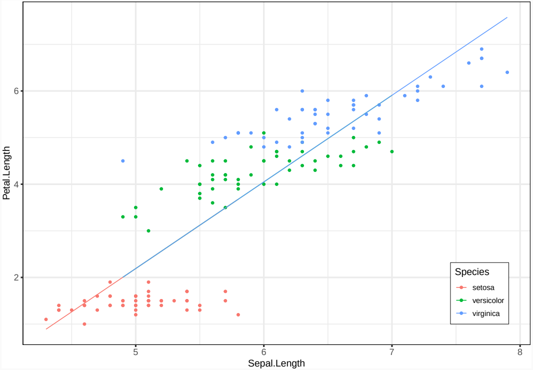
Parameter	Meaning	Where people (used to) go awry
β_0	Expected value of the DV when X1 and X2 ==0	People get this
β_1	Effect of X1 when X2 == 0	Not marginal effects!
β_2	Effect of X2 when X1 == 0	Not marginal effects!
β_3	The addition to the conditional effect when both X! and X2 are 1	People just look at the significance of the interaction parameter and do not calculate the underlying marginal or conditional effects or standard errors

In the past it was common to see standard errors wrongly calculated

- A common mistake that people make when interpreting interaction models is using the wrong standard errors.
- The standard errors that are printed in every regression table are the positive square roots of the diagonal elements of the variance- covariance matrix of β
- This does not matter anymore because of `margins()`

Interaction and dummmy variable, Iris flower data set

$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i$$



Creating the dummy

$$\text{setosa}_i = \begin{cases} 1 \text{ if species of flower i = setosa}, \forall i \in [1, 150] \\ 0 \text{ otherwise} \end{cases}$$

$$\text{versicolor}_i = \begin{cases} 1 \text{ if species of flower i = versicolor}, \forall i \in [1, 150] \\ 0 \text{ otherwise} \end{cases}$$

Our formula is then

$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa}_i + \beta_3 \text{ versicolor}_i$$

By substitution we get three lines with same slope

If it is setosa

$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa }_i + \beta_3 \text{ versicolor }_i$$
$$= \beta_0 + \beta_1 sepal.length_i + \beta_2 1 + \beta_3 0$$
$$= \beta_0 + \beta_1 sepal.length_i + \beta_2$$
$$= (\beta_0 + \beta_2) + \beta_1 sepal.length_i$$

If it is versicolor

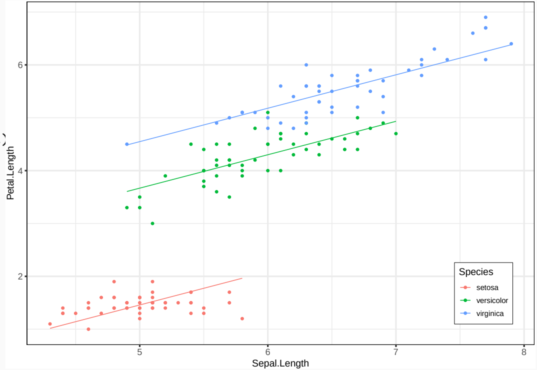
$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa }_i + \beta_3 \text{ versicolor }_i$$
$$= \beta_0 + \beta_1 sepal.length_i + \beta_2 0 + \beta_3 1$$
$$= \beta_0 + \beta_1 sepal.length_i + \beta_3$$
$$= (\beta_0 + \beta_3) + \beta_1 sepal.length_i$$

If it is virginica

$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa }_i + \beta_3 \text{ versicolor }_i$$
$$= \beta_0 + \beta_1 sepal.length_i + \beta_2 0 + \beta_3 0$$
$$= \beta_0 + \beta_1 sepal.length_i$$
$$= \beta_0 + \beta_1 sepal.length_i$$

Same slope, different intercepts

```r
iris$pred <- predict(lm(Petal.Length ~ Sepal.Length+Species,
                        data = iris))

# plot in ggplot
ggplot(iris, aes(x = Sepal.Length, y = Petal.Length, color = Species)) +
  geom_point() +
  geom_line(aes(Sepal.Length, pred )) +
  theme_luis() +
  theme( legend.position = c(0.9, 0.15))
```



Now we can add an interaction

$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa }_i + \beta_3 \text{ versicolor }_i$$
$$+ \beta_4 sepal.length_i \text{ setosa }_i + \beta_5 sepal.length_i \text{ versicolor }_i$$

- this will result in three unique lines depending on the species of the flower.
- both the intercepts and the slopes will be allowed to be different.

Does it make sense to retain the interaction?

```
inter <- lm(Petal.Length ~ Sepal.Length + Species + Sepal.Length:Species, data = iris)
```

Does it make sense to retain the interaction?

```r
broom::tidy(inter) |> kable()
```

```
Registered S3 methods overwritten by 'broom':
  method            from
  tidy.glht         jtools
  tidy.summary.glht jtools
```

term	estimate	std.error	statistic	p.value
(Intercept)	0.803	0.531	1.512	0.133
Sepal.Length	0.132	0.106	1.244	0.216
Speciesversicolor	-0.618	0.684	-0.904	0.368
Speciesvirginica	-0.193	0.658	-0.293	0.770
Sepal.Length:Speciesversicolor	0.555	0.128	4.330	0.000
Sepal.Length:Speciesvirginica	0.618	0.121	5.111	0.000

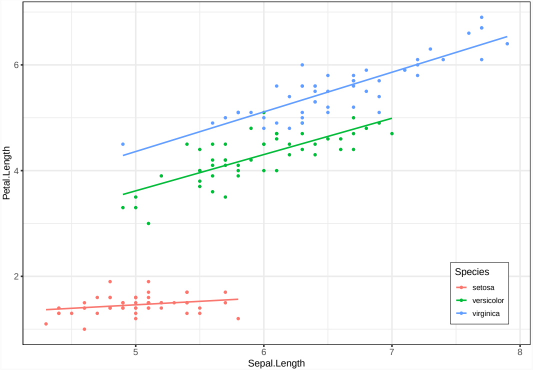
```r
anova(nospecies, w_species, inter)
```

```
Analysis of Variance Table

Model 1: Petal.Length ~ Sepal.Length
Model 2: Petal.Length ~ Sepal.Length + Species
Model 3: Petal.Length ~ Sepal.Length + Species + Sepal.Length:Species
  Res.Df   RSS Df Sum of Sq      F              Pr(>F)
1    148 111.5
```

Now we can add an interaction

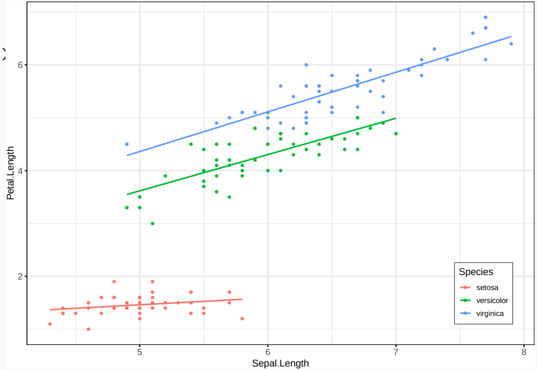
$$\widehat{petal.length}_i = \beta_0 + \beta_1 sepal.length_i + \beta_2 \text{ setosa}_i + \beta_3 \text{ versicolor}_i$$
$$+ \beta_4 sepal.length_i \text{ setosa}_i + \beta_5 sepal.length_i \text{ versicolor}_i$$

- this will result in three unique lines depending on the species of the flower.
- both the intercepts and the slopes will be allowed to be different.
- ggplot geom_smooth does this by default if `color` is used



Now we can add an interaction

```r
ggplot(iris, aes(x = Sepal.Length, y = Petal.Length, color = Species)
  geom_point() +
  geom_smooth(method = "lm", se = FALSE) +
  theme_luis()+
  theme( legend.position = c(0.9, 0.15))
```



VISUALIZING INTERACTIONS

Mtcars data set - regression of speed on wt*cyl

```
fit1 <- glm(qsec ~ wt*as.factor(cyl), data = mtcars)
broom::tidy(fit1) |> kable()
```

- Note: not significant, but we will return to this later
  `summary(margins(fit1))`

term	estimate	std.error	statistic	p.value
(Intercept)	14.83	1.500	9.884	0.000
wt	1.89	0.639	2.951	0.007
as.factor(cyl)6	-9.78	4.395	-2.226	0.035
as.factor(cyl)8	-1.44	2.273	-0.632	0.533
wt:as.factor(cyl)6	2.26	1.464	1.546	0.134
wt:as.factor(cyl)8	-1.04	0.764	-1.360	0.185

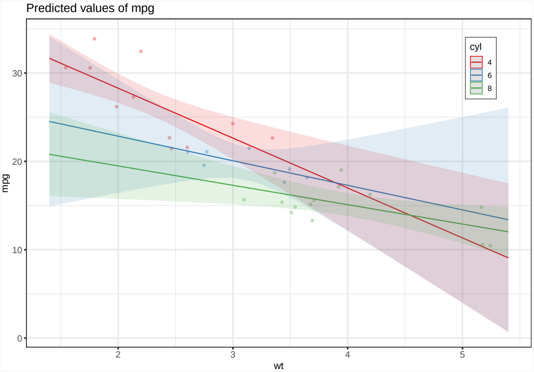
Regression of mpg on wt*cyl

```
fit <- glm(mpg ~ wt*as.factor(cyl), data = mtcars)
broom::tidy(fit) |> kable()
```

term	estimate	std.error	statistic	p.value
(Intercept)	39.57	3.19	12.39	0.000
wt	-5.65	1.36	-4.15	0.000
as.factor(cyl)6	-11.16	9.36	-1.19	0.244
as.factor(cyl)8	-15.70	4.84	-3.25	0.003
wt:as.factor(cyl)6	2.87	3.12	0.92	0.366
wt:as.factor(cyl)8	3.46	1.63	2.12	0.043

Regression of mpg on wt*cyl

```r
pred <- ggpredict(fit, terms = c("wt", "cyl"))
plot(pred, add.data = TRUE)+
  theme_luis()+
  theme( legend.position = c(0.1, 0.15))
```

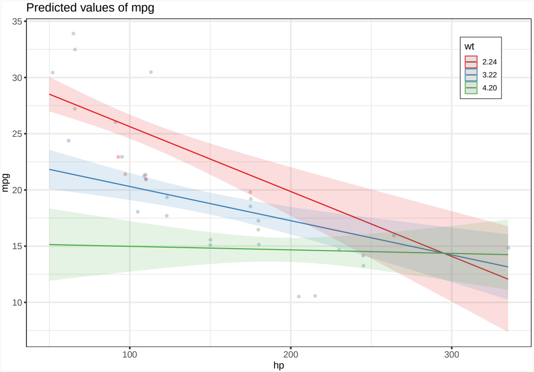


AN INTERACTION MAY NOT BE SIG ACROSS THE ENTIRE RANGE OF THE PREDICTOR

Let's see mpg ~ hp + wt

```
fit <- glm(mpg ~ hp*wt, data = mtcars)
pred <- ggpredict(fit, terms = c("hp", "wt"))

# plot(pred, add.data = TRUE) +
#   theme_luis() +
#   theme( legend.position = c(0.9, 0.85))

broom::tidy(fit) |> kable()
```

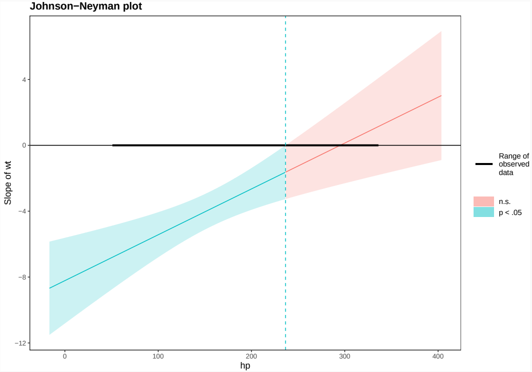
term	estimate	std.error	statistic	p.value
(Intercept)	49.808	3.605	13.82	0.000
hp	-0.120	0.025	-4.86	0.000
wt	-8.217	1.270	-6.47	0.000
hp:wt	0.028	0.007	3.75	0.001



Now, from what point the slope becomes non signficant?

JOHNSON-NEYMAN INTERVAL

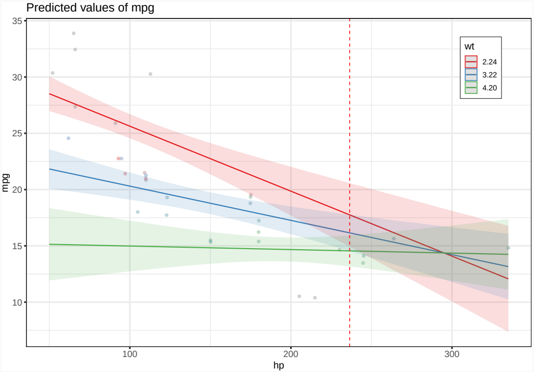
```r
jn <- johnson_neyman(fit, wt, hp , plot = TRUE)
jn
```



Now, from what point the slope becomes non signficant?

JOHNSON-NEYMAN INTERVAL - Overlayed on data

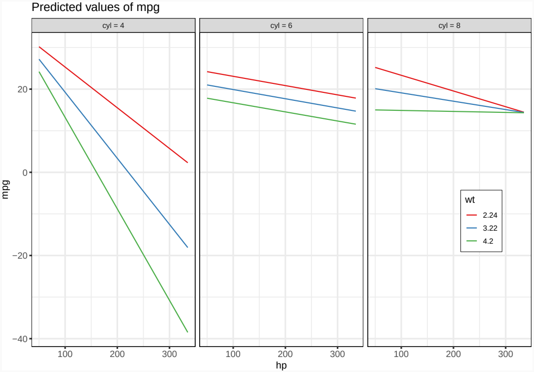
```
fit <- glm(mpg ~ hp*wt, data = mtcars)
pred <- ggpredict(fit, terms = c("hp", "wt"))
jn <- johnson_neyman(fit, wt, hp , plot = TRUE)
jn_bound <- as.numeric(jn$bounds[1])

plot(pred, add.data=T) +
  geom_vline(xintercept = jn_bound, linetype = "dashed",
             color = "red")+
  theme_luis() +
  theme( legend.position = c(0.9, 0.85))
```



Three-way interactions

```
fit <- glm(mpg ~ hp*wt*as.factor(cyl), data = mtcars)
```

```
dat <- ggpredict(fit, terms = c("hp", "wt", "cyl"))
plot(dat, ci = FALSE)+
  theme( legend.position = c(0.9, 0.85))
```



The summary

```
broom::tidy(fit) |> kable()
```

term	estimate	std.error	statistic	p.value
(Intercept)	34.972	10.883	3.213	0.004
hp	0.042	0.129	0.324	0.750
wt	0.054	5.010	0.011	0.991
as.factor(cyl)6	-2.340	77.278	-0.030	0.976
as.factor(cyl)8	5.829	42.608	0.137	0.893
hp:wt	-0.062	0.058	-1.068	0.298
hp:as.factor(cyl)6	-0.064	0.660	-0.098	0.923
hp:as.factor(cyl)8	-0.120	0.227	-0.527	0.604
wt:as.factor(cyl)6	-3.321	27.372	-0.121	0.905
wt:as.factor(cyl)8	-6.168	12.492	-0.494	0.627
hp:wt:as.factor(cyl)6	0.062	0.236	0.264	0.794
hp:wt:as.factor(cyl)8	0.080	0.078	1.026	0.317

MARGINAL EFFECTS

- Marginal effects: the effect of a predictor on the response, averaged over all values of the other predictors.
- also, the instantaneous effect of “x” on “y”. Calculated with a derivative of x in respect to y, expressed mathematically as dy/dx .
  - the “instantaneous rate of change of y with respect to x.

What are the marginal effects for `mpg ~ hp*wt`

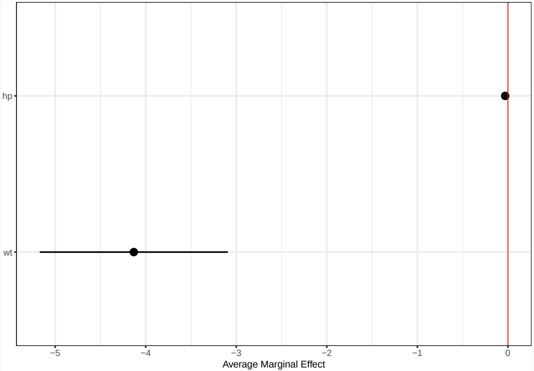
```r
fit_m <- margins(fit2)
summary(fit_m)
```

```
 factor     AME     SE       z      p   lower   upper
     hp -0.0305 0.0075 -4.0661 0.0000 -0.0452 -0.0158
     wt -4.1316 0.5296 -7.8021 0.0000 -5.1696 -3.0937
```

- Interpretation: for each unit increase of mpg an average reduction of `hp` 3% is expected.

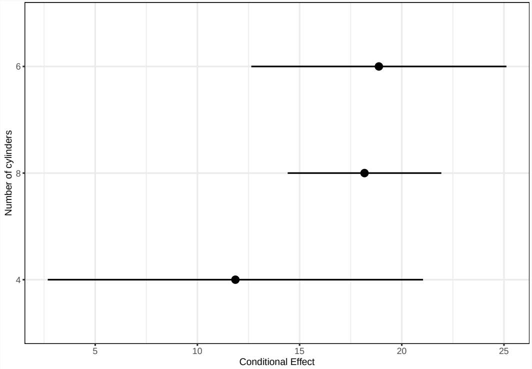
Plotting the marginal effects

```
fit_mo <- as_tibble(summary(fit_m))
p <- ggplot(data = fit_mo, aes(x = reorder(factor, AME),
                               y = AME, ymin = lower, ymax = upper))

p + geom_hline(yintercept = 0, color = "gray80") +
    geom_pointrange(size = 1) + coord_flip() +
    labs(x = NULL, y = "Average Marginal Effect") +
  geom_hline(yintercept = 0, color = "red") +
    theme_luis()
```



Can we plot the conditional effects too?

```
glm(mpg ~ hp*wt*as.factor(cyl), data = mtcars) %>%
  cplot(x = "cyl", draw = F) %>%
  ggplot( aes(x = reorder(xvals, yvals),
                    y = yvals, ymin = lower, ymax = upper))+
                    geom_pointrange(size = 1 ) + coord_flip() +
    labs(x = "Number of cylinders", y = "Conditional Effect") +
    theme_luis()
```



- At the beginning I said, keep the interaction that is significant, but[1]
    - In the transformation from log-odds to probabilities (AME) the interaction may not be significant anymore, or worse
    - The interaction may be significant in the AME, but not in conditional effects

[1] Bruin, "Deciphering Interactions in Logistic Regression," *Introduction to SAS. UCLA: Statistical Consulting Group.*; Vanhove, 2019, "Interactions in Logistic Regression Models," (2019).

Example when the conditional is not significant while the AME is

speed on wt*cyl

```r
fit1 <- glm(qsec ~ wt*as.factor(cyl), data = mtcars)
broom::tidy(fit1) |> kable()
```

term	estimate	std.error	statistic	p.value
(Intercept)	14.83	1.500	9.884	0.000
wt	1.89	0.639	2.951	0.007
as.factor(cyl)6	-9.78	4.395	-2.226	0.035
as.factor(cyl)8	-1.44	2.273	-0.632	0.533
wt:as.factor(cyl)6	2.26	1.464	1.546	0.134
wt:as.factor(cyl)8	-1.04	0.764	-1.360	0.185

```r
summary(margins(fit1)) |> kable()
```

factor	AME	SE	z	p	lower	upper
cyl6	-2.50	0.825	-3.03	0.002	-4.12	-0.884
cyl8	-4.78	0.823	-5.81	0.000	-6.39	-3.169
wt	1.93	0.406	4.74	0.000	1.13	2.721

```r
library(umx)

model <- "
  mpg ~ hp*hp + wt*wt
  moderation := hp*wt
"

m1 <- umxRAM(model, data = mtcars)
m1 <- umxCI(m1)
umxSummary(m1, std = T)
```

Table 10: Parameter loadings for model 'm1'

	name	Std.Estimate	Std.SE	CI	type
5	hp_with_wt	0.66	0.1	0.66 [0.46, 0.85]	Manifest Cov
1	hp	-0.36	0.1	-0.36 [-0.56, -0.17]	Manifest path
2	wt	-0.63	0.09	-0.63 [-0.81, -0.45]	Manifest path
7	one_to_mpg	6.28	0.7	6.28 [4.91, 7.64]	Mean
8	one_to_hp	2.17	0.32	2.17 [1.54, 2.81]	Mean
9	one_to_wt	3.34	0.45	3.34 [2.45, 4.23]	Mean
3	mpg_with_mpg	0.17	0.06	0.17 [0.06, 0.28]	Residual
4	hp_with_hp	1.00	0	1 [1, 1]	Residual
6	wt_with_wt	1.00	0	1 [1, 1]	Residual

Model Fit: Chi2(0) = 0, p = 1.000; CFI = 1; TLI = 1; RMSEA = 0 Algebra'moderation' = 0.123CI95[0.074, 0.173]. p-value < 0.001

CONCLUSION

- We started reviewing multiple regression
- Then discussed the syntax and interpretation of parameters when an interaction term is included
- Finally, we discussed how to extract the marginal effects of the interaction term
- Luckly the package margins() makes this extremely simple.

ACKNOWLEDGEMENTS

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Contact

- Thank you