# CS5014 Machine Learning

## Lecture 2 Maths background review

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21 Jan 2021

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# So why this review session ?

Maths is useful

- rigorous and concise way of communicating results
- help us understand why *and* why not algorithms work
- be able to derive your own model and algorithms!

Refresher on essential concepts

- only a refresher; we expect you have learnt them
  - don't expect to know everything after this lecture
- not complete and not rigorous

Self-assessment for yourself

- identify rusty area
- do self studies afterwards
- maths learning should be never-ending :-)

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# Mathematics for machine learning

Linear algebra

- leap forward from elementary algebra: 1-d to multi-dimensional
- number line to a number plane (space)

Probability theory and statistics

- study of uncertainty: uncertainty is the norm
  - e.g. rain tomorrow? blood pressure measurement (reading error)?
- how to generalise your results
  - from one sample to the universe: vaccine trial

Calculus

- study of continuous (real-valued) functions (using approximation, say *polynomial*)
  - $y = sin(x)$ is well approximated by $y = x$ when $x \approx 0$
- useful when we do optimisation

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# Useful textbook and references (read the italic entries!)

**Linear algebra**

- *Learning from Data Supplementary Mathematics (Vector and Linear Algebra) by David Barber*; https://api.semanticscholar.org/CorpusID:18857001
- *Chapter 2 of Deep Learning by Ian Goodfellow, Yoshua Bengio and Aaron Courville* https://www.deeplearningbook.org/contents/linear_algebra.html
- Introduction to Linear Algebra by Gilbert Strang; http://math.mit.edu/~gs/linearalgebra/
- The Matrix Cookbook by Kaare Brandt Petersen, Michael Syskind Pedersen; https://www2.imm.dtu.dk/pubdb/pubs/3274-full.html
  - useful as a reference manual

**Probability theory**

- *Chapter 2.1-2.3 Information Theory, Inference, and Learning Algorithms by David J.C. MacKay*
  http://www.inference.org.uk/itprnn/book.pdf
- *Chapter 3.1-3.9 of Deep Learning by Ian Goodfellow, Yoshua Bengio and Aaron Courville*
  https://www.deeplearningbook.org/contents/prob.html
- Introduction to Probability Models by Sheldon Ross
  - chapter 1; chapter 2.1-2.5, 2.8; chapter 3.1-3.5

**Calculus**

- Use your book of choice; read multivariate calculus part as well
- Appendix of Bayesian Reasoning and Machine Learning by David Barber
  http://web4.cs.ucl.ac.uk/staff/D.Barber/textbook/200620.pdf

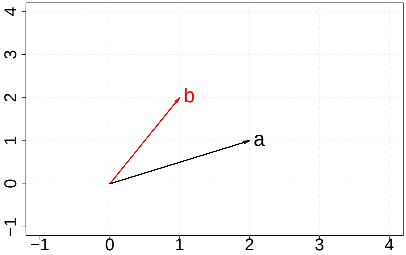
# Linear algebra: Basic concepts

- vectors
- norms and distances
- linear independence, span, subspace
- matrices, linear transformation
- matrix operations
- rank

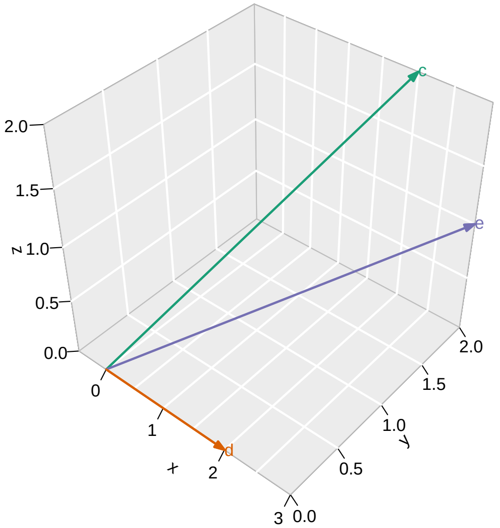
# Vector

A vector is a collection of $n$ salars

- $\boldsymbol{a} \in R^n$, default option is column vector i.e. $n \times 1$
- represents a **displacement** in $R^n$
- e.g. $\boldsymbol{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (or $\boldsymbol{a} = [2, 1]^T$ to save space)



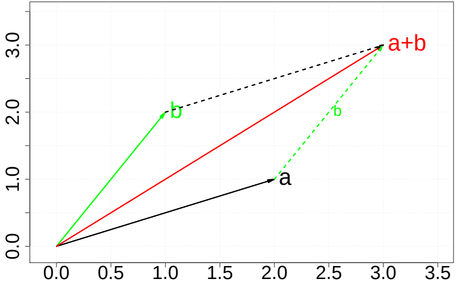
Some 3-d vectors $\boldsymbol{c} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \boldsymbol{d} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$



# Vector addition

$$\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_d + b_d \end{bmatrix}$$

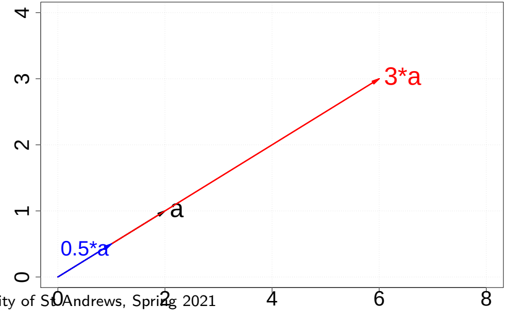
- generalisation from scalar arithmetics; remember 2+1 on a number axis ?
- parallelogram rule



# Vector scaling/multiplication

$$k \cdot \boldsymbol{a} = k \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} = \begin{bmatrix} k \times a_1 \\ k \times a_2 \\ \vdots \\ k \times a_d \end{bmatrix}, k \in R \text{ or a scalar}$$

- geometrically, scaling means shrinking or streching a vector
  - the direction does not change but length changes
- and obviously $n \cdot \boldsymbol{a} = \boldsymbol{a} + \ldots + \boldsymbol{a} = \sum_n \boldsymbol{a}$
- $0 \cdot \boldsymbol{a} = \mathbf{0}$



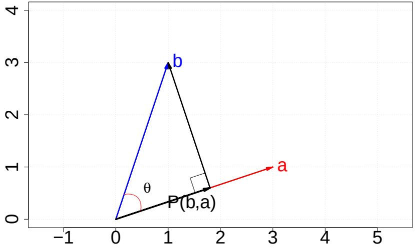
# Inner product

$$\boldsymbol{a}^T\boldsymbol{b} = [a_1, a_2 \ldots, a_d] \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix} = \sum_{i=1}^{d} a_i \times b_i$$

- $\boldsymbol{a}^T\boldsymbol{b} = \boldsymbol{b}^T\boldsymbol{a}$ and the result is a scalar
- $\boldsymbol{a}^T(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}^T\boldsymbol{b} + \boldsymbol{a}^T\boldsymbol{c}$
- $(k\boldsymbol{a})^T\boldsymbol{b} = \boldsymbol{a}^T(k\boldsymbol{b}) = k(\boldsymbol{a}^T\boldsymbol{b})$
- $\boldsymbol{a}^T\boldsymbol{a} = \sum_{i=1}^{d} a_i^2$ is squared Euclidean distance between $\boldsymbol{a}$ and $\mathbf{0}$
- $\boldsymbol{a}^T\boldsymbol{a} \geq 0$ and $\boldsymbol{a} = \mathbf{0}$ if and only if $\boldsymbol{a}^T\boldsymbol{a} = 0$

Another interpretation:

$$\boldsymbol{a}^T\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|cos\theta$$




- $\theta$ is the angle between $\boldsymbol{a}, \boldsymbol{b}$
  - $\boldsymbol{a}^T\boldsymbol{b} = 0$ if and only if $\boldsymbol{a} \perp \boldsymbol{b}$
- $\|\boldsymbol{a}\|cos\theta$ is the projected length of $\boldsymbol{a}$ on $\boldsymbol{b}$
- $\|\boldsymbol{b}\|cos\theta$ is the projected length of $\boldsymbol{b}$ on $\boldsymbol{a}$
- $P(\boldsymbol{b}, \boldsymbol{a})$ denotes the projected vector of $\boldsymbol{b}$ to $\boldsymbol{a}$
  - so $\|\boldsymbol{b}\|cos\theta = \|P(\boldsymbol{b}, \boldsymbol{a})\|$
- and (prove it or convince yourself!)

$$P(\boldsymbol{b}, \boldsymbol{a}) = \|\boldsymbol{b}\|cos\theta * \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a}$$

# Matrix

A rectangular array of real numbers $A \in R^{m \times n}$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} — & \tilde{\boldsymbol{a}}_1 & — \\ \vdots & \vdots & \vdots \\ — & \tilde{\boldsymbol{a}}_m & — \end{bmatrix}$$

- can be viewed as a collection of n column vectors $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n]$;
- or row vectors $\boldsymbol{A} = [\tilde{\boldsymbol{a}}_1^T, \tilde{\boldsymbol{a}}_2^T, \dots, \tilde{\boldsymbol{a}}_m^T]^T$
- sometimes written as $\boldsymbol{A} = (a_{ij})$ $i = 1, \dots m, j = 1, \dots, n$

# Matrix operations

- addition: $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{C} = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$
- scaling: $k\boldsymbol{A} = \boldsymbol{C}$ where $c_{ij} = k * a_{ij}$
- transpose: $\boldsymbol{A}^T = \boldsymbol{C}$ where $c_{ij} = a_{ji}$
- multiplication: Let $\boldsymbol{A} \in R^{m\times s}, \boldsymbol{B} \in R^{s\times n}$

  $$\boldsymbol{AB} = \boldsymbol{C}, \boldsymbol{C} \in R^{m\times n}$$

  where

  $$c_{ij} = \sum_{k=1}^{s} a_{ik} b_{jk}$$

  or $c_{ij} = \tilde{\boldsymbol{a}}_i^T \boldsymbol{b}_j$
  - $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
  - $\boldsymbol{AB} \neq \boldsymbol{BA}$
  - $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
  - $\boldsymbol{I}$ identity matrix: $\boldsymbol{IA} = \boldsymbol{A}$ or $\boldsymbol{AI} = \boldsymbol{A}$
- inverse (only applies to square matrix): $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{AA}^{-1} = \boldsymbol{I}$

# Examples

$$\begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = ?$$

it is not allowed as the dimensions do not match

$$\begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 1 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 6 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

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# Example

$$\begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{2 \times 5 + 3 \times 1} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 7 & 10 \\ 34 & 16 & 20 \\ 6 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 3 \\ 6 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 25 \\ 10 & 9 \end{bmatrix}$$

# Example

The inverse of $\boldsymbol{A} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ is $\boldsymbol{A}^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix}$ as $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$

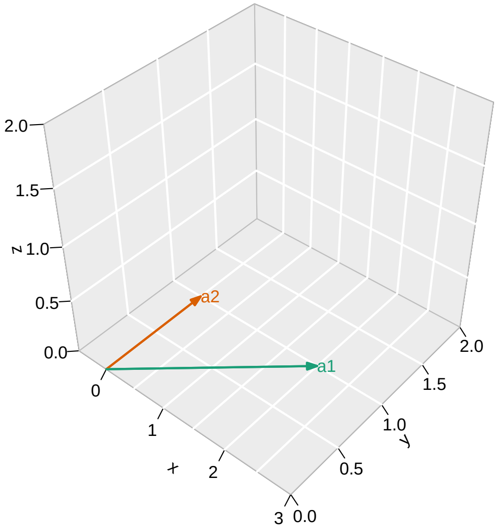
The inverse of $\boldsymbol{I}$ is itself $\boldsymbol{I}^{-1} = \boldsymbol{I}$

# Span, linear independence

- **linear combination** is just sum of some scaled vectors
  - $\lambda_1 \cdot \boldsymbol{a}_1 + \lambda_2 \cdot \boldsymbol{a}_2 + \ldots + \lambda_n \boldsymbol{a}_n$, $\boldsymbol{a}_i \in R^m$ for $i = 1, \ldots, n$
  - $\boldsymbol{a}_i$ are vectors (of the same length) and $\lambda_i$ are the scalars
- **span** is the set of all possible linear combination

$$\text{Span}(\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\}) = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{a}_i | \lambda_i \in R, i = 1, \ldots, n \right\}$$

- what is the span of $\{[1, 0]^T, [0, 1]^T\}$?
- how about $\{[2, 1]^T, [0, 1]^T\}$?
- how about $\{[2, 1]^T, [4, 2]^T\}$?
- how about $\{[2, 1, 0]^T, [0, 1, 0]^T\}$?
  - it is a **subspace** (bottom plane) in $R^3$


2.0
1.5
z 1.0
0.5
0.0
a2
a1
0
1
2
3
x
0.0
0.5
1.0
1.5
2.0
y

- **linear independence**: $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ is linear independent if there exist no $\lambda_1, \ldots, \lambda_n$ (except all being 0) such that

$$\lambda_1 \cdot \boldsymbol{a}_1 + \lambda_2 \cdot \boldsymbol{a}_2 + \ldots + \lambda_n \boldsymbol{a}_n = \mathbf{0}$$

  - how about $\{[2,3]^T, [4,6]^T\}$ ?
  - are $\{[1,0]^T, [0,1]^T\}$ LI?
  - essentially a way to tell whether there is any *redundant* vectors in the set
- **rank** of a matrix is defined as the maximum number of linearly independent column vectors

# Example

The column vectors of the matrix

$$[\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3] = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

are not linearly independent, as

$$\lambda_1 \boldsymbol{a}_1 + \lambda_2 \boldsymbol{a}_2 + \lambda_3 \boldsymbol{a}_3 = \mathbf{0}$$

holds for $\lambda_1 = \lambda_2 = 1, \lambda_3 = -2$. In other words, one of them is redundant. And $\text{rank}(\boldsymbol{A}) = 2$

# Matrix vector multiplication

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} | \\ \boldsymbol{a}_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \boldsymbol{a}_2 \\ | \end{bmatrix} + \dots + x_n \begin{bmatrix} | \\ \boldsymbol{a}_n \\ | \end{bmatrix}$$

- another view of the multiplication
- *linear combination* of the column vectors of $\boldsymbol{A}$
  - $x_1 \cdot \boldsymbol{a}_1 + x_2 \cdot \boldsymbol{a}_2 + \ldots + x_n \boldsymbol{a}_n$
  - $\boldsymbol{a}_i$ are the column vectors and $x$ are the scalars
- so ... $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ essentially solves for what ?

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- so ... $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$ essentially solves for what ?
  - $\boldsymbol{y}$ is in the column space of $\boldsymbol{A}$ or not ...
  - if not, then there is no solution
  - if yes, there will be some solution(s)? unique solution or ?

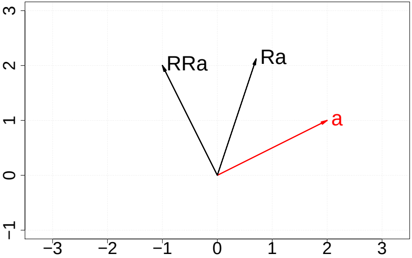
# Matrix vector multiplication (some interpretations)

So $\boldsymbol{Ax}$ is a linear transformation: $\boldsymbol{x} \to \boldsymbol{y}$

- Rotation: rotate $\boldsymbol{x}$ anti-clockwise by $\theta$

$$R\boldsymbol{x} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.71 & -0.71 \\ 0.71 & 0.71 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

say $\theta = \pi/4$




- $\boldsymbol{R}$ is a rotation or orthogonal matrix if $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$ (what does it imply?)
  - $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$
    - preserves length $(\boldsymbol{Rx})^T(\boldsymbol{Rx}) = \boldsymbol{x}^T\boldsymbol{R}^T\boldsymbol{Rx} = \boldsymbol{x}^T\boldsymbol{x}$

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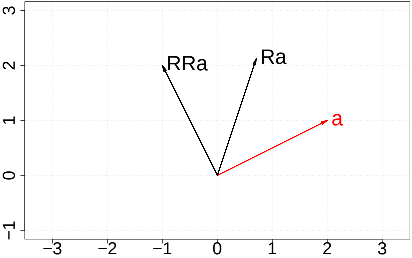


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- Projection (an example): project $\boldsymbol{x}$ to $\boldsymbol{a}$

$$P(\boldsymbol{x}, \boldsymbol{a}) = \|\boldsymbol{x}\| cos\theta * \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} = \frac{\boldsymbol{a}^T \boldsymbol{x}}{\boldsymbol{a}^T \boldsymbol{a}} \boldsymbol{a}$$

$$= \frac{\boldsymbol{a} \cdot \boldsymbol{a}^T \boldsymbol{x}}{\boldsymbol{a}^T \boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}} \boldsymbol{x}$$

  - $\boldsymbol{P} = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}}$ is a projection matrix (what is the shape of $\boldsymbol{P}$?);
  - it transforms $\boldsymbol{x}$ to its projection
  - what if we project it again (and again and again ...) ? i.e. $\boldsymbol{P}(\boldsymbol{P}\boldsymbol{x})$
    - it remains unchanged, $\boldsymbol{P}\boldsymbol{P}\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$
    - or $\boldsymbol{P}\boldsymbol{P} = \boldsymbol{P}$

$$\frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}} \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}} = \frac{\boldsymbol{a}\boldsymbol{a}^T \boldsymbol{a}\boldsymbol{a}^T}{(\boldsymbol{a}^T \boldsymbol{a})^2} = \frac{\boldsymbol{a}\boldsymbol{a}^T}{\boldsymbol{a}^T \boldsymbol{a}}$$

    - mathematics is the subject of making sense :-)

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# Probability theory

- Random variable
- Probability distribution
- Probability mass function and density function
- Probability rules
- Expectation, variance, covariance
- Conditional expectation

# Random variable and probability distribution

**Random variable** $X$ associates with a **probability distribution** $P(X)$

- formally, a r.v. is a mapping from sample space $\Omega$ to a target space $\mathcal{T}$
- e.g. toss a fair coin twice, r.v. $X$ is the number of heads turned up
  - the sample space is $\Omega = \{HH, TT, HT, TH\}$
  - target space is $T = \{0, 1, 2\}$
  - the probability distribution is

$$P(X) = \begin{cases} 0.25 & X = 0 \\ 0.5 & X = 1 \\ 0.25 & X = 2 \end{cases}$$

- the distribution $P$ must satisfy

$$P(X = x) > 0, \text{ and } \sum_{x \in T} P(X = x) = 1$$

# Random variable - discrete r.v.

If r.v. $X$'s target space $\mathcal{T}$ is discrete

- $X$ is called **discrete random variable**
- the probability distribution $P$ is called **probability mass function** (p.m.f.)
- and

$$0 \leq P(X = x) \leq 1, \text{ and } \sum_{x \in T} P(X = x) = 1$$

# Example - discrete r.v.

**Bernoulli distribution** Tossing a coin, $\mathcal{T} = H, T$,

$$P(X = H) = p, P(X = T) = 1 - p, 0 \leq p \leq 1$$

**Binomial distribution** Tossing a coin $N$ times, the r.v. $X$ that the number of head shows up is

$$P(X = k) = \binom{N}{k} \cdot p^k (1 - p)^{N-k}$$

(convince yourself why)
**Multinoulli distribution** Throw a fair 6-facet die, $\mathcal{T} = 1, 2, \ldots, 6$, the distribution is

$$P(X = i) = 1/6$$

Verify the above $P$s satisfy the requirements of p.m.f.

# Random variable - continuous r.v.

If r.v. $X$'s target space $\mathcal{T}$ is continuous

- $X$ is called **continuous random variable**
- the probability distribution $p$ is called **probability density function** (p.d.f.)
- and satisfies

$$p(x) \geq 0, \text{ and } \int_{x \in T} p(x) dx = 1$$

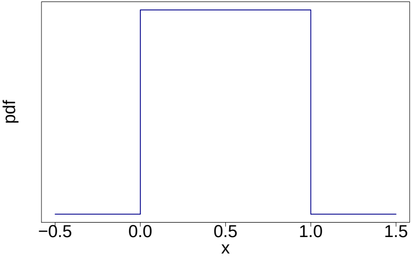
- pdf is not probability as $p(x)$ can be greater 1;
- for $\forall x \; P(X = x) = 0$
- calculate probability over an interval: e.g.

$$P(X \in [a, b]) = \int_a^b p(x) dx$$

# Example - continuous r.v.

**Uniform distribution** $\mathcal{T} = [0, 1]$, $X$ has equal chance to take any value between 0 and 1; the pdf is

$$p(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$




Easy to verify $\int_0^1 p(x)dx = \int_0^1 dx = 1$

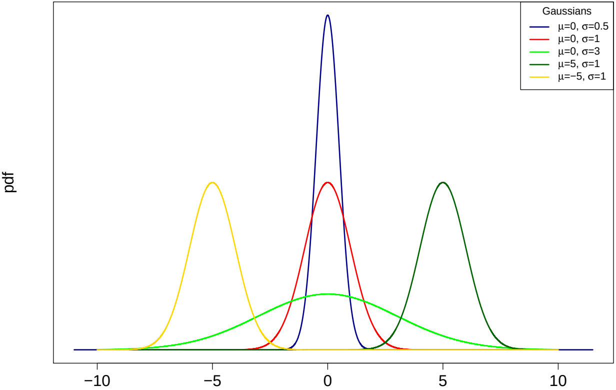
What's the probability that $0 < X < 0.5$ ?

# Example - continuous r.v.

**Gaussian distribution** $\mathcal{T} = R$, or $X \in R$ the pdf is

$$p(x) = \mathcal{N}\left(x; \mu, \sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

$(\frac{x-\mu}{\sigma})^2$ is a distance measure: how far $x$ is away from $\mu$ (measured by $\sigma$ as a unit)



# Question

Calculate quickly:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = ?$$

For $X \sim \mathcal{N}(\mu, \sigma)$, what is $P(X < \mu) = ?$

# Joint distribution

- r.v. $\boldsymbol{X} = [X_1, X_2, \dots, X_n]^T$ can be multidimensional (each $X_i$ is r.v.)
  - essentially a *random vector*

- Still satisfies the same requirements

$$\forall \boldsymbol{x}, 0 < P(\boldsymbol{X} = \boldsymbol{x}) < 1, \sum_{x_1} \sum_{x_2} \dots \sum_{x_n} P(\boldsymbol{X} = [x_1, x_2, \dots, x_n]) = 1$$

or

$$\forall \boldsymbol{x}, p(\boldsymbol{X} = \boldsymbol{x}) > 0, \int \int \dots \int p(\boldsymbol{X} = \boldsymbol{x}) dx_1 dx_2 \dots dx_n = 1$$

- for bivariate case, i.e. $n = 2$, $X_1, X_2$ are **independent** if $P(\boldsymbol{X}) = P(X_1)P(X_2)$ (e.g. rolling two dice independently)

# Example: discrete joint distribution

The joint distribution of $X$ snow or not, $Y \in \{$spring, summer, autumn, winter$\}$ represents the season that $x$ belongs to :

| | $y =$ Spring | $y =$ Summer | $y =$ Autumn | $y =$ winter |
|---|---|---|---|---|
| $x = F$ | 0.05 | 0.25 | 0.075 | 0 |
| $x = T$ | 0.2 | 0 | 0.175 | 0.25 |

It is easy to verify that

$$\sum_x \sum_y p(x, y) = 1$$

# Example: continuous joint distribution

If $X, Y$'s joint p.d.f is

$$p(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}[(\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2]}$$

$X, Y$ are bivariate Gaussian distributed (X,Y are *independent*).

# Probability rules

There are only two probability rules (use integration instead of sum for continuous r.v.):

1. Product rule:

$$p(x, y) = p(y|x)p(x) = p(x|y)p(y)$$

2. Sum rule (marginalisation):

$$p(x) = \sum_y p(x, y),\ p(y) = \sum_x p(x, y)$$

# Conditional probability

Conditional probability (distribution) by product rule:

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

- probability distribution of $x$ conditional on the value of $y$

| | $y =$ Spring | $y =$ Summer | $y =$ Autumn | $y =$ winter |
|---|---|---|---|---|
| $x = F$ | 0.05 | 0.25 | 0.075 | 0 |
| $x = T$ | 0.2 | 0 | 0.175 | 0.25 |

- $P(Y = \text{Spring})$ ? use sum rule $P(Y = \text{Spring}) = \sum_{x=\{T,F\}} P(X = T, Y = \text{Spring}) = 0.05 + 0.2 = 0.25$
- $P(X = T|Y = \text{Spring})$ ?
  $P(x = T|y = \text{Spring}) = \frac{P(x=T, y=\text{Spring})}{P(y=\text{Spring})} = \frac{0.05}{0.25} = 0.2$

# Expectation and variance

**Expection** of a r.v. is defined as

$$E[X] = \sum_x xP(x) \text{ or } E[X] = \int xP(x)dx$$

**Variance** of a r.v. is defined as

$$var[X] = \sum_x xP(x) \text{ or } E[X] = \int xP(x)dx$$

# Reference