Valuation spaces and metric properties of surface singularities

Lorenzo Fantini

Goethe-Universität Frankfurt am Main

August 25th, 2020

Seminario de Singularidades, Fortaleza

Includes joint works with C. Favre and M. Ruggiero, A. Belotto da Silva and A. Pichon.

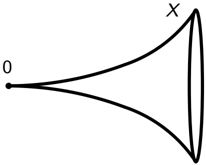
Slides: `https://lorenzofantini.eu/fantini-fortaleza.pdf`

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety,
$0 \in X$ isolated singularity

$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$



Topology: the Conical Structure Theorem

$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$

Metrics on $(X, 0)$

$d_{\mathrm{outer}}(x,y) = ||x-y||_{\mathbb{C}^N}$

$d_{\mathrm{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \big\{\mathrm{length}(\gamma)\big\}$

- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

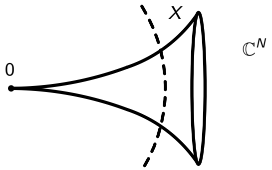
(Birbrair–Mostowski, 2010)

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety,
$0 \in X$ isolated singularity

$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

Metrics on $(X, 0)$

$d_{\mathrm{outer}}(x, y) = ||x - y||_{\mathbb{C}^N}$

$$d_{\mathrm{inner}}(x, y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \{\mathrm{length}(\gamma)\}$$

- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

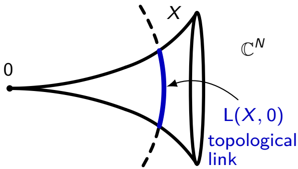
(Birbrair–Mostowski, 2010)

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety,
$0 \in X$ isolated singularity

$$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$$



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

Metrics on $(X, 0)$

$$d_{\mathrm{outer}}(x,y) = ||x-y||_{\mathbb{C}^N} \qquad d_{\mathrm{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \{\mathrm{length}(\gamma)\}$$

- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

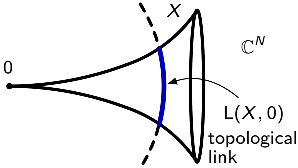
(Birbrair–Mostowski, 2010)

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety,
$0 \in X$ isolated singularity

$$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$$



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

Metrics on $(X, 0)$

$d_{\mathrm{outer}}(x,y) = ||x-y||_{\mathbb{C}^N}$

$$d_{\mathrm{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x, \gamma(1)=y}} \{\mathrm{length}(\gamma)\}$$

- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

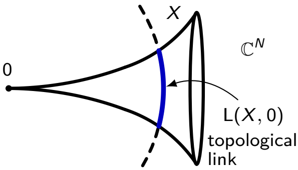
(Birbrair–Mostowski, 2010)

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety, $\qquad (X,0) \hookrightarrow (\mathbb{C}^N, 0)$

$0 \in X$ isolated singularity



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

Metrics on $(X,0)$

$$d_{\mathrm{outer}}(x,y) = ||x-y||_{\mathbb{C}^N} \qquad d_{\mathrm{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \{\mathrm{length}(\gamma)\}$$

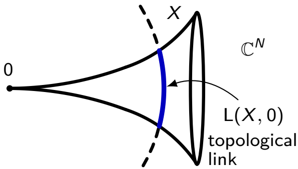
- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

(Birbrair–Mostowski, 2010)

Local structure of a singularity

A long history: Wirtinger 1895, Milnor 1968...

X complex variety, $\qquad (X,0) \hookrightarrow (\mathbb{C}^N, 0)$
$0 \in X$ isolated singularity



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

Metrics on $(X, 0)$

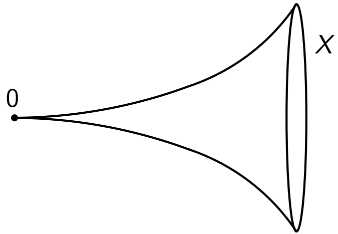
$$d_{\mathrm{outer}}(x,y) = ||x-y||_{\mathbb{C}^N} \qquad d_{\mathrm{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \{\mathrm{length}(\gamma)\}$$

- Belotto–F–Pichon 2019: inner metric germs, not bi-Lipschitz classes!
- Belotto–F–Pichon 2020: we study LNE surfaces, i.e. $\quad d_{\mathrm{outer}} \overset{\text{bi-Lip}}{\approx} d_{\mathrm{inner}}$

(Birbrair–Mostowski, 2010)

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

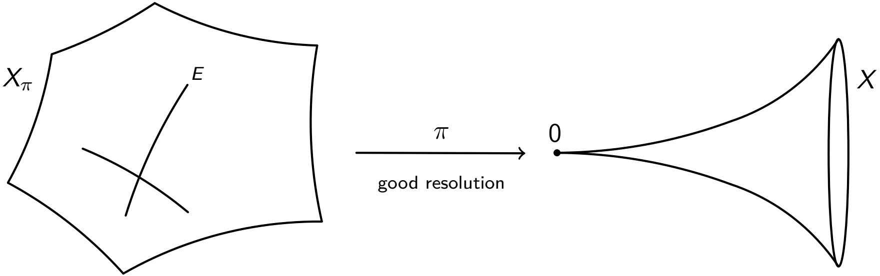
$$d_{\text{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

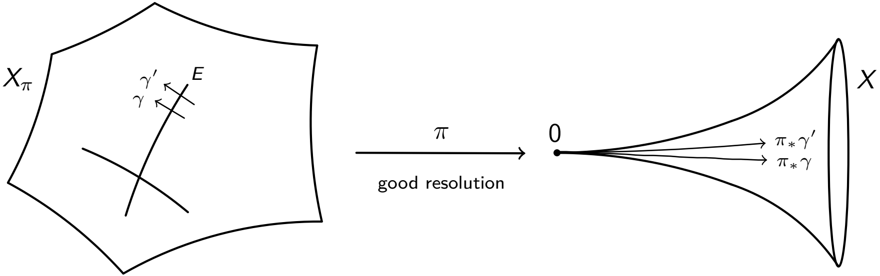
$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

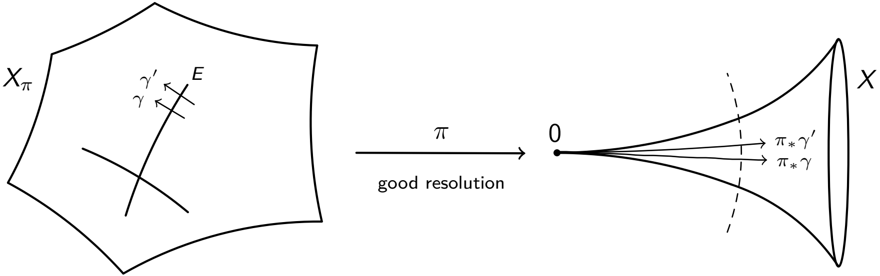
$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

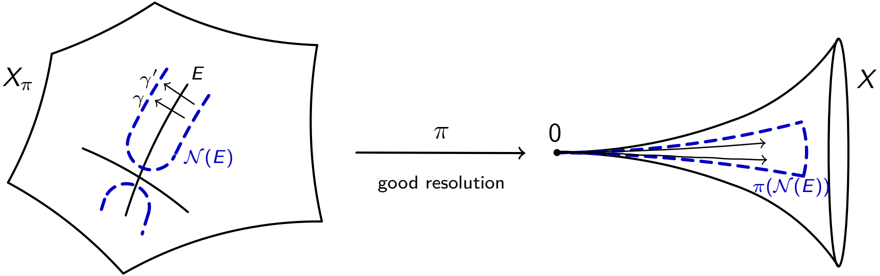
$$d_{\text{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

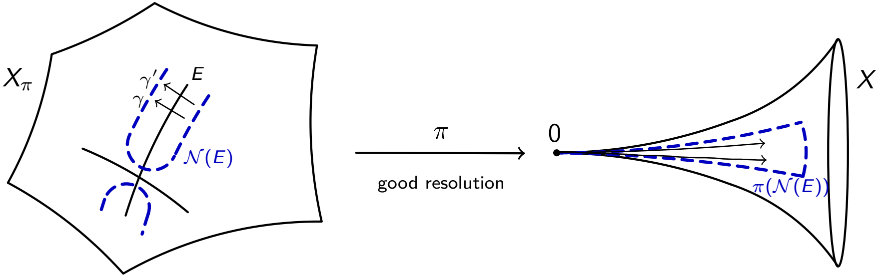
$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Inner rates

I will focus on the case of surfaces.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

Example:

$$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

Example:

$$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

$$\downarrow (y, z)$$

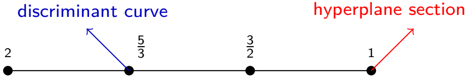
discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

Example:

$$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

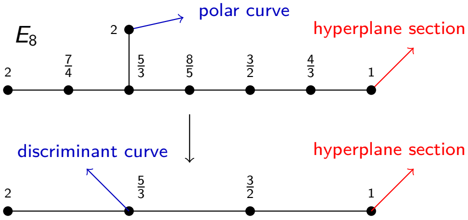


Example:

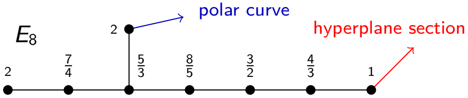
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$\downarrow (y,z)$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$



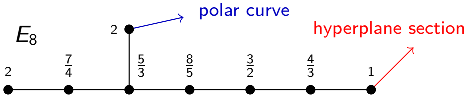
Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$



Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
- How to compute them?

Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$



Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
- How to compute them?

factoring through $\mathrm{Bl}_0(X)$ and through the Nash transform

Theorem (Belotto–F–Pichon, 2019)

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$. Then all the inner rates of $(X, 0)$ are completely determined by:

- the topology of $(X, 0)$, i.e. the weighted dual graph Γ_π;
- the arrows of a **generic** hyperplane section;
- the arrows of the polar curves of a **generic** projection $(X, 0) \to (\mathbb{C}^2, 0)$.

This is a consequence of an explicit formula that we will see later.
To get this, we use a new tool: the non-archimedean link of the singularity $(X, 0)$.

The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{M}_{X,0}} \{v(f)\} = 1 \right\}$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi \colon (Y, D) \to (X, 0)$ is a modification (Y normal, D snc divisor), then $\mathrm{ord}_{\mathrm{E}}$ is a semivaluation ("divisorial valuation")

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson).

The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{m}_{X,0}} \{v(f)\} = 1 \right\}$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi \colon (Y, D) \to (X, 0)$ is a modification (Y normal, D snc divisor), then ord_E is a semivaluation ("divisorial valuation")

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson).

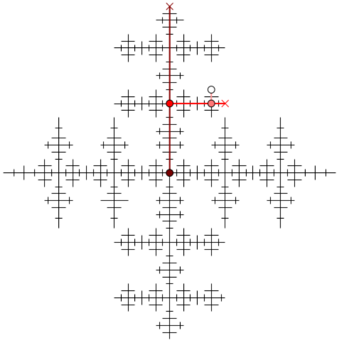
The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{M}_{X,0}}\{v(f)\} = 1 \right\}$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi \colon (Y, D) \to (X, 0)$ is a modification (Y normal, D snc divisor), then ord_E is a semivaluation ("divisorial valuation")



It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson).

The non-archimedean link of a singularity

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \;\middle|\; \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

Moreover, we have:
$H^i_{\mathrm{sing}}\big(\mathrm{NL}(X,0),\mathbb{Q}\big) \cong W^0 H^i_{\mathrm{sing}}\big(\mathrm{L}(X,0),\mathbb{Q}\big)$.

The non-archimedean link of a singularity

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\dots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \;\middle|\; \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

Moreover, we have:
$H^i_{\mathrm{sing}}\big(\mathrm{NL}(X,0),\mathbb{Q}\big) \cong W^0 H^i_{\mathrm{sing}}\big(\mathrm{L}(X,0),\mathbb{Q}\big)$.

The non-archimedean link of a singularity

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

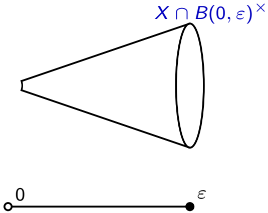
$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \;\middle|\; \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

Moreover, we have:
$H^i_{\mathrm{sing}}\big(\mathrm{NL}(X,0),\mathbb{Q}\big) \cong W^0 H^i_{\mathrm{sing}}\big(\mathrm{L}(X,0),\mathbb{Q}\big)$.



The non-archimedean link of a singularity

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

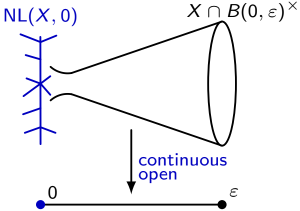
$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \;\middle|\; \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

Moreover, we have:
$H^i_{\mathrm{sing}}\big(\mathrm{NL}(X,0),\mathbb{Q}\big) \cong W^0 H^i_{\mathrm{sing}}\big(\mathrm{L}(X,0),\mathbb{Q}\big)$.



Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{m}_{X,0}}\{v(f)\} = 1 \right\}$$

If $\pi \colon X_\pi \to X$ is a good resolution of $(X,0)$ with dual graph Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset \pi^{-1}(0)$ that corresponds to v.

This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X,0)$$

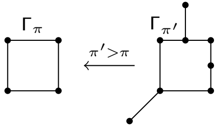
Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0) = \left\{ v\colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation } \Big|\ \min_{f\in\mathfrak{M}_{X,0}}\{v(f)\} = 1 \right\}$$

If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$ with dual graph Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset \pi^{-1}(0)$ that corresponds to v.



This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X,0)$$

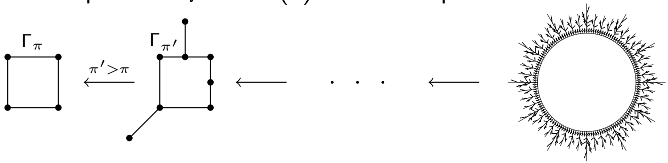
Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0) = \Big\{ v\colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation } \Big|\ \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\} = 1 \Big\}$$

If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$ with dual graph Γ_π, there exists a natural embedding:

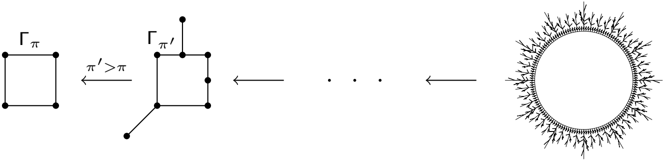
$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset \pi^{-1}(0)$ that corresponds to v.



This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X,0)$$



Moreover, any proper birational map $f\colon (Y,D) \to (X,0)$ induces an isomorphism $\mathsf{NL}(Y,D) \cong \mathsf{NL}(X,0)$.

Theorem (Berkovich, Thuillier)

If $\pi\colon (Y,D) \to (X,0)$ is a good resolution of $(X,0)$, then the retraction $\mathsf{NL}(X,0) \to \Gamma_\pi = \mathrm{Dual}(D)$ extends to a strong deformation retraction.

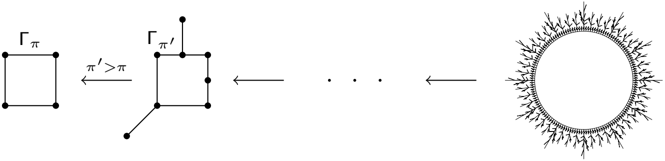
In particular, $\mathsf{NL}(X,0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual graph of a good resolution of $(X,0)$ does not depend on the choice of the resolution.

Remark: $\mathrm{Dual}(D)$ (and hence $\mathsf{NL}(X,0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



Moreover, any proper birational map $f\colon (Y, D) \to (X, 0)$ induces an isomorphism $\mathrm{NL}(Y, D) \cong \mathrm{NL}(X, 0)$.

Theorem (Berkovich, Thuillier)

If $\pi\colon (Y, D) \to (X, 0)$ is a good resolution of $(X, 0)$, then the retraction $\mathrm{NL}(X, 0) \to \Gamma_\pi = \mathrm{Dual}(D)$ extends to a strong deformation retraction.

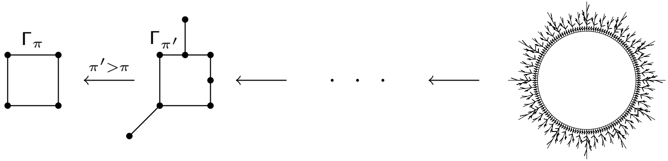
In particular, $\mathrm{NL}(X, 0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual graph of a good resolution of $(X, 0)$ does not depend on the choice of the resolution.

Remark: $\mathrm{Dual}(D)$ (and hence $\mathrm{NL}(X, 0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



Moreover, any proper birational map $f\colon (Y,D) \to (X,0)$ induces an isomorphism $\mathrm{NL}(Y,D) \cong \mathrm{NL}(X,0)$.

Theorem (Berkovich, Thuillier)

If $\pi\colon (Y,D) \to (X,0)$ is a good resolution of $(X,0)$, then the retraction $\mathrm{NL}(X,0) \to \Gamma_\pi = \mathrm{Dual}(D)$ extends to a strong deformation retraction.

In particular, $\mathrm{NL}(X,0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual graph of a good resolution of $(X,0)$ does not depend on the choice of the resolution.

Remark: $\mathrm{Dual}(D)$ (and hence $\mathrm{NL}(X,0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).

Berkovich space structure

$NL(X, 0)$ inherits from the Berkovich spaces X^{an} a non-archimedean analytic structure.

Moreover, any good resolution π induces a decomposition of $NL(X, 0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

$NL(X, 0)$ looks like a fractal :

Application 2 (F–Favre–Ruggiero 2018)

Let $(X, 0)$ normal k-surface singularity.

$NL(X, 0)$ is self-similar $\iff$ $(X, 0)$ est is sandwich

Berkovich space structure

NL$(X,0)$ inherits from the Berkovich spaces X^{an} a non-archimedean analytic structure.

Moreover, any good resolution π induces a decomposition of NL$(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

divisorial valuations of the minimal resolution

NL$(X,0)$ looks like a fractal :

Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

NL$(X,0)$ is self-similar $\iff$ $(X,0)$ est is sandwich

Berkovich space structure

> $\mathrm{NL}(X,0)$ inherits from the Berkovich spaces X^{an} a non-archimedean analytic structure.

Moreover, any good resolution π induces a decomposition of $\mathrm{NL}(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

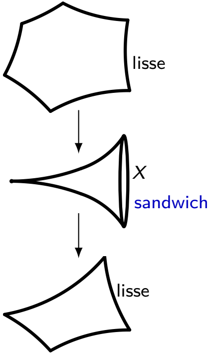
divisorial valuations of the minimal resolution

$\mathrm{NL}(X,0)$ looks like a fractal :

Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

$\mathrm{NL}(X,0)$ is self-similar $\iff$ $(X,0)$ est is sandwich



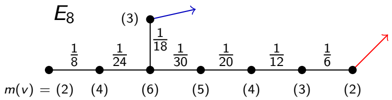
Back to the inner rates: the Laplacian formula

Natural metric on Γ_π:

$$l([v,v']) = \frac{1}{m(v)m(v')}$$

where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X,0)$



The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I}\colon \mathrm{NL}(X,0) \longrightarrow \mathbb{R}_{\geq 1}$$

Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperplane arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$

Back to the inner rates: the Laplacian formula

Natural metric on Γ_π:

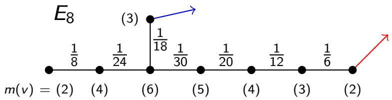
$$l([v,v']) = \frac{1}{m(v)m(v')}$$

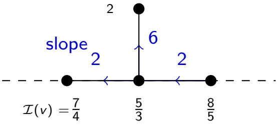
where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X,0)$

The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I}\colon \mathrm{NL}(X,0) \longrightarrow \mathbb{R}_{\geq 1}$$





Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperplane arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$

Back to the inner rates: the Laplacian formula

Natural metric on Γ_π:

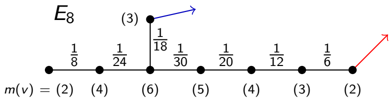
$$l([v, v']) = \frac{1}{m(v)m(v')}$$

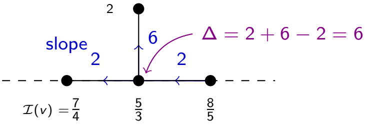
where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X, 0)$

The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I}\colon \mathrm{NL}(X,0) \longrightarrow \mathbb{R}_{\geq 1}$$





Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

$$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperplane arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$$

Back to the inner rates: the Laplacian formula

Natural metric on Γ_π:

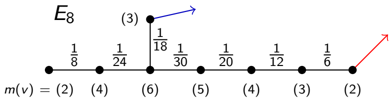
$$l([v,v']) = \frac{1}{m(v)m(v')}$$

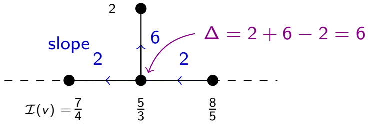
where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X,0)$

The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I}\colon \mathrm{NL}(X,0) \longrightarrow \mathbb{R}_{\geq 1}$$





Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v)$ = sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperplane arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$

Back to the inner rates: the Laplacian formula

Natural metric on Γ_π:

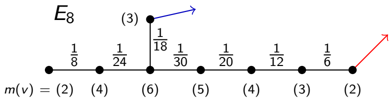
$$l([v,v']) = \frac{1}{m(v)m(v')}$$

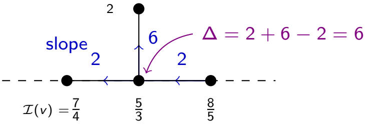
where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X,0)$

The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I}\colon \mathrm{NL}(X,0) \longrightarrow \mathbb{R}_{\geq 1}$$





Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{$hyperplane arrows at $v\} - \#\{$polar arrows at $v\}\big)$

Back to the inner rates: the Laplacian formula

Theorem (Belotto–F–Pichon, 2019)

$$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperpl. arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$$

Proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2, 0)$ to the singular case, studying the topology and monodromy of the Milnor fiber of a generic linear form, Dehn twists.

Applications:

- Simple explicit computation of the inner rates
- Lê–Greuel–Teissier Formula
- We obtain strong restrictions on the relative positions of arrows.

Back to the inner rates: the Laplacian formula

Theorem (Belotto–F–Pichon, 2019)

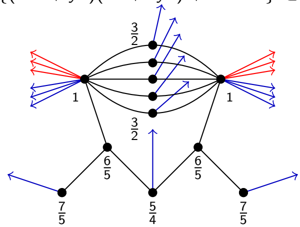
$$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperpl. arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$$

Proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2, 0)$ to the singular case, studying the topology and monodromy of the Milnor fiber of a generic linear form, Dehn twists.

Applications:

- Simple explicit computation of the inner rates
- Lê–Greuel–Teissier Formula
- We obtain strong restrictions on the relative positions of arrows.

$\{(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0\} \subset \mathbb{C}^3$



Back to the inner rates: the Laplacian formula

Theorem (Belotto–F–Pichon, 2019)

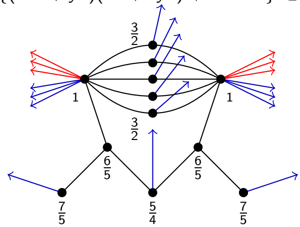
$$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperpl. arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$$

Proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2, 0)$ to the singular case, studying the topology and monodromy of the Milnor fiber of a generic linear form, Dehn twists.

Applications:

- Simple explicit computation of the inner rates
- Lê–Greuel–Teissier Formula
- We obtain strong restrictions on the relative positions of arrows.

$\{(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0\} \subset \mathbb{C}^3$



Back to the inner rates: the Laplacian formula

Theorem (Belotto–F–Pichon, 2019)

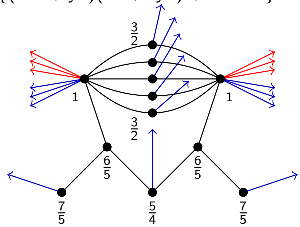
$\Delta_{\Gamma_\pi}(\mathcal{I})(v) = m(v)\big(K_{\Gamma_\pi}(v) + 2\#\{\text{hyperpl. arrows at } v\} - \#\{\text{polar arrows at } v\}\big)$

Proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2, 0)$ to the singular case, studying the topology and monodromy of the Milnor fiber of a generic linear form, Dehn twists.

Applications:

- Simple explicit computation of the inner rates
- Lê–Greuel–Teissier Formula
- We obtain strong restrictions on the relative positions of arrows.

$\{(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0\} \subset \mathbb{C}^3$



Polar exploration

Question

Given a resolution graph Γ, which configurations of arrows (both hyperplane and polar) on Γ can be realized by a surface singularity $(X, 0)$?

First answer (Belotto–F–Pichon 2020)

There are only finitely many possible configurations of arrows on Γ.

In fact, very little possible solutions:
if we fix the hyperplane arrows, the polar arrows are limited by the Laplacian formula.

Polar exploration

Question

Given a resolution graph Γ, which configurations of arrows (both hyperplane and polar) on Γ can be realized by a surface singularity $(X, 0)$?

First answer (Belotto–F–Pichon 2020)

There are only finitely many possible configurations of arrows on Γ.

In fact, very little possible solutions:
if we fix the hyperplane arrows, the polar arrows are limited by the Laplacian formula.

Polar exploration

Laplacian formula (effective)

$$I \cdot \underline{a} = \underline{k} + \underline{l} - \underline{p}$$

$(n = \#V(\Gamma_\pi)$ equations in $2n$ unknown$)$

where:

π factors through $\mathrm{Bl}_0 X$

$I =$ self-intersection matrix of π

$a_i = m_i \mathcal{I}(E_i)$

$k_i = -\chi(E_i^\circ)$

$l_i = ((X \cap H)^*, E_i)$

$p_i = (\Pi^*, E_i)$

$\implies a \in$ Red Cone

geometry of Lipman cones:

$I^{-1}(\mathbb{Z}^n_{\geq 0}) \subset \mathbb{Z}^n_{>0}$

Example: only one possibility for E_8.

Polar exploration

Laplacian formula (effective)

$$I \cdot \underline{a} = \underline{k} + \underline{l} - \underline{p}$$

($n = \#V(\Gamma_\pi)$ equations in $2n$ unknown)

where:

π factors through $\mathrm{Bl}_0 X$

$I =$ self-intersection matrix of π

$a_i = m_i \mathcal{I}(E_i)$

$k_i = -\chi(E_i^\circ)$

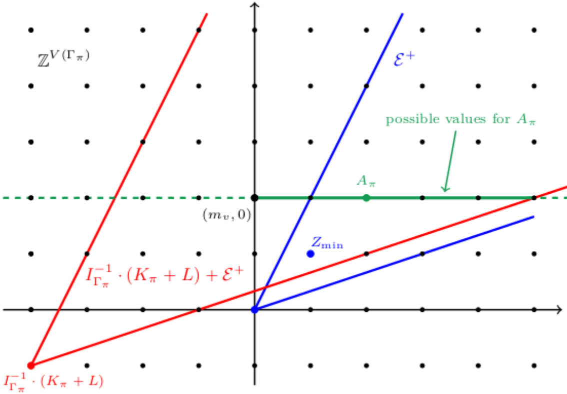
$l_i = ((X \cap H)^*, E_i)$

$p_i = (\Pi^*, E_i)$

$\implies a \in$ Red Cone

geometry of Lipman cones:

$I^{-1}(\mathbb{Z}^n_{\geq 0}) \subset \mathbb{Z}^n_{>0}$



Example: only one possibility for E_8.

Motivation: there are two ways to resolve the singularities of surfaces:

- via point blowups (Zariski 1939) $\longrightarrow$ hyperplane sections
- via Nash transformations (Spivakovsky 1990) $\longrightarrow$ polar curves

Lê Dũng Tráng (∼2000): Are these two methods dual?
Duality between hyperplane sections and polar curves?

In both methods a crucial role is played by minimal singularities.

Theorem (Spivakovsky 1990)

Let $(X, 0)$ be a surface germ with a minimal singularity.
Then the topological type of $(X, 0)$ determines the weighted dual graph of the minimal resolution of $(X, 0)$ that factors through the blowup at 0 and by its Nash transform, together with the arrows of its hyperplane sections and those of its polar curves.

We have an analogous result for the discriminant curve of a generic projection.
(Bondil 2003 in the minimal case, Belotto–F–Pichon 2020 in the LNE case)

Motivation: there are two ways to resolve the singularities of surfaces:

- via point blowups (Zariski 1939) $\longrightarrow$ hyperplane sections
- via Nash transformations (Spivakovsky 1990) $\longrightarrow$ polar curves

Lê Dũng Tráng ($\sim$2000): Are these two methods dual?
Duality between hyperplane sections and polar curves?

In both methods a crucial role is played by minimal singularities.

Theorem (Spivakovsky 1990)

Let $(X, 0)$ be a surface germ with a minimal singularity.
Then the topological type of $(X, 0)$ determines the weighted dual graph of the minimal resolution of $(X, 0)$ that factors through the blowup at 0 and by its Nash transform, together with the arrows of its hyperplane sections and those of its polar curves.

We have an analogous result for the discriminant curve of a generic projection.
(Bondil 2003 in the minimal case, Belotto–F–Pichon 2020 in the LNE case)

Motivation: there are two ways to resolve the singularities of surfaces:

- via point blowups (Zariski 1939) $\longrightarrow$ hyperplane sections
- via Nash transformations (Spivakovsky 1990) $\longrightarrow$ polar curves

Lê Dũng Tráng (∼2000): Are these two methods dual?
Duality between hyperplane sections and polar curves?

In both methods a crucial role is played by minimal singularities.

Theorem (Spivakovsky 1990)

Let $(X, 0)$ be a surface germ with a minimal singularity.
Then the topological type of $(X, 0)$ determines the weighted dual graph of the minimal resolution of $(X, 0)$ that factors through the blowup at 0 and by its Nash transform, together with the arrows of its hyperplane sections and those of its polar curves.

We have an analogous result for the discriminant curve of a generic projection.
(Bondil 2003 in the minimal case, Belotto–F–Pichon 2020 in the LNE case)

Motivation: there are two ways to resolve the singularities of surfaces:

- via point blowups (Zariski 1939) $\longrightarrow$ hyperplane sections
- via Nash transformations (Spivakovsky 1990) $\longrightarrow$ polar curves

Lê Dũng Tráng (∼2000): Are these two methods dual?
Duality between hyperplane sections and polar curves?

In both methods a crucial role is played by minimal singularities.

Theorem (Spivakovsky 1990, Belotto–F–Pichon 2020)

Let $(X, 0)$ be a surface germ with a ~~minimal~~ LNE singularity.
Then the topological type of $(X, 0)$ determines the weighted dual graph of the minimal resolution of $(X, 0)$ that factors through the blowup at 0 and by its Nash transform, together with the arrows of its hyperplane sections and those of its polar curves and all its inner rates.

We have an analogous result for the discriminant curve of a generic projection.
(Bondil 2003 in the minimal case, Belotto–F–Pichon 2020 in the LNE case)