Inner metric structure of complex surface singularities

Lorenzo Fantini

Université Aix-Marseille

Lille, May 16th, 2019

Joint work with André Belotto da Silva and Anne Pichon (arXiv:1905.01677)

Local structure of a singularity

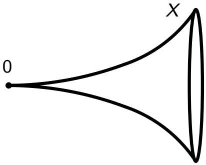
A long history: Wirtinger 1895, Milnor 1968...

X complex variety,
$0 \in X$ isolated singularity

$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$

Topology: the Conical Structure Theorem

$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \text{Cone}\big(X \cap S(0,\varepsilon)\big)$



Inner metric on $(X, 0)$

$$d_{\text{inner}}(x,y) = \inf_{\substack{\gamma\colon [0,1]\to X,\\ \gamma(0)=x,\gamma(1)=y}} \{\text{length}(\gamma)\}$$

I'm interested in the metric germ, not only on its bi-Lipschitz class !

(Mostovski 1985/$\mathbb{C}$, Parusiński 1987/$\mathbb{R}$, Birbrair–Neumann–Pichon 2014)

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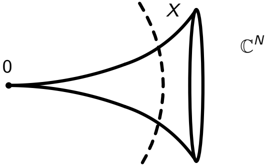

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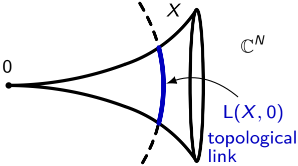

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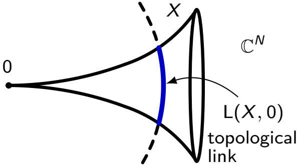

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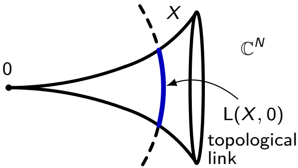

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Inner rates

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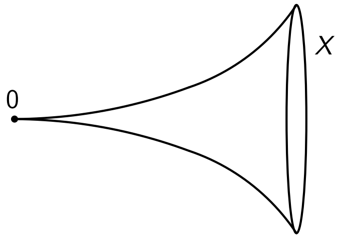

The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curve $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\mathcal{N}(E)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the inner metric structure of the germ

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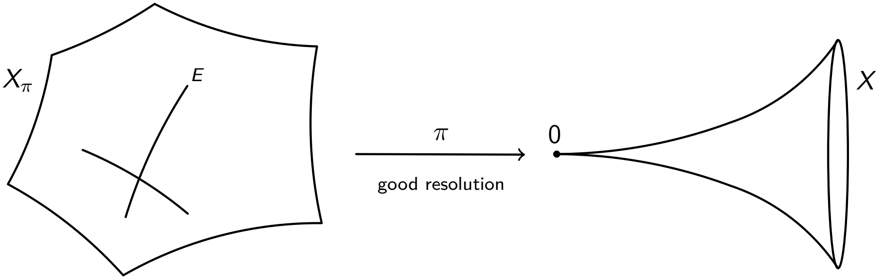

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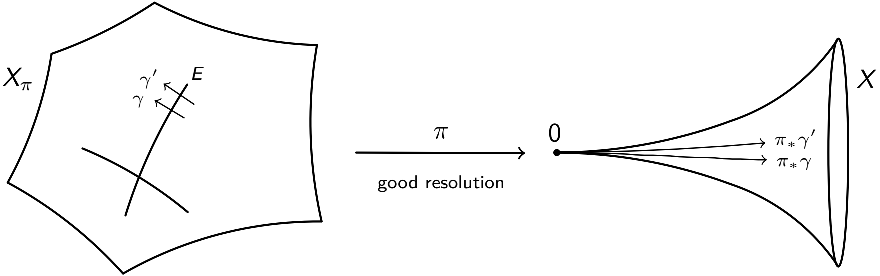

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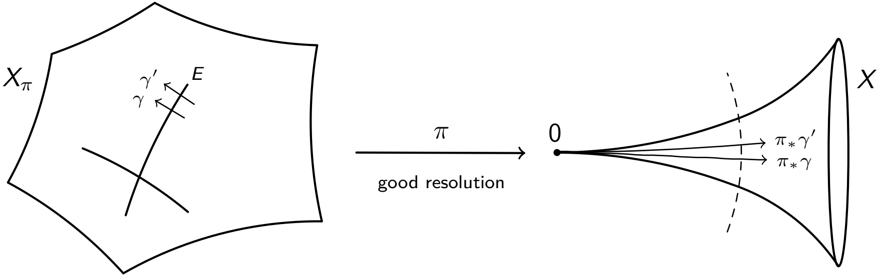

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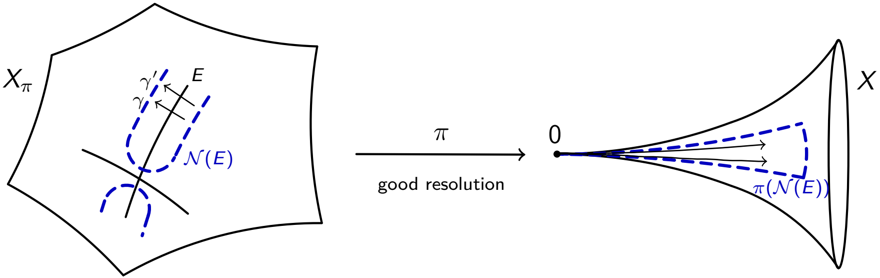

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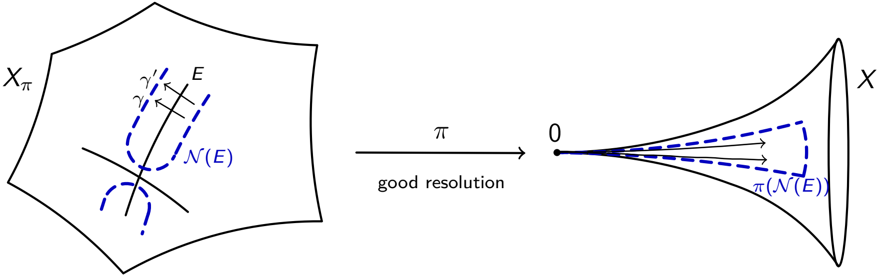

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$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$$\downarrow (y, z)$$

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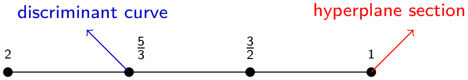

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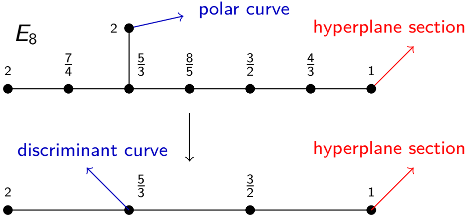

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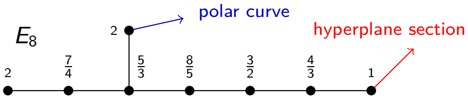

Classical questions:

- How does the geometry $(X, 0)$ influence the inner rates?
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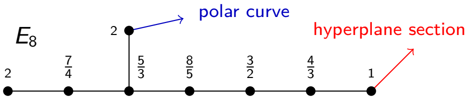

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factoring through $\mathrm{Bl}_0(X)$ and through the Nash transform

Théorème (Belotto–F–Pichon, 2019)

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$. Then all the inner rates of $(X, 0)$ are completely determined by:

- the topology of $(X, 0)$, i.e. the weighted dual graph Γ_π;
- the arrows of a **generic** hyperplane section;
- the arrows of the polar curves of a **generic** projection $(X, 0) \to (\mathbb{C}^2, 0)$.

This is a consequence of an explicit formula that we will see later.
Analogous to the study of weight functions on curves (Baker–Nicaise 2016).

The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{M}_{X,0}} \{v(f)\} = 1 \right\}$

e.g. divisorial valuation $\frac{\mathrm{ord}_E}{m(E)}$

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson).

Non-archimedean avatar of the usual link:

Theorem (F–Favre)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

Moreover, we have:
$H^i_{\mathrm{sing}}\big(\mathrm{NL}(X,0), \mathbb{Q}\big) \cong W^0 H^i_{\mathrm{sing}}\big(\mathrm{L}(X,0), \mathbb{Q}\big)$.

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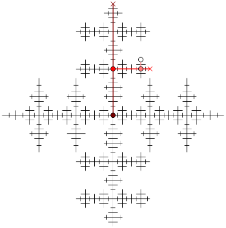

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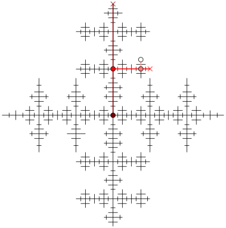

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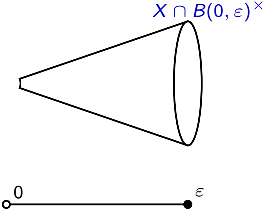

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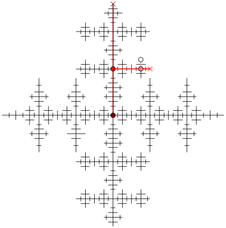

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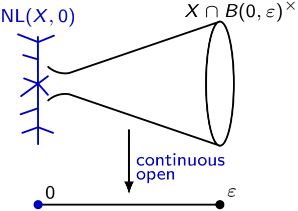

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Skeletons and combinatorics of $\mathrm{NL}(X,0)$

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If $\pi\colon X_\pi \to X$ is a good resolution of $(X,0)$ with dual graph Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset \pi^{-1}(0)$ that corresponds to v.

This induces a canonical homeomorphism:

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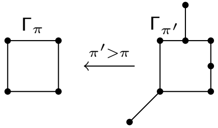

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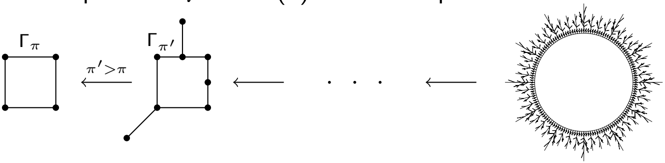

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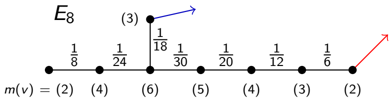
$$l([v, v']) = \frac{1}{m(v)m(v')}$$

where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X, 0)$



The inner rates $\mathcal{I}(E)$ extend to a continuous and piecewise linear map:

$$\mathcal{I} \colon \mathrm{NL}(X, 0) \longrightarrow \mathbb{R}_{\geq 1}$$

Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

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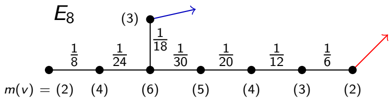

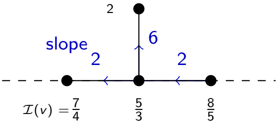

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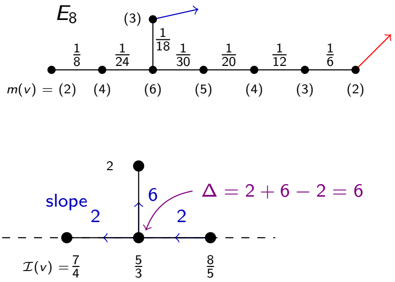

Laplacian of $\mathcal{I}$ on Γ_π: $\Delta_{\Gamma_\pi}(\mathcal{I})(v) =$ sum of the outgoing slopes of $\mathcal{I}|_{\Gamma_\pi}$ at v

Canonical divisor of Γ_π: $K_{\Gamma_\pi}(v) = \mathrm{val}_{\Gamma_\pi}(v) + 2g(v) - 2 = -\chi(\check{E}_v)$

Theorem (Belotto–F–Pichon, 2019)

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Natural metric on Γ_π:

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where $m(v)$ is the multiplicity of E_v in $\pi^{-1}(0)$.

$\longrightarrow$ Metric on $\mathrm{NL}(X,0)$

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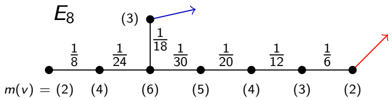

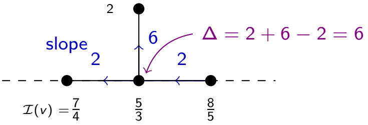

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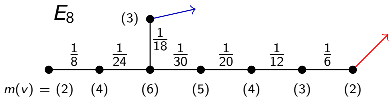

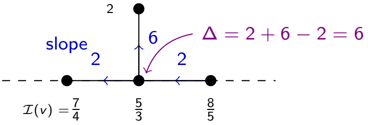

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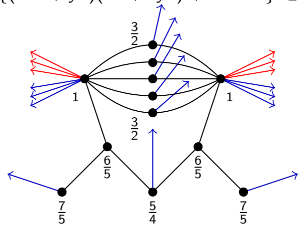
Applications:

- Simple explicit computation of the inner rates
- Lê–Greuel–Teissier Formula
- We obtain strong restrictions on the relative positions of arrows.

$\{(zx^2 + y^3)(x^3 + zy^2) + z^7 = 0\} \subset \mathbb{C}^3$



Two possible proofs:

- Lifting the formula for $NL(\mathbb{C}^2, 0)$ to the singular case: topology and monodromy of the Milnor fiber of a generic linear form, Dehn twists.
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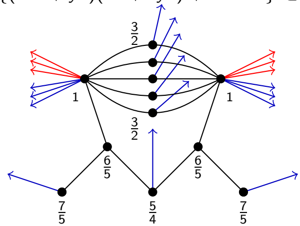

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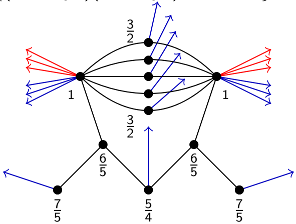

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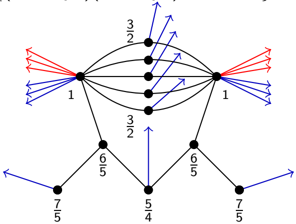

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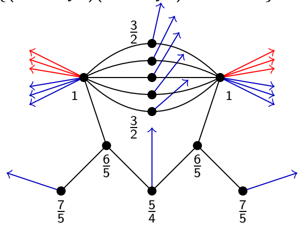

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