Coverings of the Rational Double Points in Characteristic p

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The rational double points of surfaces in characteristic zero are related to the finite subgroups G of SL_2 [6, 7]. Namely, if V denotes the affine plane with its linear G-action, then the variety $X=V/G$ has a singularity at the origin, which is the one corresponding to G. Let p be a prime integer. If p divides the order of G, this subgroup will degenerate when reduced modulo p, and the smooth reduction of V will usually not be compatible with an equisingular reduction of X. Nevertheless, it turns out that every rational double point in characteristic p has a finite (possibly ramified) covering by a smooth scheme. In this paper we prove the existence of such a covering by direct calculation, and we compute the local fundamental groups of the singularities.

1. Generalities on Coverings

We are interested in the local behavior of singularities and so we work with a scheme of the form $X=\operatorname{Spec} A$, where A is the henselization of the local ring of a normal algebraic surface over an algebraically closed field k. We could also work with complete local rings.

In general, U will denote the complement of the closed point of X: $U=X-x_0$. By *fundamental group* of X, we mean $\pi=\pi_1(U)$. This is the group which classifies finite étale coverings of U, or equivalently, normal, pure 2-dimensional schemes Y, finite over X, which are étale except above x_0.

Let us call a *covering* of X any finite surjective map $Y\to X$ such that Y is irreducible and normal, and let us call the covering *unramified* if it is étale above U, i.e., is unramified in codimension 1 on X.

Proposition (1. 1). *Let the solid arrows in the diagram be given coverings of X.*

$$\begin{tikzcd} V \arrow[rr, dashed] \arrow[dr] & & Y \arrow[dl] \\ & X & \end{tikzcd}$$

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Assume that V is smooth, i. e., $V \approx \operatorname{Spec} k\{x,y\}$, where $k\{x,y\}$ denotes the henselization of the polynomial ring, and that Y is unramified. Then a dotted arrow exists, i. e., V dominates Y.

Proof. This follows from purity of the branch locus: The scheme $V \times_X Y$ is étale over V except at the closed point, and therefore its normalization decomposes completely into a sum of copies of V. Each copy determines the graph of a map $V \to Y$.

As an immediate consequence, we have

Corollary (1. 2). (i) *If X admits a smooth covering $V \to X$, then the fundamental group π of X is finite.*

(ii) *If in addition V/X is totally ramified along some curve of X, then* $\pi = 0$.

In characteristic zero, the converse of (1. 2i) is true. Mumford [9] proved that if $\pi = 0$ then X is smooth[1]. If π is finite, then the universal cover $\tilde{U}$ of U is finite over U. The normalization V of X in $K(\tilde{U})$ is a singularity with trivial fundamental group, hence is smooth.

Mumford's theorem is easily seen to be false in characteristic $p \neq 0$. Some rational double points furnish examples (cf. sections 3–5). But it is natural to ask whether the converse of (1. 2) continues to hold:

Question (1. 3). *Suppose the fundamental group π of X is finite. Does there exist a covering $V \to X$ which is smooth?*

This would be a very beautiful fact, if true. Our calculations provide some slight positive evidence, since we exhibit smooth coverings for the rational double points. But even for these special singularities, we do not know a conceptual proof of their existence.

Now suppose that X is a rational double point [3]. Let $X' \xrightarrow{\varphi} X$ be the minimal resolution of the singularity of X. It is known [2, 2. 7] that rational double points X are characterized by the existence of a double differential ω whose divisor on X' is zero. This fact restricts the possible unramified coverings:

Proposition (1. 4). *An unramified covering Y of a rational double point is either smooth, or is a rational double point.*

Proof. Let $Y'' \to Y$ be a resolution of Y. The differential ω is regular on X', and hence has no pole along any prime divisor of $K(X)$ centered at x_0. (In other words, ω is regular on every resolution of X.) Since every prime divisor of Y centered at the closed point lies over some prime divisor of X, it follows that ω is regular on Y''.

1) Actually, Mumford [9] works with the classical topology. However, he has extended his result to the algebraic context (unpublished).

Moreover, ω has no zeros on $Y-y_0$ since $Y-y_0$ is étale over U. Therefore the divisor K of ω on Y'' is supported on the exceptional curves of the map $Y''\to Y$. Let Z be the fundamental cycle on Y'' [3, p. 132]. Then since $K\geq 0$, $(Z\cdot K)\leq 0$. Also, $(Z^2)<0$. Hence $p(Z)=0$, and $(Z^2)=-1$ or -2. The proposition follows from [3, thm. 4].

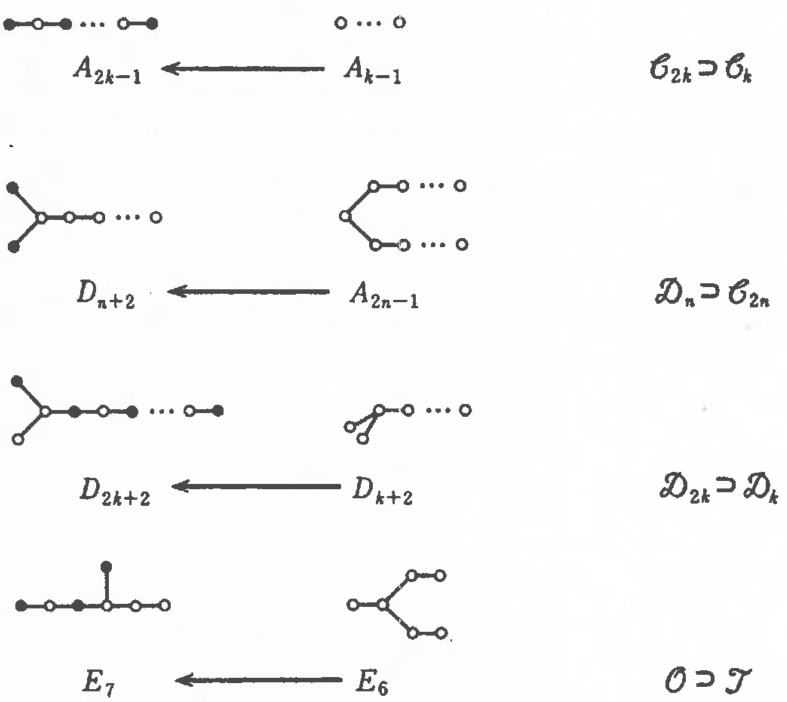
Proposition (1. 5). *With the above notation, let $Y''\to Y$ be the minimal resolution of the singularity of Y, and let Y' be the normalization of X' in $K(Y)$. Then Y' dominates Y'':*

$$\begin{tikzcd} Y' \arrow[rr] \arrow[dr] & & Y'' \arrow[dl] \\ & Y & \end{tikzcd}$$

and the curves in Y' which contract on Y'' are the ones ramified over X'. In particular, Y is smooth ($Y=Y''$) if and only if every curve is ramified.

Proof. The divisor of ω on Y'' is zero since it is concentrated on the closed fibre of Y''/Y and Y is a rational double point (or $Y=Y''$). Let D be a prime divisor of $K(Y)$ lying over a prime divisor C of $K(X)$. Then the order of zero of ω on D will be zero only if ω has order zero on C and the extension is not ramified on D. The prime divisors of X on which ω has order zero are those corresponding to the curves of X', and similarly, the prime divisors of Y on which ω has order zero correspond to the curves of Y''. The proposition now follows from Zariski's Main Theorem.

Example.



(1. 6) Double covers in characteristic zero

The unramified double covers $Y\to X$ of the rational double points in characteristic zero are listed here. The dark vertices of the Dynkin diagram stand for the ramified curves on X', whose inverse images are contracted on Y''. The corresponding subgroups of SL_2 are on the right. We use script letters to denote them, viz. $\mathcal{C}_n$=cyclic group of order n, $\mathcal{D}_n$=binary dihedral group of order $4n$, and $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ the binary tetrahedral, octahedral and icosahedral groups. The fact that the ramified curves must alternate with unramified ones is easily checked.

2. Reduction modulo p

Any rational double point in characteristic p can be obtained by specialization from one in characteristic zero. This is clear from the equations given in the next section. The main facts that we need about specialization of unramified coverings were proved in [4, Sect. 4], where the case of complete local rings was treated. The henselian case is similar, and can be deduced from [4] using standard algebraization techniques.

Let S=Spec R, where R is an unequal characteristic discrete valuation ring with algebraically closed residue field k. Let X/S be a henselian local scheme, essentially of finite type, which represents an equisingular family of two-dimensional singularities, in the sense of [10]. Suppose that the closed fibre X_0 is a rational double point. Then the generic geometric fibre $X_{\bar\eta}$ will have a single rational double point of the same type. Let $X_{\bar\eta}$ denote the henselization at that point. It follows from [4] that

(2.1) (i) *The unramified coverings of X_0 which extend to X are those which are tamely ramified along every prime divisor of X_0.*

(ii) *If $Y \to X$ is a cover such that Y_0 is unramified and reducible, then Y is reducible.*

(iii) *There is a natural 1–1 correspondence between unramified Galois covers of order prime to p of X_0 and of $X_{\bar\eta}$.*

Let us return to the case of a single rational double point X in characteristic p, dropping the subscript 0. In this case, any unramified Galois covering Y of X whose order is divisible by p must be wildly ramified along some prime divisor centered at x_0. For, let G be the Galois group and let $H \subset G$ be a subgroup of order p. This subgroup corresponds to an intermediate covering $Y \to Z \to X$, and Y/Z is cyclic of order p. It is enough to show that the covering Y/Z is ramified on some prime divisor. Let $Z' \to Z$ be its minimal resolution, and let $C \subset Z'$ be the exceptional set. This is a simply connected union of rational curves, and therefore it has no étale cover. It follows that Z' has no étale cover either. Hence Y/Z is ramified on some component of C.

Corollary (2.2). *Let X be a rational double point in characteristic p. The unramified Galois covers of X which lift to characteristic zero are those of order prime to p.*

Let us call the fundamental group π of X *tame* if it is not wildly ramified, or equivalently, if it has order prime to p. In this case, it will be the maximal quotient of order prime to p of the corresponding characteristic zero group, and so it is known.

The A_n singularities have tame fundamental groups. They can be described in all characteristics by the equations

(2. 3) $$xy+z^{n+1}=0.$$

In characteristic zero, A_{n-1} corresponds to the representation of the cyclic group $\mathcal{C}_n$ by the matrices

(2. 4) $$\begin{bmatrix} \zeta^i & 0 \\ 0 & \zeta^{n-i} \end{bmatrix} \qquad \zeta = e^{2\pi i/n}.$$

If we let $\mathcal{C}_n$ act by (2. 4) on a vector space V with basis $\{v_1, v_2\}$, then the invariant functions are generated by

(2. 5) $$v_1^n = x, \qquad v_2^n = y, \qquad v_1v_2 = z.$$

The equations (2. 5) define a covering of the A_n singularity (2. 3) by a smooth scheme V, uniformly over Spec $\boldsymbol{Z}$. When p divides n, this covering becomes inseparable modulo p, and the fundamental group of the characteristic p singularity reduces, therefore, to $\mathcal{C}_{\bar{n}}$, where $n=p^e\bar{n}$, and $p\nmid\bar{n}$. The inseparable phenomena can be explained in this case if we replace the cyclic group $\mathcal{C}_n$ by the group scheme μ_n of nth roots of unity, operating in the analogous way. Such an explanation will *not* be possible for the other rational double points.

One can begin the analysis of the other singularities by passing to a covering of degree prime to p, which will be a simpler rational double point. This reduces the problem to those types whose characteristic zero groups have no quotient of order prime to p, so that the fundamental group is completely wild. Excluding A_n, which was treated above, we are left with the following cases :

$$\begin{aligned} &D_n, \ p=2 \\ &E_6, \ p=3 \\ &E_7, \ p=2 \\ &E_8, \ p=2, 3, 5. \end{aligned}$$

(2. 6) The singularities having no non-trivial tame covering

Corollary (2. 7). *The fundamental group of a rational double point X in characteristic p is tame if $p\neq 2, 3, 5$.*

3. List of the Singularities and their Equations

Lipman [8] classified the E_8 singularities in all characteristics, and we need to extend his classification to the other rational double points[1]. Following tradition, we will omit the rather tedious verification of these results. We do not know an a priori reason for the fact that there are only finitely many singularities of each type.

In a family of singularities X^r_n, the index n is upper semi-continuous, while the co-index r is lower semi-continuous. The number to the right of the equation is the dimension of the space of deformations of the singularity, which can be used to check that the cases listed are all different.

For convenience, we include the nonsingular local scheme Spec $k\{x,y\}$ in our lists, and denote it by A_0.

A_n	:	$z^{n+1}+xy$		$n\geq 0$
D^0_{2n}	:	$z^2+x^2y+xy^n$	$4n$	$n\geq 2$
D^r_{2n}	:	$z^2+x^2y+xy^n+xy^{n-r}z$	$4n-2r$	$r=1,\cdots,n-1$
D^0_{2n+1}	:	$z^2+x^2y+y^nz$	$4n$	$n\geq 2$
D^r_{2n+1}	:	$z^2+x^2y+y^nz+xy^{n-r}z$	$4n-2r$	$r=1,\cdots,n-1$
E^0_6	:	$z^2+x^3+y^2z$	8	
E^1_6	:	$z^2+x^3+y^2z+xyz$	6	
E^0_7	:	$z^2+x^3+xy^3$	14	
E^1_7	:	$z^2+x^3+xy^3+x^2yz$	12	
E^2_7	:	$z^2+x^3+xy^3+y^3z$	10	
E^3_7	:	$z^2+x^3+xy^3+xyz$	8	
E^0_8	:	$z^2+x^3+y^5$	16	
E^1_8	:	$z^2+x^3+y^5+xy^3z$	14	
E^2_8	:	$z^2+x^3+y^5+xy^2z$	12	
E^3_8	:	$z^2+x^3+y^5+y^3z$	10	
E^4_8	:	$z^2+x^3+y^5+xyz$	8	

Rational double points in characteristic 2

A_n, D_n	:	classical forms.	
E^0_6	:	$z^2+x^3+y^4$	9
E^1_6	:	$z^2+x^3+y^4+x^2y^2$	7
E^0_7	:	$z^2+x^3+xy^3$	9
E^1_7	:	$z^2+x^3+xy^3+x^2y^2$	7

1) See also Arnold [1]. But note that in non-zero characteristics the classification of singularities does not lead to the same answers as the classification of germs of maps. They have continuous parameters. Moreover, if $p=2$, the classification of singularities depends on their dimension, which is 2 in our case.

E_8^0	: $z^2+x^3+y^5$	12
E_8^1	: $z^2+x^3+y^5+x^2y^3$	10
E_8^2	: $z^2+x^3+y^5+x^2y^2$	8

Rational double points in characteristic 3

A_n, D_n, E_6, E_7 : classical forms.

E_8^0	: $z^2+x^3+y^5$	10
E_8^1	: $z^2+x^3+y^5+xy^4$	8

Rational double points in characteristic 5

Rational double points in characteristic $p>5$: classical forms only

4. The Computations in Characteristic 2

The singularities D_N^r, $N=2n$ or $2n+1$ can be analyzed by the substitution

$$(4.1)\qquad \begin{aligned} &u^2+y^{n-r}u+y = 0 &&\text{if } r>0, \text{ or} \\ &u^2+y = 0 &&\text{if } r=0. \end{aligned}$$

Set

$$\bar{v} = z+xu,$$

so that

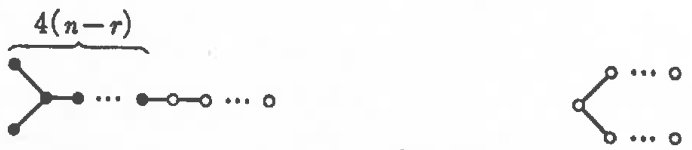
$$\begin{aligned} &\bar{v}^2+xy^{n-r}\bar{v}+xy^n = 0 &&\text{if } N \text{ even}, r>0 \\ &\bar{v}^2+xy^n = 0 &&\text{if } N \text{ even}, r=0 \\ &\bar{v}^2+xy^{n-r}\bar{v}+y^nz = 0 &&\text{if } N \text{ odd}, r>0 \\ &\bar{v}^2+y^nz = 0 &&\text{if } N \text{ odd}, r=0. \end{aligned}$$

Case 1. $N = 2n$, $2r \geq n$.

Set $\bar{v} = y^{n-r}v$, so that

$$v^2+xv+xy^{2r-n} = 0.$$

Substitution for y using (4.1) leads to an equation of type $A_{8r-4n-1}$, if $2r>n$, and of type A_0 if $2r=n$. It is an unramified double cover having ramification on the resolution as indicated :

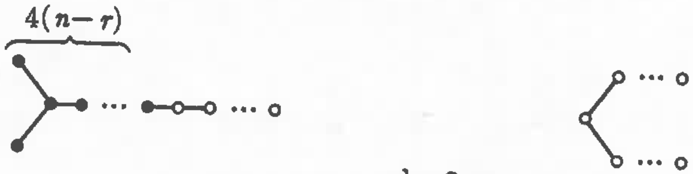


$$D_{2n}^r \xleftarrow{\ \text{deg } 2\ } A_{8r-4n-1} \quad (2r\geq n).$$

Since the fundamental group of A_k is tame and the fundamental group π of D_N^r has no non-trivial tame quotient (2.6), π is the dihedral group of order $2m$, where

$m=\overline{2r-n}$ is the greatest divisor of $2r-n$ prime to 2, or is 1 if $2r=n$.

Case 2. $N=2n+1$, $2r \geq n$.
Set $v=y^{n-r}\bar{v}$, $v_1=v+y^{2r-n}u$, and $x_1=x+y^r$, to obtain an equation of type $A_{8r-4n+1}$, which is an unramified double cover of D_N^r:



$$D^r_{2n+1} \xleftarrow{\ \deg 2\ } A_{8r-4n+1} \quad (2r \geq n).$$

As in case 1, π is the dihedral group of order $2(4p-2n+1)$.

Case 3. $N=2n$, $2r<n$.
This case leads to a ramified double cover of D^r_{2n} by A_0, and $\pi=0$.

Case 4. $N=2n+1$, $2r<n$.
This leads to a ramified double cover of D^r_{2n+1} by A_1, and $\pi=0$.

The E_6^r singularities $(r=0,1)$ have a tame cyclic cover of degree 3 defined by the equation

$$u^3=z.$$

The resulting cover is D_4^r, which has fundamental group 0 or $\mathscr{C}_2$ according as r is 0 or 1. In both cases, π is cyclic, and the double cover of E_6^1 is

$$E_6^1 \xleftarrow{\ \deg 2\ } A_2$$

The singularity E_7^0 has the purely inseparable cover of degree 2 by A_0 defined by $u^2=x$. The cases E_7^r with $r=1,2$ can be treated together if the equation for E_7^2 is replaced by the equivalent one

$$z^2+x^3+xy^3+y^3z+x^2yz.$$

The substitution

$$u^2+xyu+x=0$$

defines a ramified double cover by A_0 if $r=0$, or D_5^0 if $r=1$. It follows that $\pi=0$ for E_7^r, and $r=0,1,2$.

The case E_7^3 has an unramified double cover by D_4^1, given by the substitution

$$u^2+yu+x=0:$$

$$E_7^3 \xleftarrow{\text{deg } 2} D_4^1$$

and the fundamental group is cyclic of order 4.

It remains to treat the E_8 singularities. The case E_8^0 has a purely inseparable cover by A_0 defined by $u^2=x$, E_8^1 has a ramified double cover by A_0, defined by

$$u^2+y^3u+x = 0,$$

and E_8^2 has an unramified double cover by A_0 defined by

$$u^2+y^2u+x = 0$$

(see [5]). The substitution $u^2=x$ defines a purely inseparable cover of E_8^3 by E_7^2, which has trivial fundamental group. Thus the fundamental group of E_8^3 is trivial, too.

The most interesting case is that of E_8^4, which has an unramified double cover by E_6^1, defined by the equation

$$u^2+yu+x = 0:$$

$$E_8^4 \xleftarrow{\text{deg } 2} E_6^1$$

Thus its fundamental group π has order 12. It has no quotient of order 3 (2. 6), and therefore is either the dihedral group or the *metacyclic group* with generators σ, τ:

(4. 2) $$\sigma^3 = 1, \qquad \tau^4 = 1, \qquad \tau\sigma\tau^{-1} = \sigma^2.$$

The dihedral group is isomorphic to $\mathcal{S}_3 \oplus \mathcal{C}_2$, and so if this is π there must be a second double cover Y of $E_8^4=X$, whose fundamental group is $\mathcal{S}_3$. Consider the subset of the Dynkin diagram on which this second cover is ramified. We can contract the curves corresponding to this subset on the minimal resolution X' of X, to obtain a scheme $\check{X}$ having finitely many rational double points, and such that the normalization $\check{Y}$ of $\check{X}$ in $K(Y)$ is unramified at these points. Moreover, $\check{Y}$ will be the minimal resolution of Y, by (1. 5). Looking over the singularities having double covers by A_0 and whose Dynkin diagrams are subsets of E_8, we find the two possibilities D_4 and E_8. The second is ruled out because Y, having fundamental group $\mathcal{S}_3$, is not A_0. In the case D_4, the cover is by an E_6 singularity, as in the above figure. There is only one double cover of D_4^1. Thus in this case the tensor product of the two given covers of $\check{X}$ would be unramified on X', and hence would split completely, contradicting the assumption that they are different. Therefore, π is the metacyclic group.

We collect the results together :

A_n	:	π is tame
D^r_{2n}, $0 \le 2r < n$	:	$\pi = 0$
D^r_{2n}, $n \le 2r < 2n$	:	π is dihedral, of order $2m$, where m is the greatest divisor of $2r-n$ prime to 2, or is 1 if $2r=n$
D^r_{2n+1}, $0 \le 2r < n$	:	$\pi = 0$
D^r_{2n+1}, $n \le 2r < 2n$	:	π is dihedral, of order $2(4r-2n+1)$
E^0_6	:	$\pi = \mathcal{C}_3$ is tame
E^1_6	:	$\pi = \mathcal{C}_6$
E^r_7, $r = 0, 1, 2$	:	$\pi = 0$
E^3_7	:	$\pi = \mathcal{C}_4$
E^r_8, $r = 0, 1$	:	$\pi = 0$
E^2_8	:	$\pi = \mathcal{C}_2$
E^3_8	:	$\pi = 0$
E^4_8	:	π is the metacyclic group (4. 2) of order 12.

The fundamental groups in characteristic 2

Note that the order of π is always smaller than the order of the corresponding group in characteristic zero. But for D^r_N, the prime factors do not always correspond. Therefore any relation between smooth coverings in characteristics 0 and p must be very subtle.

5. The Computations in Characteristics 3 and 5

We look for a p-cyclic covering of X using Artin-Schreier theory, and the étale cohomology sequence

$$0 \longrightarrow Z/p \longrightarrow \mathcal{O}^+ \xrightarrow{F-1} \mathcal{O}^+ \longrightarrow 0, \qquad (F=p\text{th power}) \tag{5. 1}$$

on the open set $U=X-x_0$. The cohomology sequence of (5. 1) shows that p-cyclic étale covers of U are given by elements $\bar{\alpha} \in H^1(U, \mathcal{O})$ such that $F\bar{\alpha}-\bar{\alpha}=0$. We take as affine cover the open sets $U_0=U-\{x=0\}$ and $U_1=U-\{y=0\}$, and try the cohomology class defined by the cocycle

$$\alpha = x^{-1}y^{-1}z$$

on $U_0 \cap U_1$. In other words, we ask whether $\alpha^p-\alpha$ is a coboundary $\beta_0-\beta_1$. This is so in the following cases :

$$E^1_6,\ p = 3: \quad \alpha^3-\alpha = (x^{-3}yz)-(-y^{-3}z)$$
$$E^1_7,\ p = 3: \quad \alpha^3-\alpha = (x^{-2}z)-(-y^{-3}z)$$
$$E^2_8,\ p = 3: \quad \alpha^3-\alpha = (y^2x^{-3}z)-(-y^{-3}z)$$
$$E^1_8,\ p = 5: \quad \alpha^5-\alpha = x^{-5}(y^5+x^2y^3+2x^3+2xy^4)z-(-y^{-5}xz).$$

Thus these singularities have p-cyclic unramified covers defined by the equations

$$u^3-u+y^{-3}z=0 \qquad (p=3)$$
$$u^5-u+y^{-5}xz=0 \qquad (p=5)$$

or,

$$u^3-y^2u+z=0 \qquad \text{if } p=3$$

and

$$u^5-y^4u+xz=0 \qquad \text{if } p=5.$$

In the case of $X=E_8^1$, $p=5$, every curve of the minimal resolution X' must be ramified. For, the singularities defined by contracting the ramification curves must have a p-cyclic cover, and there are no wild covers in characteristic 5 except for E_8 (see (2. 6)). Thus the cover is nonsingular, by (1. 5) :

$$E_8^1 \xleftarrow{\text{deg } 5} A_0\,,\ p=5.$$

A similar argument applies to E_6^1 in characteristic 3. The other two cases must be computed, and they lead to

$$E_6^1 \xleftarrow{\text{deg } 3} A_0\,,\ p=3$$

$$E_7^1 \xleftarrow{\text{deg } 3} A_1\,,\ p=3$$

$$E_8^2 \xleftarrow{\text{deg } 3} D_4\,,\ p=3\,.$$

The fundamental group of E_8^2, $p=3$, is an extension of the cyclic group $\mathcal{C}_3$ by the quaternion group $\mathcal{D}_2$, and it has no quotient prime to 3. These properties characterize the binary tetrahedral group : $\pi=\mathcal{T}$. For, the extension splits, hence is determined by a non-trivial operation of $\mathcal{C}_3$ on $\mathcal{D}_2$, and there is essentially only one such operation. The fundamental groups are cyclic in the other cases considered above.

The only remaining singularity to consider in characteristic 5 is E_8^0, which has a purely inseparable cover by A_0, defined by $u^5=y$. In characteristic 3, the singularities E_6^0, E_8^0 have purely inseparable covers by A_0 defined by $u^3=y$, and E_7^0 has a tame double cover by E_6^0. Finally, the substitution

$$u^3 = x^2 + y^2$$

defines a purely inseparable cover of E_8^1 by E_6^0.

A_n, D_n	:	π is tame
E_6^0	:	$\pi = 0$
E_6^1	:	$\pi = \mathcal{C}_3$
E_7^0	:	$\pi = \mathcal{C}_2$ is tame
E_7^1	:	$\pi = \mathcal{C}_6$
$E_8^r, r = 0, 1$	:	$\pi = 0$
E_8^2	:	$\pi = \mathcal{T}$ is the binary tetrahedral group.

The fundamental groups in characteristic 3

A_n, D_n, E_6, E_7	:	π is tame
E_8^0	:	$\pi = 0$
E_8^1	:	$\pi = \mathcal{C}_5$

The fundamental groups in characteristic 5

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