

A zoo of singularity links: topological, non-archimedean, and logarithmic

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Includes joint works with C. Favre and M. Ruggiero, A. Belotto da Silva and A. Pichon.

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Slides: <https://lorenzofantini.eu/fantini-tenerife.pdf>

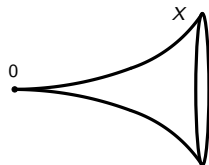
The topological link of a singularity

X complex variety,
 $0 \in X$ isolated singularity

$$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$$

Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0, \varepsilon) \stackrel{\text{homeo}}{\sim} \text{Cone}(X \cap S(0, \varepsilon))$$



$L(X, 0) = X \cap S(0, \varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$ -manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

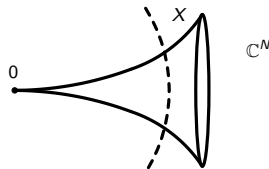
$(X, 0)$ surface, $\pi: (X_{\pi}, E_{\pi}) \rightarrow (X, 0)$ minimal good resolution of $(X, 0)$. Then:
(oriented) top. type of $L(X, 0) \iff$ weighted dual graph Γ_{π} of π .

Remark: This is also equivalent to the (oriented) top. type of $(X, 0)$ (in dim 2!).
[“ \implies ” Conical Structure Theorem, “ \impliedby ” in F–Pichon 2025]

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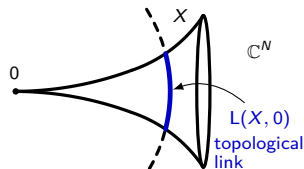
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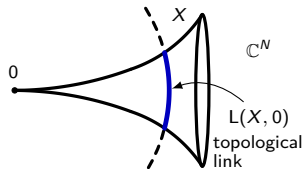
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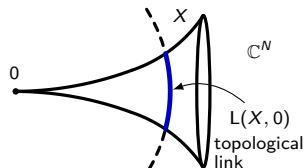
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The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$$\mathrm{NL}(X, 0) = \{v: \widehat{\mathcal{O}_{X,0}} \rightarrow \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \mid \min_{f \in \mathfrak{m}_{X,0}} \{v(f)\} = 1\}$$

$$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi: (Y, D) \rightarrow (X, 0)$ is a modification (Y normal, D snc divisor), then ord_E is a semivaluation (divisorial valuation)

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}_{\mathbb{C}}^2, 0) \cong$ valutive tree
(Favre–Jonsson 2004).

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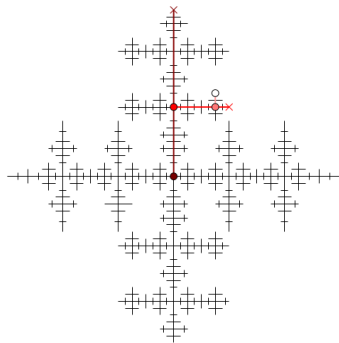
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Intuition

$NL(X, 0)$ is a non-archimedean version of the usual link $L(X, 0)$

Indeed, denote by (z_1, \dots, z_N) a set of local coordinates of \mathbb{C}^N at 0. Then:

$$NL(X, 0) = \left\{ x \in X^{\text{an}} \mid \max_i |z_i(x)| = \varepsilon \right\}$$

Theorem (F–Favre, Poineau 2025)

$L(X, 0)$ degenerates towards $NL(X, 0)$.

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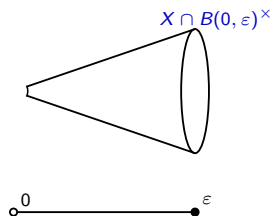
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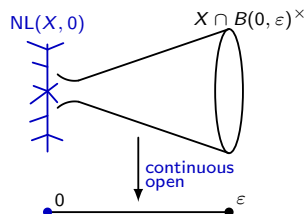
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Skeletons and combinatorics of $NL(X, 0)$

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If $\pi: (X_\pi, E_\pi) \rightarrow (X, 0)$ is a good resolution of $(X, 0)$ with dual graph Γ_π , there exists a **natural embedding**:

$$\Gamma_\pi \hookrightarrow NL(X, 0)$$

It sends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset E_\pi = \pi^{-1}(0)$ that corresponds to v .



This induces a canonical homeomorphism:

$$\varprojlim_{\pi} \Gamma_\pi \xrightarrow{\sim} NL(X, 0)$$

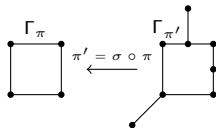
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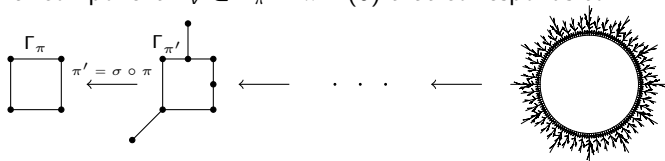
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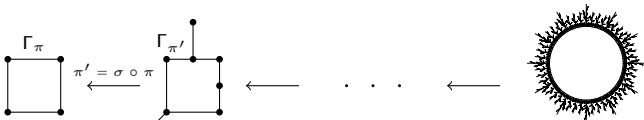
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Moreover, any proper birational map $f: (Y, D) \rightarrow (X, 0)$ induces an isomorphism $NL(Y, D) \cong NL(X, 0)$.

Theorem (Berkovich, Thuillier)

If $\pi: (Y, D) \rightarrow (X, 0)$ is a good resolution of $(X, 0)$, then the retraction $NL(X, 0) \rightarrow \Gamma_\pi$ extends to a strong deformation retraction.

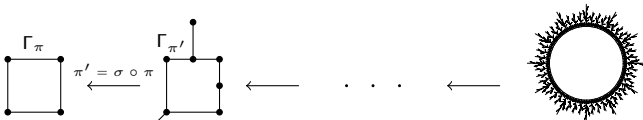
In particular, $NL(X, 0)$ has the homotopy type of Γ_π .

NB: this works over any perfect field k , in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual graph of a good resolution of $(X, 0)$ does not depend on the choice of the resolution.

Remark: Γ_π (and hence $NL(X, 0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



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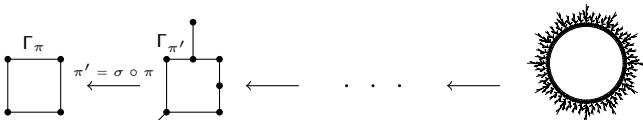
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Berkovich space structure

$NL(X, 0)$ inherits a **non-archimedean analytic structure** from the Berkovich space X^{an} .

Moreover, any good resolution π induces a decomposition of $NL(X, 0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k -surface.

$NL(X, 0)$ looks like a fractal:

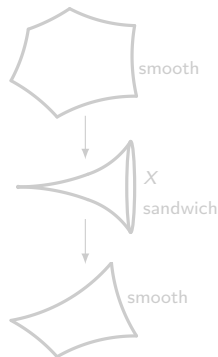
Application 2 (F–Favre–Ruggiero 2018)

Let $(X, 0)$ normal k -surface singularity.

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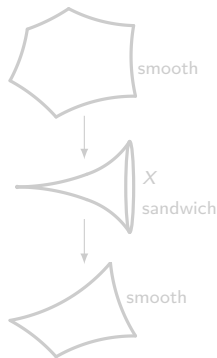
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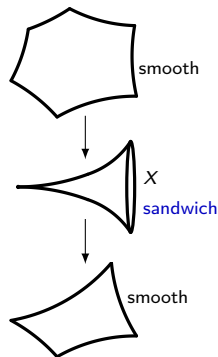
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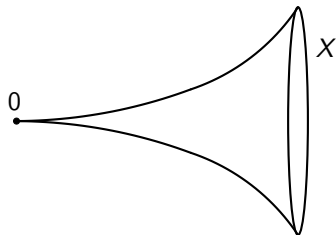


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A metric enters the game: Inner rates

Assume now that $(X, 0)$ is a **normal complex surface**.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

$$d_{\text{inner}}(\pi_*\gamma \cap S_{\mathbb{C}^n}(0, \varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0, \varepsilon)) \approx \varepsilon^{\mathcal{I}(E)}$$

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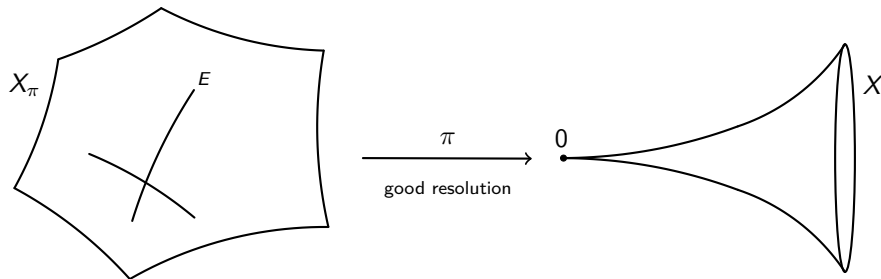
The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi(N(E))$ of $(X, 0)$



Fine understanding of the structure of the metric germ (X, d_{inner})

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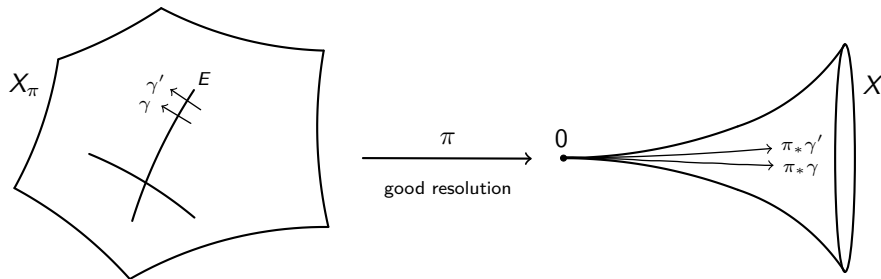
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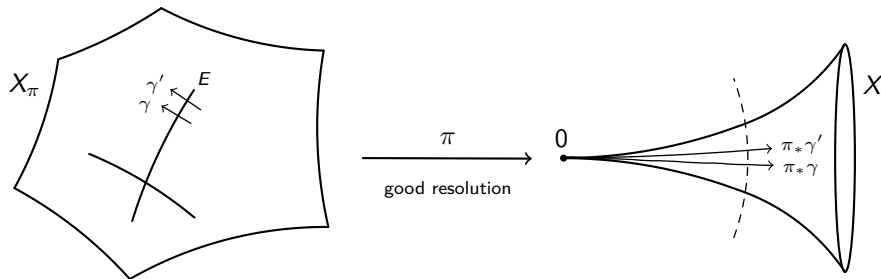
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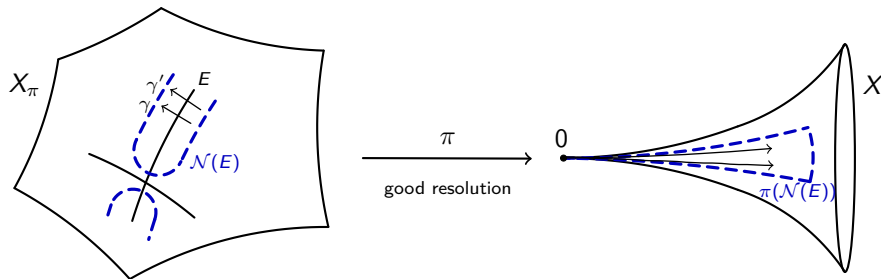
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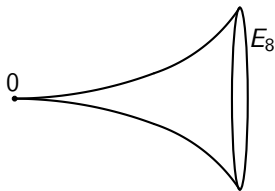
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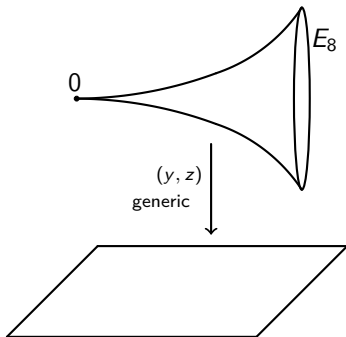


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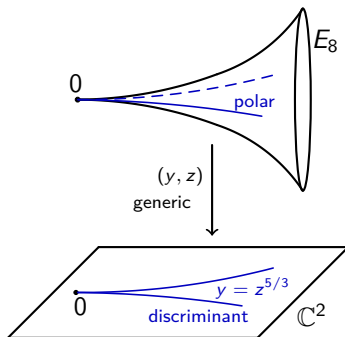


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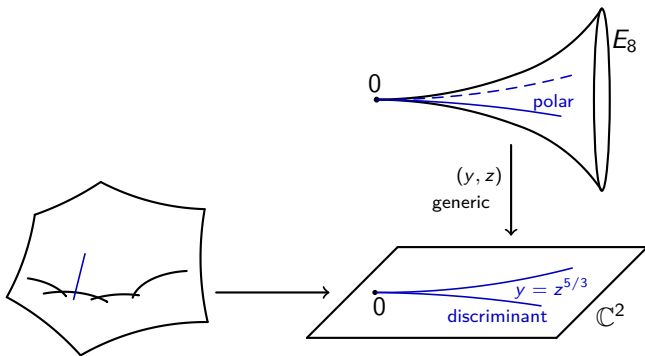
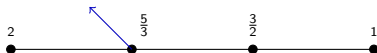
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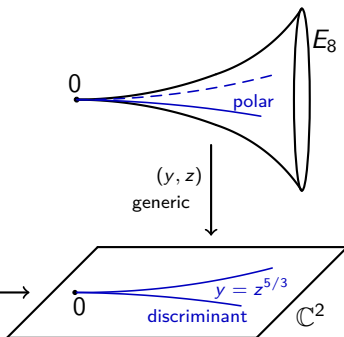
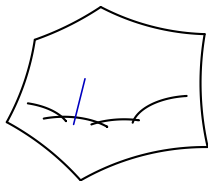
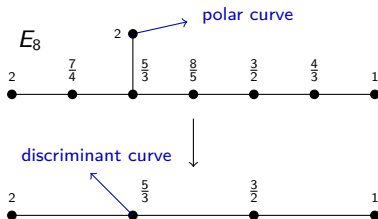


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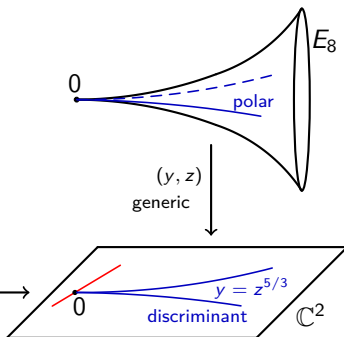
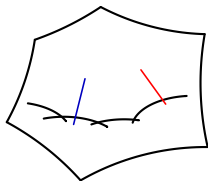
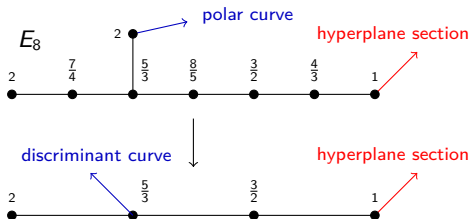


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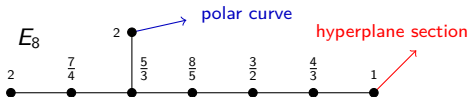
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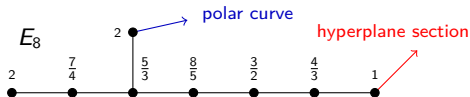


Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
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Theorem (Belotto–F–Pichon 2022, qualitative version)

Let $\pi: X_\pi \rightarrow X$ be a good resolution of $(X, 0)$ that factors through $\text{Bl}_0(X)$ and through the Nash transform.

Then all the inner rates of $(X, 0)$ are completely determined by:

- the topology of $(X, 0)$, i.e. the weighted dual graph Γ_π ;
- the **arrows** of a **generic** hyperplane section;
- the **arrows** of the polar curves of a **generic** projection $(X, 0) \rightarrow (\mathbb{C}^2, 0)$.

Our proof: study of the inner rate function on $\text{NL}(X, 0)$

In fact, $\mathcal{I}(E)$ only depends on the divisorial valuation ord_E . We get:

$$\text{inner rate function} \quad \mathcal{I}: \text{NL}(X, 0) \rightarrow \mathbb{R}_{\geq 1}$$

Our result is a precise formula for the Laplacian of \mathcal{I} on $\text{NL}(X, 0)$.

We gave a topological proof:
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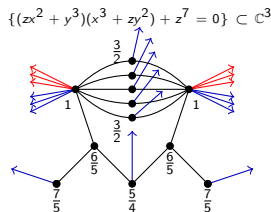
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Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g): (X, 0) \rightarrow (\mathbb{C}^2, 0)$.

$$\begin{array}{ccc} X_\pi & \xrightarrow[\text{(through Nash)}]{\pi} & (X, 0) \\ & \searrow \ell \downarrow \text{generic} & \\ & & (\mathbb{C}^2, 0) \end{array}$$

Idea:

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^* \omega)_p = \text{Jac}(\ell \circ \pi) dx \wedge dy = x^{\hat{k}_v^{\log}} f(x, y) \frac{dx}{x} \wedge dy$$

Then we have:

① $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)

② $f(x, y) = 0$ is a local equation for the strict transform of the polar curve of ℓ

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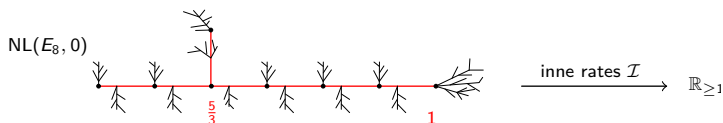
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Nobody is perfect (not even NL)

When we study the **Lipschitz geometry** of $((X, 0), d_{\text{inner}})$ (that is, its equivalence class up to bi-Lipschitz homeomorphisms) we reach the limitations of $\text{NL}(X, 0)$.

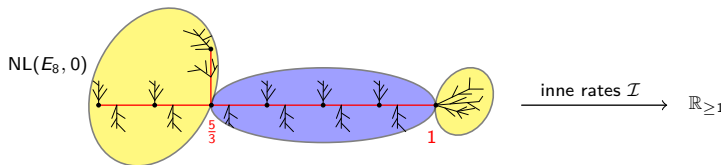


We can see the complete inner geometry invariant of surfaces (Birbrair–Neumann–Pichon 2014) as a canonical decomposition of $\text{NL}(X, 0)$.

However, the data of $\text{NL}(X, 0)$ and \mathcal{I} is too simple to retrieve this decomposition! We need to keep track of the topology of $(X, 0)$.

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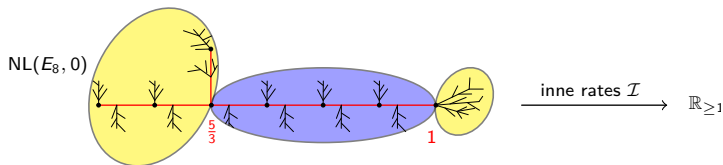


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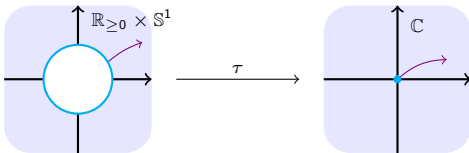
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The logarithmic link: a new name for an old object

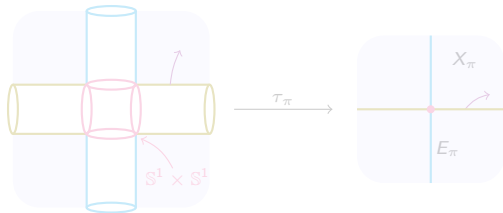
Idea going back to [A'Campo's](#) 1975 paper “La fonction zeta d'une monodromie”:

Real oriented blowups (a.k.a. polar coordinate changes) give **objects at radius 0**.

$$\begin{aligned}\tau: \mathbb{R}_{\geq 0} \times \mathbb{S}^1 &\longrightarrow \mathbb{C} \\ (r, s) &\longmapsto rs\end{aligned}$$



This can be done globally: pick a good resolution $\pi: (X_\pi, E_\pi) \rightarrow (X, 0)$, and perform the real oriented blowup of X_π along E_π :



$$E_\pi^{\log} = \tau_\pi^{-1}(E_\pi) \text{ logarithmic link}$$

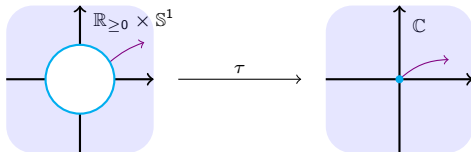
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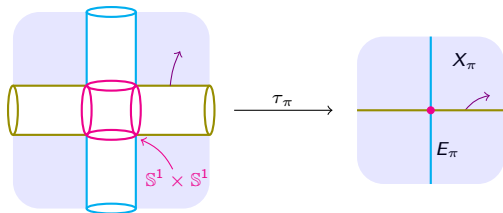
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Can be done functorially, using **logarithmic geometry** and **Kato–Nakayama spaces**.

- Yields a true manifold $\simeq L(X, 0)$, but **canonical** (at radius 0)
- Its points lift to subanalytic arcs, giving a **ultrametric distance** (given by the orders of contacts w.r.t. the given metric)

Remark: Constructions recently used with great success in singularity theory:

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With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne’s talk to hear more!) SPONSORED CONTENT

Key: interplay between the archimedean and non-archimedean topologies: jumps in the (usual) topology of non-archimedean balls.

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¡Gracias!