A zoo of singularity links: topological, non-archimedean, and logarithmic

Lorenzo Fantini

École polytechnique

Includes joint works with C. Favre and M. Ruggiero, A. Belotto da Silva and A. Pichon.

Mid-Atlantic Geometry and Singularities Conference
Universidad de La Laguna, Tenerife

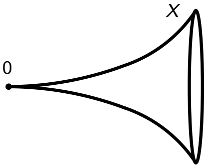
May 16th, 2025

Slides: `https://lorenzofantini.eu/fantini-tenerife.pdf`

The topological link of a singularity

X complex variety,
$0 \in X$ isolated singularity

$(X,0) \hookrightarrow (\mathbb{C}^N,0)$



Topology: the Conical Structure Theorem

$0<\varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

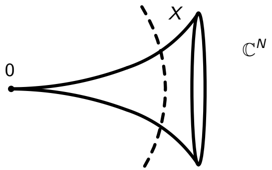
(oriented) top. type of $L(X,0)$ $\iff$ weighted dual graph Γ_π of π.

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).
[" $\implies$ " Conical Structure Theorem, " $\impliedby$ " in F–Pichon 2025]

The topological link of a singularity

X complex variety,
$0 \in X$ isolated singularity

$(X,0) \hookrightarrow (\mathbb{C}^N, 0)$



Topology: the Conical Structure Theorem

$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

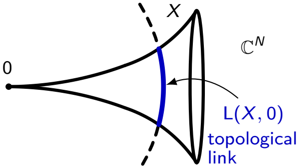
(oriented) top. type of $L(X,0)$ $\iff$ weighted dual graph Γ_π of π.

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).
[“ $\implies$ ” Conical Structure Theorem, “ $\impliedby$ ” in F–Pichon 2025]

The topological link of a singularity

X complex variety,
$0 \in X$ isolated singularity

$(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

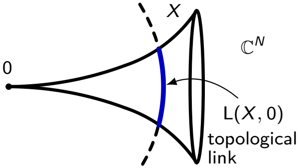
(oriented) top. type of $L(X,0)$ $\iff$ weighted dual graph Γ_π of π.

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).
[" $\implies$ " Conical Structure Theorem, " $\impliedby$ " in F–Pichon 2025]

The topological link of a singularity

X complex variety,
$0 \in X$ isolated singularity

$(X,0) \hookrightarrow (\mathbb{C}^N, 0)$



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

(oriented) top. type of $L(X,0)$ $\iff$ weighted dual graph Γ_π of π.

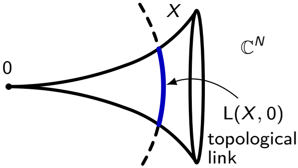
with genus and self-intersection of the E_i

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).
[" $\implies$ " Conical Structure Theorem, " $\impliedby$ " in F–Pichon 2025]

The topological link of a singularity

X complex variety, $\qquad (X,0) \hookrightarrow (\mathbb{C}^N,0)$

$0 \in X$ isolated singularity



Topology: the Conical Structure Theorem

$$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

(oriented) top. type of $L(X,0) \quad \Longleftrightarrow \quad$ weighted dual graph Γ_π of π.

with genus and self-intersection of the E_i

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).

[" $\Longrightarrow$ " Conical Structure Theorem, " $\Longleftarrow$ " in F–Pichon 2025]

The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$$\mathrm{NL}(X,0)=\left\{v\colon \widehat{\mathcal{O}_{X,0}}\to\mathbb{R}_+\cup\{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\}=1\right\}$$

$v(0)=+\infty,\ v(fg)=v(f)+v(g),\ v(f+g)\geq\min\{v(f),v(g)\}$

- if $(\gamma,0)\subset(X,0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi\colon(Y,D)\to(X,0)$ is a modification (Y normal, D snc divisor), then $\mathrm{ord}_{\mathrm{E}}$ is a semivaluation (divisorial valuation)

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}},0)\cong$ valuative tree
(Favre–Jonsson 2004).

The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\} = 1 \right\}$$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi\colon (Y, D) \to (X, 0)$ is a modification (Y normal, D snc divisor), then ord_E is a semivaluation (divisorial valuation)

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson 2004).

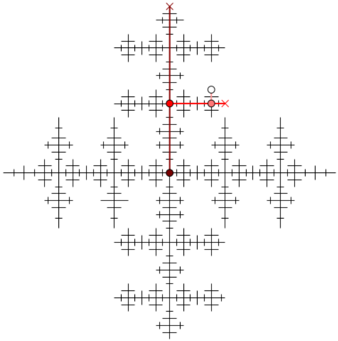
The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\} = 1 \right\}$$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X, 0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi \colon (Y, D) \to (X, 0)$ is a modification (Y normal, D snc divisor), then $\mathrm{ord}_{\mathrm{E}}$ is a semivaluation (divisorial valuation)



It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree (Favre–Jonsson 2004).

The non-archimedean link deserves its name

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \,\middle|\, \max_i |z_i(x)| = \varepsilon \right\}$$

Theorem (F–Favre, Poineau 2025)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.

The non-archimedean link deserves its name

Intuition

$\mathsf{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathsf{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \,\middle|\, \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre, Poineau 2025)

$\mathrm{L}(X,0)$ degenerates towards $\mathsf{NL}(X,0)$.

The non-archimedean link deserves its name

Intuition

$\mathsf{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

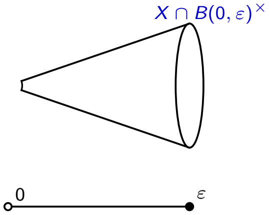
Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathsf{NL}(X,0)=\left\{x\in X^{\mathrm{an}}\;\middle|\;\max_i|z_i(x)|=\varepsilon\right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre, Poineau 2025)

$\mathsf{L}(X,0)$ degenerates towards $\mathsf{NL}(X,0)$.



The non-archimedean link deserves its name

Intuition

$\mathrm{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

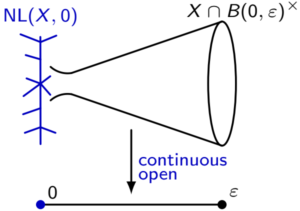
Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathrm{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \,\middle|\, \max_i |z_i(x)| = \varepsilon \right\}$$

Berkovich space of X w.r.t. the trivial absolute value

Theorem (F–Favre, Poineau 2025)

$\mathrm{L}(X,0)$ degenerates towards $\mathrm{NL}(X,0)$.



Skeletons and combinatorics of NL$(X, 0)$

$$\mathrm{NL}(X,0) = \left\{ v\colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\} = 1 \right\}$$

If $\pi\colon (X_\pi, E_\pi) \to (X, 0)$ is a good resolution of $(X, 0)$ with dual complex Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X, 0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset E_\pi = \pi^{-1}(0)$ that corresponds to v.

This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X, 0)$$

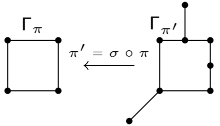
Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{M}_{X,0}} \{v(f)\} = 1 \right\}$$

If $\pi\colon (X_\pi, E_\pi) \to (X,0)$ is a good resolution of $(X,0)$ with dual complex Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset E_\pi = \pi^{-1}(0)$ that corresponds to v.



This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X,0)$$

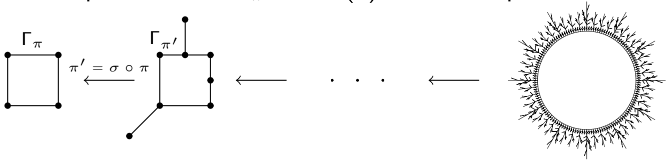
Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0)=\left\{v\colon \widehat{\mathcal{O}_{X,0}}\to\mathbb{R}_+\cup\{+\infty\}\text{ semivaluation}\;\middle|\;\min_{f\in\mathfrak{m}_{X,0}}\{v(f)\}=1\right\}$$

If $\pi\colon(X_\pi,E_\pi)\to(X,0)$ is a good resolution of $(X,0)$ with dual complex Γ_π, there exists a natural embedding:

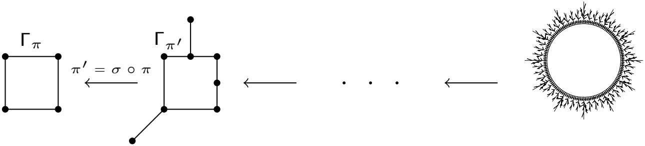
$$\Gamma_\pi\hookrightarrow\mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v\subset E_\pi=\pi^{-1}(0)$ that corresponds to v.



This induces a canonical homeomorphism:

$$\varprojlim_\pi\Gamma_\pi\xleftarrow{\ \sim\ }\mathrm{NL}(X,0)$$



Theorem (Berkovich, Thuillier)

If $\pi\colon (X_\pi, E_\pi) \to (X,0)$ is a good resolution of $(X,0)$, then the retraction $\mathrm{NL}(X,0) \to \Gamma_\pi$ extends to a strong deformation retraction.

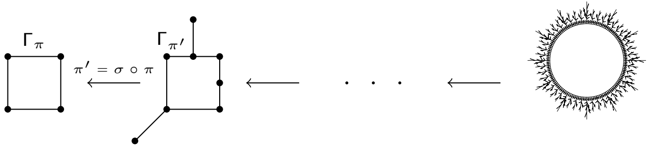
In particular, $\mathrm{NL}(X,0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual complex of a good resolution of $(X,0)$ does not depend on the choice of the resolution.

Remark: Γ_π (and hence $\mathrm{NL}(X,0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



Theorem (Berkovich, Thuillier)

If $\pi\colon (X_\pi, E_\pi) \to (X, 0)$ is a good resolution of $(X, 0)$, then the retraction $\mathrm{NL}(X, 0) \to \Gamma_\pi$ extends to a strong deformation retraction.

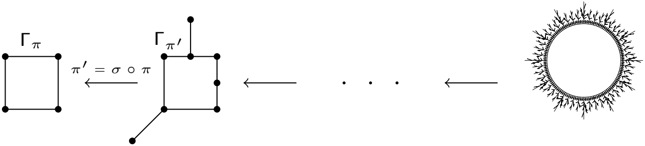
In particular, $\mathrm{NL}(X, 0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual complex of a good resolution of $(X, 0)$ does not depend on the choice of the resolution.

Remark: Γ_π (and hence $\mathrm{NL}(X, 0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



Theorem (Berkovich, Thuillier)

If $\pi \colon (X_\pi, E_\pi) \to (X, 0)$ is a good resolution of $(X, 0)$, then the retraction $\mathrm{NL}(X, 0) \to \Gamma_\pi$ extends to a strong deformation retraction.

In particular, $\mathrm{NL}(X, 0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual complex of a good resolution of $(X, 0)$ does not depend on the choice of the resolution.

Remark: Γ_π (and hence $\mathrm{NL}(X, 0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).

Berkovich space structure

NL$(X,0)$ inherits a non-archimedean analytic structure from the Berkovich space X^{an}.

Moreover, any good resolution π induces a decomposition of NL$(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

NL$(X,0)$ looks like a fractal:

Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

NL$(X,0)$ is self-similar $\iff$ $(X,0)$ is sandwich

Berkovich space structure

NL$(X,0)$ inherits a non-archimedean analytic structure from the Berkovich space X^{an}.

Moreover, any good resolution π induces a decomposition of NL$(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

divisorial valuations of the minimal resolution

NL$(X,0)$ looks like a fractal:

Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

NL$(X,0)$ is self-similar $\iff$ $(X,0)$ is sandwich

Berkovich space structure

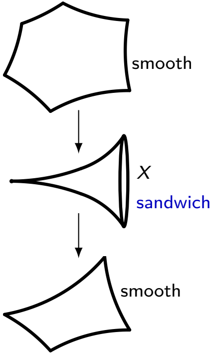
> $\mathrm{NL}(X,0)$ inherits a non-archimedean analytic structure from the Berkovich space X^{an}.

Moreover, any good resolution π induces a decomposition of $\mathrm{NL}(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

divisorial valuations of the minimal resolution



$\mathrm{NL}(X,0)$ looks like a fractal:

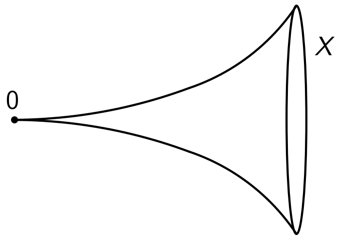
Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

$\mathrm{NL}(X,0)$ is self-similar $\iff$ $(X,0)$ is sandwich

A metric enters the game: Inner rates

Assume now that $(X,0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

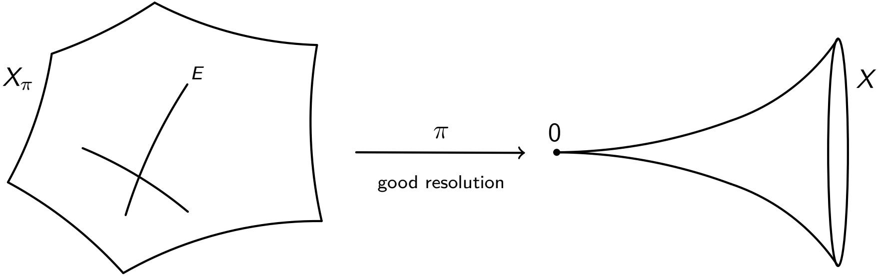
$$d_{\text{inner}}\left(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\right) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

A metric enters the game: Inner rates

Assume now that $(X, 0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

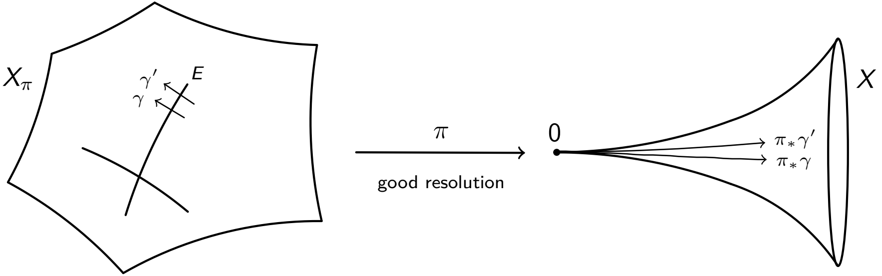
$$d_{\mathrm{inner}}\left(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\right) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

A metric enters the game: Inner rates

Assume now that $(X,0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X,0)$, with respect to the inner metric:

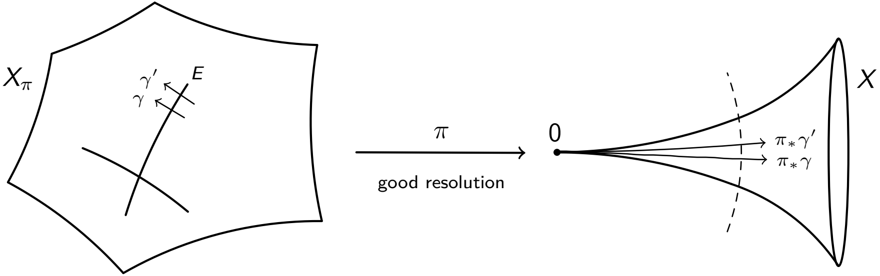
$$d_{\mathrm{inner}}\big(\pi_*\gamma\cap S_{\mathbb{C}^n}(0,\varepsilon),\pi_*\gamma'\cap S_{\mathbb{C}^n}(0,\varepsilon)\big)\approx\varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X,0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X,d_{inner})

A metric enters the game: Inner rates

Assume now that $(X, 0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

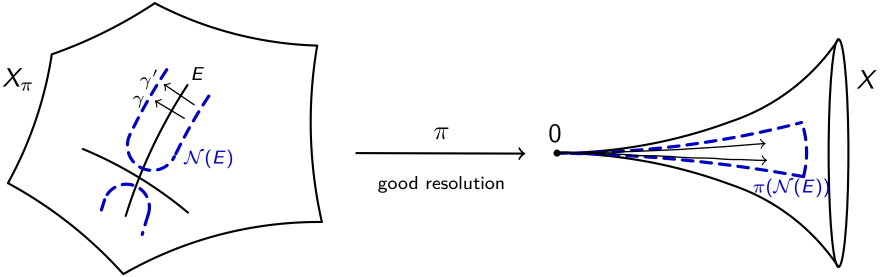
$$d_{\mathrm{inner}}\left(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\right) \approx \varepsilon^{\mathcal{I}(E)}$$

length of shortest path on X

Interpretation:
The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

A metric enters the game: Inner rates

Assume now that $(X, 0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

$$d_{\mathrm{inner}}\left(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\right) \approx \varepsilon^{\mathcal{I}(E)}$$

length of shortest path on X

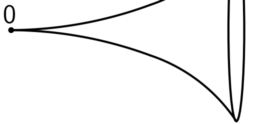
Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$$\downarrow (y, z)$$

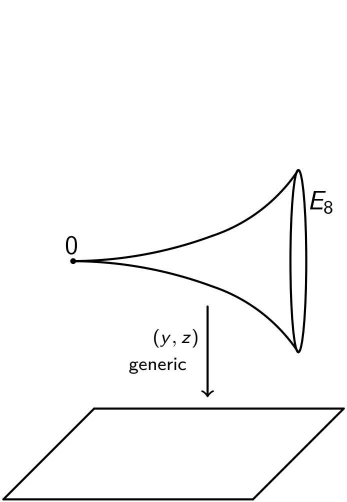
discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$



Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$$\downarrow (y, z)$$

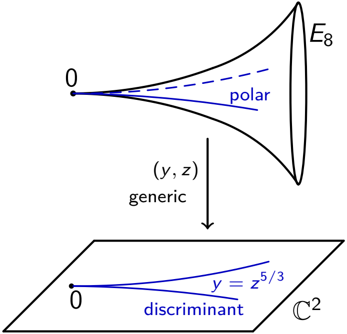
discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$



Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

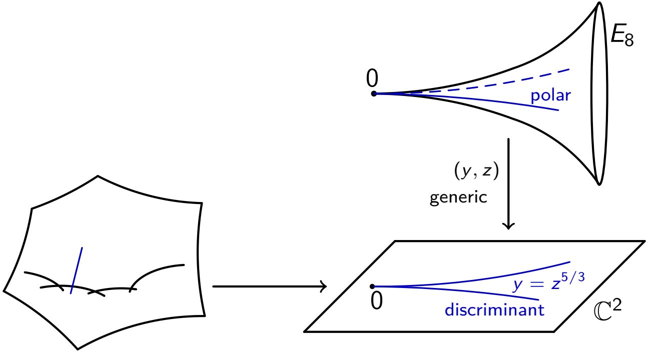


Example:

$$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

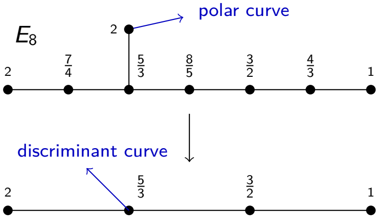


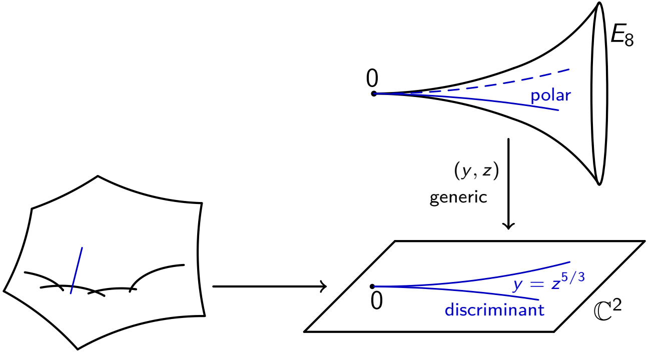
Example:

$$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$

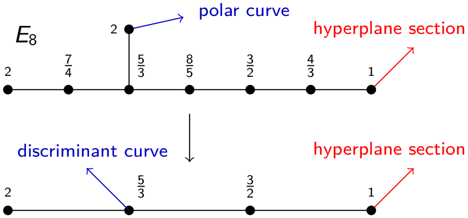


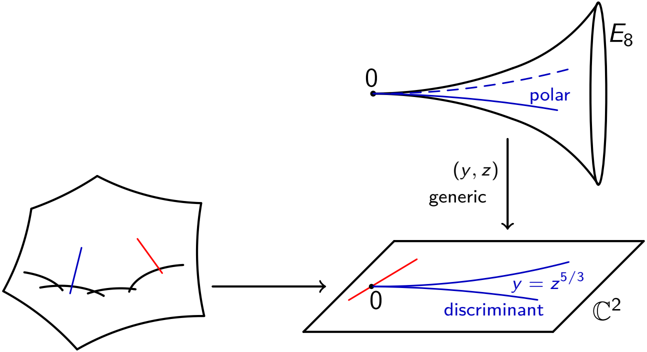


Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

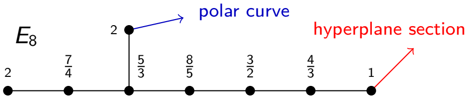
$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$





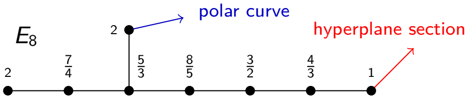
Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$



Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
- How to compute them?

Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$



Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
- How to compute them?

Theorem (Belotto–F–Pichon 2022, qualitative version)

Let $\pi \colon X_\pi \to X$ be a good resolution of $(X, 0)$ that factors through $\mathrm{Bl}_0(X)$ and through the Nash transform.
Then all the inner rates of $(X, 0)$ are completely determined by:

- the topology of $(X, 0)$, i.e. the weighted dual graph Γ_π;
- the arrows of a **generic** hyperplane section;
- the arrows of the polar curves of a **generic** projection $(X, 0) \to (\mathbb{C}^2, 0)$.

Our proof: study of the inner rate function on $\mathrm{NL}(X,0)$

In fact, $\mathcal{I}(E)$ only depends on the divisorial valuation ord_E. We get:

inner rate function $\quad \mathcal{I}\colon \mathrm{NL}(X,0) \to \mathbb{R}_{\geq 1}$

Our result is a precise formula for the Laplacian of $\mathcal{I}$ on $\mathrm{NL}(X,0)$.

We gave a topological proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2,0)$ to the singular case, by studying the topology and monodromy of the Milnor fiber of a generic linear form.

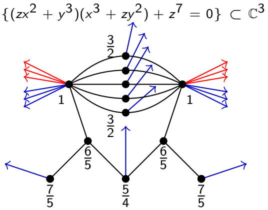
Our proof: study of the inner rate function on $\mathrm{NL}(X, 0)$

In fact, $\mathcal{I}(E)$ only depends on the divisorial valuation ord_E. We get:

inner rate function $\quad \mathcal{I}\colon \mathrm{NL}(X, 0) \to \mathbb{R}_{\geq 1}$

Our result is a precise formula for the Laplacian of $\mathcal{I}$ on $\mathrm{NL}(X, 0)$.

We gave a topological proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2, 0)$ to the singular case, by studying the topology and monodromy of the Milnor fiber of a generic linear form.



Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g)\colon (X, 0) \to (\mathbb{C}^2, 0)$.

Idea:

$$\begin{tikzcd} X_\pi \arrow[r, "\pi"', "(\textit{through Nash})"] & (X,0) \arrow[d, "\ell", "\textit{generic}"'] \\ & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^*\omega)_p = \mathrm{Jac}(\ell \circ \pi)\, dx \wedge dy = x^{\hat{k}_v^{\log}} f(x,y) \frac{dx}{x} \wedge dy$$

Then we have:

1. $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)
2. $f(x, y) = 0$ is a local equation for the strict transform of the polar curve of ℓ

And the Laplacian formula follows from adjunction.

Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g)\colon (X, 0) \to (\mathbb{C}^2, 0)$.

Idea:

$$\begin{tikzcd} X_\pi \arrow[r, "\pi", "(\textit{through Nash})"'] & (X,0) \arrow[d, "\ell", "\textit{generic}"'] \\ & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^*\omega)_p = \mathrm{Jac}(\ell \circ \pi)\, dx \wedge dy = x^{\hat{k}_v^{\log}} f(x, y) \frac{dx}{x} \wedge dy$$

Then we have:

1. $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)
2. $f(x, y) = 0$ is a local equation for the strict transform of the polar curve of ℓ

And the Laplacian formula follows from adjunction.

Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g) \colon (X, 0) \to (\mathbb{C}^2, 0)$.

Idea:
$$\begin{tikzcd} X_\pi \arrow[r, "\pi", "(\textit{through Nash})"'] & (X,0) \arrow[d, "\ell", "\textit{generic}"'] \\ & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^*\omega)_p = \mathrm{Jac}(\ell \circ \pi)\, dx \wedge dy = x^{\hat{k}_v^{\log}} f(x,y) \frac{dx}{x} \wedge dy$$

Mather log discrepancy of v

Then we have:

1. $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)
2. $f(x, y) = 0$ is a local equation for the strict transform of the polar curve of ℓ

And the Laplacian formula follows from adjunction.

Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g) \colon (X, 0) \to (\mathbb{C}^2, 0)$.

Idea:
$$\begin{tikzcd} X_\pi \arrow[r, "\pi", "(\textit{through Nash})"'] & (X,0) \arrow[d, "\ell", "\textit{generic}"'] \\ & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^*\omega)_p = \mathrm{Jac}(\ell \circ \pi)\, dx \wedge dy = x^{\hat{k}_v^{\log}} f(x, y)\, \frac{dx}{x} \wedge dy$$

Mather log discrepancy of v

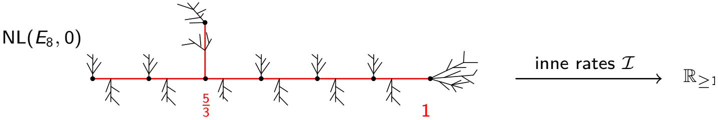
Then we have:

1. $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)
2. $f(x, y) = 0$ is a local equation for the strict transform of the polar curve of ℓ

And the Laplacian formula follows from adjunction.

Nobody is perfect (not even NL)

When we study the Lipschitz geometry of $\big((X,0), d_{\mathrm{inner}}\big)$ (that is, its equivalence class up to bi-Lipschitz homeomorphisms) we reach the limitations of $\mathrm{NL}(X,0)$.

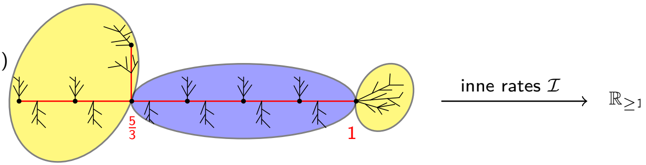


We can see the complete inner geometry invariant of surfaces (Birbrair–Neumann–Pichon 2014) as a canonical decomposition of $\mathrm{NL}(X,0)$.

However, the data of $\mathrm{NL}(X,0)$ and $\mathcal{I}$ is too simple to retrieve this decomposition! We need to keep track of the topology of $(X,0)$.

Nobody is perfect (not even NL)

When we study the Lipschitz geometry of $\big((X,0), d_{\mathrm{inner}}\big)$ (that is, its equivalence class up to bi-Lipschitz homeomorphisms) we reach the limitations of $\mathsf{NL}(X,0)$.

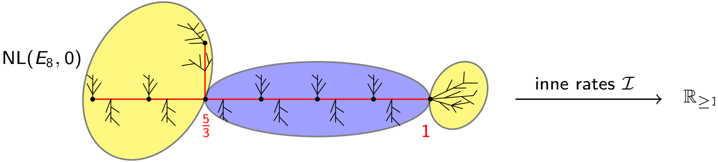


We can see the complete inner geometry invariant of surfaces (Birbrair–Neumann–Pichon 2014) as a canonical decomposition of $\mathsf{NL}(X,0)$.

However, the data of $\mathsf{NL}(X,0)$ and $\mathcal{I}$ is too simple to retrieve this decomposition! We need to keep track of the topology of $(X,0)$.

Nobody is perfect (not even NL)

When we study the Lipschitz geometry of $\big((X,0), d_{\mathrm{inner}}\big)$ (that is, its equivalence class up to bi-Lipschitz homeomorphisms) we reach the limitations of $\mathsf{NL}(X,0)$.



We can see the complete inner geometry invariant of surfaces (Birbrair–Neumann–Pichon 2014) as a canonical decomposition of $\mathsf{NL}(X,0)$.

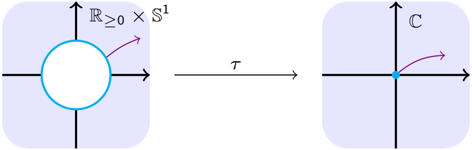
However, the data of $\mathsf{NL}(X,0)$ and $\mathcal{I}$ is too simple to retrieve this decomposition! We need to keep track of the topology of $(X,0)$.

The logarithmic link: a new name for an old object

Idea going back to A'Campo's 1975 paper "La fonction zeta d'une monodromie":

Real oriented blowups (a.k.a. polar coordinate changes) give objects at radius 0.

$$\begin{aligned} \tau\colon \mathbb{R}_{\geq 0} \times \mathbb{S}^1 &\longrightarrow \mathbb{C} \\ (r, s) &\longmapsto rs \end{aligned}$$



This can be done globally and in any dimension: pick a good resolution $\pi\colon (X_\pi, E_\pi) \to (X, 0)$, and perform the real oriented blowp of X_π along E_π:

$E_\pi^{\log} = \tau_\pi^{-1}(E_\pi)$ logarithmic link

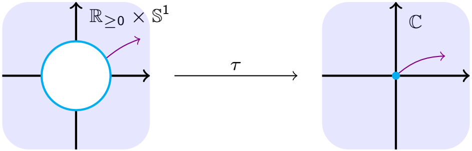
It is a manifold with corners, homeomorphic to $\mathrm{L}(X, 0)$.

The logarithmic link: a new name for an old object

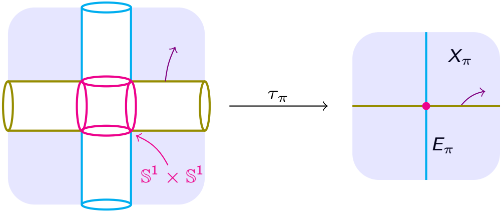
Idea going back to A'Campo's 1975 paper "La fonction zeta d'une monodromie":

> Real oriented blowups (a.k.a. polar coordinate changes) give objects at radius 0.

$$\begin{aligned} \tau\colon \mathbb{R}_{\geq 0} \times \mathbb{S}^1 &\longrightarrow \mathbb{C} \\ (r, s) &\longmapsto rs \end{aligned}$$



This can be done globally and in any dimension: pick a good resolution $\pi\colon (X_\pi, E_\pi) \to (X, 0)$, and perform the real oriented blowp of X_π along E_π:



$E_\pi^{\log} = \tau_\pi^{-1}(E_\pi)$ logarithmic link

It is a manifold with corners, homeomorphic to $\mathrm{L}(X, 0)$.

The logarithmic link of a singularity

Can be done functorially, using logarithmic geometry and Kato–Nakayama spaces.

- Yields a true manifold $\simeq \mathrm{L}(X, 0)$, but canonical (at radius 0)
- Its points lift to subanalytic arcs, giving a ultrametric distance (given by the orders of contacts w.r.t. the given metric)

Remark: Constructions recently used with great success in singularity theory:

- de Bobadilla–Pełka (equimultiplicity of μ-constant families with isolated singularities, 2024),
- Cueto–Popescu-Pampu–Stepanov (Proof of Neumann and Wahl's Milnor fiber conjecture on splice surface singularities, 2023)

With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne's talk to hear more!) SPONSORED CONTENT

Key: interplay between the archimedean and non-archimedean topologies: jumps in the (usual) topology of non-archimedean balls.

The logarithmic link of a singularity

Can be done functorially, using logarithmic geometry and Kato–Nakayama spaces.

- Yields a true manifold $\simeq \mathsf{L}(X,0)$, but canonical (at radius 0)
- Its points lift to subanalytic arcs, giving a ultrametric distance (given by the orders of contacts w.r.t. the given metric)

Remark: Constructions recently used with great success in singularity theory:

- de Bobadilla–Pełka (equimultiplicity of μ-constant families with isolated singularities, 2024),
- Cueto–Popescu-Pampu–Stepanov (Proof of Neumann and Wahl's Milnor fiber conjecture on splice surface singularities, 2023)

With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne's talk to hear more!)[SPONSORED CONTENT]

Key: interplay between the archimedean and non-archimedean topologies: jumps in the (usual) topology of non-archimedean balls.

The logarithmic link of a singularity

Can be done functorially, using logarithmic geometry and Kato–Nakayama spaces.

- Yields a true manifold $\simeq \mathrm{L}(X,0)$, but canonical (at radius 0)
- Its points lift to subanalytic arcs, giving a ultrametric distance (given by the orders of contacts w.r.t. the given metric)

Remark: Constructions recently used with great success in singularity theory:

- de Bobadilla–Pełka (equimultiplicity of μ-constant families with isolated singularities, 2024),
- Cueto–Popescu-Pampu–Stepanov (Proof of Neumann and Wahl's Milnor fiber conjecture on splice surface singularities, 2023)

With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne's talk to hear more!) SPONSORED CONTENT

Key: interplay between the archimedean and non-archimedean topologies: jumps in the (usual) topology of non-archimedean balls.

The logarithmic link of a singularity

Can be done functorially, using logarithmic geometry and Kato–Nakayama spaces.

- Yields a true manifold $\simeq \mathrm{L}(X,0)$, but canonical (at radius 0)
- Its points lift to subanalytic arcs, giving a ultrametric distance (given by the orders of contacts w.r.t. the given metric)

Remark: Constructions recently used with great success in singularity theory:

- de Bobadilla–Pełka (equimultiplicity of μ-constant families with isolated singularities, 2024),
- Cueto–Popescu-Pampu–Stepanov (Proof of Neumann and Wahl's Milnor fiber conjecture on splice surface singularities, 2023)

With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne's talk to hear more!) SPONSORED CONTENT

Key: interplay between the archimedean and non-archimedean topologies: jumps in the (usual) topology of non-archimedean balls.

¡Gracias!