A zoo of singularity links: topological, non-archimedean, and logarithmic

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École polytechnique

Includes joint works with C. Favre and M. Ruggiero, A. Belotto da Silva and A. Pichon.

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Universidad de La Laguna, Tenerife

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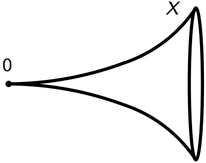
Slides: `https://lorenzofantini.eu/fantini-tenerife.pdf`

The topological link of a singularity

X complex variety, $(X, 0) \hookrightarrow (\mathbb{C}^N, 0)$
$0 \in X$ isolated singularity



Topology: the Conical Structure Theorem

$0 < \varepsilon \ll 1 \implies X \cap B(0,\varepsilon) \overset{\text{homeo}}{\sim} \mathrm{Cone}\big(X \cap S(0,\varepsilon)\big)$

$\mathrm{L}(X,0) = X \cap S(0,\varepsilon)$ real $(2\dim_{\mathbb{C}} X - 1)$-manifold, well-defined up to homeom.

Theorem (Neumann, 1981)

$(X,0)$ surface, $\pi\colon (X_\pi, E_\pi) \to (X,0)$ minimal good resolution of $(X,0)$. Then:

(oriented) top. type of $L(X,0)$ $\iff$ weighted dual graph Γ_π of π.

Remark: This is also equivalent to the (oriented) top. type of $(X,0)$ (in dim 2!).
[" $\Longrightarrow$ " Conical Structure Theorem, " $\Longleftarrow$ " in F–Pichon 2025]

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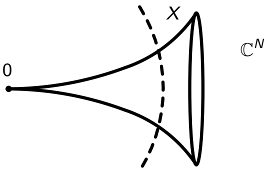

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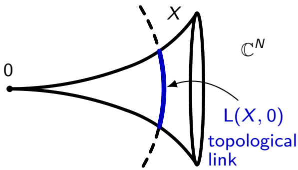

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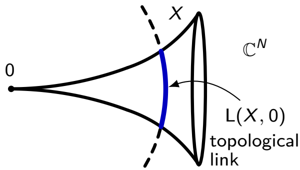

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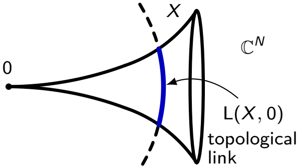

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The non-archimedean link of a singularity

Definition (Boucksom–Favre–Jonsson, F)

$\mathrm{NL}(X,0) = \left\{ v \colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f \in \mathfrak{m}_{X,0}} \{v(f)\} = 1 \right\}$

$v(0) = +\infty, v(fg) = v(f) + v(g), v(f+g) \geq \min\{v(f), v(g)\}$

- if $(\gamma, 0) \subset (X,0)$ curve germ, the order of vanishing at 0 along γ is a semivaluation
- if $\pi \colon (Y,D) \to (X,0)$ is a modification (Y normal, D snc divisor), then $\mathrm{ord_E}$ is a semivaluation (divisorial valuation)

It's a nice topological space, compact.

Example: $\mathrm{NL}(\mathbb{A}^2_{\mathbb{C}}, 0) \cong$ valuative tree
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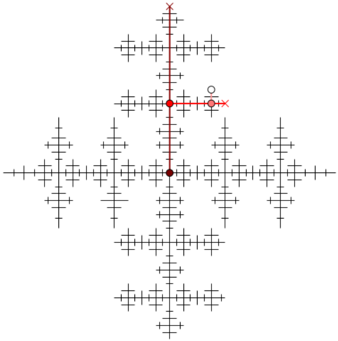

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Intuition

$\mathsf{NL}(X,0)$ is a non-archimedean version of the usual link $\mathrm{L}(X,0)$

Indeed, denote by $(z_1,\ldots,z_N)$ a set of local coordinates of $\mathbb{C}^N$ at 0. Then:

$$\mathsf{NL}(X,0) = \left\{ x \in X^{\mathrm{an}} \;\middle|\; \max_i |z_i(x)| = \varepsilon \right\}$$

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$\mathrm{L}(X,0)$ degenerates towards $\mathsf{NL}(X,0)$.

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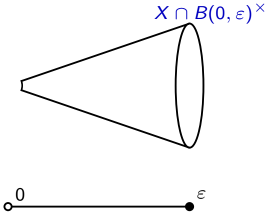

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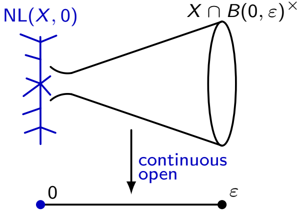

Skeletons and combinatorics of $\mathrm{NL}(X,0)$

$$\mathrm{NL}(X,0) = \left\{ v\colon \widehat{\mathcal{O}_{X,0}} \to \mathbb{R}_+ \cup \{+\infty\} \text{ semivaluation} \;\middle|\; \min_{f\in\mathfrak{m}_{X,0}}\{v(f)\} = 1 \right\}$$

If $\pi\colon (X_\pi, E_\pi) \to (X,0)$ is a good resolution of $(X,0)$ with dual graph Γ_π, there exists a natural embedding:

$$\Gamma_\pi \hookrightarrow \mathrm{NL}(X,0)$$

It seends a vertex v of Γ_π to the divisorial valuation associated with the exceptional component $E_v \subset E_\pi = \pi^{-1}(0)$ that corresponds to v.

This induces a canonical homeomorphism:

$$\varprojlim_\pi \Gamma_\pi \xleftarrow{\sim} \mathrm{NL}(X,0)$$

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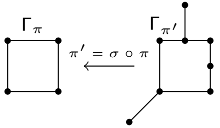

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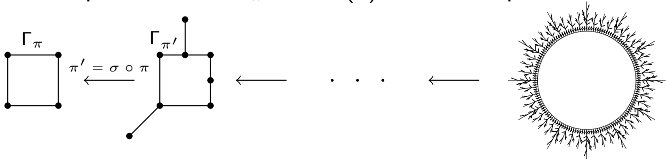

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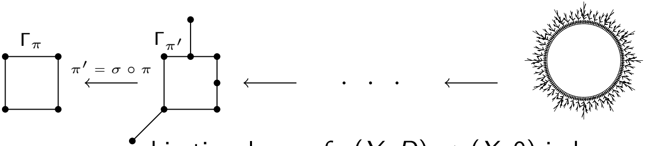


Moreover, any proper birational map $f\colon (Y, D) \to (X, 0)$ induces an isomorphism $\mathrm{NL}(Y, D) \cong \mathrm{NL}(X, 0)$.

Theorem (Berkovich, Thuillier)

If $\pi\colon (Y, D) \to (X, 0)$ is a good resolution of $(X, 0)$, then the retraction $\mathrm{NL}(X, 0) \to \Gamma_\pi$ extends to a strong deformation retraction.

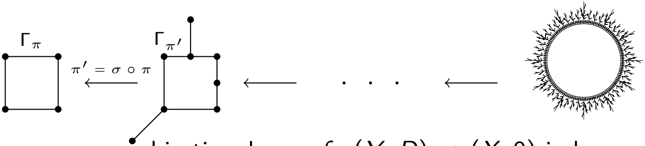
In particular, $\mathrm{NL}(X, 0)$ has the homotopy type of Γ_π.

NB: this works over any perfect field k, in any dimension.

Corollary (Stepanov 2006, Thuillier 2007)

The homotopy type of the dual graph of a good resolution of $(X, 0)$ does not depend on the choice of the resolution.

Remark: Γ_π (and hence $\mathrm{NL}(X, 0)$) can be essentially arbitrary (Kollár), but it is contractible for isolated log terminal singularities (de Fernex–Kollár–Xu).



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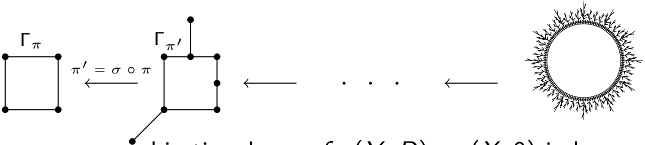

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Berkovich space structure

$\mathrm{NL}(X,0)$ inherits a non-archimedean analytic structure from the Berkovich space X^{an}.

Moreover, any good resolution π induces a decomposition of $\mathrm{NL}(X,0)$ into (non-archimedean) discs and annuli. (Similar to the topological link!)

Application 1 (F 2014)

Non-archimedean characterization of Nash's essential valuations of a k-surface.

$\mathrm{NL}(X,0)$ looks like a fractal:

Application 2 (F–Favre–Ruggiero 2018)

Let $(X,0)$ normal k-surface singularity.

$\mathrm{NL}(X,0)$ is self-similar $\Longleftrightarrow$ $(X,0)$ is sandwich

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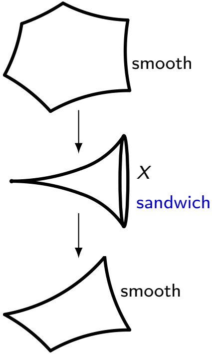

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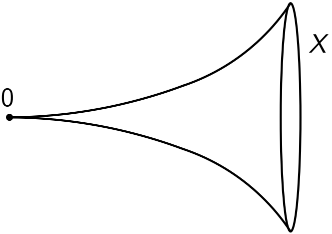
A metric enters the game: Inner rates

Assume now that $(X, 0)$ is a normal complex surface.



The inner rate $\mathcal{I}(E)$ of E is the contact order between the two curves $\pi_*\gamma$ and $\pi_*\gamma'$ on $(X, 0)$, with respect to the inner metric:

$$d_{\mathrm{inner}}\big(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\big) \approx \varepsilon^{\mathcal{I}(E)}$$

Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

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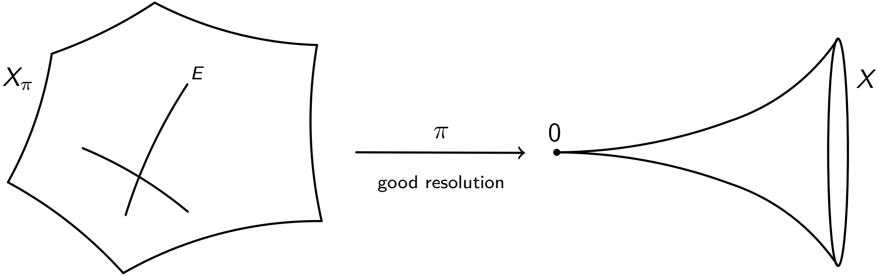

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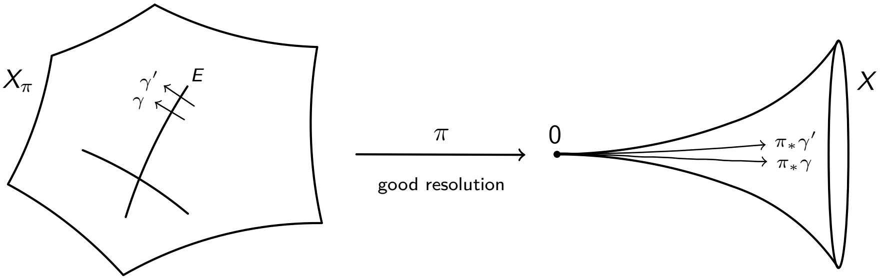

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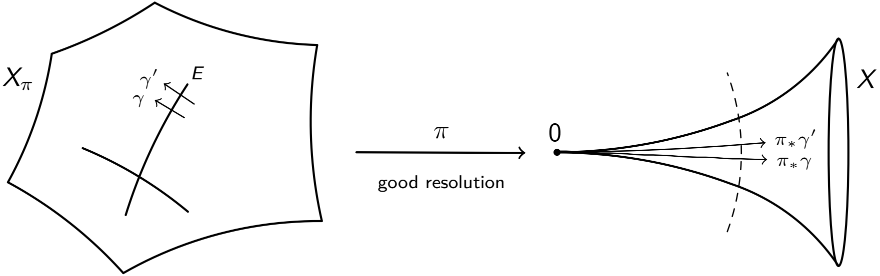

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length of shortest path on X

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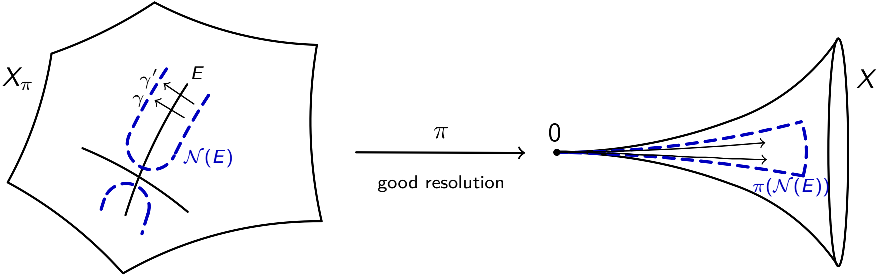

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$$\underbrace{d_{\text{inner}}}_{\text{length of shortest path on } X}\left(\pi_*\gamma \cap S_{\mathbb{C}^n}(0,\varepsilon), \pi_*\gamma' \cap S_{\mathbb{C}^n}(0,\varepsilon)\right) \approx \varepsilon^{\mathcal{I}(E)}$$

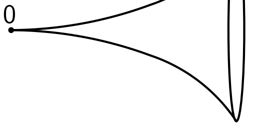
Interpretation:

The inner rate $\mathcal{I}(E)$ measures the size of a small area $\pi\big(N(E)\big)$ of $(X, 0)$ $\longleftarrow$ Fine understanding of the structure of the metric germ (X, d_{inner})

Example:
$E_8 = \{x^2 + y^3 + z^5 = 0\} \subset \mathbb{C}^3$

$$\downarrow (y, z)$$

discriminant $\{y^3 + z^5 = 0\} \subset \mathbb{C}^2$



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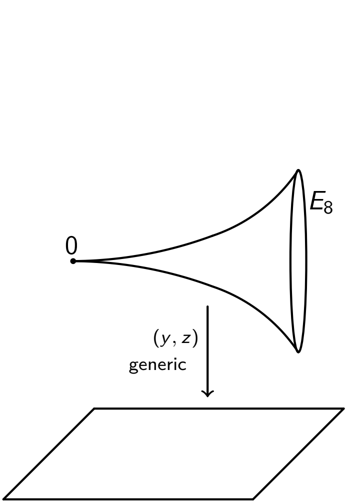

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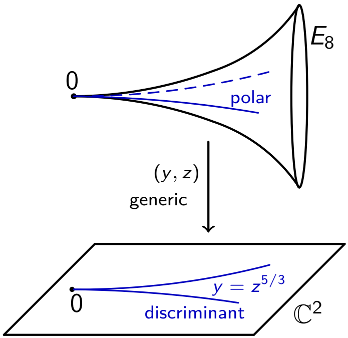

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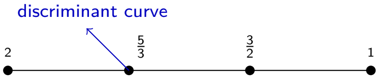

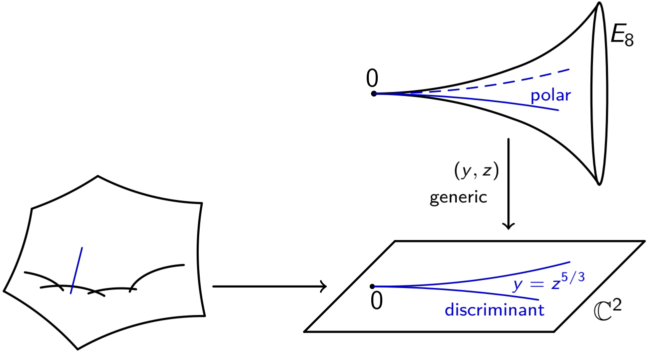

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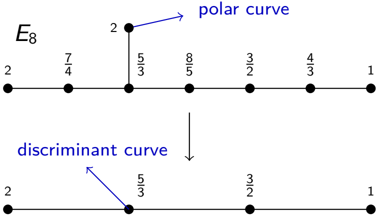

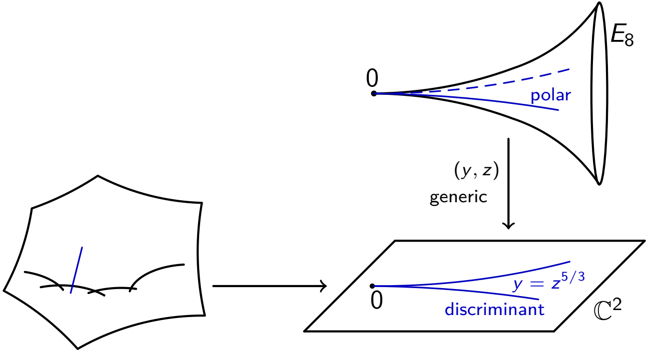

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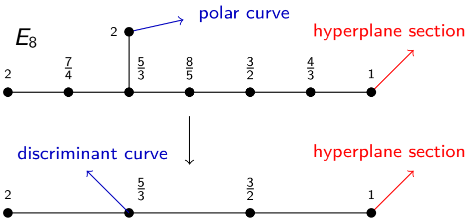

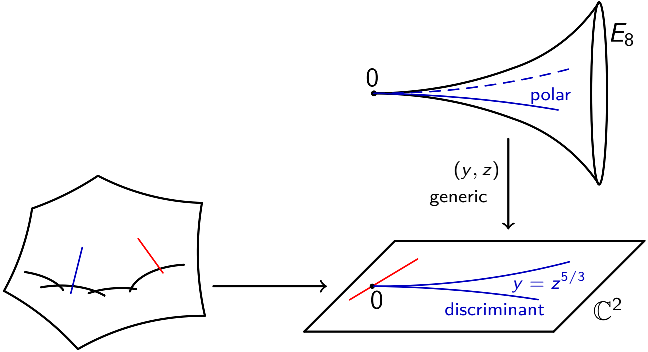

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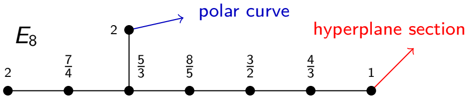

Fundamental questions (going back to Birbrair–Neumann–Pichon 2014):

- How does the geometry $(X, 0)$ influence the inner rates?
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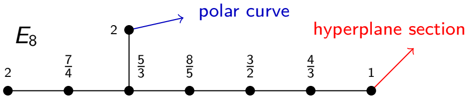

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Theorem (Belotto–F–Pichon 2022, qualitative version)

Let $\pi\colon X_\pi \to X$ be a good resolution of $(X, 0)$ that factors through $\mathrm{Bl}_0(X)$ and through the Nash transform.
Then all the inner rates of $(X, 0)$ are completely determined by:

- the topology of $(X, 0)$, i.e. the weighted dual graph Γ_π;
- the arrows of a **generic** hyperplane section;
- the arrows of the polar curves of a **generic** projection $(X, 0) \to (\mathbb{C}^2, 0)$.

In fact, $\mathcal{I}(E)$ only depends on the divisorial valuation ord_E. We get:

inner rate function $\quad \mathcal{I}\colon \mathrm{NL}(X,0) \to \mathbb{R}_{\geq 1}$

Our result is a precise formula for the Laplacian of $\mathcal{I}$ on $\mathrm{NL}(X,0)$.

We gave a topological proof: lifting the formula for $\mathrm{NL}(\mathbb{C}^2,0)$ to the singular case, by studying the topology and monodromy of the Milnor fiber of a generic linear form.

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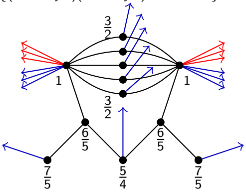
$\{(zx^2+y^3)(x^3+zy^2)+z^7=0\} \subset \mathbb{C}^3$



Cherik's proof: relation to differential forms

Theorem (Cherik, 2024)

Analogous formula for a finite morphism $\Phi = (f, g)\colon (X, 0) \to (\mathbb{C}^2, 0)$.

Idea:
$$\begin{tikzcd} X_\pi \arrow[r, "\pi"', "(\textit{through Nash})"] & (X,0) \arrow[d, "\ell", "\textit{generic}"'] \\ & (\mathbb{C}^2, 0) \end{tikzcd}$$

$\omega = \ell^* dz_1 \wedge dz_2$ "generic 2-form" on X

Given a free point p of $E_v \subset \pi^{-1}(0)$, we can find local coordinates (x, y) at p such that E_v is locally defined by $x = 0$ and:

$$(\pi^*\omega)_p = \mathrm{Jac}(\ell \circ \pi)\, dx \wedge dy = x^{\hat{k}_v^{\log}} f(x,y)\, \frac{dx}{x} \wedge dy$$

Then we have:

1. $\mathcal{I}(v) = \frac{\hat{k}_v^{\log}}{m_v} - 1$ (local computation)
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And the Laplacian formula follows from adjunction.

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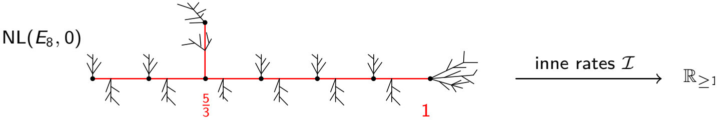
Nobody is perfect (not even NL)

When we study the Lipschitz geometry of $\big((X,0), d_{\mathrm{inner}}\big)$ (that is, its equivalence class up to bi-Lipschitz homeomorphisms) we reach the limitations of $\mathsf{NL}(X,0)$.



We can see the complete inner geometry invariant of surfaces (Birbrair–Neumann–Pichon 2014) as a canonical decomposition of $\mathsf{NL}(X,0)$.

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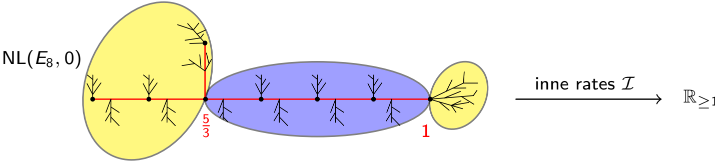

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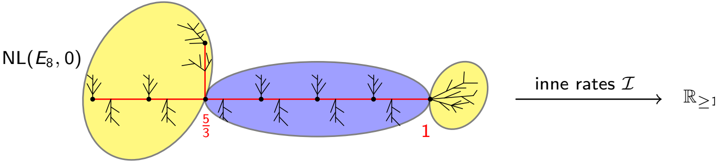

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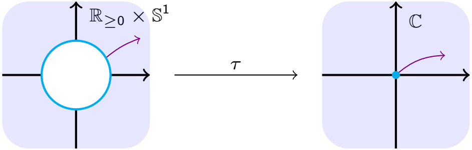
The logarithmic link: a new name for an old object

Idea going back to A'Campo's 1975 paper "La fonction zeta d'une monodromie":

Real oriented blowups (a.k.a. polar coordinate changes) give objects at radius 0.

$$\tau \colon \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \longrightarrow \mathbb{C}$$
$$(r, s) \longmapsto rs$$



This can be done globally: pick a good resolution $\pi \colon (X_\pi, E_\pi) \to (X, 0)$, and perform the real oriented blowp of X_π along E_π:

$E_\pi^{\log} = \tau_\pi^{-1}(E_\pi)$ logarithmic link

It is a manifold with corners, homeomorphic to $\mathrm{L}(X, 0)$.

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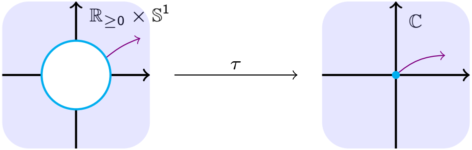

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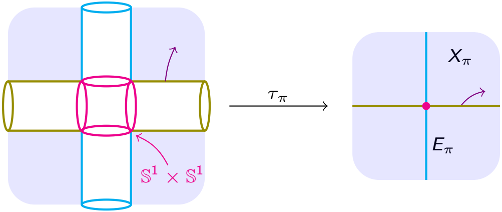

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The logarithmic link of a singularity

Can be done functorially, using logarithmic geometry and Kato–Nakayama spaces.

- Yields a true manifold $\simeq \mathrm{L}(X, 0)$, but canonical (at radius 0)
- Its points lift to subanalytic arcs, giving a ultrametric distance (given by the orders of contacts w.r.t. the given metric)

Remark: These constructions have already been used with great success:

- de Bobadilla–Pełka (equimultiplicity of μ-constant families with isolated singularities, 2024),
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With Anne Pichon, we use logarithmic links to produce canonical bilipschitz models for complex surfaces (and more!).

(Stay for Anne's talk to hear more!)[Sponsored Content]

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¡Gracias!