CHAPTER 1

BASIC CONCEPTS OF SETS AND FUNCTIONS

1.1 Introduction

In this chapter, we shall begin with concepts of sets, relations and functions, which are essential in the study of linear algebra. Towards the end of this chapter, a particular kind of function called binary operation and a brief description of algebraic structures with special emphasis on groups and rings shall be given.

1.2 Sets

The term set is used to describe any **well defined** collection of objects, called the **elements** or **members** of the set. By well defined collection, we mean to say whether any given object belongs to the collection or not.

The following are examples of the set :

(a) The collection of vowels in English alphabets.
(b) The collection of all positive integers which divide 24.
(c) The collection R of all real numbers.
(d) The collection of all prime numbers between 1 and 100.

The following collections are not sets.

(a) The collection of eminent poets of India.
(b) The collection of big rivers in India.

Capital letters A, B, C,... are generally used to denote sets. The statement 'x is an element of A' is written as $x \in A$ [The symbol $\in$ stands for, "belongs to, or "is an element of" or "is a member of"]

The statement 'x is not element of A' is written as $x \notin A$ (x does not belong to A)

1.2.1 Description of Set :

There are two ways of describing a set.

1. Tabular Form : A set may be specified by listing its elements. **For example** if the elements of a set A are 2, 3, 5, 7, 11, 13, 17, 19, then we write

$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$

This form is called **Tabular form** of the set or **Roster form** of the set.

2. Property Form or Rule Form : In this method a set is defined by specifying a property that elements of the set have in common. The set is described as

$$A = \{x : P(x)\}$$

The symbol ':' stands for 'such that'. We can also replace it by '|'.

For example, the set $\{0, \pm 2, \pm 4, ...\}$ can be written as

$$B = \{ x \mid x \text{ is an even integer}\}.$$

This is known as **Property form** or **Rule form** or **Set builder form.**

1.2.2. Definition :

A set is said to be **finite** if it consists of finite number of elements, otherwise the set is said to be **infinite.**

For example : $\{1, 2, 3\}$ is a finite set and the set N of natural numbers is an infinite set.

A set containing no element is called a **null set** or **empty set**. It is denoted by ϕ or $\{ \}$.

For example : $\{x : x^2 + 4 = 0, x \text{ is real}\}$ is an empty set, since no real x satisfies $x^2 + 4 = 0$.

A set containing only one element is called **Singleton Set.**

For example : $\{0\}$, $\{5\}$ are singleton Sets.

1.2.3 Equality of sets :

Two sets A and B are said to be equal if and only if (iff) every element of A is an element of B and consequently every element of B is an element of A. It is written as A = B.

For Example : $\{a, b, c, d\} = \{b, d, c, a\}$

1.2.4 Subsets :

If A and B are two sets such that every element of A is also an element of B then A is said to be a subset of B i.e., B is called superset of A. It is denoted by $A \subseteq B$ (read as A is a subset of B) or $B \supseteq A$ (read as B is a superset of A)

$$A \subset B \text{ iff } x \in A \Rightarrow x \in B.$$

For example : If $A = \{x : x^3 + x = 0,\ x \in R\}$ and $B = \{ x : x^2 + 2x = 0,\ x \in R \}$, then $A \subset B$ since $A = \{0\}$, $B = \{0, -2\}$

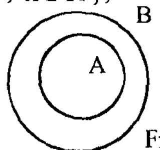


Fig. 1.2.

We note the following :

1. Every set A is a subset of itself i.e., $A \subseteq A$
2. Empty set ϕ is a subset of any set A i.e., $\phi \subset A$.
3. If A is a subset of B and $A \neq B$ then A is said to be a **propersubset** of B.
4. If A, B, C are three sets such that $A \subset B$ and $B \subset C$ then $A \subset C$.
5. If a set contains n elements, then number of subsets is 2^n.

Example 1.2.1 : Find the set of all real numbers x satisfying

(a) $\frac{2}{|x-3|} > 5$

(b) $x(x+2) < 0$

Solution :

(a) $$\frac{2}{|x-3|} > 5 \Rightarrow |x-3| < \frac{2}{5}$$
$$\Rightarrow -\frac{2}{5} < x-3 < \frac{2}{5}$$
$$\Rightarrow -\frac{2}{5}+3 < x < \frac{2}{5}+3$$
$$\Rightarrow \frac{13}{5} < x < \frac{17}{5}$$

Required set $= \left\{x \in R \mid \frac{13}{5} < x < \frac{17}{5}\right\}$

(b) We have $x(x+2) < 0$

$\Rightarrow$ $x > 0, x+2 < 0$ or
$x < 0, x+2 > 0$

$\Rightarrow$ $x > 0, x < -2$ or
$x < 0, x > -2$

$x > 0, x < -2$ is impossible

$\therefore -2 < x < 0$

$\therefore$ The required set $= \{x \in R : -2 < x < 0\}$

Example 1.2.2 : Find the set of complex numbers z satisfying

(a) $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$

(b) $|z| < 5$

Solution :

(a) The equation $z\bar{z} + b\bar{z} + \bar{b}z + c = 0$ can be re-written as

$$(z+b)(\bar{z}+\bar{b}) = b\bar{b} - c$$
$$\Rightarrow |z+b|^2 = b\bar{b} - c$$
$$\Rightarrow |z+b| = \sqrt{(b\bar{b}-c)}$$

This equation represents a circle whose centre is the point $-b$ and radius $\sqrt{(b\bar{b}-c)}$, provided $b\bar{b} - c > 0$ and $c \in R$.

(b) $$|z| < 5$$
$$\Rightarrow |a+ib| < 5$$
$$\Rightarrow a^2 + b^2 < 25$$

This inequation shows that the set of all complex numbers are represented by the points (a, b) which lies inside a circle of radius 5 and having centre at origin.

Problem Set 1 (A)

1. Which of the following statements are true ?

   (a) $N \subseteq Z$ (b) $Z \supseteq N$ (c) $\{0\} \subseteq \{2, 3, 4\}$

   (d) $\{3, 4, 7\} \not\subset \{3, 4, 8\}$ (e) $\phi \in \{\phi\}$ (f) $\phi \in \{\{\phi\}\}$

2. If A, B and C are three sets such that $A \subset B, B \subset C$ and $C \subset A$, then show that $A = B = C$

3. Determine the subsets of the following sets :

   (a) $\{1, 2, 3\}$

   (b) $\{\{1, 2, 3\}\}$

4. State which of the following statements are true for the following sets A, B, C and D.

   $A = \{x : x \text{ is an integer and } 10 < x < 15\}$

   $B = \{x : x \text{ is an even integer and } 10 < x < 15\}$

   $C = \{x : x \text{ is a positive divisor of } 15\}$

   $D = \{1, 3, 5, 15\}$

   (a) $A \subset B$ (b) $C \supset D$ (c) $C = D$ (d) $A \not\subset D$ (e) $B \subset A$ (f) $0 \in C$

5. Find the set of real numbers x satisfying the following :

   (a) $|x - 2| < 2$

   (b) $x^2 \geq 9$

   (c) $(x - 2)(x - 3) \geq 0$

   (d) $\dfrac{|x - 2|}{x - 2} > 0$

   (e) $|3x - 2| \leq \dfrac{1}{2}$

   (f) $|x + 1| \geq 3$

1.3 Set Operations

Like operations in algebra, there are operations on sets.

Those operations are

(i) Union (ii) Intersection (iii) Difference (iv) Symmetric Difference.

If we have two sets A and B then **union of A and B** is the set of all those elements which are in A or in B or in both. It is denoted $A \cup B$. Thus

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

If $A_1, A_2, \ldots, A_k$ are k sets then their union is given by

$$A_1 \cup A_2 \cup \ldots \cup A_k = \bigcup_{i=1}^{k} A_i = \{x \mid x \in A_i \text{ for some } i = 1, 2, \ldots, k\}$$

If A and B are two sets then the **intersection of A and B** is the set of elements common to A and B. It is denoted by $A \cap B$.

Thus $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

If $A_1, A_2, \ldots, A_k$ are k sets then their intersection is given by

$$A_1 \cap A_2 \cap \ldots \cap A_k = \bigcap_{i=1}^{k} A_i = \{x \mid x \in A_i \text{ for each } i = 1, 2, \ldots, k\}$$

The **difference of two sets A and B** is defined to be the set of elements which belongs to A but not to B. It is denoted by $A - B$.

Thus $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

We can read the difference of A and B as complement of B w.r.t A.

When we deal with elements and subsets of a single set U then U is treated as Universal set.

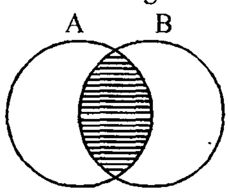
Let $A \subseteq U$.

Then the set of all elements of U which are not in A is called **complement** of A and is denoted by A' or A^C or $U - A$. Thus $A' = \{x \in U,\ x \notin A\}$

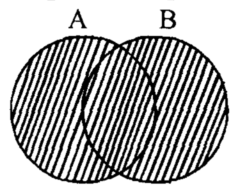
The difference of $A \cup B$ and $A \cap B$ is called **Symmetric difference** of two sets A and B. It is denoted by $A \Delta B$. Thus $A \Delta B = (A \cup B) - (A \cap B)$.

Symmetric difference of A and B can be read as complement of $A \cap B$ in $A \cup B$.

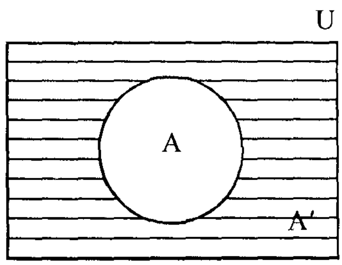
The Venn diagrams for the above operation are given in Figure 1.3.



$A \cap B$ is the shaded portion

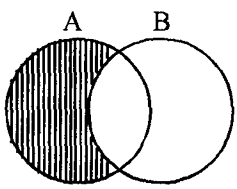


$A \cup B$ is the shaded portion

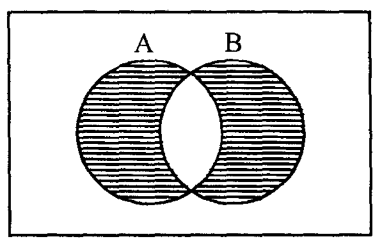


A' is the shaded portion

$(A \subseteq U)$



$A - B$ is the shaded portion



$A \Delta B$ is the shaded portion

Fig. 1.3

Example 1.3.1 :

$$\text{Let } A = \{1, 2, 3, 4\}$$
$$B = \{2, 4, 6, 8\}$$
$$\text{Then } A \cup B = \{1, 2, 3, 4, 6, 8\}$$
$$A \cap B = \{2, 4\}$$
$$A - B = \{1, 3\}$$
$$B - A = \{6, 8\}$$
$$A \,\Delta\, B = (A \cup B) - (A \cap B) = \{1, 3, 6, 8\}$$
$$= (A - B) \cup (B - A)$$

Example 1.3.2 : If U = {a, b, c, d, e}

and A = {a, e} then $A' = \{b, c, d\}$

1.4 Cartesian Product

Let A and B be two sets. Let $a \in A, b \in B$. Then (a, b) is an **ordered pair.** The Cartesian product of A and B is the set of all possible ordered pairs (a, b) where $a \in A, b \in B$.

Thus $A \times B = \{(a, b) : a \in A, b \in B\}$

and $B \times A = \{(b, a) : b \in B, a \in A\}$

Since $(a, b) \neq (b, a)$, therefore $A \times B \neq B \times A$.

Two ordered pairs (a, b) and (c, d) are said to be **equal** if a = c and b = d.

Example 1.4.1 : If A = {p, q, r} and B = {x, y} then

$$A \times B = \{(p, x), (q, x), (r, x), (p, y), (q, y), (r, y)\}$$

Example 1.4.2 : If R is the set of all real numbers, then what is R × R and R × R × R ?

Solution : $R \times R = \{(x, y) : x \in R, y \in R\}$

Thus R × R represents the set of points in two dimensional space.

$$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$$

Then R × R × R represents the set of points in three dimensional space.

Problem Set 1 (B)

1. Let $A = \{x \mid x \text{ is a multiple of } 3, 1 \leq x \leq 12\}$.

   $B = \{x \mid x \text{ is an even number}, 1 \leq x \leq 12\}$.

   $C = \{x \mid x \text{ is an odd number}, 1 \leq x \leq 12\}$.

   Find

   (a) $A \cup B$ (b) $A \cap B$ (c) $B \cap C$
   (d) $(A \cup B) \cap C$ (e) $B - A$ (f) $B - C$ (g) $A \,\Delta\, B$

2. Find the sets P, Q, R, if
   $P \cap Q = \{2, 3\}, P \cap R = \{2, 4\},$
   $P \cup Q = \{1, 2, 3, 4\}$ and $P \cup R = \{2, 3, 4, 5\}$
3. If $A = \{x : (x - 2)(x - 5) = 0\}$
   $B = \{ x : (x - 2)(x - 3) = 0 \}$
   find $A \times B$, $B \times A$ and $A \times A$.
4. Prove that
   $A \times (B \cup C) = (A \times B) \cup (A \times C)$ for the sets A, B and C.
5. If A, B, C and D are any sets then prove that
   $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
6. If A, B, and C are any sets then prove that
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (c) $A - (B \cup C) = (A - B) \cap (A - C)$
   (d) $(A \cup B \cup C)' = A' \cap B' \cap C'$
   (e) $(A \cap B \cap C)' = A' \cup B' \cup C'$

1.5 Relations on Sets

Let A and B be two non-empty sets. Any subset R of $A \times B$ is called a relation from A to B. Then R is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. If $(a, b) \in R$, we write it as **a R b**.

For example :

If $A = \{1, 2, 3\}$, $B = \{2, 4\}$ then $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$

If we consider the relationship $x < y$, where $x \in A$, $y \in B$ then the relation R from A to B is given by

$$R = \{(1, 2), (1, 4), (2, 4)\}$$

If R is a relation on a set A, then R is a subset of $A \times A$.

Examples of some relations are :

Let P denote the set of all persons. The relations on P are given by

$R_1 = \{(a, b) \mid a, b \in P,\ a \text{ is brother of } b\}$
$R_2 = \{(a, b) \mid a, b \in P, a \text{ is younger to } b\}$
$R_3 = \{(a, b) \mid a, b \in P, a \text{ is mother of } b\}$ and so on.

Let N denote the set of natural numbers. The following are the relation on N.

$R_1 = \{(a, b) \mid a, b \in N, a < b\}$
$R_2 = \{(a, b) \mid a, b \in N, a = b\}$
$R_3 = \{(a, b) \mid a, b \in N, a = b^2\}$ and so on.

1.5.1 Definition :

A relation R on a set A satisfies certain properties. The properties are defined as follows :

(i) R is **reflexive** on a set A if a R a for all $a \in A$ i.e., $(a, a) \in R$ for all $a \in A$.

(ii) R is **symmetric** on a set A if $a \, R \, b \Rightarrow b \, R \, a$ i.e., $(a, b) \in R \Rightarrow (b, a) \in R$, for all $a, b \in A$.

(iii) R is **transitive** on a set A if a R b, $b R c \Rightarrow a R c$ i.e., $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$, for all a, b, c, $\in$ A.

(iv) A relation R on a set A is called an **equivalence relation** if R is reflexive, symmetric and transitive.

Example 1.5.1 : Show that the relation $(a, b) \, R \, (c, d) \Leftrightarrow a^2 + b^2 = c^2 + d^2$ is a equivalence relation on the plane.

Solution : Since (a, b) R (a, b) $\Leftrightarrow a^2 + b^2 = a^2 + b^2$

$\therefore$ R is reflexive.

Again (a, b) R (c, d)

$\Rightarrow a^2 + b^2 = c^2 + d^2$

$\Rightarrow c^2 + d^2 = a^2 + b^2$

$\Rightarrow (c, d) \, R \, (a, b)$

$\therefore$ R is reflexive.

Further (a, b) R (c, d) and (c, d) R (p, q)

$\Rightarrow a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = p^2 + q^2$

$\Rightarrow a^2 + b^2 = p^2 + q^2$

$\Rightarrow (a, b) \, R \, (p, q)$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive and hence an equivalence relation.

Example 1.5.2 : The relation R in the set N of natural numbers is defined as $a \, R \, b$ iff $a \mid b$, $a, b \in N$. Then R is reflexive and transitive but not symmetric.

Example 1.5.3 : If R is a relation in N × N defined by (a, b) R (c, d) iff ad = bc. Then R is an equivalence relation.

Example 1.5.4 : If R is a relation in Z defined by

x R y iff $x \le y$. Then R is reflexive, transitive but not symmetric.

Example 1.5.5 : If R is a relation in N defined by $x R y$ iff $x = y^3, x, y \in N$.

Then R is neither reflexive nor symmetric nor transitive.

Problem Set 1 (C)

1. Classify the following relations as reflexive, symmetric and/or transitive on the set Z of all integers.

   (a) $x R y \Leftrightarrow x \neq y$

   (b) $x R y \Leftrightarrow xy \geq 1$

   (c) $x R y \Leftrightarrow x \equiv y \pmod 3$

   (d) $x R y \Leftrightarrow x$ is a multiple of y

   (e) $x R y \Leftrightarrow x \geq y^2$

   (f) $x R y \Leftrightarrow x \leq y+1$

   (g) $x R y \Leftrightarrow |x+y| = 2$

   (h) $x R y \Leftrightarrow x = y$

2. Which of the relations defined in the set of real numbers are equivalence relations ?

   (a) $x R y \Leftrightarrow |x| = |y|$

   (b) $x R y \Leftrightarrow x \geq y$

   (c) $x R y \Leftrightarrow |x| > |y|$

3. Show that the relation R on the set of all triangles in the plane defined by $R = \{(a, b) : a \text{ is similar to } b\}$ is an equivalence relation.

1.6. Functions (or mapping) :

A function is a special case of relation.

Let A and B be two non-empty sets. A function f from A to B is a subset of $A \times B$ such that for each element $x \in A$, there is a unique element $y \in B$ and $(x, y) \in f$. Symbolically it is represented by

$$f : A \to B.$$

With the help of diagram, we can represent a function.

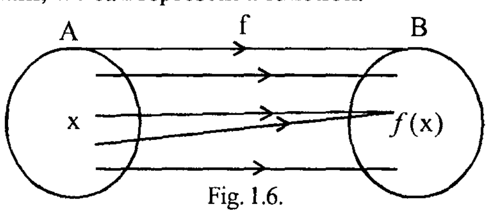


Fig. 1.6.

If $(x, y) \in f$, we write $y = f(x)$ and it is called image of x. The set of all the images of the elements of A under the function f is called range of f or image set and is denoted by $f(A)$.

Symbolically $f(A) = \{f(x) : x \in A\}$

Here A is called domain of f.

For example :

Let A = {a, b, c}, B = (3, 4}

$\therefore$ f = {(a, 3), (b, 3), (c, 4) is a function from A to B.

1.6.1 Definition :

A function $f : A \to B$ is called **one-one (injective)** if distinct elements in A have distinct images in B.

Symbolically, f is one-one if for every $x_1, x_2 \in A$, such that

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Equivalently, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Example 1.6.1 : Let A= {1, 2, 3}, B = (a, b, c, d}

Define f = {(1, a), (2, b), (3, d)}

$\therefore$ Clearly f is one-one as different elements 1, 2, 3 in A are assigned to the different elements a, b, d respectively in B.

Example 1.6.2 : Let $f : R \to R$ be defined by $f(x) = 2x$.

$\therefore$ f is one-one, since $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

Example 1.6.3 : $f : R \to R$ be defined by $f(x) = x^4$.

f is not one-one since $f(1) = 1$ and $f(-1) = 1$ i.e., 1 and (–1) have same images 1 under f.

1.6.2. Definition : A function f from A to B is called **onto (surjective)** if every element $y \in B$ is the image of atleast one element $x \in A$, such that $(x, y) \in f$. In this case, range of f = B i.e., R (f) = B.

Example 1.6.4 : Let $f : R \to R$ be defined by $f(x) = 2x + 5$.

$$\text{Take } x = \frac{y-5}{2} \in R$$

$$f(x) = f\left(\frac{y-5}{2}\right) = 2\left(\frac{y-5}{2}\right) + 5 = y$$

$\therefore$ f is onto, since range of f = R.

Example 1.6.5 : Let $f : R \to R$ be defined by $f(x) = x^2$. f is not onto since we cannot find a real number whose square is negative and range of $f \neq R$.

1.6.3 Definition : Bijective function

A function f from A to B is called bijective or one-to-one correspondence if f is both injective and surjective i.e., f is both one one and onto.

Example 1.6.6 : Let f: {1, 2, 3,...} $\to$ {2, 4, 6, 8,...} be defined by $f(x) = 2x$, $x \in \{1, 2, 3, ...\}$. Show that f is bijective.

Solution :

Let A = {1, 2, 3,...}
B = {2, 4, 6, 8,...}

Let $x, y \in A$

$\therefore\ x \neq y$

$\Rightarrow\ 2x \neq 2y$

$\Rightarrow\ f(x) \neq f(y)$

Thus different elements in A have different f-images in B.

$\therefore\ f$ is one-one.

Let $y \in B$. Take $x = \frac{y}{2} \in A$.

$$\therefore\ f(x) = f\left(\frac{y}{2}\right) = 2.\frac{y}{2} = y$$

i.e., any arbitrary element y in B is the f- image of the element $\frac{y}{2} \in A$. Hence f is onto.

Since f is one-one and onto, therefore f is bijective.

Example 1.6.7 : Let A = {–2, –1, 0, 1, 2}
Let $f : A \to R$ be defined by $f(x) = x^2 + 1$. Find the range of f.

Solution : The range of f consists of those elements of R which appears as f-images of different elements of A. So, we calculate f-image of each element of A.

$$f(-2) = (-2)^2 + 1 = 5,\ f(-1) = (-1)^2 + 1 = 2,$$
$$f(0) = 1,\ f(1) = 2,\ f(2) = 2^2 + 1 = 4 + 1 = 5$$
$$\therefore\ R(f) = \{5, 2, 1\}$$

Example 1.6.8 : Let C be the set of complex numbers. Show that the function $f : C \to R$ given by $f(z) = |z|,\ z \in C$ is neither one-one nor onto.

Solution : Let z = x + iy be a complex number where $x, y \in R$.

$$\therefore\ |z| = \sqrt{(x^2 + y^2)} \geq 0$$

If $p < 0$ and $p \in R$, then there exists no complex number $z \in C$ such that $|z| = p$. Thus p is not the f-image of any complex number $z \in C$.

$\therefore\ f$ is not onto.

Again, Let $z_1 = 3 + 4i \in C$
and $z_2 = 3 - 4i \in C$
Then $|z_1| = 5 = |z_2|$

$\therefore\ z_1 \neq z_2 \Rightarrow f(z_1) = f(z_2)$

Thus two different complex numbers z_1 and $z_2 \in C$ have same f-image in R.

$\therefore\ f$ is not one-one.

Example 1.6.9 : Determine the largest subset A of R for the function $f: A \to R$ defined by

(i) $f(x)=\sqrt{(x^3-x^2)}$

(ii) $f(x)=\dfrac{x}{x^2+1}$

Solution : (i) $f(x)=\sqrt{(x^3-x^2)}$

We have to find $A \subset R$ such that $f: A \to R$, i.e., we find the domain of f.
The domain of f is the set of all x such that $x^3 - x^2 \geq 0$

$$\Rightarrow x^2(x-1) \geq 0$$
$$\Rightarrow x-1 \geq 0 \Rightarrow x \geq 1$$
$$\therefore A = \{0\} \cup [1, \infty)$$

(ii) $f(x)=\dfrac{x}{x^2+1}$

We find $A \subset R$ such that $f: A \to R$, i.e., we find the domain of f.
Obviously A = R.

Example 1.6.10 : Find the largest subset A of C for the function $f: A \to C$ defined by $f(z)=\dfrac{z}{|z|}$.

Solution : Here $A = \{z \mid z \in C \text{ and } z \neq 0\}$.

1.6.4 Definition : Two sets A and B are said to be in one-one correspondence if there exists a function $f: A \to B$ which is bijective.

1.6.5 Definition : Let A be any set. If A is in one-one correspondence with the set {1, 2,..., n}, for some positive integer n, then A is said to be a **finite set.** A set which is not finite is called **infinite set.**

For example : Let A = {x : (x–3) (x–4) (x–5) (x–6) = 0} and B the set of vertices of a rectangle. Both A and B are finite sets because they have one-one correspondence with the set {1, 2, 3, 4}. N, Z, Q, R, C are infinite sets.

Problem Set 1 (D)

1. Decide whether or not the following are functions from X to Y where X = {1, 2, 3, 4} and Y = {a, b, c}

   (i) f_1 = {(1, a), (2, a), (3, b), (4, c)}

   (ii) f_2 = {(1, a), (1, b), (2, a), (3, b), (4, b)}

   (iii) f_3 = {(1, b), (2, b), (3, b), (4, c)}

   (iv) f_4 = {(1, a), (2, b), (3, c)}

2. Determine the largest subset A of R for the function $f : A \to R$ defined by

   (i) $f(x)=\dfrac{1}{\sqrt{1-x}}$

   (ii) $f(x)=\sqrt{x-\dfrac{x}{1-x}}$

   (iii) $f(x)=\sqrt{x^2-1}$

   (iv) $f(x)=\sqrt{1-|x|}$

   (v) $f(x)=\sqrt{(x^3-x)}$

   (vi) $f(x)=\dfrac{1}{x^3-x}$

3. Find the range of the following functions :

   (i) $f : R \to R$ be defined by $f(x)=\sin x$

   (ii) $f : R-\{0\} \to R$ be defined by $f(x)=\dfrac{|x|}{x}$

   (iii) $f : R^+ \cup \{0\} \to R$ be defined by $f(x)=\sqrt{x}$

   (iv) $f : R \to R$ be defined by $f(x)=\sqrt{25-x^2}$

   (v) $f : R \to R$ be defined by $f(x)=\dfrac{e^{-x}}{1+[x]}$

   (vi) $f : R-\{4\pm 2\sqrt{5}\} \to R$ be defined by $f(x)=\dfrac{x+2}{x^2-8x-4}$

4. Classify the following functions as one-one, onto, one-one and onto functions.

   (i) $f : Z^+ \to Z^+$ defined by $f(x) = x^2$

   (ii) $f : N \to N$ defined by $f(n)=n^2+n+1$

   (iii) $f : R \to R$ defined by $f(x)=\cos x$

   (iv) $f : \{x : x \in R \text{ and } x \neq 0\} \to R$ defined by $f(x)=\dfrac{1}{x}$.

1.7 Different types of functions :

(i) Identity function : If A is any set and the function $f : A \to A$ is defined by $f(x) = x$ i.e., every element of the set A is the image of itself, then the function is called the **identity function.** It is always one-one onto or a bijective function.

Example 1.7.1 : If A = {1, 2, 3, 6} and $f : A \to A$ is defined by $f(1) = 1, f(2) = 2, f(3) = 3, f(6) = 6$ then f = {(1, 1), (2, 2), (3, 3), (6, 6)} is an identity function.

(ii) Constant function : A mapping in which every element of the domain is assigned to the same element of the co-domain is called a constant mapping. The range of the constant mapping is a set with single element. Thus, the mapping $f : A \to B$ is constant mapping if $\forall\, x \in A$,

$$f(x) = k,\ k \in B \text{ is a constant.}$$

Example 1.7.2 : The mapping $f : R \to R$ where $f(x) = 5\ \forall\, x \in R$ is a constant mapping.

(iii) Inclusion function : If $A \subset B$, then the mapping $f : A \to B$ defined by $f(x) = x, \forall\, x \in A$ is called an inclusion mapping from A to B.

Clearly, the inclusion mapping is one-one into and the inclusion mapping of a set A to itself is the identity mapping on A.

Example 1.7.3 : If A = {1, 2, 5} and B = {1, 2, 4, 5, 8}, then the mapping given by

f = {(1, 1), (2, 2), (5, 5)} is the inclusion mapping from A to B, because $A \subset B$ and $f(x) = x, \forall\, x \in A$.

(iv) Equal function : If f and g are two functions defined on the same domain D and if $f(x) = g(x)\ \forall x \in D$, then the function f and g are said to equal, i.e., $f = g$.

Example 1.7.4 : If $f : R \to R$ and $g : R \to R$, where R is the set of ral numbers, and f is defined by $f(x) = x^2$ and g is defined by $g(y) = y^2$, then f = g.

Here x and y are variables which only define the function..

(v) Inverse function : If $f : A \to B$ be an one-one onto function, then the function $f^{-1} : B \to A$ which associates to element $y \in B$, the element $x \in A$ whose f-image is $y \in B$, is called the inverse of the function $f : A \to B$, and is denoted by $x = f^{-1}(y)$.

Clearly $f^{-1}(y) = x \Leftrightarrow y = f(x) :\ y \in B, x \in A$

In the form of ordered pairs,

If $f = \{(x, y) : x \in A, y \in B\}$

then $f^{-1} = \{(y, x) : x \in A, y \in B\}$.

Example 1.7.5 : If $f : Z \to Z$, where $f(x) = x^2$ and Z is the set of integers, then

$$f(3) = 9, \text{ since } 3^2 = 9$$

$f(-3) = 9$, since $(-3)^2 = 9$ and there is no other integer whose square is 9.

$$f^{-1}(9) = \{3, -3\}.$$

1.8 Composite function or function of a function :

Let us consider three sets X, Y, Z and two mapping f and g such that

$f : X \to Y$ and $g : Y \to Z$.

Then under the mapping f, any element $x \in X$ is mapped to an element $y \in Y$, where $y = f(x), \forall\, x \in X$.

Also under the mapping g, every element of $y \in Y$ is mapped to an element $z \in Z$, where $z = g(y), \forall\, y \in Y$.

Thus $z = g(y) = g(f(x))$.

In this way we have a mapping which maps the element $x \in X$ to the element g (f (x)) $\in Z$, which is called composition of the mapping f and g or composite or product of f and g in that order and is denoted by *gof* or merely by *g f.*

Hence we have the following definition :

If $f: X \to Y$ and $g: Y \to Z$, then the composite mapping is a mapping from X to Z given by $(gof)\,x = g(f(x)), \forall\, x \in X$.

Example 1.8.1 : If $f(x) = x^2, g(x) = x + 3, x \in R$, then

$$(gof)\,x = g(f(x)) = g(x^2) = x^2 + 3$$

$$(fog)\,x = f(g(x)) = f(x+3) = (x+3)^2 = x^2 + 6x + 9.$$

The operation gof is illustrated by the following diagram.

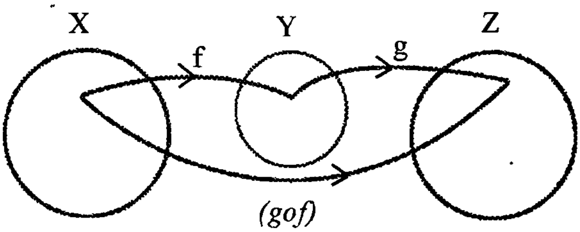


Fig. 1.8

Theorem 1.8.1 :

(a) If $f: A \to B$ be a one-one onto mapping, then f^{-1} of $= I_A$ and $fof^{-1} = I_B$ where I_A and I_B are the identity mapping of the sets A and B respectively.

(b) If $f: A \to B$ and I_A and I_B are the identity mappings of A and B respectively, then

$$f\,0\,I_A = f = I_B\, of$$

Proof : (a) Let the mapping $f: A \to B$ be given by

$$f(a) = b, \text{ where } a \in A, b \in B.$$

Then the mapping $f^{-1}: B \to A$ is given by

$$f^{-1}(b) = a.$$

$$\text{Now } (f^{-1} of)(a) = f^{-1}[f(a)]$$
$$= f^{-1}(b)$$
$$= a$$

i.e., $(f^{-1} of)(a) = a$.

Thus the mapping f^{-1} *of* maps the element $a \in A$ onto itself.

Hence $f^{-1} of = I_A$.

$$\text{Again } (f of^{-1})(b) = f[f^{-1}(b)]$$
$$= f(a)$$
$$= b$$

i.e., $(fof^{-1})\, b = b$

Thus the mapping fof^{-1} maps the element $b \in B$ onto itself.

Hence $fof^{-1} = I_B$.

(b) Since $f : A \to B, I_A \ A \to A$ and $I_B : B \to B$ therefore

$$f o I_A : A \to B \text{ and } I_B \, o f : A \to B.$$

Now for any $a \in A$.

$$f o I_A : A \to B \text{ and } I_B \, o f : A \to B.$$

Now for any $a \in A$.

$$(f o I_A)(a) = f[f_A(a)]$$
$$= f(a)$$

(Since I_A is the identity mapping of A)

i.e., $(f o I_A)(a) = f(a)$.

Hence $f o I_A = f$

Again $(I_B o f)(a) = I_B[f(a)] = f(a)$. Since I_B is the identity mapping of B and $f(a) \in (B)$ i.e., $(I_B \, o f)(a) = f(a)$.

Hence $I_B \, o f = f$.

Thus $f o \, I_A = f = I_B \, o f$.

Theorem 1.8.2 : The necessary and sufficient condition that a mapping be invertible is that it is one-one and onto.

Necessary condition : Let $f : A \to B$ and $g : B \to A$ be inverse mapping, and

$f(x_1) = f(x_2), \forall x_1, x_2, \in A$.

Then $x_1 = g[f(x_1)]$

$= g[f(x_2)]$ $\qquad [\because f(x_1) = f(x_2)]$

$= x_2$

$\Rightarrow f$ is one-one,

i.e., if $(x_1, f(x_1))$ and $(x_2, f(x_2))$ be two ordered pairs and $f(x_1) = f(x_2)$, then $x_1 = x_2$ and therefore f is one-one.

Again, let $y \in B$, then by definition $g(y) \in A$

and $f[g(y)] = y$, ($\because f$ and g are inverse mappings)

Now let g (y) = x. Then, we have $f(x) = f(g(y)) = y$.

i.e., for each $y \in B$, there exists one $x \in A$. Consequently, f is onto.

Sufficient Condition : Let $f : A \to B$ be one-one and onto. Then, by definition, since f is onto, we find that for each $y \in B$,

$$f(x) = y, \forall x \in A \qquad \text{...(i)}$$

Again, only one element $x \in A$ will satisfy $f(x) = y$, as f is one-one.

Let there be another mapping $g : B \to A$ such that

$$g(y) = x \ \forall y \in B \qquad \text{...(ii)}$$

Then, $f[g(y)] = f(x) = y$, $\qquad$ [from (i)]

Again, let $g[f(x)] = x_1$. Then,

$$f(x_1) = f[g[f(x)]] \qquad \text{...(iii)}$$
$$= f[g(y)], \qquad [\because f(x) = y]$$
$$= y, \qquad [\text{ from (ii) }]$$
$$= f(x), \qquad [\text{ from (i) }]$$

$\Rightarrow f(x_1) = f(x)$

Since f is one-one, so we have

$$x_1 = x = g[f(x)]$$

Hence f and g are inverse mapping.

Theorem 1.8.3 : If $f : A \to B$ and $g : B \to C$ are two mappings, gof is one-one and onto, then prove that f is one-one and g is onto.

Solution : Proof : (i) Let any two elements $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$.

$\therefore \ (gof)(a_1) = g(f(a_1))$

$= g(f(a_2)), \qquad [\because f(a_1) = f(a_2)]$

$= (gof)(a_2)$

$\because$ gof is one-one,

$\therefore (gof)(a_1) = (gof)(a_2) \Rightarrow a_1 = a_2$

$\therefore f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$\therefore f$ is one-one.

(ii) Since $gof : A \to C$ is onto, $\forall\, c \in C$, there exists an element $a \in A$ such that $(gof)(a) = c$.

$\therefore \ c = g(f(a)), \forall\, c \in C$

Now we have

$$f(a) = b \in B, \forall\, a \in A.$$

$\therefore \exists$ an element $b \in B$, such that

$$c = g(b), \forall\, c \in C.$$

Hence $g : B \to C$ is onto.

Theorem 1.8.4 : If $f : X \to Y$ and $A, B \subset Y$, Prove that

(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Solution : (i) Let x be an arbitrary element of $f^{-1}(A \cup B)$. Then,

$$x \in f^{-1}(A \cup B) \Rightarrow f(x) \in A \cup B$$
$$\Rightarrow f(x) \in A \text{ or } f(x) \in B$$
$$\Rightarrow x \in f^{-1}(A) \text{ or } x \in f^{-1}(B)$$
$$\Rightarrow x \in f^{-1}(A) \cup f^{-1}(B)$$

$\therefore f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B) \qquad \text{...(1)}$

Again let y be an arbitrary element of $f^{-1}(A) \cup f^{-1}(B)$. Then ,

$$y \in f^{-1} \cup f^{-1}(B) \Rightarrow y \in f^{-1}(A) \text{ or } y \in f^{-1}(B)$$
$$\Rightarrow f(y) \in A \text{ or } f(y) \in B$$
$$\Rightarrow f(y) \in A \cup B$$
$$\Rightarrow y \in f^{-1}(A \cup B)$$

$\therefore f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$ ...(2)

From (1) and (2), we get

$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(ii) Let x be an arbitrary element $f^{-1}(A \cap B)$. Then,

$$x \in f^{-1}(A \cap B) \Rightarrow f(x) \in A \cap B$$
$$\Rightarrow f(x) \in A \text{ and } f(x) \in B$$
$$\Rightarrow x \in f^{-1}(A) \text{ and } x \in f^{-1}(B)$$
$$\Rightarrow x \in f^{-1}(A) \cap f^{-1}(B)$$

$\therefore f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$ ... (3)

Again, let y be an arbitrary element of $f^{-1}(A) \cap f^{-1}(B)$. Then,

$$y \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow y \in f^{-1}(A) \text{ and } y \in f^{-1}(B)$$
$$\Rightarrow f(y) \in A \text{ and } f(y) \in B$$
$$\Rightarrow f(y) \in A \cap B$$
$$\Rightarrow y \in f^{-1}(A \cap B)$$

$\therefore f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$ ...(4)

From (3) and (4) it follows that

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Problem Set 1 (E)

1. Let f R $\rightarrow$ R be defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$
2. Let $f, g : R \rightarrow R$ be defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$ respectively. Find (a) fog (b) gof (c) fof (d) gog.
3. Let $f(x) = x^2 + 3$ and $g(x) = \sqrt{x}$. Then find the domain of fog and gof.
4. Let $f(x) = 2x^3 + 1$ and $g(x) = \sqrt[3]{\dfrac{x-1}{2}}$. Show that f and g are inverses of each other.
5. If the function f, g, h are defined from R to R by $f(x) = x^2 - 1$,

$$g(x) = \sqrt{x^2+1} \text{ and } h(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}, \text{ then}$$

find ho(fog) and determine whether fog is invertible or not.

6. If $f:[1,\infty)\to[2,\infty)$ is given by $f(x)=x+\frac{1}{x}$,
then find $f^{-1}(x)$.

7. Let $f:R\to R$ and $g:R\to R$ be defined by $f(x)=x+1$, $g(x)=x^2-2$. Find $(gof)^{-1}([-2,-1])$.

8. If $f(x)=\frac{x}{\sqrt{1+x^2}}$, then show that $(fofof)(x)=\frac{x}{\sqrt{1+3x^2}}$.

9. Let f and g be two functions defined by $f(x)=\frac{x}{x+1}$ and $g(x)=\frac{x}{1-x}$. Find $(fog)^{-1}(x)$ if $x\neq 1$.

1.9 Binary Operations :

There are four fundamental operations namely addition, subtraction, multiplication and division.

The main feature of these operations is that given any two numbers a and b, we associate another number a + b or a − b or ab of $\frac{a}{b}$, $b\neq 0$. Thus addition, multiplication, subtraction and division are examples of binary operation. For the general definition to cover all these four operations, the set of numbers is to be replaced by an arbitrary set X and then general binary operation is nothing but association of any pair of elements a, b from X to another element of X. This gives rise to a general defintion as follows :

1.9.1 Definition (Binary operations) : Given a non-empty set X, any function $f:X^2\to X$ is called a binary operation on X. It is denoted by '*' or '0' i.e. $*$ (a, b) is denoted by $a * b$.

Example 1.9.1 : Addition –A function $f:R^2\to R$ defined by $f(a,\ b)=a+b\ \ \forall\, a,b\in R$ is the binary operation, called addition on R, the set of all real numbers.

Example 1.9.2 : Multiplication – A function $f:R^2\to R$ defined by $f:(a,b)=ab\ \forall\, a,b\in R$ is the binary operation called multiplication on R.

Example 1.9. 3 : Division – A function $f:R^2\to R$ defined by $f(a,b)=\frac{a}{b}$ is not a function and hence not a binary operation, as for $b=0, \frac{a}{b}$ is not defined.

Example 1.9.4 : Show that f: $R^2\to R$ given by $f(a,b)=a+b^2$ is a binary operation.

Solution : Since f carries each pair (a, b) to a unique element $a+b^2$ in R, f is a binary operation on R.

Types of Binary Operation :

(i) Commutative Binary Operation – A binary operation $*$ on a non-empty set S is called a commutative operation if

$$a * b = b * a,\ \forall\, a,b\in S.$$

Example 1.9.5 : Addition '+' and multiplication '.' on real numbers are commutative binary operation because $x + y = y + x$ and $x \cdot y = y \cdot x$ for every pair (x, y) of real numbers.

(ii) Associative Binary Operation : A binary operation $*$ on a set S is called associative if $a * (b * c) = (a * b) * c, \ \forall a, b, c \in S$.

Example 1.9.6 :

(1) Addition and multiplication of real numbers are associative binary operations, because

$$a + (b + c) = (a + b) + c$$
$$\text{and } a \cdot (b \cdot c) = (a \cdot b) \cdot c, \ \forall a, b, c \in R.$$

(2) Operation of division '$\div$' in real numbers is not associative in the set of non-zero real numbers, because

$$a \div (b \div c) \neq (a \div b) \div c.$$

For example,

$$36 \div (12 \div 3) = 36 \div 4 = 9$$
$$\text{and } (36 \div 12) \div 3 = 3 \div 3 = 1$$
$$\text{and so } 36 \div (12 \div 3) \neq (36 \div 12) \div 3.$$

(iii) Distributive Binary Operation : If o and $*$ be two operations defined on the set S, then the operation $*$ is said to be distributive over operation o if

$$a * (b o c) = (a * b) \, o \, (a * c), \ \forall a, b, c \in R.$$

Example 1.9.7 : On the set of natural numbers N the operation '.' is distributive over '+', because

$$a \cdot (b + c) = a \cdot b + a \cdot c, \ \forall a, b, c \in N.$$

(iv) Existence of Identity : If $*$ be a binary operation on a set S, then an element $e \in S$ is said to be an identity element if

$$x * e = e * x = x, \ \forall x \in S.$$

Example 1.9.8 :

(1) Real number 0 (zero) is the identity element of the binary operation 'addition' defined on the set R, because

$$x + 0 = 0 + x = x, \forall x \in R.$$

(2) Real number 1 is the identity element of the binary operation 'multiplication' defined on the set R, because

$$x \cdot 1 = 1 \cdot x = x, \forall x \in R.$$

Note : '$-$' and '$\div$' donot have identity element.

(v) Existence of Inverse Element : If $*$ be a binary operation on a set S, and $e \in S$ be the identity element for the operation $*$, and a, b two elements in S such that

$$a * b = b * a = e,$$

then b is called the inverse of a and is denoted by a^{-1}.

Example 1.9.9 : In the set of real numbers 0 is the additive identity and the number $-a$ is the additive inverse of a, because

$$a + (-a) = (-a) + a = 0.$$

In the set of rational numbers, 1 is the multiplicative identity. Then for any rational number $\frac{p}{q}$ where p, q are integers and $q \neq 0$, the multiplicative inverse of $\frac{p}{q}$ is $\frac{q}{p}$, because $\frac{p}{q} \cdot \frac{q}{p} = 1$.

Example 1.9.10 : The Binary operation $*$ on Z is defined by $m * n = m + n - mn$. Show that $*$ is commutative andassociative.

Solution : Now $m * n = m + n - mn$ ... (1)

$\therefore$ $n * m = n + m - nm = m + n - mn$ ... (2)

(by the commutative properties of addition and multiplication of integers).

$\therefore$ From (1) and (2), we have $m * n = n * m$

Thus the binary operation $*$ defined as above is commutative.

Now $m * (n * l)$

$= m + (n * l) - m (n * l)$

$= m + (n + l - nl) - m (n + l - nl)$

$= m + n + l - nl - mn - ml + mnl$ ... (3)

and $(m * n) * l$

$= (m * n) + l - (m * n) l$

$= m + n - mn + l - (m + n - mn) l$

$= m + n - mn + l - ml - nl + mnl$

$= m + n + l - nl - mn - ml + mnl$ ... (4)

From (3) and (4), we have $m * (n * l) = (m * n) * l$

Hence the binary operation $*$ is associative also.

Example 1.9.11 : Let an operation o be defined by $aob = a + b + ab$. Then

(i) Prove that o is a binary operation on the set R of all real numbers other than -1.

(ii) the operation o is associative and commutative.

(iii) find the identity element in R for the operation o if it exists.

(iv) find the inverse of $a \in R$ with respect to the operation o it it exists.

Solution : (i) For all $a, b \in R, a + b + ab \in R$. Hence the operation o, given by $aob = a + b + ab$, is a binary operation on R.

(ii) For all $a, b, c \in R$

$(aob)\, oc = (a + b + ab)\, oc$

$= (a + b + ab) + c + (a + b + ab) c$

$= a + b + c + ab + bc + ca + abc$ ... (1)

and $(ao)\, boc = ao\, (b + c + bc)$

$= a + (b + c + bc) + a (b + c + bc)$

$= a + b + c + ab + bc + ca + abc$ ... (2)

From (1) and (2) we have $(aob)\, oc = ao\, (boc)$

Hence the operation o is associative.

Again for all $a, b \in R$

$$aob = a + b + ab \quad \text{... (3)}$$

$$\text{and } boa = b + a + ba = a + b + ab \quad \text{... (4)}$$

From (3) and (4), we find that aob = boa

Hence the operation o is commutative.

(iii) e will be the identity element in R for the operation o if for every $a \in R$, $aoe = a$.

$$\text{But} \quad aoe = a \Rightarrow a + e + ae = a$$

$$\Rightarrow e(1 + a) = 0$$

$$\Rightarrow e = 0 \text{ (since } -1 \notin R \text{ and so } 1 + a \neq 0\text{)}$$

Hence 0 is the identity element in R for the operation o.

(iv) a' will be the inverse of $a \in R$ with respect to the operation o if

$aoa' = 0$, (since the identity element in R is 0)

But $aoa' = 0 \Rightarrow a + a' + aa' = 0$

$$\Rightarrow a'(1+a) = -a$$

$$\Rightarrow a' = \frac{-a}{1+a}$$

Hence, since $-1 \notin R$ and so $1 + a \neq 0$, $\frac{-a}{1+a} \in R$ is the inverse of $a \in R$ with respect to the operation o.

Problem Set 1 (F)

1. If I denotes the set of integers, then show that each of the following is a binary operation on I.

   (i) $a * b = a + b - 1, \quad a, b \in I$

   (ii) $a * b = a + ab, \quad a, b \in I$

   (iii) $a * b = a^2 + b^2, \quad a, b \in I$

   (iv) $a * b = ab + a + b, \quad a, b \in I$

   (v) $a * b = a + ab, \quad a, b \in I.$

2. Show that each of the following are not binary operations on I.

   (i) $a * b = a + b, \quad a, b \in I$

   (ii) $a * b = \dfrac{1}{(a-b)}, \quad a, b \in I$

   (iii) $a * b = \sqrt{a^2 + b^2}, \quad a, b \in I$

   (iv) $a * b = \sqrt{|ab|}, \quad a, b \in I$

   (v) $a * b = \dfrac{\cos \pi a}{\sin \pi b}, \quad a, b \in I$

3. Let $A = \{3^n : n \in N\}$. Is A closed under
   (a) multiplication?
   (b) addition?
4. $*$ is a binary operation defined on Q. Find which of the binary operations are commutative and which of them are associative.
   (i) $a * b = a - b$
   (ii) $a * b = a^2 + b^2$
   (iii) $a * b = a + b + ab$
   (iv) $a * b = ab^2$
   (v) $a * b = \frac{a+b}{2}$
   (vi) $a * b = a\,|\,b\,|$
5. $*$ is a binary operation on N. Find which of the binary operations are commutative and which of them are associative.
   (i) $a * b = a^b$
   (ii) $a * b =$ gcd of a and b
   (iii) $a * b = (a+b)^2$
   (iv) $a * b = a + b + ab$
6. Find the identity element (if it exists) for each of the following binary operations.
   (i) $a * b = a + b + 1, \quad a, b \in I$
   (ii) $a * b = (a+b)^2, \quad a, b \in I$
   (iii) $a * b = \frac{ab}{2}, \quad a, b \in Q^+$
   (iv) $a * b = \frac{3ab}{7}, \quad a, b \in R$
   (v) $a * b = \sqrt{a^2 + b^2}, \quad a, b \in R$
   (vi) $a * b = a + b - 4, \quad a, b \in I$
7. Let $*$ be a binary operation on $Q - \{0\}$, defined by
   $a * b = \frac{ab}{4}, a, b \in Q - \{0\}$. Find the identity element and inverse element in $Q - \{0\}$.
8. Let $*$ be a binary operation on $Q - \{1\}$ defined by $a * b = a + b - ab$, $a, b \in Q - \{1\}$. Find the identity element w.r.t $*$ on Q. Also prove that each element of $Q - \{1\}$ is invertible and also find inverse element in $Q - \{1\}$.

1.10 Algebraic Structures :

1.10.1 Defintion : A non-empty set over which one or more binary operations are defined is called an **algebraic structure**. We will discuss some algebraic structures like Group, Ring, Field. These terms are very often used in linear algebra.

1.10.2 Group :

Definition : Let G be a non-empty set over which a binary operation 'o' is defined. G is said to be a group under the operation 'o', if the following axioms are satisfied.

G1 Closure property : $a \text{ o } b \in G \ \forall a, b \in B$.

G2 Associativity : $a \text{ o } (b \text{ o } c) = (a \text{ o } b) \text{ o } c \ \forall a, b, c \in G$

G3 Existence of identity : There exists an element $e \in G$ such that $eoa = a = aoe \ \forall a \in G$. The element e is called the identity element of G.

G4 Existence of inverse : Each element of G possesses inverse i.e., for all $a \in G$, there exists an element $b \in G$ such that $boa = e = aob$. The element b is called inverse of a and we write $b = a^{-1}$.

Abelian or Commutative Group : A group G is said to be Abelian or Commutative group if in addition, to the above four axioms G1–G4, the following axiom is satisfied.

G5 Commutativity : $aob = boa \ \forall a, b \in G$

Example 1.10.1 :

(i) The set of all integers under addition i,e (Z, +) is a commutative group.

(ii) The set of all real numbers under addition i.e., (R, +) is an abelian or commutative group.

(iii) (N, +) is not a group since G3 and G4 axioms are not satisfied.

(iv) $Q - \{0\}$ and $R - \{0\}$ both are commutative groups under usual multiplication.

(v) $Z - \{0\}$ under multiplication is not a group since G4 is not satisfied.

1.10.3 Rings :

Defintion : A non-empty set R over which two binary operations addition '+' and multiplication '.' are defined, is called a ring if following axioms are satisfied.

R_1 $(a + b) + c = a + (b + c) \ \forall a, b, c \in R$. (associative law under addtion)

R_2 $a + b = b + a \ \forall a, b \in R$. (Commutative law under addtion)

R_3 $\exists$ an element 0 in R such $0 + a = a + 0 = a \ \forall a \in R$. (Existence of additive identity)

R_4 For each $a \in R$, there exists an element $-a \in R$ such that $(-a) + a = 0 = a + (-a)$ (Existence of additive inverse)

R_5 $a . (b . c) = (a . b) . c \ \forall a, b, c \in R$ (Associative law under multiplication)

R_6 For all $a, b, c \in R$, $a . (b + c) = a . b + a . c$ [Distributive law]

Commutative ring – A ring (R, +, .) is said to be a commutative ring of $a . b = b . a \ \forall a, b \in R$.

Example 1.10.2 :

(i) (Z, +, .) is a commutative ring

(ii) (Q, +, .) is a commutative ring

(iii) (R, +, .) is a commutative ring

(iv) (N, +, .) is not a ring since the ring axioms R_3 and R_4 are not satisfied.

1.10.4 Field :

Definition : A non-empty set F overwhich two binary operations addition '+' and multiplication '.' are defined i.e., the algebraic structure (F, +, .) is said to be a field if following axioms are satisfied.

F1 $a + (b + c) = (a + b) + c \ \forall a, b, c \in F$

F2 $a + b = b + a \ \forall a, b \in F$

F3 $\exists \ 0 \in F$ such that $a + 0 = 0 + a = a \ \forall \ a \in F$.

F4 $\forall a \in F, \exists -a \in F$ such that $a + (-a) = (-a) + a = 0$

F5 $a . (b . c) = (a . b) . c \ \forall a, b, c \in F$

F6 $a . b = b . a \ \forall a, b \in F$

F7 $\exists \ 1 \in F$ such that $\forall a \in F, 1 . a = a . 1 = a$

F8 $\forall a \in F - \{0\}, \exists \ a^{-1} \in F - \{0\}$ such that $a . a^{-1} = a^{-1} . a = 1$

F9 $a . (b + c) = a . b + a . c \ \forall a, b, c \in F$.

In other words, a field is a commutative ring such that its non-zero elements form a group under multiplication.

Example 1.10.3 :

(i) (R, +, .) is a field

(ii) (Q, +, .) is a field

(iii) (Z, +, .) is not a field, since the field axiom F8 is not satisfied.

Problem Set 1 (G)

1. Prove that in a group G, the identity element is unique.
2. Prove that in a group G, the inverse of any element of G is unique.
3. Prove that $\{1, \omega, \omega^2\}$ under multiplication is a group (ω being imaginary cube root of unity)
4. Prove that $\{-1, -i, 1, i\}$ is a group under multiplication.
5. Let G be an abelian group with '+' as binary operation, prove that

   (a) $-(-a - b) = a + b$ (b) $a - 0 = a$

   (c) $-(a - b) = b - a$ (d) $(c - b) - (a - b) = c - a$
6. Let A be a ring with '+' and '.' as binary operations. Prove that for $a, b, c, d \in A$.

   (a) $(-a) . (-b) = a . b$

   (b) $(a + b) . (c + d) = a . c + b . d + a . d + b . c$.
7. Prove that $a . 0 = 0 \ \forall a$ in a ring A.

□□□