Module 5:
Integrals Calculus

Integral Calculus

Traditionally it is divided to:

- **Differential calculus**: A subfield of calculus that studies the rates at which quantities change.
- **Integral Calculus**: A subfield of calculus that studies the area under a curve.

Today’s Outline

I. Antiderivatives

II. Definite Integrals

III. Fundamental Theorem of Calculus

IV. Integration by Substitution

V. Integration by Parts

VI. Other Integration Methods

VII. Tutorials

Conceptual Example

- The function $y = f(x)$, is defined as:

$$y = f(x) = x^3$$

- Hence, it's derivative is:

$$y' = f'(x) = 3x^2$$

$$\boxed{y = f(x) = x^3} \xrightarrow{\text{Derivative}} \boxed{y' = f'(x) = 3x^2}$$

Conceptual Example cont

- Now, if $y' = f(x) = 3x^2$ is a given function on $I = \mathbb{R}$
- We wish to find $y = F(x)$; the antiderivative of $f(x)$

$$\boxed{y' = f(x) = 3x^2} \xrightarrow{\text{Antiderivative}} \boxed{y = F(x) = x^3}$$

Antiderivatives

$\boxed{y = f(x)}$ $\xrightarrow{\text{Antiderivative}}$ $\boxed{y = F(x)}$

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example

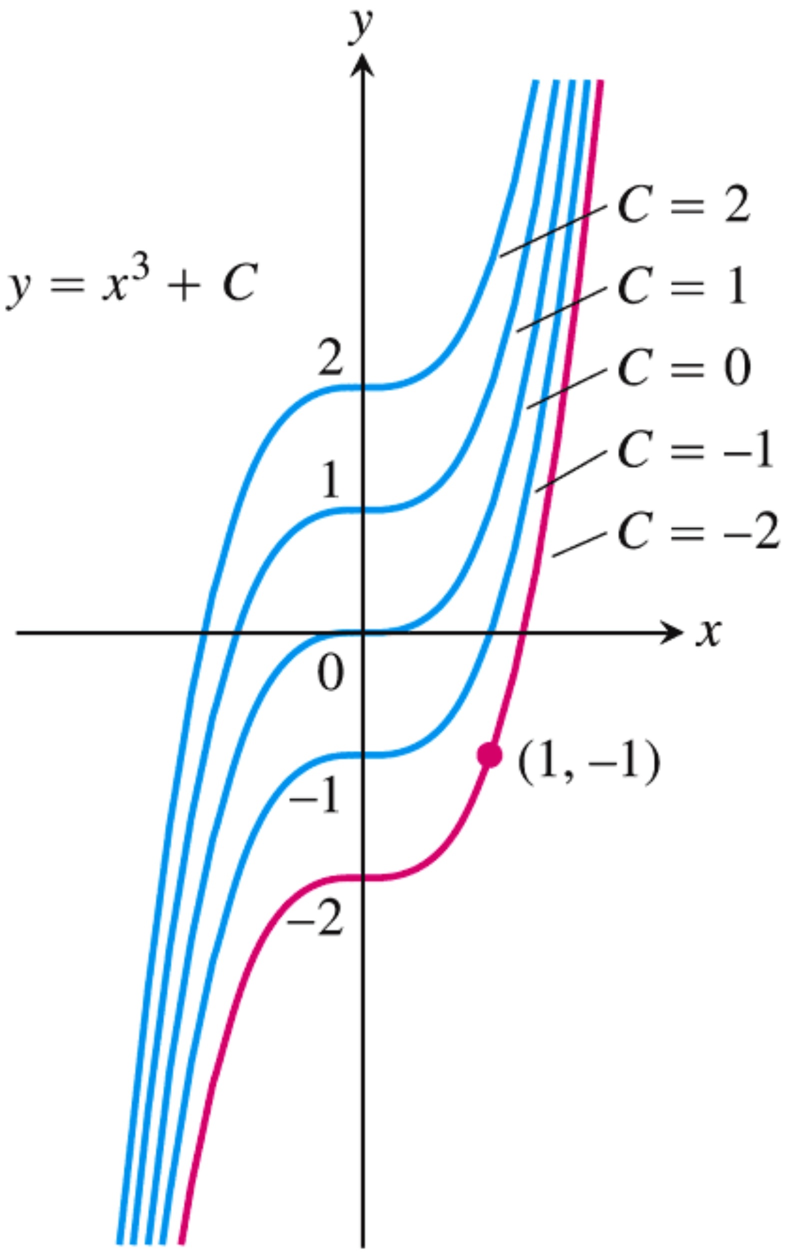
- Show that all the bellow functions are antiderivatives of $y = 3x^2$

a) $y = x^3 \rightarrow y' = 3x^2$

b) $y = x^3 + 1 \rightarrow$ "

c) $y = x^3 - 1 \rightarrow$ "

d) $y = x^3 + c$ (c is a constant) $\rightarrow$ "



$$\boxed{y = 3x^2}$$

Anti Derivative

$$\boxed{y = x^3 + C}$$

C: Constant

FIGURE 4.1 The curves $y = x^3 + C$ fill the coord This Figure. hout overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

Antiderivatives cont

We can deduce that:

THEOREM 1 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

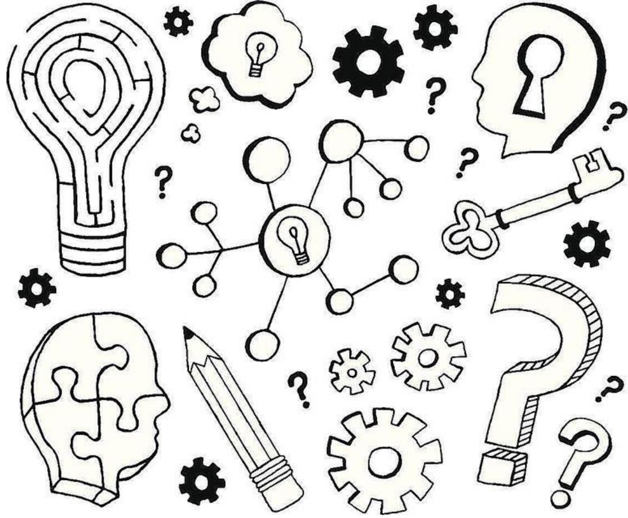
$$F(x) + C$$

where C is an arbitrary constant.

- General Solution
- Particular Solution

$\boxed{y = f(x)}$ Antiderivative $\boxed{y = F(x) + C}$

Activity (Individual, 10')



1. Show that the antiderivatives of $y = f(x) = x^{n}$; is F(x) as below.

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

$$F'(x) = \frac{d}{dx}\left(\frac{x^{n+1}}{x+1} + c\right) = \frac{(n+1)\,x^{n}}{n+1} + 0 = x^{n} = f(x)$$

2. If we graph the function F(x), what does the constant C indicate?

C is a vertical shift. It moves the whole curve of F(x) up or down but doesn't change its shape.
Every C gives a new curve parallel to others

3. How many curves will we have?

Infinitely many - one for every real number C.
Together, they form a family of antiderivatives of x^n

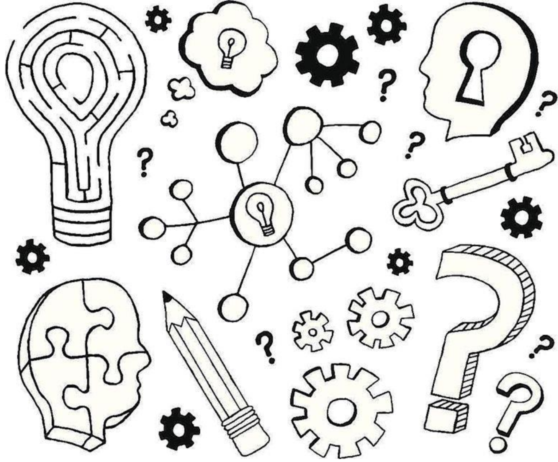
4. If we graph f(x) for a given n, how many curves will we have?

Just one. Differentiation collapses all shifted F(x) curves back into the same f(x).
That's why there's only one derivative, but infinitely many antiderivatives.

5. What is a particular solution, and how can it be obtained from the general solution?

A particular solution is obtained by giving C a specific numerical value, usually after you're told that $F(x_0)=y_0$.

$$F'(x) = \frac{x^{n+1}}{x+1} + c \quad , \; F(1) = 5 \Rightarrow C = 5 - \frac{1}{n+1}$$



Solution

- Show that the antiderivatives of $y = x^{\mathrm{n}}$; is:

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

We should show that F’(x)= x^{n}

- F’(x)=[(n+1)/(n+1)] $x^{(n+1)-1}$ + (dc/dx)
- Hence, F’(x)=x^n

$$\boxed{F(x) = \frac{x^{n+1}}{n+1} + C} \quad \begin{array}{c} \xrightarrow{\text{Derivative}} \\ \xleftarrow{\text{Antiderivative}} \end{array} \quad \boxed{F'(\mathrm{x}) = x^{\mathrm{n}}}$$

Indefinite Integral

DEFINITION The collection of all antiderivatives of f represents the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

$$\boxed{\int f(x)\,\mathrm{d}x = F(x) + c}$$

Reflection (Individual, 15')

- Reflect on Tables 4.1 & 4.2. Show that the second column is the antiderivative of the first column.

(Hint: Take the derivative of the second column)

TABLE 4.1 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	**8.**	e^{kx}	$\frac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\frac{1}{k}\cos kx + C$	**9.**	$\frac{1}{x}$	$\ln\lvert x\rvert + C, \quad x \neq 0$
3.	$\cos kx$	$\frac{1}{k}\sin kx + C$	**10.**	$\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	**11.**	$\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	**12.**	$\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	**13.**	a^{kx}	$\left(\frac{1}{k\ln a}\right)a^{kx} + C, \quad a > 0, \ a \neq 1$
7.	$\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$			

TABLE 4.2 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

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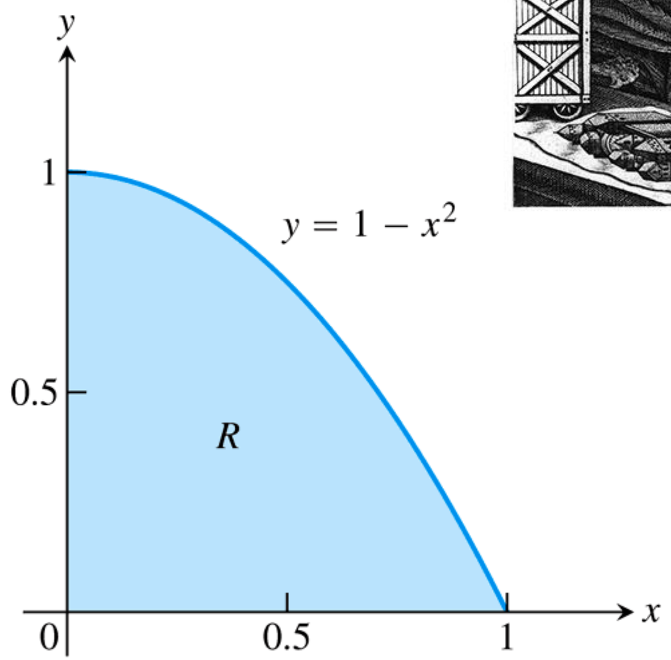
Conceptual Example

What is the are of R?

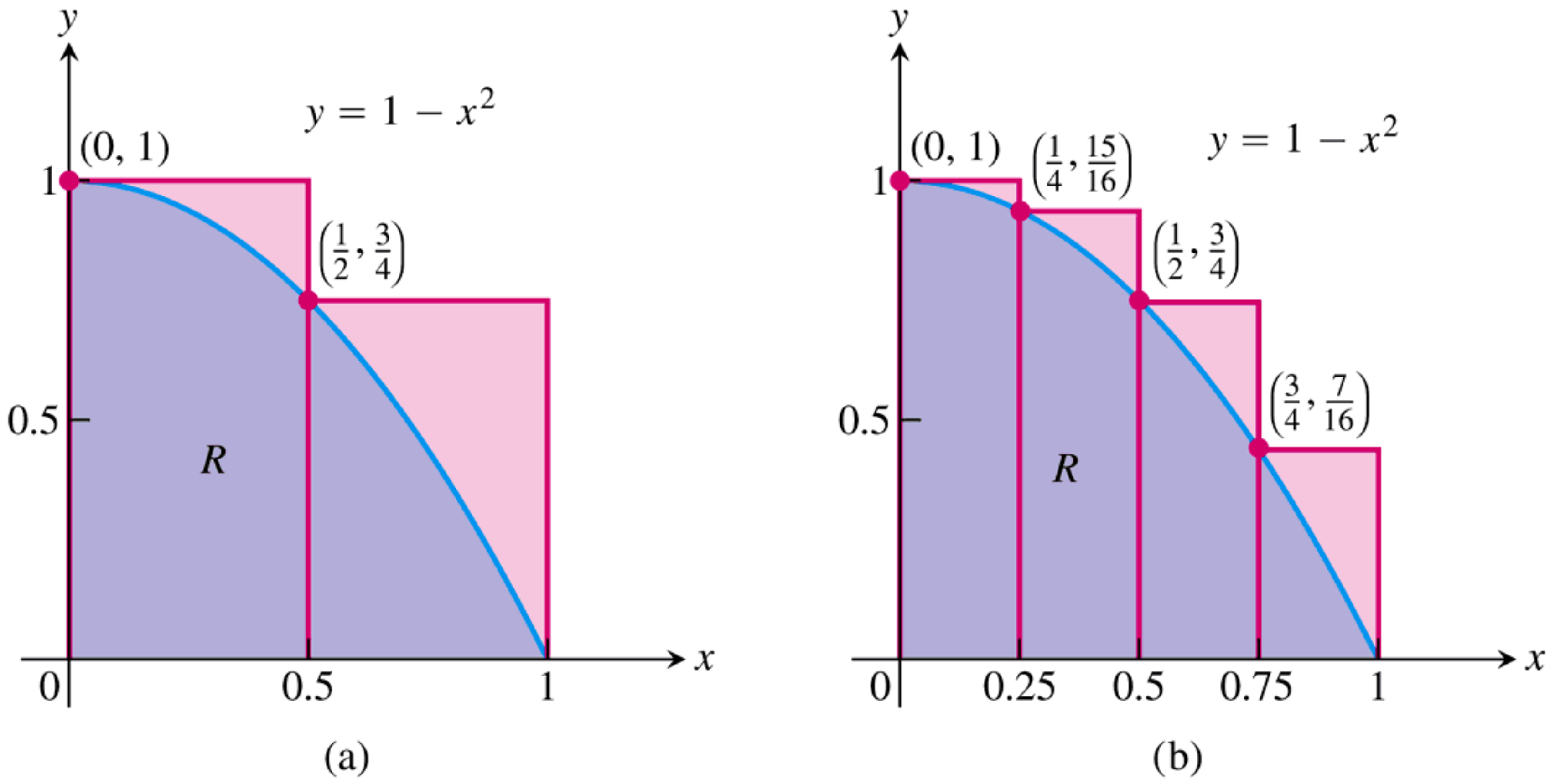
a) We can approximate area R by dividing it to n rectangles

b) Measuring the area of each rectangles → R1, R2, …

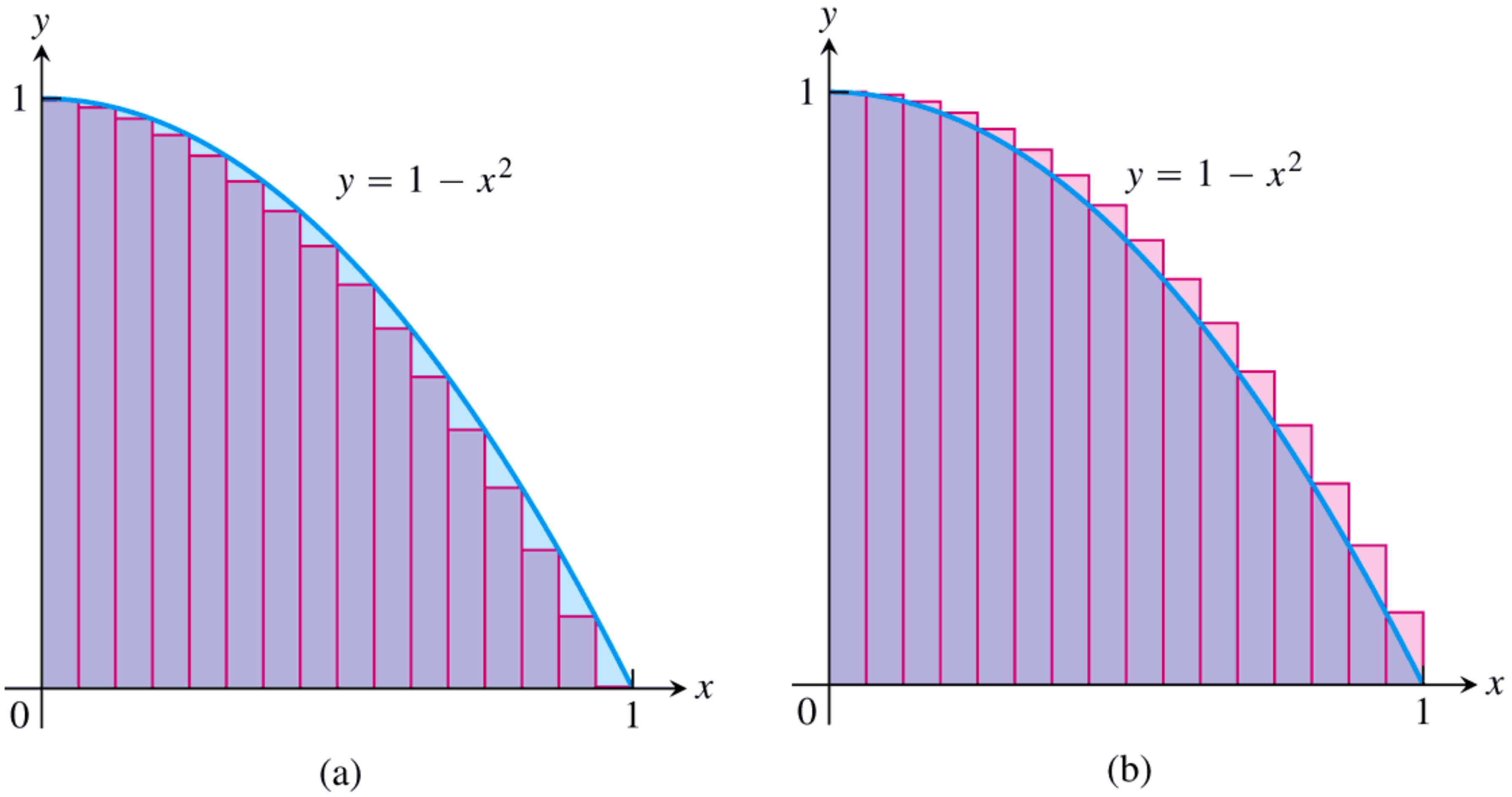
c) And R1+R2+R3+…+Rn → R



The area of the region R cannot be found by a simple formula.

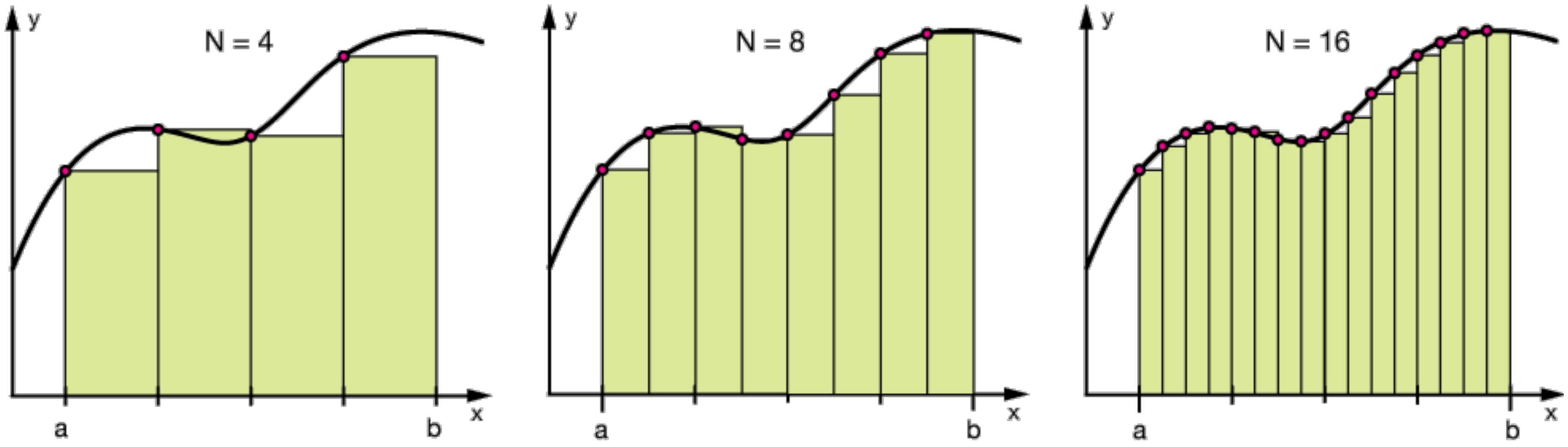


(a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.



(a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$.
(b) An upper sum using 16 rectangles.

Area Under a Curve



$$S = \lim_{n \to \infty} \sum_{i=1}^{n} R_i$$

Riemann Sum

- Riemann Completed Archimedes Method
- A Riemann sum is an approximation of the area of a region
- Area under a curve → A
- Named after German mathematician Bernhard Riemann.
- The sum is calculated by dividing the region into rectangles / trapezoids
- The area for each of these shapes is calculated: R1, R2,..., Rn
- These areas are added together: S = R1+ R2+ R3+...+ Rn = $\sum_{i=1}^{n} R_i$
- S : Riemann sum

Area under the Curve vs. Riemann Sum

- The Riemann sum S, differs from the actual area A
- The error: Error = $A - [\sum_{i=1}^{n} R_i]$= A - S
- This error can be reduced by using more rectangles → n→∞
- As n→ ∞ : $\Delta x \to \mathbf{0}$ and Error→0, hence A →S
- So A is defined as:

$$\boxed{A = \lim_{n \to \infty} \sum_{i=1}^{n} R_i}$$

Area Under a Curve cont

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} R_i$$

$$R_i = f(c_i)(\Delta x_i)$$

$$n \to \infty \equiv \Delta x \to 0$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$

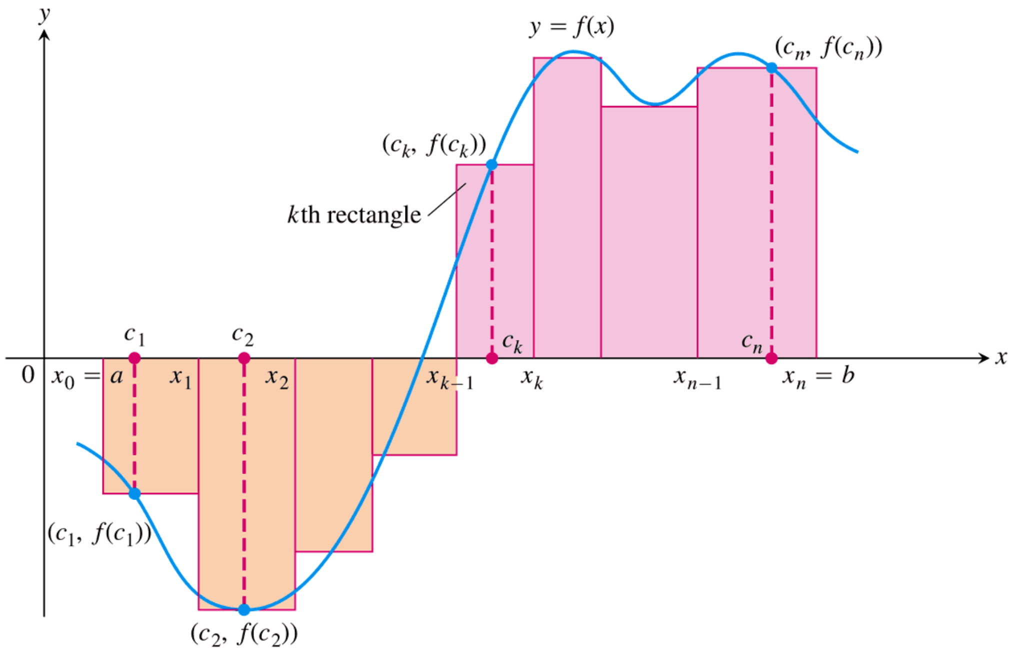
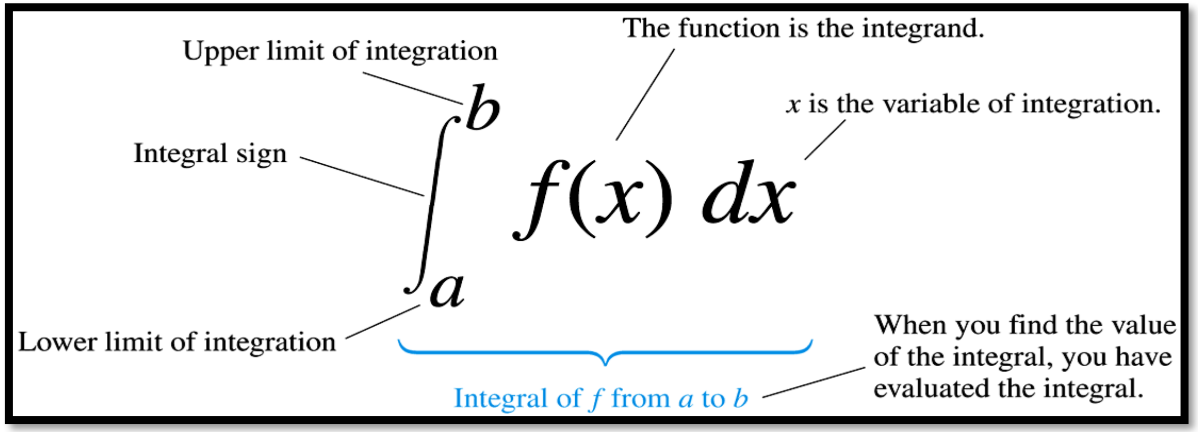


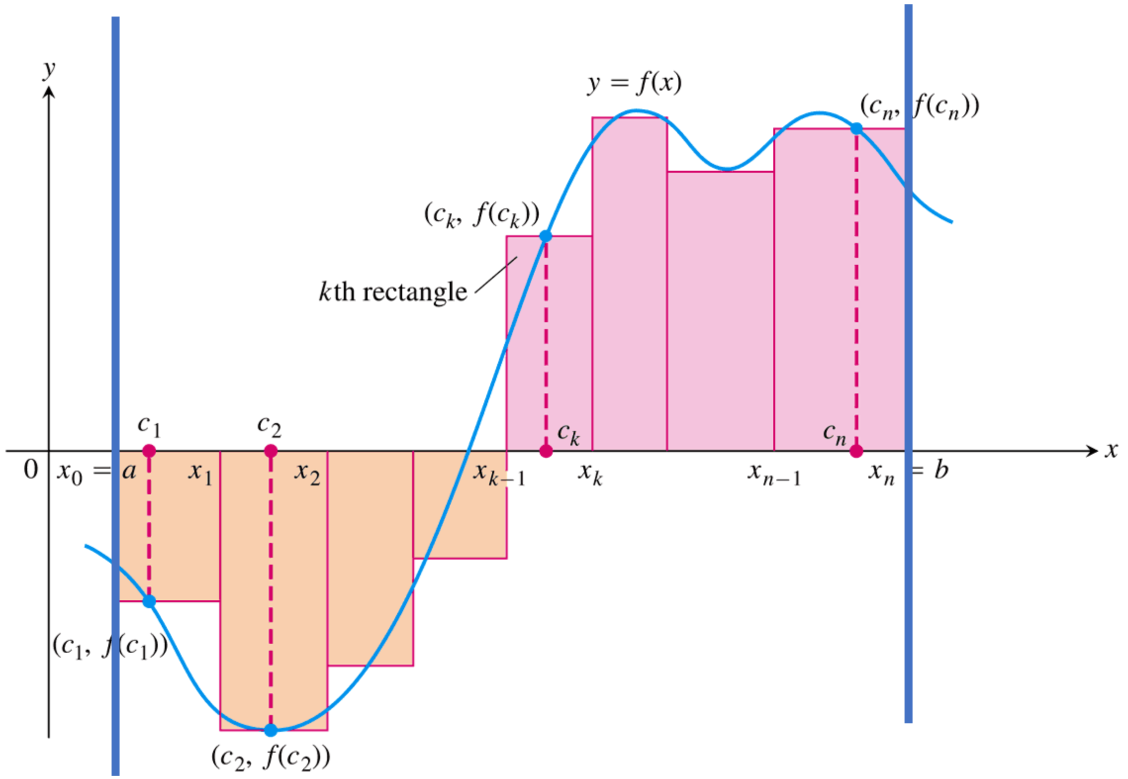
FIGURE 4.10 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

Definite Integral

A is the definite integral of f over
the interval I= [a,b]

$$A = \underset{\Delta x \to 0}{Lim} \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$



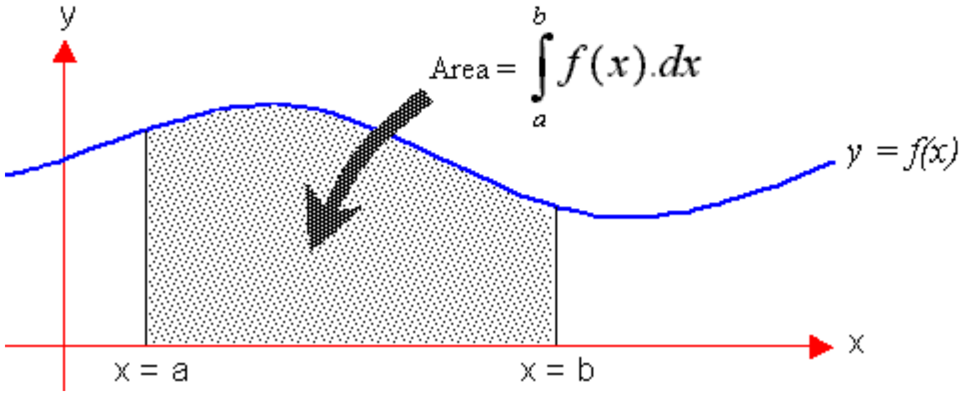


The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

Definite Integrals continued

DEFINITION If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$



Definite Integrals continued

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Activity (Individual, 15’)

- Justify the below Table by matching it with the relevant Figure.

TABLE 4.5 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Definition when $f(a)$ exists
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value $\max f$ and minimum value $\min f$ on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$.	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

Reversing limits flips the sign of the area.

The area under a single point is zero — no width means no area.

Stretching the function vertically by k multiplies the area by k.

Areas add or subtract depending on the sign.

The total area from a to c is the sum of partial areas from a→b and b→c.

The integral's value is bounded between the rectangle of min height and that of max height.

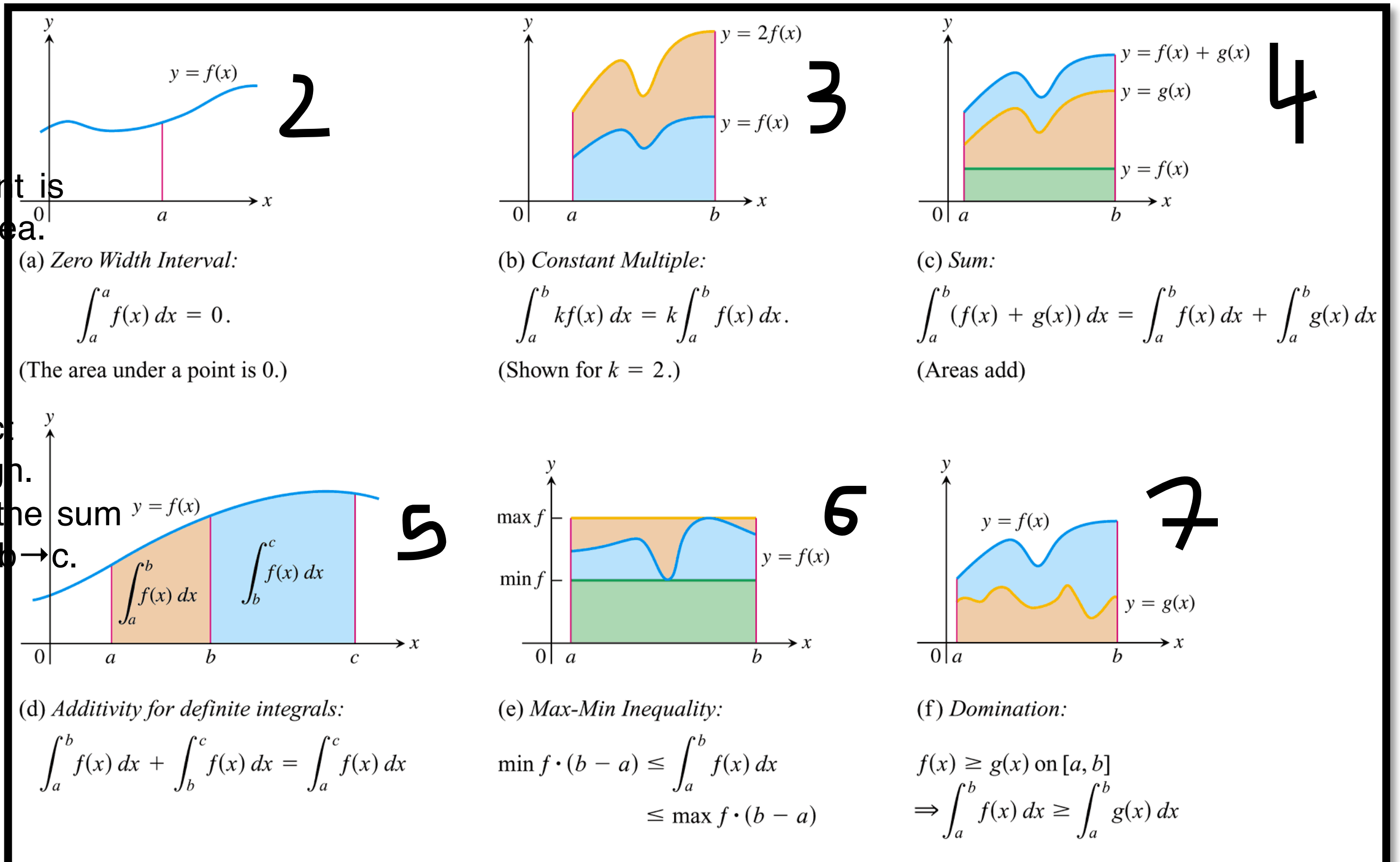


FIGURE 4.12 Geometric interpretations of Rules 2–7 in Table 4.5.

The area under a higher curve is always greater.

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Fundamental Theorems of Calculus

THEOREM 5—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

THEOREM 5 (Continued)—The Fundamental Theorem of Calculus, Part 2 If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a).}$$

Y = F(x) is the antiderivative of f(x)

Reflection (Individual, 10')

- Reflect on Tables 8.1.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$ $(n \neq -1)$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C$
    $= \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C$
    $= -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C$ $(a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C$ $(u > a > 0)$

Time for a break – 20’

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Conceptual Example

Solve the following Integral

$$I = \int 2x\,(\sin x^2)\,\mathrm{d}x$$

- Let $x^2 = u$ → Then $2x\,\mathrm{d}x = \mathrm{d}\,\mathrm{u}$
- So by replacing → $I = \int \sin u\, du = -\cos u + C \Rightarrow \boxed{I = -\cos x^2 + C}$

Integration By Substitution

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

- Substitution: g(x) = u
- g'(x) dx = du
- f(g(x)) g'(x) dx → f(u) du

Example

Solve the following Integral

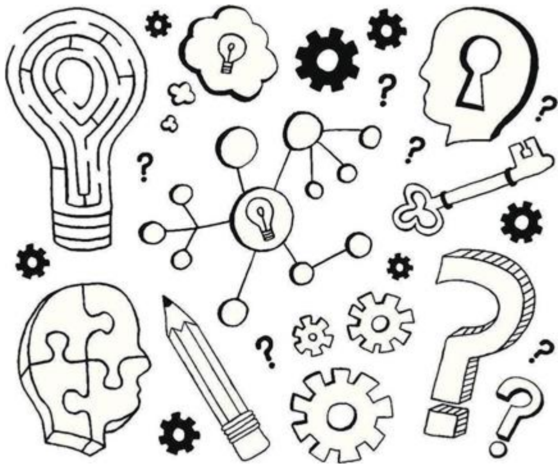
$$I = \int_0^4 e^{-2x}\, \mathrm{d}x$$

- Let $u = -2x \rightarrow \mathrm{d}u = \mathrm{d}(-2x) = -2\,(dx) \rightarrow \mathrm{d}x = -\frac{\mathrm{d}u}{2}$
- $u(0) = 0$
- $u\,(4) = -8$
- $I = \int_0^{-8} e^{\mathrm{u}}\, \mathrm{d}u/(-2) \rightarrow I = (-1/2) \int_0^{-8} e^{\mathrm{u}}\, \mathrm{du} \rightarrow I = \left(-\frac{1}{2}\right)[e^{-8} - e^{0}] = \frac{(1-e^{-8})}{2}$

Activity (Individual, 15’)

Read the following article and solve its examples.

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf



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Integration By Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx\,.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

Integration By Parts cont

- The Product Rule states that, if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

- Integrating the above, and reordering the terms:

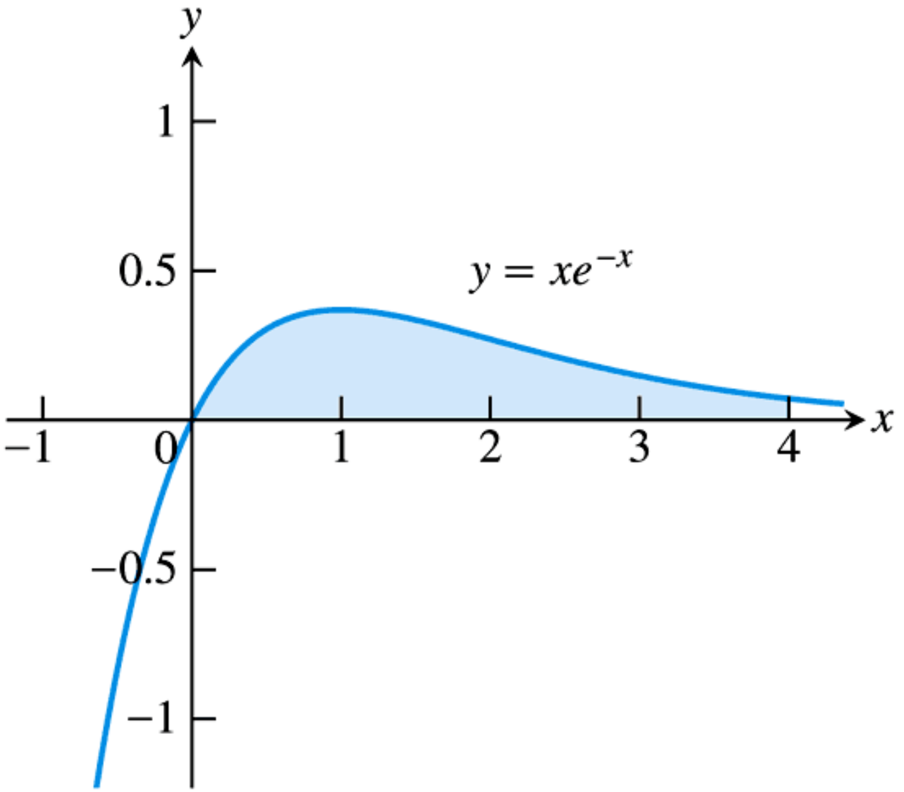
$$f(x)g(x) = \boxed{\int f(x)g'(x)\,dx} + \int g(x)f'(x)\,dx \Rightarrow$$

$$\boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,\mathrm{d}x$$



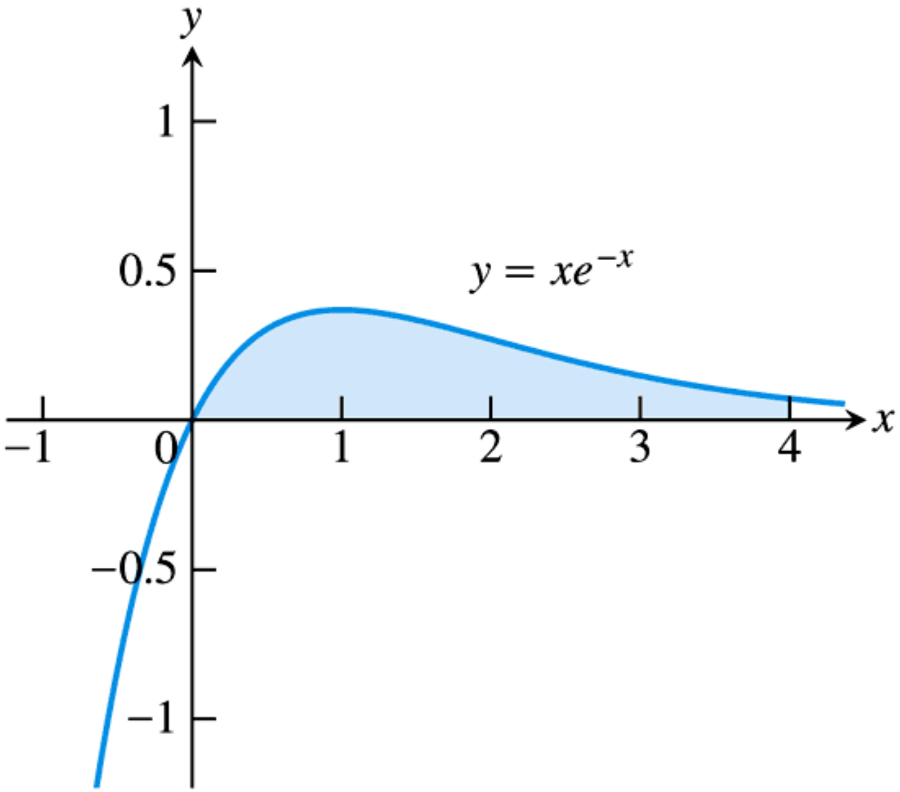
The region in Example

Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,\mathrm{dx}$$

- Let $\mathrm{x} = \mathrm{u}$ ⇨ $\mathrm{dx} = \mathrm{du}$
- Let $e^{-x}\,\mathrm{dx} = \mathrm{dv}$ ⇨ $\mathrm{V} = -e^{-x}$
- SO: $UV = -xe^{-x}$
- SO: $\int V\,\mathrm{d}u = \int -\mathrm{e}^{-x}\,\mathrm{d}x$



The region in Example

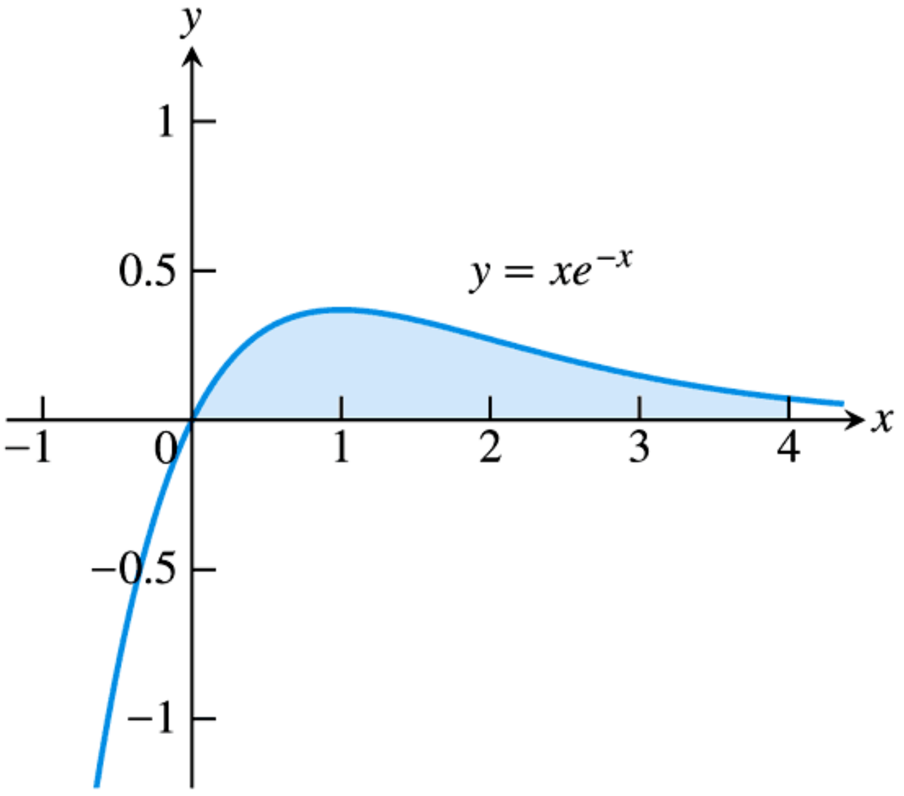
Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,\mathrm{d}x$$

- I = $-xe^{-x}$ - $\int -\mathrm{e}^{-x}\,\mathrm{d}x =$
- I = $-xe^{-x}$ - $\mathrm{e}^{-x} = -\mathrm{e}^{-x}\ (x+1)$
- $\mathrm{I}\ (4) = 5\ (-\mathrm{e}^{-4}\ ) =$ -5 e^{-4}
- $\mathrm{I}\ (0) = 1\ (-\mathrm{e}^{-0}\ ) =$ -1

$\mathrm{I} =$ -5 $\mathrm{e}^{-4} + 1$



The region in Example

Integration By Parts- Summary

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

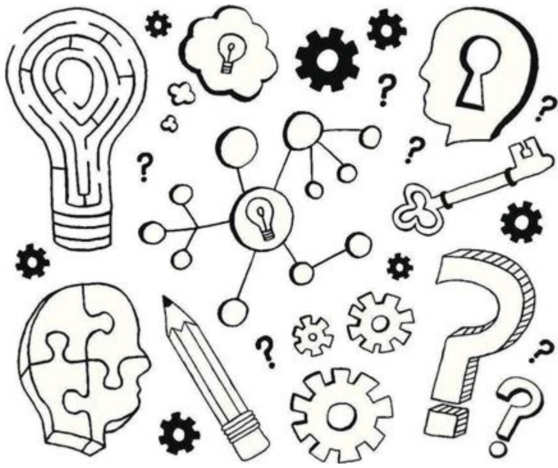
Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

Activity (Individual, 15’)

Read the following article and solve its examples.

https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parts-2009-1.pdf



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Numerical Integration

- There are several other methods for evaluation the integral of a function.
- Such as :
  - Integration Using Trigonometric Identities.
  - Integration by Partial Fraction
  - Integration Using Trigonometric substitution
  - More & More
- Despite these methods, many integrals can not be solved explicitly.
- Using numerical methods and programing, most integrals can be solved.

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Exercise 1

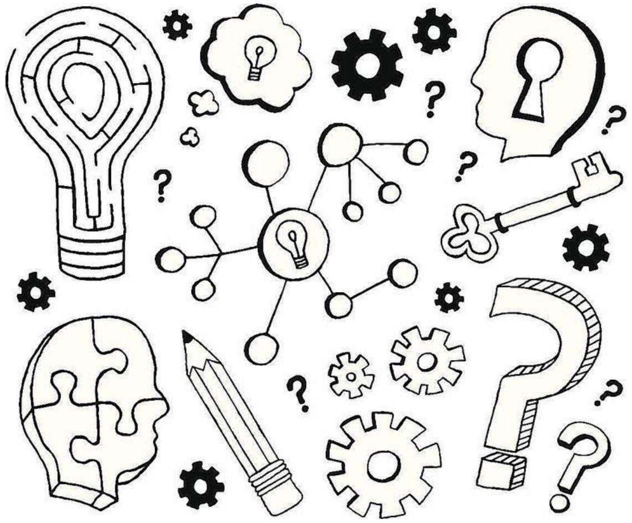
Solve the following Indefinite integrals:

i. $\int \left[(2x-9)/\left(\sqrt{x^2-9x+1}\,\right)\right] dx$

ii. $\int \sqrt{8x-x^2}\,dx$

iii. $\int (Sec\theta + tan\theta)^2\,d\theta$

iv. $\int \sqrt{1+\cos 4x}\,d\theta$



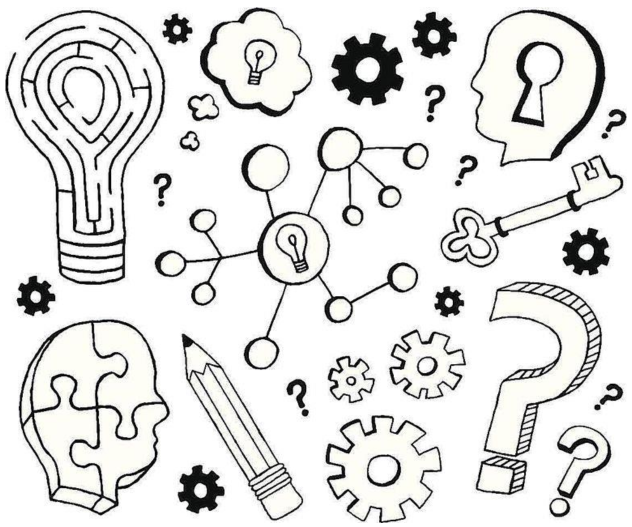
Exercise 1 cont

Procedures for Matching Integrals to Basic Formulas

	Procedure	Example
i	Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
ii	Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
iii	Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$ $= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x \tan x - 1$
iv	Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$

Exercise 2

Solve the following integrals:



i. $\int x\, sinx\, dx$

ii. $\int \ln x\, dx$

Solution 2-i

- Let $\boxed{\begin{array}{c} u = x \\ \Downarrow \\ du = dx \end{array}}$ $\boxed{\begin{array}{c} dv = \sin x\, dx \\ \Downarrow \\ v = -\cos x \end{array}}$
- Then,
- Using Formula 2, we have:

$$\int x \sin x\, dx = \int \overset{u}{x} \overbrace{\sin x\, dx}^{dv} = \overset{u}{x} \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \overset{du}{dx}$$

$$= -x \cos x + \int \cos x\, dx$$

$$= -x \cos x + \sin x + C$$

Solution 2-ii

- Let $\boxed{u = \ln x}$ $\boxed{dv = dx}$
- Then, $\boxed{du = \frac{1}{x}dx}$ $\boxed{v = x}$

$$\int \ln x \, dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$

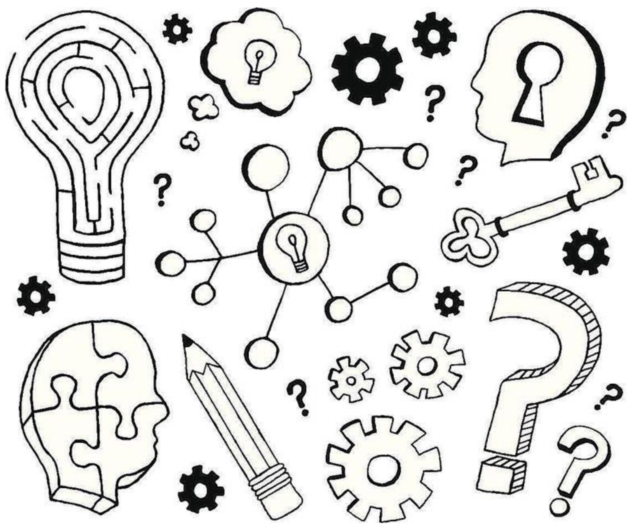
$$= x \ln x - x + C$$

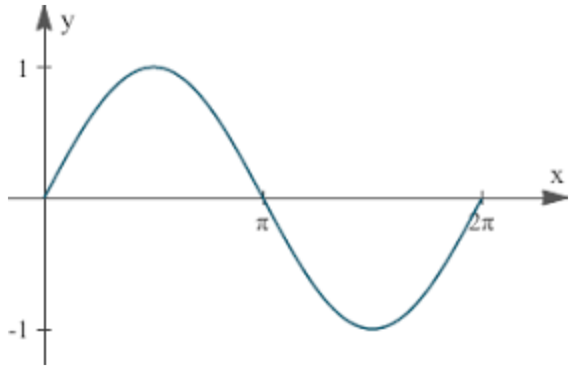
Exercise 3

Find the area under the curve $y = \sin x$

i. On I=$[0,\pi]$

ii. On I=$[0,2\pi]$





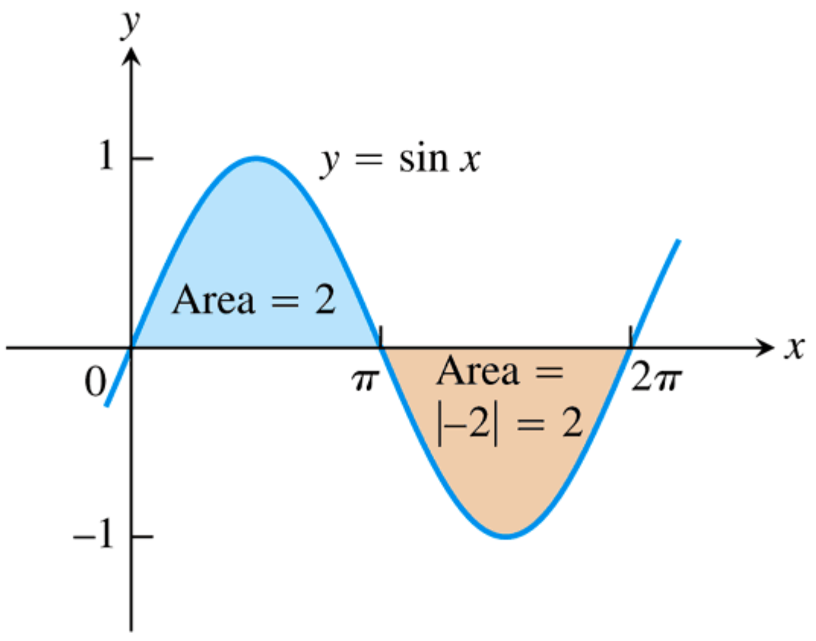
Note: Antiderivative

of Sin x is −cos(x) +C

Exercise 3- Solution

- $I = \int_0^{\pi} \sin x \, \mathrm{d}x$ = - $[\cos \pi - \cos 0]$ = 2

- $I = \int_0^{2\pi} \sin x \, \mathrm{d}x$ = 2 $\int_0^{\pi} \sin x \, \mathrm{d}x = 4$



The total area between $y = \sin x$ and the x-axis for $0 \leq x \leq 2\pi$ is the sum of the absolute values of two integrals

Source of the slides:

Thomas Calculus – 11e

Stewart Calculus

https://www.slideserve.com/search/presentations/derivatives-and-integrals