CHAPTER 3

VECTOR SPACES AND SUBSPACES

3.1 Introduction

In the previous chapter (chapter – 2) much has been discussed about two dimensional (plane) and three dimensional (space) vectors. The set of all these vectors under addition forms a commutative group. In algebra we know that the structure (F, +, .) is said to be a field if

(i) F is a nonempty set

(ii) F is a commutative group under addition +

(iii) The set of all nonzero elements of F forms a commutative group under multiplication.

(iv) Multiplication is distributive over addition.

The set of all real numbers R under ordinary addition and multiplication is a field i.e., (R, +, .) is a field.

With respect to a set of vectors we can have a field (F, +, .) such that the elements of F are regarded as scalars. For example with regard to the set of vectors $v_2 = \{(x_1, x_2) \mid x_1, x_2 \in R\}$, we consider the real field (R, +, .) so that the elements of R (real number set) are regarded as scalars. In chapter – 2 we have discussed some properties of vectors under scalar multiplication (multiplication of a vector by a scalar)

In this chapter we extend the idea of vectors of two and three dimensions to higher dimension. An n-dimensional vector is $v = (x_1, x_2, ..., x_n), x_i \in F \; \forall \; i$. It has n components and each component x_i is an element of a field. $v_n = \{(x_1, x_2, ..., x_n), x_i \in R \; \forall \; i\}$ is the set of all n-tuple of real numbers. Each vector of this set is an n-dimensional vector.

Taking into account the properties of vectors under addition and scalar multiplication we define a new structure called vector space.

3.2 Vectorspaces :

Definition : Let (F, +, .) be a field (whose elements will be termed as scalars). Let V be a non-empty set over which there is a binary operation called addition and a scalar multiplication with respect to F. Then V is said to be a **vector space** (or **linear space**) over F if following axioms are satisfied.

VA. (V, +) is an abelian group.

SM. The scalar multiplication (the multiplication of a vector by a scalar) satisfies :

(a) $\alpha(u+v)=\alpha u+\alpha v$ and $(\alpha+\beta)u=\alpha u+\beta u$
for all scalars $\alpha, \beta \in F$ and all $u, v \in V$.

(b) $\alpha(\beta u)=(\alpha\beta) u=\beta(\alpha u)$ for all scalars $\alpha, \beta \in F$ and all $u \in V$.

(c) $1u=u$ for all $u \in V$.

Example 3.2.1 : V_2 is a vector space over the field (R, +, .) of real numbers.

Proof: In chapter 2 we know that $(V_2, +)$ is an abelian group.

For all $\alpha, \beta \in R$ and $u, v \in V_2$

$\alpha(u+v)=\alpha u+\alpha v, (\alpha+\beta) u=\alpha u+\beta u$

and $1.u = u$.

So V_2 is a vector space over R.

Similarly basing on the properties of vector addition and scalar multiplication mentioned in chapter 2 it is easy to show that V_3 is also a vector space over the field of real numbers.

Example 3.2.2 : Let V_n be the set of all n-tuple of real numbers. Let us define addition and scalar multiplication on V_n as follows :

Addition – For $u=(x_1, x_2, \dots, x_n)$,

$v=(y_1, y_2, \dots, y_n) \in V_n$.

$u+v=(x_1+y_1, x_2+y_2, \dots, x_n+y_n) \in V_n$

This is called co-ordinatewise addition.

Scalar multiplication – If α be any scalar and $u=(x_1, x_2, \dots, x_n) \in V_n$, then the scalar multiplication αu is defined as

$$\alpha u=(\alpha x_1, \alpha x_2, \dots \alpha x_n)$$

This is called co-ordinatewise scalar multiplication

(i) Since for all $u, v \in V_n$, $u+v \in V_n$, V_n is closed under addition.

(ii) For $u, v, w \in V_n$ with

$u=(x_1, x_2, \dots, x_n)$,

$v=(y_1, y_2, \dots, y_n)$ and

$w=(z_1, z_2, \dots z_n)$,

$u+(v+w)$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n))$

$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \ldots, (x_n + y_n) + z_n)$

[Since associative law under addition holds in R]

$= (u + v) + w.$

Hence associative law under addition holds in V_n.

(iii) For $u = (x_1, x_2, \ldots, x_n)$,

$v = (y_1, y_2, \ldots, y_n) \in V_n$,

$u + v = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

$= (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n)$

[$\because$ Commutative law under addition holds in R]

$= v + u$

Hence commutative law under addition holds in V_n.

(iv) $\exists$ zero vector

$\theta = (0, 0, \ldots, 0) \in V_n$ such that for all $u = (x_1, x_2, \ldots, x_n) \in V_n$,

$\theta + u = (0 + x_1, 0 + x_2, \ldots, 0 + x_n)$

$= (x_1, x_2, \ldots, x_n)$

$= (x_1 + 0, x_2 + 0, \ldots, x_n + 0)$

$= u + \theta$

This shows that there exists additive identity 0 in V_n.

(v) For all $u = (x_1, x_2, \ldots x_n) \in V_n$

we find $-u = (-x_1, -x_2, \ldots, -x_n) \in V_n$

such that $u + (-u) = (x_1 + (-x_1), x_2 + (-x_2), \ldots, x_n + (-x_n))$

$= (0, 0, \ldots, 0) = \theta$

Also $-u + u = (-x_1 + x_1, -x_2 + x_2, \ldots, -x_n + x_n) = (0, 0, \ldots 0, 0) = \theta$

So $u + (-u) = -u + u = \theta$.

$-u$ is called additive inverse of u.

It exists for all $u \in V_n$.

Hence $(V_n, +)$ is a commutative group.

So the property VA holds.

Under scalar multiplication following properties holds. Here scalar is taken from the real field (R, +, .)

SM (a). For all $\alpha, \beta \in R$, $u, v \in V_n$, with

$u = (x_1, x_2, \ldots, x_n)$ and

$v = (y_1, y_2, \ldots, y_n)$

$\alpha(u + v)$

$= \alpha(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

$= (\alpha(x_1 + y_1), \alpha(x_2 + y_2), \ldots, \alpha(x_n + y_n))$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, ..., \alpha x_n + \alpha y_n)$
$= (\alpha x_1, \alpha x_2, ..., \alpha x_n) + (\alpha y_1, \alpha y_2, ..., \alpha y_n)$
$= \alpha u + \beta v$
$(\alpha + \beta) u$
$= ((\alpha + \beta) x_1, (\alpha + \beta) x_2, ..., (\alpha + \beta) x_n)$
$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, ..., \alpha x_n + \beta x_n)$
$= (\alpha x_1, \alpha x_2, ..., \alpha x_n) + (\beta x_1, \beta x_2, ..., \beta x_n)$
$= \alpha u + \beta v.$

SM (b). For all $\alpha, \beta \in R, u = (x_1, x_2, ..., x_n) \in V_n$,
$(\alpha\beta) u = ((\alpha\beta) x_1, (\alpha\beta) x_2, ..., (\alpha\beta) x_n)$
$= (\alpha(\beta x_1), \alpha(\beta x_2), ..., \alpha(\beta x_n))$
$= \alpha(\beta x_1, \beta x_2, ..., \beta x_n)$
$= \alpha (\beta u)$

[Since associative law under multiplication holds in R].

Again $\beta(\alpha u) = \beta (\alpha x_1, \alpha x_2, ..., \alpha x_n)$
$= (\beta(\alpha x_1), \beta (\alpha x_2), ..., \beta (\alpha x_n))$
$= ((\beta\alpha) x_1, (\beta\alpha) x_2, ..., (\beta\alpha) x_n)$
$= ((\alpha\beta) x_1, (\alpha\beta) x_2, ..., (\alpha\beta) x_n)$
$= (\alpha\beta) u$

So $(\alpha\beta) u = \alpha(\beta u) = \beta (\alpha u)$

SM (c) : We take $1 \in R$ and any $u = (x_1, x_2, ..., x_n) \in V_n$.

Then $1u = (1.x_1, 1.x_2, ..., 1.x_n) = (x_1, x_2, ..., x_n) = u$

As per difinition, all the properties namely VA, VM (a), VM (b) and VM (c) are satisfied.

Thus V_n is a vector space over the field of real numbers, (R, +, .)

Some notations :

C [a, b] = The set of all real-valued continuous functions defined on the closed interval [a, b].

$C^{(1)}$ [a, b] = The set of all real valued functions defined on [a, b] whose first derivatives are continuous on [a, b].

$C^{(n)}$ [a, b] = The set of all real valued functions defined on [a, b] differentiable n times whose n-th derivatives are continuous. These functions are called n-times continuously differentiable functions.

$C^{(\infty)}$ [a, b] = The set of all functions defined on [a, b] having derivatives of all orders on [a, b].

F(I) = The set of all real valued functions defined on the interval I.

P(I) = The set of all polynomials P with real co-efficients defined on the interval I.

Example 3.2.3 : C [a, b] is a real vector space (a vector space over the field of real numbers) under point wise addition and scalar multiplication.

Proof – Point-wise addition :

For $f, g \in C[a, b]$, $f + g$ is defined as $(f + g)(x) = f(x) + g(x)\ \forall\, x \in [a, b]$.

Pointwise Scalar multiplication :

For all $f \in C[a, b]$ and all scalars α,

αf is defined as $(\alpha f)(x) = \alpha f(x),\quad \forall\, x \in [a, b]$.

From calculus we know that the sum of two continuous functions is also a continuous function.

Such is the case in scalar multiple of a continuous function.

Let us verify the vector space properties for C [a, b],

Property VA :

(i) From above discussion we find

$\forall\, f, g, h \in C[a, b], f + g \in C[a, b]$

(ii) $\forall\, f, g, \in C[a, b]$,

$$\begin{aligned}
&(f + (g + h))(x) \\
&= f(x) + (g + h)(x) \\
&= f(x) + (g(x) + h(x)) \\
&= (f(x) + g(x)) + h(x)
\end{aligned}$$

[$\because$ Associative law under addition holds in R]

$$\begin{aligned}
&= (f + g)(x) + h(x) \\
&= ((f + g) + h)(x) \qquad \forall\, x \in [a, b].
\end{aligned}$$

So $f + (g + h) = (f + g) + h$.

$\therefore$ Associative law under addition holds in C [a, b].

(iii) $\forall\, f, g \in C[a, b]$,

$$\begin{aligned}
&(f + g)(x) \\
&= f(x) + g(x) \\
&= g(x) + f(x) \qquad [\because \text{Commutative law under addition holds in R}]. \\
&= (g + f)(x) \qquad \forall\, x \in [a, b].
\end{aligned}$$

So $f + g = g + f$.

$\therefore$ Commutative law under addition holds in C [a, b].

(iv) From calculus we know that the function $\theta : [a, b] \to [a, b]$ defined by $\theta(x) = 0\ \forall\, x \in [a, b]$ is continuous.

So $\theta \in C[a, b]$.

$\forall\, f \in C[a, b]$ we have $(\theta + f)(x)$

$$\begin{aligned}
&= \theta(x) + f(x) = 0 + f(x) \\
&= f(x) \qquad \forall\, x \in [a, b].
\end{aligned}$$

Thus $\theta + f = f$.

Similarly $f + \theta = f$.

θ is additive identity in C [a, b].

(v) For any $f \in C[a, b]$ we can find $-f \in C[a, b]$ such that

$(-f+f)(x) = -f(x)+f(x)$

$= 0$

$= \theta(x), \quad \forall x \in [a, b].$

So $-f+f = \theta$

Similarly we can show $f+(-f) = \theta$

Thus $-f+f = f+(-f) = \theta$

$-f$, additive inverse of f, exists in C [a, b].

From (i) to (v), we see that C [a, b] is a commutative group under addition.

SM (a): For all scalars α, β and all $f, g \in C[a, b]$,

$$\begin{aligned}
&(\alpha(f+g))(x) \\
&= \alpha(f+g)(x) \\
&= \alpha(f(x)+g(x)) \\
&= \alpha f(x)+\alpha g(x) && \text{[Distributive law holds in R]} \\
&= (\alpha f)(x)+(\alpha g)(x) \\
&= (\alpha f+\alpha g)(x) && \forall x \in [a, b].
\end{aligned}$$

So $\alpha(f+g) = \alpha f+\alpha g$.

Again $((\alpha+\beta)f)(x)$

$$\begin{aligned}
&= (\alpha+\beta) f(x) \\
&= \alpha f(x)+\beta f(x) && \text{[Distributive in R]} \\
&= (\alpha f+\beta f)(x) && \forall x \in [a, b].
\end{aligned}$$

So $(\alpha+\beta) f = \alpha f+\beta f$.

SM (b): For all scalars α, β and any vector $f \in C[a, b]$,

$$\begin{aligned}
&((\alpha\beta)f)(x) \\
&= (\alpha\beta) f(x) \\
&= \alpha(\beta f(x)) && \text{[Associative law under multiplication in R]} \\
&= \alpha(\beta f)(x) && \forall x \in C[a, b].
\end{aligned}$$

So $(\alpha\beta) f = \alpha(\beta f)$

Again $((\alpha\beta) f)(x)$

$$\begin{aligned}
&= (\alpha\beta) f(x) \\
&= (\beta\alpha) f(x) \\
&= \beta(\alpha f(x)) && \text{[Commutative law under multiplication]} \\
&= (\beta(\alpha f))(x) && \forall x \in C[a, b],
\end{aligned}$$

So $(\alpha\beta) f = \beta(\alpha f)$.

$\therefore (\alpha\beta) f = \alpha(\beta f) = \beta(\alpha f)$.

SM (c): For $1 \in R$, for all $f \in C[a, b]$,

$$(1f)(x)$$
$$= 1\,f(x) \qquad \text{[1 being multiplicative identity in R]}$$
$$= f(x), \quad \forall\, x \in [a, b]$$

So $1.f = f$.

Thus C [a, b] is a real vector space (a vector space over the field of real numbers).

Example 3.2.4 : As in the previous example it can be shown that $C^{(1)}[a, b], C^{(n)}[a, b], C^{(\infty)}[a, b]$ are real vector spaces.

Example 3.2.5 : The set of all polynomial P(I) with real co-efficients defined in the interval I is a real vector space.

Proof : First let us define addition and scalar multiplication in P(I) over R.

Addition : For $p, q \in P(I)$ with

$$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$
$$q(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n,\ x \in I,$$
$$(p + q)(x)$$
$$= p(x) + q(x)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)\,x^2 + ... + (a_n + b_n)\,x^n$$

Thus p + q is also a polynomial defined on I.

Scalar multiplication :

For any scalar (a real number) α and any $p \in P(I)$ with

$$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n,$$

We have $(\alpha p)(x)$

$$= \alpha p(x)$$
$$= \alpha\,(a_0 + a_1x + a_2\,x^2 + ... + a_n\,x^n)$$
$$= (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)\,x^2 + ... + (\alpha a_n)\,x^n,\ x \in I$$

It is a polynomial of real co-efficients defined on I.

Let us verify vector space properties as follows.

Property VA :

(i) Closure : By definition of addition, $\forall\, p, q \in P(I),\ p + q \in P(I)$

(ii) Associative law :

$\forall\, p, q, r \in P(I)$, with

$$p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$$
$$q(x) = b_0 + b_1x + b_2x^2 + ... + b_nx^n$$
$$r(x) = c_0 + c_1x + c_2x^2 + ... + c_nx^n,\ x \in I,$$

$$
\begin{aligned}
&(p+(q+r))(x)\\
&=p(x)+(q+r)(x)\\
&=p(x)+(q(x)+r(x)\\
&=(a_0+(b_0+c_0))+(a_1+(b_1+c_1))x\\
&\qquad +(a_2+(b_2+c_2))x^2+\ldots+(a_n+(b_n+c_n))\,x^n.\\
&=((a_0+b_0)+c_0)+((a_1+b_1)+c_1)x\\
&\qquad +((a_2+b_2)+c_2)\,x^2+\ldots+((a_n+b_n)+c_n)x^n\\
&=(p(x)+q(x))+r(x)\\
&=((p+q)+r)\,(x), \quad x\in I.
\end{aligned}
$$

So $p+(q+r)=(p+q)+r$.

(iii) Commutative law :

$\forall\, p, q \in P(I)$ with

$p(x)=a_0+a_1x+\ldots+a_n\, x^n$

$q(x)=b_0+b_1x+\ldots+b_n\, x^n,\ x\in I,$

$$
\begin{aligned}
&(p+q)\,(x)\\
&=p(x)+q(x)\\
&=(a_0+b_0)+(a_1+b_1)\,x+\ldots+(a_n+b_n)\,x^n\\
&=(b_0+a_0)+(b_1+a_1)\,x_1+\ldots+(b_n+a_n)\,x^n
\end{aligned}
$$

[Commutative law under addition in R]

$=(q+p)\,(x), \qquad \forall\, x\in I$

So $p+q=q+p$.

(iv) Existence of additive identity :

$\exists\,\theta$, a polynomial having zero coefficients in P(I) such that for all $p\in P(I)$ with $p(x)=a_0+a_1x+\ldots+a_n\, x^n$,

$$
\begin{aligned}
&(\theta+p)\,(x)\\
&=\theta\,(x)+p(x)\\
&=0+p\,(x)\\
&=p(x), \quad \forall\, x\in I.
\end{aligned}
$$

So $\theta+p=p$.

Similarly $p+\theta=p$.

Thus $\theta+p=p+\theta=p$.

θ is called additive identity.

(v) Existence of additive inverse :

$\forall\, p \in P(I)$ with $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$,

there exists $-p \in P(I)$ where

$-p(x) = (-a_0) + (-a_1)x + (-a_2)x^2 + \ldots + (-a_n)x^n$

such that

$(p + (-p))(x)$

$= p(x) + (-p(x))$

$= (a_0 + (-a_0)) + (a_1 + (-a_1))x + (a_2 + (-a_2))x^2 + \ldots + (a_n + (-a_n))x^n$

$= 0 + 0x + 0x^2 + \ldots + 0x^n$

$= \theta(x), \quad \forall\, x \in I.$

So $p + (-p) = \theta$.

Similarly $(-p) + p = \theta$.

Thus $p + (-p) = (-p) + p = \theta$.

$-p$ is called additive inverse of p.

From (i) to (v) we see that P (I) is a commutative group under addition.

Under scalar multiplication following properties are satisfied.

SM (a) : Let α, β be scalars and $p, q \in P(I)$. Then

$(\alpha(p+q))(x)$

$= \alpha(p+q)(x)$

$= \alpha(p(x) + q(x))$

$= \alpha p(x) + \alpha q(x)$

$= (\alpha p + \alpha q)(x), \quad \forall\, x \in I.$

So $\alpha(p+q) = \alpha p + \alpha q$.

Again $((\alpha+\beta)p)(x)$

$= (\alpha+\beta)p(x)$ (Distributive law)

$= \alpha p(x) + \beta p(x)$

$= (\alpha p + \beta p)(x), \quad \forall\, x \in I.$

So $(\alpha+\beta)p = \alpha p + \beta p$.

SM (b) : Let α, β be any scalars and $p \in P(I)$.

$((\alpha\beta)p)(x) = (\alpha\beta)p(x)$

$= \alpha(\beta p(x))$

$= \alpha(\beta p)(x), \quad \forall\, x \in I.$

So $(\alpha\beta)p = \alpha(\beta p)$.

$((\alpha\beta)p)(x)$

$= (\alpha\beta)p(x)$

$= (\beta\alpha)\, p(x)$

$= \beta(\alpha p(x))$

$= \beta\,(\alpha p)(x), \quad \forall\, x \in I.$

So $(\alpha\beta)\, p = \alpha\,(\beta p) = \beta\,(\alpha p)$.

SM (c) : $\exists$ scalar 1 such that for all $p \in P\,(I)$,

$(1.p)\,(x) = 1.p\,(x)$

$= p(x), \quad \forall\, x \in P\,(I).$

So $1.p = p$

Hence P(I) is a vector space over the real field.

Theorem 3.2.1 : In any vector space V,

(a) $\alpha\theta = \theta$ for every scalar α

(b) $0u = \theta$ for every $u \in V$

(c) $(-1)\,u = -u$ for every $u \in V$

Proof :

(a) $\alpha\theta = \alpha\,(\theta + \theta)$ (θ being additive identy)

$= \alpha\theta + \alpha\theta$ (by SM (a))

Now $\theta = -\alpha\theta + \alpha\theta$ [$-\alpha\theta$ being additive inverse of $\alpha\theta$]

$= -\alpha\theta + (\alpha\theta + \alpha\theta)$ [putting $\alpha\theta = \alpha\theta + \alpha\theta$]

$= (-\alpha\theta + \alpha\theta) + \alpha\theta$ [Associative law]

$= \theta + \alpha\theta$ [$-\alpha\theta$ being additive inverse of $\alpha\theta$]

$= \alpha\theta$ [θ being additive identity]

(b) $0u = (0+0)\,u$ (0 being additive identity in the scalar field)

$= 0u + 0u$ [by SM (a)]

Adding $-(0u)$ on both sides, we get

$\theta = -(0u) + (0u + 0u)$

$= (-0u + 0u) + 0u$ (Associative laws)

$= \theta + 0u$ [$-0u$ being additive inverse of $0u$]

$= 0u$ [θ being additive identity of V]

(c) $(-1)u + u = (-1)u + 1.u$ [by SM (c)]

$= (-1+1)u$ [by SM (a)]

$= 0u$ [-1 being additive identity of 1]

$= \theta$ (by (b))

So (-1) u is additive inverse of u.

But $-$ u is additive inverse of u.

By uniqueness of additive inverse of u,

$(-1)\,u = -u$

Notation – It is convenient to write $u - v$ in stead of $u + (-v)$.

Problem Set 3 (A)

1. Let $u_1 = (2, 3, 1, 5)$
$u_2 = (1, 0, 4, 6)$
$u_3 = (0, -1, -3, 4)$
$u_4 = (2, 6, -1, 5)$
be vectors in V_4.

Evaluate

(a) $u_1 - u_2$
(b) $2u_1 - u_2 + 3u_4$
(c) $u_1 + u_2 + 2u_3 + 5u_4$
(d) $4u_1 + (2u_3 - u_4)$

2. Let $u_1 = (1, 2, 4, 6, 8)$
$u_2 = (1, -1, 3, -2, -4)$
$u_3 = (0, 1, 0, -1, 2)$
$u_4 = (1, 1, -4, 3, 6)$
$u_5 = (-1, 0, -2, 0, -3)$
be vectors of V_5.

Evaluate :

(a) $u_1 + u_3 + u_5$
(b) $u_2 + u_4 - u_3 - u_5$
(c) $2u_1 - 3u_2 + 4u_3 - 5u_4$
(d) $u_1 - 2u_2 + 3u_3 - 4u_5$

3. Which of the following subsets of V_4 are are vector spaces for co-ordinate wise addition and scalar multiplication ?
The set of all vectors $(x_1, x_2, x_3, x_4) \in V_4$ such that

(a) $x_4 = 0$ (b) $x_1 = 1$
(c) $x_3 > 0$ (d) $x_4^2 \geq 0$
(e) $x_1^2 < 0$ (f) $3x_1 + 5x_3 = 0$
(g) $x_1 + \frac{2}{3}x_2 - 3x_3 + x_4 = 1$

4. Which of the following subsets of P are vector spaces ?
The set of all polynomials p such that

(a) degree of $p \leq 4$ (b) degree of $p \geq 5$
(c) degree of $p = 3$ (d) $p(1) = 0$

(e) $p(2) = 1$

(f) $p'(1) = 0$

(g) p has integral co-efficients.

(h) p has rational co-efficients.

5. Which of the following subsets of C [0, 1] are vector spaces ?
the set of all functions $f \in C[0,1]$ such that

(a) $f\left(\frac{1}{2}\right) = 0$ (b) $f\left(\frac{3}{4}\right) = 1$

(c) $f'(x) = x\, f(x)$ (d) $f(0) = f(1)$

(e) $f(x) = 0$ at finite number of points in [0, 1]

(f) f has a local minima at $x = \frac{1}{4}$

(g) f has a local extrema at $x = \frac{1}{2}$

6. In any vector space prove that $\alpha u = 0$ iff $\alpha = 0$ or $u = 0$.

7. Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows :
$u + v = u.v$ for all $u, v \in R^+$,
$\alpha u = u^{\alpha}$ for all $u \in R^+$ and real scalar α
Prove that R^+ is a real vector space.

8. Let V be a real vector space and X an arbitrary set. Let V^x be the set of all functions $f : X \to V$. Prove that V^x is a real vector space for pointwise addition and scalar multiplication.

9. If V is a vector space over the field F, then prove that

(i) $u + (v - u) = u$

(ii) $\alpha u = 0 \Rightarrow$ either $\alpha = 0$ or $u = \theta$

(iii) $\alpha u = \beta u \Rightarrow \alpha = \beta,\ u \neq 0.$

(iv) $\alpha u = \alpha v \Rightarrow u = v, \alpha \neq 0.$ where $u, v \in V,\ \alpha, \beta \in F.$

3.3 Subspace :

3.3.1 Definition : A non-empty subset W of a vector space V over a field F is called a subspace of V if W is a vector space over the field F with respect to the same addition and scalar multiplication defined over V.

Example 3.3.1 : Let W be the set of vectors of the form (x, 2x, 3x) in V_3. Then W is a subspace of V_3.

Proof : Let u = (x, 2x, 3x) and v = (y, 2y, 3y)

and α be any scalar (real number).

Then u + v = (x + y, 2x + 2y, 3x + 3y) = (x + y, 2(x + y), 3 (x + y))

It is of the form (x, 2x, 3x).

So $u + v \in W$.

$\alpha u = (\alpha x, 2\alpha x, 3\alpha x)$

This is also of the form (x, 2x, 3x)

So $\alpha u \in W$.

(0, 0, 0) is zero element (additive identity) of W is obtained taking x = 0,

(x, 2x, 3x) = (0, 0, 0)

For all scalars α, β and $u, v \in W$,

$$\alpha (u + v)$$
$$= \alpha (x + y, 2(x + y), 3(x + y))$$
$$= (\alpha(x + y), 2\alpha(x + y), 3\alpha (x + y))$$

$$\alpha u + \alpha v = (\alpha x, 2\alpha x, 3\alpha x) + (\alpha y, 2\alpha y, 3\alpha y)$$
$$= (\alpha x + \alpha y, 2\alpha x + 2\alpha y, 3\alpha x + 3\alpha y)$$
$$= (\alpha(x + y), 2\alpha(x + y), 3\alpha(x + y))$$

So $\alpha(u + v) = \alpha u + \alpha v$.

Similarly $(\alpha + \beta)\, u = \alpha u + \beta u$ can proved.

$\alpha(\beta u) = (\alpha\beta)u = \beta\,(\alpha u)$ can also be proved in the similar manner.

Finally 1.u

$$= 1\,(x, 2x, 3x)$$
$$= (1.x, 1.2x, 1.3x)$$
$$= (x, 2x, 3x) = u.$$

So W is a vector space with respect to addition and scalar multiplication defined on V_3.

Hence W is a subspace of V_3.

Example 3.3.2 : The set of all scalar multiples of a given element u_0 of a vector space V forms a subspace of V.

Proof : Let V be a vector space over the field F.

Let $W = \{\alpha u_0 \mid \alpha \in F\}$

Clearly $W \subset V$ and $W \neq \phi$.

For $u, v \in W$, $u = \alpha_1 u_0$, $v = \alpha_2 u_0$,
for some $\alpha_1, \alpha_2 \in F$.
Then $u + v = \alpha_1 u_0 + \alpha_2 u_0 = (\alpha_1 + \alpha_2)\, u_0$ [vector space property SM(a)]
$u + v$ is a scalar multiple of u_0.
So $u + v \in W$.
So W is closed under addition.
Associative and commutative laws hold automatically.
Taking $\alpha = 0, \alpha u_0 = \theta \in W$.
θ, the additive identy exists in W.
For $u \in W$, $u = \alpha u_0$
for some α.
Then $-u = -(\alpha u_0) = (-\alpha)\, u_0 \in W$
$-u$, the additive inverse of u exists in W. So W is a commutive group under addition.
Students should verify other properties under scalar multiplication.

Theorem 3.3.1 : A non-empty subset S of a vector space V is a subspace of V iff following conditions are satisfied.

(a) If $u, v \in S$, then $u + v \in S$

(b) If $u \in S$ and α a scalars, then $\alpha u \in S$.

or

A non-empty subset S of a vector space V is a subspace of V iff it is closed under addition and scalar multiplication.

Proof : Condition necessary – Let S be a subspace of V. Then S is a vector space under the same operations as those of V. Hence S satisfies the conditions (a) and (b).

Condition sufficient – Let S be a non-empty subset of V satisfying the conditions (a) and (b).

For all $u \in S$, taking $\alpha = 0$,

$\alpha u = \theta \in S$ (by condition (b))

Similarly for any $u \in S$, taking $\alpha = -1$,

$\alpha u = (-1)\, u = -u \in S$ (condition (b))

Thus S is closed under addition, additive identy exists in S and the inverse of every element of S exists in S.

All other properties like commutative law, associative law under addition and SM (a), SM (b) and SM (c) are inherited from V.

Hence S is a subspace of V.

Example 3.3.3 : $\{(x_1, x_2, x_3) \mid \frac{x_2}{x_3} = \sqrt{2}\}$ is not a subspace of V_3.

Proof : Let $S = \{(x_1, x_2, x_3) \mid \frac{x_2}{x_1} = \sqrt{2}\}$

Let $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3) \in S$ and
α be any scalar.

Then $\frac{x_2}{x_3} = \sqrt{2}$ and $\frac{y_2}{y_3} = \sqrt{2} \Rightarrow x_2 = \sqrt{2}\, x_3, y_2 = \sqrt{2}\, y_3$

Now $u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$.

$$\frac{x_2 + y_2}{x_3 + y_3} = \frac{\sqrt{2}\, x_3 + \sqrt{2}\, y_3}{x_3 + y_3} = \sqrt{2}$$

So $u + v \in S$

Again $\alpha u = (\alpha x_1, \alpha x_2, \alpha x_3)$

$$\frac{\alpha x_2}{\alpha x_3} = \frac{\sqrt{2}\, x_3}{x_3} = \sqrt{2}. \qquad (\alpha \neq 0)$$

So $\alpha u \in S$

But for $\alpha = 0$, $\alpha u = (0, 0, 0) \notin S$ since $\frac{0}{0} \neq \sqrt{2}$.

Hence S is not a subspace of V_3.

Example 3.3.4 : $S = \{p \in P \mid p(x_0) = 0\}$ is a subspace of P, the set of all polynomials with real co-efficients.

Proof : Clearly $S \subset P$.

Let $p, q \in S$

$\Rightarrow p(x_0) = 0 = q(x_0)$

$\Rightarrow p(x_0) + q(x_0) = 0$

$\Rightarrow (p + q)(x_0) = 0$

$\Rightarrow p + q \in S$.

Again if α be any scalar,

$(\alpha p)(x_0) = \alpha p(x_0)$
$= \alpha 0 = 0$

$\Rightarrow \alpha p \in S$.

Hence S is a subspace of P.

Example 3. 3. 5 : $\{f \in C[a, b] \mid f'(x) = 0 \text{ for all } x \in (a, b)\}$ is a subspace of C [a, b].

Proof : Let $S = \{f \in C(a, b) \mid f'(x) = 0 \text{ for all } x \in (a, b)\}$

clearly $S \subset C(a, b)$

Let $f, g \in S$

$\Rightarrow f'(x) = 0,\ g'(x) = 0 \qquad \forall\, x \in (a, b)$

$\Rightarrow f'(x) + g'(x) = 0 \qquad \forall\, x \in (a, b)$

$\Rightarrow (f + g)'(x) = 0 \qquad \forall\, x \in (a, b)$

$\Rightarrow f + g \in S \qquad (\because f + g \in C(a, b))$

Let α be any scalar and $f \in S$.

$\Rightarrow f'(x) = 0 \qquad \forall x \in (a, b).$

$\Rightarrow \alpha f'(x) = \alpha 0 = 0 \qquad \forall x \in (a, b)$

$\Rightarrow (\alpha f)'(x) = 0 \qquad \forall x \in (a, b)$

$\Rightarrow \alpha f \in S \qquad (\because \alpha f \in C(a, b))$

Hence S is a subspace of C (a, b).

Problem Set 3 (B)

1. Show that following subsets of V_3 are subspaces.
   (a) $\{(x_1, x_2, x_3) \mid \sqrt{2}\, x_1 = \sqrt{3}\, x_2\}$
   (b) $\{(x_1, x_2, x_3) \mid x_1 = \sqrt{2}\, x_2 \text{ and } x_3 = 3x_2\}$
   (c) $\{(x_1, x_2, x_3) \mid x_1 = 2x_2 \text{ and } x_3 = 3x_2\}$
   (d) $\left\{(x_1, x_2, x_3) \mid x_1 - 2x_2 = x_3 - \frac{3x_2}{2}\right\}$
2. Show that following subsets of V_3 are not subspaces.
   (a) $\left\{(x_1, x_2, x_3) \mid \frac{x_1}{x_2} = \sqrt{3}\right\}$
   (b) $\{(x_1, x_2, x_3) \mid x_1 x_2 = 0\}$
   (c) $\{(x_1, x_2, x_3) \mid x_3 \text{ is an integer}\}$
   (d) $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$
   (e) $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \geq 0\}$
3. Show that following subsets of p are subspaces of P.
   (a) $\{p \in P \mid \text{degree of } p \leq 2\}$
   (b) $\{p \in P \mid \text{degree of } p \leq 4 \text{ and } p'(0) = 0\}$
   (c) $\{p \in P \mid \text{degree of } p \leq 5\}$
   (d) $\{p \in P \mid \text{degree of } p \text{ not exceeding } 3\}$
   (e) $\{p \in P \mid p(1) = 0\}$
4. Prove that the following subsets are not subspaces of P ?
   (a) $\{p \in P \mid \text{degree of } p \geq 5\}$
   (b) $\{p \in P \mid \text{degree of } p > 4\}$
   (c) $\{p \in P \mid \text{degree of } p = 4\}$
   (d) $\{p \in P \mid \text{degree of } p \geq 4 \text{ and} \leq 5\}$

5. Prove that the following sets are subspaces of C (a, b).

(a) $\{f \in C(a,b) \mid f(x_0)=0, x_0 \in (a,b)\}$

(b) $\{f \in C(a,b) \mid f'(x)=x^2 f(x)\}$

(c) $\{f \in C(a,b) \mid 2f'''(x)+3xf''(x)-f'(x)+x^2 f(x)=0\}$

(d) $\{f \in C(a,b) \mid \int_a^b f(x)\,dx=0\}$

6. Prove that a non-empty set S is a sub space of a vector space V iff $\alpha u+\beta v=S$ for all $u, v \in S$ and all scalars α, β.

7. Let $W=\{(x_1, x_2,\ldots, x_n) \in V_n \mid x_1=0\}$

Prove that W is a subspace of V_n.

8. Prove that intersection of any two subspaces of a vector space is a subspace.

9. Prove that the intersection of a family of subspace of a vector space is a subspace.

10. Let W be the set of all vectors $(x_1, x_2,\ldots, x_n)$ of V_n satisfying the three equations :

$\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_n x_n=0$

$\beta_1 x_1+\beta_2 x_2+\ldots+\beta_n x_n=0$

$\gamma_1 x_1+\gamma_2 x_2+\ldots+\gamma_n x_n=0$, is a subspace of V_n.

11. Find the intersection of the given sets U and W and determine whether it is a subspace :

(a) $U=\{(x_1, x_2) \in V_2 \mid x_1 \geq 0\}$
$W=\{(x_2, x_2) \in V_2 \mid x_1 \leq 0\}$

(b) $U=\{f \in C(-2,2) \mid f(-1)=0\}$
$W=\{f \in C(-2,2) \mid f(1)=0\}$

(c) $U=\left\{f \in C(-2,2) \mid \lim_{x\to 2} f(x)=0\right\}$
$W=\left\{f \in C(-2,2) \mid \lim_{x\to 2} f(x)=1\right\}$

(d) $U=P, W=\{f \in C(-\infty,\infty) \mid f(x)=f(-x)\}$

3.4 Span of a set

In example 3.3.2 we see that the set of all scalar multiples of a given vector x_0 of a vector space V is a subspace of V. We denote it by $[x_0]$. A general version of the fact is given in the following definitions.

3.4.1 Definition : Let $u_1, u_2,\ldots, u_n$ be n vectors of a vector space V and $\alpha_1, \alpha_2,\ldots, \alpha_n$ be n scalars. Then $\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n$ is called a linear combination of $u_1, u_2,\ldots, u_n$. It is also called a **linear combination** of the set $S=\{u_1, u_2,\ldots, u_n\}$. This being a linear combination of a finite set is also called a **finite linear combination**. There can be many linear combinations of a given set.

3.4.2 Definition : (span) The span of a set S of a vector space V is the set of all finite linear combinations of S.

In otherwords if $S=\{u_1, u_2, \ldots . u_n\}$ be a subset of V, the span of S denoted by [S] is given symbolically as

$$[S]=\{\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n \mid \alpha_1, \alpha_2, \ldots \alpha_n \text{ any scalars and } u_1, u_2, \ldots u_n \in S\}$$

Example 3.4.1 : Let $V = V_3$, $S = \{(1, 0, 0), (0, 1, 0)\}$. Any finite combination of S is of the form

$\alpha(1,0,0)+\beta(0,1,0)=(\alpha, \beta, 0)$

for any scalars α, β.

Thus the span of S is given by $[S]=\{(\alpha, \beta, 0) \mid \alpha, \beta \text{ any scalars}\}$

It can be seen that [S] is a subspace of V_3. We shall prove this fact in the following theorem.

Theorem 3.4.1 : Let S be a non-empty subset of a vector space V. Then [S], the span of S, is a sub space of V.

Proof : According to Theorem 3.3.1, it is sufficient to show that [S] is closed under addition and scalar multiplication.

Let $u, v \in [S]$, then $u=\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n$

for some scalars α_i, some u_i's $\in S$ and a positive integer n

and $v=\beta_1 v_1+\beta_2 v_2+\ldots+\beta_m v_m$ for some scalars β_i, some v_i's $\in S$, and a positive integer m.

Hence $u+v=\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n+\beta_1 v_1+\beta_2 v_2+\ldots+\beta_m v_m$.

This is again a finite linear combination of S and so $u + v \in [S]$.

Similarly $\alpha u=(\alpha \alpha_1) u_1+(\alpha \alpha_2) u_2+\ldots+(\alpha \alpha_n) u_n$.

This is again a finite linear combination of S. So $\alpha u \in [S]$.

Hence [S] is a sub space of V.

Note : A non-trivial subspace always contains an infinite number of elements. So $[S]\ (\neq V_0)$ always contains an infinite number of elements. If $S=\phi$, by convention, we take $[\phi]=V_0$.

Theorem 3.4.2 : If S is a non-empty subset of a vector space V, then [S] is the smallest subspace of V containing S.

Proof : Clearly [S] is a subspace of V by Theorem 3.4.1. It contains S because any element u_0 of S can be written in the form $1.u_0$ which is a finite linear combination of S.

To prove that [S] is the smallest subspace containing S, we shall show that if there exists another subspace T containing S, then T contains [S].

Let T be a subspace containing S. We have to prove that $[S] \subset T$.

Let $u \in [S]$

$\Rightarrow$ $u=\alpha_1 u_1+\alpha_2 u_2+\ldots+\alpha_n u_n$ which α_i's are scalars and u_i's are vectors of S and n is a positive integer.

$\Rightarrow$ u is a finite linear combination of T

[Since u_i's $\in S$ and $S \subset T$, u_i's $\in T$]

$\Rightarrow$ $u \in T$ [$\because$ T is a subspace of V]

Hence proved.

Example 3.4.2 : Write the vector $(2, -5, 3)$ as a linear combination of vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Solution :

Let $(2, -5, 3) = \alpha_1 (1, 0, 0) + \alpha_2 (0, 1, 0) + \alpha_3 (0, 0, 1)$

$$\Rightarrow (2, -5, 3) = (\alpha_1, \alpha_2, \alpha_3)$$

$$\Rightarrow \alpha_1 = 2,\ \alpha_2 = -5,\ \alpha_3 = 3$$

So $(2, -5, 3) = 2(1, 0, 0) - 5(0, 1, 0) + 3(0, 0, 1)$

Example 3.4.3 : Let $S = \{(1, 1, 2), (1, -1, 3), (2, 1, -4)\}$ determine whether $(2, -1, 3) \in [S]$.

Solution :

Let $(2, -1, 3) = \alpha (1, 1, 2) + \beta (1, -1, 3) + \gamma (2, 1, -4)$

$$\Rightarrow (2, -1, 3) = (\alpha + \beta + 2\gamma, \alpha - \beta + \gamma, 2\alpha + 3\beta - 4\gamma)$$

$$\Rightarrow \alpha + \beta + 2\gamma = 2,\ \alpha - \beta + \gamma = -1,\ 2\alpha + 3\beta - 4\gamma = 3$$

$$\Rightarrow \alpha = \frac{16}{17},\ \beta = \frac{28}{17},\ \gamma = -\frac{5}{17}$$

So $(2, -1, 3) = \frac{16}{17}(1, 1, 2) + \frac{28}{17}(1, -1, 3) + \left(\frac{-5}{17}\right)(2, 1, -4)$ a finite linear combination of vectors of S.

Hence $(2, -1, 3) \in [S]$.

Example 3.4.4 : Show that in V_2,

$(1, 3) \in [(1, 2), (0, 1)]$ but does not belong to $[(1, 2), (2, 4)]$.

Solution : Let $(1, 3) = \alpha (1, 2) + \beta (0, 1)$

$$\Rightarrow (1, 3) = (\alpha, 2\alpha + \beta)$$

$$\Rightarrow \alpha = 1, 2\alpha + \beta = 3$$

$$\Rightarrow \alpha = \beta = 1$$

Thus $(1, 3) = 1.(1, 2) + 1.(0, 1)$

Hence $(1, 3) \in [(1, 2), (0, 1)]$

Further if $(1, 3) \in [(1, 2), (2, 4)]$, then

$(1, 3) = \alpha (1, 2) + \beta (2, 4) = (\alpha + 2\beta, 2\alpha + 4\beta)$

$$\Rightarrow \alpha + 2\beta = 1,\ 2\alpha + 4\beta = 3$$

The two equations are inconsistent.

Hence $(1, 3) \notin [(1, 2), (2, 4)]$.

Example 3.4.5 : In the vector space V_3,

Let $u_1 = (1, 2, 1), u_2 = (3, 1, 5), u_3 = (3, -4, 7)$

$S = \{u_1, u_2\}$ and $T = \{u_1, u_2, u_3\}$ Show that $[S] = [T]$

Proof : Let $u \in [T]$.

$\Rightarrow$ $u = \alpha u_1 + \beta u_2 + \gamma u_3$ for some scalars α, β, γ.

Let $u_3 = xu_1 + yu_2$

$\Rightarrow$ $(3, -4, 7) = x(1, 2, 1) + y(3, 1, 5)$

$\Rightarrow$ $(3, -4, 7) = (x + 3y, 2x + y, x + 5y)$

$\Rightarrow$ $x + 3y = 3,\ 2x + y = -4,\ x + 5y = 7$

$\Rightarrow$ $x = -3, y = 2$

So $u_3 = -3u_1 + 2u_2$.

$\therefore$ $u = \alpha u_1 + \beta u_2 + \gamma(-3u_1 + 2u_2)$

$= (\alpha - 3\gamma) u_1 + (\beta + 2\gamma) u_2$.

This is a linear combination of u_1 and u_2.

So $u \in [S]$. $\therefore [T] \subseteq [S]$ ... (a)

Let $u \in [S]$

$\Rightarrow$ $u = \alpha' u_1 + \beta' u_2$

$\Rightarrow$ $u = \alpha' u_1 + \beta' u_2 + \gamma' u_3$ $[\gamma' = 0]$

$\Rightarrow$ $u \in [T]$

So $[S] \subseteq [T]$

From (a) and (b) we get

$[S] = [T]$

Example 3.4.6 : If S and T are subsets of a vector space V, then prove that

(a) $S \subset [T] \Rightarrow [S] \subset [T]$

(b) $S \subset T \Rightarrow [S] \subset [T]$

Proof : (a) Let $S \subset [T]$

To show that $[S] \subset [T]$

Let $u \in [S]$

$\Rightarrow$ $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n$

$= \sum_{i=1}^{n} \alpha_i u_i$, u_i's $\in S$.

Since $[T]$ is a subspace of V, u_i's $\in S$, and $S \subset [T]$, a finite linear combination of S belongs to $[T]$.

Thus $u \in [T]$

This shows $[S] \subset [T]$.

(b) Let $S \subset T$. To show that $[S] \subset [T]$

Let $u \in [S]$

$\Rightarrow u = \sum_{i=1}^{n} \alpha_i u_i$, u_i's $\in S$

$\Rightarrow u \in [T]$. $[\because S \subset T, u_i$'S $\in T]$

$\therefore$ $[S] \subset [T]$.

Problem Set 3 (C)

1. Let $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$, Determine which of the following vectors are in [S].
   (a) (0, 0, 0) (b) (1, 1, 0) (c) (2, – 1, – 8)
   (d) $\left(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$ (e) (1, 0, 1) (f) (1, – 3, 5)
2. Let $S = \{x^3, x^2 + 2x, x^2 + 2, 1 - x\}$. Determine which of the following polynomials are in [S].
   (a) $2x^3 + 3x^2 + 3x + 7$ (b) $x^4 + 7x + 2$
   (c) $3x^2 + x + 5$ (d) $x^3 - \frac{3}{2}x^2 + \frac{x}{2}$
   (e) $3x + 2$ (f) $x^3 + x^2 + 2x + 3$
3. Write the following vectors as a linear combination of the vectors (1, – 3, 2), (2, – 4, – 1) and (1, – 5, 1).
   (a) (2, – 5, 3) (b) (4, 3, 2) (c) (1, 2, 3) (d) (2, 1, 4)
4. If S and T are subsets of a vector space V, then prove that
   (a) S is a subspace of V iff [S] = S.
   (b) [[S]] = [S]
   (c) $[S \cup T] = [S] + [T]$
5. Let $v_1, v_2, \ldots, v_n$ be n elements of a vector space V. then prove that
   (a) $[v_1, v_2, \ldots, v_n] = [\alpha_1 v_1, \alpha_2 v_2, \ldots, \alpha_n v_n], \alpha_i \neq 0$
   (b) $[v_1, v_2] = [v_1 - v_2, v_1 + v_2]$
   (c) If $v_k \in [v_1, v_2, \ldots, v_{k-1}]$, then $[v_1, v_2, \ldots, v_{k-1}, v_k, v_{k+1}, \ldots, v_n]$
   $= [v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n]$
6. Let S be a non-empty subset of a vector space V and $u, v \in V$. If $u \in [S \cup \{v\}]$ but $u \notin [S]$, then prove that $v \in [S \cup \{u\}]$.

3.5 Addition of Sets

3.5.1 Definition (Addition of sets) :

Let A and B be two subsets of a vector space V. Then the sum of A and B, written as A + B, is the set of all vectors of the form u + v, $u \in A$ and $v \in B$, i.e.,

$$A + B = \{u + v \mid u \in A, v \in B\}$$

Example 3.5.1 : In V_2 let A = {(1, 2), (0, 1)} and B = {(1, 1), (–1, 2), (2, 5)}. Then

$$\begin{aligned} A + B &= \{(1, 2) + (1, 1), (1, 2) + (-1, 2), (1, 2) + (2, 5), (0, 1) \\ &\quad + (1, 1), (0, 1) + (-1, 2), (0, 1) + (2, 5)\} \\ &= \{(2, 3), (0, 4), (3, 7), (1, 2), (-1, 3), (2, 6)\} \end{aligned}$$

Example 3.5.2 : In V_2 let A = {(2, 3)}, B = {t (3, 1) | t is a scalar}

$$\text{Then } A+B=\{(2,3)+t\,(3,1)\,|\,t \text{ is a scalar}\}$$
$$=\{(2+3t, 3+t)\,|\,t \text{ is a scalar}\}$$

Geometrically this set consists of all points having co-ordinates (x, y) where x = 2 + 3t, y = 3 + t.

Eleminating t we get $y-3=\dfrac{x-2}{3}$.

$\Rightarrow\ x-3y+7=0$

This shows that A + B represents a straight line in a plane.

Example 3.5.3 : In V_3 let $A=\{\alpha(1,2,0)\,|\,\alpha \text{ a scalar}\}$

$$B=\{\beta\,(0,1,2)\,|\,\beta \text{ a scalar}\}$$

$$\text{Then } A+B=\{\alpha(1,2,0)+\beta\,(0,1,2)\,|\,\alpha,\beta \text{ scalar}\}$$
$$=\{(\alpha, 2\alpha+\beta, 2\beta)\,|\,\alpha,\beta \text{ scalars}\}$$

Geometrically A + B is the set of points (x, y, z) with

$x=\alpha,\ y=2\alpha+\beta,\ z=2\beta,\quad \alpha,\beta$ are scalars.

Eliminating α,β, we get

$4x-2y+z=0$. This represents a plane through origin.

Theorem 3.5.1 : If U and W are two subspaces of a vector space V, then U + W is a subspace of V and U + W = $[U\cup W]$.

Proof : Clearly $U+W\subset[U\cup W]$, as each vector of U + W is a finite linear combination of $U\cup W$. Let us prove that $[U\cup W]\subset U+W$.

Let $v\in[U\cup W]$

$$\Rightarrow\ v=\sum_{i=1}^{m}\alpha_i\,u_i+\sum_{j=1}^{n}\beta_j\,w_j$$

u_i's $\in$ U and w_j's $\in$ W $\quad \alpha_i, \beta_j$ are scalars

$$\Rightarrow\ v=u+w \qquad \left[u=\sum_{i=1}^{m}\alpha_i u_i\in U,\right.$$
$$w=\sum_{j=1}^{n}\beta_j w_j\in W$$
since U and W are subspaces]

$\Rightarrow\ v\in U+W$

Hence $[U\cup W]\subset U+W$.

Thus $U+W=[U\cup W]$.

Since span of a subset of a vector space is its subspace, U + W is a subspace of V.

Note : U + W is the smallest subset of V containing $U\cup W$.

Example 3.5.4 : Let $V = V_3$, $U = x-\text{axis}$, and $W = y-\text{axis}$,

i.e., $U = \{(u_1, 0, 0) \mid u_1 \text{ being any scalar}\}$

$W = \{(0, u_2, 0) \mid u_2 \text{ being any scalar}\}$

$U + W = \{(u_1, 0, 0) + (0, u_2, 0) \mid u_1, u_2 \text{ are scalars}\}$

$= \{(u_1, u_2, 0) \mid u_1, u_2 \text{ are scalars}\}$

This set represents xy-plane. i.e.,

[x-axis $\cup$ y-axis] = x-axis + y-axis = xy-plane.

3.5.2 Definition (Direct sum)

A vector space V is said to be a direct sum of its two subspaces U and W if V = U + W and $U \cap W = V_0 = [\theta]$

The direct sum of U and W is written as $U \oplus W$.

Example 3.5.5 :

$U = \text{xy-plane} = \{(x, y, 0) \mid x, y \in R\}$

$W = \text{yz-plane} = \{(0, y, z) \mid y, z \in R\}$

U and W are two subspaces of V_3.

Any element $v \in V_3$ can be written as

$$v = (x, y, z)$$

$$= \left(x, \frac{y}{2}, 0\right) + \left(0, \frac{y}{2}, z\right)$$

$$= u + w \qquad \left[\begin{array}{l} u = \left(x, \frac{y}{2}, 0\right) \in U \\ w = \left(0, \frac{y}{2}, z\right) \in W \end{array}\right]$$

Thus $V_3 = U + W$

$U \cap W = xy-\text{plane} \cap yz-\text{plane} = y-\text{axis} \neq V_0$

$V_3 \neq U \oplus W$

Note : We can also write

$$v = (x, y, z) = \left(x, \frac{2}{3}y, 0\right) + \left(0, \frac{1}{3}y, z\right)$$

i.e., in many ways we can express a vector of V_3 as a sum of a vector of U and a vector of W.

Example 3.5.6 :

Let $U = zx-\text{plane} = \{(x, 0, z) \mid x, z \in R\}$

$W = y-\text{axis} = \{(0, y, 0) \mid y \in R\}$

Clearly $V_3 = U + W$

Again $U \cap W = zx-\text{plane} \cap y-\text{axis} = \text{origin} = V_0$.

So $V_3 = U \oplus W$.

Note : $V_3 =$ xy-plane $\oplus$ z-axis.

$V_3 =$ yz-plane $\oplus$ x-axis.

Theorem 3.5.2 : Let U and W be two subspaces of a vectorspace V and Z = U + W. Then $Z = U \oplus W$ iff any vector $z \in Z$ can be expressed uniquely as the sum $z = u + w, u \in U, w \in W$.

Proof : Condition necessary :

Let $Z = U \oplus W$

We have two show that any vector $z \in Z$ can be expressed uniquely as the sum $z = u + w, u \in U, w \in W$.

Since $Z = U \oplus W$, therefore $Z = U + W$.

Thus any vector $z \in Z$ can be expressed as $z = u + w$, $u \in U$ and $w \in W$.

Let us show that this representation of z is unique.

Suppose it is possible to have another representation

$z = u' + w'$ for some $u' \in U$ and $w' \in W$.

Then $u + w = u' + w'$

$\Rightarrow u - u' = w' - w$

But $u - u' \in U$ and $w' - w \in W$

Thus $u - u' = w' - w \in U \cap W = \{\theta\}$

$\Rightarrow u - u' = w' - w = \theta.$

$\Rightarrow u = u', w' = w.$

Hence no other representation of z is possible. Thus the representation $z = u + w$ is unique.

Condition sufficient :

Let $Z = U + W$ and any vector $z \in Z$ has unique representation as $z = u + w, u \in U$ and $w \in W$.

Let us show that $Z = U \oplus W$.

Since Z = U + W, only it remains to show that $U \cap W = V_0 = \{\theta\}$.

Let of possible $U \cap W$ contains a non-zero vector v. Then $v \in U$ and $v \in W$ and $v = v + \theta \in U + W$ with $v \in U$ and $\theta \in W$.

Also $v = \theta + v \in U + W$ with $\theta \in U$ and $v \in W$.

This shows that v has two representations in U + W. This contradicts our hypothesis that any vector of U + W has unique represention. So $U \cap W$ contains no non-zero vector. Hence $U \cap W = \{\theta\}$ and $Z = U \oplus W$.

3.5.3 Definition (Parallel) :

If U is a subspace of a vector space V and v a vector of V, then {v} + U, also written as v + U, is called a translate of U (by v) or a parallel of U (through v) or a linear variety.

U is called the base space of the linear variety and v a leader.

Example 3.5.7 :

Let $U = \{(x, y) \in V_2 \mid y = x\}$, a line through origin in V_2.

Clearly U is a subspace of V_2.

Consider the point v = (1, 0)

Then $u + U = \{(x+1, y) \in V_2 \mid y = x\}$

$= \{(x, y) \in V_2 \mid y = x - 1\}$

$= \{(x, y) \in V_2 \mid y = x - 1\}$

This represents the line y = x − 1 which is parallel to the line y = x passing through (1, 0).

Example 3.5.8 :

Let $U = \{(x, 0, z) \in V_3 \mid x,\ z \in R\}$

This is y = 0 plane.

Consider the point

$v = (1, 1, 1) \in V_3$

$(1, 1, 1) + U$

$= \{(x + 1, z + 1) \in V_3 \mid x, z \in R\}$

$= \{(x, 1, 2) \in V_3 \mid x, z \in R\}$

$= \{(x, 1, 2) \in V_3 \mid x, z \in R\}$

This represents the plane y = 1 which is parallel to the plane y = 0.

Example 3.5. 9 :

Describe A + B for the given subsets A and B for V_2 and determine whether it is a subspae or a parallel or just a subset of V_2.

(a) $A = \{(1, 2), (0, 1)\},\ B = \{(1, 0), (3, -1)\}$

(b) $A = \{(2, 4)\},\ B = \{(x, y) \mid 2x + 3y = 1\}$

(c) $A = \{t\,(1, 0) \mid t \text{ a scalar}\}$

$B = [(1, 2)]$

Solution :

(a) $A + B = (2, 2), (1, 1), (4, 1), (3, 0)\}$ clearly it is a subset of V_2.

This is also a non-trivial finite subset. hence it cannot be a subspace. Neither A nor B is a subspace. Thus it cannot be a parallel.

So A + B is just a subset.

(b) $A + B = \{(x + 2, y + 4) \mid 2x + 3y = 1\}$

$= \{(x, y) \mid 2\,(x - 2) + 3\,(y - 4) = 1\}$ $\quad [x = x + 2, y = y + 4]$

$= \{(x, y) \mid 2\,x + 3y = 17\}$

Since the lines 2x + 3y = 1 and 2x + 3y = 17 are parallel, A + B is a parallel with base B and leader A.

(c) $A = \{t\,(1, 0) \mid t \text{ a scalar}\}$

$B = [(1, 2)] = \{t\,(1, 2) \mid t \text{ a scalar}\}$

$A + B = \{t_1(1, 0) + t_2\,(1, 2) \mid t_1, t_2 \text{ are scalars}\}$

$= \{(t_1 + t_2, 2t_2) \mid t_1, t_2 \text{ are scalars}\}$

For $u, v \in A + B$,

$u = (t_1' + t_2', 2t_2')$

$v = (t_1'' + t_2'', 2t_2'')$

$u + v = ((t + t_1'') + (t_2' + t_2'),\ 2\,(t_2' + t_2'')$

$= (t_1 + t_2, 2t_2)$ $\quad [t_1 = t_1' + t_1'',\ t_2 = t_2' + t_2'']$

Thus $u + v \in A + B$

If α be a scalar,

$$\alpha u = (\alpha(t_1' + t_2'), 2\alpha t_2')$$
$$= (\alpha t_1' + \alpha t_2', 2\alpha t_2') \qquad [t_1 = \alpha t_1', \quad t_2 = \alpha t_2']$$
$$= (t_1 + t_2, 2t_2).$$

Thus $\alpha u \in A + B$

Hence $A + B$ is a subspace of V_2. Since x and y co-ordinates of $A + B$ are arbitrary $A + B = V_2$.

Example 3.5.10 : Describe A + B for given subsets A and B of V_3. Determine whether A + B is a subspace or a parallel or just a subset of V_3.

(a) $A = \{(1, 2, 1)\}, B = \{t\,(1, 2, 0) \mid t \text{ a scalar}\}$

(b) $A = [(1, 2, 3)], B = [(3, 1, 0)]$

Solution :

(a) $A + B = \{(1 + t, 2 + 2t, 1) \mid t \text{ a scalar}\}$

Taking $x = 1 + t$, $y = 2 + 2t$, $z = 1$, we have

$2\,(x - 1) = y - 2$, $z = 1$ (Eliminating t)

This represents a straight line on $z = 1$ plane.

$B = \{t\,(1, 2, 0) \mid t \text{ a scalar}\}$

$= \{(t, 2t, 0) \mid t \text{ a scalar}\}$

Taking $x = t$, $y = 2t$, $z = 0$, and eliminating t we get

$$2x - y = 0, \quad z = 0.$$

This represents a straight line on $z = 0$ plane.

Clearly A + B is a parallel with base B and leader A.

(b) $A = [(1, 2, 3)]$

$= [t_1 (1, 2, 3) \mid t_1 \text{ a scalar}\}$

$B = [(3, 1, 0)] = \{t_2\,(3, 1, 0) \mid t_2 \text{ a scalar}\}$

$A + B = \{t_1(1, 2, 3) + t_2\,(3, 1, 0) \mid t_1 \text{ and } t_2 \text{ are scalars}\}$

$= \{(t_1 + 3t_2, 2t_1 + t_2, 3t_1) \mid t_1, t_2 \text{ are scalars}\}$

Let $u, v \in A + B$

Then $u = (t_1' + 3t_2', 2t_1' + t'_2, 3t_1')$ for some scalars t_1', t_2'.

$v = (t_1'' + 3t_2'', 2t_1'' + t_2'', 3t_1'')$ for some scalars t_1'', t_2''.

$$u + v = (t_1' + 3t_2' + t_1'' + 3t_2'', 2t_1' + t_2' + 2t_1'' + t_2'', 3t_1' + 3t_1'')$$
$$= ((t_1' + t_1'') + 3(t_2' + t_2''), 2\,(t_1' + t_1'') + (t_2' + t_2''), 3\,(t_1' + t_1''))$$
$$= (t_1 + 3t_2, 2t_1 + t_2, 3t_1) \qquad [t_1 = t_1' + t_1'', \quad t_2 = t_2' + t_2'']$$

Thus $u + v \in A + B$

Again for some scalar α,

$$\alpha u = (\alpha(t_1' + 3t_2'), \alpha(2t_1' + t_2'), 3\alpha\, t_1')$$
$$= (\alpha t_1' + 3\alpha t_2', 2\alpha t_1' + \alpha t_2', 3\alpha t_1')$$
$$= (t_1 + 3t_2, 2t_1 + t_2, 3t_1) \qquad [t_1 = \alpha t_1', t_2 = \alpha t_2']$$

Thus $\alpha u \in A + B$

Hence A + B is a subspace of V_3.

Since x, y and z cordinates of elements of A + B are arbitrary, $A + B = V_3$.

Problem Set 3 (D)

1. Describe A + B for the given subsets A and B of V_2 and determine in each case whether it is a subspace or a parallel or just a subset of V_2.

(a) $A = \{(3, 4), (1, 1)\}$ $B = \{(1, -1), (2, 1)\}$

(b) $A = \{(1, -2), (5, 1)\}$, $B = \{(3, 5), \left(\frac{1}{2}, 3\right), (\sqrt{2}, \pi)\}$

(c) $A = \left\{\left(\frac{1}{2}, \frac{2}{3}\right)\right\}$, $B =$ Segment joining (1, 1) and (2, 3)

(d) $A = \{(2,3)\}, B = \{t\,(3, 4) \mid 1 \le t \le 2\}$

(e) $A = \{(3, 7)\}, B = \{t\,(-1, 2) \mid 0 \le t \le 1\}$

(f) $A = \left\{\left(\frac{1}{2}, 2\right)\right\}, B = \{t\,(3, 0) \mid t \ge 0\}$

(g) $A = \{(1, 5)\}, B = \{(x, y) \mid x^2 + y^2 = 1\}$

(h) $A = \{t\,(3, 4) \mid 0 \le t \le 1\}, B = \{t\,(2, 5) \mid 1 \le t \le 2\}$

(i) $A = \{t(1, 0) \mid 0 \le t \le 1\}, B = \{t\,(0, 1) \mid 2 \le t \le 4\}$

(j) $A =$ line $x = 3t, y = 4t$, $B =$ line $2x + 5y = 0$

2. Describe A + B for the given subsets A and B of V_3. Determine in each case whether A + B is a sub space or a parallel or just a subset of V_3.

(a) $A = \{3, 1, -1\}$, $B = \{(x, y, z) \mid x + y + z = 0\}$

(b) $A = \{(1, -3, 4)\}$, $B = [(1, 2, 3), (0, 0, 1)]$

(c) $A = \left[\left(\frac{1}{2}, \frac{2}{3}, 1\right)\right]$, $B =$ Plane $2x + 3y + z = 0$

(d) $A = [(5, 0, 2)]$, $B = [(1, 2, 3), (0, 1, 2)]$

(e) $A = [(1, 0, -1)]$, $B = [(2, 5, 8), (2, 3, 4)]$

3.6 Linear Dependence and Independence

3.6.1 Definition (Linear Dependence)

A finite set of vectors $\{u_1, u_2, ..., u_n\}$ is said to be linearly dependent (L.D.) if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$ not all zero such that

$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = 0$$

Example 3.6.1 : Prove that the vectors (1, 0, 1), (1, 1, 0) and (–1, 0, –1) are LD.

Proof : Let α, β, γ be scalars such that

$$\alpha(1,0,1) + \beta(1,1,0) + \gamma(-1,0,-1) = (0,0,0)$$
$$\Rightarrow (\alpha+\beta-\gamma, \beta, \alpha-\gamma) = (0,0,0)$$
$$\Rightarrow \alpha+\beta-\gamma = 0, \beta = 0, \alpha-\gamma = 0$$
$$\Rightarrow \alpha = \gamma, \beta = 0$$

If $\alpha = \gamma = 1$, then

$$\alpha(1,0,1) + \beta(1,1,0) + \gamma(-1,0,-1) = (0,0,0)$$

So the given vectors are L.D. since the scalars α, γ are non-zero.

Alternative Method –

For scalars α, β, γ

$$\alpha(1,0,1) + \beta(1,1,0) + \gamma(-1,0,-1) = (0,0,0)$$
$$\Rightarrow (\alpha+\beta+\gamma, \beta, \alpha-\gamma) = (0,0,0)$$
$$\Rightarrow \alpha+\beta+\gamma = 0,\ \beta = 0,\ \alpha-\gamma = 0 \qquad ...(1)$$

The determinant of co-efficients of α, β, γ is

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 0$$

So the system of equating (1) has non-zero solution.

Hence the given set of vectors is LD.

Example 3.6.2 : Prove that the set of vectors $u_1 = (0, 2, -4)$, $u_2 = (1, -2, -1)$, $u_3 = (1, -4, 3)$ of V_3 are LD.

Proof : Let α, β, γ be scalars such that $\alpha u_1 + \beta u_2 + \gamma u_3 = \theta$

$$\Rightarrow \alpha(0,2,-4) + \beta(1,-2,-1) + \gamma(1,-4,3) = (0,0,0)$$
$$\Rightarrow (\beta+\gamma, 2\alpha-2\beta-4\gamma, -4\alpha-\beta+3\gamma) = (0,0,0)$$
$$\Rightarrow \beta+\gamma = 0,\ 2\alpha-2\beta-4\gamma = 0,\ -4\alpha-\beta+3\gamma = 0$$
$$\Rightarrow \alpha = -\beta = \gamma$$

Taking a non zero value 1 for γ,

i.e., taking $\alpha = 1, \beta = -1, \gamma = 1$, $\alpha u_1 + \beta u_2 + \gamma u_3 = \theta$

So the set of vectors $\{u_1, u_2, u_3\}$ is LD.

3.6.2 Definition (Linear Independence)

A finite set of vectors $\{u_1, u_2, \ldots u_n\}$ is said to be linearly independent (LI) if relation of the form,

$$\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = \theta$$
$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

where $\alpha_1, \alpha_2 \ldots \alpha_n$ are scalars.

An infinite set is LI if every finite subset if it is LI.

By convention, the empty set is considered to be LI.

Example 3.6.3 : Prove that the vectors (1, 0, 1), (1, 1, 0) and (1, 1, –1) are LI.

Proof : Consider a ralation of the form

$$\alpha(1,0,1)+\beta(1,1,0)+\gamma(1,1,-1)=(0,0,0)$$

where α, β, γ are scalars.

Then $(\alpha+\beta+\gamma, \beta+\gamma, \alpha-\gamma)=(0,0,0)$

$$\Rightarrow \alpha+\beta+\gamma=0,\ \beta+\gamma=0,\ \alpha-\gamma=0$$
$$\Rightarrow \alpha=\beta=\gamma=0$$

Hence by definition, the given vectors are LI.

Alternative Method :

For scalars α, β, γ,

$$\alpha(1,0,1)+\beta(1,1,0)+\gamma(1,1,-1)=(0,0,0)$$
$$\Rightarrow (\alpha+\beta+\gamma, \beta+\gamma, \alpha-\gamma)=(0,0,0)$$
$$\Rightarrow \alpha+\beta+\gamma=0,\ \beta+\gamma=0,\ \alpha-\gamma=0$$

The determinant of the coefficients of α, β, γ

$$D=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}=-1\neq 0$$

So $\alpha=\beta=\gamma=0$

Hence given vectors are L.I.

Example 3.6.4 : Prove that the system of vectors

$$u_1=(0,1;-2),\ u_2=(1,-1,1) \text{and } u_3=(1,2,1) \text{ are L.I.}$$

Proof : For some scalars α, β, γ

Let $\alpha u_1+\beta u_2+\gamma u_3=\theta$

$$\Rightarrow \alpha(0,1,-2)+\beta(1,-1,1)+\gamma(1,2,1)=(0,0,0)$$
$$\Rightarrow (\beta+\gamma, \alpha-\beta+2\gamma, -2\alpha+\beta+\gamma)=(0,0,0)$$
$$\Rightarrow \beta+\gamma=0, \alpha-\beta+2\gamma=0, -2\alpha+\beta+\gamma=0 \qquad \ldots(1)$$
$$\Rightarrow \alpha=\beta=\gamma=0$$

So u_1, u_2, u_3 are L.I.

Alternatively, the system equations (1) has unique (trivial) solution since the determinant of coefficients of α, β, γ is

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix} = -6 \neq 0$$

Hence $\alpha = \beta = \gamma = 0$

So the given vectors are L.I.

Example 3.6.5 : Prove that the vectors $(x_1, y_1), (x_2, y_2)$ in V_2 are linearly dependent iff $x_1y_2 - x_2y_1 = 0$.

Proof : For some scalars α, β,

$\alpha(x_1, y_1) + \beta(x_2, y_2) = (0,0)$

$\Rightarrow (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (0, 0)$

$\Rightarrow \alpha x_1 + \beta x_2 = 0$

$\alpha y_1 + \beta y_2 = 0$

This system has non-zero solution iff the determinant of the coefficients of α, β,

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = 0$$

$\Rightarrow x_1y_2 - x_2y_1 = 0$ Hence proved.

Example 3.6.6 : Check the linear dependence or linear independence of the following set of vectors.

$$\{(1, 0, 1), (1, 1, 0), (1, -1, 1), (1, 2, -3)\}$$

Solution : From some scalars $\alpha, \beta, \gamma, \delta$,

let $\alpha(1, 0, 1) + \beta(1, 1, 0) + \gamma(1, -1, 1) + \delta(1, 2, -3) = (0, 0, 0)$

$\Rightarrow (\alpha + \beta + \gamma + \delta, \beta - \gamma + 2\delta, \alpha + \gamma - 3\delta) = (0, 0, 0)$

$\Rightarrow \alpha + \beta + \gamma + \delta = 0, \; \beta - \gamma + 2\delta = 0, \; \alpha + \gamma - 3\delta = 0$

$\Rightarrow \alpha = 5\delta, \; \beta = -4\delta, \; \gamma = -2\delta$ for each choice δ.

Taking $\delta = 1$, we obtain nonzero values of α, β, γ such as $\alpha = 5, \beta = -4, \gamma = -2$

so that $5(1, 0, 1) - 4(1, 1, 0) - 2(1, -1, 1) + 1(1, 2, -3) \doteq (0, 0, 0)$

Hence the given set of vectors is L.D.

Note : A set of more than 3 vectors of V_3 is always L.D.

Example 3.6.7 Check whether the set

$$\{e^x, e^{2x}\} \text{ in } C^{(\infty)}(-\infty, \infty) \text{ is L.D. or L.I.}$$

Solution :

Suppose $\alpha e^x + \beta e^{2x} = 0(x) = 0 \quad \forall x \in C^{(\infty)}(-\infty, \infty)$

on differentiation, ... (A)

$\alpha e^x + 2\beta e^{2x} = 0$ .... (B)

Subtracting (A) from (B), we get

$\beta e^{2x} = 0$

$\Rightarrow \beta = 0 \qquad (\because e^{2x} \neq 0)$

Putting $\beta = 0$ in (A), we get

$$\alpha e^x = 0$$

$\Rightarrow \alpha = 0$

$\therefore \alpha = \beta = 0.$

Hence the given set is L.I.

Example 3.6.8 : Check the linear dependence or linear independence of the set $\{x, |x|\}$ in C (–1, 1)

Solution : Suppose $\alpha x + \beta |x| = 0$

Since $|x|$ is not differentiable at zero, we cannot use the method used in Example 3.5.7. $\alpha x + \beta |x| = 0$ holds for all x in (–1, 1) choosing two different values of x,

say $x = \frac{1}{2}$, $x = -\frac{1}{2}$, we get

$$\frac{\alpha}{2} + \frac{\beta}{2} = 0 \text{ and } -\frac{\alpha}{2} + \frac{\beta}{2} = 0.$$

Solving these two equations we get $\alpha = \beta = 0$.

Hence the given set is LI.

3.6.3 Definition : Given a vector $v \neq \theta$, the set of all scalar multiples of v is called a line through v.

Geometrically, in the cases of V_1, V_2 and V_3 it is nothing but the straight line through origin and v.

Example 3.6.9 : S = {t (1, 2, 3) | t any scalar}

Here (x, y, z) = t (1, 2, 3)

$\Rightarrow x = t, y = 2t, z = 3t$

$\Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{3}.$

This represents a straight line in V_3.

So S represents a straight line in V_3 passing through origin.

Example 3.6.10 :

$S = \{v\,t \mid v = (1, 1) \text{ and } t \text{ any scalar}\}.$

$= \{(t, t) \mid t \text{ is any scalar}\}$

Here (x, y) = (t, t)

$\Rightarrow x = y = t$

$\Rightarrow x = y$

This represents equation of a straight line in V_2 through origin.

Hence S is a straight line through origin.

3.6.4 Definition : (Collinearity)

Two vectors v_1 and v_2 are **collinear** if one of them lies on the line through the other.

Example 3.6.11 : The vectors v and 2v of a vector space V are collinear.

sin x, 2 sin x are collinear.

The facts sin x and cos x are not collinear since one is not a scalar multiple of other.

3.6.5 Definition : Given two vectors v_1 and v_2 which are not collinear, their span, namely, $[v_1,v_2]$, is called the plane through v_1 and v_2.

Geometrically, in the cases of V_2 and V_3 it is nothing but the plane passing through origin, v_1 and v_2.

Example 3.6.12 : The functions sin x, cos x in F(I) are not collinear, because neither of the two lies in the line through the other. Their span, namely

$[\sin x, \cos x] = \{\alpha \sin x + \beta \cos x \mid \alpha, \beta \text{ any scalars}\}$

is a plane through the vectors sin x and cos x.

3.6.6 Definition : Three vectors v_1, v_2 and v_3 are **co-planar** if one of them lies in the plane through the other two.

Clearly, 0 is coplanar with every pair of non-collinear vectors.

Example 3.6.13 : The functions sin x, cos x, tan x in F(I) are oviously not coplanar, because none of them lies in the plane through the other two. In other words, none of them can be expressed as a linear combinations of the other two.

Example 3.6.14 : The functions $\sin^2 x$, $\cos^2 x$, cos 2x are coplanar. Because $\sin^2 x = -\cos 2x + \cos^2 x$, a linear combination of $\cos 2x$ and $\cos^2 x$.

(i.e. $\sin^2 x$ lies in the plane containing cos 2x and $\cos^2 x$).

Theorem 3.6.1 : Let V be a vector space. Then

(a) The set $\{v\}$ is LD iff $v = \theta$.

(b) The set $\{v_1, v_2\}$ is LD iff v_1 and v_2 are collinear, i.e. one of them is a scalar multiple of the other.

(c) The set $\{v_1, v_2, v_3\}$ is LD iff v_1, v_2, v_3 are coplanar, i.e. one of them is a linear combination of the other two.

Proof :

(a) $\{v\}$ is LD iff there exists a non zero scalar α such that $\alpha v = 0$. This is possible if $v = \theta$.

(b) Suppose $\{v_1, v_2\}$ is LD.

Then these exists scalars α and β not both zero such that

$\alpha u_1 + \beta v_2 = \theta$.

Suppose $\alpha \neq 0$. then $v_1 = -\dfrac{\beta}{\alpha} v_2$.

This means v_1 is a scalar multiple of v_2. i.e., v_1 lies on the line through v_2.

Hence v_1 and v_2 are collinear.

Conversely, if v_1 and v_2 are collinear, then $v_1 = \alpha v_2$ for some scalar α, i.e., $1.v_1 - \alpha v_2 = \theta$. Hence v_1 and v_2 are LD.

(c) Let $\{v_1, v_2, v_3\}$ is LD.

$\Rightarrow \alpha v_1 + \beta v_2 + \gamma v_3 = \theta$

where α, β, γ are not all zero.

$\Rightarrow v_1 = -\frac{\beta}{\alpha} v_2 - \frac{\gamma}{\alpha} v_3$ (considering $\alpha \neq 0$)

$\Rightarrow v_1 \in [v_2, v_3]$

$\Rightarrow v_1$ lies in the plane through v_2 and v_3.

$\Rightarrow v_1, v_2, v_3$ are coplanar.

Conversely, let v_1, v_2, v_3 are coplanar one of them, say $v_1 \in [v_2, v_3]$.

$\Rightarrow v_1 = \alpha_2 v_2 + \alpha_3 v_3$ for some scalars α_2 and α_3.

$\Rightarrow 1.v_1 - \alpha_2 v_2 - \alpha_3 v_3 = \theta$

$\Rightarrow v_1, v_2, v_3$ are LD $(\because 1 \neq 0)$

Example 3. 6. 15 : Consider three vectors (1, 1, 1), (1, –1, 1) and (3, – 1, 3). It can be verified that they are LD and $1(1, 1, 1) + 2(1, -1, 1) - 1(3, -1, 3) = \theta$.

Hence by Theorem 3.6.1, the plane through (1, 1, 1) and (3, –1, 3) contains the point (1, –1, 1)

The plane through (1, 1, 1) and (3, –1, 3) is

$$[(1, 1, 1), (3, -1, 3)]$$

$= \{\alpha(1,1,1) + \beta(3,-1,3) \mid \alpha, \beta \text{ any scalars}\}$

$= \{(\alpha + 3\beta, \alpha - \beta, \alpha + 3\beta) \mid \alpha, \beta \text{ any scalars}\}$

This set contains (1, –1, 1) if $\alpha + 3\beta = 1,\ \alpha - \beta = -1,\ \alpha + 3\beta = 1$

$\Rightarrow \alpha = -\frac{1}{2}, \beta = \frac{1}{2}$

We could have proved this fact from the relation

$1(1,1,1) + 2(1,-1,1) - 1(3,-1,3) = \theta$

$\Rightarrow (1,-1,1) = -\frac{1}{2}(1,1,1) + \frac{1}{2}(3,-1,3)$

$\Rightarrow (1,-1,1) \in [(1,1,1), (3,-1,3)]$

Theorem 3.6.2 : In a vector space V,

(a) any set of vectors containing the zero vector is LD

(b) if $v \in [v_1, v_2, \ldots, v_n]$, then $\{v_1, v_2, \ldots v_n, v\}$ is L.D.

(c) If the set $\{v_1, v_2, \ldots, v_n\}$ is L.I. and $v \notin [v_1, v_2, \ldots, v_n]$, then the set $\{v, v_1, v_2, \ldots, v_n\}$ is L.I.

Proof : (a) Let $\{v, v_1, \ldots, v_n\}$ be a set of vectors with $v_i = \theta$

Then $0.v_1 + 0.v_2 + 0.v_{i-1} + 1.v_i + 0.v_{i+1} \ldots + 0.v_n = \theta$

Since one of the scalars of the left hand side is different from zero $(\because 1 \neq 0)$, the set is LD.

(b) let $v \in [v_1, v_2, \ldots, v_n]$

$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ for some scalars $\alpha_1, \alpha_2, \ldots \alpha_n$.

$\Rightarrow 1.v - \alpha_1 v_1 - \alpha_2 v_2 \ldots - \alpha_n v_n = \theta$

$\Rightarrow \{v, v_1, v_2, \ldots, v_n\}$ is L.D. $(\because$ the scalars $1 \neq 0)$

(c) The set $\{v_1, v_2, ..., v_n\}$ is LI and $v \notin [v_1, v_2, ..., v_n]$

Let us show that $\{v, v_1, v_2, ..., v_n\}$ is LI.

Now suppose $\alpha v + \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \theta$ ...(1)

for some scalars $\alpha, \alpha_1, \alpha_2, ..., \alpha_n$.

If $\alpha \neq 0$, then

$$v = \left(-\frac{\alpha_1}{\alpha}\right)v_1 + \left(-\frac{\alpha_2}{\alpha}\right)v_2 + ... + \left(-\frac{\alpha_n}{\alpha}\right)v_n$$

$\Rightarrow v \in [v_1, v_2, ..., v_n]$

This contradicts our assumption.

Thus $\alpha = 0$. Putting this in (1) we get $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \theta$

$\Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ ($\because v_1, v_2, ..., v_n$ are LI)

Thus $\alpha = 0$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$

This proves $\{v, v_1, ... v_n\}$ is LI.

Theorem 3.6.3 : (a) If a set is LI, then any subset of it is LI

(b) If a set is LD, then any superset of it is LD.

Proof : (a) Let $A = \{v_1, v_2, ..., v_n\}$ be LI.

Let $S = \{v_1, v_2, ..., v_i\}$ $(i \leq n)$

So that $S \subset A$.

Suppose for some scalars $\alpha_1, \alpha_2, ..., \alpha_i$, $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_i v_i = \theta$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_i v_i + 0.v_{i+1} ... + 0.v_n = \theta$

$\Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_i = 0$ ($\because$ A is LI).

Hence S is LI.

(b) Let $A = \{v_1, v_2, ..., v_n\}$ be LD set.

Suppose $S = \{v_1, v_2, ..., v_m\}$ $(n \leq m)$ be show that S is LD.

Since A is LD, there exists scalars $\alpha_1, \alpha_2, ... \alpha_n$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \theta.$$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n + \alpha_{n+1} v_{n+1} + ... + \alpha_m v_m = \theta$

[taking $\alpha_{n+1} = ... = \alpha_m = 0$]

Now since $\alpha_1, \alpha_2, ..., \alpha_{n+1}, ..., \alpha_m$ are not all zero, the set S is LD.

Theorem 3.6.4 : The set of non-zero vectors $\{v_1, v_2, ..., v_n\}$ of a vector space V is L.D. iff one of them say v_k, $2 \leq k \leq n$ can be expressed as a linear combination of the vectors which preceed it. i.e., $v_k \in [v_1, v_2, ... v_{k-1}]$.

Proof : Condition necessary –

Let the set of n vectors $\{v_1, v_2, ..., v_n\}$ is LD.

Then there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$ not all zero such that

$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \theta$ ...(1)

Let α_k be the last non-zero co-efficient.

If $k = 1$, then $\alpha_1 v_1 = \theta$. But $\alpha_1 \neq 0$.

$\therefore\ v_1 = \theta$

This contradicts the fact that v_i's are non zero vectors.

Thus $2 \leq k \leq n$.

By assumption of k, $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k = \theta$.

$$\Rightarrow v_k = \left(-\frac{\alpha_1}{\alpha_k}\right) v_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) v_2 + \ldots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) v_{k-1}$$

$$\Rightarrow v_k \in [v_1, v_2, \ldots, v_{k-1}]$$

Condition sufficient :

Let for some k with $2 \leq k \leq n$,

$$v_k \in [v_1, v_2, \ldots, v_{k-1}]$$

$$\Rightarrow v_k = \alpha_1 v_1 + \alpha_2 v_2 \ldots + \alpha_{k-1} v_{k-1}$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{k-1} v_{k-1} + (-1) v_k = \theta$$

$\Rightarrow \{v_1, v_2, \ldots, v_k\}$ is L.D. $\quad (\because -1 \neq 0)$

$\Rightarrow \{v_1, v_1, \ldots, v_n\}$ is L.D. $\quad [\because$ any superset of LD set is LD]

Corollary 3.6.5 : A finite subset $S = \{v_1, v_2, \ldots v_n\}$ of a vector space V containing a non-zero vector has a linearly independent subset A such that [A] = [S].

Proof : We assume $v_1 \neq 0$. If S is L.I, then there is nothing to prove, as we have A = S. If not, then by Theorem 3.6.4, there exists a vector v_k such that

$$v_k \in [v_1, v_2, \ldots, v_{k-1}].$$

Discard v_k. Then we get the set $S_1 = \{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$

We claim that $[S_1] = [S]$.

Let us justify this as follows.

Since $S_1 \subset S$,

$[S_1] \subset [S]$

Let $v \in [S]$.

$$\Rightarrow v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k + \ldots + \alpha_n v_n \quad \ldots(1)$$

Since $v_k = \beta_1 v_1 + \beta_2 v_2 \ldots + \beta_{k-1} v_{k-1} \quad \ldots(2)$

From (1) and (2) we get

$$v = \alpha_1 v_1 + \ldots + \alpha_{k-1} v_{n-1} + \alpha_k (\beta_1 v_1 + \beta_2 v_2 \ldots + \beta_{k-1} v_{k-1}) + \alpha_{k+1} v_{k+1} \ldots + \alpha_n v_n$$

$$= (\alpha_1 + \alpha_k \beta_1) v_1 + \ldots + (\alpha_{k-1} + \alpha_k \beta_{x-1}) v_{\backslash-1} + \alpha_{k+1} v_{k+1} + \ldots + \alpha_n v_n$$

$\Rightarrow$ $v \in [S_1]$.

So $[S] \subset [S_1]$

Hence $[S_1] = [S]$

If S_1 is LI, then can take $A = S_1$. So that [A] = [S].

If not, the repeat the foregoing process. Ultimately we get a linearly independent subset A of S such that [A] = [S].

Example 3.6.16 : Show that the ordered set $\{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$ is LD and locate one of this vectors that belongs to the span of the previous ones.

Solution : Consider the sets

$S_1 = \{(1, 1, 0)\}$

$S_2 = \{(1, 1, 0), (0, 1, 1)\}$

$S_3 = \{(1, 1, 0), (0, 1, 1), (1, 0, -1)\}$

$S_4 = \{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$

Clearly S_1 is LI since

a singleton set having a non-zero vector is LI.(Theorem 3.6.1)

Now $\alpha(1,1,0) + \beta(0,1,1) = (0,0,0)$

$\Rightarrow (\alpha, \alpha+\beta, \beta) = (0,0,0)$

$\Rightarrow \alpha = \beta = 0$

So S_2 is alos LI.

Again $\alpha(1,1,0) + \beta(0,1,1) + \gamma(1,0,-1) = (0,0,0)$

$\Rightarrow (\alpha+\gamma, \alpha+\beta, \beta-\gamma) = (0,0,0)$

$\Rightarrow \alpha+\gamma = 0,\ \alpha+\beta = 0,\ \beta-\gamma = 0$

$\Rightarrow \beta = \gamma = -\alpha$

Taking $\alpha = 1$, we have $\beta = \gamma = -1$,

We get

$1\,(1, 1, 0) - 1\,(0, 1, 1) - 1\,(1, 0, -1) = (0, 0, 0)$

So S_3 is LD.

Also $(1, 0, -1) = 1\,(1, 1, 0) - 1\,(0, 1, 1)$

$\Rightarrow (1,0,-1) \in [(1,1,0), (0,1,1)]$

If we take

$S_4 = \{(1,1,0), (0,1,1), (1,0,-1), (1,1,1)\}$,

Then S_4 is LD since $S_4 \supset S_3$. (Theorem 3.6.3)

Example 3.6.17 : Find the largest linearly independent subset whose span is S_4 in Example 3.6.16

Solution : In Example 3.6.16 we have shown that $(1, 0, -1) \in [(1,1,0), (0,1,1)]$

$\Rightarrow (1,0,-1) \in [(1,1,0), (0,1,1), (1,1,1)]$

Discard (1, 0, –1).

Then the span of the remaining set $A = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ is same as $[S_4]$.

Let us check for the linear independence of A.

Suppose $\alpha(1,1,0)+\beta(0,1,1)+\gamma(1,1,1)=(0,0,0)$

$$\Rightarrow (\alpha+\gamma, \alpha+\beta+\gamma, \beta+\gamma)=(0,0,0)$$
$$\Rightarrow \alpha+\gamma=0, \alpha+\beta+\gamma=0, \beta+\gamma=0$$
$$\Rightarrow \alpha=\beta=\gamma=0.$$

Hence A is the largest LI subset of S_4 such that $[A] = [S_4]$

3.6.7 Definition : An infinite subset S of a vector space V is said to be linearly independent (LI) if every finite subset of S is LI.

S is said to be linearly dependent (LD) if it is not LI.

Example 3.6.18 : The subset $S = \{1, x, x^2,...\}$ of P is L.I. where P is the set of all polynomials with real co-efficients.

Proof : Suppose $\alpha_1 x^{k_1} + \alpha_2 x^{k_2} + ... \alpha_n x^{k_n} = \theta$ with $k_1, k_2,...,k_n$ being distinct non-negative integers.

This equality is an algebraic identity since the right hand side is the zero polynomial. So either by giving various values to x or by repeated differentiation of both sides of the identity, we get

$$\alpha_1 = \alpha_2 = ... = \alpha_n = 0.$$

Example 3.6.19 : If u_1, u_2 are vectors of a vector space V and a, b are scalars, then the set $\{u_1, u_2, au_1 + bu_2\}$ is L.D.

Proof : We can take $\alpha_1 = -a$,

$$\alpha_2 = -b$$
$$\alpha_3 = 1$$

So that $\alpha_1 u_1 + \alpha_2 u_2 + 1.(a u_1 + b u_2)$

$$= -au_1 - bu_2 + au_1 + bu_2 = \theta.$$

Since $\alpha_1, \alpha_2, \alpha_3$ are not all zero the set $\{u_1, u_2, au_1 + bu_2\}$ is LD.

Example 3.6.20 : Test whether the set $\left\{x^2 - 4, x + 2, x - 2, \dfrac{x^2}{3}\right\}$ in P is L.D.

Solution : Taking $\alpha = 1$, $\beta = 1$, $\gamma = -1$ and $\delta = -3$, we get

$$\alpha(x^2 - 4) + \beta(x + 2) + \gamma(x - 2) + \delta\frac{x^2}{3} = 0$$

Since $\alpha, \beta, \gamma, \delta$ are non-zero the given set is L.D.

Problem Set 3 (E)

1. Which of the following set of vectors of V_3 are L.I ?
   (a) $\{1, 2, 1), (-1, 1, 0), (5, -1, 2)\}$
   (b) $\{(1, 0, 0), (1, 1, 1), (1, 2, 3)\}$
   (c) $\{(1, 1, 2), (-3, 1, 0), (1, -1, 1), (1, 2, -3)\}$
   (d) $\{(1, 5, 2), (0, 0, 1), (1, 1, 0\}$

2. Which of the following sets of vectors of V_4 are L.D ?
   (a) $\{(1,0,0,0),(1,1,0,0),(1,1,1,1),(0,0,0,1),(2,1,-1,0)\}$
   (b) $\{(1,-1,2,0),(1,1,2,0),(3,0,0,1),(2,1,-1,0)\}$
   (c) $\{(1,1,1,0),(3,2,2,1),(1,1,3,-2),(1,2,6,-5),(1,1,2,1)\}$
   (d) $\{(1,2,3,0),(-1,7,3,3),(1,-1,1,-1)\}$
3. Which of the following subsets of S of P are L.I ?
   (a) $S=\{x^2-1,\ x+1,\ x-1\}$
   (b) $S=\{1, x+x^2, x-x^2, 3x\}$
   (c) $S=\{x, x^3-x, x-x^2, x+x^2+x^4+\frac{1}{2}\}$
   (d) $S=\{x^2, x^3+1, x^4\}$
4. Which of the following subsets S of $C(0,\infty)$ are L.I ?
   (a) $S=\{x, \sin x, \cos x\}$ (b) $S=\{\sin^2 x, \cos 2x, 1\}$
   (c) $S=\{\sin x, \cos x, \sin(x+1)\}$ (d) $S=\{l\text{n}x, l\text{n}x^2, l\text{n}\, x^3\}$
   (e) $\{n^2 e^n, n e^x, (x^2+x-1)e^x\}$
5. Show that the set $S=\{\sin x, \sin 2x, \ldots, \sin nx\}$ is a linearly independent subset of $C[-\pi,\pi]$ for every positive integer n.
6. If u, v, w are three linearly independent vectors of a vector space V, then prove that u + v, v + w and w + u are also L.I.

3.7. Dimension and Basis

3.7.1. Definition : (Basis) A subset B of a vector space V is said to be a basis for v if
(a) B is L.I and
(b) $[B]=V$, i.e. B generates V.

Example 3.7.1 : Let $V=V_3$, $B=\{i,j,k\}$

where $i=(1,0,0)$, $j=(0,1,0)$ and $k=(0,0,1)$

Now $\alpha i+\beta j+\gamma k=(0,0,0)$

$$\Rightarrow \alpha(1,0,0)+\beta(0,1,0)+\gamma(0,0,1)=(0,0,0)$$
$$\Rightarrow (\alpha,\beta,\gamma)=(0,0,0)$$
$$\Rightarrow \alpha=\beta=\gamma=0$$

So B is L.I.

Clearly $[B]\subset V$

Let $u=(x_1,x_2,x_3)\in V$.

$$\Rightarrow u=x_1 i+x_2 j+x_3 k$$
$$\Rightarrow u\in[B].$$

So $V\subset[B]$

Hence $[B]=V$.

By definition B is a basis of V_3.

Note : It can be shown that $B=\{i,j\}$ where $i=(1,0)$, $j=(0,1)$ forms a basis of V_2.

Example 3.7.2 : Show that $B_1 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ is a basis of V_3.

Proof : Now $\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1)=(0,0,0)$

$$\Rightarrow (\alpha+\beta, \alpha+\gamma, \beta+\gamma)=(0,0,0)$$
$$\Rightarrow \alpha+\beta=0,\ \alpha+\gamma=0, \alpha+\gamma=0$$
$$\Rightarrow \alpha=\beta=\gamma=0$$

So B_1 is L.I.

Let $u=(x_1, x_2, x_3) \in V_3$.

Let $u=\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1)=(\alpha+\beta, \alpha+\gamma, \beta+\gamma)$

$$\Rightarrow (x_1, x_2, x_3)=(\alpha+\beta, \alpha+\gamma, \beta+\gamma)$$
$$\Rightarrow \alpha+\beta=x_1,\ \alpha+\gamma=x_2,\ \beta+\gamma=x_3$$
$$\Rightarrow \alpha=\frac{x_1+x_2-x_3}{2},\quad \beta=\frac{x_1+x_3-x_2}{2},\ \gamma=\frac{x_2+x_3-x_1}{2}$$

Thus $$u=(x_1, x_2, x_3)=\left(\frac{x_1+x_2-x_3}{2}\right)(1,1,0)+\left(\frac{x_1+x_3-x_2}{2}\right)(1,0,1)$$
$$+\left(\frac{x_2+x_3-x_1}{2}\right)(0,1,1)$$

Hence $[B_1]=V_3$.

So B_1 is a basis of V_3.

Note : Examples 3.7.1. and Example 3.7.2 show that V_3 has two different bases. There can be more than that. From this we infer that the basis of a vector space V need not be unique.

Example 3.7.3 : Prove that the set $S = \{(1, 0, 0), (1, 1, 0), (1, 1, 1), (0, 1, 0)\}$ spans V_3 but is not a basis.

Proof : Let $(x_1, x_2, x_3) \in V_3$ and that

$$(x_1, x_2, x_3)=\alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)+\sigma(0,1,0)$$
$$\Rightarrow (x_1, x_2, x_3)=(\alpha+\beta+\gamma,\ \beta+\gamma+\delta, \gamma)$$
$$\Rightarrow x_3=\gamma,\ \alpha+\beta+\gamma=x_1,\ \beta+\gamma+\delta=x_2$$

Choosing $\delta=0$, we get $\beta=x_2-x_3,\ \gamma=x_3, \alpha=x_1-x_2$

Thus $(x_1, x_2, x_3)=(x_1-x_2)(1,0,0)+(x_2-x_3)(1,1,0)$
$$+x_3(1,1,1)+0(0,1,0)$$

Hence $V_3 \subset [S]$

But $[S] \subset V_3$. Since V_3 is a vector space.

So $V_3=[S]$.

Again $\alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)+\delta(0,1,0)=(0,0,0)$

$$\Rightarrow (\alpha+\beta+\gamma, \beta+\gamma+\delta,\ \gamma)=(0,0,0)$$
$$\Rightarrow \alpha+\beta+\gamma=0,\ \beta+\gamma+\delta=0,\ \gamma=0$$
$$\Rightarrow \alpha=\beta=\delta, \gamma=0$$

Putting $\delta=1$, we get $\alpha=1,\ \beta=-1$

Thus $1(1, 0, 0)+(-1)(1, 1, 0)+0(1, 1, 1)+1(0, 1, 0)=(0, 0, 0)$

So S is LD.

So S is not a basis.

3.7.2. Definition (Dimension)

If a vector space V has a basis consisting of a finite number of elements, the space is said to be finite dimensional. The number of elements in a basis is called the dimension of the space and is written as dim V.

If dim V = n, then V is said to be n-dimensional.

If V is not finite-dimensional, it is called infinite-dimensional.

If $V = V_0 = \{\theta\}$, its dimension is said to be zero.

The following theorem shows that if a set B of n elements generates V, then no linearly independent set can have more than n vectors.

Theorem 3.7.1 : In a vector space V if $\{v_1, v_2,..., v_n\}$ generates V and if $\{w_1, w_2,..., w_m\}$ is L.I, then $m \leq n$.

Proof : Let us construct the set $S_1 = \{w_m, v, v_2,..., v_n\}$.

S_1 has following properties.

(i) $[S_1] = V$ since $\{v_1, v_2,...v_n\}$ spans V and $w_m \in V$.

(ii) S_1 is LD since $w_m \in V = [v_1, v_2,..., v_n]$.

(iii) $w_m \neq 0$ ($\because$ no vector of a LI set is zero) By Theorem 3.6.4 there exists a vector v_i with $2 \leq i \leq n$ such that $v_i \in [w_m, v_1, v_2,..., v_{i-1}]$.

Let $S_1' = S_1 - v_i = \{w_m, v_1, v_2,..., v_{i-1}\}$

Also $[S_1'] = V$

Now consider the set

$S_2 = w_{m-1}\ S_1' = \{w_{m-1}, w_m, v_1, v_2,..., v_{i-1}\}$

since $[S_1'] = V$, $[S_2] = V$.

Further, S_2 is LD,

since $w_{m-1} \in V = [S_1']$ and $w_{m-1} \neq 0$.

Therefore, by another application of Theorem 3.5.10, we form S_2' like S_1'. Then construct the set $S_3 = w_{m-2}\ S_2'$ and continue the process of constructing new sets S and S'. Since the set of w's is LI and every time the discarded element must be a v.

If all the w's are used up in this process, then $m \leq n$. Otherwise the set $\{w_{m-n}, w_{m-n+1},...,w_{m-1}, w_m\}$ would be LD. This contradicts the linear independence of w's.

Carollary 3.7.2 : If V has a basis of n elements, then every set of p vectors with p > n, is LD.

Proof : Let $B = \{v_1, v_2,..., v_n\}$ be a basis of V. Let $A = \{u_1, u_2,..., u_p\}$ be a set of p vectors with p > n. If A is LI, then $p \leq n$ by Theorem 3.7.1. Hence A is LD.

Corollary 3.7.3 : If V has a basis of n elements, then every other basis for V also has n elements.

Proof : Let $B_1 = \{v_1, v_2,..., v_n\}$ and $B_2 = \{w_1, w_2,...w_m\}$ be two bases of V.

Then B_1 and B_2 are LI and $[B_1] = [B_2] = V$.

Some $[B_1] = V$ and B_2 is LI, then by Theorem 3.7.1., $m \leq n$. Since $[B_2] = V$ and B_1 is LI, by the same theorem, $n \leq m$.

Thus m = n.

Note : It follows from Carollary 3.7.3 that, if a vector space V is n-dimensional, (a) there exists n linearly independent vectors in V and (b) every set of n + 1 vectors in V is LD.

Example 3.7.4 : V_2 is two dimensional since $B_1 = \{(1, 0), (0, 1)\}$ is a basis of V_2. V_3 is three dimensional since $B_2 = \{i, j, k\}$ where $i = (1, 0, 0)$, $j = (0, 1, 0)$, and $k = (0, 0, 1)$ is a basis of V_3.

Consider the vectors

$$e_1 = (1, 0, 0, \ldots, 0)$$
$$e_2 = (0, 1, 0, \ldots, 0)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$e_n = (0, 0, 0, \ldots, 1)$$

With e_i as the vector, all of whose co-ordinates are zero except the i-th, which is 1. It is easy to see that $e_1, e_2, \ldots, e_n$ are LI and every n-tuple is a linear combination of $e_1, e_2, \ldots, e_n$. Thus the set $\{e_1, e_2, \ldots, e_n\}$ is a basis ov V_n. This basis is called the standard basis of V_n. In particular the standard basis of V_3 is $\{e_1, e_2, e_3\}$

where $e_1 = (1, 0, 0) = i$
$e_2 = (0, 1, 0) = j$
$e_3 = (0, 0, 1) = k$

Example 3.7.5 : For the 3-dimensional space V_3 over the field of real numbers R, determine if the set $\{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$ is a basis.

Solution : For $\alpha, \beta, \gamma \in R$.

$$\alpha(2,-1,0) + \beta(3,5,1) + \gamma(1,1,2) = (0,0,0)$$
$$\Rightarrow (2\alpha + 3\beta + \gamma, -\alpha + 3\beta + \gamma, \beta + 2\gamma) = (0, 0, 0)$$
$$\Rightarrow 2\alpha + 3\beta + \gamma = 0, \ -\alpha + 5\beta + \gamma = 0, \ \beta + 2\gamma = 0 \qquad \ldots(1)$$

Now the determinant of the co-efficients

$$\begin{vmatrix} 2 & 3 & 1 \\ -1 & 5 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 23 \neq 0$$

So there exists unique (zero) solution of the system of equations (1).

i.e. $\alpha = \beta = \gamma = 0$

Hence the given set is LI.

Again for $(x_1, x_2, x_3) \in V_3$.

$$(x_1, x_2, x_3) = \alpha(2,-1,0) + \beta(3,5,1) + \gamma(1,1,2)$$
$$\Rightarrow (x_1, x_2, x_3) = (2\alpha + 3\beta + \gamma, -\alpha + 5\beta + \gamma, \beta + 2\gamma)$$
$$\Rightarrow 2\alpha + 3\beta + \gamma = x_1, \ -\alpha + 5\beta + \gamma = x_2, \ \beta + 2\gamma = x_3$$

$$\Rightarrow \alpha = \frac{9}{23}x_1 - \frac{5}{23}x_2 - \frac{2}{23}x_3$$

$$\beta = \frac{2}{23}x_1 + \frac{4}{23}x_2 + \frac{3}{23}x_3$$

$$\gamma = \frac{-1}{23}x_1 - \frac{2}{23}x_2 + \frac{13}{23}x_3$$

Thus $(x_1, x_2, x_3) \in [(2,-1,0),(3,5,1),(1,1,2)]$

Hence $V_3 = [(2,-1,0),(3,5,1),(1,1,2)]$

So the given set is a basis of V_3.

Example 3.7.6 : Find the dimension of P_n, the set of all polynomials of degree $\leq$ n.

Solution : We know that every polynomial of P_n is a linear combination of the functions $1, x, x^2, ..., x^n$

Again $\alpha_1 + \alpha_2 x + \alpha_3 x^2 + ... + \alpha_{x+1} x^n = 0$

This being an identity,

$$\alpha_1 = \alpha_2 = ... = \alpha_{n+1} = 0.$$

So $\{1, x, x^2, ..., x^n\}$ is L.I.

Again for $p(x) \in P_n$,

$p(x) = a_0 + a_1 x + ... + a_n x^n$, a linear combination of $\{1, x, x^2, ..., x^n\}$

So $[1, x, x^2, ..., x^n] = P_n$

Thus this set is a basis of P_n.

The basis of P_n is having n + 1 elements.

So dim P_n = n + 1.

Theorem 3.7.4 : In an n-dimensional vector space V, any set of n linearly independent vectors is a basis.

Proof : Suppose $B = \{v_1, v_2, ..., v_n\}$ is a set of n linearly independent vectors.

To prove that B is a basis we have only to show that [B] = V.

Take $v \in V$. The set $B' = \{v_1, v_2, ..., v_n, v\}$ is LD since V is n-dimensional. Hence by Theorem 3.6.4, one of the vectors of B' say u is in the span of its predecessors. But this u cannot be any one of $v_1, v_2, ..., v_n$. Because if so, it will contradict the linear independence of $v_1, v_2, ..., v_n$. Thus $v \in [v_1, v_2, ..., v_n]$.

Therefore $[B] = V$

Hence B is a basis of V.

Note : If three vectors of V_3 are L.I, then the set consisting of these vectors is a basis of V_3. Using this principle, after showing B to be LI in Example 3.6.2, we can declare that B is a basis of V_3.

Example 3.7.7 : Prove that the set $\{(1, 1, 1), (1, -1, 1), (0, 1, 1)\}$ is a basis of V_3.

Proof : Suppose $\alpha(1,1,1)+\beta(1,-1,1)+\gamma(0,1,1)=(0,0,0)$

$$\Rightarrow (\alpha+\beta, \alpha-\beta+\gamma, \alpha+\beta+\gamma)=(0,0,0)$$
$$\Rightarrow \alpha=\beta=\gamma=0$$

So the given set is LI and it consists of three elements of V_3, by Theorem 3.7.4., the set forms a basis.

Theorem 3.7.5 : In a vector space V let $B=\{v_1, v_2, ..., v_n\}$ spans V. Then the following two conditions are equivalent.

(a) $\{v_1, v_2, v_3, ..., v_n\}$ is a LI set.

(b) If $v \in V$, then the expression $v=\alpha_1 v_1+\alpha_2 v_2+...+\alpha_n v_n$ is unique

Proof : Let us first prove $(a) \to (b)$ given that $B=\{v_1, v_2, ..., v_n\}$ spans V. If B is LI, then B is a basis of V.

We shall prove that any expression for v in terms of $v_1, v_2, ..., v_n$ is unique.

If $v=\alpha_1 v_1+\alpha_2 v_2+...+\alpha_n v_n$ ... (1)

and also $v=\beta_1 v_1+\beta_2 v_2+...+\beta_n v_n$ ...(2)

Then $\alpha_1 v_1+\alpha_2 v_2+...+\alpha_n v_n=\beta_1 v_1+\beta_2 v_2+...+\beta_n v_n$

$$\Rightarrow (\alpha_1-\beta_1)v_1+(\alpha_2-\beta_2)v_2+...+(\alpha_n-\beta_n)v_n=\theta$$

Since B is L.I,

$$\alpha_1-\beta_1=0=\alpha_2-\beta_2=...=\alpha_n-\beta_n$$
$$\Rightarrow \alpha_i-\beta_i=0,\ i=1,2,...,n.$$
$$\Rightarrow \alpha_i=\beta_i \text{ for all i.}$$

Hence the expression (1) is unique.

Let us prove $(b) \to (a)$

Let $v \in V$ has unique expression $v=\alpha_1 v_1+\alpha_2 v_2+....+\alpha_n v_n$.

We shall prove that the set B is L.I.

Let $\beta_1 v_1+\beta_2 v_2+...+\beta_n v_n=\theta$

Again $\theta=0v_1+0v_2...+0v_n$.

Since the expression of θ, a linear combination of elements of B is unique, we must have $\beta_1=0=\beta_2=...=\beta_n$.

Hence $B=\{v_1, v_2, ..., v_n\}$ is L.I.

Note : The Theorem 3.7.5 can be restated as 'A set B is a basis for a vector space V iff $[B]=V$ and the expression for any $v \in V$ interms of elements of B is unique.'

3.7.3 Definition (Co-ordinate)

Let $B=\{v_1, v_2, ..., v_n\}$ be an basis (ordered) for V. then a vector $v \in V$ can be written as $v=\alpha_1 v_1+\alpha_2 v_2+...+\alpha_n v_n$. The vector $(\alpha_1, \alpha_2, ..., \alpha_n)$ is called the co-ordinate vector of v relative to the ordered basis B. It is denoted by $[v]_B$. $\alpha_1, \alpha_2, ..., \alpha_n$ are called co-ordinates of the vector v relative to the ordered basis B.

The co-ordinates of a vector relative to the standard basis are simply called co-ordinates of the vector.

Example 3.7.8 : Find the co-ordinates of the vector (1, 2, 3, 4) of V_4 relative to the standard basis of V_4.

Solution : The standard basis of V_4 is $\{e_1, e_2, e_3, e_4\}$ where $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$. Since $(1, 2, 3, 4) = 1.e_1 + 2.e_1 + 3.e_3 + 4.e_4$ the co-ordinate vector of (1, 2, 3, 4) relative to the standard basis is (1, 2, 3, 4). Therefore 1, 2, 3 and 4 are co-ordinates of the vector (1, 2, 3, 4).

Exampe 3.7. 9 : Find the co-ordinates of (1, 2, 3, 4) relative to the ordered basis B = {(0, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 1), (1, 1, 1, 1)}

Solution : Let

$$\begin{aligned}(1,2,3,4) &= \alpha(0,0,0,1)+\beta(0,0,1,1)+\gamma(0,1,1,1)+\delta(1,1,1,1)\\ &\Rightarrow (1,2,3,4)=(\delta,\gamma+\delta,\beta+\gamma+\delta,\alpha+\beta+\gamma+\delta)\\ &\Rightarrow \delta=1,\ \gamma+\delta=2,\ \beta+\gamma+\delta=3,\ \alpha+\beta+\gamma+\delta=4\\ &\Rightarrow \alpha=\beta=\gamma=\delta=1.\end{aligned}$$

Hence the co-ordinates of (1, 2, 3, 4) relative to the basis B are 1, 1, 1, 1. Also $[(1,2,3,4)]_B = (1,1,1,1)$.

In the following theorem we shall show how a linearly independent set of a vector space is extended to its basis.

Theorem 3.7.6 : Let the set $\{v_1, v_2, ..., v_k\}$ be a linearly independent subset of an n-dimensional vector space V. Then we can find vectors $v_{k+1}, ..., v_n$ in V such that the set $\{v_1, ..., v_k, v_{k+1}, ..., v_n\}$ is a basis of V.

Proof : By Theorem 3.6.3, $k \leq n$. If k = n, then by Theorem 3.7.4, $\{v_1, v_2, ..., v_{k(=n)}\}$ is a basis for V. If k < n, then $\{v_1, v_2, ..., v_k\}$ is not a basis (corollary 3.7.2). But the set $\{v_1, v_2, ..., v_k\}$ is LI.

Therefore $[v_1, v_2, ..., v_k] \neq V$.

Hence $[v_1, v_2, ..., v_k]$ is a proper subset of V. Thus there exists a non-zero vector v_{k+1} in V such that $v_{k+1} \notin [v_1, v_2, ..., v_k]$.

Hence $\{v_1, v_2, ..., v_k, v_{k+1}\}$ is LI (Theorem 3.6.2).

Now if k + 1 = n, we are done.

If not, we repeat the foregoing process until we get n-linearly independent vectors $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$. This forms a basis for V by Theorem 3.7.4.

Note : Starting from any non-zero vector, any number of bases can be formed extending it.

Example 3.7.10 : Extend the set {(3, –1, 2)} to two different bases for V_3.

Solution : Clearly {(3, –1, 2)} is L.I.

Consider $[(3, -1, 2)] = \{(3\alpha, -\alpha, 2\alpha) \mid \alpha \text{ scalar}\}$

Let us search for a vector which is not in the above span.

We see that for each α, the first co-ordinate of any vector of the span is 3α.

Let us choose a vector in which this is not true. Taking $\alpha = 1$, we can choose such a vector as (1, – 1, 2)

Now by Theorem 3.6.2, the set {(3, –1, 2), (1, –1, 2)} is L.I.

Again consider

$$[(3,-1,2),(1,-1,2)]$$
$$=\{(3\alpha+\beta,-\alpha-\beta,2\alpha+2\beta)\mid\alpha,\ \beta\ \text{scalars}\}\quad ...(1)$$

Let us now search for a vector which is not in the current span. We see that for any choice of α, β, the 3rd co-ordinate of a vector in the span is always $2(\alpha+\beta)$. Let us choose a vector in V_3 not having this property. Taking $\alpha=\beta=1$, we can choose such a vector (4, –2, 1).

Since $(4,-2,1)\notin[(3,-1,2),(1,-1,2)]$, by Thoerem 3.6.2.

$$\{(3,-1,2),(1,-1,2),(4,-2,1)\}\text{ is L.I.}$$

By Theorem 3.7.4, $\{(3,-1,2),(1,-1,2),(4,-2,1)\}$ is a basis of V_3.

Let us form another basis of V_3.

After getting a LI set $\{(3,-1,2),(1,-1,2)\}$, let us find a vector which is not in its span.

We see in (1) that for every choice of α, β, the 2nd co-ordinate of a vector in the span is always $-(\alpha+\beta)$. Let us choose a vector which is not having the property. Taking $\alpha=\beta=1$, (4, 1, 4) is such a vector.

Thus $\{(3,-1,2),(1,-1,2),(4,1,4)\}$ is LI and it is another basis of V_3.

Example 3.7.11 : Let $\{(1,1,1,1),(1,2,1,2)\}$ be a linearly independent subset of a vector space V_4. Extend it to a basis for V_4.

Solution : We have

$$[(1,1,1,1),(1,2,1,2)]=\{\alpha(1,1,1,1)+\beta(1,2,1,2)\mid\alpha,\beta\text{ are scalars}\}$$
$$=\{(\alpha+\beta,\alpha+2\beta,\alpha+\beta,\alpha+2\beta)\mid\alpha,\beta\text{ are scalars}\}$$

Since first and third co-ordinates are equal for all vectors of the span, we find that (0, 3, 2, 3) is not in the span. Thus by hypothesis, $\{(1,1,1,1),(1,2,1,2),(0,3,2,3)\}$ is L.I.

Now $[(1,1,1,1),(1,2,1,2),(0,3,2,3)]$

$$=\{\alpha(1,1,1,1)+\beta(1,2,1,2)+\gamma(0,3,2,3)\mid\alpha,\beta,\gamma\text{ are scalars}\}$$
$$=\{(\alpha+\beta,\alpha+2\beta+3\gamma,\alpha+\beta+2\gamma,\alpha+2\beta+3\gamma)\mid\alpha,\beta,\gamma\text{ are scalars}\}$$

Taking $\alpha=\beta=\gamma=1$, (2, 6, 4, 6) is in the span

but (2, 6, 4, 5) is not in the span.

Hence the set $\{(1,1,1,1),(1,2,1,2),(0,3,2,3),(2,6,4,5)\}$ is L.I.

Thus by Theorem 3.7.4 it is a basis of V_4.

Theorem 3.7.7 : Let U be a subspace of a finite-dimensional vector space V. Then dim $U\le$ dim V. Equality holds only when U = V.

Proof : Let $B=\{v_1,v_2,...,v_n\}$ be a basis for V. This generates V and having n elements. Any set of linearly independent vectors in V and therefore any set of linearly independent vectors in U cannot have more than n vectors. Therefore dim $U\le$ dim V.

When dim U = dim V, a basis B_1 of U, is a set of n linearly independent vectors of V whose dimension is also n. By Theorem 3.7.4., it follows that B_1 is a basis for V. This means $V=[B_1]=U$.

Theorem 3.7.8 : If U and W are two subspaces of a finite dimensional vector space V, then $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$.

Proof : Let $\dim U = m$, $\dim W = p$, $\dim(U \cap W) = r$ and $\dim V = n$

By Theorem 3.7.7, $m \le n, p \le n$ and $r \le n$.

Let $B_1 = \{v_1, v_2, ..., v_r\}$ be a basis of $U \cap W$. This is a linearly independent set in $U \cap W$. Since $U \cap W \subset U$ and $U \cap W \subset W$, B_1 is also a linearly independent set in U and W. So it can be extended to a basis of U, say $B_2 = \{v_1, v_2, ..., v_r, u_{r+1}, ..., u_m\}$ and to a basis of W, say

$$B_3 = \{v_1, v_2, ..., v_r, w_{r+1}, ..., w_p\}$$

Let us construct the set

$$B = \{v_1, v_2, ..., v_r, u_{r+1}, ..., u_m, w_{r+1}, ..., w_p\}.$$

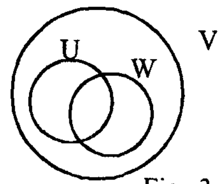

Fig. 3.1

Let us prove that this is a basis of U + W.

For this purpose, let us show that (a) B is L.I in U + W and (b) [B] = U + W.

To prove (a), assume that

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=r+1}^{p} \gamma_i w_i = \theta \qquad ...(1)$$

$$\Rightarrow \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = -\sum_{i=r+1}^{p} \gamma_i w_i = v \text{ say} \qquad ...(2)$$

The vector $v \in U$ since the left hand side of Equation (2) is in U.

Also $v \in W$, since the right hand side of Equation (2) is in W.

Thus $v \in U \cap W$. Therefore, v can be expressed uniquely in terms of vectors of its basis B_1.

Thus $v = \sum_{i=1}^{r} \delta_i v_i$ for some scalar δ_i.

Hence $\sum_{i=1}^{r} \delta_i v_i + \sum_{i=r+1}^{p} \gamma_i w_i = \theta \quad ...(3)$

But B_3 is L.I.

So each of the δ_i's and γ_i's is zero.

Putting $\gamma_{r+1} = \gamma_{r+2} = ... = \gamma_p = 0$ in the equation (2), we find that

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i = \theta$$

Again B_2 is L.I.

So each of α_i's and β_i's is zero.

Thus the equation (1) implies that each scalar involved is zero. Hence B is L.I. This proves (a).

To prove (b), let $z \in U + W$.

Then $z = u + w$, for some $u \in U$ and $w \in W$.

This gives

$$z = \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=1}^{r} \alpha_i ' v_i + \sum_{i=r+1}^{p} \beta_i ' w_i \qquad .....(4)$$

For some sclars α_i 's, β_i 's, α_i 's and β_i 's.

Simplifying the expression (4), we see that $z \in [B]$. Hence $U + W \subset [B]$.

Again for $z \in [B]$

$$z = \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i + \sum_{i=r+1}^{p} \gamma_i w_i = u + w \in U + W$$

where $u = \sum_{i=1}^{r} \alpha_i v_i + \sum_{i=r+1}^{m} \beta_i u_i \in U$

$$w = \sum_{i=r+1}^{p} \gamma_i w_i \in W$$

So $[B] \subset U + W$.

Hence $[B] = U + W$.

This shows that B is a basis of U + W.

Therefore, dim (U + W)

$= | B |$

$= r + (m - r) + (p - r)$

$= m + p - r$

$= \dim U + \dim W - \dim (U \cap W)$.

Corollary 3.7.9 : If U and W are subspaces of a finite dimensional vector space V such that $U \cap W = \{\theta\}$, then $\dim (U \oplus W) = \dim U + \dim W$.

Proof : $U \cap W = \{\theta\}$.

$\Rightarrow \dim (U \cap W) = 0$.

From this result and Theorem 3.7.8, the proof of the corollary follows.

Example 3.7.12 : Verify Theorem 3.7.8 and Corollary 3.7.9 with the help of a suitable example.

Solution : Take U = xy-plane, and

W = yz-plane in V_3.

Then $U \cap W = y-$axis

Clearly U, W and $U \cap W$ are subspaces of V_3 with dim U = dim W = 2 and $\dim U \cap W = 1$.

We have $U + W = V_3$

so dim (U + W) = 3

$\dim U + \dim W - \dim U \cap W = 2 + 2 - 1 = 3$.

Hence Theorem 3.7.8 is verified.

On the other hand if we take U = xy-plane and W = z-axis, then $U \cap W = \{\theta\}$ and $U + W = V_3$.

$\dim (U \oplus W) = 3$

dim U + dim W = 2 + 1 = 3.

Hence Corollary 3.7.9 is verified.

Example 3.7.13 : Construct two subspaces A and B of V_4 such that dim A = 2, dim B = 3 and dim $(A \cap B) = 1$

Solution : Let A = [(1, 0, 0, 0) (0, 1, 0, 0)]

B = [(1, 0, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)]

Let $(p, q, r, s) \in A \cap B$

$\Rightarrow (p, q, r, s) \in A \wedge (p, q, r, s) \in B$

$\Rightarrow (p, q, r, s) = a\,(1, 0, 0, 0) + b\,(0, 1, 0, 0)$

and $(p, q, r, s) = \alpha(1, 0, 0, 0) + \beta(0, 0, 1, 0) + \gamma\,(0, 0, 0, 1)$

$\Rightarrow a = p, \quad b = q, \quad r = 0, \quad s = 0$

and $\alpha = p, \quad \beta = r, \quad \gamma = s, \quad q = 0$

$\Rightarrow (p, q, r, s) = p\,(1, 0, 0, 0)$

$\Rightarrow (p, q, r, s) \in [(1, 0, 0, 0)]$

$\therefore\ A \cap B \subset [(1, 0, 0, 0)]$

Further, $[(1, 0, 0, 0)] \subset A$ and $[(1, 0, 0, 0)] \subset B$

$\Rightarrow [(1, 0, 0, 0)] \subset A \cap B.$

Hence $A \cap B = [(1, 0, 0, 0)]$

Thus dim A = 2, dim B = 3 and dim $A \cap B = 1$

Example 3.7.14 : Given $S_1 = \{(1, 2, 3), (0, 1, 2), (3, 2, 1)\}$

and $S_2 = \{(1, -2, 3), (-1, 1, -2), (1, -3, 4)\}$

determine the dimension and basis for

(a) $[S_1] \cap [S_2]$

(b) $[S_1] + [S_2]$

Solution : (a) Let $\alpha\,(1, 2, 3) + \alpha\,(0, 1, 2) + \gamma\,(3, 2, 1) = 0$

for some scalars α, β and γ.

$\Rightarrow (\alpha + 3\gamma, 2\alpha + \beta + 2\gamma, 3\alpha + 2\beta + \gamma) = (0, 0, 0)$

$\Rightarrow \alpha + 3\gamma = 0,\ 2\alpha + \beta + 2\gamma = 0,\ 3\alpha + 2\beta + \gamma = 0$

$\Rightarrow \alpha = -3\gamma,\ \beta = 4\gamma,\ \gamma$ is arbitrary.

In particular, take $\gamma = -1$, then $\alpha = 3$, $\beta = -4$

Thus $3(1, 2, 3) - 4\,(0, 1, 2) - (3, 2, 1) = (0, 0, 0)$

$$\Rightarrow (1, 2, 3) = \frac{4}{3}(0, 1, 2) + \frac{1}{3}(3, 2, 1)$$

$\Rightarrow (1, 2, 3) \in [(0, 1, 2), (3, 2, 1)]$

$\therefore\ [S_1] = [(0, 1, 2), (3, 2, 1)]$

Let $a(1, -2, 3) + b\,(-1, 1, -2) + c(1, -3, 4) = (0, 0, 0)$

$\Rightarrow (a - b + c, -2a + b - 3c, 3a - 2b + 4c) = (0, 0, 0)$

$\Rightarrow a - b + c = 0,\ -2a + b - 3c = 0,\ 3a - 2b + 4c = 0$

$\Rightarrow a = -2c,\ b = -c,\ c$ is arbitrary.

Putting c = 1, a = – 2, b = – 1

Thus –2 (1, –2, 3) – (–1, 1, –2) + (1, –3, 4) = (0, 0, 0)

$\Rightarrow$ (1, –3, 4) = 2 (1, –2, 3) + (–1, 1, – 2)

$\Rightarrow$ $(1, -3, 4) \in [(1, -2, 3), (-1, 1, -2)]$

$\therefore$ $[S_2] = [(1, -2, 3), (-1, 1, -2)]$

Let $x \in [S_1] \cap [S_2]$

$$\Rightarrow x = \alpha(0, 1, 2) + \beta(3, 2, 1)$$
$$= \gamma(1, -2, 3) + \delta(-1, 1, -2)$$
$$\Rightarrow (3\beta, \alpha + 2\beta, 2\alpha + \beta) = (\gamma - \delta, -2\gamma + \delta, 3\gamma - 2\delta)$$
$$\Rightarrow 3\beta = \gamma - \delta,\ \alpha + 2\beta = -2\gamma + \delta,\ 2\alpha + \beta = 3\gamma - 2\delta$$
$$\Rightarrow (\alpha + 2\beta) + (2\alpha + \beta) - 3\beta = (-2\gamma + \delta) + (3\gamma - 2\delta) - \gamma + \delta$$
$$\Rightarrow 3\alpha = 0$$
$$\Rightarrow \alpha = 0$$

$\therefore$ $x = \beta(3, 2, 1) \in [(3, 2, 1)]$

$\therefore$ $[S_1] \cap [S_2] = [(3, 2, 1)]$

Hence the basis of $[S_1] \cap [S_2]$ is $\{(3, 2, 1)\}$.

(b) $$\dim\{[S_1] + [S_2]\} = \dim[S_1] + \dim[S_2] - \dim\{[S_1] \cap [S_2]\}$$
$$= 2 + 2 - 1$$
$$= 3$$

$\therefore$ $[S_1] + [S_2] = V_3$.

Let $y \in [S_1] + [S_2]$

$\therefore$ $y = \alpha(0, 1, 2) + \beta(3, 2, 1) + \gamma(1, -2, 3) + \delta(-1, 1, -2)$

$\alpha, \beta, \gamma, \delta$ are scalars.

Putting $\beta = \gamma = \delta = 0$, and $\alpha = 1$,

we get $(0, 1, 2) \in [S_1] + [S_2]$.

Similarly taking $\alpha = \gamma = \delta = 0$ and $\beta = 1$,

we get $(3, 2, 1) \in [S_1] + [S_2]$

Also $(1, -2, 3) \in [S_1] + [S_2]$

{(0, 1, 2), (3, 2, 1), (1, –2, 3)} is L.I.

Because $\begin{vmatrix} 0 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -2 & 3 \end{vmatrix} = -24 \neq 0.$

So B = {(0, 1, 2), (3, 2, 1), (1, –2, 3)} is a basis of $[S_1] + [S_2]$.

Example 3.7.15 Let $B_1 = \{u_1, u_2, \ldots, u_n\}$ and $B_2 = \{v_1, v_2, \ldots, v_n\}$ be ordered bases for an n-dimensional vector space V such that $\{u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\}$ is L.D. Then prove that there exists a non-zero vector $u \in V$ such that $[u]_{B_1} = [u]_{B_2}$.

Solution : Since $\{u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\}$ is L.D., therefore there exists scalars $c_1, c_2, \ldots, c_n$ not all zero such that

$$c_1(u_1 - v_1) + c_2(u_2 - v_2) + \ldots + c_n(u_n - v_n) = 0 \qquad \ldots(1)$$

Suppose $c_1 \neq 0$

From (1),

$$c_1u_1 + c_2u_2 + ... + c_nu_n$$
$$= c_1v_1 + c_2v_2 + ... + c_nv_n$$

Let $\quad u = c_1u_1 + c_2u_2 + ... + c_nu_n$
$$= c_1v_1 + c_2v_2 + ... + c_nv_n$$

$\therefore$ u is a non-zero vector since $c_1 \neq 0$ and $u_1 \in B_1$, a basis.

$\therefore$ $[u]_{B_1} = (c_1, c_2, ..., c_n) = [u]_{B_2}$.

Hence proved.

Problem Set 3 (F)

1. Which of the following subsets of S form a basis for V_2 ?
   (a) $S = \{(1, 2)\}$
   (b) $S = \{(1, 2), (0, 1)\}$
   (c) $S = \{(1, 2), (1, 1), (2, 1)\}$
   (d) $S = \{(1, 3), (3, 1)\}$
2. Which of the following subsets of S form a basis for V_3 ?
   (a) $S = \{(1, 2, 3)\}$
   (b) $S = \{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$
   (c) $S = \{(3, 2, 1), (1, 2, 3), (-1, 0, 1)\}$
   (d) $S = \{(1, 1, 1), (1, 2, 3), (-1, 0, 1)\}$
   (e) $S = \{(0, 0, 1), (1, 0, 1), (1, -1, 1), (3, 0, 1)\}$
   (f) $S = \left\{\left(1, \frac{2}{5}, -1\right), (0, 1, 2), \left(\frac{3}{4}, -1, 1\right)\right\}$
3. Which of the following subsets S form a basis for the given vector space V ?
   (a) $S = \{(1, -1, 0, 1), (0, 0, 0, 1), (2, -1, 0, 1), (3, 2, 1, 0)\}$, $V = V_4$
   (b) $S = \{(0, 1, 2, 1), (1, 2, -1, 1), (2, -3, 1, 0), (4, -2, -7, -5)\}$, $V = V_4$
   (c) $S = \{x-1, x^2 + x - 1, x^2 - x + 1\}$, $V = P_2$
   (d) $S = \{1, x, (x - 1)x, x(x - 1)(x - 2)\}$, $V = P_3$
   (e) $S = \{1, \sin x, \sin^2 x, \cos 2x\}$, $V = C[-\pi, \pi]$
4. Determine the dimension of the subspace [S] of V_2 for each S in Problem 1.
5. Determine the dimension of the subspace [S] of V_3 for each S in Problem 2.
6. Determine the dimension of the subspace [S] of V_3 for each S in Problem 3.

7. Find the co-ordinates of the following vectors of V_3 relative to the ordered basis B = {(2, 1, 0), (2, 1, 1), (2, 2, 1)}

   (a) (1, 2, 1) (b) (–1, 3, 1)

   (c) (x_1, x_2, x_3) (d) $(-\sqrt{2}, \pi, e)$

   (e) $\left(-\frac{1}{2}, \frac{11}{3}, 5\right)$ (f) (2, 0, – 1)

8. Find the co-ordinates of the following polynomials relative to the ordered basis $\{1 - x, 1 + x, 1 - x^2\}$ of P_2.

   (a) $3 + 7x + 2x^2$ (b) $x - 3x^2$ (c) $x^2 + 2x - 1$

9. Find an ordered basis for V_4 relative to which the vector (–1, 3, 2, 1) has the co-ordinates 4, 1, –2 and 7.

10. Let U and W be two distinct (n–1) dimensional subspaces of an n-dimensional vector space V. Then prove that dim $(U \cap W) = n - 2$.

11. Extend the following subsets S of a vector space V to its basis

   (a) $S = \{(1, 2)\}, V = V_2$

   (b) $S = \{(1, -1, 1)\}, V = V_3$

   (c) $S = \{(1, 3, 2)\}, V = V_3$

   (d) $S = \{(1, 5, 1), (1, -1, 2)\}, V = V_3$

   (e) $S = \{(1, 1, -1, 2)\ (1, 2, 1, -1)\}, V = V_4$

   (f) $S = \{(-1, 2, -1, 3), (1, -2, 3, 2), (1, 2, 4, 1)\}, V = V_4$

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