Week 10

Foundations of

Probability, Theorem & Statistics For AI

Today's Outline

I. Different Types of Phenomena

II. Nondeterministic Random Phenomena and Probability Theorem

III. Probability Experiment and Events

IV. Classical Probability

V. Axiomatic Probability

VI. Probability of Composite Events

VII. Statistics Reasoning

VIII. Tutorial

Phenomena

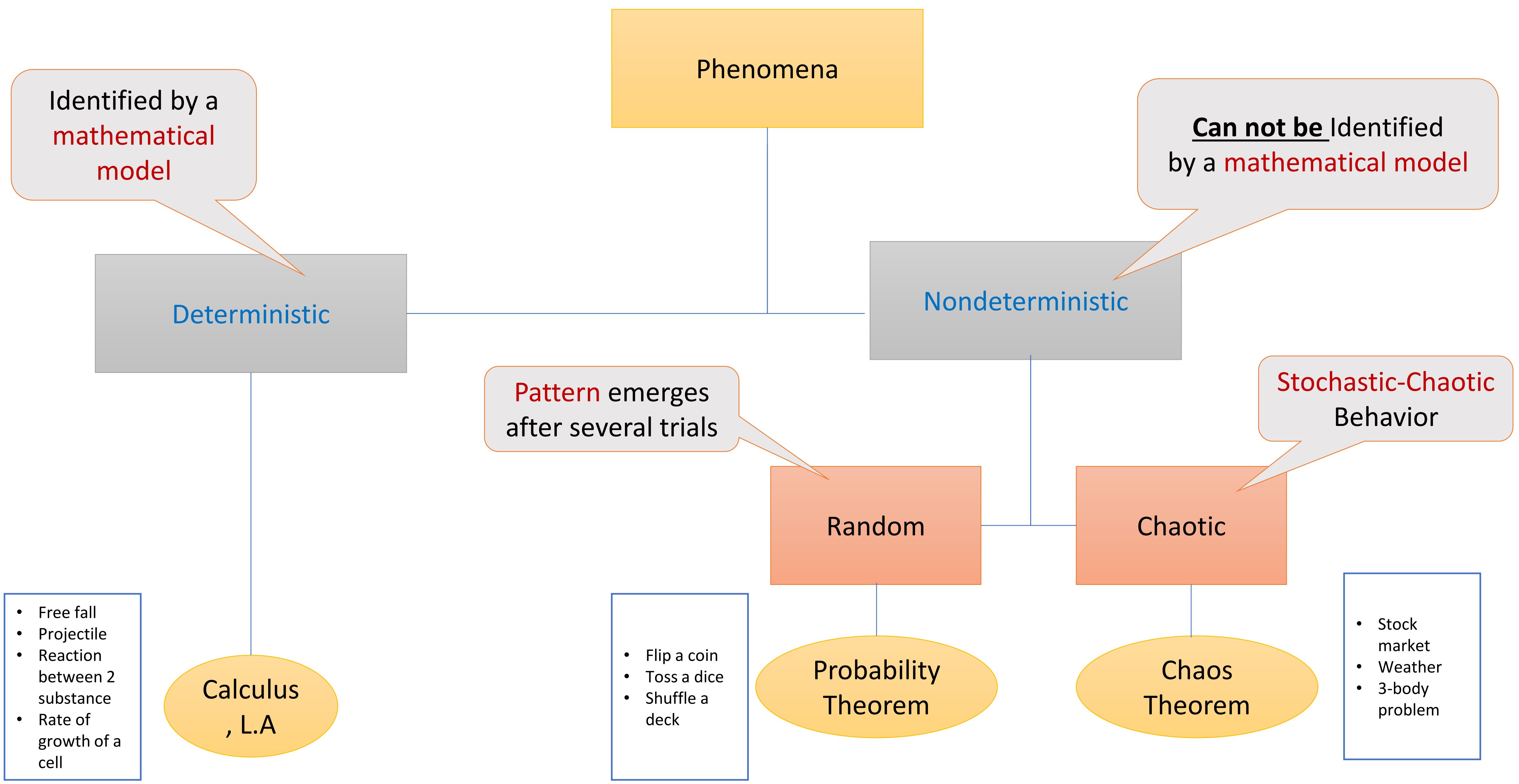
- A **phenomenon** → observable fact .
- Considering the nature of the phenomena, it is divided to:

    I. **Deterministic**

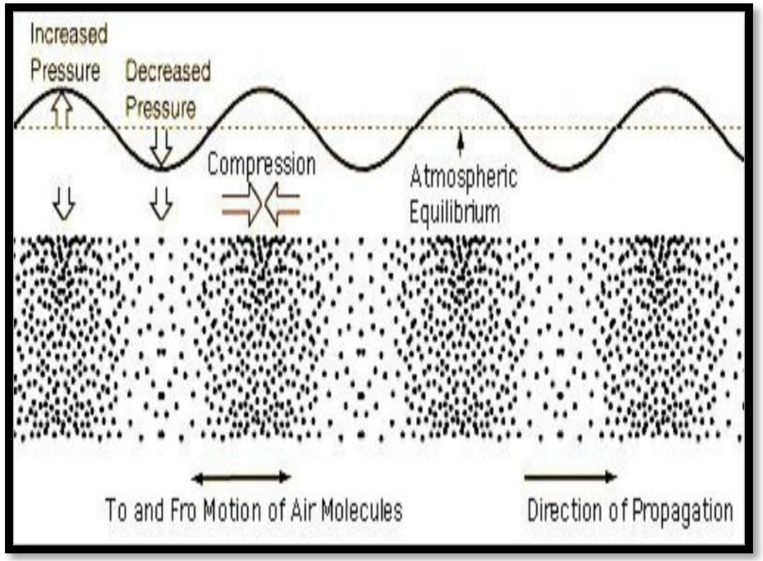
    II. **Nondeterministic**

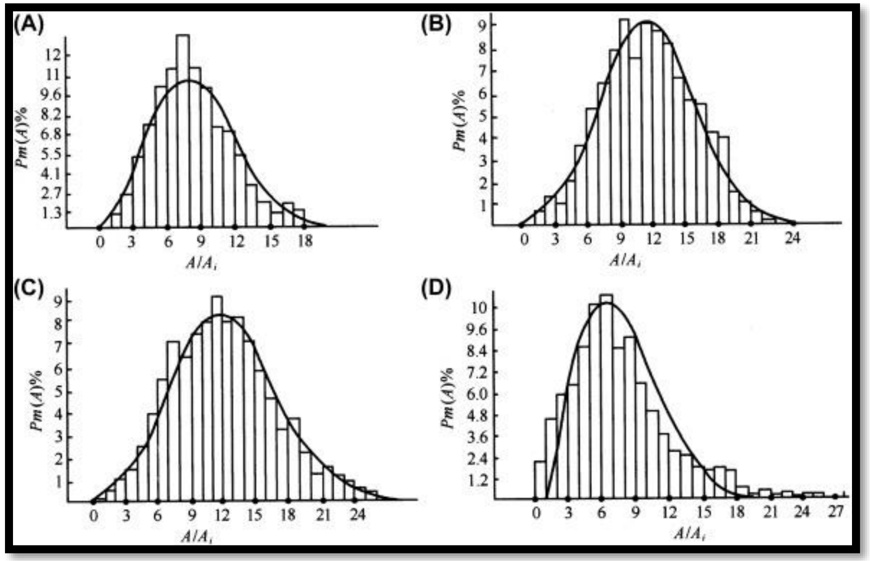
        a. Nondeterministic **Random** phenomena

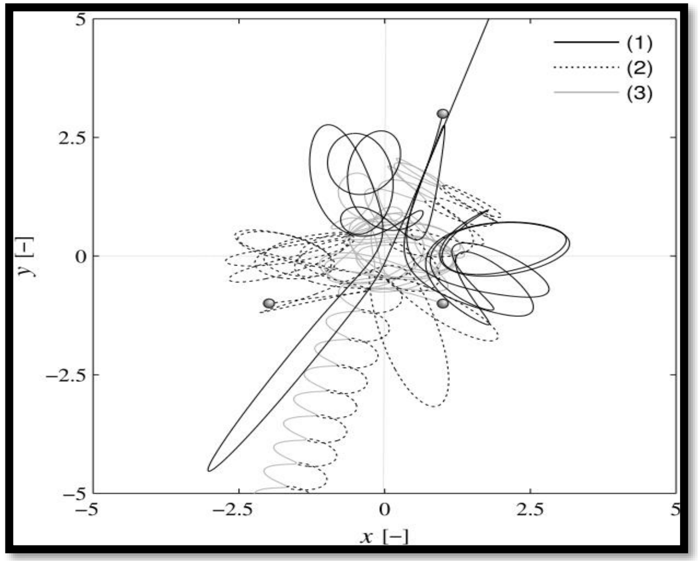
        b. Nondeterministic **Chaotic** phenomena


Phenomena
Identified by a mathematical model
Deterministic
Can not be Identified by a mathematical model
Nondeterministic
Pattern emerges after several trials
Stochastic-Chaotic Behavior
Random
Chaotic
Free fall
Projectile
Reaction between 2 substance
Rate of growth of a cell
Calculus , L.A
Flip a coin
Toss a dice
Shuffle a deck
Probability Theorem
Chaos Theorem
Stock market
Weather
3-body problem

Phenomena- Different Types







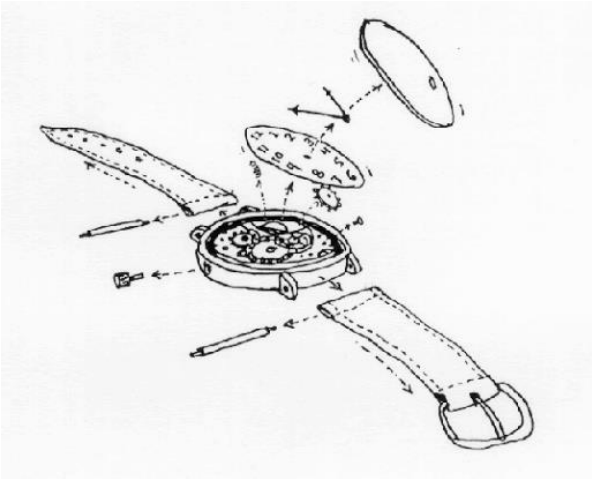
Deterministic Phenomena

- Mathematical model that allows “**perfect**” prediction of the phenomena’s outcome.
- Based on mathematical modeling in Calculus & Algebra, etc.
- Deterministic approach → Certainty of the outcome
- Deterministic approach → Based on clock work universe Philosophy

Example:

- Physics: Free Fall, Projectile motion,
- Chemistry: Reaction between two substances
- Biology: Rate of growth of a cell
- Other exact sciences

- Derive the **system’s state equation** from D.E → f[(x1,x2,x3,….,xn),t]
- Given the initial condition f[(x1,x2,x3,….,xn)at (t0)]
- The state function at any given time is calculated, i.e. f[(x1,x2,x3,….,xn) @ t+Δt]



Nondeterministic Phenomena

- No mathematical model exists for "perfect" prediction of the phenomena's outcome.
- Uncertainty of the outcome
- Nondeterministic Phenomena are divided into:
  - Random Nondeterministic Phenomena
  - Chaotic Nondeterministic Phenomena

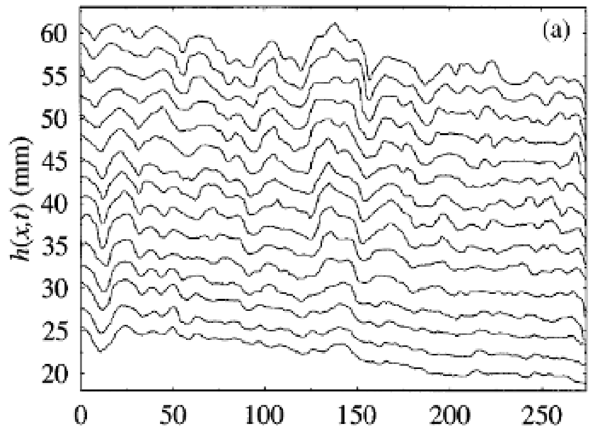
Example:

- Flipping a coin
- Shuffling a deck of cards
- Weather forecast
- Earthquake
- Stock market

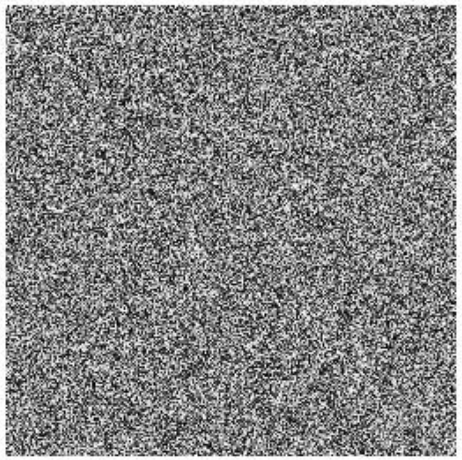
- System's state equation can not be **derived** via conventional models
- System's state equation can not be **predicted** via conventional models

Random phenomena

- Unable to predict the outcomes for each event
- But in the long-run
- The outcomes exhibit statistical regularity
- Modeled by Probability Theorem



System.Random
numbers 0...n of seed 0



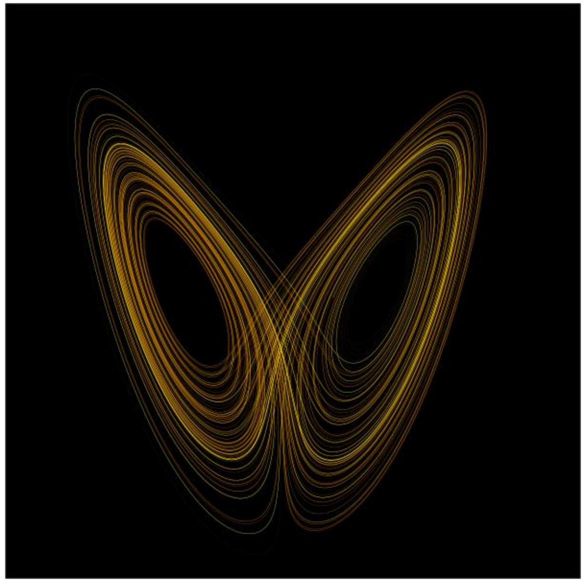
Sequence of 65536 random values.

Example:

- Flip a coin
- Tossing a dice
- Shuffling a deck of card

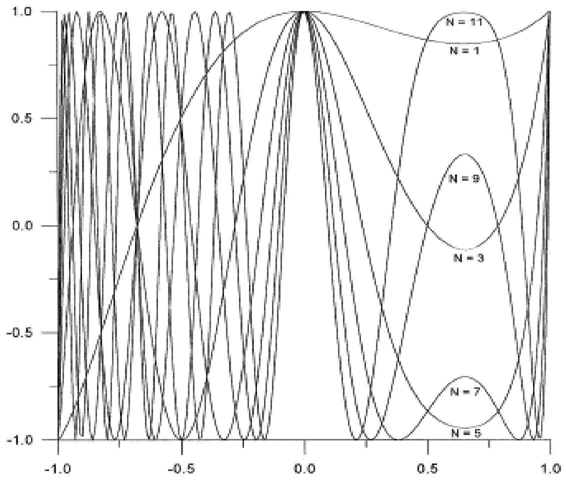
Chaotic phenomena

- Unpredictable outcomes
- No exhibition of statistical regularity in the outcomes.
- Stochastic Behaviour → Fractal geometry



Examples:

- 3- Body Problem
- Earthquake
- Stock Market



Random Phenomena- Example 1

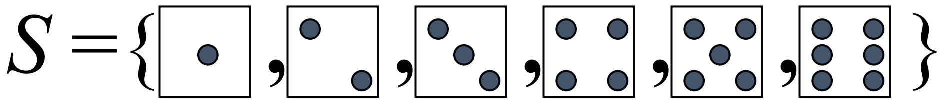
Tossing a coin → outcomes

S ={**Head, Tail**}

- Unable to predict on each toss whether is Head or Tail.
- In the long run → **50% chance** for H and T **each**

Random Phenomena- Example 2

Rolling a die → outcomes



- Unable to predict outcome for each roll
- In the long run → each outcome will occur 1/6 of the time.

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Probability Theorem

- **Probability** is a branch of mathematics
- It is concerned with the study of nondeterministic random phenomena.
- Probability means the **mathematical chance** that something might happen.
- It is concerned with the numerical descriptions of how likely an event is to occur.

Fortuna and the wheel of chance

Probability Theorem- Applications

Some Applications:

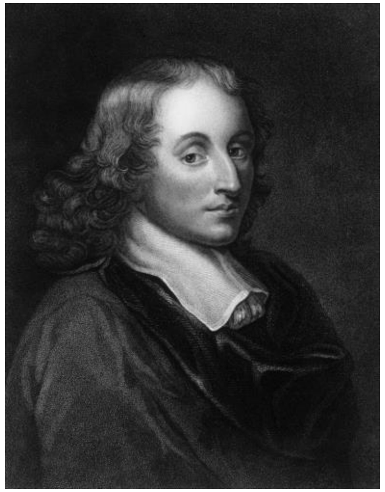
- Numerous day- to-day applications (such as *weather forecasts*, *Sports Strategies*, *Games*)
- Business and Management
- Finance, Banking and Insurance
- Exact Sciences
- Engineering
- Computer Science
- More & More

Origins of Probability Theory

- **Modern foundation** of Probability began by analysis of chance in games.
- De Mere (1654) – asked Blaise Pascal *if he can bet on the following*:

To throw a **pair of dice** 24 times → at least **one double 6** → Win

- Pascal & Fermat begin a series of correspondences on this subject



Probabilistic Reasoning- Conceptual Example

- Suppose I exactly know the proportions of *top selling cars*, in Australia
- Example: S = {Toyota Hilux, Ford Ranger, Toyota Corolla, Hyundai i30, …}
- Then I can estimate the probability that the first car I see in the street, is a Ford Ranger.
- This is known as “Probabilistic Reasoning”

Probabilistic Reasoning

- Probabilistic reasoning is using logic and probability to predict the outcomes and handle *uncertain situations*.
- Humans have an intuitive understanding of Probability Reasoning.
- Probabilistic Reasoning:

    i. Know the population

    ii.Know the structure of population

    iii. Predict the chances of an outcome → sample.

Different Types of Probability Theorem

Different Types of Probability:

1) **Theoretical** (Laplace) → Predict a numerical value for probability of an event

2) **Experimental**: The actual result of an experiment

3) **Subjective**: Derived from individuals judgment.

4) **Axiomatic**: Based on the works of the Russian mathematician A.N. Kolmogorov

Reflection I (Individual, 10’)

1. What are the different types of phenomena? Define each and give some examples

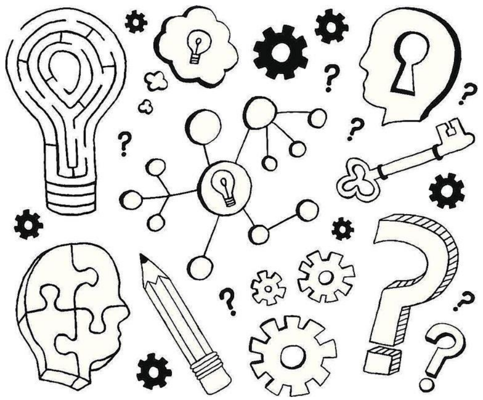
2. What is the main distinguishable characteristic for each phenomena.

3. How is each phenomena modeled?

4. What is Probability Theorem?Why is it important?

5. What are the application of Probability Theorem? Give 3 examples.

6. What is probabilistic reasoning?



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Definitions

Sample Space: The set of all possible outcomes.

Probability Experiment

- Flip a coin
- Toss a dice

All outcomes → Sample Space

- S={H, T}
- S={1,2,3,4,5,6}

Event: $E_i \subset S$

- E1= H in a flip of a coin
- E2= Even number is rolled

Each Experiment → one particular Outcome

Outcome: The result of a single trial/ experiment

Event: An outcome of a probability experiment (Consists of one or more single outcomes).

Events: **Simple** or **composite**

Probability Experiments

- Probability Experiments An action through which specific results are obtained.
- Results → Observed and/or measured
- These results → may have more than one possible outcome.
- These Results → well-defined set of possible outcomes.
- This experiment can be infinitely repeated

Probability Experiments

- Rolling a die → observing the number that is rolled
- Tossing a coin → observing the side
- Weather → Forecast of the weather

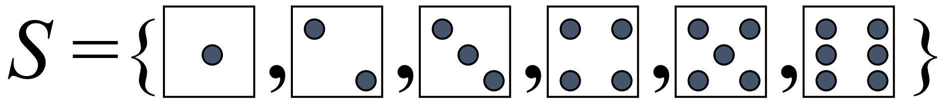
The Sample Space

- Sample Space *(for a random phenomena):* the set of all possible outcomes.
- Notation: S, Ω

- Examples

1. Tossing a coin → S ={**Head, Tail**}

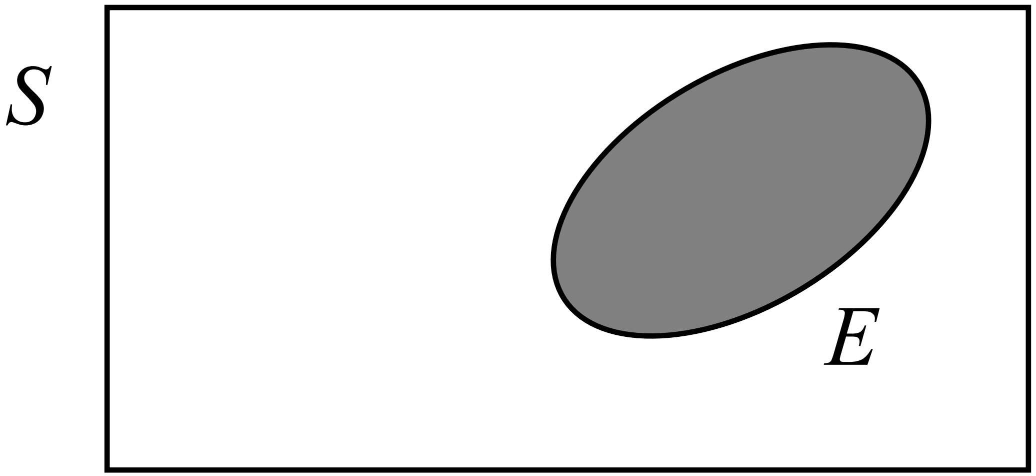
2. Rolling a die → S = {1,2,3,4,5,6}



Event

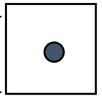
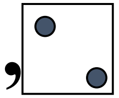
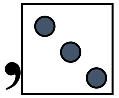
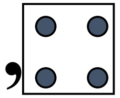
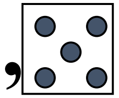
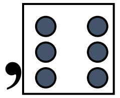
- An outcome of a probability experiment / Random phenomena.
- Consists of one or more single outcomes.
- When an experiment is performed, a particular event either happens, or it doesn't!
- Is any subset of the sample space, *S*.
- Usually denoted by a capital letter.

$$E \in S$$

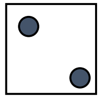
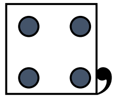
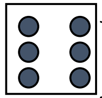


Example) Rolling a die

- Outcomes → Sample Space S

$S = \{$  ,  ,  ,  ,  ,  $\}$ $= \{1, 2, 3, 4, 5, 6\}$

- E = the event that an even number is rolled

$= \{2, 4, 6\}$ $= \{$  ,  ,  $\}$

Simple Event

- An event that cannot be decomposed is called a simple event.
- Each Simple event is denoted by E with a subscript: E_i
- Each event is composed of one or more simple events.
- Each simple event will be assigned a probability → measuring "how often" it occurs. $E_i \rightarrow P_i$

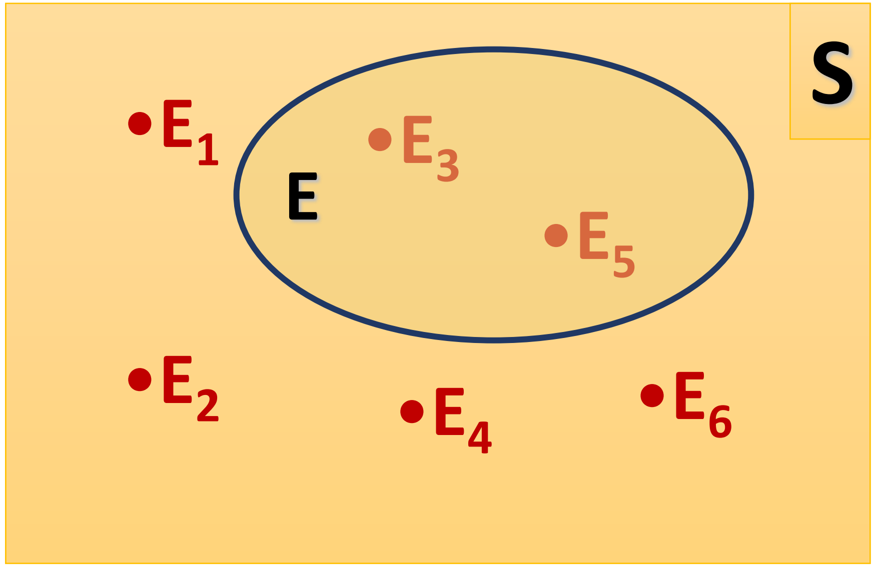
$$S=\bigcup_{i=1}^{n} E_i$$

Sample Space = **Union** of all simple events

Simple Event cont

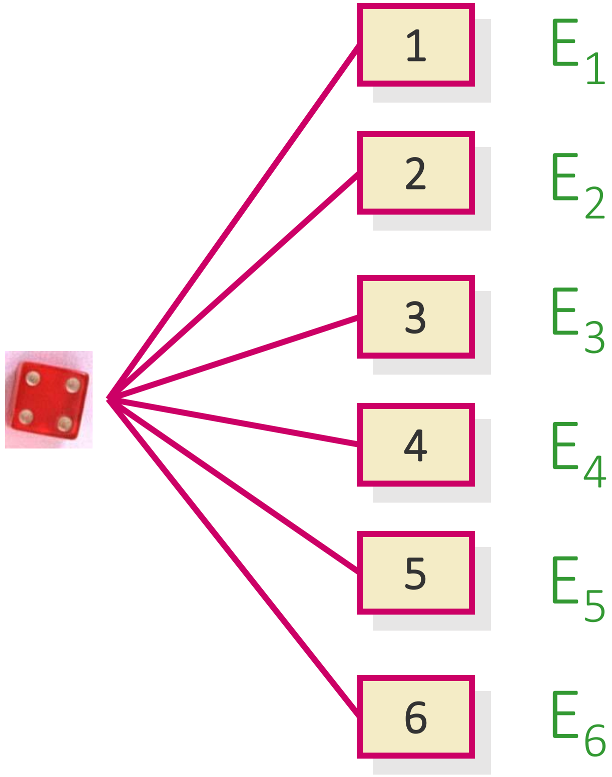
- Simple Events: E_1 & E_2 & E_3 & E_4 & E_5 & E_6
- Sample Space → $S=\{E_1,E_2,E_3,E_4,E_5,E_6\}$
- Composite Event→ $E=\{E_3,E_5\}$

- $(for, \exists i) E_i \epsilon E$
- $(for, \forall i) E_i \epsilon S$
- $E \subset S$



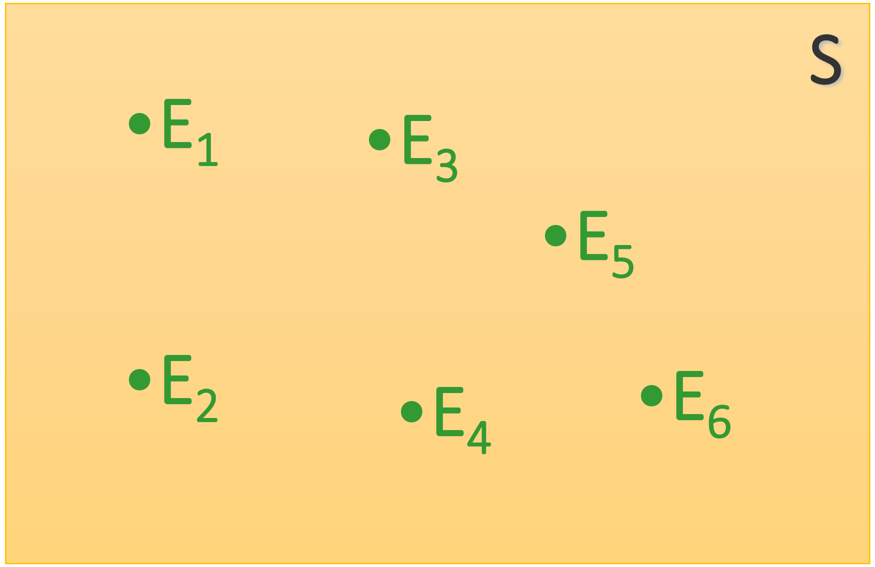
Example) The die tossed:

Simple events:



Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



Special Events

1. The Null Event (The empty event) : ϕ

➢ $\phi = \{\ \} =$ the event that contains no outcomes

2. The Entire Event (The Sample Space): S

➢ $S =$ the event that contains all outcomes

- The empty event ϕ → never occurs.
- The entire event S →100% occurs.

Relationship Between Events

Two Events A and B may be:

1. Dependent: outcome of one event, **affects** the other
2. Independent : outcome of one event, **does not affects** the other
3. Mutual Exclusive: They can not occur at the same time: $A \cap B = \emptyset$
4. We don’t know

Dependent Events

- Two events, A and B, are dependent if the outcome of the first **event** affects the outcome of the second **event**
- If A **occurs** → The probability of B happening changes
- This is known as "Conditional Probability"

Example:

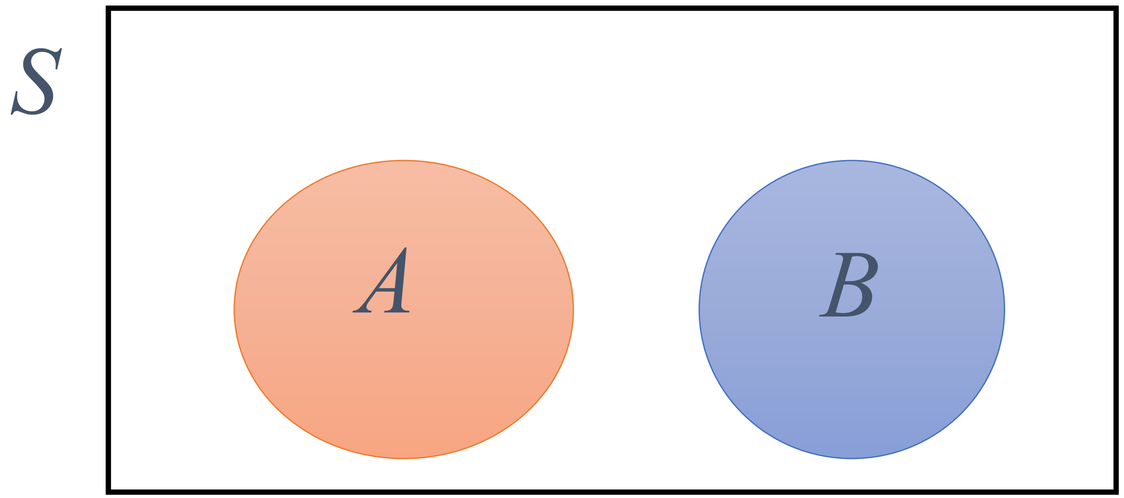
- Event A: Having a sleeplessness night
- Event B: Having a car accident
- Event A & B are dependent

Independent Events

- Two events, A and B, are independent if the fact that A occurs does not affect the probability of B occurring.
  - Example: *Landing on heads after tossing a coin* *AND* *rolling a 5 on a die*.
  - Example: *Choosing a marble from a jar* *AND* *coin landing on heads.*

Mutually Exclusive Events

- If they cannot both occur at the same time → They have no outcomes in common.
- Example: The set of outcomes of a single coin toss: S={H,T}
- Example: The birth of a son or a daughter are mutually exclusive events.



$$A \cap B = \emptyset$$

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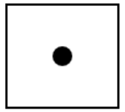
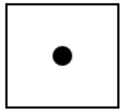
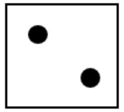
Conceptual Example

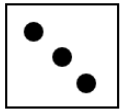
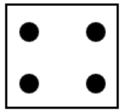
Example): A die is rolled → Sample Space → S={1,2,3,4,5,6}

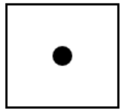
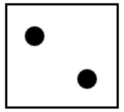
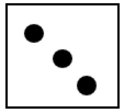
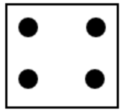
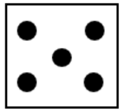
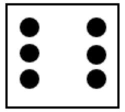
- Find the probability rolling a "5"? → Event A → $P(A) = \frac{1}{6} \approx 0.167$

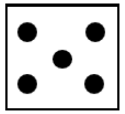
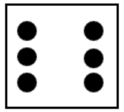
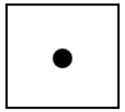
"Probability of Event *A*."

 $\frac{1}{6}$

 or  $\frac{2}{6}$ $\boxed{\frac{1}{6}+\frac{1}{6}}$

 or  $\frac{2}{6}$ $\boxed{\frac{1}{6}+\frac{1}{6}}$

 or  or  or  or  or  $\frac{6}{6} = 1$ $\boxed{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}}$

 or  or  $\frac{3}{6} = 0.5$ $\boxed{\frac{1}{6}+\frac{1}{6}+\frac{1}{6}}$

Classical Probability for Finite Sample Space

- The probability of the event A is defined as:

$$\mathrm{P}(A) = \frac{\text{Number of outcomes } of\ the \text{ event } A}{\text{Total number of outcomes in sample space}}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{n(A)}{N}$$

$*$ **Note**: the symbol $n(A) =$ no. of elements of A

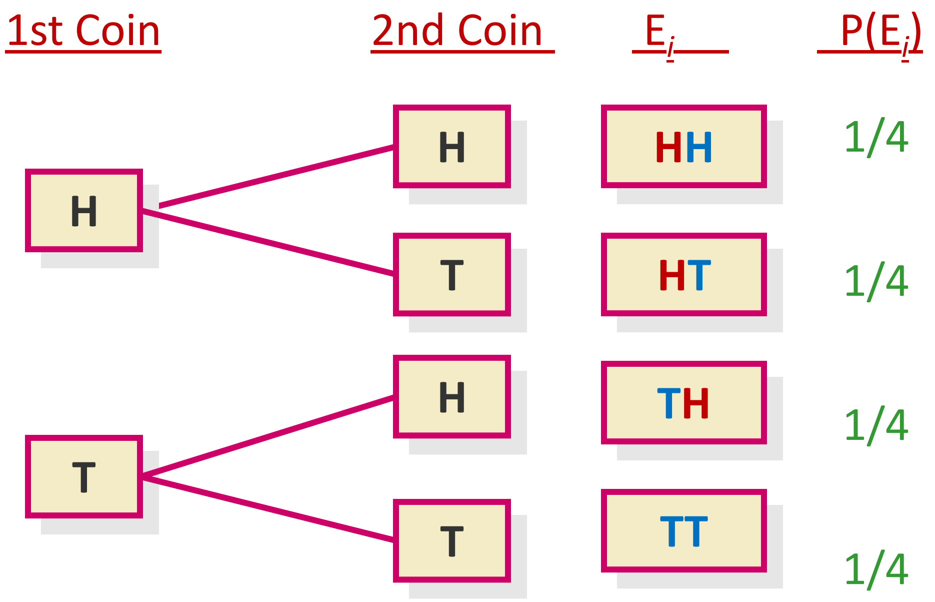
Classical Probability for an Infinite Sample Space

- In many random phenomena, the *sample space is very large*
  I. That is the number of trials are too many: $n \rightarrow \infty$
  II. The sample trial is continues
    - Example 1: Tossing n coins simultaneously
    - Example 2: Repeating a trial for n times
    - Example 3: Sample space is the pixels in a page

$$\boxed{P(a) = \lim_{n \to \infty} \frac{N(a)}{n}}$$

Example 1)

- Toss a fair coin twice.
- What is the probability of observing at least one head?



- S={HH,HT,TH,TT} → N=4
- Number of events that at least one coin is H=4

P(at least 1 head) =

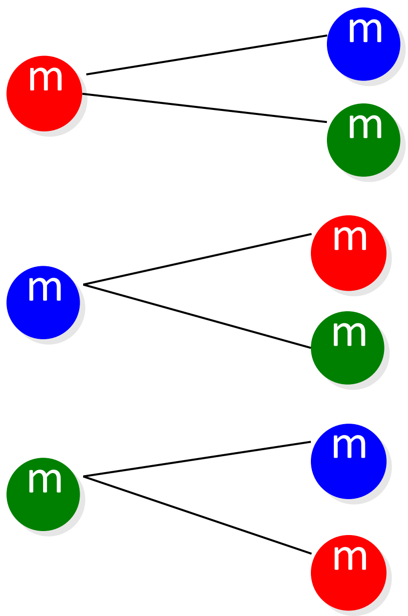
$P(E_1) + P(E_2) + P(E_3)$ =

1/4 + 1/4 + 1/4 = 3/4

Example 2)

- A bowl contains three M&Ms®, one red, one blue and one green.
- A child selects two M&Ms at random.
- What is the probability that at least one is red?

1st M&M　　2nd M&M



E_i	$P(E_i)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

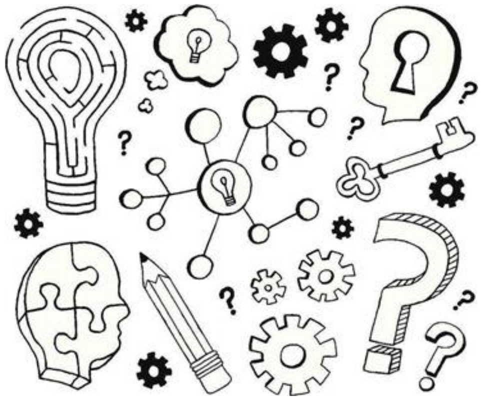
- S={RB,RG,BR,BG,GB,GR} → N= 6
- Number of events that 1 M&M is Red =4

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

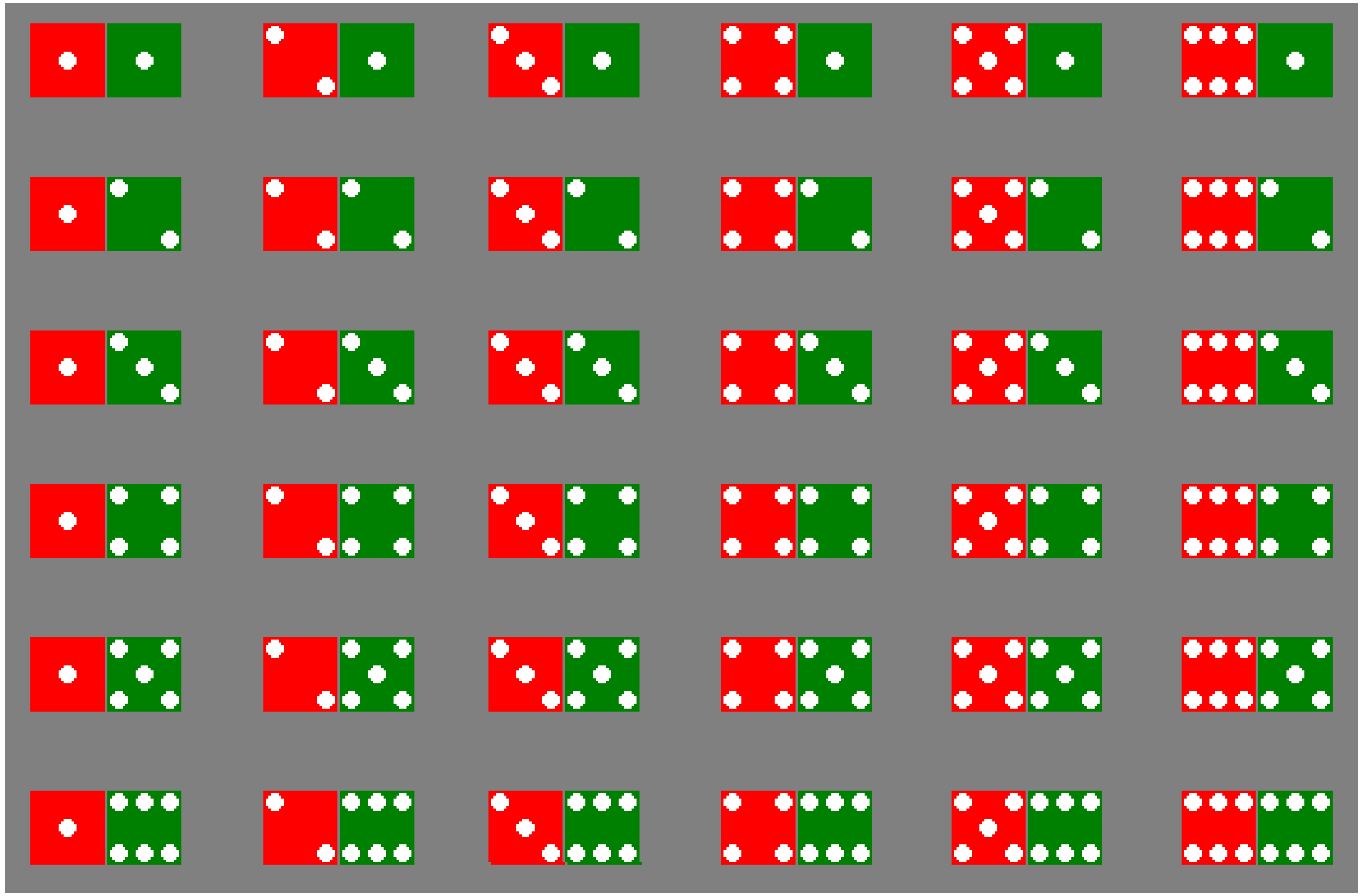
Activity 1) (Individual, 10’)



- A 1st die is tossed → Red die
- Then, a 2nd die is tossed → Green Die
- Find the probability for the following events:
    - ❑ A: Dice add to 3
    - ❑ B: Dice add to 6
    - ❑ C: Red die show 1
    - ❑ D: Green die show 1

Activity)cont.

- The sample space of throwing a pair of dice is as follow:
- Hence by counting → N= 6x6 =36



Activity)cont.

- This can be summarized in the following table

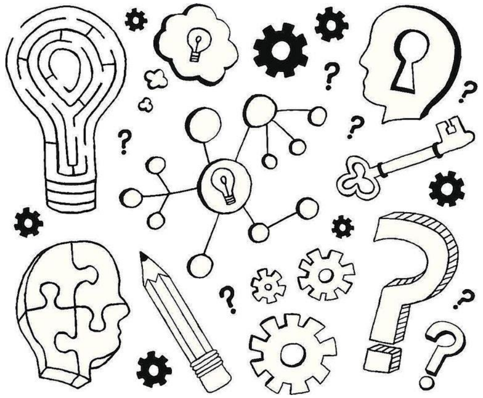
First Die	Second Die	Event	Add	First Die	Second Die	Event	Add
1	1	1,1	2	4	1	4,1	5
	2	1,2	3		2	4,2	6
	3	1,3	4		3	4,3	7
	4	1,4	5		4	4,4	8
	5	1,5	6		5	4,5	9
	6	1,6	7		6	4,6	10
2	1	2,1	3	5	1	5,1	6
	2	2,2	4		2	5,2	7
	3	2,3	5		3	5,3	8
	4	2,4	6		4	5,4	9
	5	2,5	7		5	5,5	10
	6	2,6	8		6	5,6	11
3	1	3,1	4	6	1	6,1	7
	2	3,2	5		2	6,2	8
	3	3,3	6		3	6,3	9
	4	3,4	7		4	6,4	10
	5	3,5	8		5	6,5	11
	6	3,6	9		6	6,6	12

N=36

Activity)cont.

Event	Simple events	Probability
A: Dice add to 3	(1,2),(2,1) → n(3)=2	P(A)= 2/36
B: Dice add to 6	(1,5),(2,4),(3,3), (4,2),(5,1) → n(6)=5	P(B)= 5/36
C: Red die show 1	(1,1),(1,2),(1,3), (1,4),(1,5),(1,6) → n(1,x)=6	P(C)= 6/36
D: Green die show 1	(1,1),(2,1),(3,1), (4,1),(5,1),(6,1) → n(x,1)=6	P(D)= 6/36

Reflection II (Individual, 10')



1. What is Sample Space?
2. What is an Event?
3. What is a simple Event?
4. Sample Space S has n outcomes. How many events can S have? *(Hint: try for n=2 and n=3, then generalize)*
5. Give an example of a null event?
6. Give an example of the entire event?
7. What are different relationships between events? Explain with an example.
8. What are the axioms of the probability theorem?

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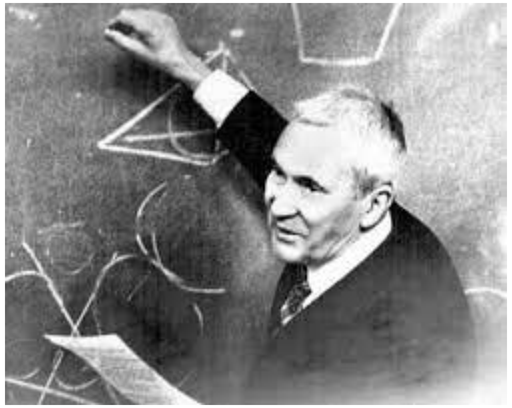
Axiomatic Probability

- The Classic Probability Theorem has *some shortcomings*.
- Using the conclusions of the Classic Probability Theorem, , Andrey . N. Kolmogorov

  → developed Axiomatic Probability (1933)
- In order to rectify the above problems, he proposed the *following 3 axioms*.

* Axiom: a statement accepted as true as the basis for argument or inference



Axiom 1

- The probability of every event (A) is at least zero.

Axiom 1: $\forall A \in S; \Pr(A) \geq 0$

Axiom 2

- The probability of the entire outcome space (Sample Space) is 1.

$$\boxed{\text{Axiom 2: } \Pr(S) = 1}$$

Axiom 3

- If two events are disjoint,
- The probability that either of the events happens
- Is the sum of the probabilities that each happens.

Axiom 3:
For every sequence of disjoint events

$$\Pr(\bigcup_i A_i) = \sum_i \Pr(A_i)$$

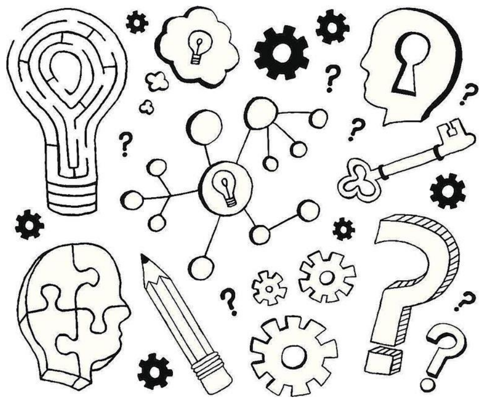
Axioms of Probability Theorem

- If Ω is the Sample Space
- If $\omega \subset \Omega$ is a single outcome
- If $A \subset \Omega$ is a set of Events

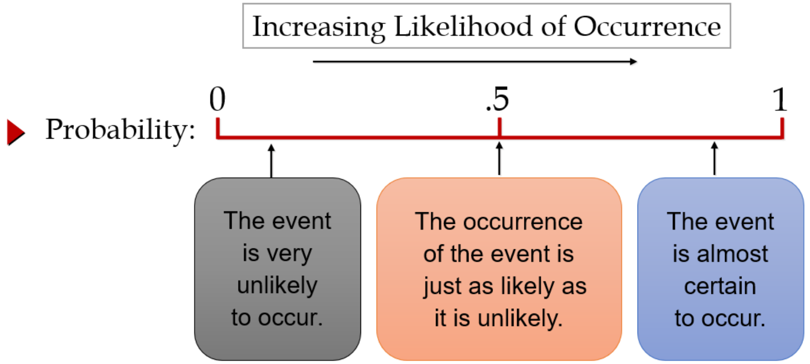
The Axioms of Probability are:

1. $P(A) \geq 0 \forall A \in \Omega$
2. $P(\Omega) = 1$
3. $A_i \cap A_j = \emptyset \forall i, j \Rightarrow P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

Reflection III(Individual, 10’)



1) What are the 3 Axioms of Probability?
2) Justify each of the bellow statements:
   a) Probability is a numerical measure of the likelihood that an event will occur.
   b) Probability values are always assigned on a scale from 0 to 1.
   c) If P(A)=0 → n(A)=0 ⟹ The event will not occur.
   d) If P(A)=1 → n(A) = N ⟹ The event will definitely occur.
   e) The following diagram:



Time for a break – 20’

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Composite Events

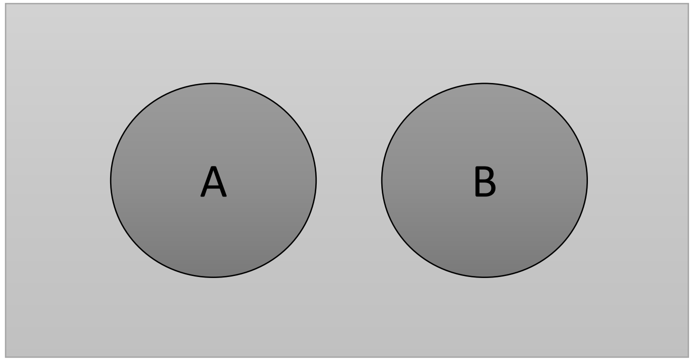
Composite events for two simple events A & B are:

i. Probability of either A **or** B happening.

ii. Probability of event A **not** happening.

iii. Probability of event A happening, **if** event B happens.

iv. Probability of A and B, **both** happening.



Probability of either A or B happening

- For any two events, A and B, the probability of their union, **P(A ∪ B)** is the Probability of either A or B or both happening.
- This is known as "The additive rule of probability"

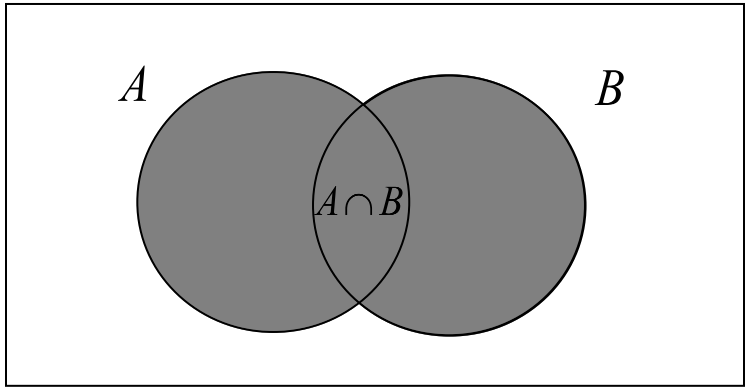
I. The Probability of A happening → P(A)

II. The Probability of B happening → P(B)

III. The Probability of A and B happening (Together) → P(A∩B)

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$$



Probability of event A not happening

i. We know that for any **event A**: $P(A \cap A^C) = \emptyset$ → $P(A \cup A^C) = P(A)+P(A^C)$

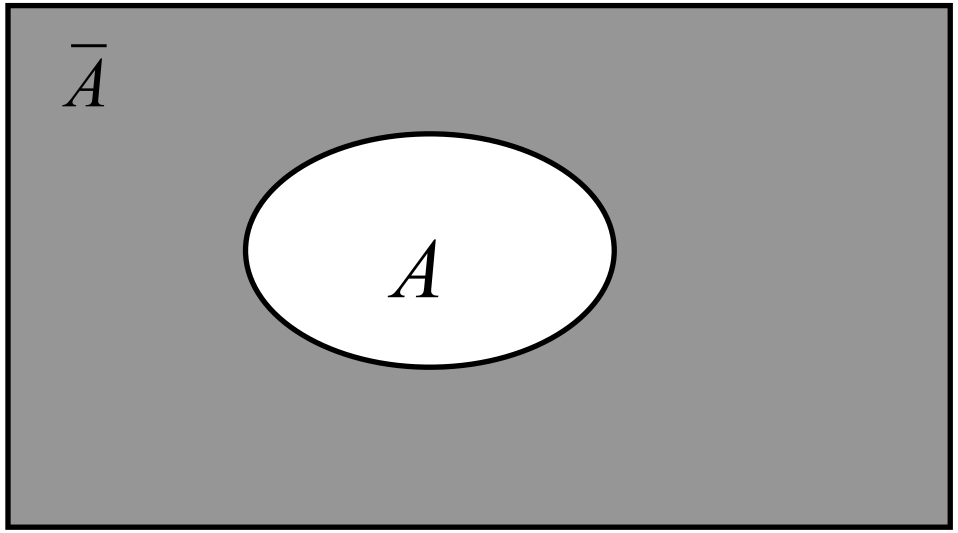
ii. Since either **A** or **A^C** must occur → $P(A \cup A^C) =1$

iii. From i & ii $P(A \cup A^C) = P(A)+ P(A^C) = 1$

$P[\text{not } A] = 1 - P[A]$

$$P(A^C) = 1 - P(A)$$

Complement of A



Probability of event A happening, if event B happens.

- If A and B are two dependent events → probability of A happening, **depends** on the probability of B happening.

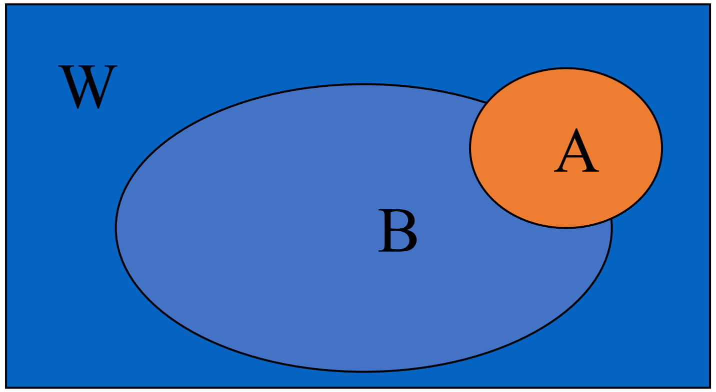
B → A ⇒ P(A)=f(P(B))

- Then the conditional probability of A [Probability of A happening, if B happens]:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad [\text{if } P(B) \neq 0] *$$

*Note:

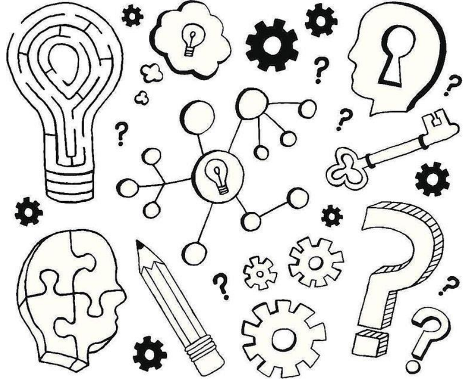
- For P(A|B), P(B) is given
- For P(B|A), P(A) is given



Summary

- If A & B are two Simple Events: $P(A)=n(A)/N$ and $P(B)=n(B)/N$
- Probability of A or B happening: $P[A \text{ or } B] = P[A] + P[B] - P[A \cap B]$
- Probability of A or B happening *(disjoint)*: $P[A \text{ or } B] = P[A] + P[B]$
- Probability of A not happening: $P[\text{"not " } A]=1-P[A]$
- Probability of B not happening: $P[\text{"not " } B]=1-P[B]$
- Probability of A if B happens: $P(A|B) = \frac{P(A \cap B)}{P(B)}$
- Probability of B if A happens: $P(B|A) = \frac{P(A \cap B)}{P(A)}$
- Probability of A and B happening *(independent)*: $P(A \cap B) = P(A)P(B)$
- Probability of A and B happening *(dependent)*: $P(A \cap B) = P(B|A)\,P(B) = P(A|B)\,P(A)$

Reflection IV (Individual, 15’)



1. What does the probability of the union of two events mean? How can we do it?
2. How can we calculate the probability of the union of two mutually exclusive events?
3. What does the probability of the complement of an event mean? How can we calculate the probability of the complement of an event?
4. What is Conditional Probability? How is it calculated?
5. What is the probability of two independent events?
6. What is the probability of event A & B, both happening?

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Statistics

- Statistics is a mathematical discipline that is concerned with:

  - Collection ,
  - Organization,
  - Analysis,
  - Interpretation,
  - Presentation,

  of Data

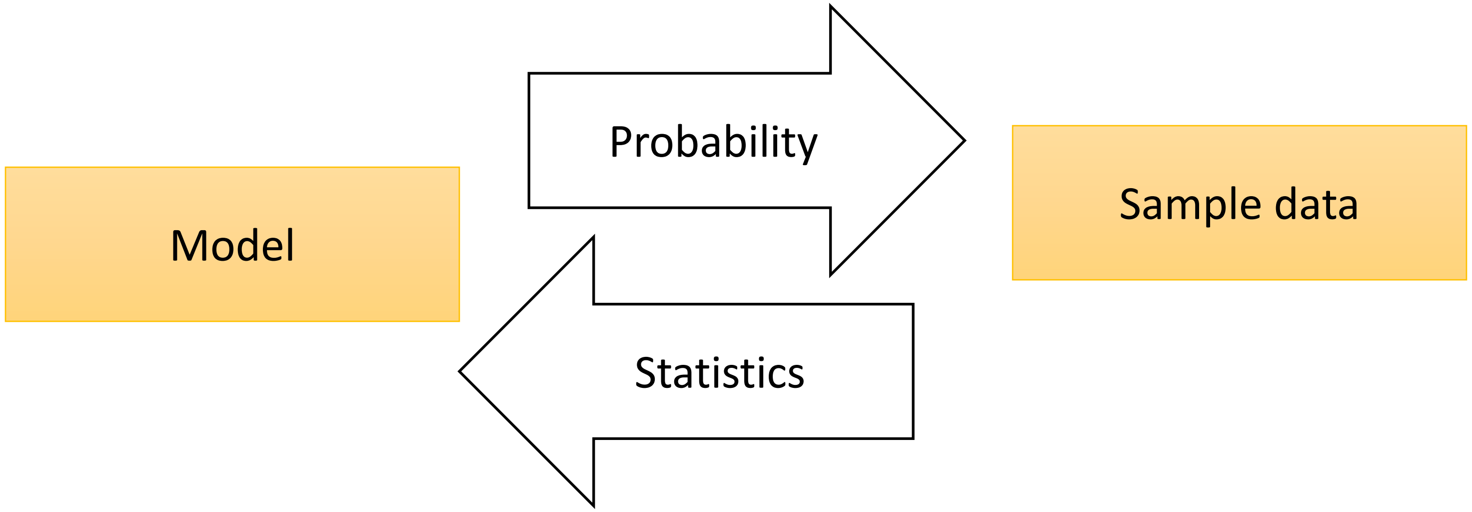
- Sir Ronald Fisher (1890 – 1962) is credited as the Father of modern statistics.

- Application: scientific, industrial, social, etc.

Statistics vs. Probability

Probability and Statistics are closely related fields, yet separate.

- **Probability** deals with predicting the **likelihood** of future events (from a known model).
- **Statistics** involves the analysis of the **frequency** of past events (to identify the model).



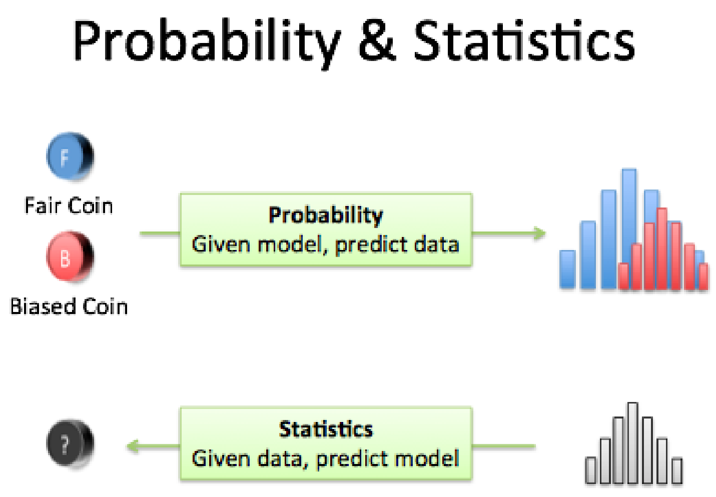
Statistics vs. Probability - Cont.

Probability Reasoning

- 10 marbles are in a bag.
- 3 Green, 3 Red, 2 Blue, 2 yellow
- If two random marbles are drawn
- What is the likelihood of both being Blue?

Statistics Reasoning

- Unknown number of marbles are in a bag.
- Randomly, draw 6 marbles from the bag
- Study this sample
- Based on the sample Conjecture on the population and colors of marbles in the bag?



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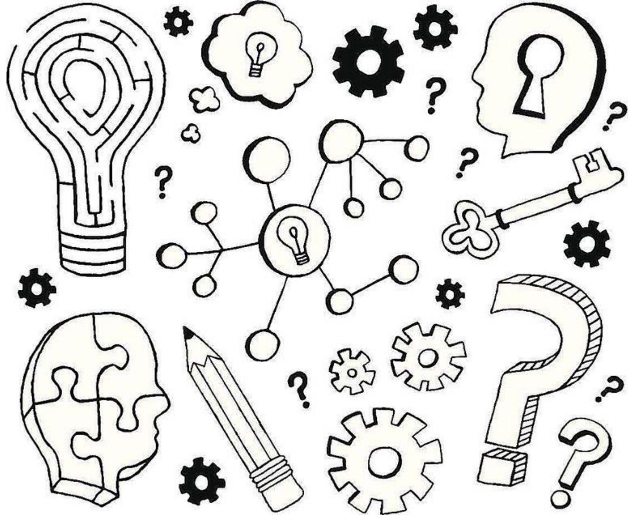
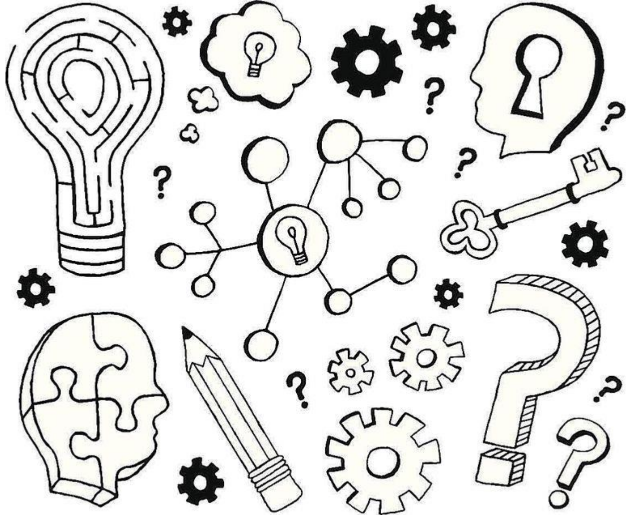
Reading & Research I



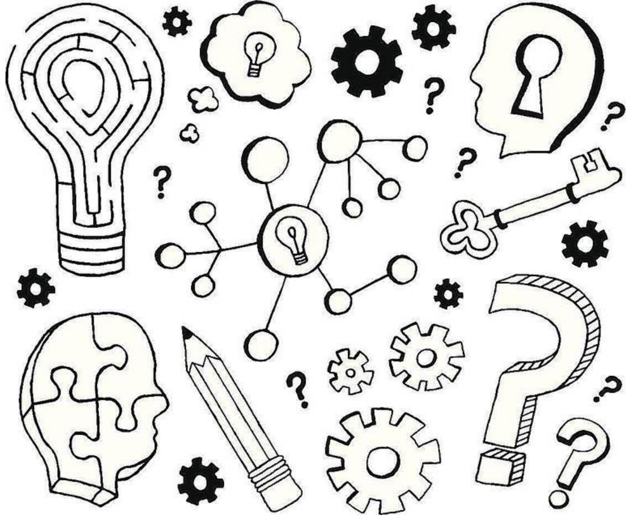
1. Read the following article about chaos theory and summaries your findings:
   https://pmc.ncbi.nlm.nih.gov/articles/PMC3202497/

Reading & Research II



1. Read the following article about probabilistic reasoning and summaries your findings:
   https://indiaai.gov.in/article/the-importance-of-probabilistic-reasoning-in-ai

Reading & Research III



1. Read the following article about probability and summaries your findings:

   https://ocw.mit.edu/courses/6-825-techniques-in-artificial-intelligence-sma-5504-fall-2002/a1456866242bef33836e0cfe56e5e75c_Lecture14FinalPart1.pdf

Source of the slides:

https://www.slideserve.com/paul2/probability

https://www.tes.com/en-au/teaching-resource/introduction-to-probability-powerpoint-6323807

First Course in Probability, “Sheldon M. Ross,2018”

https://www.slideserve.com/search/presentations/random-variables-and-distributions

MIT Open Course

And

https://www.xpowerpoint.com/gradient-methods-igl--PPT.html#