7
The Derivative

As I said before, calculus can be thought of as the study of limits. In the last chapter, we defined continuity in terms of limits. We will see that the definition of the derivative also involves limits.

The derivative is a sophisticated way to analyze a function. In algebra, the first method for graphing a function involves plotting points. After a while, plotting points becomes tiring, and new methods for graphing a function are introduced. For example, instead of plotting points, quadratic functions can be graphed using five characteristics: concavity, the axis of symmetry, the vertex, the y-intercept, and the x-intercept. The amount of work required to graph a quadratic function is decreased significantly. Using calculus, we will be able to analyze complicated functions more efficiently.

Lesson 7-1: Secant Lines and Difference Quotients

If a function $f(x)$ passes through the points $(a, f(a))$ and $(b, f(b))$, the **average rate of change** of the function between $x = a$ and $x = b$ is given by the formula:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

The average rate of change of a function gives a general idea of how a function is changing over a specific interval. The function could undergo several changes in direction between $x = a$ and $x = b$. The average rate of change will not reveal any of those changes in direction.

This formula should look familiar: The average rate of change of a function over $[a, b]$ is just the *slope* of the line that passes through the points $(a, f(a))$ and $(b, f(b))$:

$$\text{slope} = \frac{f(b) - f(a)}{b - a}$$

A line that passes through two points of the graph of a function is called a **secant** line, and the slope of the secant line between $(a, f(a))$ and $(b, f(b))$ is the average rate of change of the function $f(x)$ between $x = a$ and $x = b$.

Example 1

Find the average rate of change of the function $f(x) = x^2$ between $x = 2$ and $x = 4$, and find the equation of the secant line passing through $(2, f(2))$ and $(4, f(4))$.

Solution: Using the equation for the average rate of change, we have:

average rate of change $= \frac{f(b) - f(a)}{b - a}$

Substitute $a = 2$ and $b = 4$ into the equation for the average rate of change

average rate of change $= \frac{f(4) - f(2)}{4 - 2}$

$f(2) = 2^2 = 4$, and $f(4) = 4^2 = 16$ average rate of change $= \frac{16 - 4}{2}$

Simplify average rate of change = 6

To find the equation of the secant line, substitute the slope (the average rate of change that we just calculated) and one of the points into the point-slope formula for a line: $y - 4 = 6(x - 2)$

$y = 6x - 8$.

There are several formulas that can be used to find the slope of the secant line, but they all say basically the same thing. To find the slope of the secant line of the function $f(x)$ over the interval $[x, x + h]$ we evaluate:

$$\text{slope} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

We discussed this formula in Chapter 1. We even gave it a name: It was called the difference quotient. I assured you in Lesson 1-6 that this formula would be important in calculus. It is the basis for the derivative. We will be working with the difference quotient throughout this chapter.

Example 2

Find the average rate of change of the function $f(x) = 3x^2$ between $x = 1$ and $x = 1 + h$, where $h \neq 0$, in terms of h.

Solution: Using the equation for the average rate of change, we have:

average rate of change $= \frac{f(b) - f(a)}{b - a}$

Substitute $a = 1$ and $b = 1 + h$ into the equation for the average rate of change:

average rate of change

$$= \frac{f(1+h) - f(1)}{(1+h) - 1}$$

$f(1 + h) = 3(1 + h)^2$, and $f(1) = 3 \cdot 1^2 = 3$:

average rate of change

$$= \frac{3(1+h)^2 - 3}{h}$$

Expand $3(1 + h)^2$: average rate of change $= \frac{3(1+2h+h^2)-3}{h}$

Combine the constant terms: average rate of change $= \frac{3+6h+3h^2-3}{h}$

Factor an h out of each term in the numerator:

average rate of change

$$= \frac{6h+3h^2}{h}$$

Cancel the h in the numerator with the h in the denominator, because $h \neq 0$:

average rate of change

$$= \frac{\cancel{h}(6+3h)}{\cancel{h}}$$

$= (6 + 3h)$

The difference quotient is the slope of the secant line passing through the points $(x, f(x))$ and $(x + h, f(x + h))$. The difference quotient is also the average rate of change of the function $f(x)$ over the interval $[x, x + h]$. The average rate of change gives an idea of the overall change of a function over an interval. Knowing the overall change of a function over an interval can certainly be useful, but it may be more useful to know how the function is changing at an *instant.*

Suppose $f(x)$ represents the position of an object as a function of time, x. Then $f(b) - f(a)$ represents the change in location, or the distance traveled, between time $x = a$ and time $x = b$, and the quantity $(b - a)$ represents the length of time that has elapsed. The average rate of change of $f(x)$ over the interval $[a, b]$, $\frac{f(b)-f(a)}{b-a}$, can then be interpreted as the ratio of the distance traveled divided by the time elapsed, or the average *velocity.*

Figure 7.1 will help us visualize secant lines more clearly. The function shown in Figure 7.1 passes through the points (0, 1) and (2, 5). The secant line is the line that passes through those two points. The average rate of change of the function over the interval [0, 2] is: $\frac{f(2)-f(0)}{2-0}=\frac{5-1}{2-0}=2$. This is equivalent to saying that the slope of the secant line that passes through the points (0, 1) and (2, 5) is 2.

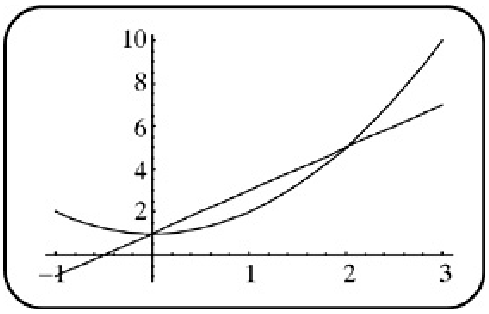

Figure 7.1.

Lesson 7-1 Review

1. Find the slope of the secant line of the given functions over the indicated intervals:

a. $f(x) = 2^x$ over [1, 3]

b. $g(x) = \sin x$ over $[0, \frac{\pi}{2}]$

c. $h(x) = \log_3 x$ over $[\frac{1}{3}, 3]$

2. Find the average rate of change of the following functions over the given intervals:

a. $g(x) = x^2 - 2$ over [0, 2]

b. $f(x) = \tan x$ over $[0, \frac{\pi}{4}]$

c. $h(x) = \log_2 x$ over [1, 4]

Lesson 7-2: Tangent Lines

Secant lines intersect a function at two distinct points, and the slope of the secant line represents the *average rate of change of a function* over a given interval. There are times when it is more useful to know the *instantaneous* rate of change of a function at a *point* $x = a$. Before we discuss what is meant by the instantaneous rate of change, we need to discuss tangent lines.

A line is **tangent** to a function at a point $x = a$ if the line just touches the graph of the function at that point. Figure 7.2 shows the function $f(x) = x^2$ with two of its tangent lines: one at $x = -1$ and the other at $x = 2$. Notice that the tangent lines just glance off of the graph of the function. In order to find the equation of a tangent line, you need to know the point where the tangent line touches the graph of the function and the slope of the tangent line. That's where calculus comes into play.

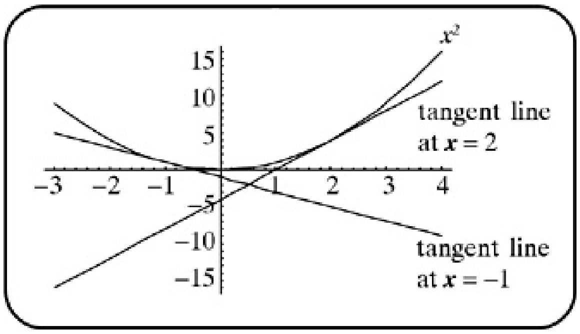

Figure 7.2.

The slope of the secant line passing through the points $(a, f(a))$ and $(a + h, f(a + h))$ is the ratio:

$$\frac{f(a+h)-f(a)}{(a+h)-a}=\frac{f(a+h)-f(a)}{h}$$

If h is small, then the secant line is very close to the tangent line. The smaller the value of h, the closer the secant line gets to the tangent line. If we let $h \to 0$, then the secant line → the tangent line, and the slope of the secant line approaches the slope of the tangent line:

$$\lim_{h\to 0} \text{(slope of secant line)} = \text{slope of tangent line}$$

If we denote the slope of the line tangent to the graph of the function $f(x)$ at $x = a$ by f'(a), we have:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

We call f'(a) the derivative of the function $f(x)$ at $x = a$. The **derivative** of a function at a point $x = a$ is the slope of the line tangent to the graph of $f(x)$ at $x = a$. The definition of the derivative of a function involves a limit. Remember that calculus is the study of limits. The notion of continuity was the first application of a limit. The derivative of a function is our second application of limits.

Lesson 7-3: The Definition of the Derivative

The derivative of a function at $x = a$ is the slope of the line tangent to the graph of $f(x)$ at $x = a$. The lines tangent to the graph of a function do not all have the same slope. Sometimes the tangent lines will have a positive slope, and other times the tangent lines will have a negative slope. The sign and the magnitude of the slope of the tangent lines depend on how the function is changing. The derivative of a function measures the rate of change of a function along its graph. In fact, the derivative of a function is itself a function. We define the function $f'(x)$ as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We can find the derivative of various functions by evaluating this limit using the techniques discussed in Chapter 5.

Example 1

Find the derivative of the function $f(x) = x^2$.

Solution: Use the definition of the derivative and evaluate the limit directly. If $f(x) = x^2$, then $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$. Substituting into the definition for the derivative, we have:

Start with the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute in for $f(x)$, and $f(x + h)$

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h}$$

The x^2 terms subtract out

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

Cancel the h's

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

Take the limit. The term $2x$ is not affected by changes in h.

$$f'(x) = \lim_{h \to 0} (2x + h)$$

$$f'(x) = 2x$$

The derivative of the function $f(x) = x^2$ is the function $f'(x) = 2x$.

Example 2

Find the derivative of the function $f(x) = \sqrt{x}$.

Solution: Use the definition of the derivative and evaluate the limit directly. If $f(x) = \sqrt{x}$, then $f(x + h) = \sqrt{x+h}$. Substituting into the definition for the derivative, we have: Start with the definition of the derivative

Start with the definition of the derivative

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Substitute in for $f(x)$ and $f(x + h)$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Multiply by the conjugate of the numerator

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

The x's subtract out

$$f'(x) = \lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

Cancel the h's

$$f'(x) = \lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

Evaluate the limit; x is not affected by changes in h

$$f'(x) = \lim_{h\to 0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

The derivative of the function $f(x) = \sqrt{x}$ is the function $f'(x) = \frac{1}{2\sqrt{x}}$.

Example 3

Find the derivative of the function $f(x) = e^{x}$.

Solution: To solve this problem, we need to make use of a previous result: $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$. Keep in mind that the variable used in the limit expression is not important; any letter of the alphabet would serve the same purpose as long as one variable is consistently replaced with another variable throughout the limit expression. This equation can also be written $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. We will make use of this limit in evaluating the limit of the difference quotient for the function $f(x) = e^x$. First, evaluate $f(x + h)$: $f(x + h) = e^{x + h}$. Substitute into the definition of the derivative and evaluate the limit.

Start with the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute in for $f(x)$ and $f(x + h)$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$$

Factor e^x out of each expression in the numerator

$$f'(x) = \lim_{h \to 0} \frac{e^x \left(e^h - 1\right)}{h}$$

The term e^x has no dependence on h, and it is not affected by the behavior of h. It can be factored out of the limit expression.

$$f'(x) = e^x \lim_{h \to 0} \frac{\left(e^h - 1\right)}{h}$$

Evaluate the limit

$$f(x) = e^x \cdot 1$$

The derivative of the function $f(x) = e^x$ is $f'(x) = e^x$.

Example 4

Find the derivative of the function $f(x) = \sin x$.

Solution: Start with the definition of the derivative and simplify. In this situation, the sum of two angles formula will come in handy. We will also make use of two results obtained in Chapter 5: $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$. These two results are instrumental in determining the derivative of the sine function.

Start with the definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substitute in for $f(x)$ and $f(x + h)$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

Use the sum formula to expand sin $(x + h)$

$$f'(x) = \lim_{h \to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$

Regroup the numerator

$$f'(x) = \lim_{h \to 0} \frac{(\sin x \cos h - \sin x) + \sin h \cos x}{h}$$

Split the numerator into two pieces

$$f'(x) = \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h} \right)$$

The sin x and cos x are not affected by h, so they can be factored out of the limit

$$f'(x) = \lim_{h \to 0} \left((\sin x) \frac{(\cos h - 1)}{h} + (\cos x) \frac{\sin h}{h} \right)$$

Evaluate each limit separately

$$f'(x) = (\sin x) \lim_{h \to 0} \frac{(\cos h - 1)}{h} + (\cos x) \lim_{h \to 0} \frac{\sin h}{h}$$

$$f'(x) = (\sin x) \cdot 0 + (\cos x) \cdot 1$$

$$f'(x) = \cos x$$

Lesson 7-3 Review

Use the definition of the derivative (the difference quotient) to find the derivative of the following functions:

1. $f(x) = x^2 + 3x$
2. $f(x) = \sqrt{x+2}$
3. $f(x) = e^{x+3}$
4. $f(x) = \frac{1}{x}$
5. $g(x) = \cos x$

Lesson 7-4: The Existence of the Derivative

In order for the derivative of a function $f(x)$ at$x = a$ to exist, $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ must exist. The denominator of the difference quotient is heading towards 0. The only way for this limit to exist is if the *numerator* also heads towards 0. The only way for the numerator to head towards 0 is if $\lim_{h\to 0} f$ $(a + h = f(a)$. This means that the function $f(x)$ is *continuous* at $x = a$. In other words, if the derivative of a function at a point $x = a$ exists, then at a *minimum*, the function must be continuous at $x = a$. In other words, the derivative of a function cannot be defined wherever the function has holes, jump discontinuities, or vertical asymptotes.

The requirement that a function be continuous is, as I mentioned, a minimum requirement. There are functions that are continuous on their domain that have points where the derivative does not exist. The function $f(x) = |x|$ is an example of a function that is continuous everywhere, yet its derivative does not exist at the point $x = 0$. The graph of the function $f(x) = |x|$ is shown in Figure 7.3 on page 123.

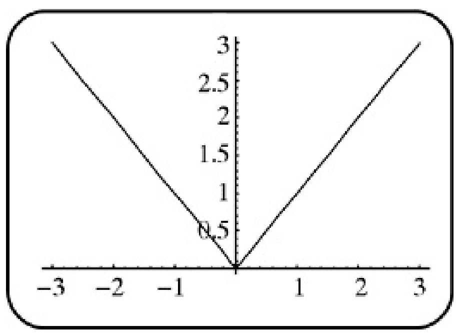

Figure 7.3.

To understand why the derivative does not exist at $x = 0$, we will try to evaluate $f'(0)$ using the difference quotient. Recall that the absolute value function is a piecewise-defined function and can be written:

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

To evaluate the derivative of this function at $x = 0$, we will need to evaluate:

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

In order to evaluate this limit, we will need to let $h \to 0$ from below and from above:

$$\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1$$

$$\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{-h}{h} = -1$$

Because the limit from below and the limit from above are different, we end up with two values for the derivative at $x = 0$: 1 and −1. It does not make sense to have the slope of the tangent line flipping between 1 and −1. Our only other option is to say that the derivative of the function $f(x) = |x|$ does not exist at $x = 0$.

The absolute value function is continuous everywhere, but its derivative does not exist at $x = 0$. Notice that the graph of $f(x) = |x|$ has a sharp corner at $x = 0$. Sharp corners and cusps are graphical indicators of problems with the derivative. Algebraically, functions that are defined piecewise may have problems with the derivative where the pieces meet. In order to determine whether or not the derivative of a piecewise-defined function exists at the seam, you must use the definition of the derivative and evaluate the limit from above and from below. If the two results are the same, then the derivative exists. Otherwise, the derivative does not.

Another problem with the existence of the derivative occurs if the tangent line is vertical. The derivative is the slope of the tangent line, and the slope of a vertical line is undefined. So if the graph of a function has a vertical tangent line at $x = a$, then $f'(a)$ will not be defined. An example of a function with a vertical tangent line is $f(x) = \sqrt[3]{x}$. The graph of $f(x) = \sqrt[3]{x}$ is shown in Figure 7.4. Notice that at $x = 0$, the tangent line is vertical: $f'(0)$ is not defined.

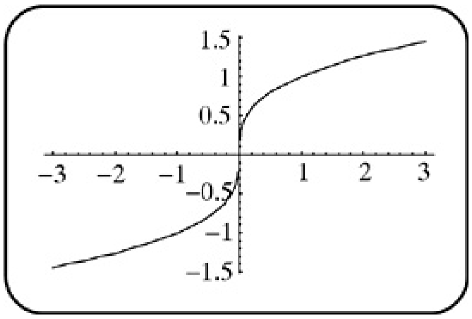

Figure 7.4.

Example 1

Does the derivative of the function $g(x) = \begin{cases} x & x < 1 \\ \sqrt{x} & x \geq 1 \end{cases}$ exist at $x = 1$

Solution: We need to evaluate $\lim_{h \to 0^+} \frac{g(1+h) - g(1)}{h}$ and $\lim_{h \to 0^-} \frac{g(1+h) - g(1)}{h}$ and compare the results.

Start with the definition of the derivative

$$\lim_{h \to 0^+} \frac{g(1+h) - g(1)}{h}$$

If $h > 0$, then $1 + h > 1$ and we use the second formula for $g(x)$

$$\lim_{h\to 0^+}\frac{\sqrt{1+h}-\sqrt{1}}{h}$$

Multiply both the numerator and denominator by the conjugate

$$\lim_{h\to 0^+}\frac{\sqrt{1+h}-\sqrt{1}}{h}\cdot\frac{\sqrt{1+h}+\sqrt{1}}{\sqrt{1+h}+\sqrt{1}}$$

Simplify

$$\lim_{h\to 0^+}\frac{(1+h)-1}{h\left(\sqrt{1+h}+\sqrt{1}\right)}$$

Cancel the *h*'s

$$\lim_{h\to 0^+}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{1+h}+\sqrt{1}\right)}$$

Evaluate the limit

$$\lim_{h\to 0^+}\frac{1}{\left(\sqrt{1+h}+\sqrt{1}\right)}$$

$$\lim_{h\to 0^+}\frac{1}{\left(\sqrt{1+h}+\sqrt{1}\right)}=\frac{1}{2}$$

Next, evaluate the limit from below:

$$\lim_{h\to 0^-}\frac{g(1+h)-g(1)}{h}$$

Start with the definition of the derivative

$$\lim_{h\to 0^-}\frac{(1+h)-\sqrt{1}}{h}$$

If $h < 0$, then $1 + h < 1$ and we use the first formula for $g(x)$. We still use $g(1) = \sqrt{1}$, however.

$$\lim_{h\to 0^-}\frac{(1+h)-\sqrt{1}}{h}$$

Simplify

$$\lim_{h\to 0^-}\frac{(1+h)-1}{h}$$

Simplify

$$\lim_{h \to 0^-} \frac{h}{h} = 1$$

The limit from below and the limit from above are not equal to each other. The derivative from above is $\frac{1}{2}$ the derivative from below is 1, which means that the derivative does not exist. The graph of $g(x)$ is shown in Figure 7.5. Even though the pieces of the function $g(x)$ line up to give a continuous function, the derivative does not exist at $x = 1$.

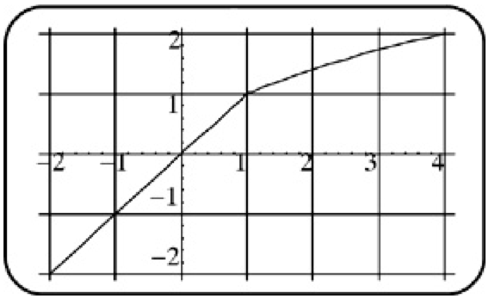

Figure 7.5.

Lesson 7-4 Review

Use the definition of the derivative (the difference quotient) to find the derivative of the following functions:

1. Does the derivative of the function $f(x) = \begin{cases} x^2 + 1 & x < 0 \\ 2x + 1 & x \geq 0 \end{cases}$ exist at $x = 0$?
2. Does the derivative of the function $f(x) = \begin{cases} x^2 & x < 0 \\ x^3 & x \geq 0 \end{cases}$ exist at $x = 0$?.

Lesson 7-5: The Derivative and Tangent Line Equations

To find the equation of a line, we need two things: a point and a slope. When finding the equation of the line tangent to the graph of $f(x)$ at $x = a$, the point that the line passes through is $(a, f(a))$. The slope of the line tangent to the graph of $f(x)$ at $x = a$ is $f(a)$. Using these two pieces of information, we can use the point-slope equation for a line:

$$y - f(a) = f'(a)(x - a)$$

Realize that the quantities $a, f(a)$ and $f'(a)$ are all *numbers*. The equation of a tangent line must actually be the equation of a line!

Example 1

Find the equation of the line tangent to the graph of $f(x) = x^2 + 1$ at $x = 3$. Write your answer in slope-intercept form.

Solution: We need a point and a slope. The point is $(3, f(3))$, or $(3, 10)$. The slope is $f'(3)$, which we will find using the definition of the derivative:

Start with the definition of the derivative

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

Substitute in for $f(3 + h)$ and $f(3)$

$$f'(3) = \lim_{h \to 0} \frac{\left((3+h)^2 + 1\right) - 10}{h}$$

Expand $(3 + h)^2$

$$f'(3) = \lim_{h \to 0} \frac{\left(9 + 6h + h^2 + 1\right) - 10}{h}$$

Simplify

$$f'(3) = \lim_{h \to 0} \frac{\left(h^2 + 6h + 10\right) - 10}{h}$$

Factor an h out of both terms in the numerator

$$f'(3) = \lim_{h \to 0} \frac{h^2 + 6h}{h}$$

Cancel the h's

$$f'(3) = \lim_{h \to 0} \frac{\cancel{h}(h+6)}{\cancel{h}}$$

Evaluate the limit

$$f'(3) = \lim_{h \to 0} (h+6) = 6$$

The slope of the tangent line is 6. Now we can substitute into the point-slope formula for a line and simplify:

The point is (3, 10) and the slope is 6

$$y - 10 = 6(x - 3)$$

$$y - 10 = 6x - 18$$

Simplify

$$y = 6x - 18$$

The equation of the line tangent to the graph of $f(x) = x^2 + 1$ at $x = 3$ is $y = 6x - 8$.

Figure 7.6 shows the graphs of $f(x) = x^2 + 1$ and the line $y = 6x - 8$. Notice that the line just touches the graph of $f(x) = x^2 + 1$ at $x = 3$.

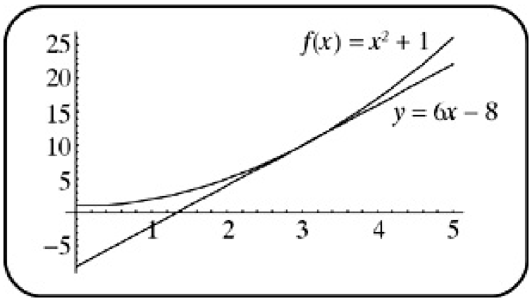

Figure 7.6.

Example 2

Find the equation of the line tangent to the graph of $f(x) = e^x$ at $x = 0$. Write your answer in slope-intercept form.

Solution: In order to find the equation of a tangent line, we need a point and a slope. The point is (0, f(0)), or (0, 1). We can use Example 3 of Lesson 7-3 to find the slope of the tangent line. We derived a formula for the derivative of $f(x) = e^x$: $f'(x) = e^x$. We can use this formula to find the derivative (or the slope of the tangent line) for any value of x: $f(0) = e^0 = 1$. So the slope is 1, and the equation of the line tangent to the graph of $f(x) = e^x$ at $x = 0$ is:

$$y - 1 = 1(x - 0)$$
$$y = x + 1$$

Figure 7.7 shows the graph of the function $f(x) = e^x$ and the line $y = x + 1$. Notice that the line just touches the graph of $f(x) = e^x$ at $x = 0$.

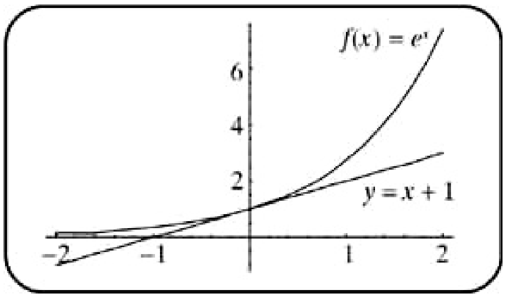

Figure 7.7.

Example 3

Find the equation of the line tangent to the graph of $g(x) = \sin x$ at $x = 0$. Write your answer in slope-intercept form.

Solution: In order to find the equation of a tangent line, we need a point and a slope. The point is (0, g(0)), or (0, 0). We can use Example 4 of Lesson 7-3 to find the slope of the tangent line. We derived a formula for the derivative of $g(x) = \sin x$: $g'(x) = \cos x$. We can use this formula to find the derivative (or the slope of the tangent line) for any value of x: $g'(0) = \cos 0 = 1$. So the slope is 1, and the equation of the line tangent to the graph of $g(x) = \sin x$ at $x = 0$ is:

$$y - 0 = 1(x - 0)$$
$$y = x$$

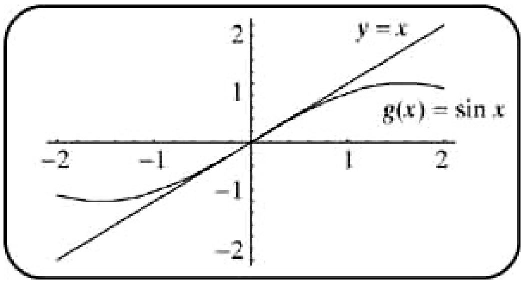

Figure 7.8.

Figure 7.8 shows the graph of the function $g(x) = \sin x$ and the line $y= x$. Notice that the line just touches the graph of $g(x) = \sin x$ at $x = 0$.

Example 4

Find the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$. Write your answer in slope-intercept form.

Solution: In order to find the equation of a tangent line, we need a point and a slope. The point is (4, f(4)), or (4, 2). We can use Example 2 of Lesson 7-3 to find the slope of the tangent line. We derived a formula for the derivative of $f(x) = \sqrt{x}$: $f'(x) = \frac{1}{2\sqrt{x}}$. We can use this formula to find the derivative (or the slope of the tangent line) for any value of x:. $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. So the slope is $\frac{1}{4}$, and the equation of the line tangent to the graph of $f(x) = \sqrt{x}$ at $x = 4$ is:

$$y - 2 = \tfrac{1}{4}(x - 4)$$
$$y - 2 = \tfrac{1}{4}x - 1$$
$$y = \tfrac{1}{4}x + 1$$

Figure 7.9 shows the graph of the function $f(x) = \sqrt{x}$ and the line $y = \frac{1}{4}x + 1$. Notice that the line just touches the graph of $f(x) = \sqrt{x}$ at $x = 4$.

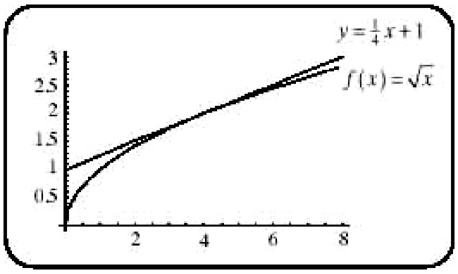

Figure 7.9.

Lesson 7-5 Review

Use the Lesson 7-3 Review problems to find the equation of the line tangent to the graphs of the following functions at the point indicated:

1. $f(x) = x^2 + 3x$, $x = 2$
2. $f(x) = \sqrt{x+2}$, $x = 2$
3. $f(x) = e^{x+3}$, $x = -3$
4. $f(x) = \frac{1}{x}$, $x = 1$
5. $g(x) = \cos x$, $x = \frac{\pi}{2}$

Lesson 7-6: Notation

Calculus has developed over the years, and, as calculus has changed, so has its notation. I have been using $f'(x)$ to denote the derivative of the function $f(x)$. We usually write $f'(a)$ when we talk about the derivative of the function $f(x)$ at the point $x = a$. We can also write the derivative of the function $f(x)$ with respect to x as $\frac{dx}{dx}$. This notation is called **Leibniz notation,** named after one of the mathematicians credited with developing calculus. The phrase "with respect to x" means that the independent variable in our function is x. This notation suggests that the derivative is a fraction. Because the derivative is a limit of a difference quotient, this notation follows naturally. If we write $\Delta f = f(x + h) - f(x)$, and $\Delta x = (x + h) - x$, then we can write the derivative as:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The derivative of the function $f(x) = x$ with respect to x can be thought of as $\frac{dx}{dx}$, and if we pretend that this is a fraction, we see that the numerator and the denominator are the same thing, so we would expect this value to be 1. In fact, we can use the definition of the derivative to show that the derivative of the function $f(x) = x$ with respect to x is, in fact, 1:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)-x}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

If we want to express the derivative of the function $f(x)$ at the point $x = a$ using Leibniz notation, we write $\left.\frac{df}{dx}\right|_{x=a}$. In other words, $f'(a) = \left.\frac{df}{dx}\right|_{x=a}$. But wait! There's more! If we write $y = f(x)$, then the derivative of $f(x)$ with respect to x can also be written as y' or $\frac{dy}{dx}$. If our function is actually a function of time, so that t is the independent variable instead of x, then we would be taking the derivative with respect to t, and we would write $f'(t)$, $\frac{dx}{dt}$ y' or $\frac{dy}{dt}$.

To make matters worse, physicists also use dot notation to represent the derivative with respect to time: $\dot{y} = \frac{dy}{dt}$. Fortunately, they reserve this notation for derivatives with respect to time only, so there is no confusion between y' and $\dot{y}$. We also write $f'(x)$ as $\frac{d}{dx}[f(x)]$, $D_x[f(x)]$, and $D_x y$. The subscript x specifies the independent variable, or the variable that the derivative is taken with respect to. Keep in mind that these variables are "dummy" variables, meaning that these variables can be interchanged. If, for whatever reason, y is the independent variable, then we could talk about $f'(y)$, or $\frac{df}{dy}$. Leibniz notation helps keep track of the players: which is the independent variable, and which variable is dependent.

The notation for the derivative has evolved over the years, and everyone has a favorite way to represent the derivative. There are times when one particular type of notation works better than another, which is why there are so many different symbols for the derivative.

Answer Key

Lesson 7-1 Review

1.
a. $f(x)=2^x$ over $[1,3]$: $\frac{f(3)-f(1)}{3-1}=\frac{8-2}{2}=3$

b. $g(x)=\sin x$ over $\left[0,\frac{\pi}{2}\right]$: $\frac{g\left(\frac{\pi}{2}\right)-g(0)}{\frac{\pi}{2}-0}=\frac{\sin\frac{\pi}{2}-\sin 0}{\frac{\pi}{2}}=\frac{1}{\frac{\pi}{2}}=\frac{2}{\pi}$

c. $h(x)=\log_3 x$ over $\left[\frac{1}{3},3\right]$: $\frac{h(3)-h\left(\frac{1}{3}\right)}{3-\frac{1}{3}}=\frac{\log_3 3-\ln\frac{1}{3}}{\frac{8}{3}}=\frac{1-(-1)}{\frac{8}{3}}=\frac{2}{\frac{8}{3}}=\frac{3}{4}$

2.
a. $g(x)=x^2-2$ over $[0,2]$: $\frac{g(2)-g(0)}{2-0}=\frac{2-(-2)}{2}=2$

b. $f(x)=\tan x$ over $\left[0,\frac{\pi}{4}\right]$: $\frac{f\left(\frac{\pi}{4}\right)-f(0)}{\frac{\pi}{4}-0}=\frac{\tan\frac{\pi}{4}-\tan 0}{\frac{\pi}{4}}=\frac{1}{\frac{\pi}{4}}=\frac{4}{\pi}$

c. $h(x)=\log_2 x$ over $[1,4]$: $\frac{h(4)-h(1)}{4-1}=\frac{\log_2 4-\log_2 1}{3}=\frac{2-0}{3}=\frac{2}{3}$

Lesson 7-3 Review

1. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2+3(x+h)-(x^2+3x)}{h}=2x+3$

2. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}=\lim_{h\to 0}\frac{\sqrt{x+h+2}-\sqrt{x+2}}{h}\cdot\frac{\sqrt{x+h+2}+\sqrt{x+2}}{\sqrt{x+h+2}+\sqrt{x+2}}=\frac{1}{2\sqrt{x+2}}$

3. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h+3}-e^{x+3}}{h}=\lim_{h\to 0}\frac{e^{x+3}\left(e^{h}-1\right)}{h}=e^{x+3}$

4. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{\frac{x}{x(x+h)}-\frac{(x+h)}{x(x+h)}}{h}=\lim_{h\to 0}\frac{-h}{hx(x+h)}=-\frac{1}{x^2}$

5. Use the sum of two angles formula for the cosine function:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}&=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\&=\lim_{h\to 0}\left[(\cos x)\frac{(\cos h-1)}{h}-(\sin x)\frac{\sin h}{h}\right]\\&=(\cos x)\cdot 0-(\sin x)\cdot 1=-\sin x\end{aligned}$$

Lesson 7-4 Review

1. Evaluate the derivative using the definition:

$$f'(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(h^2+1)-(1)}{h}=\lim_{h\to 0^-}\frac{h^2}{h}=0$$

$$f'(0)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{(2h+1)-(1)}{h}=\lim_{h\to 0^+}\frac{2h}{h}=2$$

Because these two limits are not the same, the derivative *does not* exist at $x = 0$.

2. Evaluate the derivative using the definition:

$$f'(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(h^2)-(0)}{h}=\lim_{h\to 0^-}\frac{h^2}{h}=0$$

$$f'(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{(h^2)-(0)}{h}=\lim_{h\to 0^-}\frac{h^2}{h}=0$$

Because the two limits are the same, the derivative *does* exist at $x = 0$.

Lesson 7-5 Review

1. The point is (2, 10) and $f'(x) = 2x + 3$, so the slope is $f'(2) = 7$.
   The equation of the tangent line is $y - 10 = 7(x - 2)$, or $y = 7x - 4$.

2. The point is (2, 2) and $f'(x) = \frac{1}{2\sqrt{x+2}}$, so the slope is $f'(2) = \frac{1}{4}$.
   The equation of the tangent line is $y - 2 = \frac{1}{4}(x - 2)$, or $y = \frac{1}{4}x + \frac{3}{2}$.

3. The point is $(-3, 1)$ and $f'(x) = e^{x+3}$, so the slope is $f'(-3) = 1$.
The equation of the tangent line is $y - 1 = 1(x + 3)$, or $y = x + 4$.

4. The point is $(1,1)$ and $f'(x) = -\frac{1}{x^2}$, so the slope is $f'(1) = -1$.
The equation of the tangent line is $y - 1 = -1(x - 1)$, or $y = -x + 2$.

5. The point is $(\frac{\pi}{2}, 0)$ and $f'(x) = -\sin x$, so the slope is $f'(\frac{\pi}{2}) = -1$.
The equation of the tangent line is $y - 0 = -1(x - \frac{\pi}{2})$, or $y = -x + \frac{\pi}{2}$.