CHAPTER 2

VECTORS

We shall begin with vectors. Then we will come across with plane and space vectors. It is very important to define addition and scalar multiplication of vectors and product of vectors. The importance of vectors and geometry of space will be discussed, which is an important tool in preliminary Mathematics.

2.1. Vectors

A quantity which is completely specified by its magnitude only is called a scalar, whereas a quantity which is completely specified by its magnitude and direction is called a vector.

Vectors are generally represented geometrically as directed line segments. For example, a vector can represented by a straight line AB with an arrowhead as shown in the diagram. Then the vector is written as $\overrightarrow{AB}$ having $|\overrightarrow{AB}|$ or AB as the magnitude of the vector. Here A is called **initial point** and B is called **Terminal point.** If we denote $\overrightarrow{AB}$ by $\vec{v}$ or **v** then vector $\overrightarrow{BA}$ represents the negative of $\overrightarrow{AB}$ i.e. $-\vec{v}$ or $-\mathbf{v}$.

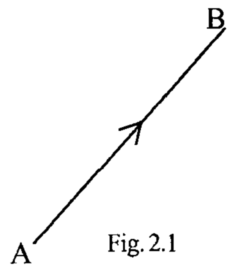

Fig. 2.1

A vector of unit magnitude is called an unit vector i.e., if $|\vec{a}| = 1$ then $\vec{a}$ is an unit vector. Generally unit vectors are used to represent the direction of any vector.

2.2. Plane and Space Vectors

A plane vector is an ordered pair of real numbers. If $\overrightarrow{AB}$ is a plane vector and co-ordinates of A are (x_1, x_2) and co-ordinates of B are (y_1, y_2) then

$$|\overrightarrow{AB}| = |\overrightarrow{BA}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

A space vector is an ordered triple (x_1, x_2, x_3) of real numbers. If co-ordinates of A and B are respectively (x_1, x_2, x_3) and (y_1, y_2, y_3) then

$$|\overrightarrow{AB}| = \sqrt{(x_1 - y_1)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} = |\overrightarrow{BA}|$$

Here, there is no distinction between a point in a plane or a point in space we can consider a vector in R^2 or R^3 i.e.,

$$u = (a_1, a_2) \in R^2 \text{ and } v = (a_1, a_2, a_3) \in R.$$

Two vectors (plane or space) $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel if the lines AB and CD are parallel.

Two parallel vectors in the same line may have same direction or opposite directions. All parallel vectors may not have same length.

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ have same length and same direction then $\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$.

If the initial points of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ coincide and terminal points of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ coincide, then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are said to be equal i.e., $\overrightarrow{AB} = \overrightarrow{CD}$.

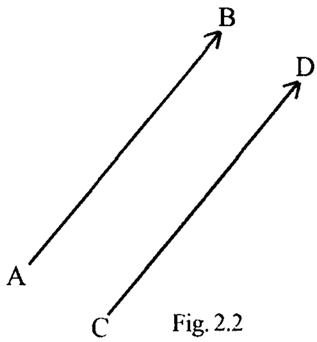

Fig. 2.2

Note : Equivalent vectors have same magnitude and direction but may have different initial and terminal points.

A vector drawn from origin O to a point P in a (plane or space) is called the position vector of the point P. If (a_1, a_2, a_3) are co-ordinates of a point P, then $\overrightarrow{OP} = (a_1, a_2, a_3)$ is the position vector of P.

Since any vector can be drawn from origin, therefore any vector (in a plane) $v = (v_1, v_2)$ can be regarded as a position vector of a point having co-ordinates (v_1, v_2). Also vector (in space) $v = (v_1, v_2, v_3)$ is the position vector of a point having co-ordinates (v_1, v_2, v_3).

A vector of unit length is called a **unit vector**. If $\overrightarrow{AB}$ is a vector, then $\dfrac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$ is a unit vector in the direction of $\overrightarrow{AB}$.

Example 2.2.1 : If $v = (3, 4)$,

$$|v| = \sqrt{3^2 + 4^2} = 5$$

Example 2.2.2 : If $v = (1, 2, 2)$,

$$|v| = \sqrt{1^2 + 2^2 + 2^2} = 3$$

Example 2.2.3. : The vector $v = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ is a unit vector since its length is 1.

2.3. Sum and scalar multiple of vectors :

The sum of two plane vectors $v_1 = (a_1, a_2)$ and $v_2 = (b_1, b_2)$ is defined as $v_1 + v_2 = (a_1 + b_1, a_2 + b_2)$. Similarly the sum of two space vectors $v_1 = (a_1, a_2, a_3)$ and $v_2 = (b_1, b_2, b_3)$ is defined as $v_1 + v_2 = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Thus the sum of two vectors of two or three dimensions is also a vector of two or three dimension.

Generally, the sum of two vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ is given by $\overrightarrow{OR}$, the diagonal of the parallelogram whose adjacent sides are OP and OQ. If P and Q have co-ordinates (a_1, a_2) and (b_1, b_2), respectively, then by elementary geometry, the co-ordinates of R are $(a_1 + b_1, a_2 + b_2)$. This fact is refelcted in Fig 2.3.

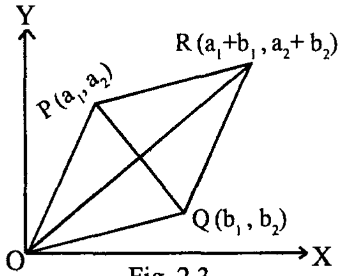

Fig. 2.3

In figure 2.3

$$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$$
$$\Rightarrow \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= (b_1, b_2) - (a_1, a_2)$$
$$= (b_1 - a_1, b_2 - a_2)$$

The scalar multiple of a vector $v = (v_1, v_2)$ with respect to a scalar λ is defined as $\lambda v = (\lambda v_1, \lambda v_2)$. An analogous situation arises in space.

Geometrically, if $v = (v_1, v_2) = \overrightarrow{OP}$, then $\lambda\,\overrightarrow{OP}$ with $\lambda > 0$ is defined as the vector whose length is λ times the length of $\overrightarrow{OP}$, i.e., $\lambda\,|\overrightarrow{OP}|$ whose direction is same as that of $\overrightarrow{OP}$. If $\lambda < 0$, then $\lambda\,\overrightarrow{OP}$ or λ v denotes a vector of length $-\lambda\,|\overrightarrow{OP}|$ in a direction opposite to that of v. This fact is reflected in Fig 2.4.

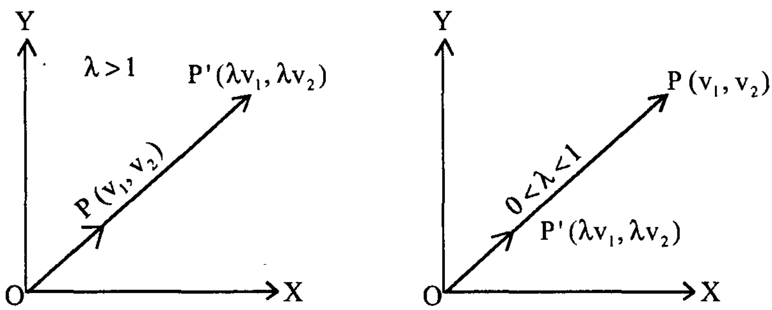

Fig. 2.4

Example 2.3.1 : The sum of vectors (3, −1) and (1, 4) is the vector

$$(3,-1)+(1,4)=(3+1,-1+4)=(4,3)$$

Example 2.3.2 : The sum of vectors (1, 3, 4) and (2, −1, −2) is the vector

$$(1,3,4)+(2,-1,-2)=(1+2,3-1,4-2)=(3,2,2).$$

Example 2.3.3 : The scalar multiple of the vector (2, 3) by the scalar 4 is the vector

$$4(2,3)=(8,12)$$

Example 2.3.4 : The scalar multiple of the vector (4, 6) by the scalar $\frac{1}{2}$ is the vector $\frac{1}{2}(4,6)=\left(\frac{4}{2},\frac{6}{2}\right)=(2,3).$

Example 2.3.5 : The vector $v=\left(\frac{\sqrt{2}}{10},\frac{-7\sqrt{2}}{10}\right)$ is a unit vector since

$$|v|=\sqrt{\left(\frac{\sqrt{2}}{10}\right)^2+\left(\frac{-7\sqrt{2}}{10}\right)^2}=1$$

Example 2.3.6 : If $u=(a_1, a_2)$, then the unit vector in the direction of u is

$$\frac{u}{|u|}=\frac{(a_1,a_2)}{\sqrt{a_1^2+a_2^2}}=\left(\frac{a_1}{\sqrt{a_1^2+a_2^2}},\frac{a_2}{\sqrt{a_1^2+a_2^2}}\right)$$

Example 2.3.7 : If $u=(a_1, a_2, a_3)$, then $|u|=\sqrt{a_1^2+a_2^2+a_3^2}$. Thus the unit vector in the direction of u is $\frac{u}{|u|}=\frac{(a_1,a_2,a_3)}{\sqrt{a_1^2+a_2^2+a_3^2}}$

$$=\left(\frac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}},\frac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}},\frac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}\right)$$

Example 2.3.8 : If u = (1, 1, 1), the unit vector in the direction of u is

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

Example 2.3.9 : If u = (4, 3), the unit vector in the direction of u is $\left(\frac{4}{5},\frac{3}{5}\right)$ since

$$|u|=\sqrt{4^2+3^2}=5.$$

In a plane the unit vector in the direction of an angle θ is $(\cos\theta, \sin\theta)$ as shown in Fig 2.5. It is also called the unit vector at an angle θ with the positive direction of the x-axis. Putting $\theta=0$, $(\cos\theta, \sin\theta)=(1, 0)$ is the unit vector in the positive direction of x-axis. Putting $\theta=\pi/2$, $(\cos\theta,\sin\theta)=(0,1)$ is the unit vector in the positive direction of y-axis. They are denoted by i and j respectively, i.e. i = (1, 0) and j = (0, 1) as in Fig. 2.6.

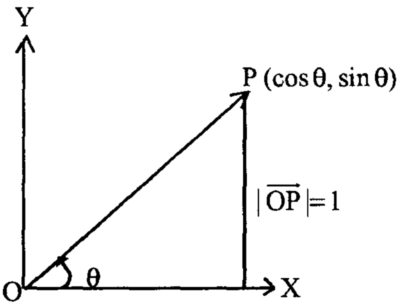

Fig. 2.5

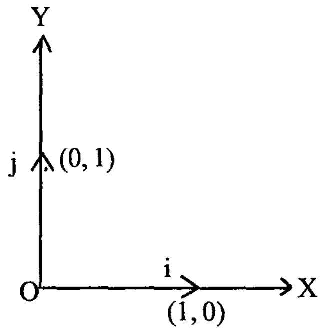

Fig. 2.6

In space the unit vectors along the positive directions of the x-axis, the y-axis and the z-axis are respectively (1, 0, 0), (0, 1, 0), and (0, 0, 1). They are denoted by i, j and k respectively, i.e. i = (1, 0, 0), j = (0, 1, 0) and k = (0, 0, 1) as in Fig 2.7.

The direction of a non-zero vector in V_2 is the radian measure $\theta, 0 \le \theta < 2\pi$ of the angle from the positive direction of the x-axis to the vector $\overrightarrow{OP}$ measured counter-clockwise. If $u = (a_1, a_2) = \overrightarrow{OP}$, then the direction θ of u is given by

$$\sin\theta = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad \cos\theta = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}$$

In V_3 the direction of a vector is given by its direction cosines which shall be studied in the article 2.5.

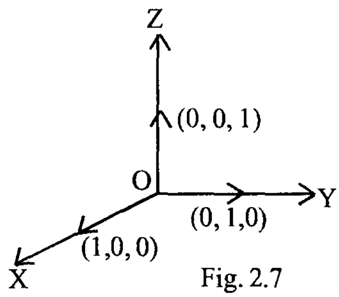

Fig. 2.7

Example 2.3.10 : Find the direction and magnitude of the vector u = (3, 4).

Solution : $|u| = \sqrt{3^2 + 4^2} = 5$

Direction of u :

$$\sin\theta = \frac{4}{\sqrt{3^2 + 4^2}} = \frac{4}{5}$$

$$\cos\theta = \frac{3}{5}$$

Example 2.3.11 : Find the unit vector in the direction $\theta = \frac{\pi}{3}$

Solution : $\sin\theta = \sin\frac{\pi}{3} = \frac{1}{2}$

$$\cos\theta = \cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$(\cos\theta, \sin\theta) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ is the required unit vector.

Example 2.3.12 : $(1, 2) = i + 2j$ since

$$\begin{aligned}(1, 2) &= (1, 0) + (0, 2)\\ &= (1, 0) + 2\,(0, 1)\\ &= i + 2j\end{aligned}$$

Example 2.3.13 : $(1, 2, 3) = i + 2j + 3k$ since

$$\begin{aligned}(1, 2, 3) &= (1, 0, 0) + (0, 2, 0) + (0, 0, 3)\\ &= (1, 0, 0) + 2\,(0, 1, 0) + 3\,(0, 0, 1)\\ &= i + 2j + 3k\end{aligned}$$

Example 2.3.14 : $(2, 4, 5) = 2i + 4j + 5k$ since

$$\begin{aligned}(2, 4, 5) &= (2, 0, 0) + (0, 4, 0) + (0, 0, 5)\\ &= 2\,(1, 0, 0) + 4\,(0, 1, 0) + 5\,(0, 0, 1)\\ &= 2i + 4j + 5k.\end{aligned}$$

Example 2.3.15 :
$$\begin{aligned}(a_1, a_2) &= (a_1, 0) + (0, a_2)\\ &= a_1\,(1, 0) + a_2\,(0, 1)\\ &= a_1 i + a_2 j\end{aligned}$$

Example 2.3.16 :
$$\begin{aligned}(a_1, a_2, a_3) &= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3)\\ &= a_1\,(1, 0, 0) + a_2 (0, 1, 0) + a_3\,(0, 0, 1)\\ &= a_1 i + a_2 j + a_3 k.\end{aligned}$$

From the above examples we observe the following.

Example 2.3.17 : Express the vector (1, 2, 3) as sum of scalar multiples of $u = (1, 1, 0)$, $v = (1, 0, 1)$, $w = (0, 1, 1)$

Solution : Let
$$\begin{aligned}(1, 2, 3) &= \alpha u + \beta v + \gamma w\\ &= \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1)\\ &= (\alpha + \beta, \alpha + \gamma, \beta + \gamma)\end{aligned}$$

$$\begin{aligned}\therefore\ & \alpha + \beta = 1\\ & \alpha + \gamma = 2\\ & \beta + \gamma = 3\\ \Rightarrow\ & \alpha + \beta + \gamma = 3\\ \Rightarrow\ & \alpha = 0, \beta = 1, \gamma = 2\\ \therefore\ & (1, 2, 3) = 0.u + 1.v + 2.w.\end{aligned}$$

Note : (a) Every plane vector is of the from $a_1 i + a_2 j$ and every vector of this form is a plane vector.

(b) Every space vector is of the form $a_1 i + a_2 j + a_3 k$ and every vector of this form is a space vector.

The numbers a_1, a_2 are called components of the plane vector

$$u = (a_1, a_2) = a_1 i + a_2 j.$$

a_1 is called the i-component (or x-component) and a_2 is called the j-component (or y-component). Also a_1, a_2 are regarded as co-ordinates of u with respect to i and j.

The numbers a_1, a_2 and a_3 are called components of the space vector

$$u = (a_1, a_2, a_3) = a_1 i + a_2 j + a_3 k.$$

a_1 is the i-component, a_2 is the j-component and a_3 the k-component. We also say that a_1, a_2, a_3 are co-ordinates of u with respect to i, j, k respectively.

2.3.1 Algebraic Properties of sum :

$V_2(V_3)$, the set of all plane (space) vectors under the operation of addition, is a commutative group. Group properties for V_2 are written below.

The reader shall check these properties for V_3.

G1 Closure :

V_2 is closed under vector addition.

Since $v_1, v_2 \in V_2 \Rightarrow v_1 + v_2 \in V_2$.

G2 Vector addition is associative :

For any three vectors $u = (a_1, a_2), v = (b_1, b_2)$

and $w = (c_1, c_2)$ in V_2,

$$\begin{aligned}(u+v)+w &= ((a_1, a_2) + (b_1, b_2)) + (c_1, c_2)\\ &= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2)\\ &= (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2))\end{aligned}$$

(Since associative law under addition holds in the set of all real numbers)

$$\begin{aligned}&= (a_1, a_2) + ((b_1 + c_1), (b_2 + c_2))\\ &= (a_1, a_2) + ((b_1, b_2) + (c_1, c_2))\\ &= u + (v + w).\end{aligned}$$

Therefore $(u + v) + w = u + (v + w)$

for all $u, v, w \in V_2$.

G3 Existence of Identity :

For any $u = (a_1, a_2)$ in V_2

we have $(0, 0) + (a_1, a_2) = (a_1, a_2) = (a_1, a_2) + (0, 0)$.

So zero vector (0, 0) plays the role of identity. The zero vector is usually denoted by 0.

G4 Existence of inverse :

Given a vector $u = (a_1, a_2)$,

the vector $x = (-a_1, -a_2)$ satisfies $x + u = 0 = u + x$.

The vector x is called additive inverse of u or the negative of u and is denoted by – u.

G5 Vector addition is commutative

Let $u = (a_1, a_2)$ and $v = (b_1, b_2)$ be two vectors in V_2. Then

$u + v = (a_1, + b_1, a_2 + b_2)$

$v + u = (b_1, + a_1, b_2 + a_2)$

But we know that for real numbers $a_1, a_2, b_1, b_2, a_i + b_i = b_i + a_i, i = 1, 2$

Therefore $u + v = v + u$ for all $u, v \in V_2$.

Thus V_2, under the operation '+' is a commutative group. i.e., $(V_2, +)$ is a commutative group.

2.3.2 Properties under Scalar multiplication :

For all vectors $u, v \in V_2$ (or in V_3) and scalars α, β we have

(a) $\alpha(u + v) = \alpha u + \beta v$

(b) $(\alpha + \beta)\, v = \alpha v + \beta v$

(c) $\alpha(\beta u) = \alpha\beta u = \beta(\alpha u)$

(d) $1 . u = u$

(e) $0 . u = \mathbf{0}$

Let us justify the above properties in V_3 only.

Let $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ be two vectors in V_3 and α be a scalar.

Then $\quad u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$$\alpha(u + v) = ((\alpha(a_1 + b_1), \alpha\,(a_2 + b_2), \alpha(a_3 + b_3))$$

(by properties of real numbers)

$$= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3)$$

$$= (\alpha a_1, \alpha a_2, \alpha a_3) + (\alpha b_1, \alpha b_2, \alpha b_3)$$

$$= \alpha\,(a_1, a_2, a_3) + \alpha\,(b_1, b_2, b_3)$$

$$= \alpha u + \alpha v.$$

Thus property of (a) is justified.

Properties (b) and (c) are trivial and their justification are left to the reader.

$$1.\ x = 1\ (a_1, a_2, a_3)$$

$$= (1a_1,\ 1a_2,\ 1a_3)$$

$$= (a_1, a_2, a_3)$$

and $0\ u = 0\ (a_1, a_2, a_3)$

$$= (0\ a_1, 0\ a_2, 0\ a_3)$$

$$= (0, 0, 0)$$

Problem Set 2 (A)

1. Find the magnitude and direction of the following vectors :

(a) $(2, -3)$ (b) $(3, -1)$ (c) $(3, 6)$ (d) $(-1, 2)$

(e) $i + j$ (f) $2i + 3j$ (g) $-i + 4j$ (h) $4i - 2j$

2. Find the magnitude of the following space vectors :

   (a) (1, 2, 3) (b) (1, 0, 2) (c) $2i - j + 2k$ (d) $-i + 2j - 4k$

3. Simplify :

   (a) 2 (1, 6) (b) 3 (1, –2, 5) (c) 3 (–1, 1) + 4 (3, 2)

   (d) 3 (–1, 2, 3) + 4 (2, – 1, 0) (e) $4(2i + 3j - k) - 3(2i + j + k)$

4. Find the unit vector u in the direction of each of the vectors in Problem 1 and 2.

5. Express the following vectors interms of the unit vectors i, j and k.

   (a) (2, 3) (b) (–1, 4) (c) (3, – 5)

   (d) (3, 5, 2) (e) (1, 0, 2) (f) (1, – 2, 0)

6. Express the following space vectors as sums of scalar multiples of $u = (1, 1, 1)$ $v = (1, 1, 0)$ $w = (2, 0, 3)$

   (a) (2, 3, 4) (b) (6, 1, 3) (c) (2, 1, 2)

   (d) $2i + j + k$ (e) $i - j + 3k$ (f) $3i - 2j - 3k$

7. Find the unit vector in the direction

   (a) $\theta = -\dfrac{\pi}{4}$ (b) $\theta = \dfrac{\pi}{6}$ (c) $\theta = \dfrac{5\pi}{4}$

8. Prove that $\alpha u = 0$ iff $\alpha = 0$ or $u = 0$.

9. If u be a non-zero vector in V_2, then prove that the set $S = \{\alpha u \mid \alpha \in R\}$ is a group under vector addition.

10. Let u and v be two non-zero vectors in V_3. Then prove that the set $S = \{\alpha u + \beta v \mid \alpha, \beta \in R\}$ is a group under vector addition.

2.4. Dot Product of Vectors

2.4.1 Definition :

Let u and v be two non-zero vectors and θ be the angle between them. Then the dot product (scalar product or inner product) written as u . v is defined by

$$u \cdot v = |u||v|\cos\theta$$

$$\text{If } \theta = \frac{\pi}{2}, \text{ then } \cos\theta = 0, \text{ So } u \cdot v = 0.$$

Thus u . v = 0 if u is perpendicular to v. i.e. if u and v are orthogonal.

2.4.2. Defnition :

Let $u = (a_1, a_2)$ and $v = (b_1, b_2)$ be two vectors of V_2. Then the dot product u . v is defined by
$u \cdot v = a_1 b_1 + a_2 b_2$
If $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ are two vectors of V_3, then u . v is defined by
$u \cdot v = a_1b_1 + b_2b_2 + a_3b_3$.

Example 2.4.1 : Let u = (2, 3), v = (5, 2). Then the angle θ between the vectors u and v is given by

$$|u||v|\cos\theta = u.v$$

$$\Rightarrow \cos\theta = \frac{u.v}{|u||v|}$$

$$= \frac{2\times5+3\times2}{\sqrt{2^2+3^2}\sqrt{5^2+2^2}} = \frac{16}{\sqrt{13}\sqrt{29}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{16}{\sqrt{13}\sqrt{29}}\right]$$

Example 2.4.2 : Let u = (1, 2, –1), v = (3, –1, 1). Then the angle θ between the vectors u and v is given by

$$\cos\theta = \frac{u.v}{|u|\,|v|} \frac{1\times3+2\times(-1)+(-1)\times1}{\sqrt{1^2+2^2+(-1)^2}\sqrt{3^2+(-1)^2+1^2}}$$

$$= \frac{0}{\sqrt{6}\sqrt{11}} = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

So u and v are perpendicular to each other i.e., u and v are orthogonal.

2.3.3 Properties of dot product :

Let u, v, w be vectors in V_3 (or in V_2) and α, β be real numbers. Then

(a) $u.u = |u|^2$, hence $u.u \geq 0$

(b) $u.u = 0$ iff $u = 0$

(c) $u.v = v.u$

(d) $u.(v+w) = u.v + u.w$

(e) $(\alpha u).v = \alpha(u.v) = u.(\alpha v)$

Let us justify these properties one by one

Let $u = (a_1, a_2, a_3)$, $v = (b_1, b_2, b_3)$, $w = (c_1, c_2, c_3)$ be vectors in V_3 and α be a scalar (a real number).

(a) $u.u = a_1^2 + a_2^2 + a_3^2 = |u|^2 \geq 0$

(b) If u = (0, 0, 0), then u . u = 0.

Conversely, If $u.u = a_1^2 + a_2^2 + a_3^2 = 0$, then $a_1 = a_2 = a_3 = 0$.

Hence u = (0, 0, 0)

(c) $u.v = a_1b_1 + a_2b_2 + a_3b_3$

$$= b_1a_1 + b_2a_2 + b_3a_3 = v.u.$$

(d) $u.(v+w) = a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$
$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$
$= u.v + u.w.$

(e) $(\alpha u).v = (\alpha a_1)b_1 + (\alpha a_2)b_2 + (\alpha a_3)b_3$
$= \alpha(a_1b_1) + \alpha(a_2b_2) + \alpha(a_3b_3)$
$= \alpha(a_1b_1 + a_2b_2 + a_3b_3)$
$= \alpha(u.v).$

Similarly it can be proved that $u.(\alpha v) = \alpha(u.v)$.

2.4.4 Schwarz Inequality

For any two vectors u and v of V_3 (or V_2),

$$|u.v| \le |u||v|$$

i.e., $(u.v)^2 \le (u.u)(v.v)$

Proof : If either u or v is a zero vector, there is nothing to prove. So we assume that u and v are non zero vectors. Let θ be the angle between u and v.

By definition we have

$$\cos\theta = \frac{u.v}{|u||v|}$$

But we know that $|\cos\theta| \le 1$.

$$\Rightarrow \frac{|u.v|}{|u||v|} \le 1$$

$$\Rightarrow |u.v| \le |u| + |v|$$

i.e., $(u.v)^2 \le |u|^2|v|^2 = (u.u)(v.u)$

Example 2.4.3 : The dot product of two vectors (3, 1) and (–1, 6) is

$$(3, 1).(-1, 6) = 3 \times -1 + 1 \times 6$$
$$= -3 + 6 = 3.$$

Example 2.4.4 : The dot product of two vectors (1, –1, 2) and (–1, 0, 1) is $(1, -1, 2).(-1, 0, 1) = 1 \times (-1) + (-1) \times 0 + 2 \times 1 = -1 + 0 + 2 = 1$

Example 2.4.5 : For any real numbers $a_1, a_2, a_3, b_1, b_2, b_3$ we have

$$(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Solution: Let $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$

By Schawerz inequality $(u.v)^2 \le (u.u)(v.v)$

$$\Rightarrow (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

Applying definition of dot product to the vectors i, j and k, we have

$i.i = j.j = k.k = 1$

$i.j = j.k = k.i = 0$

2.4.5 Scalar Projection :

Let $\overrightarrow{OP} = u$ and $\overrightarrow{OR} = v$.

θ be angle between u and v.

In $\Delta\, OPD$, the scalar projection of

$\overrightarrow{OP}$ on $\overrightarrow{OR}$ is $OD = OP\cos\theta \;= |u|\cos\theta$

$$= |u|\frac{u\,.\,v}{|u||v|}$$

$$= u\,.\,\frac{v}{|v|}$$

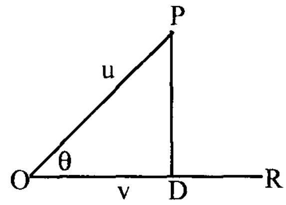

Fig. 2.8

$\therefore$ Scalar projection of u on v $= \dfrac{u\,.\,v}{|v|}$

Similarly scalar projection of v on u $= \dfrac{u\,.\,v}{|u|}$

Example 2. 3. 6 : Let $u = (1, -2, 3) = i - 2j + 3k$
and $v = (2, 1, -1) = 2i + j - k$

Then the scalar projection of u on $v = \dfrac{u\,.\,v}{|v|}$

$$= (1\times 2 + (-2)\times 1 + 3\times(-1)) \Big/ \sqrt{2^2 + 1^2 + (-1)^2} = \frac{-3}{\sqrt{6}}$$

The scalar projection of v on u $= \dfrac{u\,.\,v}{|u|} = \dfrac{-3}{\sqrt{1^2 + (-2)^2 + 3^2}} = \dfrac{-3}{\sqrt{14}}$

Problem Set 2 (B)

1. Find u . v for the vectors given :
   (a) $u = (1, -1), v = (2, 3)$
   (b) $u = i + 2j, v = 2i - 3j$
   (c) $u = -2i + 5j, v = 3i - 4j$
   (d) $u = (2, 1, 3), v = (1, 2, -1)$
   (e) $u = 6i + 4j - k, v = 2i - 3j + 5k$
   (f) $u = 2i - 3j + k, v = i + j + k$
2. Find the scalar projection of u on v for each of the bits of Q no 1.
3. Find the scalar projection of v on u for each of the bits of Q no. 1.
4. Find the angle between u and v for each of the bits of Q.No. 1.
5. Find the real number α such that the vectors u and v given as below are orthogonal.
   (a) $u = (3, \alpha, 2), \quad v = (4, 1, -1)$
   (b) $u = (1, -2, 1), \quad v = (1, 2, \alpha)$
   (c) $u = (-2, 3, 0), \quad v = (6, 4, \alpha)$
   (d) $u = (\alpha, -3, 1), \quad v = (\alpha, \alpha, 2)$

2.5 Application to Geometry

In this section we shall take up standard applications of vectors to equation of lines and planes in analytic geometry.

First we note that if P (x_1, y_1) and Q (x_2, y_2) are two points in a plane then the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\,i + (y_2 - y_1)\,j$

2.5.1 The equation of a straight line :

Consider the straight line L parallel to a given vector u and passing through a point P (Fig. 2. 9)

Let Q be any point on L.

The position vector of Q,

$r = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= v + tu$ [$\overrightarrow{OP} = v$, the position vector of P and $\overrightarrow{PQ} \parallel u$]

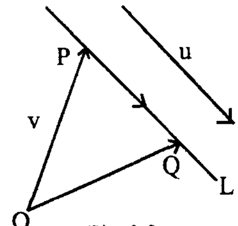

Fig. 2.9

Where t is a real number.

The equation r = v + tu ... (1) is satisfied by all points Q on the line.

So r = v + tu is the equation of the line.

Taking r = (x, y), u = (a, b) and $v = (x_1, y_1)$ the equation reduces to

$$(x, y) = (x_1, y_1) + t(a, b)$$

$$\Rightarrow \left.\begin{aligned} x &= x_1 + ta \\ y &= y_1 + tb \end{aligned}\right\} \quad \text{(comparing co-ordinates)}$$

This is **parametric equation** of a line in a plane.

Further eliminating t from the parametric equations we get

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}$$

$$\Rightarrow y - y_1 = \frac{b}{a}(x - x_1)$$

This equation is of the form $y - y_1 = m(x - x_1)$

Clearly this is cartesian equation of a line in a plane passing through the fixed point (x_1, y_1) and having slope m.

2.5.2 Equation of a line in space :

If we take r = (x, y, z), u = (a, b, c) and v = (x_1, y_1, z_1) in equation (1), then we have

$(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$

$$\Rightarrow \left.\begin{aligned} x &= x_1 + ta \\ y &= y_1 + tb \\ z &= z_1 + tc \end{aligned}\right\} \quad \text{Parametric equation of a line in space.}$$

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

This is Cartesian equation of a line in space passing through (x_1, y_1, z_1) and parallel to a fixed line (vector) (a, b, c).

Example 2.5.1 : Obtain the vector equation of a line passing through the point (1, 2) and parallel to the vector (2, 5). Then find out the Cartesian equation of the line.

Solution : Here v = (1, 2), u = (2, 5). Let r = (x, y) be position vector of any point P (x, y) on the line.

Then the vector equation of the line is r = v + tu. Where t is any real number (parameter)

Now r = v + t u

$$\Rightarrow (x, y) = (1, 2) + t(2, 5)$$

$$\Rightarrow x = 1 + 2t$$

$$y = 2 + 5t$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-2}{5}$$

$$\Rightarrow 5x - 2y = 1$$

This is Cartesian equation of the straight line.

Example 2.5.2 Obtain the vector equation and Cartesian equation of the straight line passing through the point (1, –1, 2) and parallel to the vector (1, 2, 3).

Solution :

Here u = (1, 2, 3)

v = (1, –1, 2)

The vector equation of the line is r = v + t u.

r being the position vector of a current point on the line.

Taking r = (x, y, z) the equation becomes

$$(x, y, z) = (1, -1, 2) + (1, 2, 3)\,t$$

$$\Rightarrow \left.\begin{aligned} x &= 1 + t \\ y &= -1 + 2t \\ z &= 2 + 3t \end{aligned}\right\} \quad \text{Parametric equation of the line.}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y+1}{2} = \frac{z-2}{3} \quad \text{(Eliminating t)}$$

This is Cartesian equation of the line.

2.5.3 Direction Cosines :

Let α, β, γ be inclinations of the vector $u = \overrightarrow{OA} = (a, b, c)$ to the positive directions of the co-ordinate axes OX, OY, OZ respectively as in Fig. 2.10.

Then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of the vector u = (a,b,c) = ai + bj + ck.

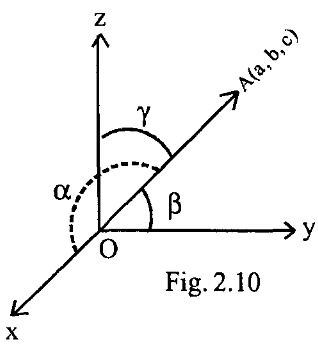

Fig. 2.10

$$\text{Clearly } \cos\alpha = \frac{u.i}{|u||i|} = \frac{a}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\beta = \frac{u.j}{|u||j|} = \frac{b}{\sqrt{a^2+b^2+c^2}}$$
$$\cos\gamma = \frac{u.k}{|u||k|} = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

It follows that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

For any real number k, the ordered set $\{k\cos\alpha, k\cos\beta, k\cos\gamma\}$ is called the set of direction ratios of the vector u or a line parallel to u. Clearly a, b, c are direction ratios of the vector ai+bj+ck.

Example 2.5.3 : Find the direction ratios and direction cosines of the vector 3i + 4j + 2k.

Solution : The direction ratios of the vector 3i + 4j + 2k are 3, 4, 2.

The direction cosines are

$$\cos\alpha = \frac{3}{\sqrt{3^2+4^2+2^2}} = \frac{3}{\sqrt{29}}$$
$$\cos\beta = \frac{4}{\sqrt{3^2+4^2+2^2}} = \frac{4}{\sqrt{29}}$$
$$\cos\gamma = \frac{2}{\sqrt{3^2+4^2+2^2}} = \frac{2}{\sqrt{29}}$$

Example 2.5.4 : Find the direction cosines of a line through the points A (1, 2, 1) and B (3, –1, –1).

Solution : $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= (3-1)\,i + (-1-2)\,j + (-1-1)\,k$$
$$= 2i - 3j - 2k$$

The direction cosines of the line are equal to direction cosines of the vector $\overrightarrow{AB}$.

So required d.c's are

$$\cos\alpha = \frac{2}{\sqrt{2^2+(-3)^2+(-2)^2}} = \frac{2}{\sqrt{17}}$$
$$\cos\beta = \frac{-3}{\sqrt{17}}$$
$$\cos\gamma = \frac{-2}{\sqrt{17}}$$

2.5.4 The equation of a plane :

Consider a plane passing through a point Q (x_1, y_1, z_1) and perpendicular to a vector $u = \overrightarrow{OA} = (a, b, c)$.

Let P (x, y, z) be any point in the plane.

Then the vector $\overrightarrow{QP} = (x - x_1) + (y - y_1)j + (z - z_1)k$ lies in the planc and is perpendicular to the vector u = ai + bj + ck

Therefore, $u\,.\,\overrightarrow{QP} = 0$

$$\Rightarrow a(x - x_1) + b\,(y - y_1) + c(z - z_1) = 0 \quad ...(2)$$

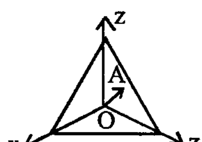

Fig. 2.11

This is satisfied by all points P on the plane and by no points outside the plane. Equation (2) of the plane can further be simplified to the form

$$ax + by + cz + d = 0$$

where $d = ax_1 - by_1 - cz_1$

Thus the equation of a plane in space is a linear equation in x, y and z.

Conversely, consider the linear equation

$$Ax + By + Cz + D = 0 \qquad \text{... (3)}$$

where A, B and C are not all zero.

Let $A \neq 0$. Then we can write the equation (3) in the form

$$A\left(x + \frac{D}{A}\right) + B(y-0) + C(z-0) = 0$$

This is the equation of a plane through the point $\left(-\frac{D}{A}, 0, 0\right)$ and perpendicular to the vector $Ai + Bj + Ck$.

Example 2.5.5 : Find the equation of a plane passing through the point (2, 3, 4) and perpendicular to the vector (4, 3, 6).

Solution : Here u = (4, 3, 6)

Let P (x, y, z) be any point on the plane Q be the given point (2, 3, 4) on the plane.

Then $\overrightarrow{QP} = (x-2)\,i + (y-3)\,j + (z-4)\,k$

The vector equation of the plane is $u \cdot \overrightarrow{QP} = 0$

$\Rightarrow 4(x-2) + 3(y-3) + 6(z-4) = 0$

This is Cartesian equation of the plane. After further simplification this equation reduces to

$4x + 3y + 6z = 41.$

Example 2.5.6 : Find the angle between two lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z+3}{2}$

and $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-1}{4}$

Solution : Here a vector parallel to the first line is $u = 3i + j + 2k$. A vector parallel to the 2nd line is $v = 2i + 3j + 4k$

The angle between two lines is same as the angle between the vectors u and v.

If θ is the angle between the lines,

$$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{3\times2 + 1\times3 + 2\times4}{\sqrt{3^2+1^2+2^2}\,\sqrt{2^2+3^2+4^2}}$$

$$= \frac{6+3+8}{\sqrt{14}\,\sqrt{29}} = \frac{17}{\sqrt{14}\,\sqrt{29}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{17}{\sqrt{14}\,\sqrt{29}}\right]$$

Example 2.5.7 : Find the angle between the planes $2x - y + 2z = 1$ and $x - y = 2$.

Solution : A vector perpendicular to the plane $2x - y + 2z = 1$ is $u = (2, -1, 2)$

A vector perpendicular to the plane $x - y = 2$ is $v = (1, -1, 0)$

The angle between two planes is same as the angle between the vectors u and v.

Thus if θ is the angle between the vectors, then

$$\cos\theta = \frac{2\times1+(-1)\times(-1)+2\times0}{\sqrt{2^2+(-1)^2+2^2}\ \sqrt{(1)^2+(-1)^2}}$$

$$= \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Problem Set 2 (C)

1. Find the direction cosines of the line passing through the points
   (a) A (2, 1, 3) and B (1, 0, 1)
   (b) A (3, 1, 2) and B (–1, 1, 3)
   (c) A (–1, 4, 5) and B (0, 0, 0)
   (d) A (1, 3, 5) and B (1, 2, 1)
2. Find the vector equation of a line passing through two given points mentioned in Q. No. 1 (a) — (d).
3. Find the angle that the following vectors make with the co-ordinate axes.
   (a) $u = 2i - 2j + k$
   (b) $u = 2i - 6j - 3k$
   (c) $u = 4i + 3j + 5k$
   (d) $u = i + j + 6k$
4. Find the angle between the lines
   (a) $\frac{x-1}{2} = \frac{y-3}{1} = \frac{z+5}{2}$ and $\frac{x}{3} = \frac{y+1}{2} = \frac{z+2}{-1}$
   (b) $\frac{x+1}{1} = \frac{y}{3} = \frac{z+1}{4}$ and $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z+5}{1}$
   (c) $\frac{x+2}{2} = \frac{y+3}{1} = \frac{z}{2}$ and $\frac{x+3}{1} = \frac{y-2}{2} = \frac{z-1}{3}$
   (d) $\frac{x}{2} = \frac{y}{4} = \frac{z}{3}$ and $\frac{x-1}{3} = \frac{y-1}{4} = \frac{z-1}{5}$

5. Find the equation of a plane passing through a point Q and perpendicular to a vector u in each of the following :

   (a) Co-ordinates of Q are (1, 2, 3) and u = (4, 1, – 1)

   (b) Co-ordinates of Q are (–1, 1, 2) and u = (2, 4, 7)

   (c) Co-ordinates of Q are (–2, 1, 1) and u = (1, –1, 4)

   (d) Co-ordinates of Q are (1, 2, 1) and u = (2, 3, –1)

6. Find the angle between the planes :

   (a) $3x - 5y + 2z = 1$ and $x + 2y + 3z = 1$

   (b) $x + 2y + z = 3$ and $2x + 3y + z = 2$

   (c) $2x + y - z = 2$ and $x + 2y - z = 1$

   (d) $x - 2y + 3z = 1$ and $2x + y - z = 2$