Week 2

Vectors, Vector Spaces

and Subspaces

Assessment 1

- Online Test
- 20 Question
- Deadline: Sunday of week 3,
- After this date, **the link for A.1 will not be accessible**
- 20%

ASSESSMENT 1 BRIEF	
Subject Code and Title	MFA501 Mathematical Foundations of AI
Assessment	Theory Test
Individual/Group	Individual
Length	60 minutes. 20 questions consisting of multiple choice questions, true/false, and fill in the blanks.
Learning Outcomes	This assessment addresses the Subject Learning Outcomes outlined at the bottom of this document.
Submission	This assessment will be made available to students from Monday, 12 noon (AEST) of Module 3. This assessment will close on Sunday, 11.55pm (AEST) of Module 3
Weighting	20%
Total Marks	100 marks

Assessment 1, Cont

Instructions:

- Review the learning resources for Module 1 & Module 2 (before undertaking the quiz).
- You only have **one attempt** at the quiz
- This quiz is made up of **20 questions**.
- You will have **60 minutes to complete** the quiz.
- This quiz will comprise of a mixture of multiple-choice, true or false and fill in the blank questions.
- The quiz attempt will **shut down at the 60 minute duration**.
- If you leave the quiz for a period of time prior to completion there **will be a forced completion.**
- Your quiz answers will be **recorded automatically** and your grade **will be generated on completion.**
- The final grade will be **20% of your overall grade.**
- The quiz will be available throughout the week of Module 3, **closing at 11.55pm AEST on Sunday evening.**
- Upon completing, students should submit their quiz for automated marking. Your **score will be available immediately** and entered in the Grade Centre.

I wish you the Best of Luck

in Assessment 1!

Today's Outline

I. Introduction to Linear Algebra

II. Information

III. Cartesian Coordinate System

IV. Vectors in Cartesian plane

V. Vectors in Cartesian space

VI. Vector Space & Subspace

VII. Tutorial

Linear Algebra

- A branch of abstract mathematics
- With a long history *(in Mathematics)*
- Deals with
  - Linear equations
  - Linear mapping
  - Vectors
  - Matrices
  - Tensors
  - Etc.
- Widely used in : *Differential equation*, *geometry*, *Calculus*, etc.

Leibnitz

Applications of Linear Algebra

- Provides a very **useful notation** for huge amount of information

I. To Compact the data set

II. Enables & facilitates operations on data

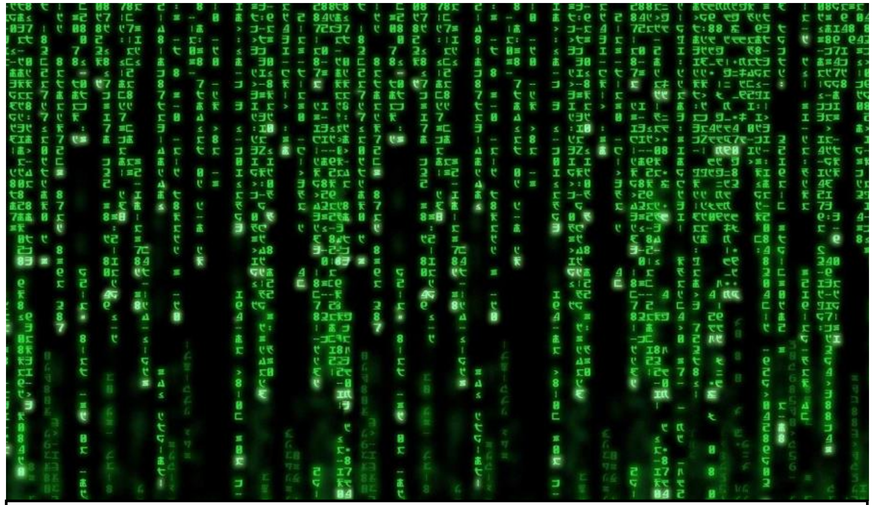
III. Provides Geometrical insight

Linear Algebra Provides a **very compact** form to express ideas in large data sets

Linear Algebra in Computer Science

Application in:

- Engineering → Modelling
- Physics → Language of nature (QM)
- Biology → Population analysis
- Data science → Compact & Manage Data
- Computer Science→ Analyse Data
- Etc.



Why do we need to compact large data sets

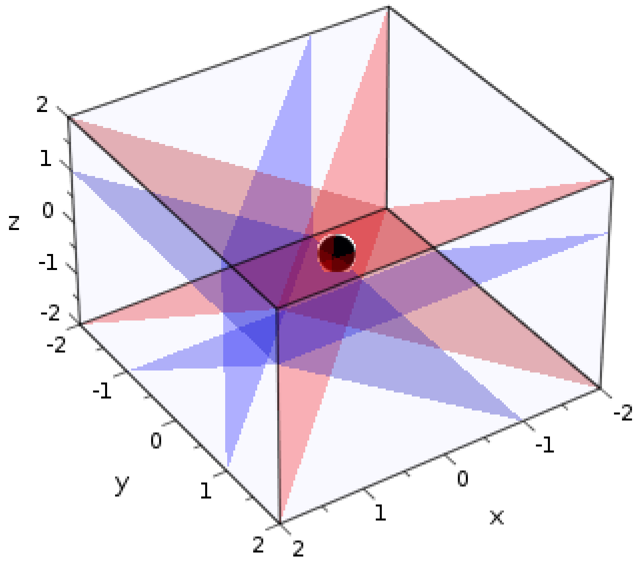
Linear Algebra in MFA501

W.2 → Vectors & vector spaces

W.3 → Introduction to Matrix theory

W.4 → Transformations

W.5 → Eigenvalues & Eigenvectors



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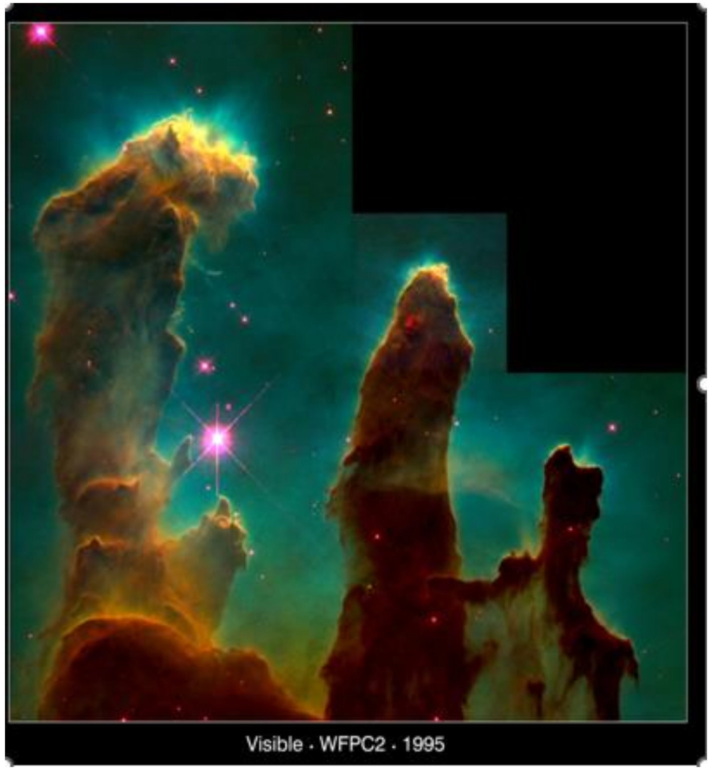
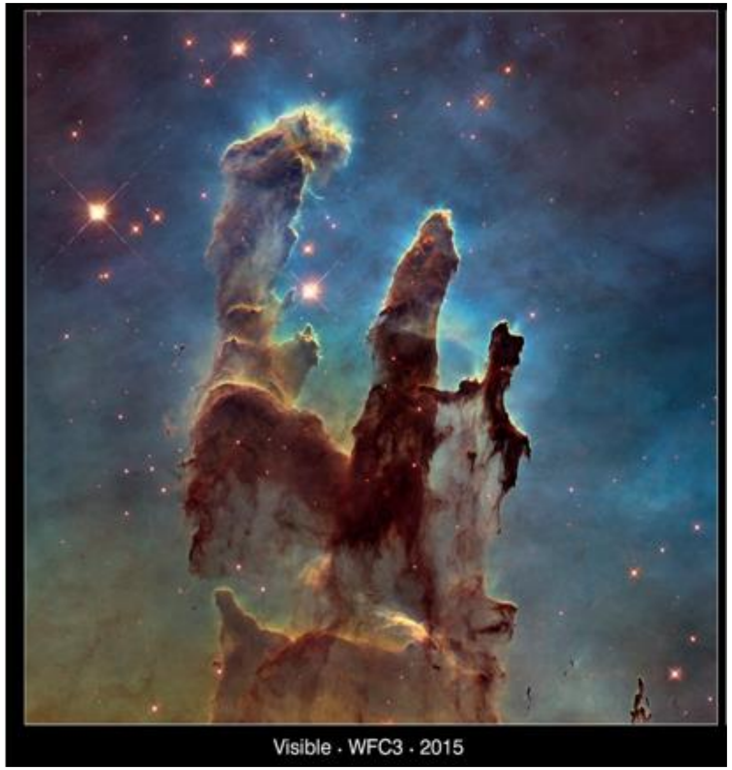
Eagle Nebula - Pillars of creation

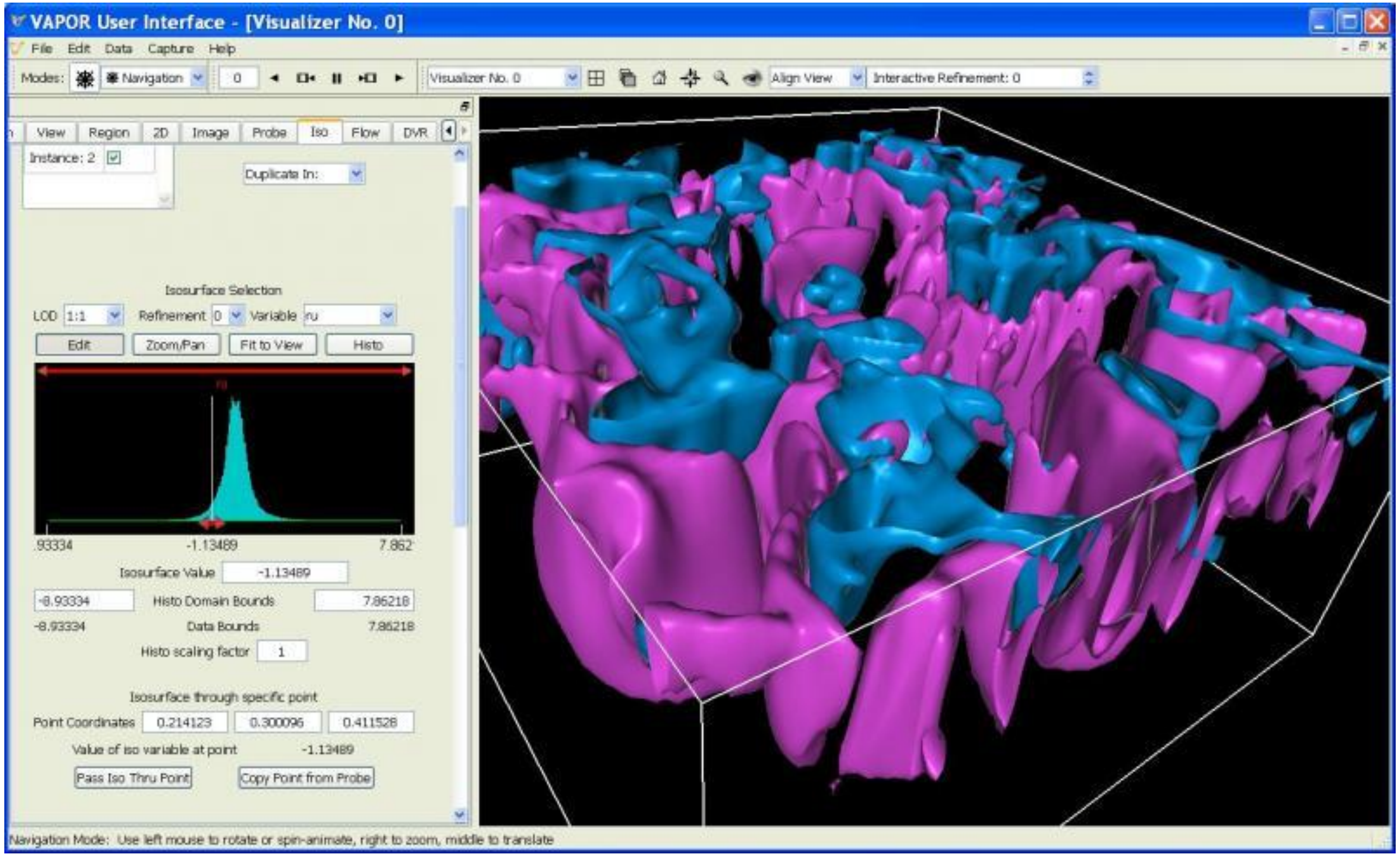




Why are these images different?

Conceptual Example cont

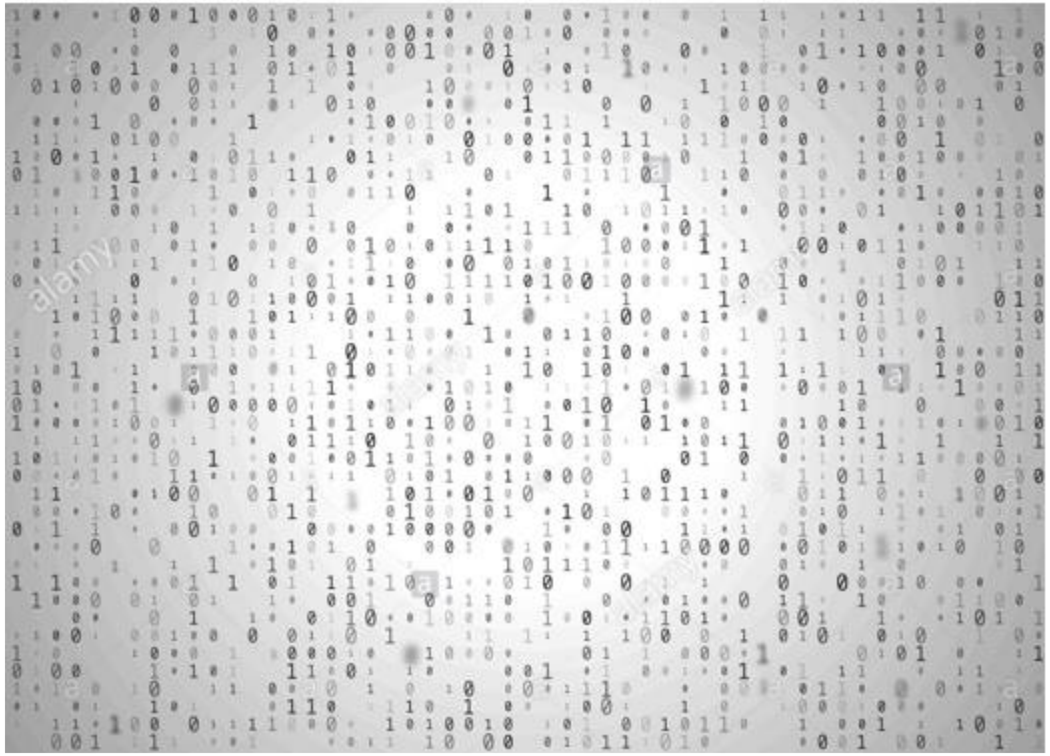
- How many pieces of information, do we need to model & analyze this image?



Introduction

Data/Information can be interpreted as:

- Scalars
- Vectors
- Matrices
- Tensors



Conceptual Example

How many pieces of information do we need to specify an event?

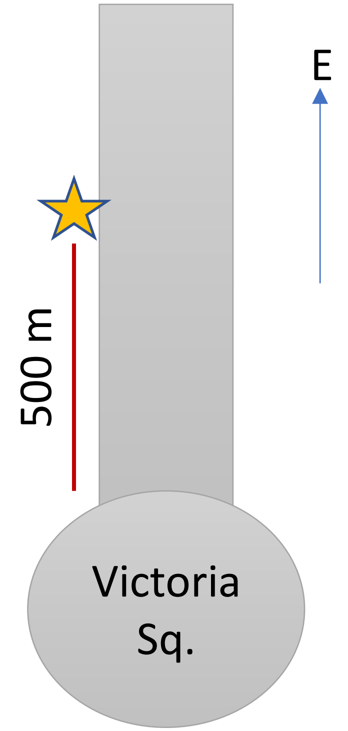
a. Walking for 500 meters → 1 piece of Information

b. Walking for 500 meters in Wakefield Street due East → 2 pieces of Information.

c. Walking for 500 meters in Wakefield Street due East, from Victoria square → 3 pieces of Information.

d. Walking for 500 meters in Wakefield Street due East, from Victoria square, in 5 minutes → 4 pieces of Information

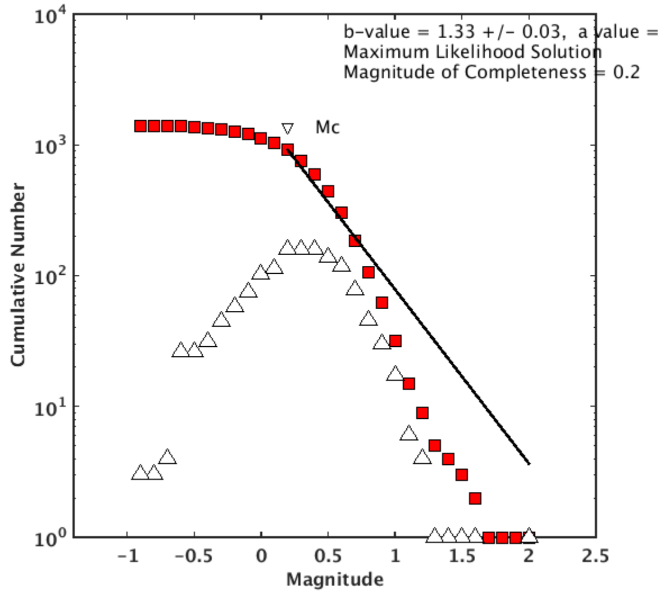
e. Add more & more information *ad infinitum* → n pieces of Information



The more information → The more detail → More precise

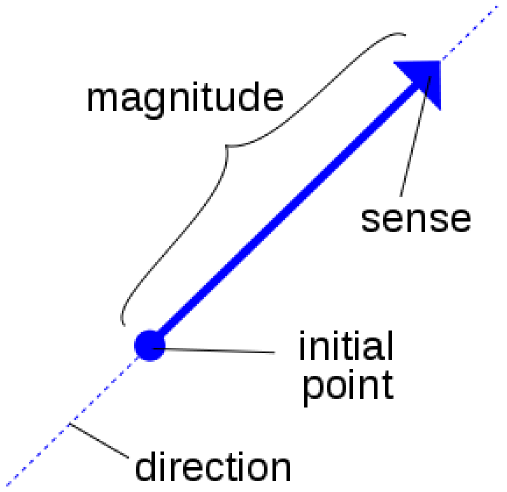
Scalars

- A physical quantity that can be described by a single element of a number field
- This Physical quantity → only Magnitude
- This quantity is *represented by* one single number
- *Such as: Mass, Time, Temperature*
- However, many physical quantities have both:
    - Magnitude
    - Direction
    - Contain more information



Vectors

- Vectors are quantities /objects that has **both** a magnitude and a direction
- Vectors are List of numbers, list of functions
- Geometrically, vector can be **represented** as a directed line segment,
  - *With its length → the magnitude of the vector*
  - *With an arrow indicating the direction.*
- The direction of the vector is from its tail to its head.
- *Such as: Force, Electric Field, Magnetic Field*



- [1,2,3,4,….]
- [sin t, cos t, tan t]

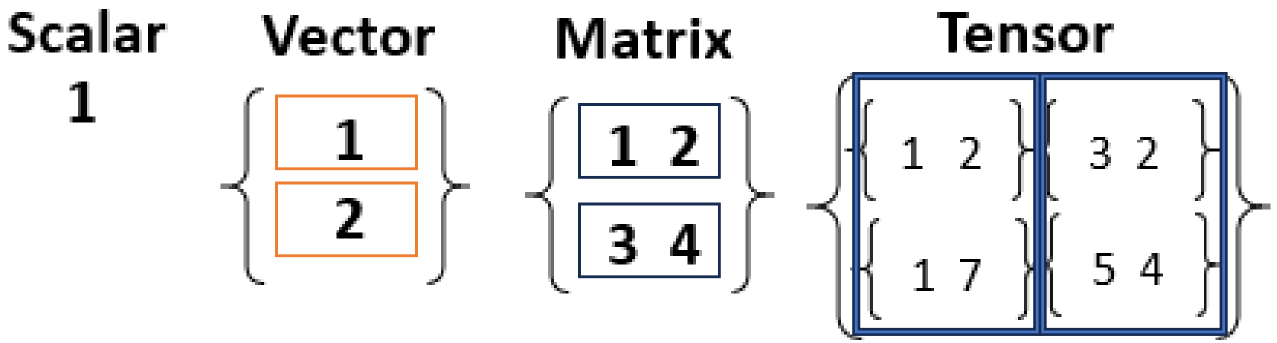
Matrices

- A spreadsheet of numbers
- A rectangular array of numbers.
- It is written within brackets.
- A matrix Contains more information in compare to vectors.
- A matrix contains several vectors

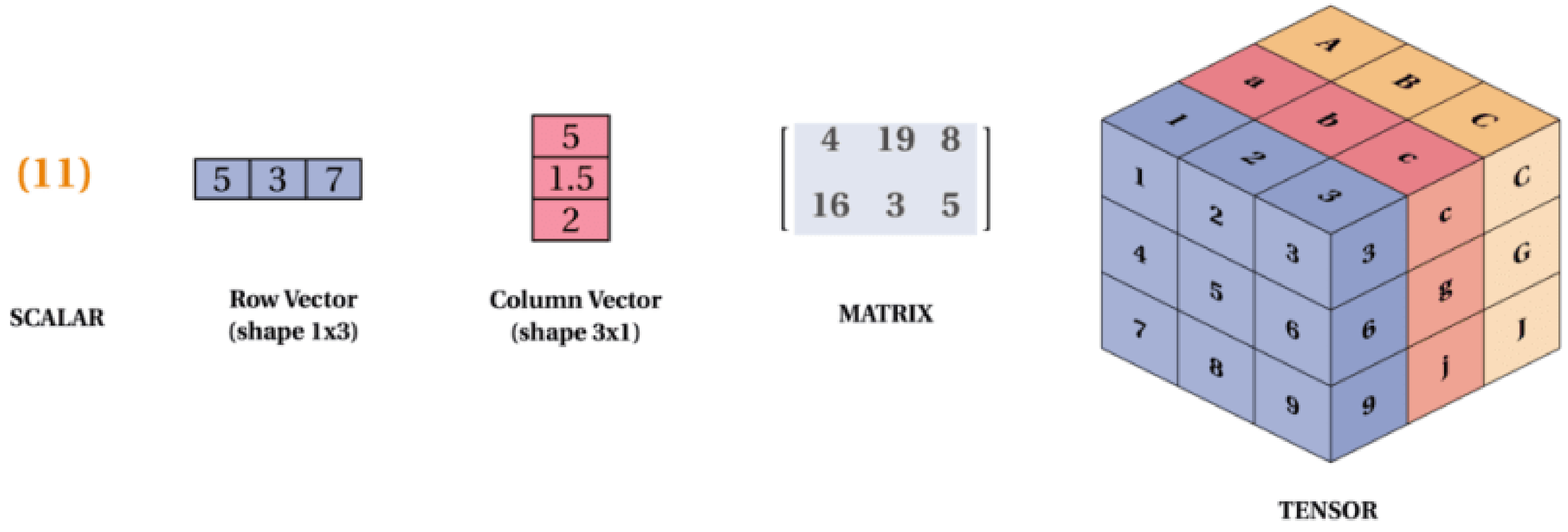
$$\begin{bmatrix} 1 & 8 & 13 & 12 \\ 14 & 11 & 2 & 7 \\ 4 & 5 & 16 & 9 \\ 15 & 10 & 3 & 6 \end{bmatrix}$$

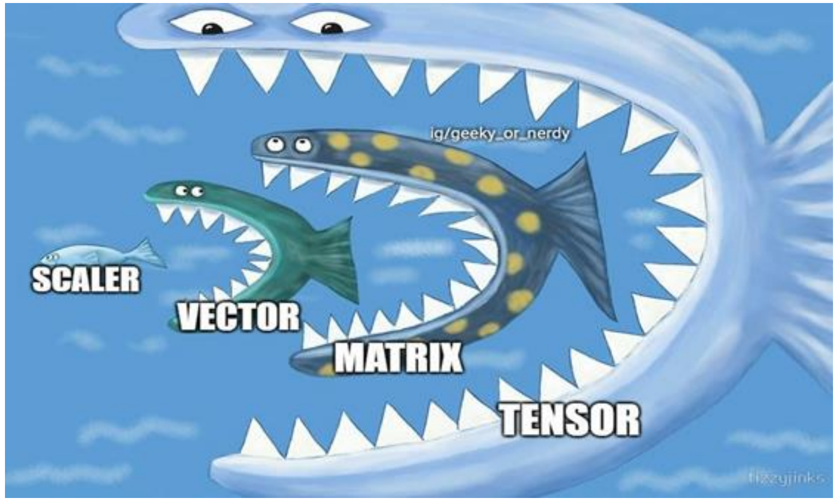
Tensors

- Tensors are a type of data structure used in linear algebra.
- A Tensor is a **generalization** of vectors and matrices.
- It is a multidimensional array
- Tensors can be interpreted as *Matrices within Matrices*

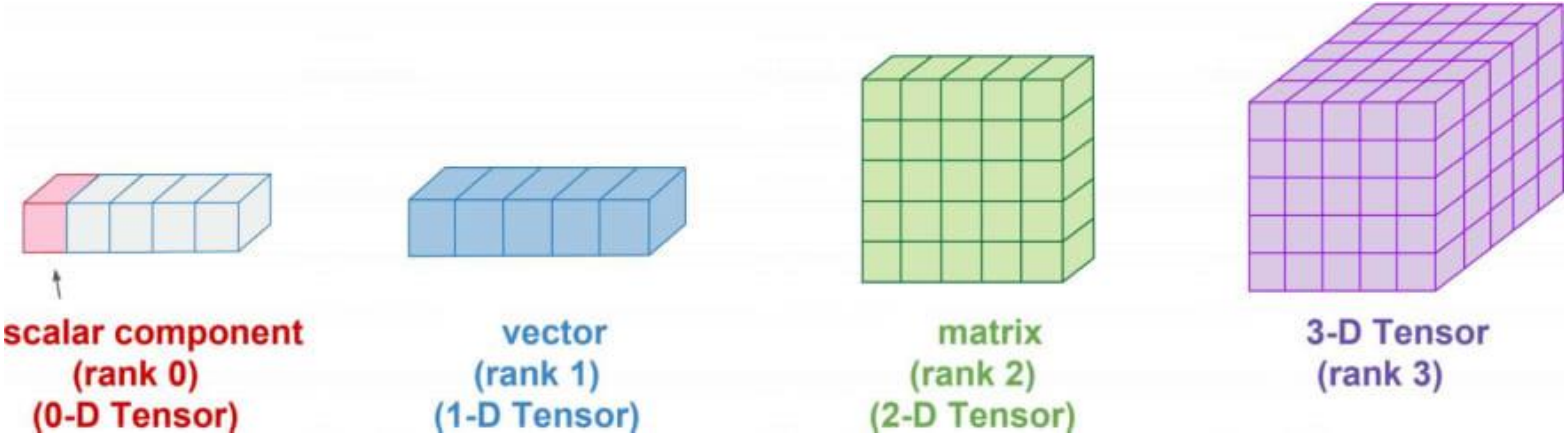


Visualizing Tensors

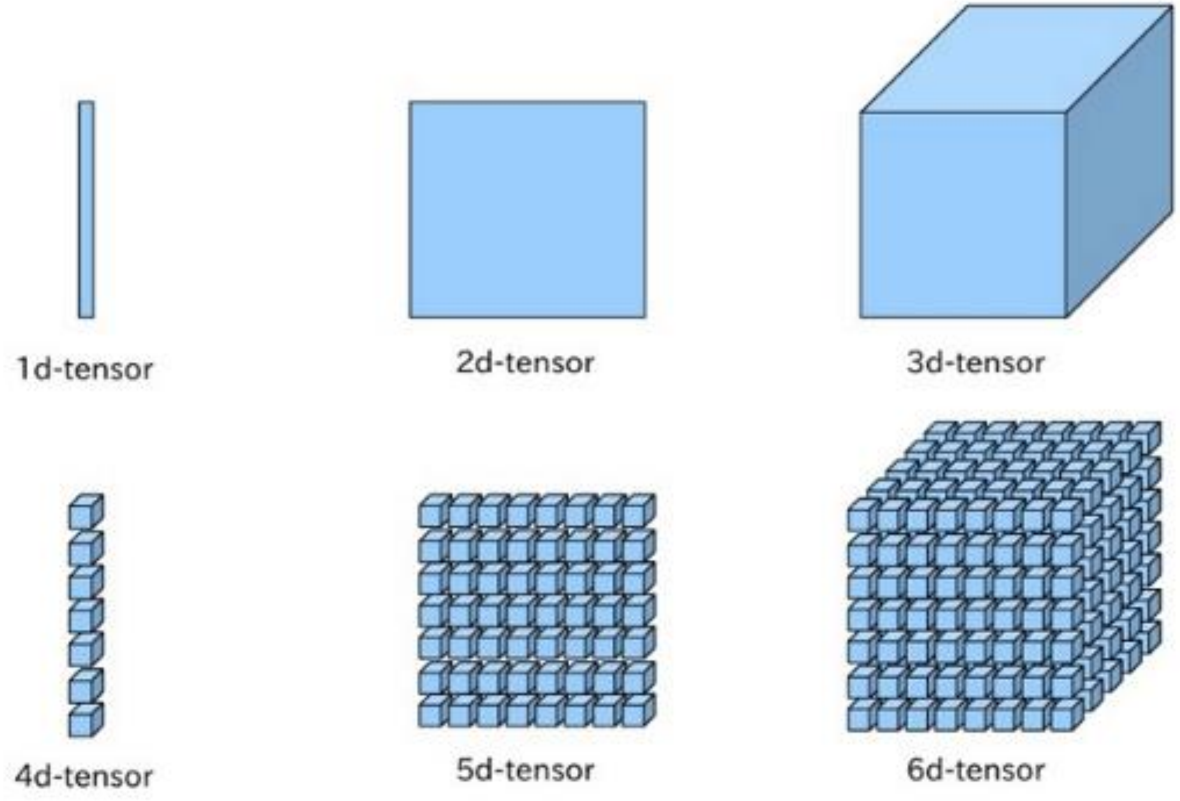




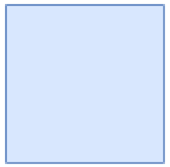
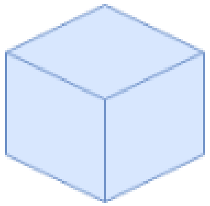
Rank of Tensors



Rank of Tensors cont



In a Glance

Type	Scalar	Vector	Matrix	Tensor
Definition	a single number	an array of numbers	2-D array of numbers	k-D array of numbers
Notation	x	$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$	$\boldsymbol{X} = \begin{bmatrix} X_{1,1} & X_{1,2} & \dots & X_{1,n} \\ X_{2,1} & X_{2,2} & \dots & X_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ X_{m,1} & X_{m,2} & \dots & X_{m,n} \end{bmatrix}$	$\mathbf{X}$ $X_{i,j,k}$
Example	1.333	$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 9 \end{bmatrix}$	$\boldsymbol{X} = \begin{bmatrix} 1 & 2 & \dots & 4 \\ 5 & 6 & \dots & 8 \\ \vdots & \vdots & \vdots & \vdots \\ 13 & 14 & \dots & 16 \end{bmatrix}$	$x = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 10 & 20 & 30 \\ 50 & 60 \\ 80 & 90 \end{bmatrix} \begin{bmatrix} 100 & 200 & 300 \\ 00 & 600 \\ 00 & 900 \end{bmatrix} \right]$
Python code example	x = np.array(1.333)	x = np.array([1,2,3, 4,5,6, 7,8,9])	x = np.array([[1,2,3,4], [5,6,7,8], [9,10,11,12], [13,14,15,16]])	x = np.array([[[1, 2, 3], [4, 5, 6], [7, 8, 9]], [[10, 20, 30], [40, 50, 60], [70, 80, 90]], [[100, 200, 300], [400, 500, 600], [700, 800, 900]]])
Visualization				 3-D Tensor

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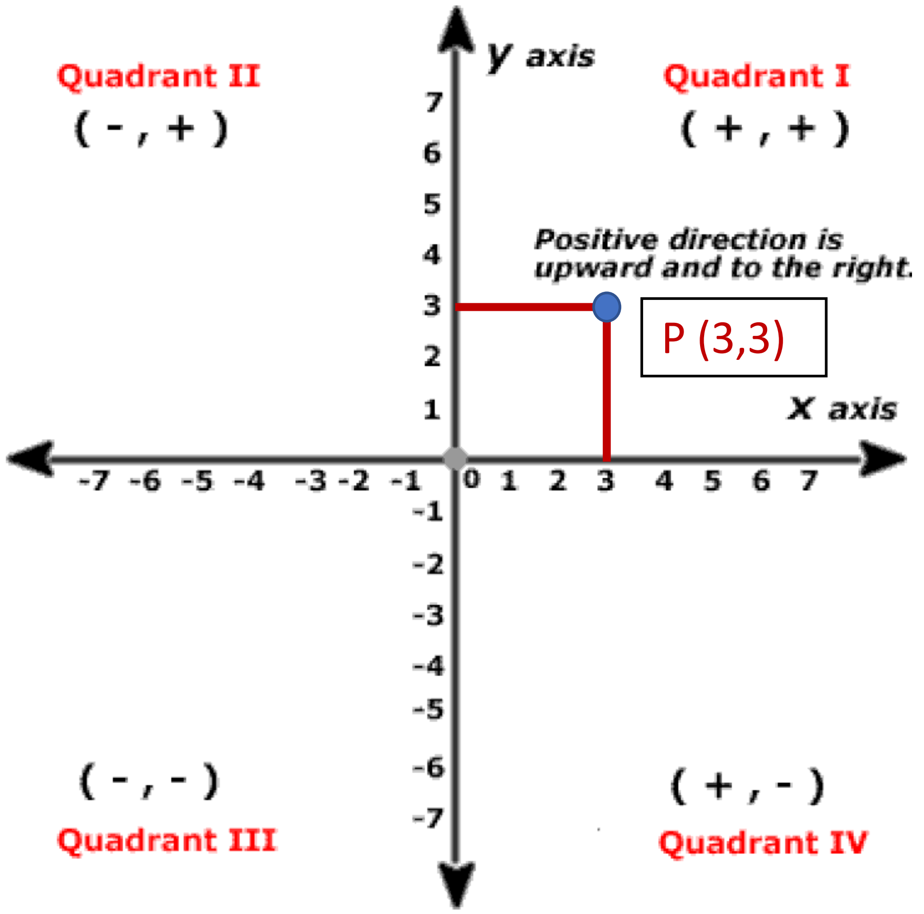
Coordinate System

- How can we give a unique address → to every point in a manifold?
- **Coordinate system** → A system that specifies an address (coordinates)
- These coordinates uniquely determines the position of each point *in that manifold.*
- This manifold can be: a *line*, a *plane*, or an *Euclidean space*.
- These coordinates are a set of ordered numbers
- It was first proposed by Descartes
- It combines Geometry & Algebra

Cartesian Coordinate System- Plane

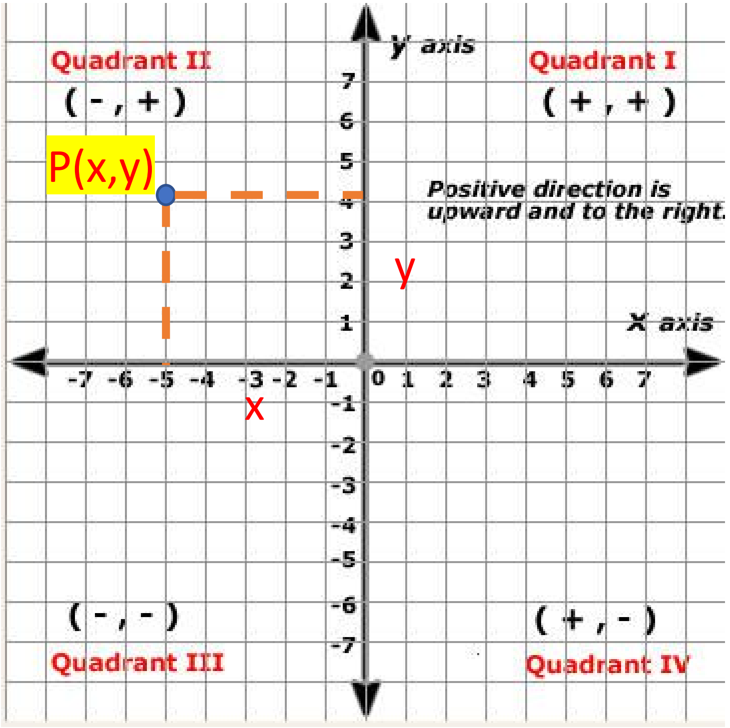


Cartesian Coordinate System - Plane

A Cartesian coordinate system in plane

- Consist of 2 axis (X & Y)
- X & Y are orthogonal
- X & Y intersect at origin O
- It divides the plane to 4 quadrants (I,II,III,IV)
- Unit vector X → *i*
- Unit Vector Y → *j*
- Each point P in the plane → specified by a

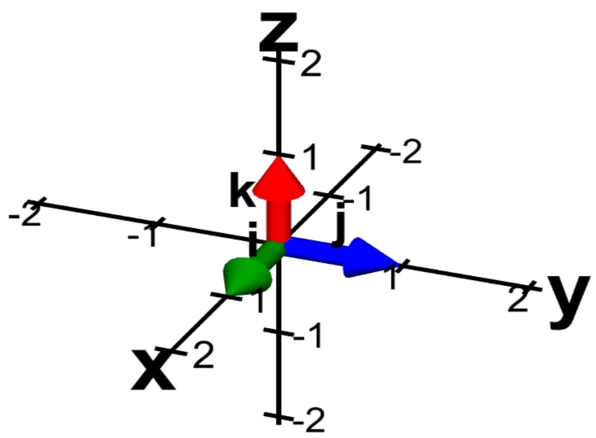
  unique ordered Pair → P(x,y)
- x =Coordinate of P on X
- y =Coordinate of P on Y

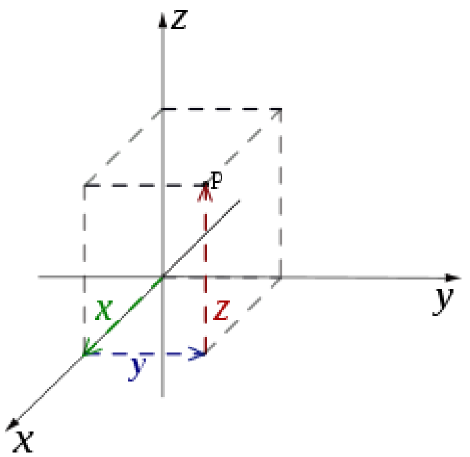


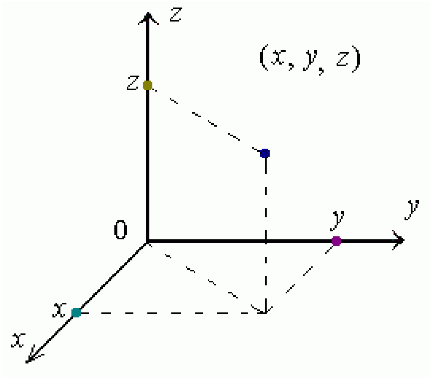
Cartesian Coordinate System - Space

A Cartesian coordinate system in Space

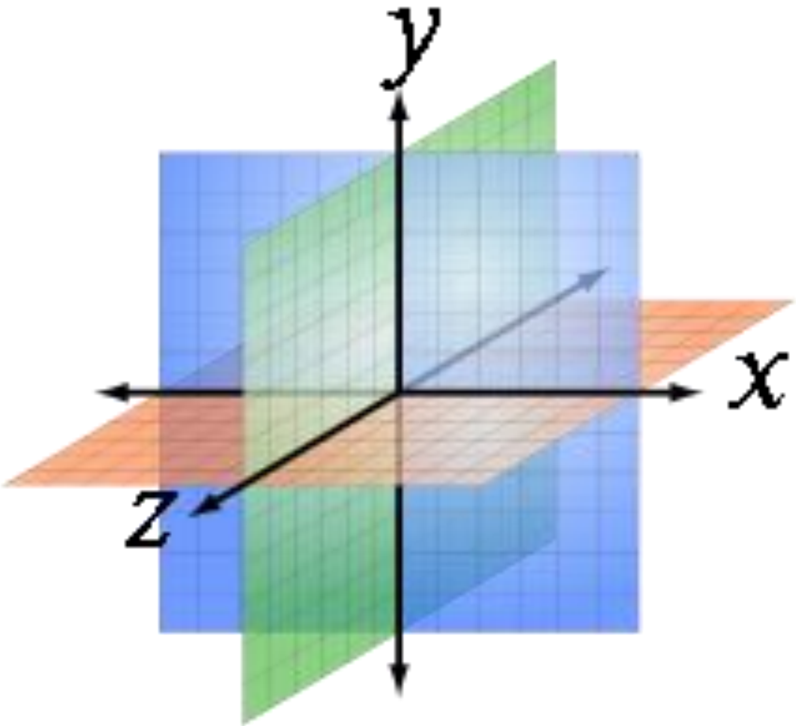
- Consist of 3 axis (X & Y & Z)
- X & Y & Z are orthogonal
- X & Y & Z intersect at origin O
- Unit vector X → *i*
- Unit Vector Y → *j*
- Unit Vector Z → *K*
- Each point P in the plane → specified by a unique ordered Triple → P(x,y,z)
- x = Coordinate of P on X
- y = Coordinate of P on Y
- z = Coordinate of P on Z

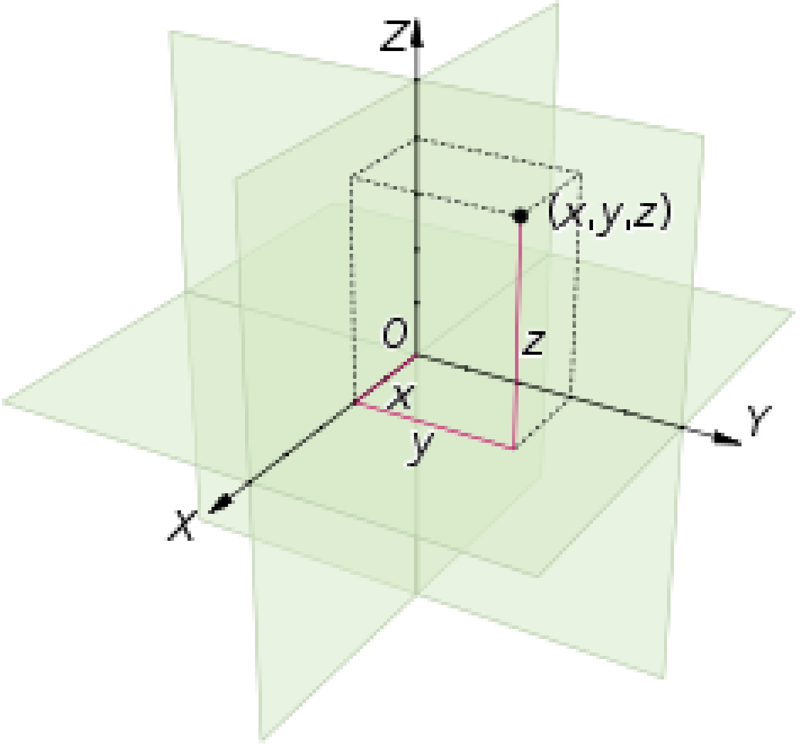






Cartesian Coordinate System- Space





N-D Coordinate System

Manifold	Number of Variables	Coordinates
Line	1	P(X)
Plane	2	P(x,y)
Space (Euclidean Space)	3	P(x,y,z)
Space-Time (Minkowski space)	4	P (x,y,z,t)
… … …	… … …	…. … …
N-Dimensional Space	n	P $(x_1, x_2, x_3, \ldots, x_n)$

ordered Pair

ordered Triple

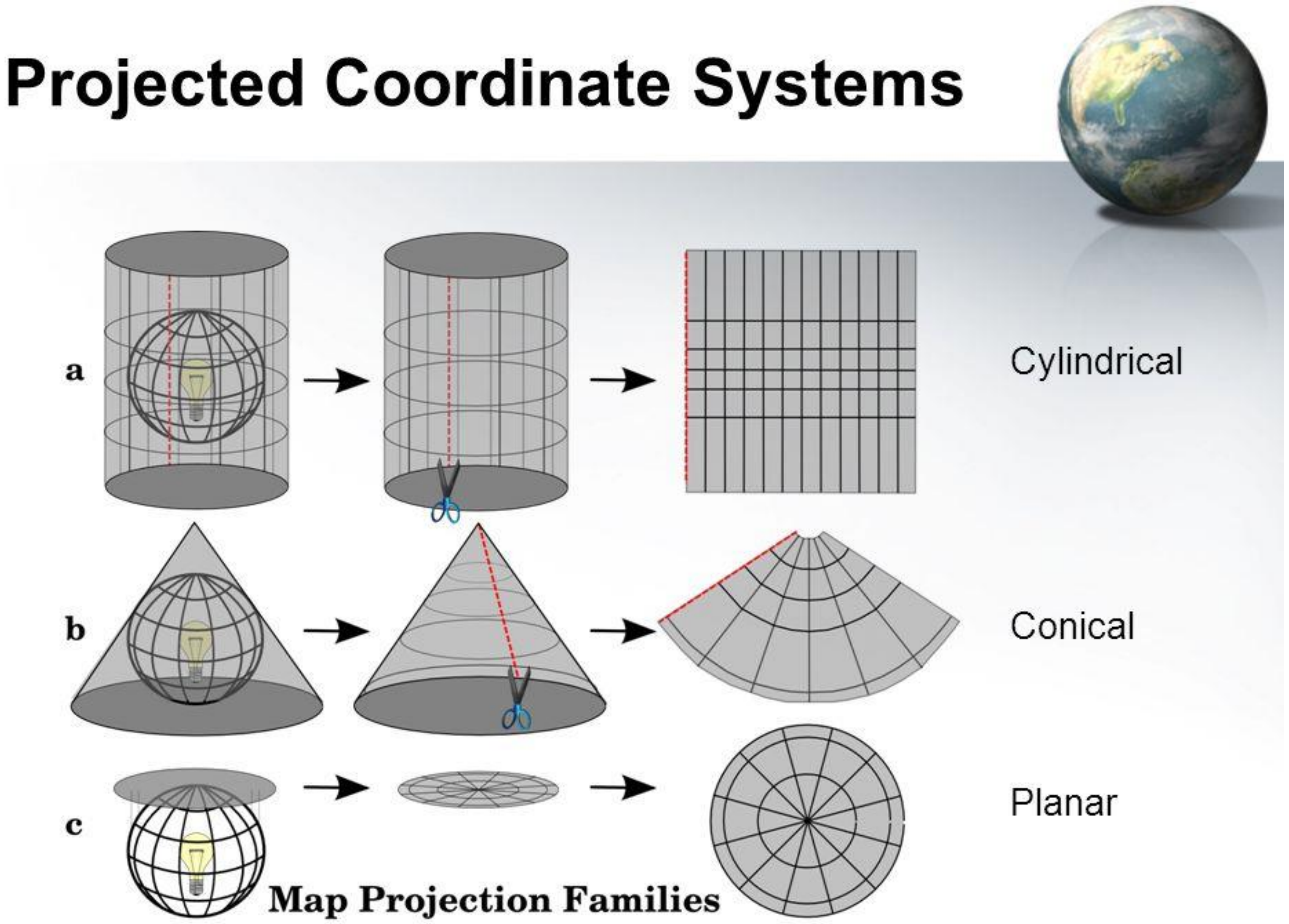
ordered n-tuples

Other Types of Coordinate Systems

Coordinate systems differ by:

- Number of Variables
- Type of projection

- Rectangular
- Cylindrical
- Spherical
- More & More



Break- 20’

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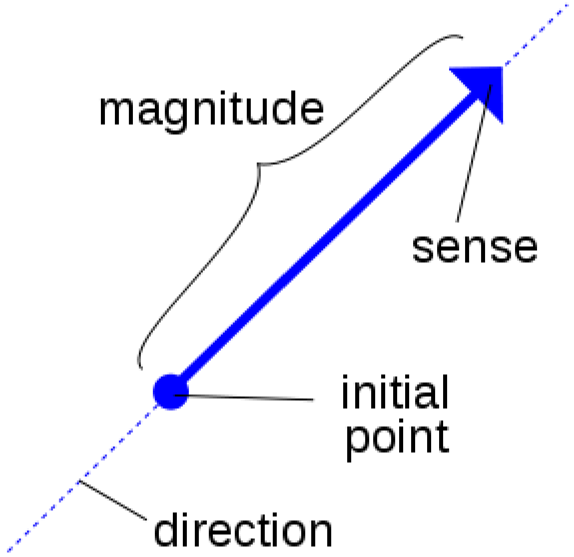
Definition of a Vector

- A vector is a quantity that has two independent properties:

  i. Magnitude

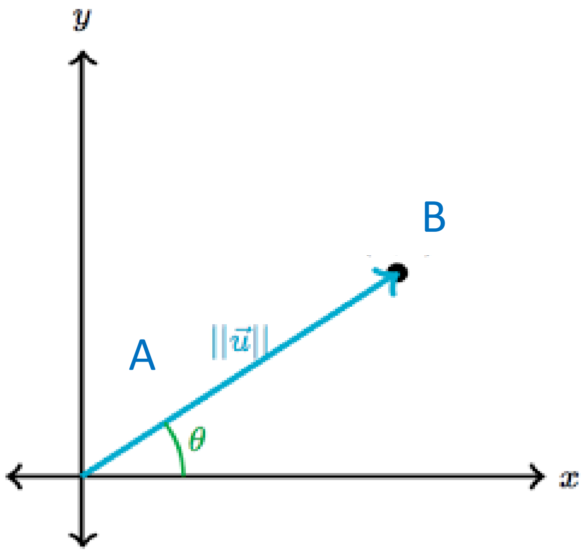
  ii. Direction

- Vectors can be represented graphically in two or three dimensions.
- Magnitude is shown as the length of a line segment.
- Direction is shown by the orientation of the line segment, and by an arrow at one end



Vector in Plane

- A vector in a plane is a directed line segment.
- The directed line segment $\overrightarrow{AB}$, has **initial point** A and **terminal Point** B
- It's length is denoted by $|\overrightarrow{AB}|$
- $\hat{e}$ is a unit vector in the direction of $\overrightarrow{AB}$



Unit Vector

Note: A Unit Vector, is a vector which has a magnitude of 1 unit & is in the direction of the vector.

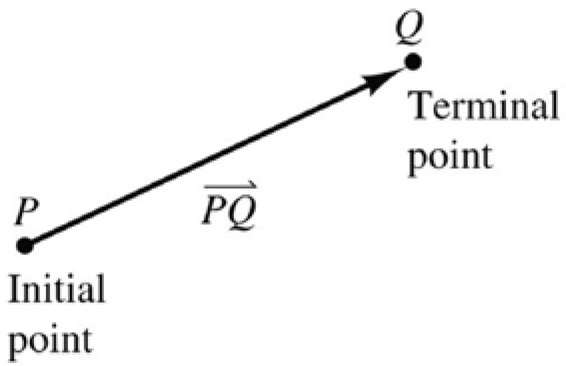
$|\hat{e}|=1$

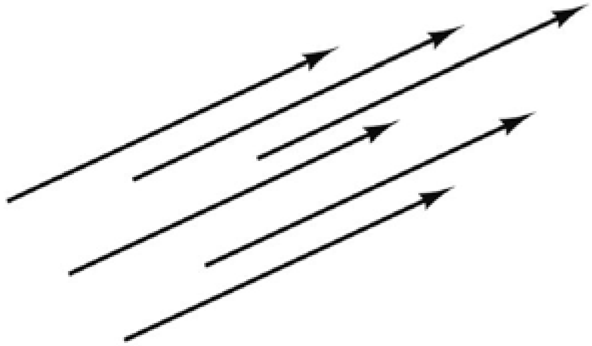
$$\boxed{\overrightarrow{AB} = |\overrightarrow{AB}|\,\hat{e}}$$

Equivalent Vectors

Two or more vectors are equivalent, if

- They have the same magnitude
- They have the same direction

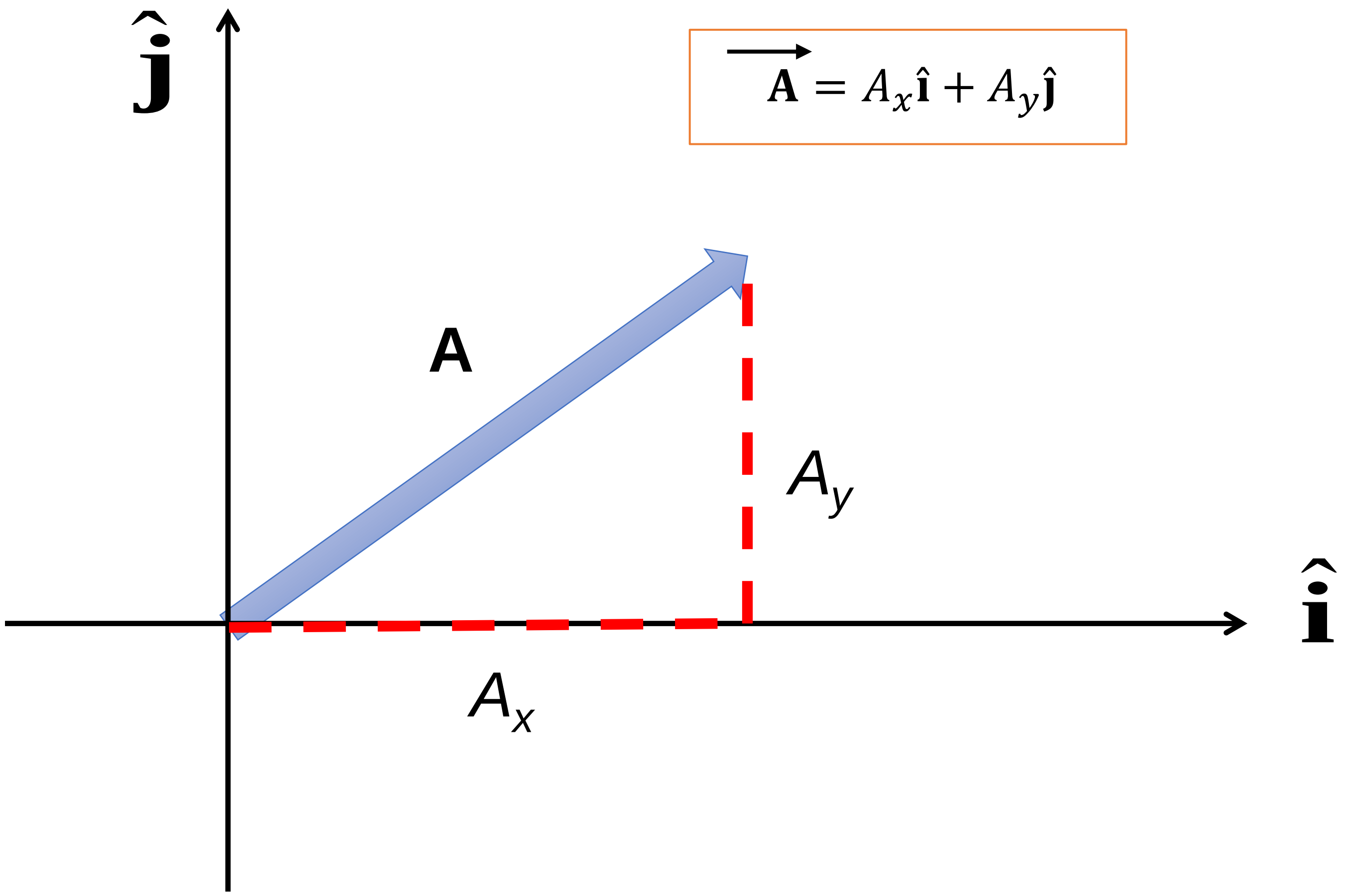




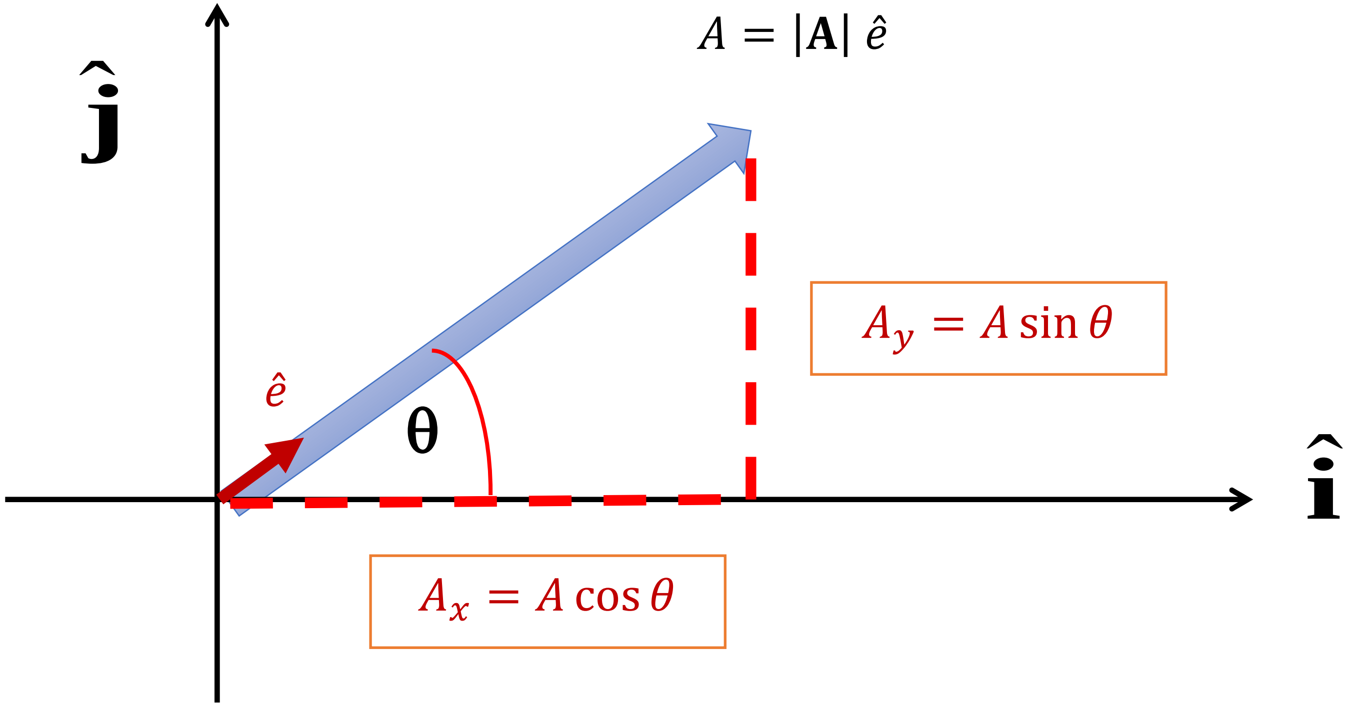
Components of a Vector in 2-D Plane

- Vectors are usually described *in terms of their components* in a **coordinate system**.
- In a **rectangular *xy*-coordinate system** (in a plane), a point is described by a pair of coordinates *(x, y)*.
- In a similar fashion, a vector in a plane is described by a pair of its *vector coordinates* as follows:

$$\boxed{\vec{\mathbf{A}}\ \langle A_x\,, A_y \rangle}$$



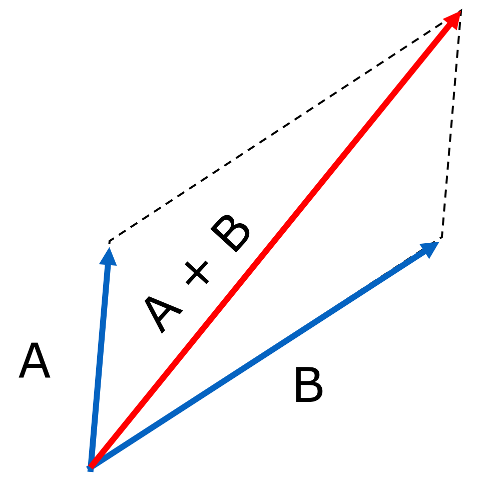
Components of a Vector in 2-D Plane cont



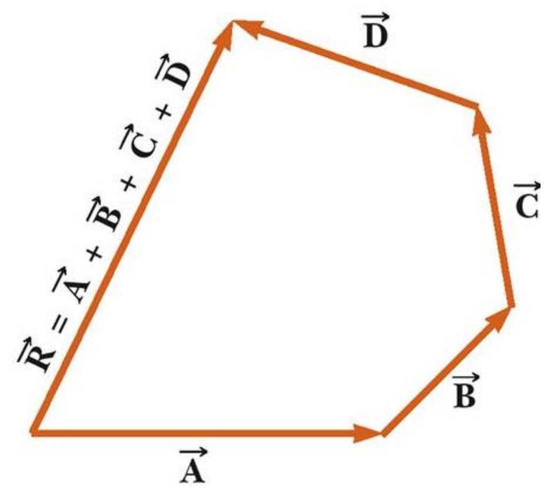
$$\text{Note}: \frac{A\sin\theta}{A\cos\theta} = \tan\theta = \frac{A_y}{A_x}$$

Adding Vectors

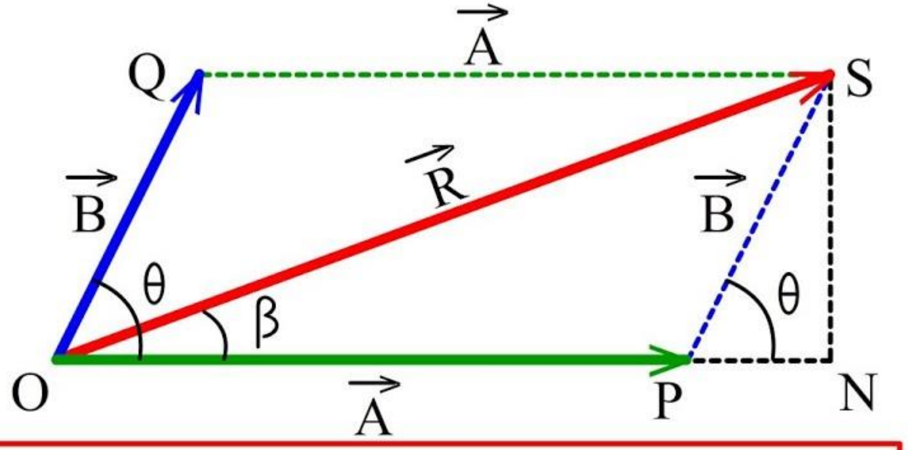
- The sum of two (or more) vectors is the single equivalent vector with the same effect as the application of the two (or more) vectors altogether.



- Note that the sum of two vectors is the diagonal of the resulting parallelogram



Adding Vectors - Parallelogram Method



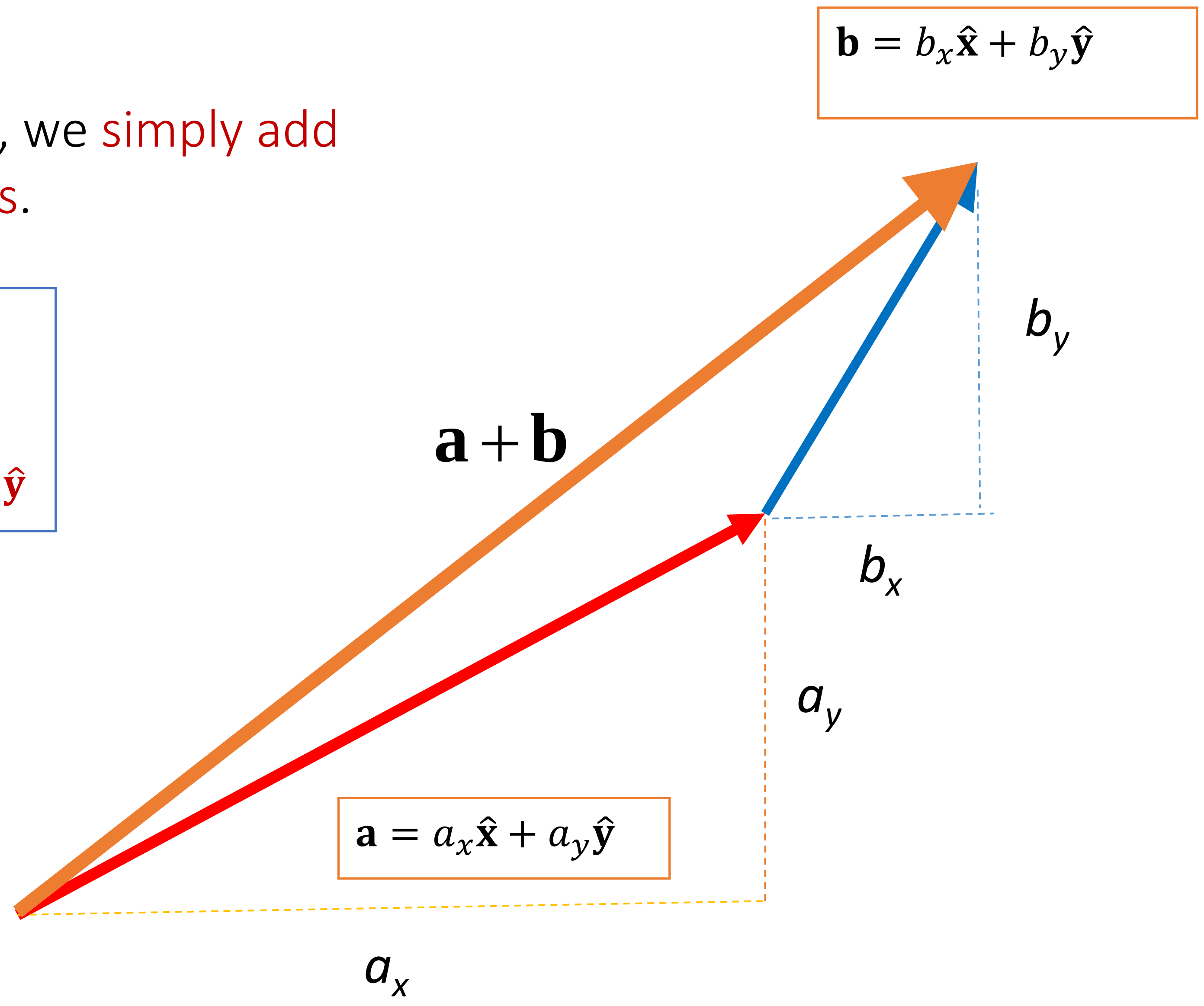
$$\boxed{R = \sqrt{A^2 + 2AB\cos\theta + B^2}}$$

$$\boxed{\beta = \tan^{-1}\left[\frac{B\sin\theta}{A + B\cos\theta}\right]}$$

Adding Vectors – Component Method

- To add one or more vectors, we simply add their respective components.

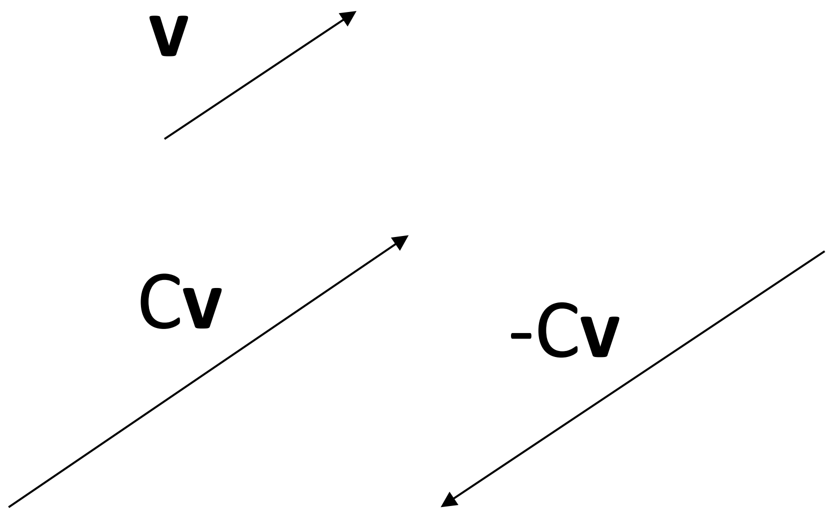
For $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$ and
$\mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}}$,
Then
$$\mathbf{a} + \mathbf{b} = (a_x + b_x)\hat{\mathbf{x}} + (a_y + b_y)\hat{\mathbf{y}}$$



Scaling a Vector

- Given a real number *c*, we can multiply a vector by *c* .
- This is defined as multiplying the magnitude of the vector by *c.*

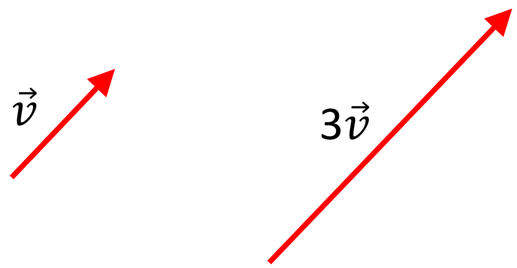
$$c\vec{v} = (c|\vec{v}|)\hat{e}$$



- The initial vector is scaled by *c.*
- Notice that multiplying a vector by a *negative real number* reverses the direction.

Scalar Multiplication

- Scalar multiplication: $c\vec{v}$?
- Stretch the vector
  - $|3\vec{v}| = 3|\vec{v}|$

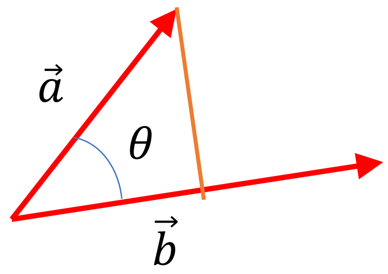


- Scalar multiplication in number representation
  - $\vec{v} = (1, 2, 1)$
  - $3\vec{v} = 3(1, 2, 1) = (3 \cdot 1,\ \ 3 \cdot 2,\ \ 3 \cdot 1) = (3, 6, 3)$
  - Multiply the number to every component

Dot Product

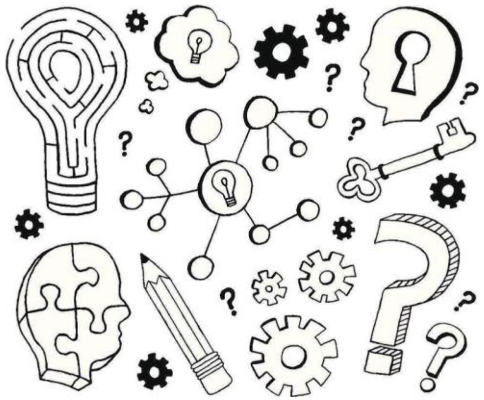
- Dot product: $\vec{a} \cdot \vec{b}$
- $\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta = a_x b_x + a_y b_y + a_z b_z$

  - Dot product in number representation- Component-wise multiplication



  - $\vec{a} = (1, 2, 0),\ \vec{b} = (2, 1, 1)$
  - $\vec{a} \cdot \vec{b} = 1 \cdot 2 + 2 \cdot 1 + 0 \cdot 1 = 4$

Exercise I (Individual, 10’)



- If vectors **u, v,** and **w** are 3 vectors in a **plane**
- If c & d are real numbers
- Investigate the following properties:
  1. $\mathbf{u} + \mathbf{v}$ is also a vector
  2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  4. There exists a vector $\mathbf{0}$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$
  5. For each $\mathbf{v}$, there is a vector $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
  6. $c\mathbf{v}$ is a vector
  7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
  8. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$
  9. $c(d\mathbf{v}) = (cd)\mathbf{v}$
  10. $1\mathbf{v} = \mathbf{v}$

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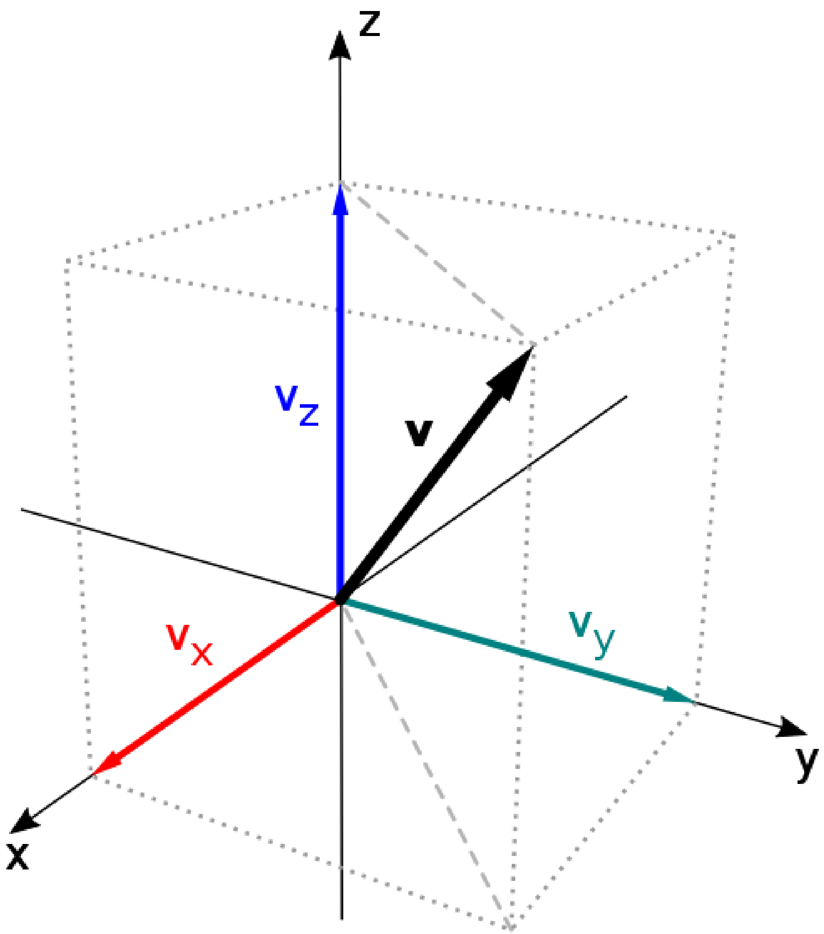
VI. Vector Space & Subspace

VII. Tutorial

Vector in a Cartesian 3D Space

- Each Vector in space has 3 coordinates

$$\overrightarrow{V} = < v_x, v_y, v_z >$$

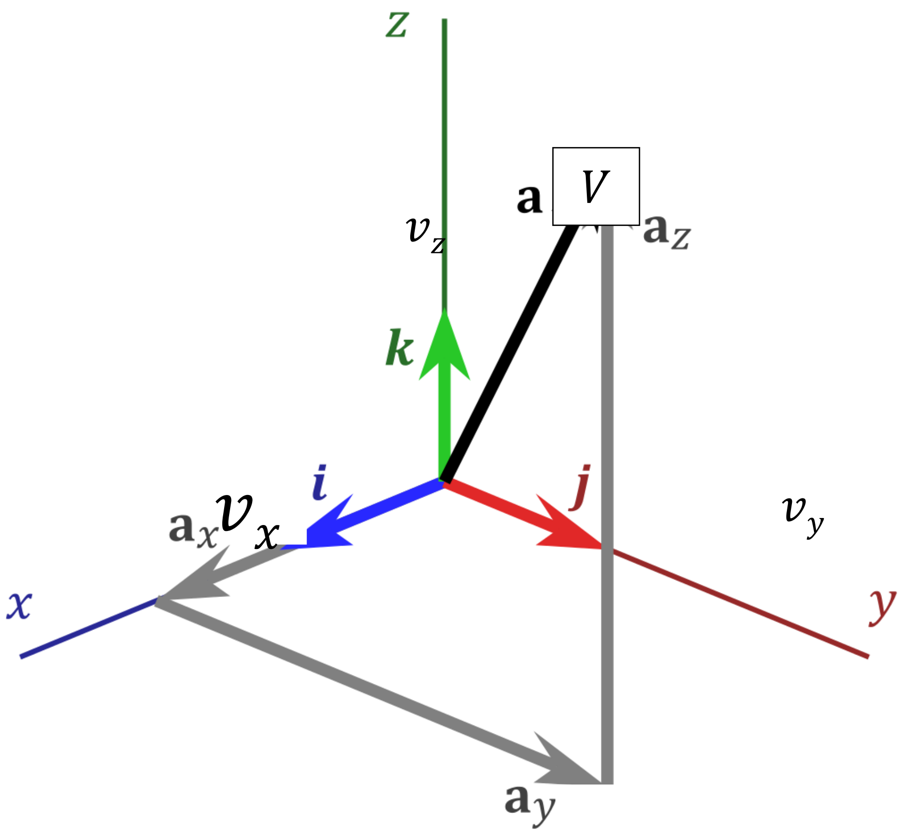


Vector in a Cartesian 3D Space

- Position Vector: A vector with it's tail at the origin
- The Position vector will be:

$$\overrightarrow{V} = v_x i + v_y j + v_z k$$

$$\overrightarrow{V} =< v_x, v_y, v_z >$$



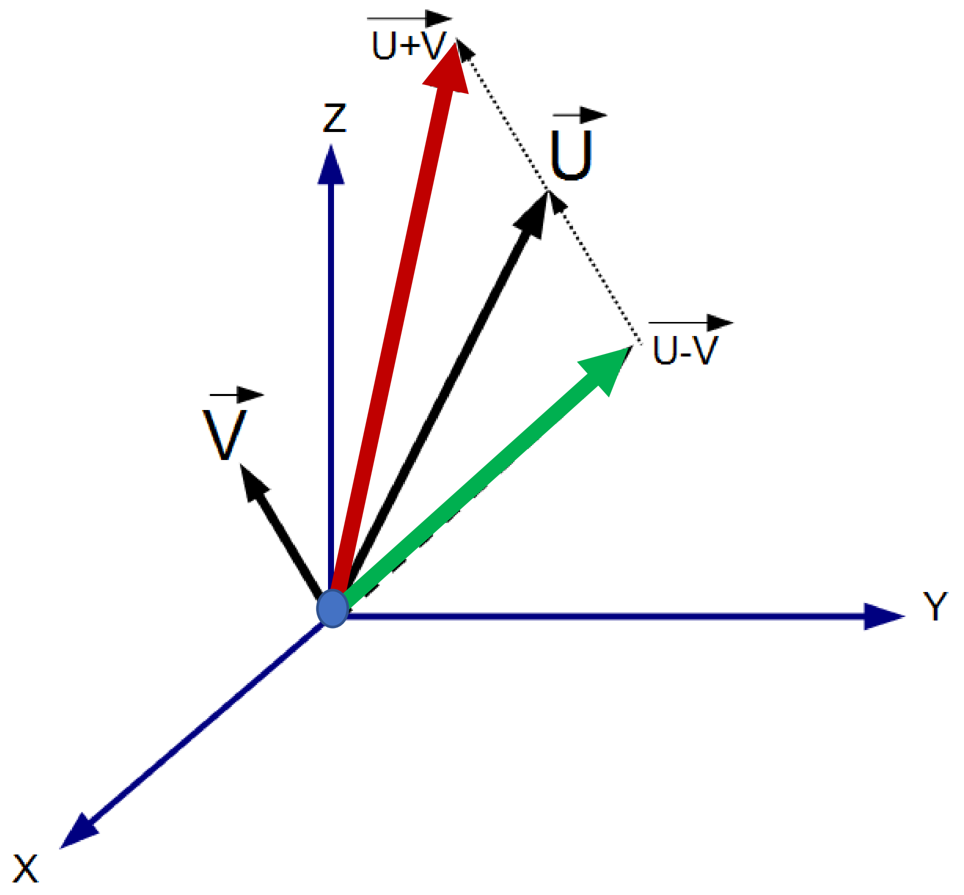
Adding Vectors in Space

- We can add 2 or more vectors in space, simply by adding their respective components.

$\overrightarrow{V} =< v_x, v_y, v_z >$

$\overrightarrow{U} =< u_x, u_y, u_z >$

$$\boxed{\overrightarrow{U}+\overrightarrow{V} =< u_x + v_x , u_y + v_y , u_z + v_z >}$$

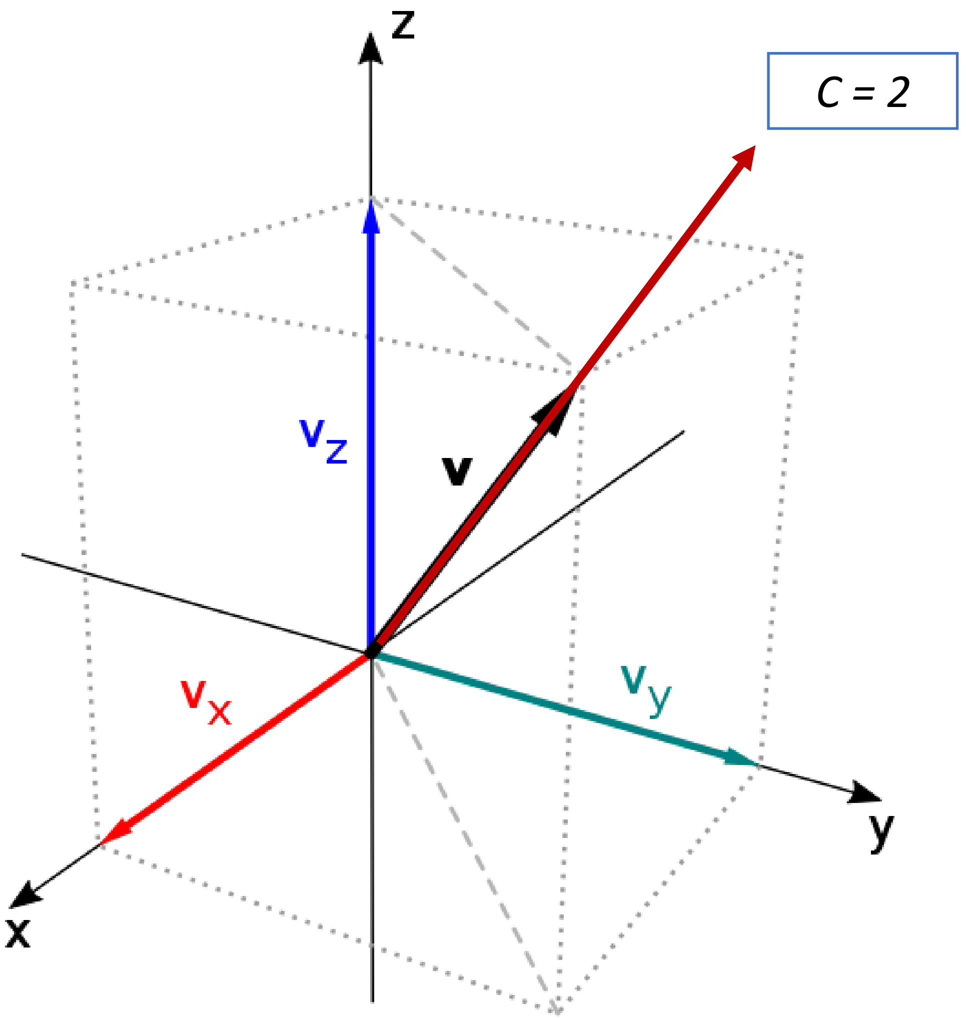


Scaling Vectors in Space

- We can scale any vector in space simply by scaling it's respective components.

$$\vec{V} = < v_x, v_y, v_z >$$

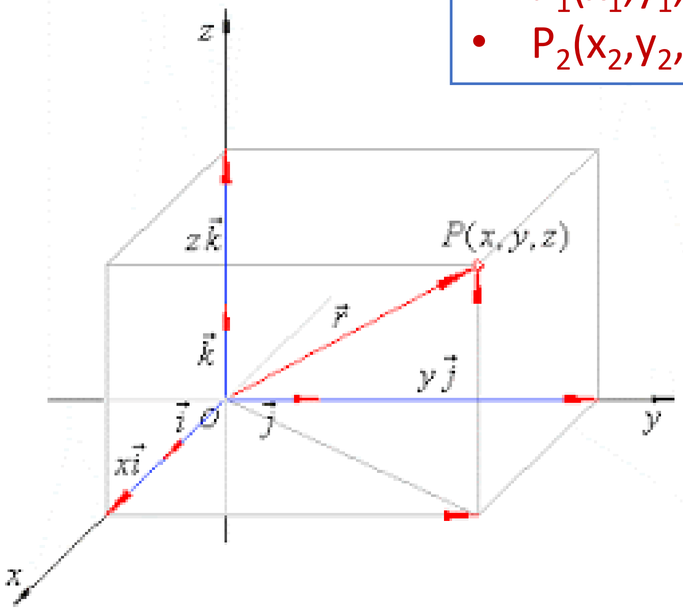
$$c\vec{V} = < cv_x, cv_y, cv_z >$$

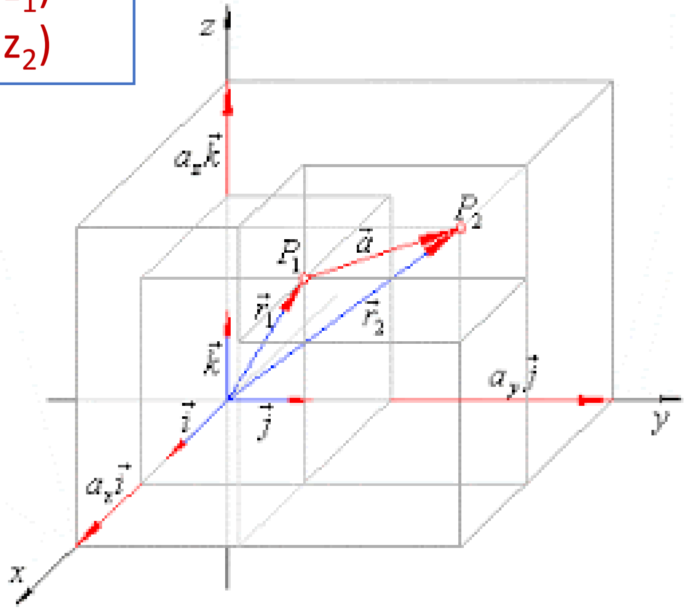


Discussion

1. What is $\overrightarrow{OP_1}$?
2. What is $\overrightarrow{OP_2}$?
3. What is $\overrightarrow{P_1P_2}$?

- $P_1(x_1,y_1,z_1)$
- $P_2(x_2,y_2,z_2)$





Exercise II (Individual, 10')

- If vectors **u, v,** and **w** are 3 vectors in **space**
- If c & d are real numbers
- Investigate the following properties:

    i. **u** + **v** is also a vector in space

    ii. **u** + **v** = **v** + **u**

    iii. (**u** + **v**) + **w** = **u** + (**v** + **w**)

    iv. There exists a vector **0** , such that **v** + **0** = **v**

    v. For each **v**, there is a vector –**v**, such that **v** + (-**v**) = **0.**

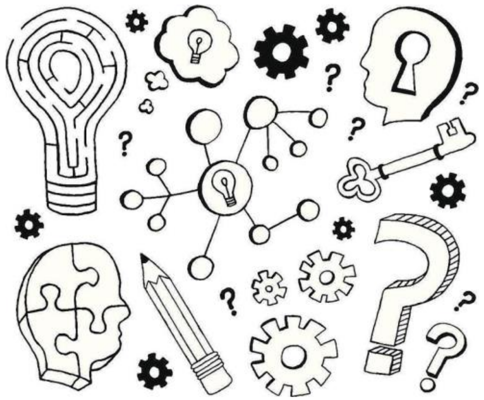
    vi. c**v** is a vector

    vii. c(**u** + **v**) = c**u** + c**v**

    viii. (c + d)**v** = c**v** + d**v**

    ix. c(d**v**) = (cd)**v**

    x. 1**v** = **v**



Today's Outline

I. Introduction to Liner Algebra

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Vector Spaces- Introduction

- Vectors in the Cartesian plane have **specific properties**. → 2D vector Space
- Vectors in the Cartesian Space have the same **specific properties**. → 3D Vector Space
- We can argue that vectors in any manifolds, → have the *same properties.*
- Hence we generalize the above properties → **vector space**

Vector Space

- A **vector space** is a set of objects called vectors,

  Which:

  i) May be added together

  ii) Multiplied by numbers called scalars.

- A **vector space** is a set of all possible vectors in a n-D space.
- A **vector space** has certain properties.

Definition of Vector Space

- Let V be a set on which two operations, called *addition* and *scalar multiplication*, have been defined
- If **u**, **v**, and **w** are vectors in V
- If c & d are real numbers
- V is called a **vector space**, if & only if the following axioms hold for V

i. **u + v** is also a vector in space *(Closure under addition Property)*

ii. **u + v = v + u** *(Commutative Property)*

iii. **(u + v) + w = u + (v + w)** *(Associative Property)*

iv. There exists a vector **0** , such **that v + 0 = v** *(Additive Identity)*

v. For each **v**, there is a vector **–v**, such that **v** + **(-v) = 0** *(Additive Inverse)*

vi. **cv** is a vector *(Closure under Scalar Multiplication)*

vii. **c(u + v)** = *c***u** + *c***v** *(Distributive from Left)*

viii. $(c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$ *(Distributive from Right)*

ix. $c(d\mathbf{v}) = (cd)\mathbf{v}$ *(Associative Property)*

x. 1**v** = **v** *(Multiplicative Identity)*

Subspace

- If V is a **vector space**
- If W is a **subset** of V
- AND W is a **vector space**
- Then W is a Subspace of V

Every Vector Space V has the

following two subspaces

- **{0}** (Zero Space)
- **{V}** (Itself)

Definition: A subset **W** of a vector space **V** is called a subspace

if **W** itself is a vector space

*(with the same scalars, addition and scalar multiplications as **V**).*

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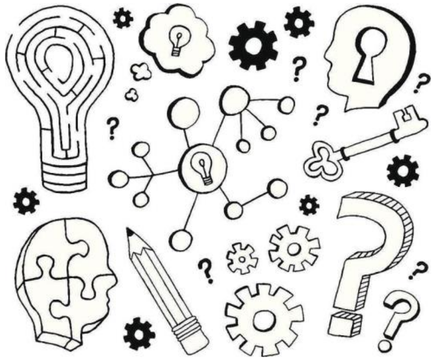
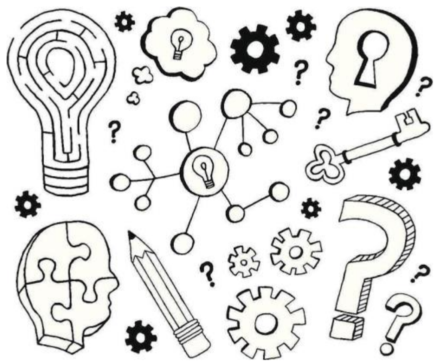
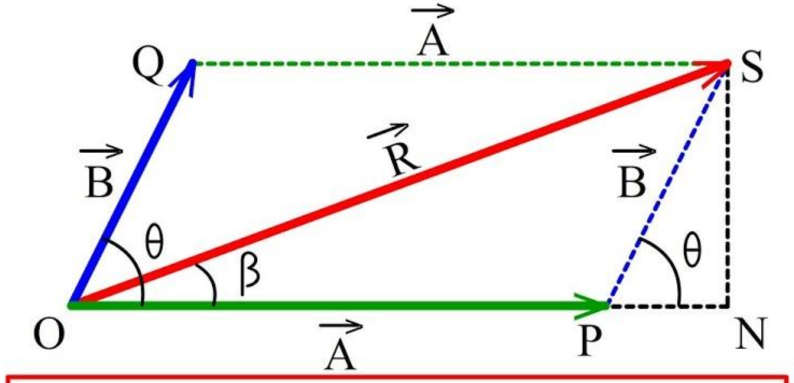
Reflection (40’, Individual)



1. What is Linear Algebra? Why do you think it is important in computer science?
2. Recall your programming experience. Where have you used Linear Algebra?
3. What is a scalar? Give 2 examples.
4. What is a vector? Give 2 examples.
5. What are matrices? Give 2 examples.
6. What are Tensors? Give 2 examples.
7. What is the relationship between scalars, vectors, matrices, and tensors?
8. What do you understand from coordinate systems? Why are they important?
9. How do coordinate systems vary? What types of coordinate systems do you know?
10. What is the application of coordinate systems in computer science?
11. What are the main operations for vectors?
12. How can you scale a vector V in a cartesian space? What is its relation with the initial vector?
13. Given $\vec{a} =< a_x, a_y, a_z>$, what is the magnitude of $\vec{a}$?
14. Proof the parallelogram property for vectors in the plane (Hint: use the figure in next slide).
15. If $\vec{a} =< a_x, a_y, a_z>$ and $\vec{b} =< b_x, b_y, b_z>$; what would be $m\vec{a} - n\vec{b}$ (where m & n are nonzero constants)?
16. **If $\vec{a} =< a_x, a_y, a_z>$ and $\vec{b} =< b_x, b_y, b_z>$** , using the definition : $\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta$; proof $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$.
17. If $\vec{a} =< 2, 3, 1>$ and $\vec{b} =< 2, 2, 0 >$; **a)** draw both vectors, **b)** what is $\vec{a} + \vec{b}$, **c)** what is $\vec{a} \cdot \vec{b}$? , **d**) what is θ?
18. What are the properties of a vector space?
19. What are subspaces? Why are they important?
20. If you are developing a Game (such as Duke Nukem 3D), what is the significance of a vector space?

Problem 14





$$\boxed{R = \sqrt{A^2 + 2AB\cos\theta + B^2}}$$

$$\boxed{\beta = \tan^{-1}\left[\frac{B\sin\theta}{A + B\cos\theta}\right]}$$

Next Class

(Week 3) Matrices

Come prepared!

Any Questions or Concerns?

Source of the slides:

https://www.slideserve.com/

And

https://fdocuments.in/

And

http://web.math.ucsb.edu/~dai/4A14KenApril14.pdf