Module 5:
Integrals Calculus

Integral Calculus

Traditionally it is divided to:

- **Differential calculus**: A subfield of calculus that studies the rates at which quantities change.

- **Integral Calculus**: A subfield of calculus that studies the area under a curve.
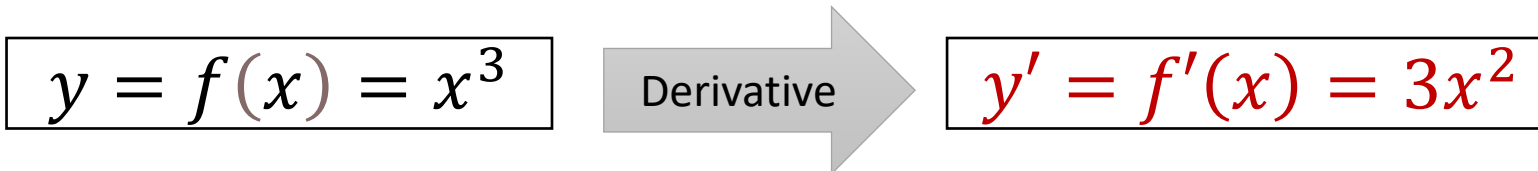
Today's Outline

Conceptual Example

- The function $y = f(x)$, is defined as:

$$y = f(x) = x^3$$

- Hence, it's derivative is:

$$y' = f'(x) = 3x^2$$
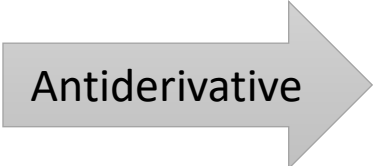
$$\boxed{y = f(x) = x^3} \quad \xrightarrow{\text{Derivative}} \quad \boxed{y' = f'(x) = 3x^2}$$

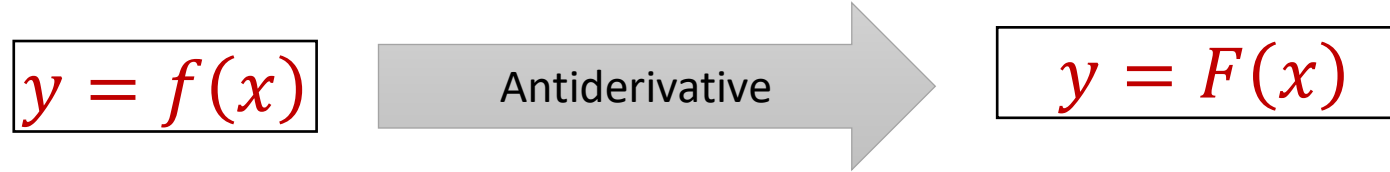# Conceptual Example _{cont}

- Now, if $y' = f(x) = 3x^2$ is a given function on $I = \mathbb{R}$

- We wish to find $y = F(x)$; the antiderivative of $f(x)$

$$y' = f(x) = 3x^2 \quad \xrightarrow{\text{Antiderivative}} \quad y = F(x) = x^3$$

Antiderivatives

$$y = f(x)$$ → Antiderivative → $$y = F(x)$$

DEFINITION   A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example

- Show that all the bellow functions are antiderivatives of $y = 3x^2$

a) $y = x^3 \rightarrow y' = 3x^2$

b) $y = x^3 + 1 \rightarrow$ "

c) $y = x^3 - 1 \rightarrow$ "
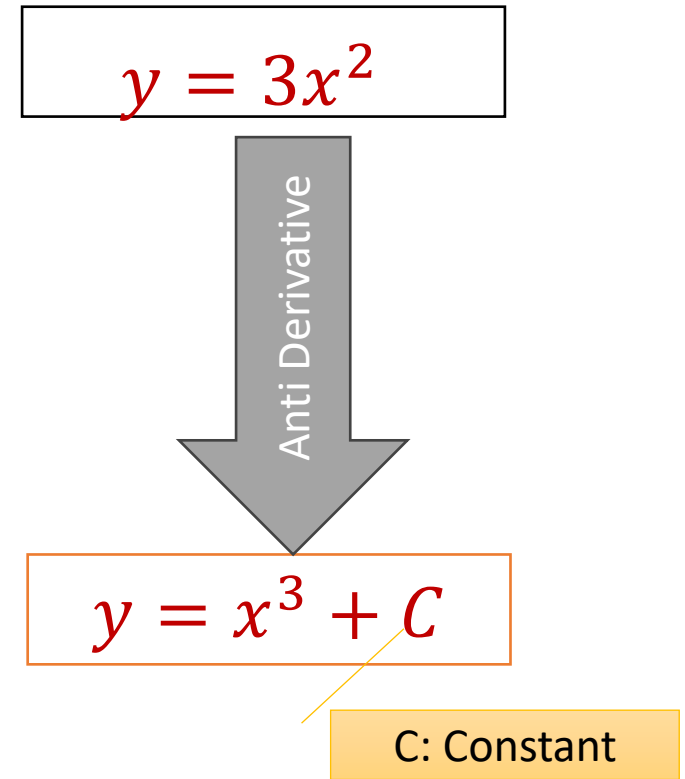
d) $y = x^3 + c$ (c is a constant) $\rightarrow$ "

$y = x^3 + C$

$C = 2$

$C = 1$

$C = 0$

$C = -1$

$C = -2$

$(1, -1)$

$$y = 3x^2$$
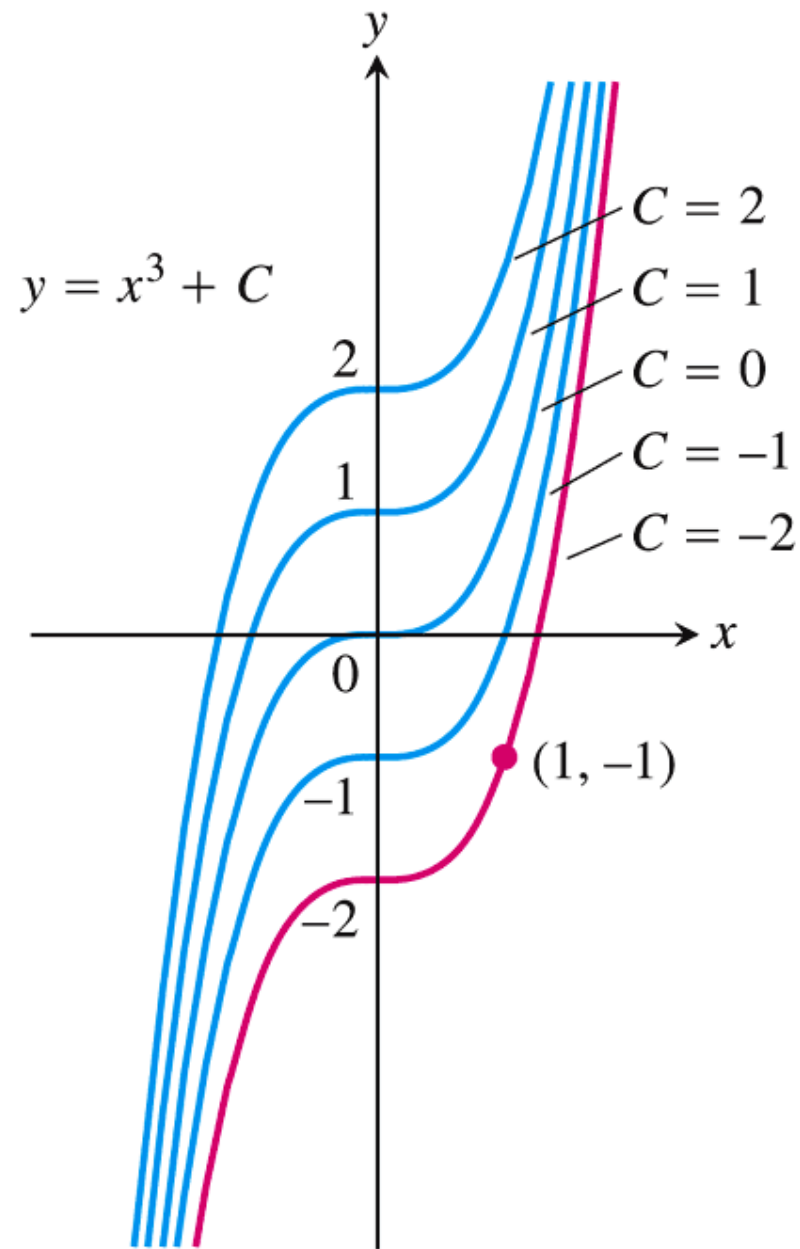
Anti Derivative

$$y = x^3 + C$$

C: Constant

FIGURE 4.1 The curves $y = x^3 + C$ fill the coord This Figure. hout overlapping. In Example 2, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

Antiderivatives _{cont}

We can deduce that:

THEOREM 1 If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

- General Solution
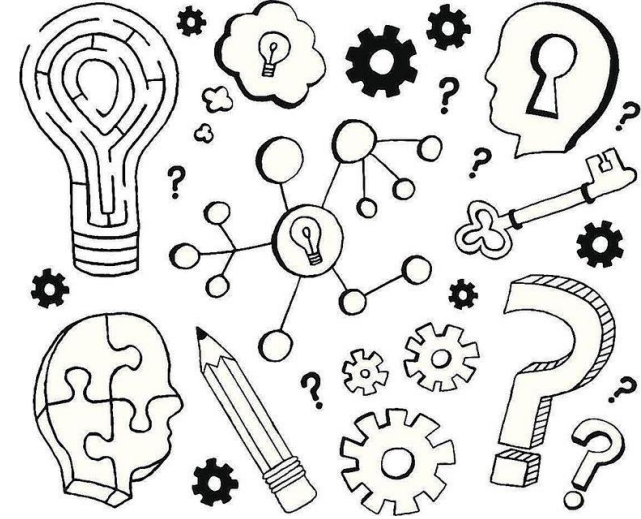
- Particular Solution

$$y = f(x)$$ Antiderivative $$y = F(x)+C$$

Activity (Individual, 10')

1. Show that the antiderivatives of $y = f(x) = x^n$; is F(x) as below.

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

$$F'(x) = \frac{d}{dx}\left(\frac{x^{\wedge}n+1}{x+1} + c\right) = \frac{(n+1)\ x^{\wedge}n}{n+1} + 0 = x^{\wedge}n = f(x)$$

2. If we graph the function F(x), what does the constant C indicate?

3. How many curves will we have?

4. If we graph f(x) for a given n, how many curves will we have?

5. What is a particular solution, and how can it be obtained from the general solution?

Solution

- Show that the antiderivatives of $y = x^n$; is:

$$F(x) = \frac{x^{n+1}}{n+1} + C$$
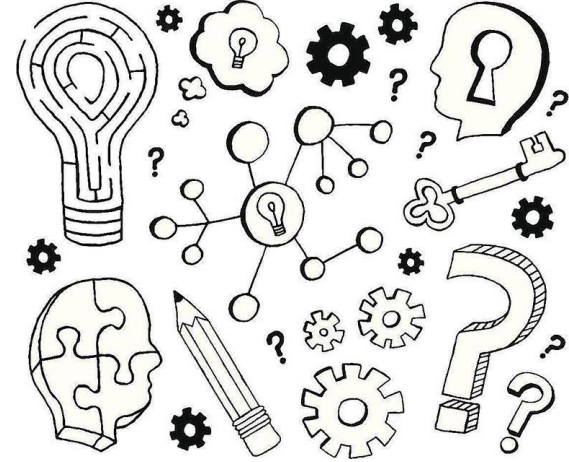
We should show that F'(x)= x^n

- F'(x)=[(n+1)/(n+1)] $x^{(n+1)-1}$ + (dc/dx)

- Hence, F'(x)=x^n

$$F(x) = \frac{x^{n+1}}{n+1} + C \qquad \xrightarrow{\text{Derivative}} \xleftarrow{\text{Antiderivative}} \qquad F'(x) = x^n$$

Indefinite Integral

DEFINITION The collection of all antiderivatives of f represents the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

$$\int f(x)\, dx = F(x) + c$$

- Reflect on Tables 4.1 & 4.2.
  Show that the second
  column is the antiderivative
  of the first column.

  (Hint: Take the derivative of the
  second column)

TABLE 4.1 Antiderivative formulas, k a nonzero constant

	Function	General antiderivative		Function	General antiderivative
1.	x^n	$\dfrac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8.	e^{kx}	$\dfrac{1}{k}e^{kx} + C$
2.	$\sin kx$	$-\dfrac{1}{k}\cos kx + C$	9.	$\dfrac{1}{x}$	$\ln|x| + C, \quad x \neq 0$
3.	$\cos kx$	$\dfrac{1}{k}\sin kx + C$	10.	$\dfrac{1}{\sqrt{1 - k^2 x^2}}$	$\dfrac{1}{k}\sin^{-1} kx + C$
4.	$\sec^2 kx$	$\dfrac{1}{k}\tan kx + C$	11.	$\dfrac{1}{1 + k^2 x^2}$	$\dfrac{1}{k}\tan^{-1} kx + C$
5.	$\csc^2 kx$	$-\dfrac{1}{k}\cot kx + C$	12.	$\dfrac{1}{x\sqrt{k^2 x^2 - 1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6.	$\sec kx \tan kx$	$\dfrac{1}{k}\sec kx + C$	13.	a^{kx}	$\left(\dfrac{1}{k \ln a}\right) a^{kx} + C, \quad a > 0, \ a \neq 1$
7.	$\csc kx \cot kx$	$-\dfrac{1}{k}\csc kx + C$			

TABLE 4.2 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C, \quad k$ a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Today's Outline
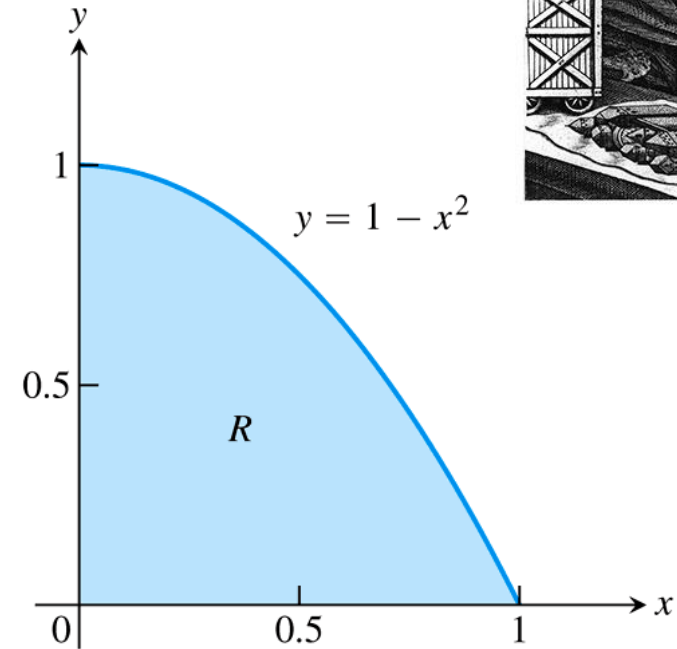
Conceptual Example

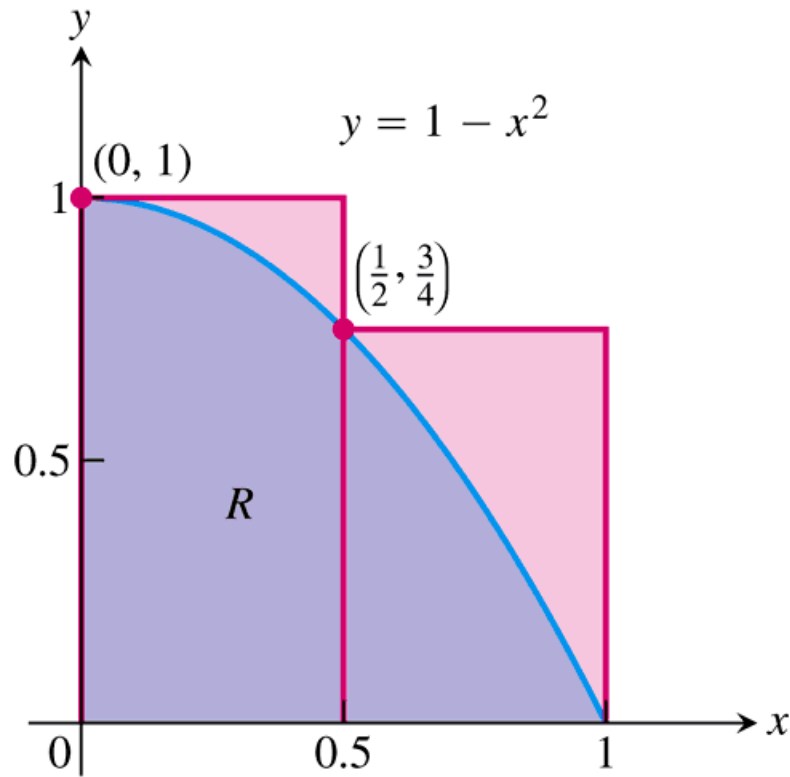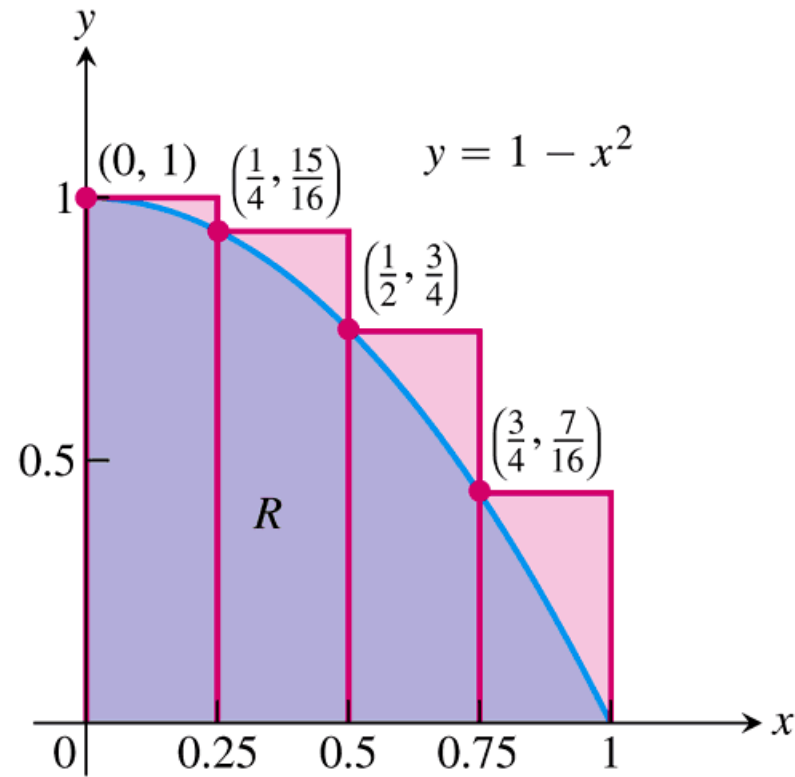**What is the are of R?**

a) We can approximate area R by dividing it to n rectangles

b) Measuring the area of each rectangles → R1, R2, …

c) And R1+R2+R3+…+Rn → R

$y = 1 - x^2$

R

The area of the region R cannot be found by a simple formula.
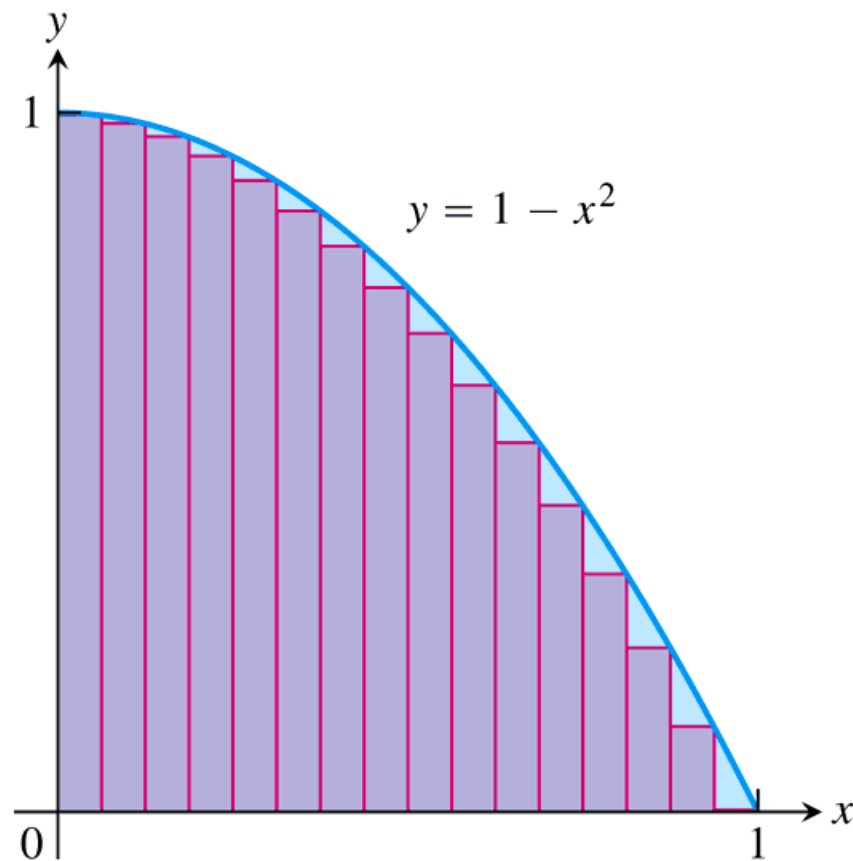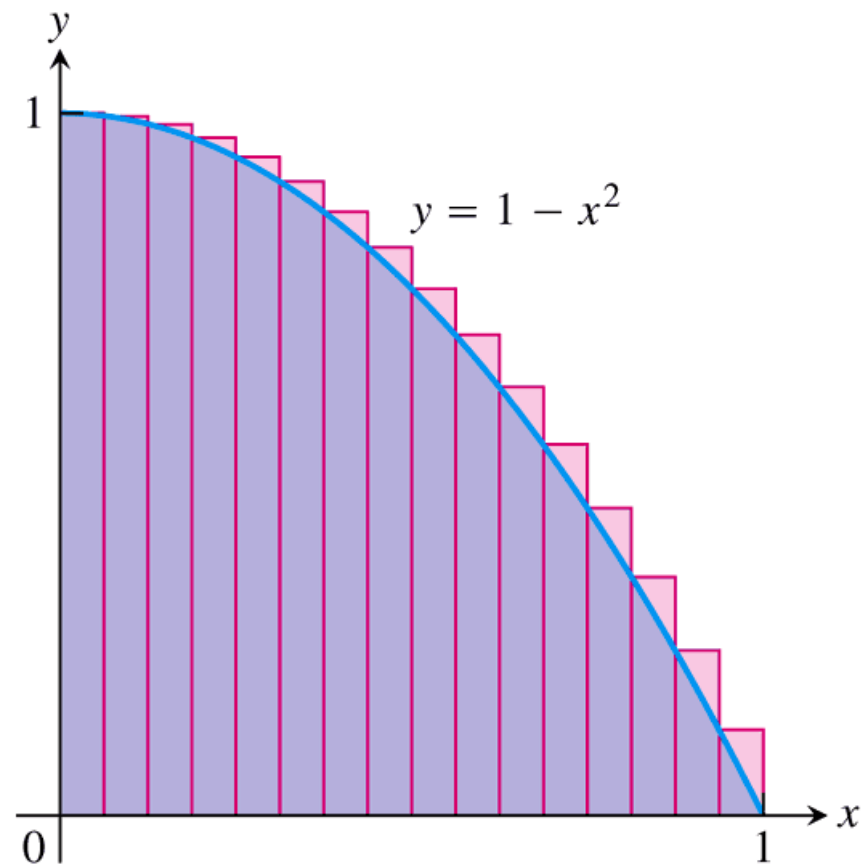
(a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.

(a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$.

(b) An upper sum using 16 rectangles.

Area Under a Curve

$$S = Lim_{n \to \infty} \sum_{i=1}^{n} R_i$$

Riemann Sum

- **Riemann** Completed Archimedes Method

- A Riemann sum is an approximation of the area of a region

- Area under a curve → A

- Named after German mathematician Bernhard Riemann.

- The sum is calculated by dividing the region into rectangles / trapezoids

- The area for each of these shapes is calculated: R1, R2,…, Rn

- These areas are added together: S = R1+ R2+ R3+…+ Rn = $\sum_{i=1}^{n} R_i$

- **S** : Riemann sum

Area under the Curve vs. Riemann Sum

- The Riemann sum S, differs from the actual area A

- The error: Error = $A - [\sum_{i=1}^{n} R_i] = A - S$

- This error can be reduced by using more rectangles → n→∞

- As n→ ∞ : $\Delta x \to 0$ and Error→0, hence A →S

- So A is defined as:

$$A = Lim_{n \to \infty} \sum_{i=1}^{n} R_i$$

Area Under a Curve cont

$$A = \underset{n \to \infty}{Lim} \sum_{i=1}^{n} R_i$$

$$R_i = f(c_i)(\Delta x_i)$$

$$n \to \infty \equiv \Delta x \to 0$$

$$A = \underset{n \to \infty}{Lim} \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$

$$A = \underset{\Delta x \to 0}{Lim} \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$

FIGURE 4.10 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

Definite Integral

A is the definite integral of f over the interval I= [a,b]

$$A = Lim \sum_{i=1}^{n} [f(ci)(\Delta x_i)]$$
$$\Delta x \to 0$$

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\int_{a}^{b} f(x)\, dx$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.
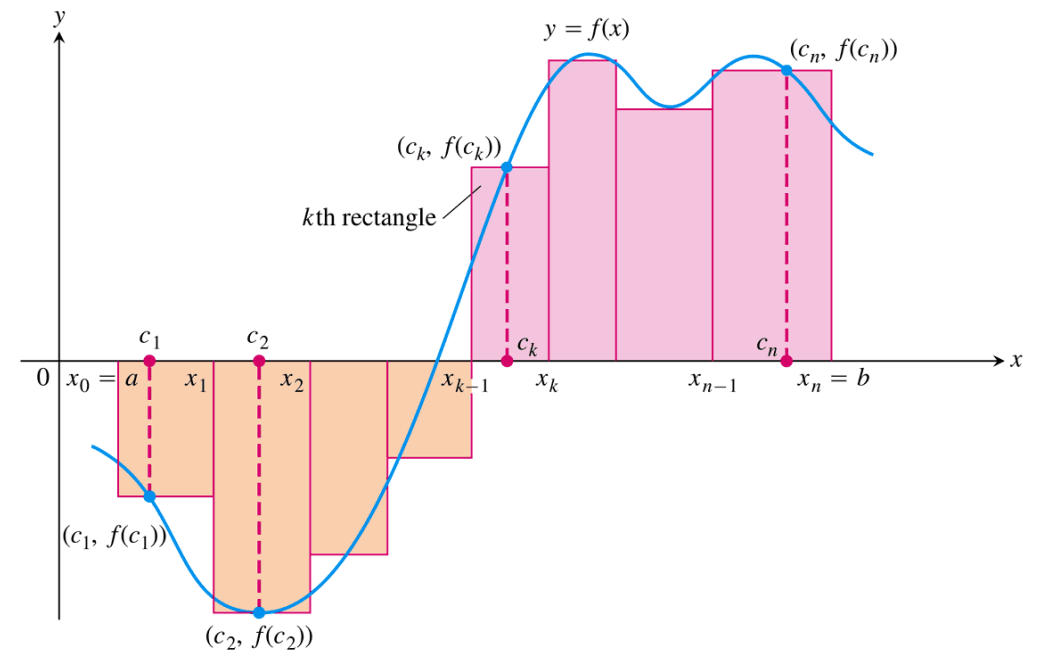
Integral of *f* from *a* to *b*

The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

Definite Integrals continued

DEFINITION   If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\, dx.$$

Definite Integrals continued

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Activity (Individual, 15')

- Justify the below Table by matching it with the relevant Figure.

TABLE 4.5 Rules satisfied by definite integrals

1. *Order of Integration:* $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ A Definition

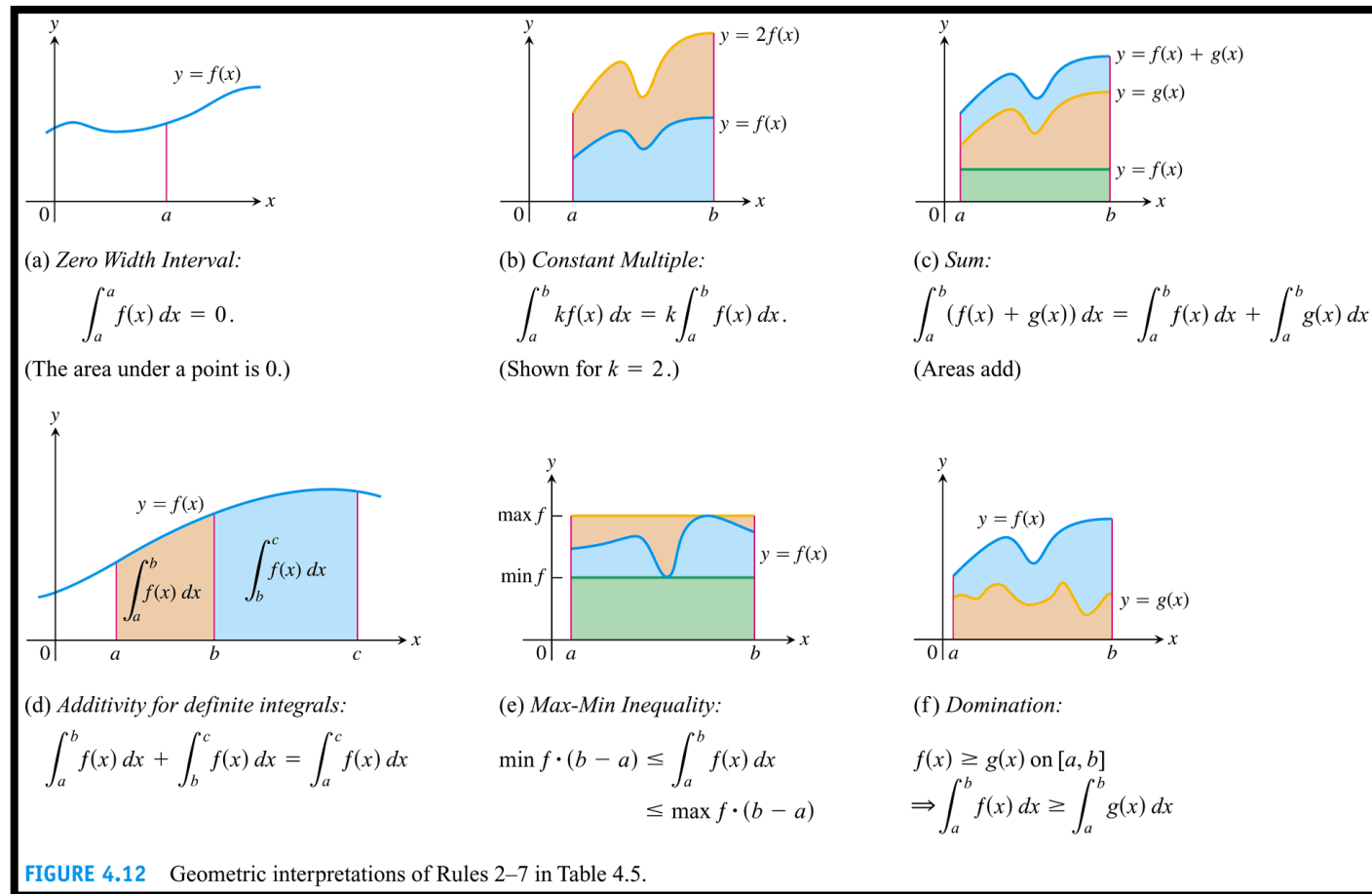2. *Zero Width Interval:* $\displaystyle\int_a^a f(x)\,dx = 0$ Definition when $f(a)$ exists

3. *Constant Multiple:* $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ Any Number k

   $\displaystyle\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ $k = -1$

4. *Sum and Difference:* $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

   $$\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a).$$

7. *Domination:* $f(x) \geq g(x)$ on $[a, b] \Rightarrow \displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

   $f(x) \geq 0$ on $[a, b] \Rightarrow \displaystyle\int_a^b f(x)\,dx \geq 0$ (Special Case)

(a) *Zero Width Interval:*

$\displaystyle\int_a^a f(x)\,dx = 0.$

(The area under a point is 0.)

(b) *Constant Multiple:*

$\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$

(Shown for $k = 2$.)

(c) *Sum:*

$\displaystyle\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(Areas add)

(d) *Additivity for definite integrals:*

$\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

(e) *Max-Min Inequality:*

$\displaystyle\min f \cdot (b - a) \leq \int_a^b f(x)\,dx \leq \max f \cdot (b - a)$

(f) *Domination:*

$f(x) \geq g(x)$ on $[a, b]$

$\Rightarrow \displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

FIGURE 4.12 Geometric interpretations of Rules 2–7 in Table 4.5.

Today's Outline

Fundamental Theorems of Calculus

THEOREM 5—**The Fundamental Theorem of Calculus, Part 1**   If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\, dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\, dt = f(x). \tag{2}$$

THEOREM 5 (Continued)—**The Fundamental Theorem of Calculus, Part 2**   If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Y = F(x) is the antiderivative of f(x)

- Reflect on Tables 8.1.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$

2. $\int k\,du = ku + C$ (any number k)

3. $\int (du + dv) = \int du + \int dv$

4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C$ $(n \neq -1)$

5. $\int \dfrac{du}{u} = \ln |u| + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = -\ln |\cos u| + C$

$= \ln |\sec u| + C$

13. $\int \cot u\,du = \ln |\sin u| + C$

$= -\ln |\csc u| + C$

14. $\int e^u\,du = e^u + C$

15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C$ $(a > 0, a \neq 1)$

16. $\int \sinh u\,du = \cosh u + C$

17. $\int \cosh u\,du = \sinh u + C$

18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$

21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C$ $(a > 0)$

22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C$ $(u > a > 0)$

Time for a break – 20'

Today's Outline

I. ~~Antiderivatives~~

II. ~~Definite Integrals~~

III. ~~Fundamental Theorem of Calculus~~

IV. Integration by Substitution

V. Integration by Parts

VI. Other Integration Methods

VII. Tutorials

Conceptual Example

Solve the following Integral

$$I = \int 2x \left(\sin x^2\right) dx$$

- Let $x^2 = u$ → Then $2x\, dx = du$

- So by replacing → $I = \int \sin u\, du = -\cos u + C \Rightarrow \boxed{I = -\cos x^2 + C}$

Integration By Substitution

THEOREM 6—The Substitution Rule   If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

- Substitution: g(x) = u

- g'(x) dx = du

- f(g(x)) g'(x) dx ➔ f(u) du
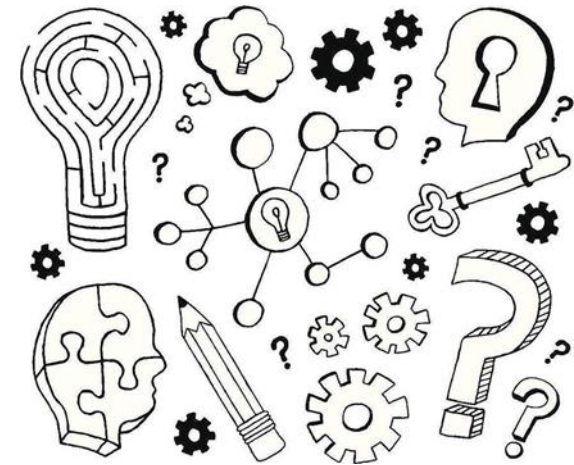
Example

Solve the following Integral

$$I = \int_0^4 e^{-2x}\, dx$$

- Let $u = -2x \rightarrow du = d(-2x) = -2\,(dx) \rightarrow dx = -\dfrac{du}{2}$

- $u(0) = 0$

- $u(4) = -8$

- $I = \int_0^{-8} e^u\, du/(-2) \rightarrow I = (-1/2) \int_0^{-8} e^u\, du \rightarrow I = \left(-\dfrac{1}{2}\right)[e^{-8} - e^0] = \dfrac{(1-e^{-8})}{2}$

Activity (Individual, 15')

Read the following article and solve its examples.

https://www.mathcentre.ac.uk/resources/workbooks/mathcentre/web-integrationbysub-tony.pdf

Today's Outline

I. ~~Antiderivatives~~

II. ~~Definite Integrals~~

III. ~~Fundamental Theorem of Calculus~~

IV. ~~Integration by Substitution~~

V. Integration by Parts

VI. Other Integration Methods

VII. Tutorials

Integration By Parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

Integration By Parts cont

- The <u>Product Rule</u> states that, if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

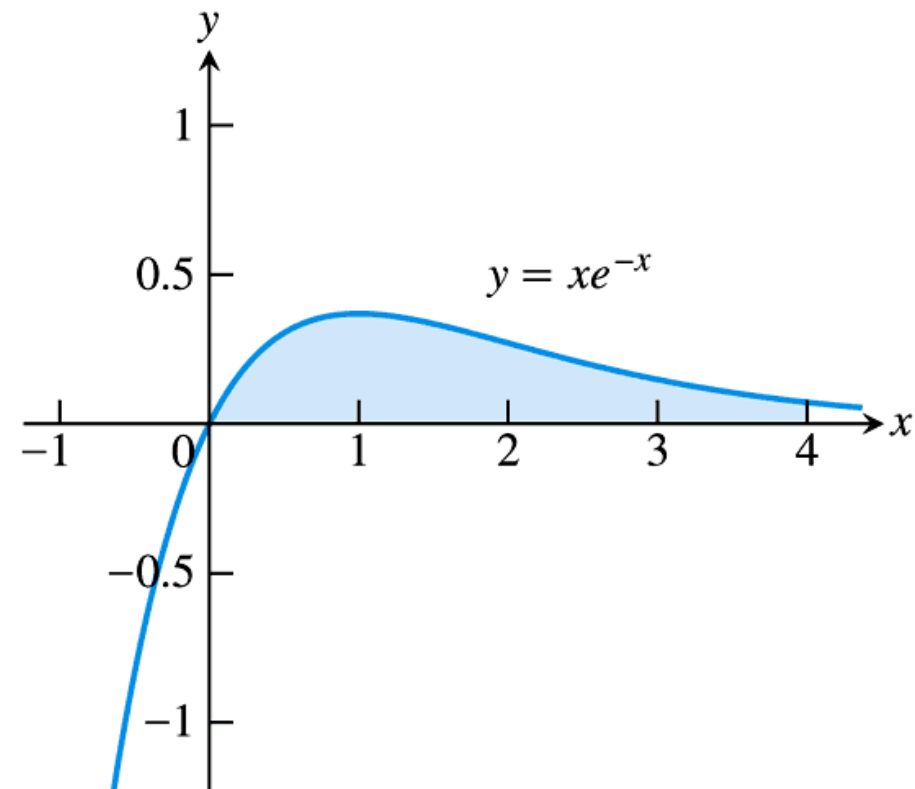- <u>Integrating</u> the above, and <u>reordering</u> the terms:

$$f(x)g(x) = \boxed{\int f(x)g'(x)\,dx} + \int g(x)f'(x)\,dx \Rightarrow$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

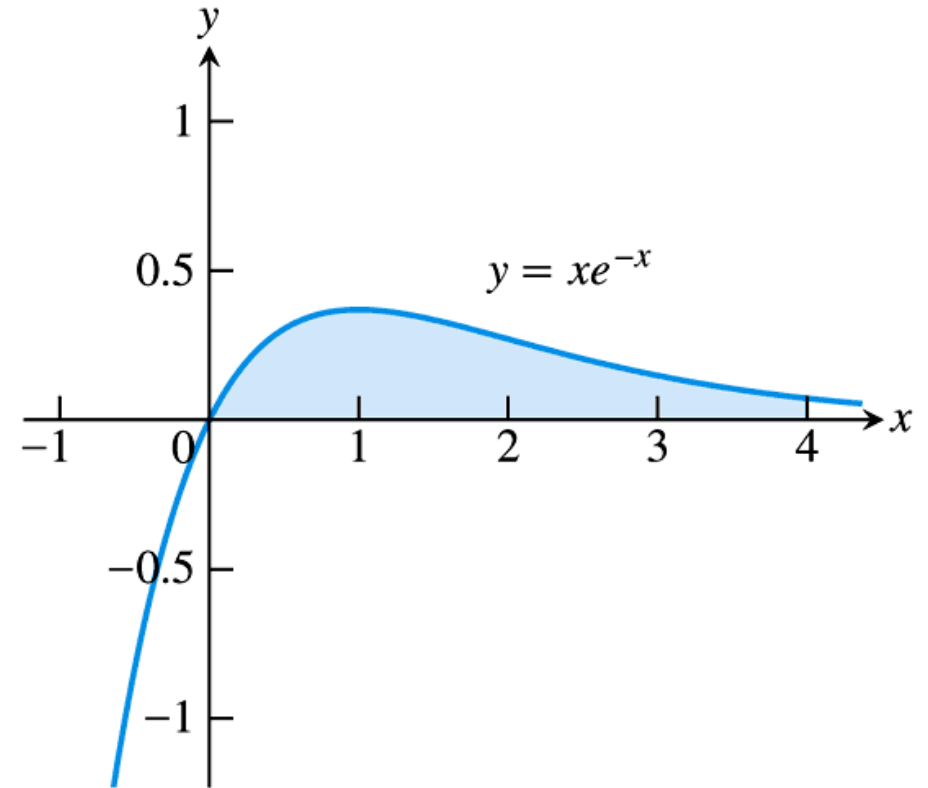# Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,\mathrm{d}x$$

$y = xe^{-x}$

The region in Example

Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,dx$$

The region in Example

- Let $x = u$ ➡ $dx = du$

- Let $e^{-x}\,dx = dv$ ➡ $V = -e^{-x}$

- SO: $UV = -xe^{-x}$
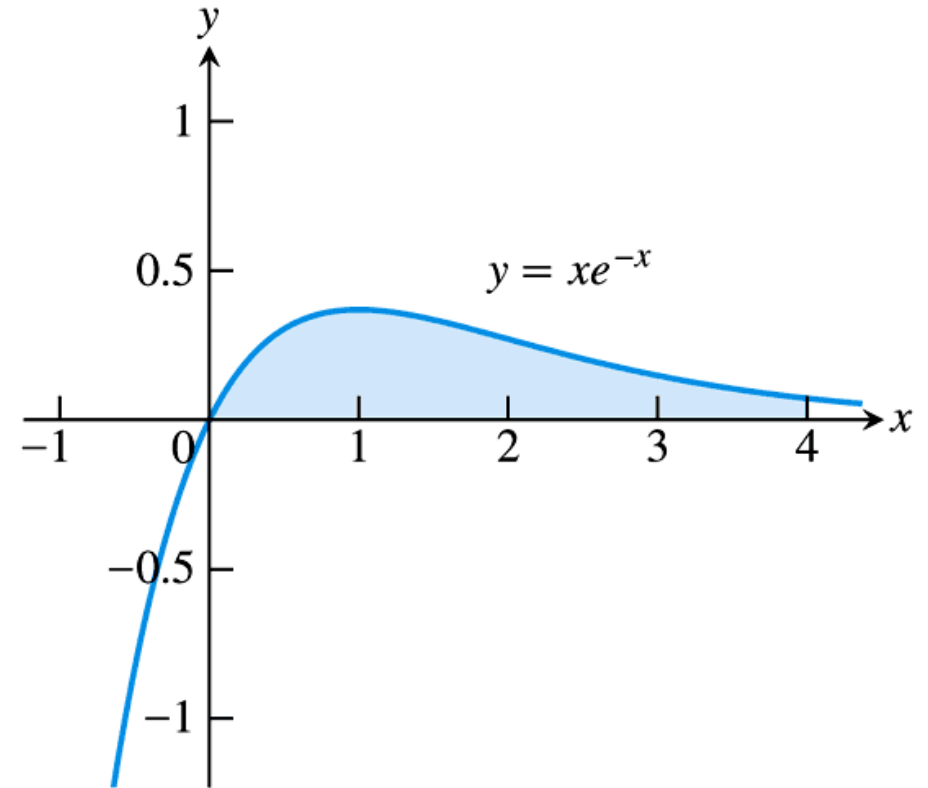
- SO: $\int V\,du = \int -e^{-x}\,dx$

Example

Solve the following Integral

$$\int_0^4 xe^{-x}\,dx$$

- I $= -xe^{-x}$ - $\int -e^{-x}\,dx =$

- I $= -xe^{-x}$ - $e^{-x} = -e^{-x}\,(x+1)$

- I $(4) = 5\,(-e^{-4}\,) = $ -5 e^{-4}

- I $(0) = 1\,(-e^{-0}\,) = $ -1

  ➡ I $=$ -5 e^{-4} +1

$y = xe^{-x}$

The region in Example

Integration By Parts- Summary

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

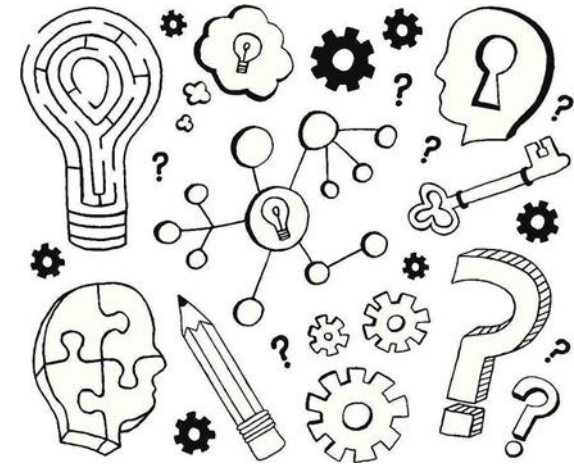**Integration by Parts Formula for Definite Integrals**

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \tag{3}$$

Activity (Individual, 15')

Read the following article and solve its examples.

https://www.mathcentre.ac.uk/resources/uploaded/mc-ty-parts-2009-1.pdf

Today's Outline

I. ~~Antiderivatives~~

II. ~~Definite Integrals~~

III. ~~Fundamental Theorem of Calculus~~

IV. ~~Integration by Substitution~~

V. ~~Integration by Parts~~

VI. Other Integration Methods

VII. Tutorials

Numerical Integration

- There are several other methods for evaluation the integral of a function.

- Such as :

  - Integration Using Trigonometric Identities.

  - Integration by Partial Fraction

  - Integration Using Trigonometric substitution

  - More & More

- Despite these methods, many integrals can not be solved explicitly.

- Using numerical methods and programing, most integrals can be solved.

Today's Outline

I. ~~Antiderivatives~~

II. ~~Definite Integrals~~

III. ~~Fundamental Theorem of Calculus~~

IV. ~~Integration by Substitution~~

V. ~~Integration by Parts~~

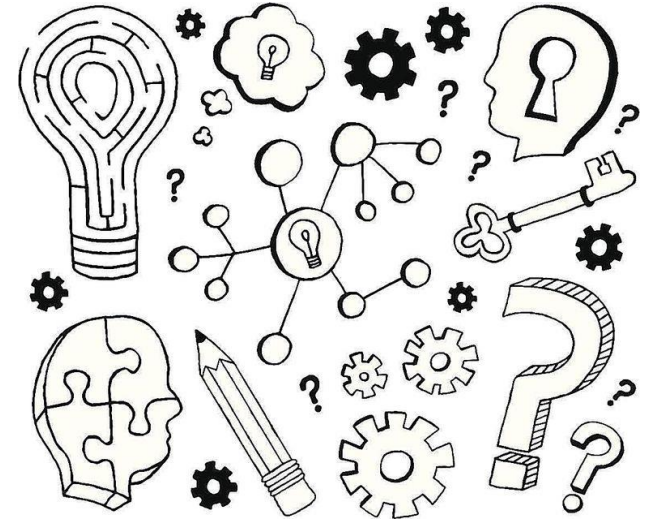VI. ~~Other Integration Methods~~

VII. Tutorials

Exercise 1

Solve the following Indefinite integrals:

i. $\int \left[(2x - 9)/\left(\sqrt{x^2 - 9x + 1} \right) \right] dx$

ii. $\int \sqrt{8x - x^2} \, dx$

iii. $\int (Sec\theta + tan\theta)^2 \, d\theta$

iv. $\int \sqrt{1 + \cos 4x} \, d\theta$

Exercise 1 _{cont}

Procedures for Matching Integrals to Basic Formulas

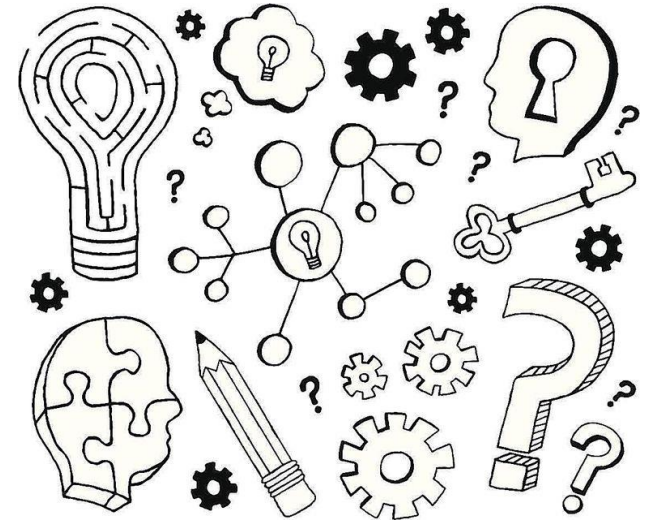| PROCEDURE	EXAMPLE
(i) Making a simplifying substitution	$\dfrac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx = \dfrac{du}{\sqrt{u}}$
(ii) Completing the square	$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$
(iii) Using a trigonometric identity	$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x$ $= \sec^2 x + 2\sec x \tan x$ $\qquad + (\sec^2 x - 1)$ $= 2\sec^2 x + 2\sec x \tan x - 1$
(iv) Eliminating a square root	$\sqrt{1 + \cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,\lvert\cos 2x\rvert$

Exercise 2

Solve the following integrals:

i. $\int x \, sinx \, dx$

ii. $\int \ln x \, dx$

Solution 2-i

- Let $\qquad$ $\boxed{u = x \ \Downarrow \ du = dx}$ $\boxed{dv = \sin x \, dx \ \Downarrow \ v = -\cos x}$
- Then,

- Using Formula 2, we have:

$$\int x \sin x \, dx = \int \overset{u}{x} \overbrace{\sin x \, dx}^{dv} = \overset{u}{x} \overbrace{(-\cos x)}^{v} - \int \overbrace{(-\cos x)}^{v} \overset{du}{dx}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

Solution 2-ii

- Let

$$u = \ln x \qquad dv = dx$$

- Then,

$$du = \frac{1}{x}\,dx \qquad v = x$$

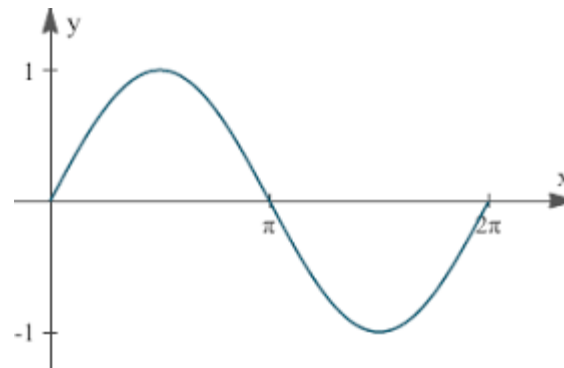$$\int \ln x\,dx = x \ln x - \int x \frac{dx}{x}$$

$$= x \ln x - \int dx$$
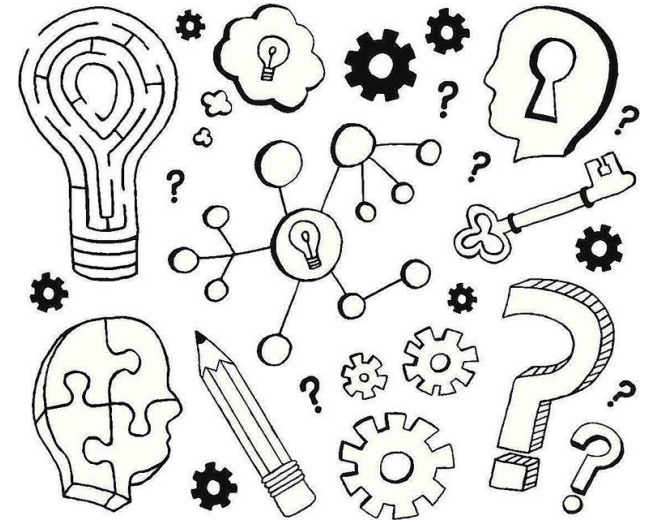
$$= x \ln x - x + C$$

Exercise 3

Find the area under the curve $y = \sin x$
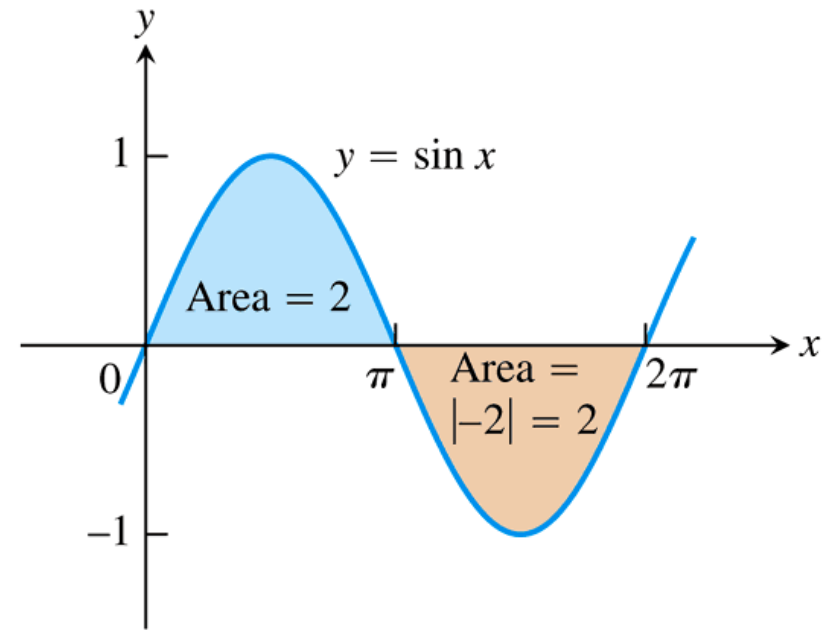
i.   On I=$[0,\pi]$

ii.   On I=$[0,2\pi]$

Note: Antiderivative

of Sin x is $-\cos(x) + C$

Exercise 3- Solution

- $I = \int_0^\pi \sin x\, dx = -[\cos\pi - \cos 0] = 2$

- $I = \int_0^{2\pi} \sin x\, dx = 2\int_0^\pi \sin x\, dx = 4$

The total area between $y = \sin x$ and the x-axis for $0 \le x \le 2\pi$ is the sum of the absolute values of two integrals

Source of the slides:

Thomas Calculus – 11e

Stewart Calculus

https://www.slideserve.com/search/presentations/derivatives-and-integrals