of zero. The point is that these absolute systems aren't used in many cases—the relative system is the one that's the most useful.

Velocity. How fast are you moving right now? Perhaps you're sitting in a comfy chair, so you'd say that your speed was zero. Maybe you're in a car and so you might say something like 65 mph. (Hopefully someone *else* is driving!) Actually, you are hurtling through space at almost 30 km *per second*! That's about the speed that Earth travels in order to make the 939-million-km trek around the sun each year. Of course, even this velocity is relative to the sun. Our solar system is moving around within the Milky Way galaxy. So then how fast are we actually moving, in absolute terms? Galileo told us back in the 17th century that this question doesn't have an answer—all velocity is relative.

Our difficulty in establishing absolute velocity is similar to the difficulty in establishing position. After all, velocity is displacement (difference between positions) over time. To establish an absolute velocity, we'd need to have some reference location that would "stay still" so that we could measure our displacement from that location. Unfortunately, everything in our universe seems to be orbiting something else.

2.5 Negating a Vector

The previous sections have presented a high-level overview of vectors. The remainder of this chapter looks at specific mathematical operations we perform on vectors. For each operation, we first define the mathematical rules for performing the operation and then describe the geometric interpretations of the operation and give some practical uses for the operation.

The first operation we'd like to consider is that of vector negation. When discussing the zero vector, we asked you to recall from group theory the idea of the *additive identity*. Please go back to wherever it was in your brain that you found the additive identity, perhaps between the metaphorical couch cushions, or at the bottom of a box full of decade-old tax forms. Nearby, you will probably find a similarly discarded obvious-to-the-point-of-useless concept: the *additive inverse*. Let's dust it off. For any group, the additive inverse of x, denoted by $-x$, is the element that yields the additive identity (zero) when added to x. Put simply, $x + (-x) = 0$. Another way of saying this is that elements in the group can be negated.

The negation operation can be applied to vectors. Every vector $\mathbf{v}$ has an additive inverse $-\mathbf{v}$ of the same dimension as $\mathbf{v}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$. (We will learn how to add vectors in Section 2.7.)

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2.5.1 Official Linear Algebra Rules

To negate a vector of any dimension, we simply negate each component of the vector. Stated formally,

Negating a vector

$$-\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_{n-1} \\ -a_n \end{bmatrix}.$$

Applying this to the specific cases of 2D, 3D, and 4D vectors, we have

Negating 2D, 3D, and 4D vectors

$$\begin{aligned} -\begin{bmatrix} x & y \end{bmatrix} &= \begin{bmatrix} -x & -y \end{bmatrix}, \\ -\begin{bmatrix} x & y & z \end{bmatrix} &= \begin{bmatrix} -x & -y & -z \end{bmatrix}, \\ -\begin{bmatrix} x & y & z & w \end{bmatrix} &= \begin{bmatrix} -x & -y & -z & -w \end{bmatrix}. \end{aligned}$$

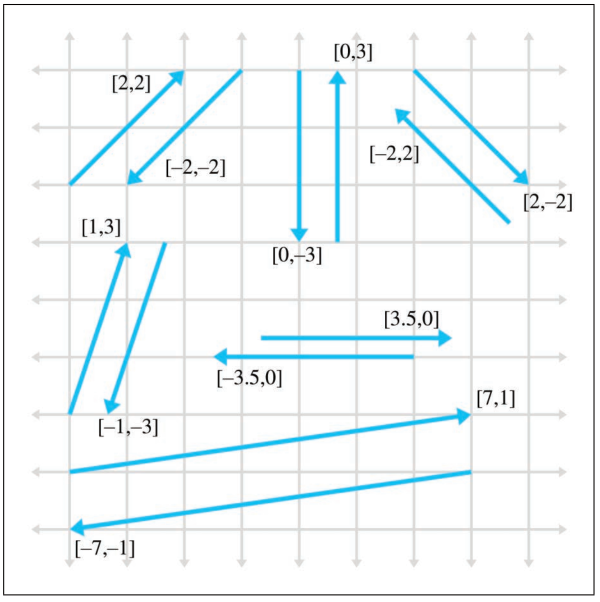

Figure 2.8. Examples of vectors and their negatives. Notice that a vector and its negative are parallel and have the same magnitude, but point in opposite directions.

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A few examples are

$$
\begin{aligned}
-\begin{bmatrix}4 & -5\end{bmatrix} &= \begin{bmatrix}-4 & 5\end{bmatrix}, \\
-\begin{bmatrix}-1 & 0 & \sqrt{3}\end{bmatrix} &= \begin{bmatrix}1 & 0 & -\sqrt{3}\end{bmatrix}, \\
-\begin{bmatrix}1.34 & -3/4 & -5 & \pi\end{bmatrix} &= \begin{bmatrix}-1.34 & 3/4 & 5 & -\pi\end{bmatrix}.
\end{aligned}
$$

2.5.2 Geometric Interpretation

Negating a vector results in a vector of the same magnitude but opposite direction, as shown in Figure 2.8.

Remember, the position of a vector on a diagram is irrelevant—only the magnitude and direction are important.

2.6 Vector Multiplication by a Scalar

Although we cannot add a vector and a scalar, we can multiply a vector by a scalar. The result is a vector that is parallel to the original vector, with a different length and possibly opposite direction.

2.6.1 Official Linear Algebra Rules

Vector-times-scalar multiplication is straightforward; we simply multiply each component of the vector by the scalar. Stated formally,

Multiplying a vector by a scalar

$$
k\begin{bmatrix}a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n\end{bmatrix} = \begin{bmatrix}a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n\end{bmatrix} k = \begin{bmatrix}ka_1 \\ ka_2 \\ \vdots \\ ka_{n-1} \\ ka_n\end{bmatrix}.
$$

Applying this rule to 3D vectors, as an example, we get

Multiplying a 3D vector by a scalar

$$
k\begin{bmatrix}x \\ y \\ z\end{bmatrix} = \begin{bmatrix}x \\ y \\ z\end{bmatrix} k = \begin{bmatrix}kx \\ ky \\ kz\end{bmatrix}.
$$

Although the scalar and vector may be written in either order, most people choose to put the scalar on the left, preferring $k\mathbf{v}$ to $\mathbf{v}k$.

A vector may also be divided by a nonzero scalar. This is equivalent to multiplying by the reciprocal of the scalar:

Dividing a 3D vector by a scalar

$$
\frac{\mathbf{v}}{k} = \left(\frac{1}{k}\right)\mathbf{v} = \begin{bmatrix}v_x/k \\ v_y/k \\ v_z/k\end{bmatrix} \qquad \text{for 3D vector } \mathbf{v} \text{ and nonzero scalar } k.
$$

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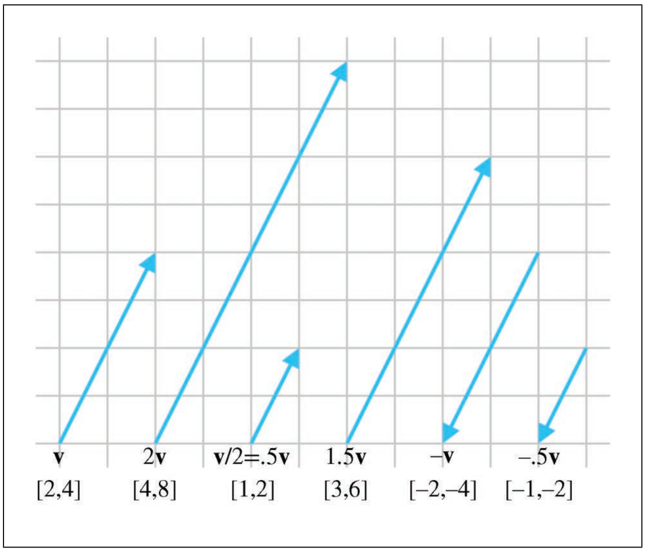

Figure 2.9
A 2D vector multiplied by various scalars

Some examples are

$$\begin{aligned} 2\begin{bmatrix}1 & 2 & 3\end{bmatrix} &= \begin{bmatrix}2 & 4 & 6\end{bmatrix}, \\ -3\begin{bmatrix}-5.4 & 0\end{bmatrix} &= \begin{bmatrix}16.2 & 0\end{bmatrix}, \\ \begin{bmatrix}4.7 & -6 & 8\end{bmatrix}/2 &= \begin{bmatrix}2.35 & -3 & 4\end{bmatrix}. \end{aligned}$$

Here are a few things to notice about multiplication of a vector by a scalar:

- When we multiply a vector and a scalar, we do not use any multiplication symbol. The multiplication is signified by placing the two quantities side-by-side (usually with the vector on the right).
- Scalar-times-vector multiplication and division both occur before any addition and subtraction. For example $3\mathbf{a}+\mathbf{b}$ is the same as $(3\mathbf{a})+\mathbf{b}$, not $3(\mathbf{a}+\mathbf{b})$.
- A scalar may not be divided by a vector, and a vector may not be divided by another vector.
- Vector negation can be viewed as the special case of multiplying a vector by the scalar -1.

2.6.2 Geometric Interpretation

Geometrically, multiplying a vector by a scalar k has the effect of scaling the length by a factor of $|k|$. For example, to double the length of a vector we

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would multiply the vector by 2. If $k < 0$, then the direction of the vector is flipped. Figure 2.9 illustrates a vector multiplied by several different scalars.

2.7 Vector Addition and Subtraction

We can add and subtract two vectors, provided they are of the same dimension. The result is a vector quantity of the same dimension as the vector operands. We use the same notation for vector addition and subtraction as is used for addition and subtraction of scalars.

2.7.1 Official Linear Algebra Rules

The linear algebra rules for vector addition are simple: to add two vectors, we add the corresponding components:

Adding two vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_{n-1} + b_{n-1} \\ a_n + b_n \end{bmatrix}.$$

Subtraction can be interpreted as adding the negative, so $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$:

Subtracting two vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} + \left(- \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \right) = \begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ \vdots \\ a_{n-1} - b_{n-1} \\ a_n - b_n \end{bmatrix}.$$

For example, given

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 7 \\ -3 \\ 0 \end{bmatrix},$$

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then

$$\mathbf{a}+\mathbf{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}+\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}1+4\\2+5\\3+6\end{bmatrix}=\begin{bmatrix}5\\7\\9\end{bmatrix},$$

$$\mathbf{a}-\mathbf{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}-\begin{bmatrix}4\\5\\6\end{bmatrix}=\begin{bmatrix}1-4\\2-5\\3-6\end{bmatrix}=\begin{bmatrix}-3\\-3\\-3\end{bmatrix},$$

$$\mathbf{b}+\mathbf{c}-\mathbf{a}=\begin{bmatrix}4\\5\\6\end{bmatrix}+\begin{bmatrix}7\\-3\\0\end{bmatrix}-\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}4+7-1\\5+(-3)-2\\6+0-3\end{bmatrix}=\begin{bmatrix}10\\0\\3\end{bmatrix}.$$

A vector cannot be added or subtracted with a scalar, or with a vector of a different dimension. Also, just like addition and subtraction of scalars, vector addition is commutative,

$$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a},$$

whereas vector subtraction is anticommutative,

$$\mathbf{a}-\mathbf{b}=-(\mathbf{b}-\mathbf{a}).$$

2.7.2 Geometric Interpretation

We can add vectors **a** and **b** geometrically by positioning the vectors so that the head of **a** touches the tail of **b** and then drawing a vector from

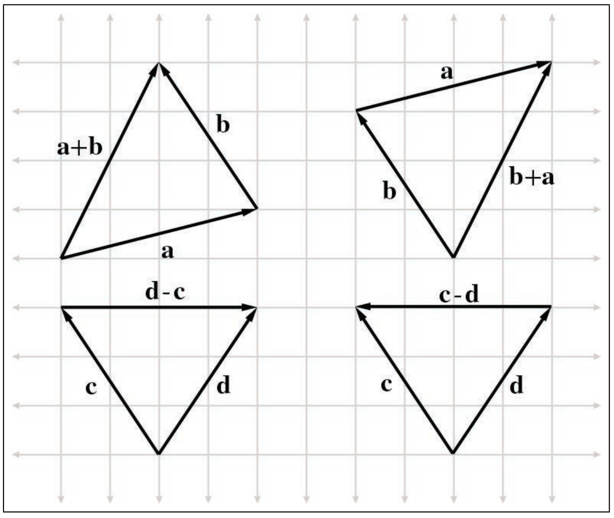

Figure 2.10
2D vector addition and subtraction using the triangle rule.

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the tail of $\mathbf{a}$ to the head of $\mathbf{b}$. In other words, if we start at a point and apply the displacements specified by $\mathbf{a}$ and then $\mathbf{b}$, it's the same as if we had applied the single displacement $\mathbf{a} + \mathbf{b}$. This is known as the *triangle rule* of vector addition. It also works for vector subtraction, as shown in Figure 2.10.

Figure 2.10 provides geometric evidence that vector addition is commutative but vector subtraction is not. Notice that the vector labeled $\mathbf{a}+\mathbf{b}$ is identical to the vector labeled $\mathbf{b}+\mathbf{a}$, but the vectors $\mathbf{d}-\mathbf{c}$ and $\mathbf{c}-\mathbf{d}$ point in opposite directions because $\mathbf{d}-\mathbf{c} = -(\mathbf{c}-\mathbf{d})$.

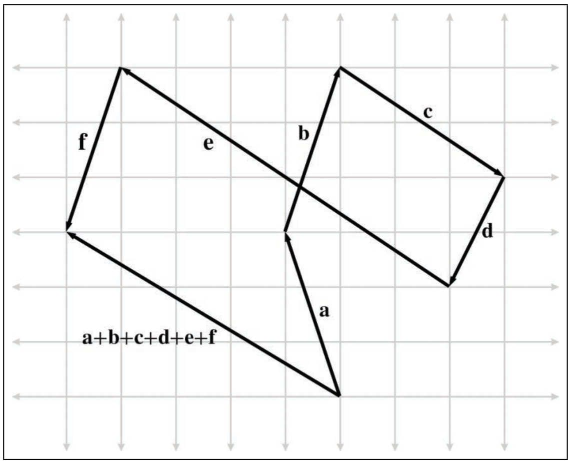

Figure 2.11
Extending the triangle rule to more than two vectors

The triangle rule can be extended to more than two vectors. Figure 2.11 shows how the triangle rule verifies something we stated in Section 2.3.1: a vector can be interpreted as a sequence of axially aligned displacements.

Figure 2.12 is a reproduction of Figure 2.5, which shows how the vector $[1, -3, 4]$ may be interpreted as a displacement of 1 unit to the right, 3 units down, and then 4 units forward, and can be verified mathematically by using vector addition:

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}.$$

This seems obvious, but this is a very powerful concept. We will use a similar technique in Section 4.2 to transform vectors from one coordinate space to another.

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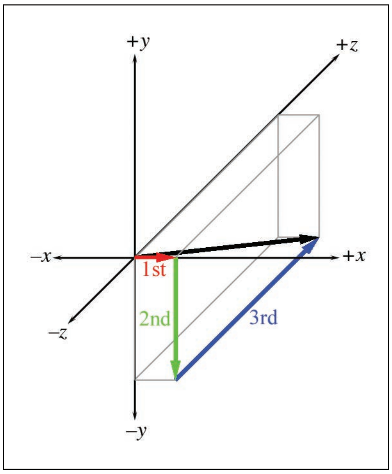

Figure 2.12
Interpreting a vector as a sequence of displacements

2.7.3 Displacement Vector from One Point to Another

It is very common that we will need to compute the displacement from one point to another. In this case, we can use the triangle rule and vector subtraction. Figure 2.13 shows how the displacement vector from **a** to **b** can be computed by subtracting **a** from **b**.

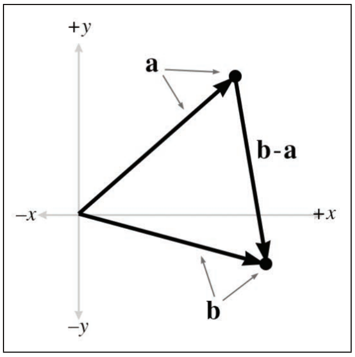

Figure 2.13
Using 2D vector subtraction to compute the vector from point **a** to point **b**

As Figure 2.13 shows, to compute the vector from **a** to **b**, we interpret the points **a** and **b** as vectors from the origin, and then use the triangle rule. In fact, this is how vectors are defined in some texts: the subtraction of two points.

Notice that the vector subtraction $\mathbf{b} - \mathbf{a}$ yields a vector *from* **a** *to* **b**. It doesn't make any sense to simply find the vector "between two points," since the language in this sentence does not specify a direction. We must always form a vector that goes *from* one point *to* another point.

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2.8 Vector Magnitude (Length)

As we have discussed, vectors have magnitude and direction. However, you might have noticed that neither the magnitude nor the direction is expressed explicitly in the vector (at least not when we use Cartesian coordinates). For example, the magnitude of the 2D vector $[3, 4]$ is neither 3 nor 4; it's 5. Since the magnitude of the vector is not expressed explicitly, we must compute it. The magnitude of a vector is also known as the *length* or *norm* of the vector.

2.8.1 Official Linear Algebra Rules

In linear algebra, the magnitude of a vector is denoted by using double vertical bars surrounding the vector. This is similar to the single vertical bar notation used for the absolute value operation for scalars. This notation and the equation for computing the magnitude of a vector of arbitrary dimension n are shown in Equation (2.2):

Magnitude of a vector of arbitrary dimension

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} {v_i}^2} = \sqrt{{v_1}^2 + {v_2}^2 + \cdots + {v_{n-1}}^2 + {v_n}^2}. \tag{2.2}$$

Thus, the magnitude of a vector is the square root of the sum of the squares of the components of the vector. This sounds complicated, but the magnitude equations for 2D and 3D vectors are actually very simple:

Vector magnitude for 2D and 3D vectors

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{{v_x}^2 + {v_y}^2} && \text{(for a 2D vector } \mathbf{v}\text{)}, \\ \|\mathbf{v}\| &= \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} && \text{(for a 3D vector } \mathbf{v}\text{)}. \end{aligned} \tag{2.3}$$

The magnitude of a vector is a nonnegative scalar quantity. An example of how to compute the magnitude of a 3D vector is

$$\begin{aligned} \left\| \begin{bmatrix} 5 & -4 & 7 \end{bmatrix} \right\| &= \sqrt{5^2 + (-4)^2 + 7^2} = \sqrt{25 + 16 + 49} = \sqrt{90} \\ &\approx 9.4868. \end{aligned}$$

Some books use a single bar notation to indicate vector magnitude: $|\mathbf{v}|$

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One quick note to satisfy all you sticklers who already know about vector norms and at this moment are pointing your web browser to gamemath.com, looking for the email address for errata. The term *norm* actually has a very general definition, and basically any equation that meets a certain set of criteria can call itself a norm. So to describe Equation (2.2) as *the* equation for the vector norm is slightly misleading. To be more accurate, we should say that Equation (2.2) is the equation for the *2-norm*, which is one specific way to calculate a norm. The 2-norm belongs to a class of norms known as the *p-norms*, and the p-norm is not the only way to define a norm. Still, omitting this level of generality isn't too harmful of a delusion; because the 2-norm measures Euclidian distance, it is by far the most commonly used norm in geometric applications. It is also widely used in situations even where a geometric interpretation is not directly applicable. Readers interested in such exotica should check out Exercise 15.

2.8.2 Geometric Interpretation

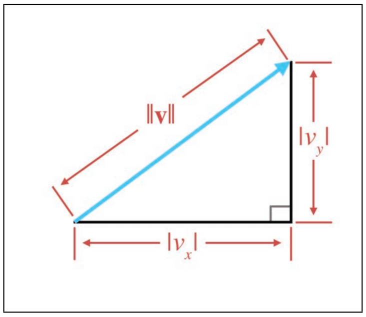

Figure 2.14
Geometric interpretation of the magnitude equation

Let's try to get a better understanding of why Equation (2.3) works. For any vector $\mathbf{v}$ in 2D, we can form a right triangle with $\mathbf{v}$ as the hypotenuse, as shown in Figure 2.14.

Notice that to be precise we had to put absolute value signs around the components v_x and v_y. The components of the vector may be negative, since they are *signed displacements*, but *length* is always positive.

The Pythagorean theorem states that for any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. Applying this theorem to Figure 2.14, we have

$$\|\mathbf{v}\|^2 = |v_x|^2 + |v_y|^2.$$

Since $|x|^2 = x^2$, we can omit the absolute value symbols:

$$\|\mathbf{v}\|^2 = {v_x}^2 + {v_y}^2.$$

Then, by taking the square root of both sides and simplifying, we get

$$\begin{aligned}\sqrt{\|\mathbf{v}\|^2} &= \sqrt{{v_x}^2 + {v_y}^2},\\ \|\mathbf{v}\| &= \sqrt{{v_x}^2 + {v_y}^2},\end{aligned}$$

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which is the same as Equation (2.3). The proof of the magnitude equation in 3D is only slightly more complicated.

For any positive magnitude m, there are an infinite number of vectors of magnitude m. Since these vectors all have the same length but different directions, they form a circle when the tails are placed at the origin, as shown in Figure 2.15.

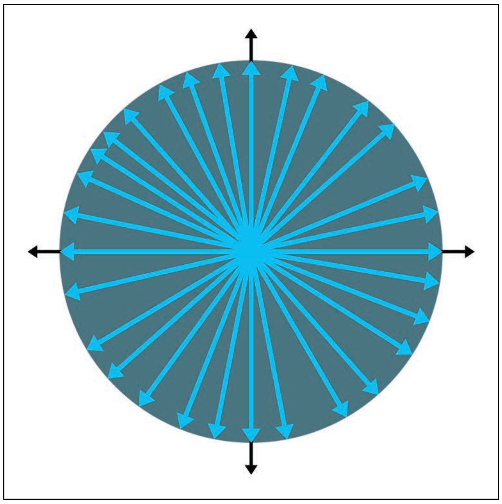

Figure 2.15
For any positive magnitude, there are an infinite number of vectors with that magnitude

2.9 Unit Vectors

For many vector quantities, we are concerned only with direction and not magnitude: "Which way am I facing?" "Which way is the surface oriented?" In these cases, it is often convenient to use unit vectors. A *unit vector* is a vector that has a magnitude of one. Unit vectors are also known as *normalized vectors*.

Unit vectors are also sometimes simply called *normals*; however, a warning is in order concerning terminology. The word "normal" carries with it the connotation of "perpendicular." When most people speak of a "normal" vector, they are usually referring to a vector that is perpendicular to something. For example, a *surface normal* at a given point on an object is a vector that is perpendicular to the surface at that location. However, since the concept of perpendicular is related only to the direction of a vector and not its magnitude, in most cases you will find that unit vectors are used for normals instead of a vector of arbitrary length. When this book refers to a vector as a "normal," it means "a unit vector perpendicular

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to something else." This is common usage, but be warned that the word "normal" primarily means "perpendicular" and not "unit length." Since it is so common for normals to be unit vectors, we will take care to call out any situation where a "normal" vector does not have unit length.

In summary, a "normalized" vector always has unit length, but a "normal" vector is a vector that is perpendicular to something and by convention usually has unit length.

2.9.1 Official Linear Algebra Rules

For any nonzero vector $\mathbf{v}$, we can compute a unit vector that points in the same direction as $\mathbf{v}$. This process is known as *normalizing* the vector. In this book we use a common notation of putting a hat symbol over unit vectors; for example, $\hat{\mathbf{v}}$ (pronounced "v hat"). To normalize a vector, we divide the vector by its magnitude:

Normalizing a vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad \text{for any nonzero vector } \mathbf{v}.$$

For example, to normalize the 2D vector $[12, -5]$,

$$\begin{aligned}\frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\left\|\begin{bmatrix}12 & -5\end{bmatrix}\right\|} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{12^2+5^2}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{\sqrt{169}} = \frac{\begin{bmatrix}12 & -5\end{bmatrix}}{13} = \begin{bmatrix}\frac{12}{13} & \frac{-5}{13}\end{bmatrix}\\ \approx \begin{bmatrix}0.923 & -0.385\end{bmatrix}.\end{aligned}$$

The zero vector cannot be normalized. Mathematically, this is not allowed because it would result in division by zero. Geometrically, it makes sense because the zero vector does not define a direction—if we normalized the zero vector, in what direction should the resulting vector point?

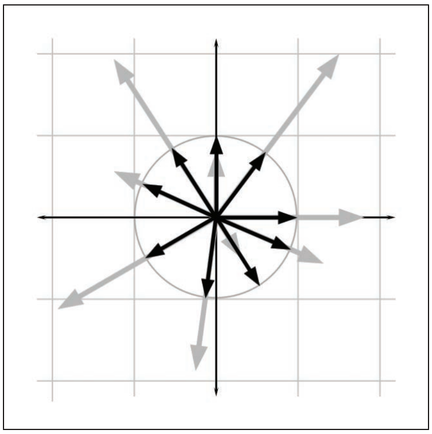

Figure 2.16
Normalizing vectors in 2D

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2.9.2 Geometric Interpretation

In 2D, if we draw a unit vector with the tail at the origin, the head of the vector will touch a unit circle centered at the origin. (A unit circle has a radius of 1.) In 3D, unit vectors touch the surface of a unit sphere. Figure 2.16 shows several 2D vectors of arbitrary length in gray, beneath their normalized counterparts in black.

Notice that normalizing a vector makes some vectors shorter (if their length was greater than 1) and some vectors longer (if their length was less than 1).

2.10 The Distance Formula

We are now prepared to derive one of the oldest and most fundamental formulas in computational geometry: the distance formula. This formula is used to compute the distance between two points.

First, let's define distance as the length of the line segment between the two points. Since a vector is a directed line segment, geometrically it makes sense that the distance between the two points would be equal to the length of a vector from one point to the other. Let's derive the distance formula in 3D. First, we will compute the vector $\mathbf{d}$ from $\mathbf{a}$ to $\mathbf{b}$. We learned how to do this in 2D in Section 2.7.3. In 3D, we use

$$\mathbf{d} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} b_x - a_x \\ b_y - a_y \\ b_z - a_z \end{bmatrix}.$$

The distance between $\mathbf{a}$ and $\mathbf{b}$ is equal to the length of the vector $\mathbf{d}$, which we computed in Section 2.8:

$$\text{distance}\,(\mathbf{a}, \mathbf{b}) = \|\mathbf{d}\| = \sqrt{{d_x}^2 + {d_y}^2 + {d_z}^2}.$$

Substituting for $\mathbf{d}$, we get

$$\text{distance}\,(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} - \mathbf{a}\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2}.$$

The 3D distance formula

Thus, we have derived the distance formula in 3D. The 2D equation is even simpler:

$$\text{distance}\,(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} - \mathbf{a}\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}.$$

The 2D distance formula

Let's look at an example in 2D:

$$\begin{aligned}\text{distance}\left(\begin{bmatrix}5 & 0\end{bmatrix}, \begin{bmatrix}-1 & 8\end{bmatrix}\right) &= \sqrt{(-1-5)^2 + (8-0)^2} \\ &= \sqrt{(-6)^2 + 8^2} = \sqrt{100} = 10.\end{aligned}$$

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Notice that it doesn't matter which point we call **a** and which point we call **b**. If we define **d** to be the vector from **b** to **a** instead of from **a** to **b**, we will derive a slightly different, but mathematically equivalent, equation.

2.11 Vector Dot Product

Section 2.6 showed how to multiply a vector by a scalar. We can also multiply two vectors together. There are two types of vector products. The first vector product is the *dot product* (also known as the *inner product*), the subject of this section. We talk about the other vector product, the *cross product*, in Section 2.12.

The dot product is ubiquitous in video game programming, useful in everything from graphics, to simulation, to AI. Following the pattern we used for the operations, we first discuss the algebraic rules for computing dot products in Section 2.11.1, followed by some geometric interpretations in Section 2.11.2.

The dot product formula is one of the few formulas in this book worth memorizing. First of all, it's really easy to memorize. Also, if you understand what the dot product does, the formula makes sense. Furthermore, the dot product has important relationships to many other operations, such as matrix multiplication, convolution of signals, statistical correlations, and Fourier transforms. Understanding the formula will make these relationships more apparent.

Even more important than memorizing a formula is to get an intuitive grasp for what the dot product *does*. If there is only enough space in your brain for either the formula or the geometric definition, then we recommend internalizing the geometry, and getting the formula tattooed on your hand. You need to understand the geometric definition in order to *use* the dot product. When programming in computer languages such as C++, HLSL, or even Matlab and Maple, you won't need to know the formula anyway, since you will usually tell the computer to do a dot product calculation not by typing in the formula, but by invoking a high-level function or overloaded operator. Furthermore, the geometric definition of the dot product does not assume any particular coordinate frame or even the use of Cartesian coordinates.

2.11.1 Official Linear Algebra Rules

The name "dot product" comes from the dot symbol used in the notation: $\mathbf{a} \cdot \mathbf{b}$. Just like scalar-times-vector multiplication, the vector dot product is performed before addition and subtraction, unless parentheses are used to override this default order of operations. Note that although we usually

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omit the multiplication symbol when multiplying two scalars or a scalar and a vector, we must not omit the dot symbol when performing a vector dot product. If you ever see two vectors placed side-by-side with no symbol in between, interpret this according to the rules of *matrix multiplication*, which we discuss in Chapter 4.[7]

The dot product of two vectors is the sum of the products of corresponding components, resulting in a *scalar*:

Vector dot product

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_{n-1} b_{n-1} + a_n b_n.$$

This can be expressed succinctly by using the summation notation

Dot product using summation notation

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i.$$

Applying these rules to the 2D and 3D cases yields

2D and 3D dot products

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_x b_x + a_y b_y && (\mathbf{a} \text{ and } \mathbf{b} \text{ are 2D vectors}), \\ \mathbf{a} \cdot \mathbf{b} &= a_x b_x + a_y b_y + a_z b_z && (\mathbf{a} \text{ and } \mathbf{b} \text{ are 3D vectors}). \end{aligned}$$

Examples of the dot product in 2D and 3D are

$$\begin{bmatrix} 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} -3 & 7 \end{bmatrix} = (4)(-3) + (6)(7) = 30,$$

$$\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = (3)(0) + (-2)(4) + (7)(-1) = -15.$$

It is obvious from inspection of the equations that vector dot product is commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. More vector algebra laws concerning the dot product are given in Section 2.13.

2.11.2 Geometric Interpretation

Now let's discuss the more important aspect of the dot product: what it means geometrically. It would be difficult to make too big of a deal

[7]One notation you will probably bump up against is treating the dot product as an ordinary matrix multiplication, denoted by $\mathbf{a}^{\mathrm{T}}\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are interpreted as column vectors, or $\mathbf{a}\mathbf{b}^{\mathrm{T}}$ for row vectors. If none of this makes sense, don't worry, we will repeat it after we learn about matrix multiplication and row and column vectors in Chapter 4.

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