Notice that it doesn't matter which point we call **a** and which point we call **b**. If we define **d** to be the vector from **b** to **a** instead of from **a** to **b**, we will derive a slightly different, but mathematically equivalent, equation.

2.11 Vector Dot Product

Section 2.6 showed how to multiply a vector by a scalar. We can also multiply two vectors together. There are two types of vector products. The first vector product is the *dot product* (also known as the *inner product*), the subject of this section. We talk about the other vector product, the *cross product*, in Section 2.12.

The dot product is ubiquitous in video game programming, useful in everything from graphics, to simulation, to AI. Following the pattern we used for the operations, we first discuss the algebraic rules for computing dot products in Section 2.11.1, followed by some geometric interpretations in Section 2.11.2.

The dot product formula is one of the few formulas in this book worth memorizing. First of all, it's really easy to memorize. Also, if you understand what the dot product does, the formula makes sense. Furthermore, the dot product has important relationships to many other operations, such as matrix multiplication, convolution of signals, statistical correlations, and Fourier transforms. Understanding the formula will make these relationships more apparent.

Even more important than memorizing a formula is to get an intuitive grasp for what the dot product *does*. If there is only enough space in your brain for either the formula or the geometric definition, then we recommend internalizing the geometry, and getting the formula tattooed on your hand. You need to understand the geometric definition in order to *use* the dot product. When programming in computer languages such as C++, HLSL, or even Matlab and Maple, you won't need to know the formula anyway, since you will usually tell the computer to do a dot product calculation not by typing in the formula, but by invoking a high-level function or overloaded operator. Furthermore, the geometric definition of the dot product does not assume any particular coordinate frame or even the use of Cartesian coordinates.

2.11.1 Official Linear Algebra Rules

The name "dot product" comes from the dot symbol used in the notation: $\mathbf{a} \cdot \mathbf{b}$. Just like scalar-times-vector multiplication, the vector dot product is performed before addition and subtraction, unless parentheses are used to override this default order of operations. Note that although we usually omit the multiplication symbol when multiplying two scalars or a scalar and a vector, we must not omit the dot symbol when performing a vector dot product. If you ever see two vectors placed side-by-side with no symbol in between, interpret this according to the rules of *matrix multiplication*, which we discuss in Chapter 4.[7]

The dot product of two vectors is the sum of the products of corresponding components, resulting in a *scalar*:

Vector dot product

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \cdots + a_{n-1}b_{n-1} + a_nb_n.$$

This can be expressed succinctly by using the summation notation

Dot product using summation notation

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i.$$

Applying these rules to the 2D and 3D cases yields

2D and 3D dot products

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_xb_x + a_yb_y && (\mathbf{a} \text{ and } \mathbf{b} \text{ are 2D vectors}), \\ \mathbf{a} \cdot \mathbf{b} &= a_xb_x + a_yb_y + a_zb_z && (\mathbf{a} \text{ and } \mathbf{b} \text{ are 3D vectors}). \end{aligned}$$

Examples of the dot product in 2D and 3D are

$$\begin{bmatrix} 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} -3 & 7 \end{bmatrix} = (4)(-3) + (6)(7) = 30,$$

$$\begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} = (3)(0) + (-2)(4) + (7)(-1) = -15.$$

It is obvious from inspection of the equations that vector dot product is commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. More vector algebra laws concerning the dot product are given in Section 2.13.

2.11.2 Geometric Interpretation

Now let's discuss the more important aspect of the dot product: what it means geometrically. It would be difficult to make too big of a deal

[7]One notation you will probably bump up against is treating the dot product as an ordinary matrix multiplication, denoted by $\mathbf{a}^{\mathrm{T}}\mathbf{b}$ if $\mathbf{a}$ and $\mathbf{b}$ are interpreted as column vectors, or $\mathbf{a}\mathbf{b}^{\mathrm{T}}$ for row vectors. If none of this makes sense, don't worry, we will repeat it after we learn about matrix multiplication and row and column vectors in Chapter 4.

out of the dot product, as it is fundamental to almost every aspect of 3D math. Because of its supreme importance, we're going to dwell on it a bit. We'll discuss two slightly different ways of thinking about this operation geometrically; since they are really equivalent, you may or may not think one interpretation or the other is "more fundamental," or perhaps you may think we are being redundant and wasting your time. You might especially think this if you already have some exposure to the dot product, but please indulge us.

The first geometric definition to present is perhaps the less common of the two, but in agreement with the advice of Dray and Manogue [15], we believe it's actually the more useful. The interpretation we first consider is that of the dot product performing a *projection*.

Assume for the moment that $\hat{\mathbf{a}}$ is a unit vector, and $\mathbf{b}$ is a vector of any length. Now take $\mathbf{b}$ and *project* it onto a line parallel to $\hat{\mathbf{a}}$, as in Figure 2.17.

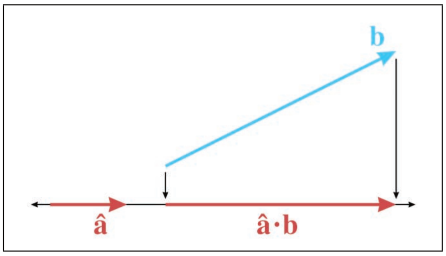


Figure 2.17
The dot product as a projection

(Remember that vectors are displacements and do not have a fixed position, so we are free to move them around on a diagram anywhere we wish.) We can define the dot product $\hat{\mathbf{a}} \cdot \mathbf{b}$ as the signed length of the projection of $\mathbf{b}$ onto this line. The term "projection" has a few different technical meanings (see Section 5.3) and we won't bother attempting a formal definition here.[8] You can think of the projection of $\mathbf{b}$ onto $\hat{\mathbf{a}}$ as the "shadow" that $\mathbf{b}$ casts on $\hat{\mathbf{a}}$ when the rays of light are perpendicular to $\hat{\mathbf{a}}$.

We have drawn the projections as arrows, but remember that the result of a dot product is a scalar, not a vector. Still, when you first learned about negative numbers, your teacher probably depicted numbers as arrows on a number line, to emphasize their sign, just as we have. After all, a scalar is a perfectly valid one-dimensional vector.

What does it mean for the dot product to measure a *signed* length? It means the value will be negative when the projection of $\mathbf{b}$ points in the opposite direction from $\hat{\mathbf{a}}$, and the projection has zero length (it is a single point) when $\hat{\mathbf{a}}$ and $\mathbf{b}$ are perpendicular. These cases are illustrated in Figure 2.18.

[8]Thus shirking our traditional duties as mathematics authors to make intuitive concepts sound much more complicated than they are.

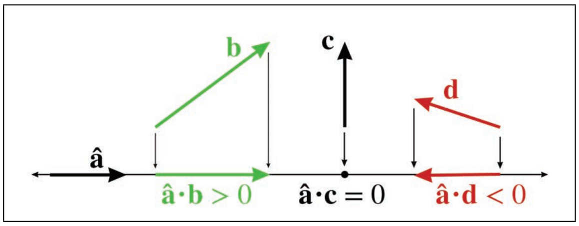


Figure 2.18. Sign of the dot product

In other words, the sign of the dot product can give us a rough classification of the relative directions of the two vectors. Imagine a line (in 2D) or plane (in 3D) perpendicular to the vector $\hat{\mathbf{a}}$. The sign of the dot product $\hat{\mathbf{a}} \cdot \mathbf{b}$ tells us which half-space $\mathbf{b}$ lies in. This is illustrated in Figure 2.19.

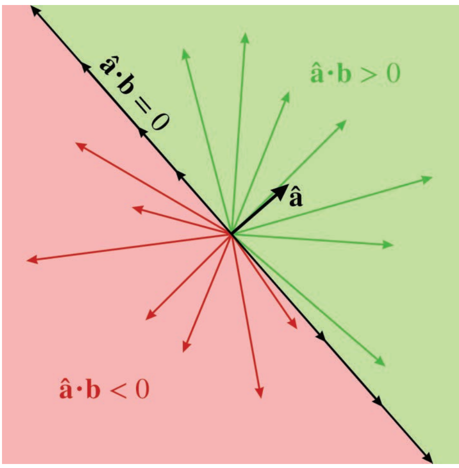


Figure 2.19
The sign of the dot product gives a rough classification of the relative orientation of two vectors.

Next, consider what happens when we scale $\mathbf{b}$ by some factor k. As shown in Figure 2.20, the length of the projection (and thus the value of the dot product) increases by the same factor. The two triangles have equal interior angles and thus are similar. Since the hypotenuse on the right is longer than the hypotenuse on the left by a factor of k, by the properties of similar triangles, the base on the right is also longer by a factor of k.

Let's state this fact algebraically and prove it by using the formula:

$$\begin{aligned}\hat{\mathbf{a}} \cdot (k\mathbf{b}) &= a_x(kb_x) + a_y(kb_y) + a_z(kb_z)\\ &= k(a_xb_x + a_yb_y + a_zb_z)\\ &= k(\hat{\mathbf{a}} \cdot \mathbf{b}).\end{aligned}$$

Dot product is associative with multiplication by a scalar

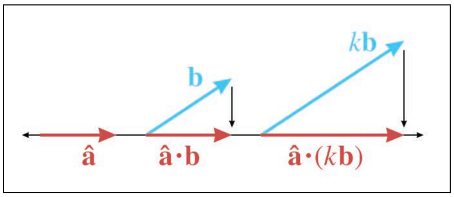


Figure 2.20
Scaling one operand of the dot product

The expanded scalar math in the middle uses three dimensions as our example, but the vector notation at either end of the equation applies for vectors of any dimension.

We've seen what happens when we scale $\mathbf{b}$: the length of its projection onto $\hat{\mathbf{a}}$ increases along with the value of the dot product. What if we scale $\mathbf{a}$? The algebraic argument we just made can be used to show that the value of the dot product scales with the length of $\mathbf{a}$, just like it does when we scale $\mathbf{b}$. In other words,

Dot product is associative with multiplication by a scalar for either vector

$$(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b}).$$

So scaling $\mathbf{a}$ scales the numeric value of the dot product. However, this scale has no affect geometrically on the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$. Now that we know what happens if we scale either $\mathbf{a}$ or $\mathbf{b}$, we can write our geometric definition without any assumptions about the length of the vectors.

Dot Product as Projection

The dot product $\mathbf{a} \cdot \mathbf{b}$ is equal to the signed length of the projection of $\mathbf{b}$ onto any line parallel to $\mathbf{a}$, multiplied by the length of $\mathbf{a}$.

As we continue to examine the properties of the dot product, some will be easiest to illustrate geometrically when either $\mathbf{a}$, or both $\mathbf{a}$ and $\mathbf{b}$, are unit vectors. Because we have shown that scaling either $\mathbf{a}$ or $\mathbf{b}$ directly scales the value of the dot product, it will be easy to generalize our results after we have obtained them. Furthermore, in the algebraic arguments that accompany each geometric argument, unit vector assumptions won't be necessary. Remember that we put hats on top of vectors that are assumed to have unit length.

You may well wonder why the dot product measures the projection of the second operand onto the first, and not the other way around. When the two vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, we can easily make a geometric argument that the projection of $\hat{\mathbf{a}}$ onto $\hat{\mathbf{b}}$ has the same length as the projection of $\hat{\mathbf{b}}$ onto $\hat{\mathbf{a}}$. Consider Figure 2.21. The two triangles have equal interior angles and thus are similar. Since $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are corresponding sides and have the same length, the two triangles are reflections of each other.

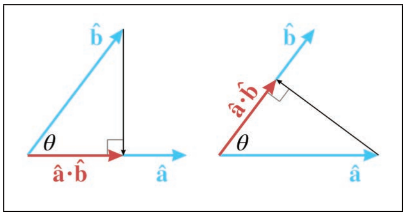


Figure 2.21
Dot product is commutative

We've already shown how scaling either vector will scale the dot product proportionally, so this result applies for $\mathbf{a}$ and $\mathbf{b}$ with arbitrary length. Furthermore, this geometric fact is also trivially verified by using the formula, which does not depend on the assumption that the vectors have equal length. Using two dimensions as our example this time,

Dot product is commutative

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y = b_x a_x + b_y a_y = \mathbf{b} \cdot \mathbf{a}.$$

The next important property of the dot product is that it distributes over addition and subtraction, just like scalar multiplication. This time let's do the algebra before the geometry. When we say that the dot product "distributes," that means that if one of the operands to the dot product is a sum, then we can take the dot product of the pieces individually, and then take their sum. Switching back to three dimensions for our example,

Dot product distributes over addition and subtraction

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \cdot \begin{bmatrix} b_x + c_x \\ b_y + c_y \\ b_z + c_z \end{bmatrix} \\
&= a_x(b_x + c_x) + a_y(b_y + c_y) + a_z(b_z + c_z) \\
&= a_x b_x + a_x c_x + a_y b_y + a_y c_y + a_z b_z + a_z c_z \\
&= (a_x b_x + a_y b_y + a_z b_z) + (a_x c_x + a_y c_y + a_z c_z) \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.
\end{aligned}$$

By replacing $\mathbf{c}$ with $-\mathbf{c}$, it's clear that the dot product distributes over vector subtraction just as it does for vector addition. Figure 2.22 shows how the dot product distributes over addition.

Now let's look at a special situation in which one of the vectors is the unit vector pointing in the $+x$ direction, which we'll denote as $\hat{\mathbf{x}}$. As shown in Figure 2.23, the signed length of the projection is simply the x-coordinate

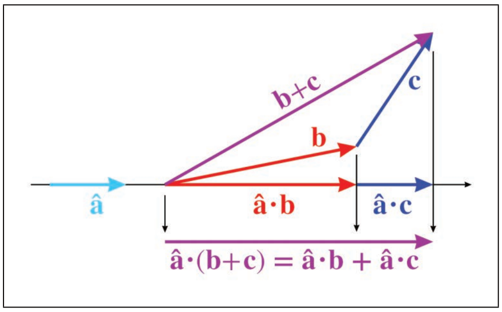


Figure 2.22
The dot product distributes over addition.

of the original vector. In other words, taking the dot product of a vector with a cardinal axis "sifts" out the coordinate for that axis.

If we combine this "sifting" property of the dot product with the fact that it distributes over addition, which we have been able to show in purely geometric terms, we can see why the formula has to be what it is.

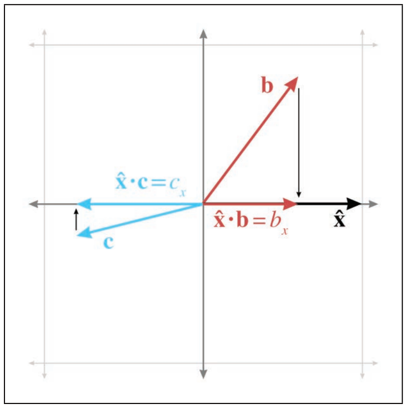


Figure 2.23
Taking the dot product with a cardinal axis sifts out the corresponding coordinate.

Because the dot product measures the length of a projection, it has an interesting relationship to the vector magnitude calculation. Remember that the vector magnitude is a scalar measuring the amount of displacement (the length) of the vector. The dot product also measures the amount of displacement, but only the displacement *in a particular direction* is counted; perpendicular displacement is discarded by the projecting process. But what if we measure the displacement in the same direction that the vector is pointing? In this case, *all* of the vector's displacement is in the direction being measured, so if we project a vector onto itself, the length of that projection is simply the magnitude of the vector. But remember that $\mathbf{a} \cdot \mathbf{b}$ is equal to the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$, scaled by $\|\mathbf{a}\|$. If we dot a vector with itself, such as $\mathbf{v} \cdot \mathbf{v}$, we get the length of the projection, which is $\|\mathbf{v}\|$, times the length of the vector we are projecting onto, which is also $\|\mathbf{v}\|$. In other words,

Relationship between vector magnitude and the dot product

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2, \qquad \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Before we switch to the second interpretation of the dot product, let's check out one more very common use of the dot product as a projection. Assume once more that $\hat{\mathbf{a}}$ is a unit vector and $\mathbf{b}$ has arbitrary length. Using the dot product, it's possible to separate $\mathbf{b}$ into two values, $\mathbf{b}_{\|}$ and $\mathbf{b}_{\perp}$ (read "$\mathbf{b}$ parallel" and "$\mathbf{b}$ perp"), which are parallel and perpendicular to $\hat{\mathbf{a}}$, respectively, such that $\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}$. Figure 2.24 illustrates the geometry involved.

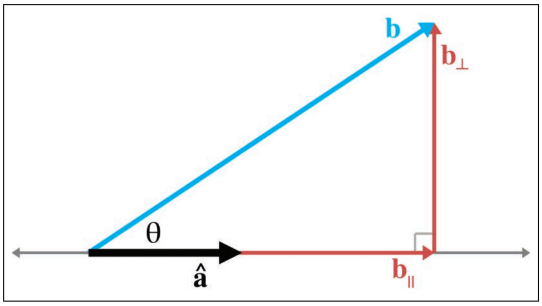


Figure 2.24
Projecting one vector onto another

We've already established that the length of $\mathbf{b}_{\|}$ will be equal to $\hat{\mathbf{a}} \cdot \mathbf{b}$. But the dot product yields a scalar, and $\mathbf{b}_{\|}$ is a vector, so we'll take the direction specified by the unit vector $\hat{\mathbf{a}}$ and scale it up:

$$\mathbf{b}_{\|} = (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}}.$$

Once we know $\mathbf{b}_{\|}$, we can easily solve for $\mathbf{b}_{\perp}$:

$$\begin{aligned} \mathbf{b}_{\perp} + \mathbf{b}_{\|} &= \mathbf{b}, \\ \mathbf{b}_{\perp} &= \mathbf{b} - \mathbf{b}_{\|}, \\ \mathbf{b}_{\perp} &= \mathbf{b} - (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}}. \end{aligned}$$

It's not too difficult to generalize these results to the case where $\mathbf{a}$ is not a unit vector.

In the rest of this book, we make use of these equations several times to separate a vector into components that are parallel and perpendicular to another vector.

Now let's examine the dot product through the lens of trigonometry. This is the more common geometric interpretation of the dot product, which places a bit more emphasis on the angle between the vectors. We've been thinking in terms of projections, so we haven't had much need for this angle. Less experienced and conscientious authors [16] might give you just one of the two important viewpoints, which is probably sufficient to interpret an equation that contains the dot product. However, a more valuable skill is to recognize situations for which the dot product is the correct tool for the job; sometimes it helps to have other interpretations pointed out, even if they are "obviously" equivalent to each other.

Consider the right triangle on the right-hand side of Figure 2.25. As the figure shows, the length of the hypotenuse is 1 (since $\hat{\mathbf{b}}$ is a unit vector) and the length of the base is equal to the dot product $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$. From elementary trig (which was reviewed in Section 1.4.4), remember that the cosine of an angle is the ratio of the length of the adjacent leg divided by the length of the hypotenuse. Plugging in the values from Figure 2.25, we have

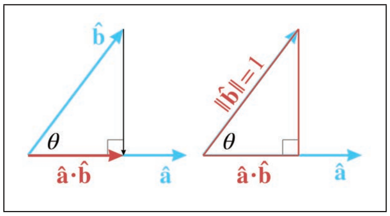


Figure 2.25
Interpreting the dot product by using the trigonometry of the right triangle

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}}{1} = \hat{\mathbf{a}} \cdot \hat{\mathbf{b}}.$$

In other words, the dot product of two unit vectors is equal to the cosine of the angle between them. This statement is true even if the right triangle in Figure 2.25 cannot be formed, when $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \leq 0$ and $\theta > 90^{\text{o}}$. Remember that the dot product of any vector with the vector $\hat{\mathbf{x}} = [1, 0, 0]$ will simply extract the x-coordinate of the vector. In fact, the x-coordinate of a unit vector that has been rotated by an angle of θ from standard position is one way to *define* the value of $\cos\theta$. Review Section 1.4.4 if this isn't fresh in your memory.

By combining these ideas with the previous observation that scaling either vector scales the dot product by the same factor, we arrive at the general relationship between the dot product and the cosine.

Dot Product Relation to Intercepted Angle

The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is equal to the cosine of the angle θ between the vectors, multiplied by the lengths of the vectors (see Figure 2.26). Stated formally,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta. \tag{2.4}$$

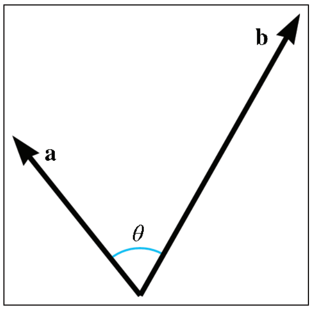


Figure 2.26
The dot product is related to the angle between two vectors.

What does it mean to measure the angle between two vectors in 3D? Any two vectors will always lie in a common plane (place them tail to tail to see this), and so we measure the angle in the plane that contains both vectors. If the vectors are parallel, the plane is not unique, but the angle is either 0^o or $\pm 180^o$, and it doesn't matter which plane we choose.

The dot product provides a way for us to compute the angle between two vectors. Solving Equation (2.4) for θ,

Using the dot product to compute the angle between two vectors

$$\theta = \arccos\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}\right). \tag{2.5}$$

We can avoid the division in Equation (2.5) if we know that **a** and **b** are unit vectors. In this very common case, the denominator of Equation (2.5) is trivially 1, and we are left with

Computing the angle between two unit vectors

$$\theta = \arccos\left(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}\right) \qquad \text{(assume } \hat{\mathbf{a}} \text{ and } \hat{\mathbf{b}} \text{ are unit vectors).}$$

If we do not need the exact value of θ, and need only a classification of the relative orientation of **a** and **b**, then we need only the *sign* of the dot product. This is the same idea illustrated in Figure 2.18, only now we can relate it to the angle θ, as shown in Table 2.1.

$\mathbf{a}\cdot\mathbf{b}$	θ	Angle is	a and b are
> 0	$0^o \le \theta < 90^o$	acute	pointing mostly in the same direction
0	$\theta = 90^o$	right	perpendicular
< 0	$90^o < \theta \le 180^o$	obtuse	pointing mostly in the opposite direction

Table 2.1. The sign of the dot product can be used as a rough classification of the angle between two vectors.

Since the magnitude of the vectors does not affect the sign of the dot product, Table 2.1 applies regardless of the lengths of **a** and **b**. However, notice that if either **a** or **b** is the zero vector, then $\mathbf{a}\cdot\mathbf{b} = 0$. Thus, when we use the dot product to classify the relationship between two vectors, the dot product acts as if the zero vector is perpendicular to any other vector. As it turns out, the cross product behaves differently.

Let's summarize the dot product's geometric properties.

- The dot product $\mathbf{a} \cdot \mathbf{b}$ measures the length of the projection of $\mathbf{b}$ onto $\mathbf{a}$, multiplied by the length of $\mathbf{a}$.
- The dot product can be used to measure displacement in a particular direction.
- The projection operation is closely related to the cosine function. The dot product $\mathbf{a} \cdot \mathbf{b}$ also is equal to $\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$, where θ is the angle between the vectors.

We review the commutative and distributive properties of the dot product at the end of this chapter along with other algebraic properties of vector operations.

2.12 Vector Cross Product

The other vector product, known as the *cross product*, can be applied only in 3D. Unlike the dot product, which yields a scalar and is commutative, the vector cross product yields a 3D vector and is not commutative.

2.12.1 Official Linear Algebra Rules

Similar to the dot product, the term "cross" product comes from the symbol used in the notation $\mathbf{a} \times \mathbf{b}$. We always write the cross symbol, rather than omitting it as we do with scalar multiplication. The equation for the cross product is

Cross product

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

For example,

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} = \begin{bmatrix} (3)(8) - (4)(-5) \\ (4)(2) - (1)(8) \\ (1)(-5) - (3)(2) \end{bmatrix} = \begin{bmatrix} 24 - (-20) \\ 8 - 8 \\ -5 - 6 \end{bmatrix} = \begin{bmatrix} 44 \\ 0 \\ -11 \end{bmatrix}.$$

The cross product enjoys the same level of operator precedence as the dot product: multiplication occurs before addition and subtraction. When dot product and cross product are used together, the cross product takes precedence: $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Luckily, there's an easy way to remember this: it's the only way it could work. The dot product returns a scalar, and so $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ is undefined, since you cannot take the cross product of a scalar and a vector. The operation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is known as the *triple product*. We present some special properties of this computation in Section 6.1.

As mentioned earlier, the vector cross product is not commutative. In fact, it is *anticommutative*: $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$. The cross product is not associative, either. In general, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. More vector algebra laws concerning the cross product are given in Section 2.13.

2.12.2 Geometric Interpretation

The cross product yields a vector that is perpendicular to the original two vectors, as illustrated in Figure 2.27.

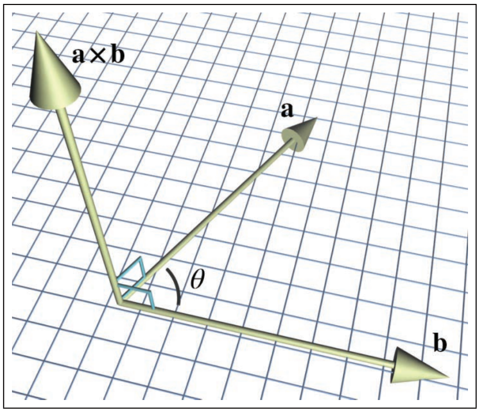


Figure 2.27
Vector cross product

The length of $\mathbf{a} \times \mathbf{b}$ is equal to the product of the magnitudes of $\mathbf{a}$ and $\mathbf{b}$ and the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta.$$

The magnitude of the cross product is related to the sine of the angle between the vectors

As it turns out, this is also equal to the area of the parallelogram formed with two sides $\mathbf{a}$ and $\mathbf{b}$. Let's see if we can verify why this is true by using Figure 2.28.

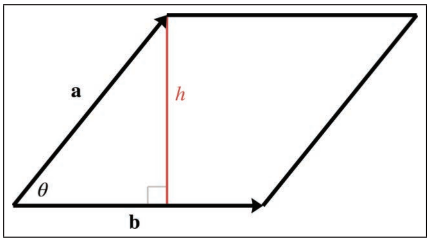


Figure 2.28
A parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$

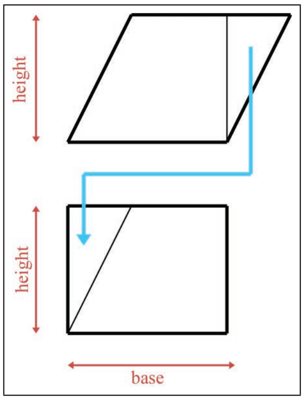


Figure 2.29
Area of a parallelogram

First, from planar geometry, we know that the area of the parallelogram is bh, the product of the base and the height. (In Figure 2.28, the base is $b = \|\mathbf{b}\|$.) We can verify this rule by "clipping" off a triangle from one end and moving it to the other end, forming a rectangle, as shown in Figure 2.29.

The area of a rectangle is given by its length and width. In this case, this area is the product bh. Since the area of the rectangle is equal to the area of the parallelogram, the area of the parallelogram must also be bh.

Returning to Figure 2.28, let a and b be the lengths of $\mathbf{a}$ and $\mathbf{b}$, respectively, and note that $\sin\theta = h/a$. Then

$$\begin{aligned} A &= bh \\ &= b(a\sin\theta) \\ &= \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta \\ &= \|\mathbf{a}\times\mathbf{b}\|. \end{aligned}$$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, or if $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then $\mathbf{a}\times\mathbf{b} = \mathbf{0}$. So the cross product interprets the zero vector as being parallel to every other vector. Notice that this is different from the dot product, which interprets the zero vector as being *perpendicular* to every other vector. (Of course, it is ill-defined to describe the zero vector as being perpendicular or parallel to any vector, since the zero vector has no direction.)

We have stated that $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$. But there are two directions that are perpendicular to $\mathbf{a}$ and $\mathbf{b}$—which of these two directions does $\mathbf{a}\times\mathbf{b}$ point? We can determine the direction of $\mathbf{a}\times\mathbf{b}$ by placing the tail of $\mathbf{b}$ at the head of $\mathbf{a}$, and examining whether we make a clockwise or counterclockwise turn from $\mathbf{a}$ to $\mathbf{b}$. In a left-handed coordinate system, $\mathbf{a}\times\mathbf{b}$ points towards you if the vectors $\mathbf{a}$ and $\mathbf{b}$ make a clockwise turn from your viewpoint, and away from you if $\mathbf{a}$ and $\mathbf{b}$ make a counterclockwise turn. In a right-handed coordinate system, the exact opposite occurs: if $\mathbf{a}$ and $\mathbf{b}$ make a counterclockwise turn, $\mathbf{a}\times\mathbf{b}$ points towards you, and if $\mathbf{a}$ and $\mathbf{b}$ make a clockwise turn, $\mathbf{a}\times\mathbf{b}$ points away from you.

Figure 2.30 shows clockwise and counterclockwise turns. Notice that to make the clockwise or counterclockwise determination, we must align the

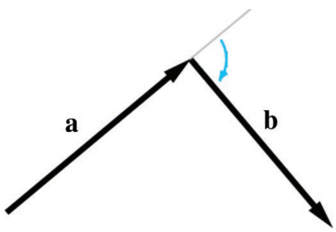
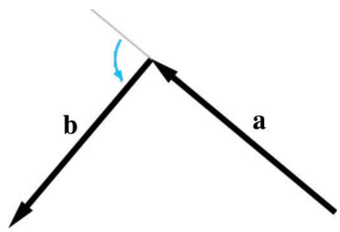
Clockwise turn	Counterclockwise turn
	
In a left-handed coordinate system, $\mathbf{a} \times \mathbf{b}$ (not shown) points towards you. In a right-handed coordinate system, $\mathbf{a} \times \mathbf{b}$ points away from you.	In a left-handed coordinate system, $\mathbf{a} \times \mathbf{b}$ (not shown) points away from you. In a right-handed coordinate system, $\mathbf{a} \times \mathbf{b}$ points towards you.

Figure 2.30. Determining clockwise versus counterclockwise turns

head of **a** with the tail of **b**. Compare this to Figure 2.26, where the tails are touching. The tail-to-tail alignment shown in Figure 2.26 is the correct way to position the vectors to measure the angle between them, but to judge whether the turn is clockwise or counterclockwise, the vectors should be aligned head-to-tail, as shown in Figure 2.30.

Let's apply this general rule to the specific case of the cardinal axes. Let $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ be unit vectors that point in the $+x$, $+y$, and $+z$ directions, respectively. The results of taking the cross product of each pair of axes are

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}}, & \qquad \hat{\mathbf{y}} \times \hat{\mathbf{x}} &= -\hat{\mathbf{z}}, \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= \hat{\mathbf{x}}, & \qquad \hat{\mathbf{z}} \times \hat{\mathbf{y}} &= -\hat{\mathbf{x}}, \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}}, & \qquad \hat{\mathbf{x}} \times \hat{\mathbf{z}} &= -\hat{\mathbf{y}}. \end{aligned}$$

Cross product of the cardinal axes

You can also remember which way the cross product points by using your hand, similar to the way we distinguished between left-handed and right-handed coordinate spaces in Section 1.3.3. Since we're using a left-handed coordinate space in this book, we'll show how it's done using your left hand. Let's say you have two vectors, **a** and **b**, and you want to figure out which direction $\mathbf{a} \times \mathbf{b}$ points. Point your thumb in the direction of **a**, and your index finger (approximately) in the direction of **b**. If **a** and **b** are pointing in nearly the opposite direction, this may be difficult. Just make sure that if your thumb points exactly in the direction of **a**; then your index finger is on the same side of **a** as the vector **b** is. With your fingers in this position, extend your third finger to be perpendicular to your thumb and index finger, similar to what we did in Section 1.3.3. Your third finger now points in the direction of $\mathbf{a} \times \mathbf{b}$.

Of course, a similar trick works with your right hand for right-handed coordinate spaces.

One of the most important uses of the cross product is to create a vector that is perpendicular to a plane (see Section 9.5), triangle (Section 9.6), or polygon (Section 9.7).

2.13 Linear Algebra Identities

The Greek philosopher Arcesilaus reportedly said, "Where you find the laws most numerous, there you will find also the greatest injustice." Well, nobody said vector algebra was fair. Table 2.2 lists some vector algebra laws that are occasionally useful but should not be memorized. Several identities are obvious and are listed for the sake of completeness; all of them can be derived from the definitions given in earlier sections.

Identity	Comments
$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$	Commutative property of vector addition
$\mathbf{a}-\mathbf{b}=\mathbf{a}+(-\mathbf{b})$	Definition of vector subtraction
$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$	Associative property of vector addition
$s(t\mathbf{a})=(st)\mathbf{a}$	Associative property of scalar multiplication
$k(\mathbf{a}+\mathbf{b})=k\mathbf{a}+k\mathbf{b}$	Scalar multiplication distributes over vector addition
$\Vert k\mathbf{a}\Vert=\vert k\vert\Vert\mathbf{a}\Vert$	Multiplying a vector by a scalar scales the magnitude by a factor equal to the absolute value of the scalar
$\Vert\mathbf{a}\Vert\geq 0$	The magnitude of a vector is nonnegative
$\Vert\mathbf{a}\Vert^2+\Vert\mathbf{b}\Vert^2=\Vert\mathbf{a}+\mathbf{b}\Vert^2$	The Pythagorean theorem applied to vector addition.
$\Vert\mathbf{a}\Vert+\Vert\mathbf{b}\Vert\geq\Vert\mathbf{a}+\mathbf{b}\Vert$	Triangle rule of vector addition. (No side can be longer than the sum of the other two sides.)
$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$	Commutative property of dot product
$\Vert\mathbf{a}\Vert=\sqrt{\mathbf{a}\cdot\mathbf{a}}$	Vector magnitude defined using dot product
$k(\mathbf{a}\cdot\mathbf{b})=(k\mathbf{a})\cdot\mathbf{b}=\mathbf{a}\cdot(k\mathbf{b})$	Associative property of scalar multiplication with dot product
$\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$	Dot product distributes over vector addition and subtraction
$\mathbf{a}\times\mathbf{a}=\mathbf{0}$	The cross product of any vector with itself is the zero vector. (Because any vector is parallel with itself.)
$\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$	Cross product is anticommutative.
$\mathbf{a}\times\mathbf{b}=(-\mathbf{a})\times(-\mathbf{b})$	Negating both operands to the cross product results in the same vector.
$k(\mathbf{a}\times\mathbf{b})=(k\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(k\mathbf{b})$	Associative property of scalar multiplication with cross product.
$\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$	Cross product distributes over vector addition and subtraction.

Table 2.2
Table of vector algebra identities

2.14 Exercises

(Answers on page 746.)

1. Let

$$\mathbf{a} = \begin{bmatrix} -3 & 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 16 \\ -1 \\ 4 \\ 6 \end{bmatrix}.$$

   (a) Identify **a**, **b**, and **c**, as row or column vectors, and give the dimension of each vector.

   (b) Compute $b_y + c_w + a_x + b_z$.

2. Identify the quantities in each of the following sentences as scalar or vector. For vector quantities, give the magnitude and direction. (Note: some directions may be implicit.)

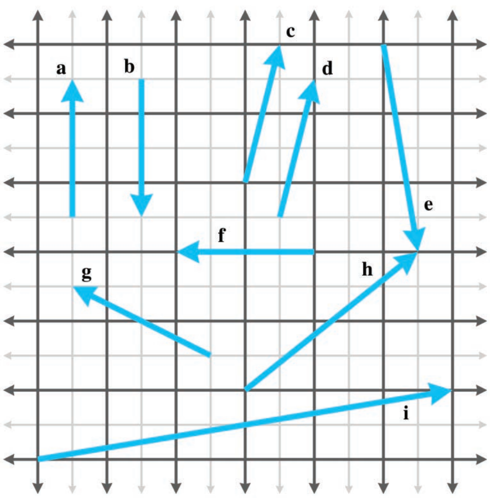
   (a) How much do you weigh?

   (b) Do you have any idea how fast you were going?

   (c) It's two blocks north of here.

   (d) We're cruising from Los Angeles to New York at 600 mph, at an altitude of 33,000 ft.

3. Give the values of the following vectors. The darker grid lines represent one unit.



4. Identify the following statements as true or false. If the statement is false, explain why.
   (a) The size of a vector in a diagram doesn't matter; we just need to draw it in the right place.
   (b) The displacement expressed by a vector can be visualized as a sequence of axially aligned displacements.
   (c) These axially aligned displacements from the previous question must occur in order.
   (d) The vector $[x, y]$ gives the displacement from the point (x, y) to the origin.
5. Evaluate the following vector expressions:
   (a) $-\begin{bmatrix}3 & 7\end{bmatrix}$
   (b) $\left\|\begin{bmatrix}-12 & 5\end{bmatrix}\right\|$
   (c) $\left\|\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}\right\|$
   (d) $3\begin{bmatrix}4 & -7 & 0\end{bmatrix}$
   (e) $\begin{bmatrix}4 & 5\end{bmatrix}/2$
6. Normalize the following vectors:
   (a) $\begin{bmatrix}12 & 5\end{bmatrix}$
   (b) $\begin{bmatrix}0 & 743.632\end{bmatrix}$
   (c) $\begin{bmatrix}8 & -3 & 1/2\end{bmatrix}$
   (d) $\begin{bmatrix}-12 & 3 & -4\end{bmatrix}$
   (e) $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$
7. Evaluate the following vector expressions:
   (a) $\begin{bmatrix}7 & -2 & -3\end{bmatrix} + \begin{bmatrix}6 & 6 & -4\end{bmatrix}$
   (b) $\begin{bmatrix}2 & 9 & -1\end{bmatrix} + \begin{bmatrix}-2 & -9 & 1\end{bmatrix}$
   (c) $\begin{bmatrix}3\\10\\7\end{bmatrix} - \begin{bmatrix}8\\-7\\4\end{bmatrix}$
   (d) $\begin{bmatrix}4\\5\\-11\end{bmatrix} - \begin{bmatrix}-4\\-5\\11\end{bmatrix}$
   (e) $3\begin{bmatrix}a\\b\\c\end{bmatrix} - 4\begin{bmatrix}2\\10\\-6\end{bmatrix}$

8. Compute the distance between the following pairs of points:

   (a) $\begin{bmatrix}10\\6\end{bmatrix}, \begin{bmatrix}-14\\30\end{bmatrix}$

   (b) $\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}-12\\5\end{bmatrix}$

   (c) $\begin{bmatrix}3\\10\\7\end{bmatrix}, \begin{bmatrix}8\\-7\\4\end{bmatrix}$

   (d) $\begin{bmatrix}-2\\-4\\9\end{bmatrix}, \begin{bmatrix}6\\-7\\9.5\end{bmatrix}$

   (e) $\begin{bmatrix}4\\-4\\-4\\4\end{bmatrix}, \begin{bmatrix}-6\\6\\6\\-6\end{bmatrix}$

9. Evaluate the following vector expressions:

   (a) $\begin{bmatrix}2\\6\end{bmatrix} \cdot \begin{bmatrix}-3\\8\end{bmatrix}$

   (b) $-7\begin{bmatrix}1 & 2\end{bmatrix} \cdot \begin{bmatrix}11 & -4\end{bmatrix}$

   (c) $10 + \begin{bmatrix}-5\\1\\3\end{bmatrix} \cdot \begin{bmatrix}4\\-13\\9\end{bmatrix}$

   (d) $3\begin{bmatrix}-2\\0\\4\end{bmatrix} \cdot \left(\begin{bmatrix}8\\-2\\3/2\end{bmatrix} + \begin{bmatrix}0\\9\\7\end{bmatrix}\right)$

10. Given the two vectors

$$\mathbf{v} = \begin{bmatrix}4\\3\\-1\end{bmatrix}, \quad \hat{\mathbf{n}} = \begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{bmatrix},$$

    separate $\mathbf{v}$ into components that are perpendicular and parallel to $\hat{\mathbf{n}}$. (As the notation implies, $\hat{\mathbf{n}}$ is a unit vector.)

11. Use the geometric definition of the dot product

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$$

    to prove the law of cosines.

12. Use trigonometric identities and the algebraic definition of the dot product in 2D

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y$$

    to prove the geometric interpretation of the dot product in 2D. (Hint: draw a diagram of the vectors and all angles involved.)

13. Calculate $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ for the following vectors:
    (a) $\mathbf{a} = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
    (b) $\mathbf{a} = \begin{bmatrix} -2 & 4 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 & -2 & -1 \end{bmatrix}$
    (c) $\mathbf{a} = \begin{bmatrix} 3 & 10 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 8 & -7 & 4 \end{bmatrix}$
14. Prove the equation for the magnitude of the cross product

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin \theta.$$

(Hint: make use of the geometric interpretation of the dot product and try to show how the left and right sides of the equation are equivalent, rather than trying to derive one side from the other.)

15. Section 2.8 introduced the norm of a vector, namely, a scalar value associated with a given vector. However, the definition of the norm given in that section is not the only definition of a norm for a vector. In general, the *p-norm* of an n-dimensional vector is defined as

$$\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Some of the more common p-norms include:

- The L^1 norm, a.k.a. Taxicab norm ($p = 1$):

$$\|\mathbf{x}\|_1 \equiv \sum_{i=1}^{n} |x_i|.$$

- The L^2 norm, a.k.a. Euclidean norm ($p = 2$). This is the most common and familiar norm, since it measures geometric length:

$$\|\mathbf{x}\|_2 \equiv \sqrt{\sum_{i=1}^{n} x_i^2}.$$

- The infinity norm, a.k.a. Chebyshev norm ($p = \infty$):

$$\|\mathbf{x}\|_\infty \equiv \max\left(|x_1|, \dots, |x_n|\right).$$

Each of these norms can be thought of as a way to assigning a length or size to a vector. The Euclidean norm was discussed in Section 2.8. The Taxicab norm gets its name from how a taxicab would measure distance driving the streets of a city laid out in a grid (e.g., Cartesia from Section 1.2.1). For example, a taxicab that drives 1 block east and 1 block north drives a total distance of 2 blocks, whereas a bird flying "as the crow flies" can fly in a straight line from start to finish and travel only $\sqrt{2}$ blocks (Euclidean norm). The Chebyshev norm is simply the absolute value of the vector component with the largest absolute value. An example of how this norm can be used is to consider the number of moves required to move a king in a game of chess from one square to another. The immediately surrounding squares require 1 move, the squares surrounding those require 2 moves, and so on.

(a) For each of the following find $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_3$, and $\|\mathbf{x}\|_\infty$:

(1) $\begin{bmatrix}3 & 4\end{bmatrix}$

(2) $\begin{bmatrix}5 & -12\end{bmatrix}$

(3) $\begin{bmatrix}-2 & 10 & -7\end{bmatrix}$

(4) $\begin{bmatrix}6 & 1 & -9\end{bmatrix}$

(5) $\begin{bmatrix}-2 & -2 & -2 & -2\end{bmatrix}$

*(b) Draw the unit circle (i.e., the set of all vectors with $\|\mathbf{x}\|_p = 1$) centered at the origin for the L^1 norm, L^2 norm, and infinity norm.

16. A man is boarding a plane. The airline has a rule that no carry-on item may be more than two feet long, two feet wide, or two feet tall. He has a very valuable sword that is three feet long, yet he is able to carry the sword on board with him.[9] How is he able to do this? What is the longest possible item that he could carry on?

17. Verify Figure 2.11 numerically.

18. Is the coordinate system used in Figure 2.27 a left-handed or right-handed coordinate system?

19. One common way of defining a bounding box for a 2D object is to specify a center point $\mathbf{c}$ and a *radius vector* $\mathbf{r}$, where each component of $\mathbf{r}$ is half the length of the side of the bounding box along the corresponding axis.

(a) Describe the four corners $\mathbf{p}_{\text{UpperLeft}}$, $\mathbf{p}_{\text{UpperRight}}$, $\mathbf{p}_{\text{LowerLeft}}$, and $\mathbf{p}_{\text{LowerRight}}$.

(b) Describe the eight corners of a bounding cube, extending this idea into 3D.

[9]Please ignore the fact that nowadays this could never happen for security reasons. You can think of this exercise as taking place in a Quentin Tarantino movie.

20. A nonplayer character (NPC) is standing at location $\mathbf{p}$ with a forward direction of $\mathbf{v}$.

    (a) How can the dot product be used to determine whether the point $\mathbf{x}$ is in front of or behind the NPC?

    (b) Let $\mathbf{p} = \begin{bmatrix}-3 & 4\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}5 & -2\end{bmatrix}$. For each of the following points $\mathbf{x}$ determine whether $\mathbf{x}$ is in front of or behind the NPC:

    (1) $\mathbf{x} = \begin{bmatrix}0 & 0\end{bmatrix}$
    (2) $\mathbf{x} = \begin{bmatrix}1 & 6\end{bmatrix}$
    (3) $\mathbf{x} = \begin{bmatrix}-6 & 0\end{bmatrix}$
    (4) $\mathbf{x} = \begin{bmatrix}-4 & 7\end{bmatrix}$
    (5) $\mathbf{x} = \begin{bmatrix}5 & 5\end{bmatrix}$
    (6) $\mathbf{x} = \begin{bmatrix}-3 & 0\end{bmatrix}$
    (7) $\mathbf{x} = \begin{bmatrix}-6 & -3.5\end{bmatrix}$

21. Extending the concept from Exercise 20, consider the case where the NPC has a limited field of view (FOV). If the total FOV angle is ϕ, then the NPC can see to the left or right of its forward direction by a maximum angle of $\phi/2$.

    (a) How can the dot product be used to determine whether the point $\mathbf{x}$ is visible to the NPC?

    (b) For each of the points $\mathbf{x}$ in Exercise 20 determine whether $\mathbf{x}$ is visible to the NPC if its FOV is $90°$.

    (c) Suppose that the NPC's viewing distance is also limited to a maximum distance of 7 units. Which points are visible to the NPC then?

22. Consider three points labeled $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the xz plane of our left-handed coordinate system, which represent waypoints on an NPC's path.

    (a) How can the cross product be used to determine whether, when moving from $\mathbf{a}$ to $\mathbf{b}$ to $\mathbf{c}$, the NPC makes a clockwise or counterclockwise turn at $\mathbf{b}$, when viewing the path from above?

    (b) For each of the following sets of three points, determine whether the NPC is turning clockwise or counterclockwise when moving from $\mathbf{a}$ to $\mathbf{b}$ to $\mathbf{c}$:

    (1) $\mathbf{a} = \begin{bmatrix}2 & 0 & 3\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}-1 & 0 & 5\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-4 & 0 & 1\end{bmatrix}$
    (2) $\mathbf{a} = \begin{bmatrix}-3 & 0 & -5\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}4 & 0 & 0\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}3 & 0 & 3\end{bmatrix}$
    (3) $\mathbf{a} = \begin{bmatrix}1 & 0 & 4\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}7 & 0 & -1\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}-5 & 0 & -6\end{bmatrix}$
    (4) $\mathbf{a} = \begin{bmatrix}-2 & 0 & 1\end{bmatrix}$, $\mathbf{b} = \begin{bmatrix}1 & 0 & 2\end{bmatrix}$, $\mathbf{c} = \begin{bmatrix}4 & 0 & 4\end{bmatrix}$

23. In the derivation of a matrix to scale along an arbitrary axis, we reach a step where we have the vector expression

$$\mathbf{p}' = \mathbf{p} + (k-1)\,(\mathbf{p}\cdot\mathbf{n})\,\mathbf{n},$$

where $\mathbf{n}$ is an arbitrary vector $[n_x, n_y, n_z]$ and k is an arbitrary scalar, but $\mathbf{p}$ is one of the cardinal axes. Plug in the value $\mathbf{p} = [1, 0, 0]$ and simplify the resulting expression for $\mathbf{p}'$. The answer is not a vector expression, but a single vector, where the scalar expressions for each coordinate have been simplified.

24. A similar problem arises with the derivation of a matrix to rotate about an arbitrary axis. Given an arbitrary scalar θ and a vector $\mathbf{n}$, substitute $\mathbf{p} = [1, 0, 0]$ and simplify the value of $\mathbf{p}'$ in the expression

$$\mathbf{p}' = \cos\theta\,(\mathbf{p} - (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n}) + \sin\theta\,(\mathbf{n}\times\mathbf{p}) + (\mathbf{p}\cdot\mathbf{n})\,\mathbf{n}.$$

What's our vector, Victor?

— Captain Oveur in *Airplane!* (1980)