CHAPTER 6

MATRICES

6.1 Introduction

Let us consider a set of simultaneous equations.

$x + y + 3z + 4t = 0$

$3x + 2y + 2z + 5t = 0$

$3x + 4y + 2z + t = 0$

We can write down the co-efficients of x, y, z and t of the above equations keeping within brackets in the following way

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 3 & 2 & 2 & 5 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

The above system of numbers arranged in a rectangular array in rows and columns and enclosed within the brackets is an example of a matrix. The horizontal lines are called rows and the vertical lines are called columns of the matrix. There are 3 rows and 4 columns in this matrix. It is termed as 3×4 matrix to be read as 3 by 4 matrix. An element of a matrix is denoted by a_{ij}. This element is actually the element of ith row and jth column.

Here $a_{14} = 3$.

An $m \times n$ matrix in general form is written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}] \quad \text{or} \quad [a_{ij}]_{m\times n} \quad \ldots (6.1)$$

If all the elements of a matrix belong to the field of real numbers, the matrix is said to be real.

Square Matrix :

When m = n, (6.1) is called a square matrix of order n or an n-square matrix.

In a square matrix, the elements $a_{11}, a_{22}..., a_{nn}$ are called diagonal elements.

The sum of the diagonal elements of a square matrix A is called the trace of A.

Thus trace of A = $a_{11} + a_{22} + ... + a_{nn}$.

6.1.1. Definition

Equal matrices : Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal (A=B) if and only if they have the same order and their corresponding elements are equal i.e. $a_{ij} = b_{ij}$

$$(i = 1, 2...m, j = 1, 2....n)$$

6.1.2. Definition

Zero matrix : A matrix, every element of which is zero is called a zero matrix. If A is zero matrix, we write A = 0.

6.1.3. Definition :

Sum of matrices : If A = $[a_{ij}]$ and B = $[b_{ij}]$ are two m × n matrices, Their sum (difference) $A \pm B$ is defined as the mxn matrix C = $[c_{ij}]$, whare the elements of C is the sum (difference) of the corresponding elements of A and B.

Thus $A \pm B = [a_{ij} \pm b_{ij}]$

Example 6.1.1 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$

and $B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$, then

$$A+B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix}$$

$$A-B = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Two matrices of the same order are said to be conformable for addition and subtraction. Two matrices of different orders cannot be added or subtracted.

The sum of k matrices A is a matrix of same order as A and each of its elements is k times the corresponding element of A.

6.1.4. Definition : (Scalar multiplication) If k be a scalar and A be an m × n matrix, then the scaler multiple of A denoted by kA = Ak is an mxn matrix C = $[c_{ij}]$ such that $c_{ij} = ka_{ij}$.

Example 6.1.2.

$$\text{If } A=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix},$$

$$\text{then } 3A=\begin{bmatrix}3 & -6\\ 6 & 9\end{bmatrix} \text{ and } -5A=\begin{bmatrix}-5 & 10\\ -10 & -15\end{bmatrix}$$

In particular, by –A, called negative of A, is meant the matrix obtained from A by multiplying each of its elements by –1 or by simply changing the sign of all its elements. For every A we have A + (–A) = 0. Where 0 indicates zero matrix of same order as A.

6.1.5 Algebraic properties under matrix addition and scalar multiplication :

Assuming that matrices A, B, C are conformable for addition, we have

(a) A +B = B + A (Commutative law)

(b) A + (B+C) = (A+B) +C (associative law)

(c) k(A+B) = kA + kB= (A+B) k, k a scalar.

(d) There exists a matıcx D such that A+D = B

6.2. Product of matrices : By the product AB in that order of 1 × m matrix

$A = [\ a_{11}\ a_{12}\ a_{13}\ ....a_{1m}]$ and the m × 1 matrix

$$B=\begin{bmatrix}b_{11}\\ b_{21}\\ b_{31}\\ \vdots\\ b_{m1}\end{bmatrix} \text{ is meant the } 1\times 1 \text{matrix}$$

$$C = [\ a_{11}\ b_{11} + a_{12}\ b_{21} +...+a_{1m}b_{m1}]$$

$$\text{i,e } [\ a_{11}\ a_{12}\ a_{13}....a_{1m}]\begin{bmatrix}b_{11}\\ b_{21}\\ b_{31}\\ \vdots\\ b_{m1}\end{bmatrix} = [\ a_{11}\ b_{11} + a_{12}\ b_{21} +....+ a_{1m}\ b_{m1}]$$

$$=\left[\sum_{k=1}^{m} a_{1k}\ b_{k1}\right]$$

Note that the operation is now by column; each element of the row is multiplied into the corresponding element of the column and the product is summed.

Example 6.2.1. : (a) $[2\quad 3\quad 4]\begin{bmatrix}1\\ -1\\ 2\end{bmatrix}=[2\times 1+3\times(-1)+4\times 2]=[7]$

(b) $[3\quad -1\quad 4]=\begin{bmatrix}-2\\ 6\\ 3\end{bmatrix}=[-6-6+12]=0$

6.2.1. Definition : (Product of matrices)

If = A = $[a_{ij}]$ be an m × p matrix and B = $[b_{ij}]$ be a p × n matrix, then the product AB in that order is defined as the m × n matrix

C = $[c_{ij}]$ where

$$C_{ij} = a_{i1}\, b_{1j} + a_{i2}\, b_{2j} + a_{i3}b_{3j} + \ldots + a_{ip}\, b_{pj}$$

$$= \sum_{k=1}^{p} a_{ik}\, b_{kj} \quad (i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n)$$

Example 6.2.2.

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + b_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

The product AB is defined or A is conformable to B for multiplication only when the number of columns of A is equal to the number of rows of B.

If A is conformable to B for multiplication, (AB is defined), B is not necessarily conformable to A for multiplication (BA may or may not be defined)

Example 6.2.3

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$$

Here both AB and BA are defined.

$$AB = \begin{bmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+12-3 \\ -2+2+0 & -4+4+0 & -6+6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So AB = 0

$$BA = \begin{bmatrix} 1-6-6 & -1+4+3 & 1-2+0 \\ 2-12-12 & -2+8+6 & 2-4+0 \\ 1-6-6 & -1+4+3 & 1-2+0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{bmatrix}$$

BA ≠ 0

Also AB ≠ BA.

Note – The above example shows that the matrix product is not commutative.

6.2.2. Algebraic properties of product :

Assumming that, A, B, C, are conformable for the indicated sums and products, We have,

(e) A (B+C) = AB +AC (Distributive lars)

(f) (A+B) C = AC + BC (Associative law)

(g) A (BC) = (AB) C

However,

(h) $AB \neq BA$

(i) AB = 0 does not necessarily imply A = O or B = O.

(j) AB = AC does not necessarily imply B = C.

6.2.3. Product by partitioning :

Let $A = [a_{ij}]$ be of order m × p and $B = [b_{ij}]$ be of order p × n. In forming the product AB, the matrix A is in fact partitioned into m martrices of order 1 × p and B into n matrices óf order p × 1. Other partitions may be used. For example, let A and B be partitioned into matrices of indicated orders by drawing the dotted lines as

$$A = \left[\begin{array}{c|c|c} (m_1 \times p_1) & (m_1 \times p_2) & (m_1 \times p_3) \\ \hline (m_2 \times p_1) & (m_2 \times p_2) & (m_2 \times p_3) \end{array}\right], \quad B = \left[\begin{array}{c|c} (p_1 \times n_1) & (p_1 \times n_2) \\ \hline (p_2 \times n_1) & (p_2 \times n_2) \\ \hline (p_3 \times n_1) & (p_3 \times n_2) \end{array}\right]$$

$$\text{or } A = \left[\begin{array}{c|c|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline B_{31} & B_{32} \end{array}\right]$$

In any such partitioning, it is necessary that the columns of A and the rows of B be partitioned in exactly same way.

$$\text{Then} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = C$$

Example 6.2.4 Compute AB, given

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 3 & 1 & 2 \end{bmatrix}$$

Answer : Let us partition A and B in the following way.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 2 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \quad \text{and}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ \hline 2 & 3 & 1 & 2 \end{array}\right]$$

We have

$$AB=\begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix}2&1\\3&2\end{bmatrix}\begin{bmatrix}1&1&1\\2&1&1\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}[2\ \ 3\ \ 1] & \begin{bmatrix}2&1\\3&2\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}[2] \\ [1\ \ 0]\begin{bmatrix}1&1&1\\2&1&1\end{bmatrix}+[1][2\ \ 3\ \ 1] & [1\ \ 0]\begin{bmatrix}0\\0\end{bmatrix}+[1][2] \end{bmatrix}$$

$$=\begin{bmatrix} \begin{bmatrix}4&3&3\\7&5&5\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix} \\ [1\ \ 1\ \ 1]+[2\ \ 3\ \ 1] & [0]+[2] \end{bmatrix}=\begin{bmatrix} \begin{bmatrix}4&3&3\\7&5&5\end{bmatrix} & \begin{bmatrix}0\\0\end{bmatrix} \\ [3\ \ 4\ \ 2] & [2] \end{bmatrix}$$

$$=\begin{bmatrix}4&3&3&0\\7&5&5&0\\3&4&2&2\end{bmatrix}$$

6.2.4. Definition (Identity matrix) :

A square matrix $I_n = [a_{ij}]_{n\times n}$ is said to be an identity matrix if all its elements on the main diagonal arè 1 and all other elements are zero. i, e, if $a_{ij} = 1$ for $i = j$

and $a_{ij} = 0$ for $i \neq j$

For example $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ are identity matrices.

6.2.5. Definition (non - singular matrices)

A square matrix A is said to be non-singular if there exists a square matrix B such that AB = I = BA where I is identity matrix.

The matrix B is called inverse of A.

It is denoted by A^{-1} ($B= A^{-1}$)

Note - Every non-singular matrix has inverse i,e, every non-singular matrix is invertible.

For example If $A=\begin{bmatrix}1&2&3\\0&1&2\\-1&1&1\end{bmatrix}$, $B=\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\-\frac{1}{2}&\frac{3}{2}&-\frac{1}{2}\end{bmatrix}$,

then $AB=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ = BA.

Therefore, B is an inverse of A and also A is an inverse of B.

Theorem 6.2.1 : If an inverse of a matrix A exists, then it is unique.

Proof : Suppose that B and C are two inverses of A. Then AB = I = BA and AC = I = CA.

Thus, we have C = CI = C(AB) = (CA)B.

= IB = B.

Hence B = C.

Problem Set 6 (A)

1. Given

$$A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix},\ B=\begin{bmatrix}3 & -1 & 2\\ 4 & -2 & 5\\ 2 & 0 & 3\end{bmatrix} \text{ and } C=\begin{bmatrix}4 & 1 & 2\\ 0 & 3 & 2\\ 1 & -2 & 3\end{bmatrix},$$

Compute the following.

(a) A+B (b) A – B (c) – 2 A

(d) 2A + 3B (e) Find matrix D such that A+D = B.

2. Taking into account the matrices of problem 1, compute

(a) AB (b) AC (c) CB

(d) (AB) C and A(BC) and see (AB) C = A(BC)

(e) A (B+C) and AB+AC and see A(B+C) = AB + AC

3. Given

$$A=\begin{bmatrix}1 & -3 & 2\\ 2 & 1 & -3\\ 4 & -3 & -1\end{bmatrix},\ B=\begin{bmatrix}1 & 4 & 1 & 0\\ 2 & 1 & 1 & 1\\ 1 & -2 & 1 & 2\end{bmatrix} \text{ and } C=\begin{bmatrix}2 & 1 & -1 & -2\\ 3 & -2 & -1 & -1\\ 2 & -5 & -1 & 0\end{bmatrix}$$

Show that AB = AC. Thus show that AB = AC does not imply B = C.

4. Compute AB, given :

(a) $A=\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 1 & 1\\ \hline 0 & 0 & 1\end{array}\right]$ and $B=\left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 1 & 1 & 0\end{array}\right]$

(b) $A=\left[\begin{array}{cc|c}1 & 0 & 3\\ 0 & 1 & 2\end{array}\right]$ and $B=\left[\begin{array}{cc}1 & 0\\ 0 & 1\\ \hline -1 & 2\end{array}\right]$

(c) $A=\left[\begin{array}{cc|cc}1 & 2 & 0 & 0\\ 0 & 1 & 0 & 0\\ \hline 0 & 0 & 0 & 1\\ 0 & 0 & 2 & 2\end{array}\right]$ and $B=\left[\begin{array}{cc|cc}0 & 0 & 0 & 1\\ 0 & 1 & 2 & 0\\ \hline 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{array}\right]$

5. Find the product

$$\begin{bmatrix}x & y & z\end{bmatrix}\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$$

6. Find x such that

$$\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\\ x\end{bmatrix}=O$$

7. Evaluate $A^2 - 3A + 9I$ if

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

8. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, prove that $A^3 = A^{-1}$

9. Give example of two real matrices such that the product AB and BA both are defined and $AB = O$ but $BA \neq O$.

10. Show by means of example that the product of two non-zero matrices can be a zero matrix.

11. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$,

where n is a positive integer.

12. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, Prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$,

when n is a positive integer. [Hint : Use method of induction]

13. If K is a positive intger, prove by induction that $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$

14. Let n be a positive intger and A, B be matrices such that AB=BA. Then prove by induction that $AB^n = B^nA$

6.3. Transpose of a matrix :

6.3.1. Definition (Transpose) : The matrix of order n×m obtained by interchanging the rows and columns of an m×n matrix A is called the transpose of A and is denoted by A^{l}(A transposed or A^T)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ is } A^{l} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

In general, if $A = [\,a_{ij}]_{m\times n}$, then

$A^T = [\,a^{l}_{ij}]_{n\times m}$ where

$a^{l}_{ij} = a_{ji}$

Theorem 6.3.1 : If A and B are two m×n matrices, then

(a) $(A+B)^T = A^T + B^T$

(b) $(\alpha A)^T = \alpha A^T$ for a scalar α

(c) $(A^T)^T = A$

(d) $(AB)^T = B^T A^T$

Proof : (a) Let $A = [a_{ij}]_{m\times n} =$ and $B = [b_{ij}]_{m\times n}$

Then $A^T = [a_{ij}^{1}]_{n\times m}$ and $B^T = [b^1_{ij}]_{n\times m}$

Then $A^T = [a^1_{ij}]_{n\times m}$ and $B^T = [b^1_{ij}]_{n\times m}$.

Where $a_{ij}^{1} = a_{ji}$

$b_{ij}^{1} = b_{ji}$

$A^T + B^T = [a'_{ij} + b'_{ij}]_{n\times m}$

$A + B = [a_{ij} + b_{ij}]_{m\times n}$

$= [C_{ij}]_{m\times n}$

Where $C_{ij} = a_{ij} + b_{ij}$

Now $(A+B)^T = [C'_{ij}]_{n\times m}$

$C'_{ij} = C_{ji} = a_{ji} + b_{ji}$

$= a'_{ij} + b'_{ij}$

Thus $(A+B)^T = A^T + B^T$

(b) Let $A = [a_{ij}]_{m\times n}$

$\alpha A = \alpha [a_{ij}]_{m\times n} = [\alpha a_{ij}]_{m\times n}$

$= [c_{ij}]_{m\times n}$

Where $c_{ij} = \alpha a_{ij}$

Again $(\alpha A)^T = [c'_{ij}]_{n\times m}$

$\alpha A^T = \alpha [a'_{ij}]_{n\times m} = [\alpha a'_{ij}]_{n\times m}$

Now $c'_{ij} = c_{ji} = \alpha a_{ji} = \alpha a'_{ij}$

So $(\alpha A)^T = \alpha A^T$.

(c) Let $A = [a_{ij}]_{m\times n}$

$A^T = [a^1_{ij}]_{n\times m}$

$= [c_{ij}]_{n\times m}$ say

Where $c_{ij} = a'_{ij} = a_{ji}$.

$(A^T)^T = [c'_{ij}]_{m\times n}$

Now $c'_{ij} = c_{ji} = c'_{ji} = a_{ij}$

Hence $(A^T)^T = A$.

(d) Let $A = [a_{ij}]_{m\times p}$, $B = [b_{ij}]_{p\times n}$

$A^T = [a'_{ij}]_{p\times m}$, $B^T = [b'_{ij}]_{n\times p}$

Where $a'_{ij} = a_{ji}$, $b'_{ij} = b_{ji}$

$AB = [c_{ij}]_{m\times n}$

Where $c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$

$B^TA^T = [d_{ij}]_{n\times m}$

Where $d_{ij} = \sum_{k=1}^{p} b'_{ik}\, a'_{kj}$

Again $(AB)^T = [c'_{ij}]$

Then $c'_{ij} = c_{ji}$

$= \sum_{k=1}^{p} a_{jk}\, b_{ki}$

$= \sum_{k=1}^{p} a'_{kj}\, b'_{ik}$

$= \sum_{k=1}^{p} b'_{ik}\, a'_{kj}$

$= d_{ij}$

Hence $(AB)^T = B^T A^T$.

Theorem : 6.3.2. If A is a non singular matrix, then A^T is also non-singular and $(A^T)^{-1} = (A^{-1})^T$

Proof : Since A is non-singular, there exists a matrix B such that $AB = I = BA$ and $B = A^{-1}$.

Therefore $(AB)^T = I^T = (BA)^T$

$\Rightarrow B^T A^T = I = A^TB^T$

Thus A^T is non-singular and B^T is inverse of A^T.

i, e, $(A^T)^{-1} = B^T = (A^{-1})^T$

Corollary : 6.3.3. The columns of a squre matrix A are LI iff its rows are LI.

Proof : The columns of A are LI.

iff A is non-singular

iff A^T is non- singular

iff the colums A^T are LI

iff the rows of A are LI.

6.3.2. Definition : (Symmetric matrices)

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i, j i.e, if $A = A^T$.

For example, $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & h & c \end{bmatrix}$ is a symmetric matrix since $A = A^T$.

Example 6.3.1 : If A is a square matrix, show that $A + A^T$ is symmetric.

Solution : $(A+A^T)^T = A^T+(A^T)^T$

$= A^T+A$

$= A+A^T$

Hence $A + A^T$ is a symmetric matrix.

6.3.3. Definition (Skew - symmetric matrices)

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if

$$a_{ij} = -a_{ji} \text{ for all } i, j$$

i.e. $A^T = -A$

For example, $A = \begin{bmatrix} 0 & 2 & -4 \\ -2 & 0 & -1 \\ 4 & 1 & 0 \end{bmatrix}$ is skew - symmetric.

Since $A^T = -A$.

Example 6.3.2 : Prove that every square matrix can be expressed in one and only one way as a sum if a symmetric and a skew symmetric matrix.

Solution : Let A be a square matrix.

It can be expressed as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = P + Q$$

when $P = \frac{1}{2}(A + A^T)$, $Q = \frac{1}{2}(A - A^T)$

Now $P^T = \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(A + A^T) = P.$

So P is a symmetric matrix

$Q^T = \frac{1}{2}(A - A^T)^T$

$= \frac{1}{2}\left(A^T - (A^T)^T\right)$

$= \frac{1}{2}(A^T - A)$

$= -\frac{1}{2}(A - A^T)$

$= -Q.$

So Q is a skew-symmetric matrix.

Hence A is expressed as a sum of symmetric and a skew symmetric matrix.

Uniqueness : Let A = R+S be another representation of A where R is a symmetric and S is a skew symmetric matrix.

Then $R^T = R$ and $S^T = -S$

Now $A^T = (R+S)^T = R^T + S^T = R - S$

Thus $R = \frac{1}{2}(A + A^T) = P$ and $S = \frac{1}{2}(A - A^T) = Q$.

So A = R + S = P + Q

Hence the representation A = P+Q is unique.

Example 6.3.3 : Express the matrix $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$ as the sum of a symmetric and skew symmetric matrix.

Solution : $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$

$$A^T = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix} = \begin{bmatrix} 4 & \frac{3}{2} & -4 \\ \frac{3}{2} & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix}$$

$$\frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & \frac{3}{2} & -4 \\ \frac{3}{2} & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

(Symmetric matrix) (Skew-Symmetric matrix)

6.3.4. Definition (Orthogonal Matrix) : A square matrix A is said to be orthogonal if $A A^T = A^T A = I$

Example 6.3.4.: If A is orthogonal matix show that A^T is also orthgonal.

Solution : Let A is orthogonal matrix.

By definition.

$$AA^T = A^T A = I$$
$$\Rightarrow (AA^T)^T = (A^T A)^T = I$$
$$\Rightarrow (A^T)^T A^T = A^T (A^T)^T = I$$
$$\Rightarrow AA^T = A^T A = I$$

Thus A^T is othogonal.

Example 6.3.5 : Prove that the matrix

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \text{ is orthogonal.}$$

Solution :

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$A A^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha . \sin\alpha + \sin\alpha . \cos\alpha \\ -\sin\alpha . \cos\alpha + \cos\alpha . \sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence A is an othogonal matrix.

6.4. Conjugate of a matrix :

We know that $z = x + iy$ is a complex number where x,y are real numbers and $\sqrt{-1} = i$. Also the conjugate of z is $\overline{z} = x - iy$.

6.4.1. Definition (Conjugate of a matrix) :

A matrix formed by replacing the elements of a matrix A by their respective conjugate complex numbers is called the conjugate of A and is denoted by $\overline{A}$.

$$\text{If } A = [a_{ij}]_{m\times n},\ \overline{A} = [\overline{a_{ij}}]_{m\times n}$$

When $\overline{a}_{ij}$ is conjugate of a_{ij} for all i, j.

$$\text{For example, if } A = \begin{bmatrix} 3+4i & 2-i & 4 \\ i & 2 & -3i \end{bmatrix}$$

$$\text{then } \overline{A} = \begin{bmatrix} 3-4i & 2+i & 4 \\ -i & 2 & 3i \end{bmatrix}$$

Theorem 6.4.1 : If A and B be two matrices and their conjugate matrices are $\bar{A}$ and $\bar{B}$ respectively, then.

(a) $(\bar{\bar{A}}) = A$ (b) $(\overline{A+B}) = \bar{A}+\bar{B}$

(c) $(\overline{kA}) = \bar{k}\,\bar{A}$ (d) $(\overline{AB}) = \bar{A}\,\bar{B}$

Proof : Let $A = [a_{ij}]_{m\times n}$, then

$\bar{A} = [\bar{a}_{ij}]_{m\times n}$ where $\bar{a}_{ij}$ is the conjugate of a_{ij}.

$(\bar{\bar{A}}) = [b_{ij}]_{m\times n}$

Where b_{ij} is conjugate of complex numbers $\bar{a}_{ij}$.

Thus $b_{ij} = \bar{\bar{a}}_{ij} = a_{ij}$ for all i, j.

Hence $(\bar{\bar{A}}) = A$

This proves (a)

(b) Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$

Then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$, $\bar{B} = [\bar{b}_{ij}]_{m\times n}$.

$A + B = [a_{ij}+b_{ij}]_{m\times n}$

$$(\overline{A+B}) = [\overline{a_{ij}+b_{ij}}]_{m\times n}$$

$$= [\overline{a_{ij}+b_{ij}}]_{m\times n}$$

$$= [\bar{a}_{ij}]_{m\times n} + [\bar{b}_{ij}]_{m\times n}$$

$$= \bar{A}+\bar{B}$$

(c) Let $A = [a_{ij}]_{m\times n}$ and k be any complex number.

Then $kA = [ka_{ij}]_{m\times n}$

$$\Rightarrow (\overline{kA}) = [\overline{ka_{ij}}]_{m\times n}$$

$$= [\bar{k}\,\overline{a_{ij}}]_{m\times n}$$

$$= \bar{k}\,[\bar{a}_{ij}]_{m\times n}$$

$$= \bar{k}\,\bar{A}.$$

(d) Let $A = [a_{ij}]_{m\times p}$, $B = [b_{ij}]_{p\times n}$

Then $\bar{A} = [\bar{a}_{ij}]_{m\times p}$. $\bar{B} = [\bar{b}_{ij}]_{p\times n}$

$AB = [c_{ij}]_{m\times n}$

$$\text{where } c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$$

$$(\overline{AB}) = [\overline{c_{ij}}]_{m\times n}$$

$$\bar{A}\;\bar{B} = [D_{ij}]_{m\times n}$$

where $D_{ij} = \sum_{k=1}^{p} \bar{a}_{ik}\, \bar{b}_{kj}$

$$= \sum_{k=1}^{p} \overline{a_{ik}\, b_{kj}} = \left(\overline{\sum_{k=1}^{p} a_{ik}\, b_{kj}}\right)$$

$$= \bar{c}_{ij}$$

Hence $(\overline{AB}) = \bar{A}\ \bar{B}$

6.4.2. Definition (Transpose of a conjugate matrix)

The transpose of a conjugate matrix A is denoted by A* defined as

$A^* = (\overline{A^T})$

Example 6.4.1 : If

$$A = \begin{bmatrix} 2+3i & 1-2i & 2+4i \\ 3-4i & 4+3i & 2-6i \\ 5 & 5+6i & 3 \end{bmatrix},$$

$$\text{then } A^T = \begin{bmatrix} 2+3i & 3-4i & 5 \\ 1-2i & 4+3i & 5+6i \\ 2+4i & 2-6i & 3 \end{bmatrix}$$

$$A^* = (\overline{A^T}) = \begin{bmatrix} 2-3i & 3+4i & 5 \\ 1+2i & 4-3i & 5-6i \\ 2-4i & 2+6i & 3 \end{bmatrix}$$

6.4.3. Definition (Hermitian matrix)

A square matrix $A = [\, a_{ij}]_{m\times n}$ is said to be Hermitian if
$A^* = A$

For example of $A = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 2 & 3-4i \\ 3+4i & 1 \end{bmatrix}$$

$$A^* = (\overline{A^T}) = \begin{bmatrix} 2 & 3+4i \\ 3-4i & 1 \end{bmatrix} = A.$$

Therorem : 6.4.2. If A, B, are square matrices conformable for addition and multiplication.

(a) $(A^*)^* = A$ (b) $(A+B)^* = A^* + B^*$

(c) $(kA)^* = \bar{k}\, A^*$, k is a scalar.

(d) $(AB)^* = B^*.A^*$.

Proof :

(a) $A^* = (\bar{A})^T$

$(A^*)^* = [\overline{\{(\bar{A})^T\}}^T] = [\bar{\bar{A}}] = A$

(b) $(A+B)^* = (\overline{A+B})^T = (\bar{A}+\bar{B})^T$

$= (\bar{A})^T + (\bar{B})^T$

$= A^* + B^*$

(c) $(kA)^* = (\overline{kA})^T = (\bar{k}\,\bar{A})^T = \bar{k}\,(\bar{A})^T = \bar{k}\,A^*.$

(d) $(AB)^* = (\overline{AB})^T = (\bar{A}\,\bar{B})^T$

$= (\bar{B})^T\,(\bar{A})^T$

$= B^*A^*$

6.4.4. Definition (Skew - Hermitian matrix)

A square matrix $A = [a_{ij}]$ is said to be skew - Hermitian matrix if $a_{ij} = -\bar{a}_{ij}$ for all i and j. i.e, if $A^* = -A$.

Example 6.4.2 : Show that $A = \begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is skew-Hermitian- matrix.

Solution :

$$\bar{A} = \begin{bmatrix} -i & 3-2i & -2+i \\ -3-2i & 0 & 3+4i \\ 2+i & -3+4i & 2i \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} -i & -3-2i & 2+i \\ 3-2i & 0 & -3+4i \\ -2+i & 3+4i & 2i \end{bmatrix}$$

$$= -\begin{bmatrix} i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$$

$$= -A$$

So A is skew-Hermitian.

Example 6.4.3 : Show that every square matrix can be expressed as R + iS uniquely where R and S are Hermitian matrices.

Solution : Let A be a square matrix.

It can be written as

$$A = \left\{\frac{1}{2}(A+A^*)\right\} + i\left\{\frac{1}{2i}(A-A^*)\right\}$$

$$= R + iS$$

where $R = \frac{1}{2}(A + A^*)$

$S = \frac{1}{2i}(A - A^*)$

Now $R^* = \frac{1}{2}(A + A^*)^*$

$= \frac{1}{2}(A^* + (A^*)^*)$

$= \frac{1}{2}(A^* + A)$

$= \frac{1}{2}(A + A^*) = R$

So R is a Hermitian matrix.

Also $S^* = -\frac{1}{2i}(A - A^*)^*$

$= -\frac{1}{2i}(A^* - (A^*)^*)$

$= -\frac{1}{2i}(A^* - A)$

$= \frac{1}{2i}(A - A^*) = S.$

So S is a Hermitian matrix.

Hence = A = R + iS, where R and S are Hermitian matrices.

Uniqueness : Let A = P + iQ be another expresson i.e, P* =P and Q* = Q.

Then A* = (P+iQ)*

= P* + (iQ)*

= P*– iQ*

= P – iQ

A = P + iQ, A* = P – iQ.

$\Rightarrow P = \frac{1}{2}(A + A^*) = R$ and $Q = \frac{1}{2i}(A - A^*) = S.$

Hence A = R + iS is the unique expression.

where R and S are Hermitian matrics.

Example 6.4.4. For any square matrix, if

AA* = I, show that

A*A= I

Solution : AA* = I

So A is invertible.

Let B be another matrix such

AB=BA=I

$$\begin{aligned}\text{Now}\quad B &= BI = B(AA^*)\\ &= (BA)A^*\\ &= IA^*\\ &= A^*\end{aligned}$$

So $B = A^*$

$\therefore\ AA^* = A^*A = I$

Example 6.4.5 : Express the matrix

$$A = \begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$$

as the sum of Hermitian matrix and skew - Hermitian matrix.

Solution :

$$\bar{A} = \begin{bmatrix} 1-i & 2 & 5+5i \\ -2i & 2-i & 4-2i \\ -1-i & -4 & 7 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

$$R = \frac{1}{2}(A + A^*) = \frac{1}{2}\begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$$S = \frac{1}{2}(A - A^*) = \frac{1}{2}\begin{bmatrix} 2i & 2+2i & 6-4i \\ -2+2i & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 1+i & 3-2i \\ -1+i & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

Thus A = R + S where R is Hermitian and S is skew - Hermitian matrix.

6.4.5. Definition : (Unitary matrix)

A square matrix A is said to be unitary matrix if

$$AA^* = A^*A = I$$

Example. 6.4.6 : Prove that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Solution :

$$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^*A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1+(1+1) & (1+i)-(1+i) \\ (1-i)-(1-i) & (1+1)+1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Hence A is unitary matrix.

Problem Set 6 (B)

1. Calculate the transpose of each of the following matrices.

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 0 & 5 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$

2. For each of the following matrices A, verify if $A^T = A$.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 5 \end{bmatrix}$ (b) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 6 & 8 \\ 8 & 12 & 16 \\ 12 & 18 & 24 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

3. For each of the following matrices A, verify if $A^T = -A$

(a) $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -3 \\ -2 & 3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$ (d) $\begin{bmatrix} a & b & c \\ -b & b & d \\ -c & -d & e \end{bmatrix}$

4. Show that following matrices are orthogonal.

(a) $A=\frac{1}{3}\begin{bmatrix}-1 & 2 & 2\\ 2 & -1 & 2\\ 2 & 2 & -1\end{bmatrix}$ (b) $A=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}$

(c) $A=\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\end{bmatrix}$ (d) $A=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$

5. Determine α, β, γ, when $\begin{bmatrix}0 & 2\beta & \gamma\\ \alpha & \beta & -\gamma\\ \alpha & -\beta & \gamma\end{bmatrix}$ is orthogonal.

6. Express the following matrices as the sum of a symmetric and a skew symmetic matrix.

(a) $\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & 1\\ 2 & -1 & 3\end{bmatrix}$ (b) $\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ (c) $\begin{bmatrix}1 & 2 & 4\\ -2 & 5 & 3\\ -1 & 6 & 3\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 5 & 3\\ -2 & 1 & 6 & 1\\ 3 & 2 & 7 & 1\\ 4 & -4 & -2 & 0\end{bmatrix}$

7. Which of the following matrics are symmetric :

(a) $\begin{bmatrix}a & h & g\\ h & b & f\\ g & f & c\end{bmatrix}$ (b) $\begin{bmatrix}1 & 3 & 4\\ 3 & 5 & -1\\ 4 & -1 & 0\end{bmatrix}$ (c) $\begin{bmatrix}2 & 4 & 8\\ 6 & 2 & 6\\ 4 & 6 & 2\end{bmatrix}$ (d) $\begin{bmatrix}0 & 1 & 2\\ 3 & 0 & 4\\ 5 & 6 & 0\end{bmatrix}$

8. Which of the following matrices are skew Symmetric :

(a) $\begin{bmatrix}-1 & 2 & -3\\ -2 & -1 & -4\\ 3 & 4 & -1\end{bmatrix}$ (b) $\begin{bmatrix}0 & -2 & 3\\ 2 & 0 & 4\\ 3 & -4 & 0\end{bmatrix}$ (c) $\begin{bmatrix}0 & -2 & 3\\ 2 & 0 & 7\\ -3 & -7 & 0\end{bmatrix}$ (d) $\begin{bmatrix}0 & a & b\\ -a & 0 & -c\\ -b & c & 0\end{bmatrix}$

9. If A and B are two symmetric matrices of the same order, show that AB is symmetric if AB = BA

10. If the matrix $A=\begin{bmatrix}1+i & 3-5i\\ 2i & 5\end{bmatrix}$, find

(a) $\bar{A}$ (b) $(\bar{A})^T$ (c) A^* (d) $(A^*)^*$

11. Which of the following matrices are Hermitian :

(a) $\begin{bmatrix} 1 & 2+i & 3-i \\ 2+i & 2 & 4-i \\ 3+i & 4+i & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 2i & 3 & 1 \\ 4 & -1 & 6 \\ 3 & 7 & 2i \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2-i & 5+2i \\ 2+i & 1 & 2-5i \\ 5-2i & 2+5i & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & i & 3 \\ -7 & 0 & 5i \\ 3i & 1 & 0 \end{bmatrix}$

12. Which of the following matrices are skew - Hermitian :

(a) $\begin{bmatrix} 2i & -3 & 4 \\ 3 & 3i & -5 \\ -4 & 5 & 4i \end{bmatrix}$ (b) $\begin{bmatrix} 3i & -1 & 2 \\ 1 & 2i & -6 \\ 4 & 6 & -3i \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1-i & 2+3i \\ -1-i & 0 & 6i \\ -2+3i & 6i & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 3 & 7+i \\ 3i & -i & 6 \\ 7-i & 8 & 0 \end{bmatrix}$

13. Give an example of a matrix which is skew - symmetic but not skew- Hermitian.

14. If A be a Hermitian matrix, show that iA is skew - Hermitian. Also show that if B be a skew - Hermitian matrix, then iB must be Hermitian.

15. If A and B are Hermitian matrices, than show that AB+BA is Hermitian and AB-BA is skew - Hermitian.

16. If A is any square matrix, then show that A + A* is Hermitian.

17. Show that the matrix B*AB is Hermitian or skew-Hermitian according as A is Hermitian or skew Hermitian.

18. Show that $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ is unitary.

19. Prove that a real matrix is unitary if it is orthogonal.

20. If A and B are unitary matrices, then show that AB is a unitary matrix.

6.5. Some types of matrices.

6.5.1. Definition : (Upper triangular matrix)

A square matrix $A = [a_{ij}]$ is said to be upper triangular of $a_{ij} = 0$ for $i > j$.

6.5.2. Definition : (Lower triangular matrix)

A square matrix $A = [a_{ij}]$ is said to be lower triangular if $a_{ij} = 0$ for $i < j$.

6.5.3. Definition : (Diagonal matrix)

A square matrix $D = [a_{ij}]$ is said to be diagonal if it is both upper triangular and lower triangular. This matrix is frequently written as $D = \text{diag}(a_{11}, a_{22}, \dots a_{nn})$

If in the diagonal matrix D,

$a_{11} = a_{22} = a_{33} = \dots = a_{nn} = k$, then

D is called a scalar matrix. If in addition, $k = 1$, then the matrix is called the identity matrix and is denoted by I_n.

Example : 6.5.1. :

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix} \text{ are upper triangular matrices.}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \text{ are lower triangular matrices.}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ is a diagonl matrix.}$$

A diagonal matrix is both lower triangular and upper triangular.

$$F = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ is a scalar matrix.}$$

6.5.4. Definition (Commutative matrices)

If A and B are square matrices such that $AB = BA$, then A and B are called commutative or are said to commute on the other hand if $AB = -BA$, then A and B are said to be anticommutative.

Example : 6.5.2 : Show that the matrices

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \text{and} \begin{bmatrix} c & d \\ d & c \end{bmatrix} \text{ commute for all values of a, b, c, d.}$$

Solution :

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix} = \begin{bmatrix} c & d \\ d & c \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

6.5.5. Definition (Periodic) :

A matrix A for which $A^{K+1} = A$ where k is a positive integer, is called periodic. If k is the least positive integer for which $A^{K+1} = A$, then A is said to be of period k.

If $k = 1$, so that $A^2 = A$, then

A is called **idempotent matrix**.

Example 6.5.3 :

Show that $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -2 \end{bmatrix}$ is idempotent.

Solution :

$$A^2 = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence A is idempotant.

6.5.6. Definition (involutory) :

A square matrix A is said to be involutory matrix if $A^2 = I$.

Example 6.5.4 : Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory (Verify that $A^2=I$)

6.5.7. Definition (Nilpotent) :

A square matrix A is said to be Nilponent of there exists a positive integer m such that $A^m = 0$.

If m is the least positive interger such that $A^m = 0$, then m is called the index of nilpotent matrix A.

Example 6.5.5. : Show that the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent and find its index.

Solution : We can see that

$$A^2 = A.A. = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$A^3 = A.A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

So A is nilpotent with index 3.

Problem Set 6(C)

1. Show that $A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$ is periodic with period 2.

2. Show that the following matrices commute :

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ -1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ -\frac{3}{5} & \frac{2}{5} & \frac{4}{5} \\ \frac{7}{15} & -\frac{1}{5} & \frac{1}{15} \end{bmatrix}$

3. Show that $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$ anti-commute and $(A+B)^2 = A^2 + B^2$.

4. Show that each of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ anti-commute with the others.

5. Show that the following matrices are idempotent.

(a) $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

6. Prove that the matrix.

$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix}$ is idempotent,

Where $\lambda_1, \lambda_2, \lambda_3$ are direction cosines.

7. If AB = A and BA= B, then show that A and B are idempotent.

8. If B is an idempotent matrix, show so that A = I-B is also idempotent and AB = BA = O.

9. Show that the following matrices are involutory.

(a) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

10. Show that $\begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is nilpotent

Find its index.

11. Show that the matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is nilpotent.

12. Prove that a matrix A is involutory iff (I-A) (I+A) = O.

13. If A is nilpotent of index 2, show that $A(I \pm A)^n = A$ for n any positive integer.

14. Show that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of order 3.

6.6. Invertible matrices :

A square matrix A is said to be invertible iff it is non-singular.

i.e, iff $\exists$ another square matrix B, such that AB=BA=I.

B is called inverse of A and it is denoted by A^{-1}.

Example 6.6.1 : Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

Solution :

Let $B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ be the matrix such that AB = BA = I

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_3 & x_2 + 2x_4 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & 2x_1 + x_2 \\ x_3 & 2x_3 + x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_3 = x_1 = 1, \quad x_2 + 2x_4 = 2x_1 + x_2 = 0$$

$$x_3 = 0, \quad x_4 = 2x_3 + x_4 = 1$$

$$\Rightarrow x_1 = 1, \quad x_2 = -2, \quad x_3 = 0, \quad x_4 = 1$$

$$\Rightarrow B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Example 6.6.2. Find the inverse of the matrix.

$$A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & 1 & 1\end{bmatrix}$$

Solution :

$$\text{Let } A^{-1}=\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}$$

Then $AA^{-1} = A^{-1}A = I$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}=\begin{bmatrix}x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\\ z_1 & z_2 & z_3\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ -1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x_1+2y_1+3z_1 & x_2+2y_2+3z_2 & x_3+2y_3+3z_3\\ y_1+2z_1 & y_2+2z_2 & y_3+2z_3\\ -x_1+y_1+z_1 & -x_2+y_2+z_2 & -x_3+y_3+z_3\end{bmatrix}$$

$$=\begin{bmatrix}x_1-x_3 & 2x_1+x_2+x_3 & 3x_1+2x_2+x_3\\ y_1-y_3 & 2y_1+y_2+y_3 & 3y_1+2y_2+y_3\\ z_1-z_3 & 2z_1+z_2+z_3 & 3z_1+2z_2+z_3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow x_1+2y_1+3z_1=x_1-x_3=1,\ x_2+2y_2+3z_2=2x_1+x_2+x_3=0$$
$$x_3+2y_3+3z_3=3x_1+2x_2+x_3=0,\ y_3+2z_3=3y_1+2y_2+y_3=0$$
$$-x_1+y_1+z_1=z_1-z_3=0,\ -x_2+y_2+z_2=2z_1+z_2+z_3=0$$
$$-x_3+y_3+z_3=3z_1+2z_2+z_3=1$$

$$\Rightarrow x_1=\frac{1}{2}\quad x_2=x_3=-\frac{1}{2}$$
$$y_1=1,\ y_2=-2,\ y_3=1$$
$$z_1=z_3=-\frac{1}{2},\ z_2=\frac{3}{2}$$

$$\Rightarrow A^{-1}=\begin{bmatrix}\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2}\\ 1 & -2 & 1\\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$$

Theorem : 6.6.1 An $n \times n$ matrix A is invertible iff the corresponding linear transformation T (via the standard bases) is non- singular.

Proof : Suppose A is invertible. Then there exists an $n\times n$ matrix B such that $AB = I_n = BA$. Let the linear transformation corresponding to B be S: $V_n \to V_n$. We know that if $M_{m,n}$ denote the set of all $m \times n$ real matrices and if U and V be real vector spaces of dimension n and m respectively relative to same fixed bases of U and V, there exists a linear map $Z : L(U, V) \to M_{m,n}$ which is one-one and onto.

L (U,V) denote the set of all linear maps from U $\to$ V.

Thus Z^{-1} (AB) = $Z^{-1}(I_n)$ = Z^{-1}(BA)

$\Rightarrow Z^{-1}(A)\, Z^{-1}(B) = I = Z^{-1}(B)\, Z^{-1}(A)$

$\Rightarrow TS = I = ST$

Thus T is non-singular.

Conversely, if T is non-singular thn there exists a linear transformation :

$S : V_n \to V_n$ such that TS = I = ST.

There fore Z (TS) = Z (I)= Z (ST)

If B be a matrix corresponding to S,

then AB = I_n = BA

Hence A is invertible.

Problem Set 6 (D)

1. Prove that following matrices are non-singular and find their inverses.

(a) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

2. Prove that following matrices are non-singular and find their inverses.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$

3. Find the values of α, β for which the following matrix is invertible.

$$\begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{bmatrix}$$

6.7. Elementary row (column) operations in matrices :

There are 3 types of elementary row (column) operations (transformations)

Type I : Interchanging two rows (columns) ($R_i \leftrightarrow R_j$ / $C_i \leftrightarrow C_j$)

Type II : Multiplying a rows (columns) a non-zero scalar.

(R_i; $\to kR_i$/$C_i \to kC_j$)

Type III : Adding to a row a scalar times another row (column)

($R_i \to R_i + kR_j$ / $C_i \to C_i + kC_j$)

6.7.1. Definition : When a matrix A is subjected to a finite number of elementary row operations, the resulting matrix B is said to be row-equivalent to A.

We write this as $B \sim A$.

It can be proved that $\sim$ is an equivalence relation.

6.7.2. Definition (Rank of a matrix) : The maximum number of linearly independent row (column) vectors of a matrix is called the rank of a matrix.

6.7.3. Definition (Submatrix) : A submatrix is obtained by deleting some rows and columns from a matrix.

6.7.4. Definition (Minor) : If from an $m \times n$ matrix A, $m - p$ rows and $n - p$ columns are removed, a square submatrix of p rows and p columns is formed. The determinant of the square submatrix p is called a minor of A of order p.

For example : In the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 3 & 1 & 0 \\ 4 & 1 & 3 & 6 \\ 8 & 1 & 2 & 0 \end{bmatrix}$$

every element is a minor of order 1.

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 6 \\ 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

are minors of order 2.

$$\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 6 \\ 1 & 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 1 & 0 \\ 4 & 3 & 6 \\ 8 & 2 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 3 \\ 8 & 1 & 2 \end{vmatrix}$$

are minors of order 3.

6.7.5. Definition (Rank of a matrix) : A positive integer r is said to be the rank of a matrix A denoted by $\rho(A)$ if

(i) There exist at least one minor in A of order r which is not zero.

(ii) Every minor in A of order greater than r is zero.

Example :

(i) Rank of $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is 2 since

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \neq 0$$

(ii) Rank of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is 3 since

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

(iii) Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix}$

We have $|A| = 0$ since first two rows are identical. So the rank of A is not 3.

A minor $\begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$, but

$\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = -2 \neq 0$. At least one minor of order 2 is not zero. Hence the rank of A is 2.

Example : 6.7.1 : Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$

Solution : $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 18 \neq 0.$

A minor of order 3 is non zero.
Hence $\rho\ (A) \geq 3$
Since A doesnot possess any 4th order minor, $\rho\ (A) \leq 3$.
So $\rho\,(A) = 3$

Example 6.7.2.: Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution : $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$

The only minor of order 3 is non-zero
So $\rho(A) < 3$ i.e., $\rho(A) \leq 2$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \neq 0.$$

A minor of order 2 is non- zero
So $\rho(A) \geq 2$
Hence $\rho\,(A) = 2$

Example 6.7.3 : Reduce the following matrix to upper triangular form using elementary row operators.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$$

Solution :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -5 & -7 \end{bmatrix} R_3 \to R_3 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

This matrix is upper triangular.

Example : 6.7.4 Transform

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} \text{ into unit matrix.}$$

Solution :

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix} R_2 \to -\frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -5 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 3R_2 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{bmatrix} R_3 \to -\frac{1}{7} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 9R_3 \\ R_2 \to R_2 + 2R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6.7.6 : Definition : (Normal form of a matrix)

Every non- zero matrix A can be reduced to the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is unit matrix of order r for some positive onteger r by elementary row (column) transformation. The form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is called the normal form of the matrix A.

Note : The rank of a matrix is r iff it can be reduced to the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Example 6.7.5 : Reduce the following matrix to normal form and find its rank.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Solution :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 1 & 2 \end{bmatrix} \quad R_2 \to \frac{-1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad C_2 \to C_2 - C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 - C_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Where } I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So the rank of the matrix is 2.

Example 6.7.6 : Reduce the following matrix into its normal form and hence find its rank.

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution :

$$A \sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 + C_1 \\ C_3 \to C_3 + 2C_1 \\ C_4 \to C_4 + 4C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 + 6C_2 \\ C_4 \to C_4 + 3C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_3 \to \frac{1}{33} C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C_4 \to C_4 - 22C_3$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence $\rho(A) = 3$

6.7.7. Definition (Echelon form of matrix)

A matrix A is said to be in row - reduced echelon form if

(a) the zero rows, if any occur below all non-zero rows.

(b) the first non zero entry in each non zero row is 1.

(c) if a column contains the first non zero entry of any row, then every other entry in that column is zero.

(d) Let there be r non zero rows. If the first non zero entry of the i-th row occurs in column k_i (i = 1, 2,..., r), then $k_1 < k_2 < \ldots < k_r$.

Example 6.7.7 : Examine that the matrix :

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & 1 & 0 & 5 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ is in row- reduced echelom form.}$$

Here (a) the zero row is below all non zero rows.

(b) first non zero entry of each row is 1

(c) C_1, C_2, C_3 contain first non zero entries of R_1, R_2, and R_3 respectly. So other entries of first column are zero.

(d) First non zero only of 1st row occurs in 1st column i.e., $k_1 = 1$ similarly $k_2 = 2$, $k_3 = 3$.

Thus $k_1 < k_2 < k_3$

Hence the given matrix is in row reduced achelon form.

Example 6.7.8 : Students should examine that the following matrix is in row-reduced echelon form.

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By careful scrutiny of the process of row-reduction the reader can convince himself about the following facts.

Fact 1 : Every matrix A is row-equivalent to a row reduced echelon matrix.

Fact 2 : If a matrix is in the row-reduced echelon from, its rank is the number of non-zero rows in it. Basing on the above facts we prove the following Theorem.

Theorem : 6.7.1. The rank of a matrix A is equal to the rank of the row-reduced echelon matrix B, obtained from A.

Proof : B has been obtained from A by a finite sequence of elementary row operations. Let us now show that these row operations do not affect the row rank of A. If any two rows are intechanged or if any row is multiplied by a non zero scalar, then the number of linearly independent rows will remain unaffected. Thus the rank of the matrix remains unchanged with respect to first two type of elementary row operations.

Suppose we add α times $(\alpha \neq 0)$ a row vector v_1 to another row vector v_2. Let the other row vectors be $v_3, v_4, \ldots v_m$. Let us examine the two set of row vectors $P = \{v_1, v_2, \ldots, v_m\}$, $Q = \{v_1, v_2 + \alpha v_1, v_3, \ldots, v_m\}$

Suppose P is L.D.

The there exists scalars not all zero such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m = 0$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \alpha_2 \alpha v_1 - \alpha_2 \alpha v_1 + \alpha_3 v_3 + \ldots + \alpha_m v_m = 0$$
$$\Rightarrow (\alpha_1 - \alpha_2 \alpha)\, v_1 + \alpha_2 (v_2 + \alpha v_1) + \alpha_3 v_3 + \ldots + \alpha_m v_m = 0$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_m$ are not all zero,

$\alpha_1 - \alpha_2 \alpha, \alpha_2, \alpha_3, \ldots, \alpha_m$ are so.

Thus Q is LD.

Let P is LI.

Let us show that Q is L.I

Now $\alpha_1 v_1 + \alpha_2 (v_2 + \alpha v_1) + \alpha_3 v_3 + \ldots + \alpha_m v_m = 0$

$$\Rightarrow (\alpha_1 + \alpha \alpha_2) v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \ldots + \alpha_m v_m = 0$$
$$\Rightarrow \alpha_1 + \alpha \alpha_2 = 0,\ \alpha_2 = 0 = \alpha_3 = \ldots = \alpha_m \qquad [\because \text{P is LI}]$$
$$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 \ldots = \alpha_m = 0$$

Thus with respect to third type of elementary row operation the linear independence of rows shall remain unaffected.

Thus the rank of the matrix will not change under three types of elementary row operation.

Hence $\rho(A) = \rho(B)$.

Theorem : 6.7.2 The rank of a matrix A is the number of nonzero rows in its row reduced echelon form.

Proof : Fact 2 tells that the rank of a matrix, in row-reduced echelon form is the number of nonzero rows in it. Theorem 6.7.1. tells that the rank of a matrix is equal to the rank of its row reduced echelon matrix. Hence proved.

Example 6.7.9. Determine the rank of the matrix $A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ by reducing it to row-reduced echelon form.

Solution :

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix} \quad R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 5R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + 9R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to -R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_3 \\ R_1 \to R_1 + R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 2R_2$$

This matrix is in echolon form.

The number of non zero rows is 3.

So $\rho(A) = 3$.

Example 6.7.10 : Find the rank of the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$

Solution : $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 3 & 2 & 5 & 1 \\ 0 & 4 & 4 & -4 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 4 & 4 & -4 \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \to \frac{1}{2} R_2.$$

The last equivalnt matrix is in echelon form. The number of non zero rows being 2, $\rho(A) = 2$

Note : We can define row rank of a matrix A to be the maximum number of linearly independent rows of the matrix. Similarly the column rank as the maximum number of LI columns of A. Infact the row rank is equal to the column rank and that is same as the rank of a matrix. This fact is classfied by the following theorem as the column rank of A is same as row rank A^T.

Theorem :6.7.3 If A is a non zero matrix,

$$\rho(A) = \rho(A^T)$$

Proof : Let $A = [a_{ij}]_{m \times n}$

Then $A^T = [a_{ij}]_{n \times m}$

Let the rank of the matrix A be r.

Let B be a submatrix of A of order r such that $|B| \neq 0$.

$\therefore$ $|B^T| \neq 0$ (by property of determinent in chapter 7)

B^T is a submatrix of order r of A^T.

So $\rho(A^T) \geq r$ ...(1)

Let C be a submatrix of order r + 1 of A.

Then $|C| = 0$ (since $\rho(A) = r$).

$\Rightarrow |C^T| = 0$

Since C^T is a submatrix of order (r+1) of A^T, the rank of A^T cannot be greater than r.

i.e., $\rho(A^T) \leq r$ ... (2)

From (1) and (2) we get

$\rho(A^T) = r$

$\therefore$ $\rho(A) = \rho(A^T)$

6.7.8. Definition (Elementary Matrix)

A matrix obtained from a unit matrix by subjecting it to any of the elementary row operation is called an elementary matrix.

For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The elementary matrix corresponding to the operation $R_3 \leftrightarrow R_1$ (interchange of 1st and 3rd row) is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The elementary matrix corresponding to the operation $R_3 \to 5\,R_3$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The elementary matrix corrsponding to the operation $R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem : 6.7.4 Every elementary row transormation of a matrix can be affected by premultiplication with the corresponding elementary matrix.

Proof : Let us verify this fact by considered the following example.

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 3 & 5 & 9 \end{bmatrix}$$

Let us apply the transformation (elementary row operation) $R_3 \to R_3 + 4R_1$ and we get a matrix B.

$$B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 11 & 17 & 25 \end{bmatrix}$$

The elementary matrix corresponding to the operation $R_3 + 4R_1$ is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\text{Now } E\ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 3 & 5 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 11 & 17 & 25 \end{bmatrix} = B$$

Now the reader can easily visualise the general proof of the following theorem.

Theorem : 6.7.5 (Gause-Jordan method of finding the inverse of a square matrix):

Those elementary row operations which reduce a given matrix A to a unit matrix, when applied to unit matrix I give the inverse of A.

Proof : Let the successive row operations (transformation) which reduce A to I result from pre-multiplication by the elementary matrices $E_1, E_2, ..., E_i$,

so that $\quad E_i\, E_{i-1} \ldots E_2\, E_1 \quad A = I$

$\Rightarrow \; E_i\, E_{i-1} \ldots E_2\, E_1 \quad AA^{-1} = I\,A^{-1}$

$\Rightarrow \; E_i\, E_{i-1} \ldots E_2\, E_1 \quad I = A^{-1}$. Hence proved.

Note : For practical evaluation of A^{-1},the two matrices A and I are written side by side and the same row transformations are performed on both. As soon as A is reduced to I, the other matrix represents A^{-1}.

Example 6.7.11 : Using Gauss - Jordan method, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution : Writing the matrices A and I side by side we get,

$$[A:I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to \frac{1}{2} R_2 \\ R_3 \to \frac{1}{2} R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 1 & 0 & \frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - \frac{3}{2} R_3 \\ R_2 \to -\frac{1}{2} R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right] = [I : A^{-1}]$$

$$\text{Hence } A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Problem Set 6(E)

1. Find the rank of the following matrices :

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 2 & 3 & 1 \\ 4 & 3 & 5 & 2 \\ 3 & 1 & 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 3 & 3 & 0 \\ 5 & 3 & 1 & 8 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 3 & 0 \\ -1 & 0 & 2 & -8 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$

2. Transform the following matrices to normal from and find the rank.

(a) $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ 1 & 3 & 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

3. Reducing to echelon form find the rank of the following matrices.

(a) $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & -2 \\ 0 & 5 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$

(i) $\begin{bmatrix} 3 & -2 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ (j) $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$

(k) $\begin{bmatrix} 0 & c & -b & a^1 \\ -c & 0 & a & b^1 \\ b & -a & 0 & c^1 \\ -a^1 & -b^1 & -c^1 & 0 \end{bmatrix}$ (l) $\begin{bmatrix} 0 & b-a & c-a & b+c \\ a-b & 0 & c-b & c+a \\ a-c & b-c & 0 & a+b \\ b+c & c+a & a+b & 0 \end{bmatrix}$

Where $aa^1 + bb^1 + cc^1 = 0$ Where a, b, c are unequal and a, b, c, are all positive numbers.

4. Find the inverse of the following matrices using Gauss-Jordan method.

(a) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \\ 8 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (h) $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -9 & 5 \end{bmatrix}$

6.8. System of linear Equations :

6.8.1. Matrix form of a given system of equations.

Let a system of m linear equations in n unknowns be

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n = b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n = b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_m = b_m$$

This system in matrix form becomes :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i, e $Ax = b$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is called co-efficient matrix. The matrix obtained by adjoining the column vector b, at the end to the matrix A, is called the augmented matrix of the system A x = b and is denoted by (A,b)

The system is called non homogeneous of $b \neq 0$.

The system is called homogeneous if $b = 0$.

Theorem : 6.8.1

(a) (Existence) The system AX = b (non homogeneous) has a solution iff the matrix A and the augmented matrix (A,b) have the same rank.

(b) (Uniqueness) If the system AX= B has a solution, then the solution is unique iff $\rho(A) = n$.

Proof : We know that since A is an m×n matrix it can be consideard as a linear transformation from V_n to V_m. Thus Ax = b has a solution iff $b \in$ range of A. This is equivalent to saying that $b \in$ span of the column vectors of A. In other words, A and the augmented matrix (A,b) have the same rank. This proves (a).

Again if A x = b has a solution, then the solution is unique iff A x = 0 has the trivial solution x = 0 as its only solution. This hapens iff the kernel of A is $\{0\}$, i.e., iff the nullity of A is zero. This is equivalent to saying that rank of A is n. This proves (b).

Theorem : 6.8.2 Let $\rho(A) = r$. Then if r = m = n, them Ax = b (Non-homogeneous) has a unique solution and also Ax = 0 (homogeneous) has unique solution, x = 0, the trivial solution.

Proof : Since $\rho(A) = r$, by Theorem 6.8.1. (b) the system Ax = b has unique solution for all $b \in V_m$. Also the kernal of A being $\{0\}$, Ax = 0 has only one solution x = 0, the trivial one.

Theorem : 6.8.3 If r = m < n, (i.e. the number of equations is less than the number of unknowns of the system) the system Ax = b for all $b \in V_m$ have infinite number of solutions. In fact, r of the unknowns can be determined in terms of remaing (n-r) unknowns whose values can be arbitararily chosen.

Proof : Since r = m, the range of A is V_m. Thus every $b \in V_m$ has an A pre-image in V_n. The kernel of A has dimenson n – r > 0 by rank-nullity theorem. So the kernel K, being a subspace other than $\{0\}$, has an infinite number of vectors in it. The solution set of Ax = b being a translate of that of Ax = 0 (i.e.K), it has infinite number of vectors in it. From the row reduction process of A, r unknowns can be determined interms of the remaing (n-r) unknowns.

Note - (i) In cases $r < m = n$, $r < m < n$ and $r < n < m$ if the system $Ax = b$ has a solution, then there is infinite number of solutions.

(ii) In the case $r = n < m$, the system $Ax = 0$ has unique (trivial) solution and if $Ax = b$ has a solution, then that solution is unique.

(iii) If a system $Ax = b$ has no solution then the system is said to be inconsistent.

Thus the system is inconsistent iff $\rho(A, b) \neq \rho(A)$.

Example 6.8.1 : Test the consistency and solve.

$$x + y + z = 6$$
$$x + 2y - 3z = -4$$
$$-x - 4x + 9z = 18$$

Solution : Augmented matrix :

$$(A, b) = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & -3 & -4 \\ -1 & -4 & 9 & 18 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -4 & -10 \\ 0 & -3 & 10 & 24 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 + 3R_2 \\ R_1 \to R_1 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 16 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & -2 & -6 \end{bmatrix} \quad R_2 \to -\frac{1}{2} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 16 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 5R_3 \\ R_2 \to R_2 + 4R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \qquad ...(1)$$

This reduced matrix is in echelon form. It has 3 non-zero rows.

So $\rho(A, b) = \rho(A) = 3$

Hence the system is consistent.

The reduced matrix (1) gives the equivalent system.

$$\left.\begin{matrix} x_1 & & & = 1 \\ & x_2 & & = 2 \\ & & x_3 & = 3 \end{matrix}\right\} \qquad ...(2)$$

Here $m = n = r = 3$ By Theorem 6.8.2 the system has unique solution as given in (2)

Example 6.8.2. Examine the system

$$2x_1 + x_3 - x_4 + x_5 = 2$$
$$x_1 + x_3 - x_4 + x_5 = 1$$
$$12x_1 + 2x_2 + 8x_3 + 2x_5 = 12$$

for consistency.

Solution : The augmented matrix

$$(A, b) = \begin{bmatrix} 2 & 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 12 & 2 & 8 & 0 & 2 & 12 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 - 12R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & -4 & 12 & -10 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ R_3 \to \frac{1}{2} R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 6 & -5 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 2R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{bmatrix} \quad R_2 \to (-1) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}$$

This shows $\rho(A, b) = \rho(A) = 3$

So the system is consistent.

Here $r = m = 3 < n = 5$.

By Theorem 6.8.3, the system has infinite number of solutions.

$n - r = 5 - 3 = 2$ unknowns can be chosen arbitrarity.

From the row reduced echelon form of augmented matrix we have the equivalent system of the given system.

$$x_1 = 1$$
$$x_2 + 4x_4 - 3x_5 = 0$$
$$x_3 - x_4 + x_5 = 0.$$

$$\Rightarrow x_1 = 1$$
$$x_2 = -4x_4 + 3x_5$$
$$x_3 = x_4 - x_5$$

Thus choosing two unknows x_4 and x_5 arbitrarily we get infinite number of solutions.

The solution set
$\{(1, -4x_4 + 3x_5, x_4 - x_5, x_4\, x_5) \mid x_4, x_5$ are arbitary scalar$\}$
$= \{(1, 0, 0, 0, 0) + x_4 (0, -4, 1, 1, 0) + x_5 (0, 3, -1, 0, 1) \mid x_4, x_5$ are arbitary scalars$\}$
$= (1, 0, 0, 0, 0) + [(0, -4, 1, 1, 0), (0, 3, -1, 0, 1)]$
It is a linear variety.

Example : 6.8.3 : Examine the consistency of the following system.

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 4 \\ 2x_1 \quad - x_3 - 3x_4 &= 4 \\ x_1 - 2x_2 - x_3 &= 0 \\ 3x_1 + x_2 - x_3 - 5x_4 &= 5 \end{aligned}$$

Solution : The augmented matrix :-

$$(A, b) = \begin{bmatrix} 1 & 2 & 4 & 1 & 4 \\ 2 & 0 & -1 & -3 & 4 \\ 1 & -2 & -1 & 0 & 0 \\ 3 & 1 & -1 & -5 & 5 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 1 & 4 \\ 0 & -4 & -9 & -5 & -4 \\ 0 & -4 & -5 & -1 & -4 \\ 0 & -5 & -13 & -8 & -7 \end{bmatrix} \quad R_2 \to -\frac{1}{4} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 1 & 4 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & -4 & -5 & -1 & -4 \\ 0 & -5 & -13 & -8 & -7 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 4R_2 \\ R_4 \to R_4 + 5R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 2 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & -\frac{7}{4} & -\frac{7}{4} & -2 \end{bmatrix} \quad \begin{matrix} R_3 \to \frac{1}{4} R_3 \\ R_4 \to -\frac{4}{7} R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 2 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{8}{7} \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + \frac{1}{2} R_3 \\ R_2 \to R_2 - \frac{9}{4} R_3 \\ R_4 \to R_4 - R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{7} \end{bmatrix} \quad R_4 \to \frac{7}{8} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 2R_4 \\ R_2 \to R_2 - R_4 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots(1)$$

This matrix is in echelon form.

This shows $\rho(A, b) = 4$ where as $\rho(A) = 3$, $\rho(A, b) \neq \rho(A)$ Hence by Theorem 6.8.1 the system is inconsistent.

Alternatively, the last row of matrix (1) produces

$0 . x_1 + 0 . x_2 + 0 . x_3 + 0 . x_4 = 1$ which is impossible.

Hence the system is inconsistent.

Problem Set 6 (F)

1. Show that the following systems are inconsistent.

(a) $2x + 6y = -11$, $6x + 20y - 6z = -3$, $6y - 18z = -1$

(b) $x + 2y - z = 3$, $3x - y + 3z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$

(c) $x - 4y + 7z = 8$, $3x + 8y - 2z = 6$, $7x - 8y + 26z = 31$.

(d) $x_1 - x_3 = 1$, $2x_1 + x_2 + x_3 = 2$, $x_2 - x_3 = 3$, $x_1 + x_2 + x_3 = 4$, $2x_2$ $= 0$

2. Show that the following systems are consistent and write the natrue of solution.

(a) $x + y + z = 8$, $x - y + 2z = 6$, $3x + 5y - 7z = 14$.

(b) $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$.

(c) $x - y + 2z = 4$, $3x + y + 4z = 6$, $x + y + z = 1$

(d) $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = 1$

(d) $x + 2y - 3z = 4$, $2x + 3y - 6z = 8$, $3x + 5y - 9z = 12$.

3. Determine whether the following systems of linear equations are consistent. Discuss completely the solution in the case of consistent systems.

(a)
$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 1 \\ 2x_1 + 2x_2 \quad\quad + 2x_4 &= 1 \\ 4x_1 + x_2 - x_3 - x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 \quad\quad &= 1 \end{aligned}$$

(b) $x_1 + 2x_2 + 4x_3 + x_4 = 4$
$2x_1 - x_3 + 3x_4 = 4$
$x_1 - 2x_2 - x_3 = 0$
$3x_1 + x_2 - x_3 - 5x_4 = 7$

(c) $2x_1 + x_3 - x_4 + x_5 = 2$
$x_1 + x_3 - x_4 + x_5 = 1$
$12x_1 + 2x_2 + 8x_3 + 2x_5 = 12$

(d) $x_1 + 2x_2 - x_3 - 2x_4 = 0$
$2x_1 + 4x_2 + 2x_3 + 4x_4 = 4$
$3x_1 + 6x_2 + 3x_3 + 6x_4 = 6$

(e) $x_1 + 2x_3 = 1$
$2x_1 + x_2 + 2x_3 = 1$
$x_2 - 2x_3 = 1$
$x_1 + x_2 = 1$
$x_1 - x_2 + 4x_3 = 1$

(f) $2x_1 + x_2 + x_3 + x_4 = 2$
$3x_1 - x_2 + x_3 - x_4 = 2$
$x_1 + 2x_2 - x_3 + x_4 = 1$
$6x_1 + 2x_2 + x_3 + x_4 = 5$

(g) $x_1 + 3x_2 - 3x_3 + 2x_4 = 1$
$4x_1 + x_2 - 2x_3 + x_4 = 1$
$6x_1 + 5x_2 + 10x_3 + 3x_4 = 15$
$x_1 + 2x_2 + 3x_3 + x_4 = 6$

(h) $x_1 - 2x_2 - x_3 = -1$
$2x_1 - x_3 - 3x_4 = 1$
$3x_1 + x_2 - x_3 - 5x_4 = 1$
$2x_1 + 3x_3 + x_4 = 0$

(i) $3x_1 + 6x_2 + 3x_3 + 6x_4 = 5$
$x_1 + 2x_2 - x_3 - 2x_4 = -1$
$3x_1 + 6x_2 + x_3 + 2x_4 = 3$
$x_1 + 2x_2 + 2x_3 + 4x_4 = 3$

(j) $x_1 + x_2 - x_3 - 6x_4 + 6x_5 = -19$
$x_1 + 7x_4 - 7x_5 = 28$
$2x_2 - 3x_3 + 18x_4 - 4x_5 = 24$

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