CHAPTER 8

# EIGEN VALUES AND EIGEN VECTORS

## 8.1 Introduction

The eigen value problem involves the evaluation of all the eigen values and eigen vectors of a square matrix or a linear transformation. The solution of this problem has variety of application in pure and applied mathematics. In this chapter, we shall study eigen values and eigen vectors of a matrix and a linear transformation. Further we shall obtain a non-singular matrix which diagonalises a given diagonalisable matrix.

## 8.2 Eigen values and Eigen vectors of a matrix :

Let A be a square matrix of order n having elements

$$a_{ij}\ (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n).$$

A scalar $\lambda$ is called an **eigen value** of A if there exists a non zero (column) vector v which satisfies the relation.

$$Av = \lambda v \qquad \ldots(1)$$

Here v is called **eigen vector** of A corresponding to eigen value $\lambda$. The set of all eigen vectors belonging to $\lambda$ is called **eigen space** of $\lambda$.

Equation (1) can be rewritten as

$$(A - \lambda I)\, v = 0 \qquad \ldots(2)$$

This matrix equation represents n homogeneous linear equations :

$$\left.\begin{aligned} (a_{11} - \lambda)\, v_1 + a_{12}\, v_2 + \ldots + a_{1n}\, v_n &= 0 \\ a_{21}\, v_1 + (a_{22} - \lambda)\, v_2 + \ldots + a_{2n}\, v_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1} v_1 + a_{n2} v_2 + \ldots + (a_{nn} - \lambda)\, v_n &= 0 \end{aligned}\right\} \quad \ldots(3)$$

Equation (3) will have a non-trivial solution if and only if det $(A - \lambda I) = 0$ i.e., $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn}-\lambda \end{vmatrix} = 0 \quad ...(4)$$

Equation (4) is called characteristic equation of matrix A. The matrix $A-\lambda I$ is called characteristic matrix of the given matrix A and $\det(A-\lambda I)$ is called characteristic polynomial of matrix A, which is a polynomial of degree n in $\lambda$. Since equation (4) has n roots $\lambda_1, \lambda_2, ...., \lambda_n$, we have n characteristic roots or eigen values.

**Note :** (1) Corresponding to n distinct eigen values, we have n independent eigen vectors. It is not possible to have a linearly independent eigen vectors corresponding to repeated eigen values i.e., when two or more eigen values are equal.

(2) eigen vector corresponding to a eigen value is not nuique.

**Example 8.2.1 :** Find the eigen values and eigen vectors of the following matrices :

(a) $\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

**Solution :** (a) Let $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

The eigen values $\lambda$ are the roots of $\det(A-\lambda I)=0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)-12=0$$

$$\Rightarrow \lambda^2-3\lambda-10=0$$

$$\Rightarrow (\lambda-5)(\lambda+2)=0$$

$$\Rightarrow \lambda=5,\ -2$$

So, the eigen values of A are $\lambda_1=5$ and $\lambda_2=-2$.

Now to obtain eigen vectors of A.

The eigen vector of A w.r.t. $\lambda_1=5$ are the non-trivial solutions of

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \left.\begin{matrix} x_1+4x_2=5x_1 \\ 3x_1+2x_2=5x_2 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1-x_2=0 \\ x_1-x_2=0 \end{matrix}$$

$$\therefore \frac{x_1}{1}=\frac{x_2}{1}=k \qquad \therefore x_1=k,\ x_2=k, \text{ where k is arbitrary}$$

For k = 1,

the corresponding eigen vector = $[(1,\ 1)]^T$

Again the eigen vector of A w.r.t $\lambda_2 = -2$ are the non-trivial solutions of

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} x_1 + 4x_2 = -2x_1 \\ 3x_1 + 2x_2 = -2x_2 \end{matrix}\right\} \quad \Rightarrow \begin{matrix} 3x_1 + 4x_2 = 0 \\ 3x_1 + 4x_2 = 0 \end{matrix}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{-3} = k$$

$$\Rightarrow x_1 = 4k, \; x_2 = -3k$$

For $k = 1$, corresponding eigen vector $= [(4, \; -3)]^T$

(b) Let $A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

The characteristic equation is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & 0-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\{-\lambda(3-\lambda) - 4\} - 2\{2(3-\lambda) - 8\} + 4\{4 - 4(-\lambda)\} = 0$$

$$\Rightarrow (3-\lambda)(-3\lambda + \lambda^2 - 4) - 2(-2 - 2\lambda) + 4(4 + 4\lambda) = 0$$

$$\Rightarrow (3-\lambda)(\lambda^2 - 4\lambda + \lambda - 4) + 4(\lambda + 1) + 16(\lambda + 1) = 0$$

$$\Rightarrow (3-\lambda).(\lambda + 1)(\lambda - 4) + 4(\lambda + 1) + 16(\lambda + 1) = 0$$

$$\Rightarrow (\lambda + 1)\{3\lambda - \lambda^2 - 12 + 4\lambda + 4 + 16\} = 0$$

$$\Rightarrow (\lambda + 1)(-\lambda^2 + 7\lambda + 8) = 0$$

$$\Rightarrow (\lambda^2 - 7\lambda - 8)(\lambda + 1) = 0$$

$$\Rightarrow (\lambda + 1)^2(\lambda - 8) = 0$$

$$\Rightarrow \lambda = -1, \; -1, \; 8$$

Hence the eigen values are $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 8$

Now to find the eigen vectors corresponding to the eigen value $\lambda_1 = -1$, we write

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (-1)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 4x_1 + 2x_2 + 4x_3 = 0 \quad \ldots (1)$$
$$2x_1 + x_2 + 2x_3 = 0 \quad \ldots (2)$$
$$4x_1 + 2x_2 + 4x_3 = 0 \quad \ldots (3)$$

Putting $x_1 = 0, \ -x_2 = 2x_3$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{-2} = \frac{x_3}{1} = k$$

$$\Rightarrow x_1 = 0, \ x_2 = -2k, \ x_3 = k$$

For k = 1, the eigen vector corresponding to $\lambda_1 = -1$ is

$$[(0, -2, 1)]^T$$

Putting $x_3 = 0, \ -x_2 = 2x_1$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-2} = \frac{x_3}{0} = k$$

$$\Rightarrow x_1 = k, \ x_2 = -2k, \ x_3 = 0$$

For k = 1, the eigen vector corresponding to $\lambda_2 = -1$ is

$$[(1, -2, 0)]^T$$

Again for $\lambda_3 = 8$, we have

$$\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 8 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 + 2x_2 + 4x_3 \\ 2x_1 + 0 + 2x_3 \\ 4x_1 + 2x_2 + 3x_3 \end{bmatrix} - \begin{bmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -5x_1 + 2x_2 + 4x_3 = 0 \quad \ldots (4)$$
$$2x_1 - 8x_2 + 2x_3 = 0 \quad \ldots (5)$$
$$4x_1 + 2x_2 - 5x_3 = 0 \quad \ldots (6)$$

From (4) and (5), we have

$$\frac{x_1}{4+32} = \frac{x_2}{8+10} = \frac{x_3}{40-4}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{2} = k$$

$\Rightarrow \ x_1 = 2k, \ x_2 = k, x_3 = 2k$, which satisfies equation (6).

For k = 1, the eigen vector corresponding to $\lambda_3 = 8$ is $[(2, 1, 2)]^T$

## 8.3 Normal Vector

### 8.3.1 Definition :

A vector $X = [x_1, \ x_2, \ldots, x_n]^T$ is called a normal vector (normalised form) if

$$\sqrt{x_1^2 + x_2^2 + \ldots.. + x_n^2} = 1$$

**Example 8.3.1** Find the eigen values and eigen vectors of the following matrix :

(a) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

Write the eigen vector is normalised form.

**Solution :**

(a) Given $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda^2 - 9 - 16 = 0$$

$$\Rightarrow \lambda^2 = 25$$

$$\Rightarrow \lambda = \pm 5$$

Hence the eigen values are 5 and – 5.

Corresponding to $\lambda_1 = 5$, the eigen vector is given by

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 + 4x_2 - 5x_1 \\ 4x_1 - 3x_2 - 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -2x_1 + 4x_2 = 0 \qquad .... (1)$$

$$\text{and } 4x_1 - 8x_2 = 0 \qquad .... (2)$$

From (1), $\dfrac{x_1}{2} = \dfrac{x_2}{1} = k$

$$\Rightarrow x_2 = k, \; x_1 = 2k$$

$\therefore$ $x_1$ and $x_2$ satisfies equation (2).

By definition of normal vector,

$$x_1^2 + x_2^2 = 4k^2 + k^2 = 5k^2 = 1$$

$$\Rightarrow k = \frac{1}{\sqrt{5}} \qquad \therefore x_1 = \frac{2}{\sqrt{5}}, x_2 = \frac{1}{\sqrt{5}}$$

corresponding eigen vector for $\lambda = 5$ is $\left[\left(\dfrac{2}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}\right)\right]^T$, which is in normalised form.

Corresponding to $\lambda_2 = -5$, the eigen vector is given by

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (-5) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_1 + 4x_2 + 5x_1 \\ 4x_1 - 3x_2 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 8x_1 + 4x_2 = 0$$

$$4x_1 + 2x_2 = 0$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{2} = k$$

$$\Rightarrow x_1 = -k,\ x_2 = 2k$$

Now $x_1^2 + x_2^2 = 1 \Rightarrow 5k^2 = 1 \Rightarrow k = \frac{1}{\sqrt{5}}$

$\therefore$ Corresponding eigen vector for $\lambda_2 = -5$ is $\left[\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right]^T$, which is in normalised form.

(b) Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ -1 & -1-\lambda & 0 & 0 \\ -2 & -2 & 2-\lambda & 1 \\ 1 & 1 & -1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} -1-\lambda & 0 & 0 \\ -2 & 2-\lambda & 1 \\ 1 & -1 & -\lambda \end{vmatrix}$$

$$-1\begin{vmatrix} -1 & 0 & 0 \\ -2 & 2-\lambda & 1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0 \qquad \text{[Expanding by } R_1]$$

$$\Rightarrow (1-\lambda).[(-1-\lambda).\{(2-\lambda)(-\lambda)+1\}] - 1[(-1).\{(2-\lambda)(-\lambda)+1\}] = 0$$

$$\Rightarrow (1-\lambda)(2\lambda + 2\lambda^2 - \lambda^2 - \lambda^3 - 1 - \lambda) + \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow (1-\lambda)(-\lambda^3 + \lambda^2 + \lambda - 1) + (\lambda - 1)^2 = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2-1)(\lambda-1) + (\lambda-1)^2 = 0$$

$$\Rightarrow (\lambda-1)^2(\lambda^2 - 1 + 1) = 0$$

$$\Rightarrow \lambda^2(\lambda-1)^2 = 0$$

$\therefore$ The eigen values are $\lambda_1 = 0$ and $\lambda_2 = 1$

For $\lambda_1 = 0$, the eigen vectors are given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 + x_2 &= 0 & \dots (1) \\ -x_1 - x_2 &= 0 & \dots (2) \\ -2x_1 - 2x_2 + 2x_3 + x_4 &= 0 & \dots (3) \\ x_1 + x_2 - x_3 &= 0 & \dots (4) \end{aligned}$$

From (1) and (4), we have $x_3 = 0$

Putting $x_3 = 0$ in equation (3), we have $x_4 = 0$

$\therefore\ x_1 = -x_2$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{0} = \frac{x_4}{0}$$

Corresponding eigen vector for $\lambda_1 = 0$ is $[(-1, 1, 0, 0)]^T$

For $\lambda_2 = 1$, the eigen vectors are given by

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x_1 + x_2 &= x_1 \\ -x_1 - x_2 &= x_2 \\ -2x_1 - 2x_2 + 2x_3 + x_4 &= x_3 \\ x_1 + x_2 - x_3 &= x_4 \end{aligned}$$

$$\Rightarrow \begin{aligned} x_2 &= 0 \\ -x_1 - 2x_2 &= 0 \\ -2x_1 - 2x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 - x_4 &= 0 \end{aligned}$$

$$\Rightarrow x_2 = 0,\ x_1 = 0, x_4 = -x_3$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{1} = \frac{x_4}{-1} = k$$

$x_1 = 0, \quad x_2 = 0, \quad x_3 = k, \quad x_4 = -k$

$\therefore$ By definition of normal vector,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

$$\Rightarrow k^2 + k^2 = 1$$

$$\Rightarrow 2k^2 = 1$$

$$\Rightarrow k = \frac{1}{\sqrt{2}}$$

$\therefore$ Corresponding eigen vector for $\lambda_2 = 1$ is $\left[\left(0, 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right]^T$, which is in normalised form.

## 8.4 Similarity of Matrices

### 8.4.1 Definition :

Basing on Theorem 5.10.3, we define similarity of matrices.

Let A and B be square matrices of order n. Then A and B are said to be similar if there exists a non-singular matrix C of order n such that AC = CB

or $A = CBC^{-1}$ or $B = C^{-1}AC$

**Theorem 8.4.1 :** The relation of similarity is an equivalence relation in the set of all n × n matrices.

**Proof** Let A and B be two n × n matrices. By definition of similarity of matrices there exists a non-singular matrix C of order n such that

$$B = C^{-1}AC$$

**Reflexive :** Since $A = I^{-1}AI$, where I is n × n unit matrix

$\therefore$ A is similar to A, because I is non-singular.

**Symmetric :** Suppose A is similar to B.

Then there exists an n × n non-singular matrix P such that

$$AP = PB$$

$$\Rightarrow A = PBP^{-1}$$

$$\Rightarrow P^{-1}AP = P^{-1}(PBP^{-1})P$$

$$\Rightarrow P^{-1}AP = B$$

$$\Rightarrow B = P^{-1}AP$$

$\Rightarrow$ B is similar to A.

**Transitive** Let A be similar to B and B be similar to C.

Then there exists non-singular matrices P and Q, such that

$$A = PBP^{-1}$$

$$\text{and } B = QCQ^{-1}$$

$$\Rightarrow A = P(QCQ^{-1})P^{-1}$$

$$= (PQ)C(Q^{-1}P^{-1})$$

$$= (PQ)C(PQ)^{-1} \quad \text{[By reversal rule]}$$

$\therefore$ A is similar to C.

Hence similarity is an equivalence relation in the set of all n × n matrices.

**Theorem 8.4.2** Similar matrices have the same determinant.

**Proof** Let A and B be two similar matrices. Then there exists a non-singular matrix P such that

$$A = PBP^{-1}$$

$$\Rightarrow \det A = \det(PBP^{-1})$$

$$\Rightarrow \det A = (\det P)(\det B)(\det P^{-1})$$

$$\Rightarrow \det A = \det(P^{-1}P)\det B$$

$$\Rightarrow \det A = (\det I)\det B$$

$$\Rightarrow \det A = \det B \quad [\because \det I = 1]$$

$\therefore$ A and B have same determinant.

**Example 8.4.1** Show that the matrix $\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$ is similar to $\begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}$ for the non singular matrix

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

**Solution :** Let $B=\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$,

$$C=\begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}, \quad P=\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\therefore \ BP=\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{20}{\sqrt{5}}+\frac{2}{\sqrt{5}} & \frac{10}{\sqrt{5}}-\frac{4}{\sqrt{5}} \\ \frac{4}{\sqrt{5}}+\frac{7}{\sqrt{5}} & \frac{2}{\sqrt{5}}-\frac{14}{\sqrt{5}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{22}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} & \frac{-12}{\sqrt{5}} \end{bmatrix}$$

$$PC=\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}=\begin{bmatrix} \frac{22}{\sqrt{5}}+0 & 0+\frac{6}{\sqrt{5}} \\ \frac{11}{\sqrt{5}}+0 & 0-\frac{12}{\sqrt{5}} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{22}{\sqrt{5}} & \frac{6}{\sqrt{5}} \\ \frac{11}{\sqrt{5}} & -\frac{12}{\sqrt{5}} \end{bmatrix}$$

$\therefore$ BP = PC i.e., $B = PCP^{-1}$

$\Rightarrow$ B and C are similar matrices.

**Theorem 8.4.3** Similar matrices have the same eigen values.

**Proof** Let an $n \times n$ matrix A be similar to an $n \times n$ matrix B. Hence, there exists an invertible matrix P such that $B = P^{-1}AP$

Let I be a unit matrix of oder n.

$$\therefore\ B - \lambda I = P^{-1}AP - \lambda I$$
$$= P^{-1}AP - P^{-1}(\lambda I)P$$
$$= P^{-1}(A - \lambda I)P$$
$$\det(B - \lambda I) = (\det P^{-1})\ \det(A - \lambda I)(\det P) \qquad \left[\because \det(AB) = (\det A)(\det B)\right]$$
$$= \det(P^{-1}P)\det(A - \lambda I)$$
$$= (\det I)\det(A - \lambda I)$$
$$= \det(A - \lambda I) \qquad [\because \det I = 1]$$

Thus the matrices A and B have same characteristic polynomial and so they will have same eigen values.

**Theorem 8.4.4 :** Corresponding to an eigen vector (non-zero) of a matrix, there exists unique eigen value.

**Proof :** Let $\lambda_1$ and $\lambda_2$ be two eigen values of the matrix A corresponding to an non-zero eigen vector v.

Then $Av = \lambda_1 v,\ Av = \lambda_2 v,\ \lambda_1 \neq \lambda_2$

$\Rightarrow Av = \lambda_1 v = \lambda_2 v$

$\Rightarrow (\lambda_1 - \lambda_2)v = 0$

$\Rightarrow v = 0 \quad [\because \lambda_1 - \lambda_2 \neq 0]$

This is a contradiction since v is a non-zero vector.

Hence, corresponding to an eigen vector $v\ (v \neq 0)$ there is only one eigen value of A.

**Note :** The converse of the above theorem is not true.

Corresponding to an eigen value $\lambda$ of a matrix A, there are different eigen vectors.

We have, if v is the eigen vector of A corresponding to eigen value $\lambda$, then $Av = \lambda v$

$$\Rightarrow A(kv) = \lambda(kv),\ k \text{ is a non zero scalar.}$$

This shows that kv is also an eigen vector of A corresponding to the same eigen value $\lambda$. In Example 8.2.1 (b), for $\lambda = -1$, there are two eigen vectors

$$\text{i.e., } [0, -2, 1]^T \text{ and } [1, -2, 0]^T$$

**Theorem 8.4.5 :** The scalar $\lambda$ is an eigen value of the matrix A iff $(A - \lambda I)$ is singular.

**Proof :** Let $\lambda$ be the eigen value of A.

By definition,

$$\det(A - \lambda I) = 0$$

$\Rightarrow (A - \lambda I)$ is singlular.

Conversely, If $\det(A - \lambda I) = 0$, then there exists a non-zero vector v such that $(A - \lambda I)\, v = 0$

$$\Rightarrow Av = \lambda v$$

$$\Rightarrow \lambda \text{ is the eigen value of A.}$$

**Note :** $\lambda = 0$ is the eigen value of the matrix A iff A is singular.

**Theorem 8.4.6 :** Let Let $\lambda_1, \lambda_2, \lambda_3 ..., \lambda_m$ be the distinct eigen values of a square matrix A of order n and $v_1, v_2, ..., v_m$ be the eigen vectors of A corresponding to $\lambda_1, \lambda_2, ....., \lambda_m$ respectively. Then the set $\{v_1, v_2, ..., v_m\}$ is linearly independent.

**Proof :** By definition of eigen vector, $Av_i = \lambda_i v_i, v_i \neq 0$

for $i = 1, 2, ..., m$

and $\lambda_i \neq \lambda_j$ for $i \neq j$

Suppose, if possible, the set $\{v_1, v_2, ...., v_m\}$ is linearly dependent.

The non-zero vector $v_1$ is linearly independent. We choose $r (\leq m)$ such that the set $\{v_1, v_2, ...., v_{r-1}\}$ is linearly independent and $\{v_1, v_2, ...., v_{r-1}, v_r\}$ is linearly dependent. Then we can express $v_r$ as a linear combination of $v_1, v_2, ...., v_{r-1}$

$$\therefore\ v_r = \alpha_1 v_1 + \alpha_2 v_2 + .... + \alpha_{r-1} v_{r-1} \qquad ....(1)$$

for the scalars $\alpha_i, i = 1, 2, ...., (r-1)$

$$\Rightarrow Av_r = \alpha_1 Av_1 + \alpha_2 Av_2 + ........ + \alpha_{r-1} Av_{r-1}$$

$$\Rightarrow \lambda_r v_r = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + .... + \alpha_{r-1} \lambda_{r-1} v_{r-1} \qquad ... (2)$$

Multiplying equation (1) by $\lambda_r$ and subtracting from (2), we get

$$0 = \alpha_1 (\lambda_1 - \lambda_r) v_1 + \alpha_2 (\lambda_2 - \lambda_r) v_2 + .... + \alpha_{r-1} (\lambda_{r-1} - \lambda_r) v_{r-1}$$

Since the set $\{v_1, v_2, .... v_{r-1}\}$ is L.I, therefore each of the coefficients in the above equation is zero i.e.,

$$\alpha_i (\lambda_i - \lambda_r) = 0, \quad \text{for } i = 1, 2, ...., (r-1).$$

But $\lambda_i - \lambda_r \neq 0$, [$\because$ eigen values are distinct]

for $i = 1, 2, ...., (r - 1)$

Therefore $\alpha_i = 0$ for $i = 1, 2, ...., (r-1)$

Putting $\alpha_i = 0$ for $i = 1, 2, ...., (r-1)$

in equation (1), we have

$v_r = 0$. This is a contradiction, since the eigen vector $v_r$ can never be zero.

Hence our assumption is wrong.

This proves that the set $\{v_1, v_2, ...., v_n\}$ must be L.I.

## 8.5 Diagonalisation

### 8.5.1 Definition :

An $n \times n$ matrix A is said to be diagonalisable if A is similar to a diagonal matrix i.e., $P^{-1}AP$ is diagonal for some non-singular $n \times n$ matrix P.

**Note :** Given an $n \times n$ matrix A, it is diagonisable if it has n distinct eigen values.

Now, we will prove some theorems which establishes the method of diagonalising a matrix.

**Theorem 8.5.1 :** Let A be an $n \times n$ matrix having n distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$. Let $v_1, v_2, \ldots v_n$ be the eigen vectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively. Let $P = (v_1, v_2, \ldots, v_n)$ be the $n \times n$ matrix having $v_1, v_2, \ldots, v_n$ as its column vectors. Then $P^{-1}AP = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

**Proof :** We have, by actual multiplication,

$$\begin{aligned} AP &= A(v_1, v_2, \ldots, v_n) \\ &= (Av_1, Av_2, \ldots, Av_n) \\ &= (\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n), \end{aligned}$$

since $Av_i = \lambda_i v_i$

for $i = 1, 2, \ldots, n$.

$$= (v_1, v_2, \ldots, v_n) \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots\ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots\ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix} = PD \text{ where}$$

$D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

By Theorem 8.4.6 the column vectors of P are L.I. Therefore P is invertible.

$$\begin{aligned} \therefore\ P^{-1}AP &= P^{-1}PD = D \\ &= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \end{aligned}$$

## 8.6. Power of a matrix

### 8.6.1 Definition :

Let A be an $n \times n$ matrix.

Consider $D = P^{-1}AP$ for some non-singular matrix P.

$\therefore\ A = PDP^{-1}$

$$\begin{aligned} \Rightarrow A^k &= (PDP^{-1})\ (PDP^{-1})\ \ldots\ldots k \text{ times} \\ &= PD(P^{-1}P)\ D(P^{-1}P) \ldots D(P^{-1}P)\ DP^{-1} \\ &= PD^k P^{-1} \end{aligned}$$

$$= P \begin{bmatrix} \lambda_1^k & 0 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2^k & 0 & \ldots\ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 & 0 & \ldots\ldots & \lambda_n^k \end{bmatrix} P^{-1}$$

**Example 8.6.1** Diagonalise the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix} \text{ and calculate } A^2.$$

**Solution :**

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}$$

Characteristic Polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & -6 \\ 2 & -2 & 3-\lambda \end{vmatrix}$$

Characteristic equation is

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & -6 \\ 2 & -2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(1-\lambda)(3-\lambda)-12\}-2\{2(3-\lambda)+12\}=0$$
$$\Rightarrow (1-\lambda)\{3-3\lambda-\lambda+\lambda^2-12\}-2\{18-2\lambda\}=0$$
$$\Rightarrow (1-\lambda)(\lambda^2-4\lambda-9)-4(9-\lambda)=0$$
$$\Rightarrow \lambda^2-4\lambda-9-\lambda^3+4\lambda^2+9\lambda-36+4\lambda=0$$
$$\Rightarrow -\lambda^3+5\lambda^2+9\lambda-45=0$$
$$\Rightarrow \lambda^3-5\lambda^2-9\lambda+45=0$$
$$\Rightarrow (\lambda^2-9)(\lambda-5)=0$$
$$\Rightarrow (\lambda+3)(\lambda-3)(\lambda-5)=0$$

$\therefore$ The eigen values of A are

$\lambda_1 = 5,\ \lambda_2 = 3,\ \lambda_3 = -3.$

These eigen values are distinct. Therefore A is diagonisable.

**Eigen vector for $\lambda_1 = 5$**

It is given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 + 2x_2 + 0 - 5x_1 \\ 2x_1 + x_2 - 6x_3 - 5x_2 \\ 2x_1 - 2x_2 + 3x_3 - 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -4x_1 + 2x_2 \quad &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ 2x_1 - 2x_2 - 2x_3 &= 0 \end{aligned}$$

From the last two equations,

$$\frac{x_1}{8-12} = \frac{x_2}{-12+4} = \frac{x_3}{-4+8}$$

$$\Rightarrow \frac{x_1}{-4} = \frac{x_2}{-8} = \frac{x_3}{4}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1} = k$$

$\therefore\ v_1 = [k,\ 2k,\ -k]^T = [1,\ 2, -1]^T$, if $k = 1$

Similarly eigen vector for $\lambda_2 = 3$ is $v_2 = [1, 1, 0]^T$ and eigen vector for $\lambda_3 = -3$ is $v_3 = [-1,\ 2,\ 1]^T$

$$\therefore\ P = [v_1, v_2, v_3].$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}, \text{ which diagonalises A.}$$

$$\therefore\ D = P^{-1}AP = -\frac{1}{4}\begin{bmatrix} 1 & -1 & 3 \\ -4 & 0 & -4 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -6 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} 5 & -5 & 15 \\ -12 & 0 & -12 \\ -3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= -\frac{1}{4}\begin{bmatrix} -20 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \text{ which is in diagonal form.}$$

$$\therefore\ A^2 = PD^2P^{-1}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & (-3)^2 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 9 & -9 \\ 50 & 9 & 18 \\ -25 & 0 & 9 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{3}{4} \\ 1 & 0 & 1 \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & -12 \\ -8 & 17 & -24 \\ 4 & -4 & 21 \end{bmatrix}$$

**Example 8.6.2 :** Show that the eigen values of a traingular matrix are the diagonal elements of the matrix.

**Solution :** Let $A = [a_{ij}]_{n\times n}$ be a upper triangular matrix, then $a_{ij} = 0,\ i > j$

$$\therefore\ A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

The characteristic equation of A is given by

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} a_{11} - \lambda & 0 & \dots & 0 \\ 0 & a_{22} - \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)\dots(a_{nn} - \lambda) = 0$$

$$\Rightarrow \lambda = a_{11},\ a_{22},\dots,\ a_{nn}$$

Similarly, we can prove that for lower triangular matrix, the eigen values are the diagonal elements of the matrix.

**Example 8.6.3** Show that the sum of the eigen values of a matrix of order 3 is the sum of the elements of the principal diagonal.

**Solution :** Consider a square matrix of order 3.

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\therefore\ \det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + \lambda^2 (a_{11} + a_{22} + a_{33}) - \dots \text{[on expanding]} \qquad \dots (1)$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A

$\therefore$ det $(A - \lambda I)$

$$= (-1)^3 (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

$$= -\lambda^3 + \lambda^2 (\lambda_1 + \lambda_2 + \lambda_3) - \ldots \qquad \ldots. (2)$$

Equating the right hand sides of (1) and (2) and comparing coefficients of $\lambda^2$ we get,

$\lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$, which is the sum of the elements of the principal diagonal.

**Example 8.6.4** If $\lambda$ is an eigen value of a non-singular matrix A then show that $\frac{1}{\lambda}$ is the eigen value of $A^{-1}$.

**Solution : First Method :**

Let v be the eigen vector corresponding to eigen value $\lambda$.

By definition, $Av = \lambda v$. Premultiplying both sides by $A^{-1}$, we get

$$A^{-1} Av = A^{-1} \lambda v$$

$$\Rightarrow Iv = \lambda (A^{-1}v)$$

$$\Rightarrow v = \lambda (A^{-1}v)$$

$$\Rightarrow \lambda^{-1}v = A^{-1}v$$

$$\Rightarrow \lambda^{-1} \text{ i.e., } \frac{1}{\lambda} \text{ is the eigen value of } A^{-1}.$$

**Second Method :**

$\lambda$ is an eigen value of the non-singular matrix A.

$$\therefore \det (A - \lambda I) = 0$$

$$\Rightarrow \det (A - \lambda AA^{-1}) = 0$$

$$\Rightarrow \det\left(\lambda A\left(\frac{1}{\lambda} I - A^{-1}\right)\right) = 0$$

$$\Rightarrow (\det \lambda A) . \det\left(\frac{1}{\lambda} I - A^{-1}\right) = 0$$

$$\Rightarrow \det\left(\frac{1}{\lambda} I - A^{-1}\right) = 0 \qquad [\because \det (\lambda A) \neq 0]$$

$$\Rightarrow \det\left(A^{-1} - \frac{1}{\lambda} I\right) = 0$$

$\therefore \frac{1}{\lambda}$ is the eigen value of $A^{-1}$.

**Example 8.6.5** If v is an eigen vector of a non-singular matrix A corresponding to the eigen value $\lambda$, then prove that v is an eigen vector of $A^{-1}$ corresponding to the eigen value $\frac{1}{\lambda}$.

**Solution :**

If v is an eigen vector of A corresponding to the eigen value $\lambda$, then $Av = \lambda v$

$$\Rightarrow \frac{1}{\lambda} A^{-1}(Av) = \frac{1}{\lambda} A^{-1}(\lambda v)$$

$$\Rightarrow \frac{1}{\lambda} Iv = \frac{1}{\lambda} \lambda A^{-1} v$$

$$\Rightarrow \frac{1}{\lambda} v = A^{-1} v$$

$$\text{i.e., } A^{-1} v = \frac{1}{\lambda} v$$

$\therefore$ v is an eigen vector of $A^{-1}$ corresponding to $\frac{1}{\lambda}$.

# Problem Set 8 (A)

1. Determine the eigen values and eigen vectors for the following matrices.

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$ (g) $\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$

2. Show that eigen values of a square matrix and its transpose are same.
3. Show that all the eigen values of identity matrix are 1 and every non-zero vector is the eigen vector of identity matrix.
4. Let A be a square matrix of order n. If $\lambda$ is the eigen value of A, then prove that $p(\lambda)$ is the eigen value of p (A), for any polynomila p.
5. Show that
   (i) If $\lambda$ is an eigen value of a square matrix A, then $\lambda^n$ is an eigen value of $A^n$, for some positive integer n.
   (ii) If $\lambda$ is an eigen value of a square matrix A, then $k + \lambda$ is an eigen value of A + kI, where k is any scalar.
   (iii) If $\lambda$ is an eigen value of A, then $k\lambda$ is an eigen value of kA, where k is any scalar.
   (iv) If $\lambda$ is an eigen value of a non-singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adj. A.

6. If v is an eigen vector of A corresponding to the eigen value $\lambda$, prove that
(i) v is an eigen vector of $A^2$ corresponding to eigen value $\lambda^2$.
(ii) v is an eigen vector of p (A) corresponding to the eigen value p ($\lambda$), for any polynomial p.

7. If $A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 8 \end{bmatrix}$, then find the eigen values of $A^2$, $A^{-1}$, $2A$.

8. Show that the following matrices are diagonalisable :

(a) $\begin{bmatrix} 20 & 18 \\ -27 & -25 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$

9. Diagonalise the following matrices :

(a) $\begin{bmatrix} 3 & 1 & 4 \\ 2 & 2 & 4 \\ 4 & 1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

10. Show that the matrix $\begin{bmatrix} \frac{29}{11} & \frac{-6}{11} \\ \frac{-12}{11} & \frac{15}{11} \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ for the non-singular matrix $\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

11. Show that the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ for the non-singular matrix $\begin{bmatrix} \frac{-1}{4} & \frac{3}{4} & -1 \\ \frac{3}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix}$

12. Find the matrix P which diagonalises the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Compute $A^4$.

13. Find eigen values and eigen vectors of the differential operator $D : P_2 \to P_2$ defined by $D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2\, x,\ a_0, a_1, a_2 \in R$

[Hints : $[D]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \{1, x, x^2\}$ is a basis for $P_2$.]

14. State whether the following statements are true (T) or false (F).
(i) The product of all eigen values of a square matrix A is equal to det A.
(ii) At least one eigen value of a non-singular matrix is zero.
(iii) The eigen values of a singular matrix are all non-zero.
(iv) If zero is the eigen value of A, then A is singular.
(v) If v is an eigen vector of A for eigen value $\lambda$, then v is an eigen vector of $A^T$ for eigen value $\lambda$.
(vi) If A and B are square matrices, then AB and BA have same eigen values.
(vii) All square matrices are diagonalisable.
(viii) The matrices A and $P^{-1}AP$ have different eigen values for any non-singular matrix P.
(ix) The square matrices A and $A^T$ have same eigen vector.
(x) If $\lambda$ is an eigen value of an idempotent matrix A then $\lambda$ must be either 0 or 1.

## 8.7 Eigen value and Eigen vector of a linear map

### 8.7.1 Definition :

An eigen value of a linear transformation $T: V \to V$ is a scalar $\lambda \in R$ such that there exists a non-zero vector $v \in V$ satisfying the equation $T(v) = \lambda v$.

The non-zero vector $v \in V$ is called an eigen vector of T w.r.t the eigen value $\lambda$.

**Example 8.7.1 :** Find the eigen value and corresponding eigen vector for the linear map $T: V_3 \to V_3$ defined by $T(x, y, z) = (3x, 3y, 3z)$

**Solution :** We have, if $v = (x, y, z) \in V_3$,

then $T(v) = T(x, y, z) = (3x, 3y, 3z)$
$$= 3\,(x, y, z)$$
$$= 3v = \lambda v$$
$$\Rightarrow \lambda = 3 \text{ is an eigen value of T.}$$

Any non-zero element of $V_3$ is the eigen vector of T, corresponding to $\lambda = 3$.

**Example 8.7.2** Find the eigen value and the eigen vector of $T: C \to C$ defined by
$$T(x, y, z) = (ix, -iy, z).$$

**Solution :** Let $\lambda \in C$ is the eigen value of T. By definition, $T(v) = \lambda v$, where $v = (x, y, z) \neq (0, 0, 0)$

$$\Rightarrow (ix, -iy, z) = \lambda\,(x, y, z)$$
$$\Rightarrow (ix, -iy, z) = (\lambda x, \lambda y, \lambda z)$$
$$\Rightarrow \lambda x = ix,\ \lambda y = -iy,\ \lambda z = z$$
$$\Rightarrow \lambda = i,\quad y = 0,\quad z = 0$$
$$\text{or}\quad \lambda = -i,\ x = 0,\quad z = 0$$
$$\text{or}\quad \lambda = 1,\ x = 0,\quad y = 0$$

For $\lambda = i$, corrseponding eigen vector is (1, 0, 0) or (x, 0, 0) where x = 0

For $\lambda = -i$, corresponding eigen vector is (0, y, 0) where $y \neq 0$.

For $\lambda = 1$, corresponding eigen vector is (0, 0, z) where $z \neq 0$.

**Note** (1) The zero vector can never be an eigen vector but zero can be an eigen value.

(2) The set of all eigen vectors of T is denoted by $I_\lambda$.

**Theorem 8.7.1 :** Let $\lambda$ be the eigen value of a linear map $T: V \to V$. Then $I_\lambda$, the set of all eigen vectors of T, corresponding to eigen value $\lambda$, is a subspace of V.

**Proof :** Let $u, v \in I_\lambda$.

By definition,

$$T(u) = \lambda u \text{ and } T(v) = \lambda v \qquad ...(1)$$

$$\therefore \quad T(u+v) = T(u) + T(v) \qquad [\because T \text{ is linear}]$$

$$= \lambda u + \lambda v \qquad [\text{By (1)}]$$

$$= \lambda(u+v)$$

$\Rightarrow$ $u + v$ is the eigen vector corresponding to eigen value $\lambda$.

$$\therefore \quad u + v \in I_\lambda$$

Again, if $\alpha$ be a scalar then

$$T(\alpha v) = \alpha T(v) \qquad [\because T \text{ is linear}]$$

$$= \alpha(\lambda v) \qquad [\text{by (1)}]$$

$$= \lambda(\alpha v)$$

$\Rightarrow$ $\alpha v$ is the eigen vector corresponding to eigen value $\lambda$

$$\therefore \quad \alpha v \in I_\lambda$$

Hence,

$$u, v \in I_\lambda \Rightarrow u + v \in I_\lambda$$

and $\quad v \in I_\lambda$ and $\alpha$ be a scalar

$$\Rightarrow \alpha v \in I_\lambda$$

$\therefore$ $I_\lambda$ is a subspace of V.

**Theorem 8.7.2** Let $T: V \to V$ be a linear map on a vector space V. Then $\lambda$ is an eigen value of T iff the map $T - \lambda I$ is singular. (I is the matrix of identity operator)

**Proof :** Let $\lambda$ be the eigen value of T.

$\Leftrightarrow$ there exists a non-zero vector v such that $T(v) = \lambda v$

$\Leftrightarrow$ $T(v) = \lambda I(v) \qquad [\because I(v) = v]$

$\Leftrightarrow$ $(T - \lambda I)(v) = 0, \quad$ where $v \neq \mathbf{0}$

$\Leftrightarrow$ $T - \lambda I$ is singular.

Now we state a theorem which shows that eigen value of a linear map T and eigen value of matrix A = [T : B], where B is an ordered basis of V and det $(A - \lambda I) = 0$ are equivalent.

**Theorem 8.7.3** Let T be a linear map on a vector space V and dim V = n and A = [T : B] where B is an ordered basis of V. Then for any $\lambda \in R$, the following statements are equivalent :

(a) $\lambda$ is an eigen value of T

(b) $\lambda$ is an eigen value of A

(c) det $(A - \lambda I) = 0$, where $I = I_n$.

The proof is left to the reader.

**Note :** The eigen values of T are given by det $(T - \lambda I) = 0$, where $A = [T]$.

**Example 8.7.3 :** Find the eigen values and eigen vectors of the linear transformation $T: V_3 \to V_3$ defined by $T(x, y, z) = (x + y + z, 2y + z, 2y + 3z)$

**Solution :** We have to find the matrix representation of T relative to the standard basis

$$B = \{(1,0,0), (0,1,0), (0,0,1)\} \text{ of } V_3.$$

$$\therefore\ A = [T] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

The eigen values of T are the values of $\lambda$ such that

$$\det(T - \lambda I) = 0 \quad \text{[by Theorem 8.7.2]}$$

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(2-\lambda)(3-\lambda)-2\} = 0$$
$$\Rightarrow (1-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$
$$\Rightarrow (1-\lambda)(\lambda - 1)(\lambda - 4) = 0$$
$$\Rightarrow \lambda = 1, 1, 4$$

$\therefore$ The eigen values of T are 1 and 4.

Now to find eigen vectors of T.

For $\lambda = 1$

The corresponding eigen vector

$v = [x \quad y \quad z]^T$ is given by $(A - A\,I)\,v = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y+z \\ y+z \\ 2y+2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow y + z = 0,\ y + z = 0,\ 2y + 2z = 0$$

$\Rightarrow$ $y = -z$, x is arbitrary. i.e., eigen vector corresponding to $\lambda = 1$ is of the form $\{(x, -z, z) \mid x \text{ is scalar}\}$.

Thus (1, 0, 0) and (0, –1, 1) are eigen vectors corresponding to eigen value $\lambda = 1$.

For $\lambda = 4$

The corresponding eigen vector $v = [x \quad y \quad z]^T$ is $(A - \lambda I)\,v = 0$

$$\Rightarrow \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3x+y+z \\ -2y+z \\ 2y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} -3x+y+z &= 0 \\ -2y+z &= 0 \\ 2y-z &= 0 \end{aligned}$$

$$\Rightarrow x=1,\ y=1,\ z=2$$

$\therefore$ (1, 1, 2) is the eigen vector corresponding to $\lambda = 4$.

**Theorem 8.7.4** Let $\lambda$ be an eigen value of a linear transformation T. Then $f(\lambda)$ is an eigen value of $f(T)$.

**Proof :** Let $v \in V$ be an eigen vector of T corresponding to eigen value $\lambda$.

Then by definition

$$T(v) = \lambda v,\ v \neq \mathbf{0} \qquad \text{... (1)}$$

To prove that

$f(\lambda)$ is an eigen value of $f(T)$,

i.e., $(f(T))(v) = (f(\lambda))\,v$.

In order to prove the theorem, first of all, we have to show that

$T^m(v) = \lambda^m v$, where m is a positive integer.

To prove by method of induction.

Let $P(m) : T^m(v) = \lambda^m v$

For m = 1,

$T(v) = \lambda v$, which is true by (1).

Let P(m) be true for m = k, where k is a positive integer.

i.e., $T^k(v) = \lambda^k v \qquad \text{... (2)}$

Now

$$\begin{aligned} T^{k+1}(v) &= (T^k\, T)(v) \\ &= T^k\,(T(v)) \\ &= T^k\,(\lambda\, v) && \text{[By (1)]} \\ &= \lambda\,(T^k\,(v)) && [\because \text{T is linear}] \\ &= \lambda\,(\lambda^k\, v) && \text{[By (2)]} \\ &= \lambda^{k+1}\, v \end{aligned}$$

$\therefore$ $P(k+1)$ is true.

Hence by axiom of induction, P(m) is true for all positive integral values of m.

$$\therefore\ T^m(v) = \lambda v \qquad \text{... (3)}$$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$

be any polynomial over T, of degree m.

Then

$$f(T) = a_0 I + a_1\, T + a_2\, T^2 + \ldots + a_m T^m$$

Now,

$$
\begin{aligned}
&(f(T))\,(v)\\
&=(a_0\, I+a_1\, T+a_2T^2+\ldots+a_mT^m)\,(v)\\
&=(a_0\, I)\,(v)+(a_1T)\,(v)+(a_2T^2)\,(v)+\ldots+(a_mT^m)\,(v)\\
&=a_0\, I(v)+a_1\, T(v)+a_2\, T^2\,(v)+\ldots+a_mT^m\,(v)\\
&=a_0v+a_1\,\lambda v+a_2(\lambda^2v)+\ldots+a_m\,(\lambda^m\, v)\\
&=(a_0+a_1\,\lambda+a_2\,\lambda^2+\ldots+a_m\lambda^m)\, v\\
&=f(\lambda)\, v, \qquad v\neq \mathbf{0}
\end{aligned}
$$

$\therefore$ $f(\lambda)$ is an eigen value of $f(T)$.

**Example 8.7.4 :** Verify that if $\lambda$ is an eigen value of T then $f(\lambda)$ is an eigen value of $f(T)$, where the matrix representation of T relative to standard basis of $V_2$ is $\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}$ and $f(x)=x^2-2x+3$.

**Solution :** The eigen values of [T] = A are the values of $\lambda$ such that

$$\det(A-\lambda I)=0 \quad \text{where } A=\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}$$

$$\Rightarrow \begin{vmatrix}3-\lambda & 3\\1 & 5-\lambda\end{vmatrix}=0$$

$$\Rightarrow (3-\lambda)(5-\lambda)-3=0$$

$$\Rightarrow \lambda^2-8\lambda+12=0$$

$$\Rightarrow (\lambda-2)(\lambda-6)=0$$

$$\Rightarrow \lambda=2,6$$

$\therefore$ 2 and 6 are eigen values of T.

Now

$$f(x)=x^2-2x+3$$

$$\therefore \quad f(T)=f(A)=A^2-2A+3I$$

$$=\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}-2\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix}+3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}12 & 24\\8 & 28\end{bmatrix}-\begin{bmatrix}6 & 6\\2 & 10\end{bmatrix}+\begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}=\begin{bmatrix}9 & 18\\6 & 21\end{bmatrix}$$

$\therefore$ The eigen values of $f(T)$ are given by

$$\det(f(A)-\lambda I)=0$$

$$\Rightarrow \begin{vmatrix}9-\lambda & 18\\6 & 21-\lambda\end{vmatrix}=0$$

$$\Rightarrow (9-\lambda)(21-\lambda)-108=0$$
$$\Rightarrow \lambda^2-30\lambda+81=0$$
$$\Rightarrow (\lambda-3)(\lambda-27)=0$$
$$\Rightarrow \lambda=3,\ 27.$$

$f(2)=4-4+3=3$

and $f(6)=36-12+3=27$

$\therefore$ 2 and 6 are eigen values of T

$\Rightarrow f(2)$ and $f(6)$ are eigen values of $f$(T). Hence verified.

## Problem Set 8 (B)

1. Let $f(x)=2x^2-3x+1$ and $g(x)=x^2-2x+3$

$A=\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$. Evaluate

(a) $f$(A) (b) g (A) (c) $f$(B) (d) g (B) (e) $f$(A + B) (f) g (A – B)

2. Let $T: V_2 \to V_2$ be defined by T (x, y) = (x, 0).
Find the eigen value and corresponding eigen vector of T.

3. Find the eigen values and eigen vectors of the identity matrix.

4. Show that the matrices

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ have same eigen values.

5. Let T be a linear map on $V_3$ and $A=[T]_B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Verify that if $\lambda$ is an eigen value of T then $f(\lambda)$ the eigen value of $f$(T), where $f(x) = x^2 - x + 1$

6. Prove that $A=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ is a zero of $f(t) = t^2 - 4t - 5$

[Hints : $f(A)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ]

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