Week 4

Linear Transformations

Today’s Outline

I. Image Processing

II. Transformation

III. Two types of Transformation

IV. Linear Transformation

V. Geometrical Interpretation of Matrices

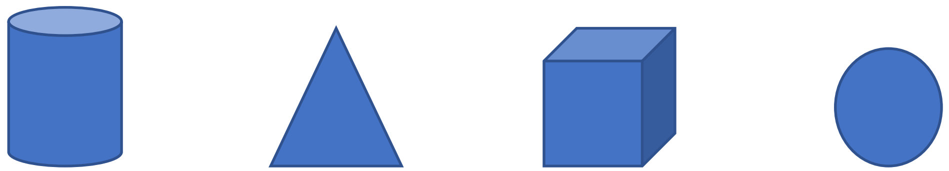
VI. Matrix Transformation

VII. Different types of Matrix Transformation

VIII. Tutorial

Geometric objects

- There is **No single mathematical** definition of geometric object.
- However, it is rather safe to simply rely on our intuition.
- Geometric objects may be defined by
  1. Geometric Properties
  2. Equations
  3. Inequalities



Geometric objects- Examples

- $x^2 + y^2 = 1 \rightarrow$ circle inside $\mathbb{R}^2$;
- $x^2 + y^2 \leq 1 \rightarrow$ disc inside $\mathbb{R}^2$;
- $x^2 + y^2 \leq 1 \rightarrow$ And a cylinder in $\mathbb{R}^3$;
- $x + y + z = 0 \rightarrow$ a plane in $\mathbb{R}^3$;
- $x^2 + y^2 + z^2 = 1 \rightarrow$ a sphere in $\mathbb{R}^3$.

Real-World objects

- There are also many geometric objects which are not naturally described by any equation or inequality. For example, real world objects
- To solve this problem in Computer science → Pixels

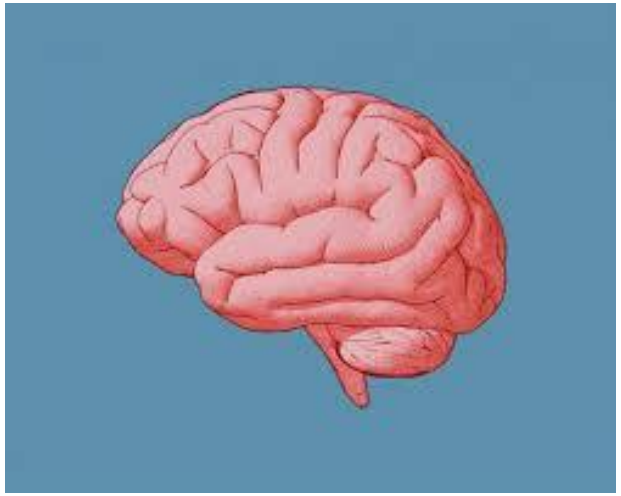
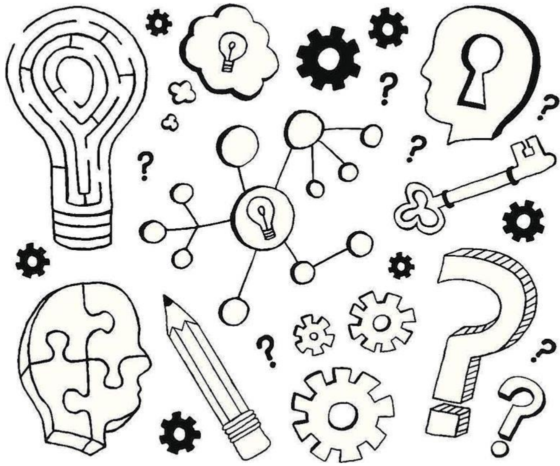


Image Processing

- Image processing involves performing operations on an image .
- To make it better or to gain information from it.
- We will discuss the foundations of mathematics behind image processing.

Reading (Individual, 10’)



Read the abstract of the bellow article and reflect on it.

What do you comprehend from this abstract?

https://www.sciencedirect.com/science/article/abs/pii/0097849378900079

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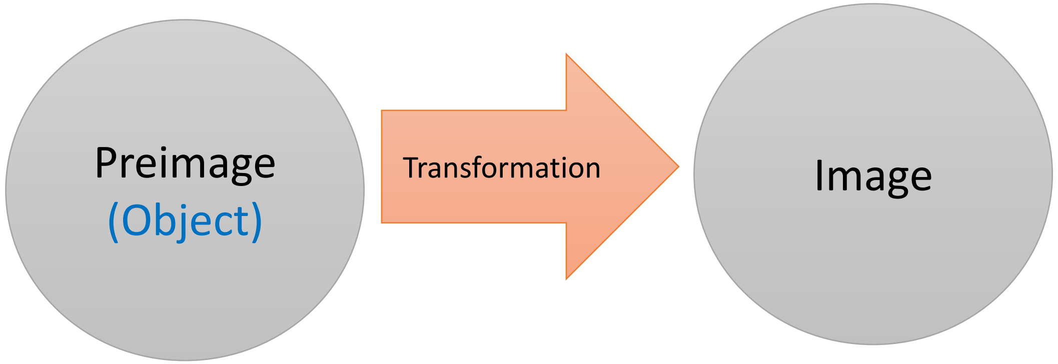
What is Transformation?

Transformation :

- The operation of changing
- One configuration into another configuration
- Under a mathematical rule.

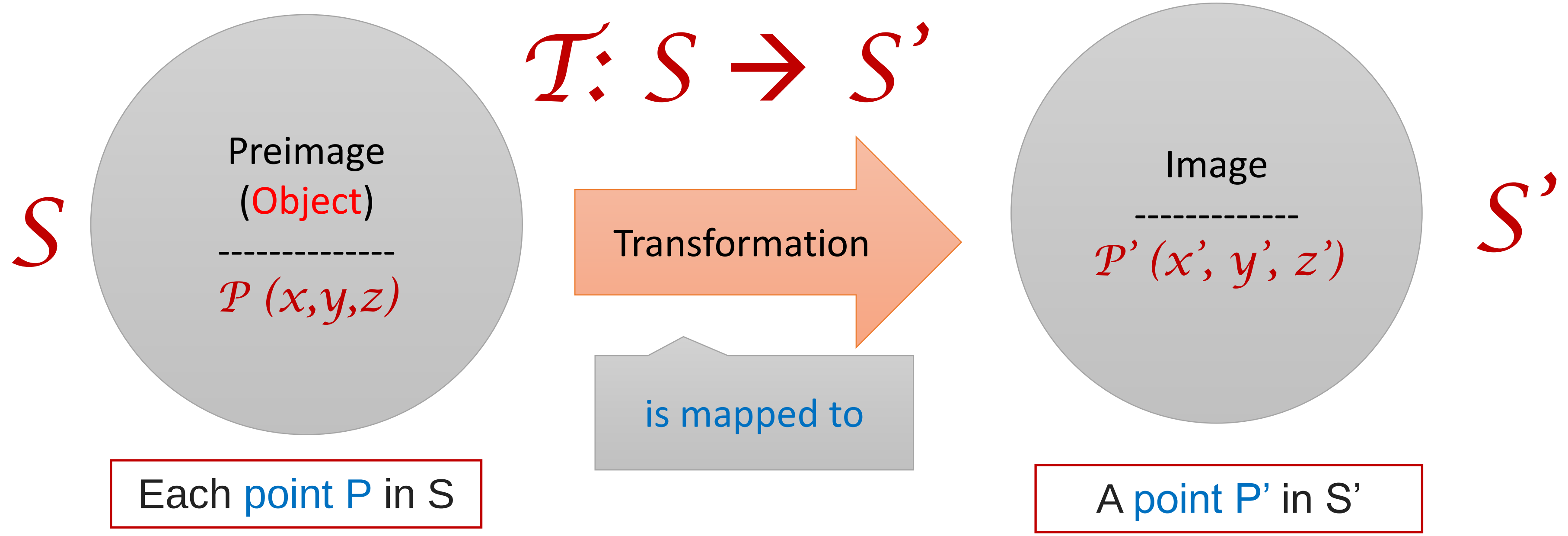


Each point in the object

is mapped to

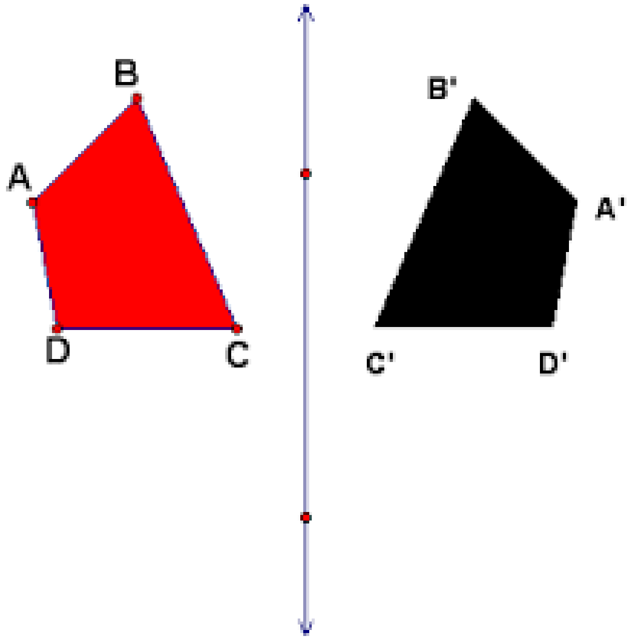
another point in the image

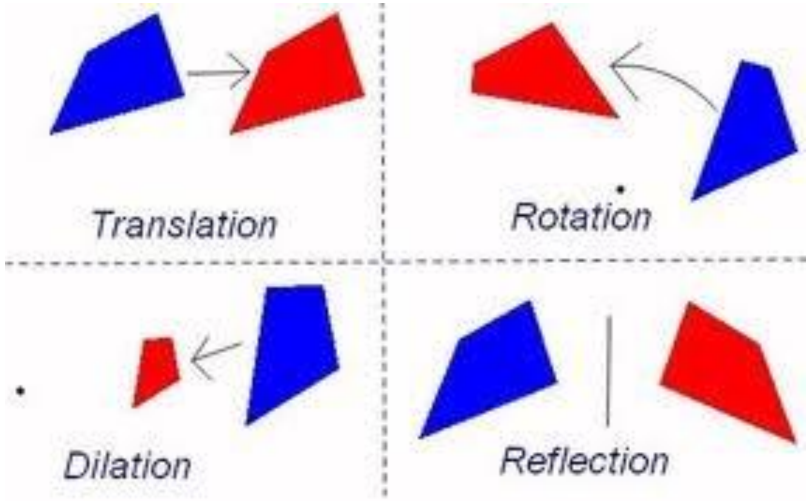
What is Transformation? cont



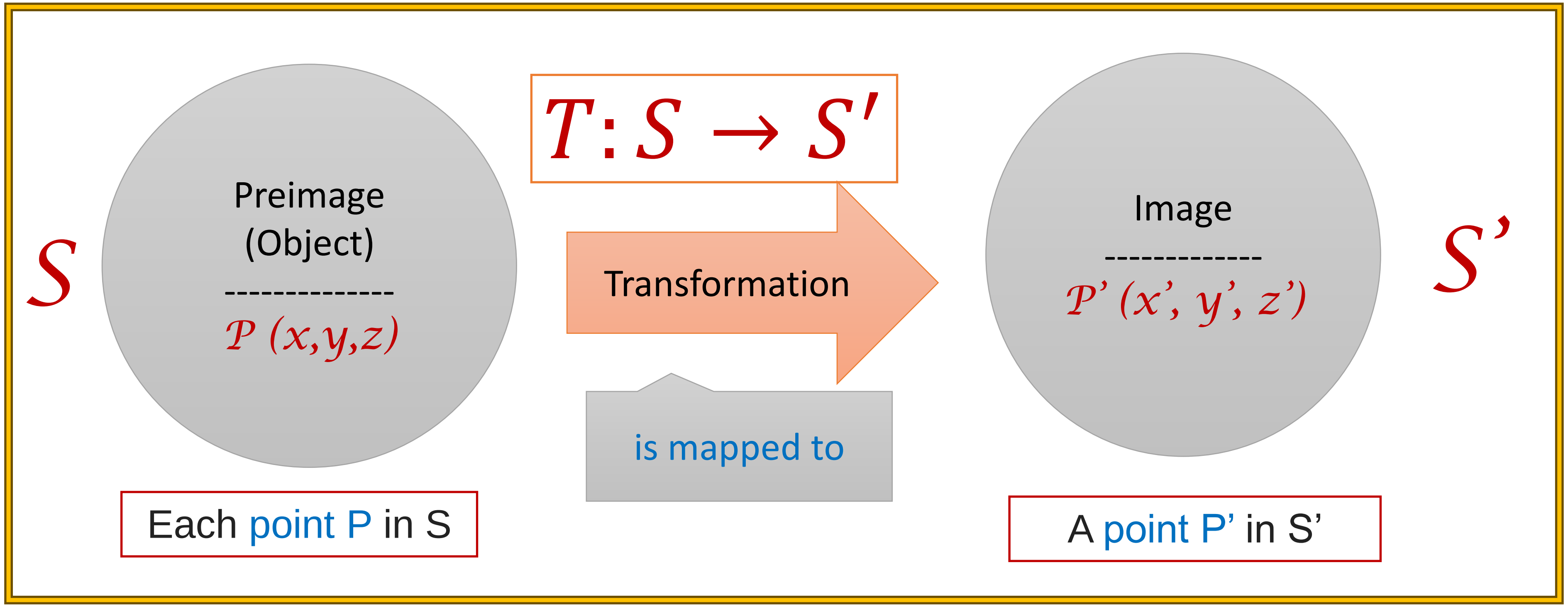
Transformation T is an **operator** which corresponds a point in S to a point in S'

Pre-Image & Image





Transformation- Definition



$$\forall\ P(x,y,z) \in S \overset{T}{\leftrightarrow} \exists P'(x',y',z') \in S'$$

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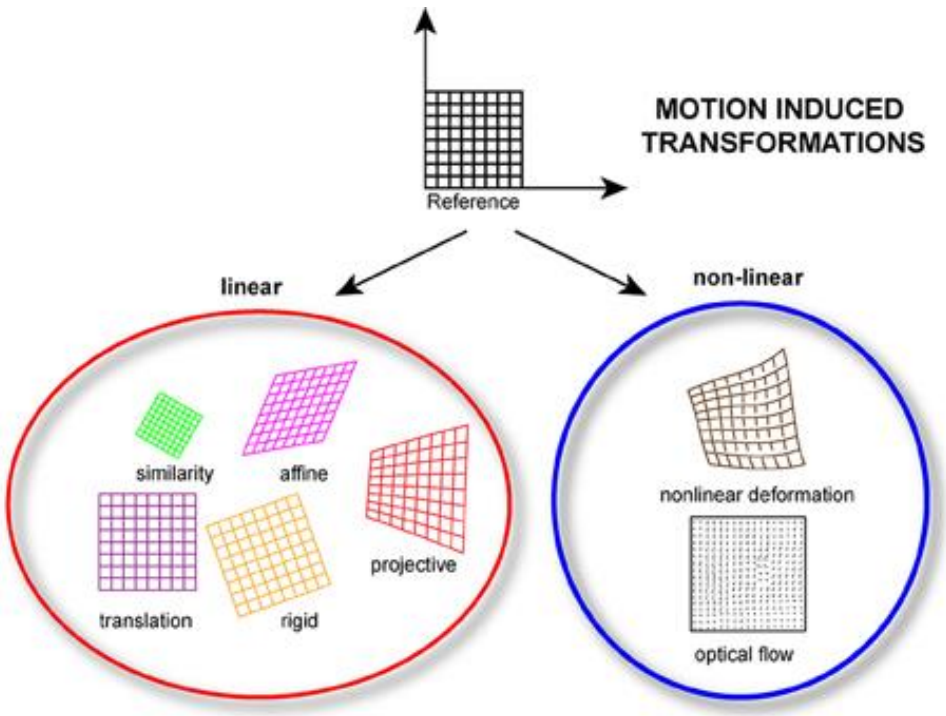
2-Types of Transformation

Transformations are categorised into :

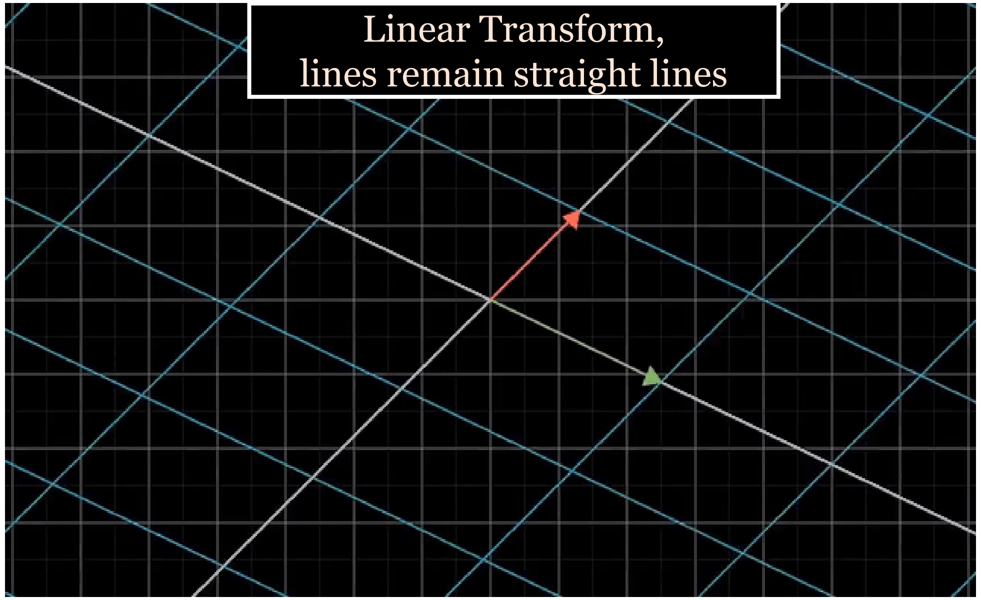
- **Linear** transformation
- **Nonlinear** transformation

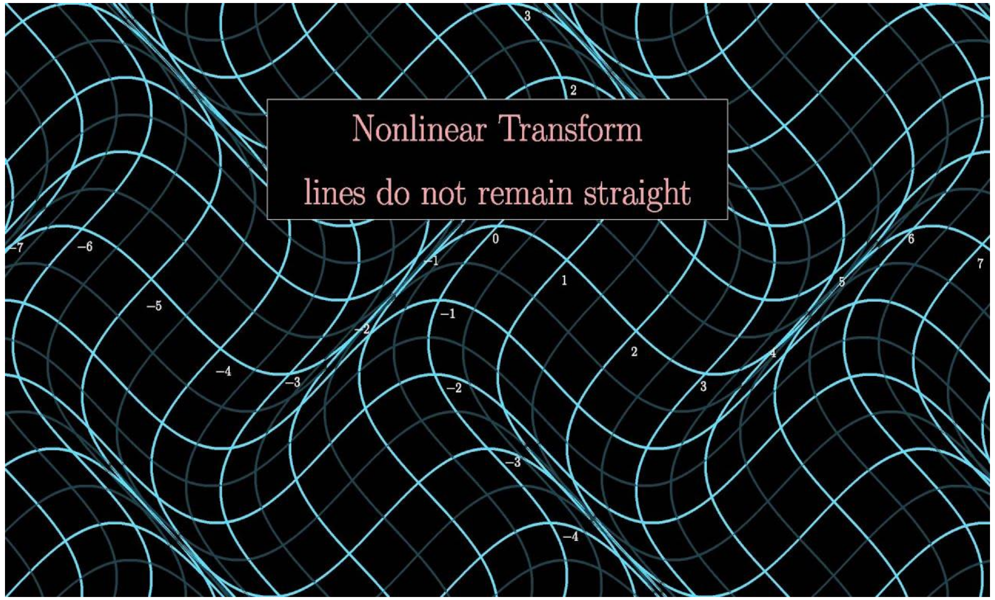


Linear vs. Nonlinear Transformation

Transformations are categorised into Linear & nonlinear.

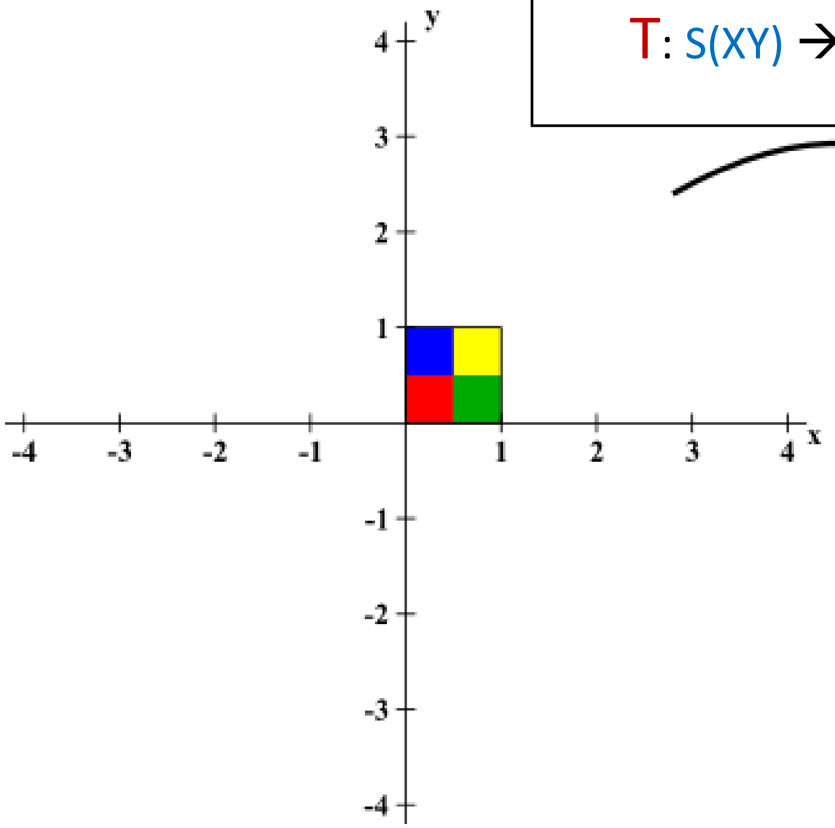
- A linear transformation **preserves** linear relationships between variables.
- A nonlinear transformation **changes** linear relationships between variables

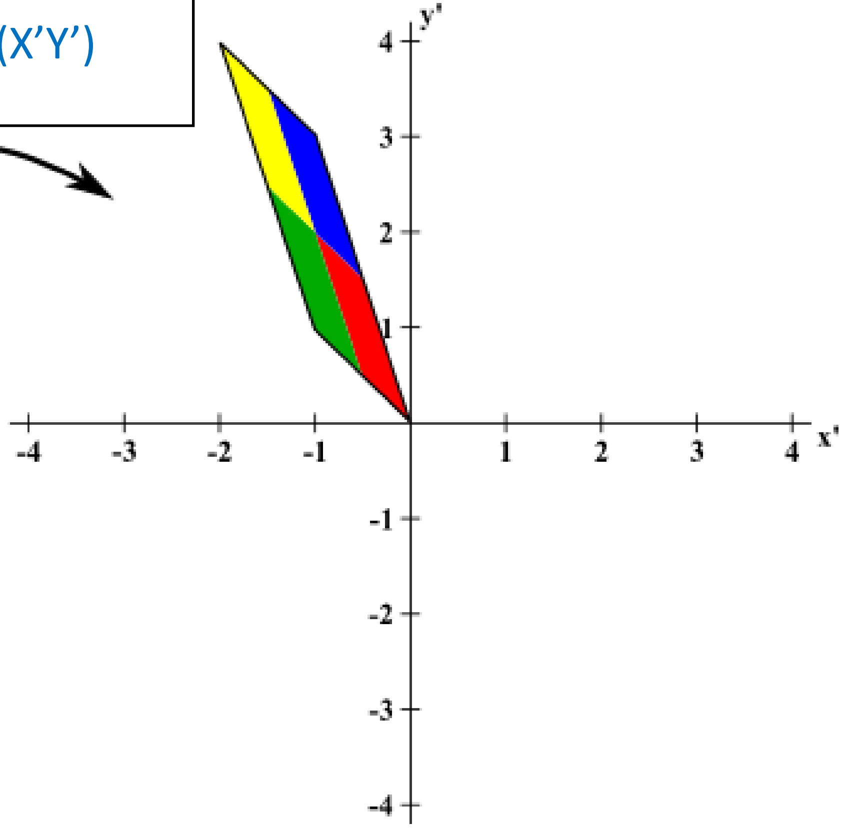




Example of Linear Transformation

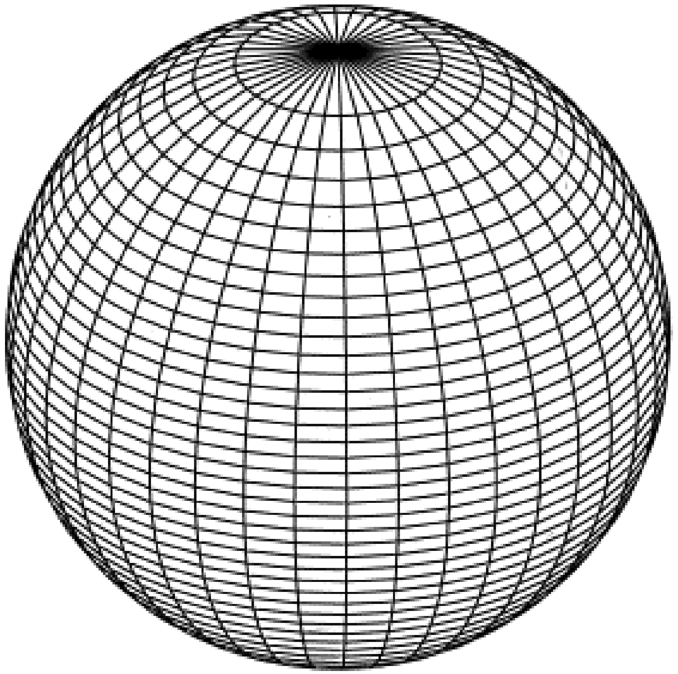
T: S(XY) → S'(X'Y')



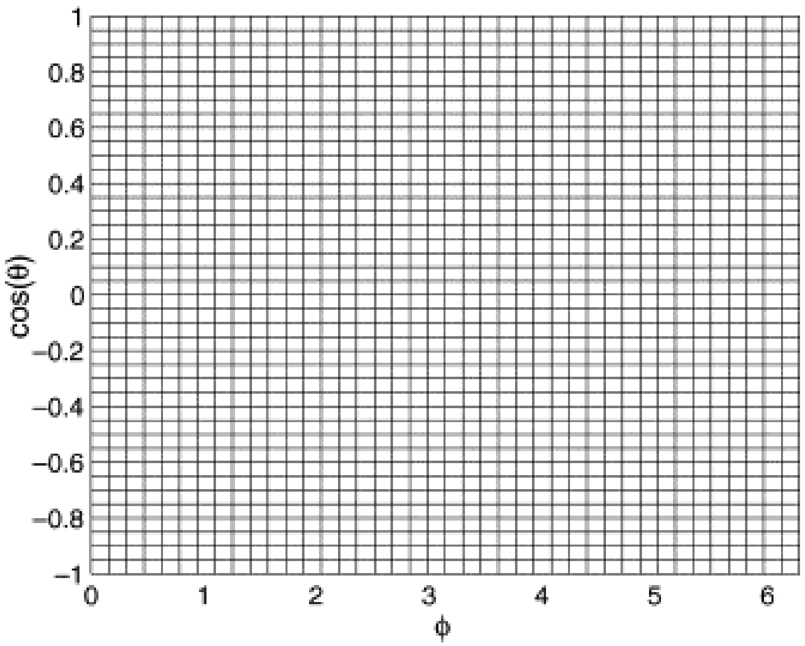


- The **linear relationship** between **variables** is preserved

Example of Nonlinear Transformation



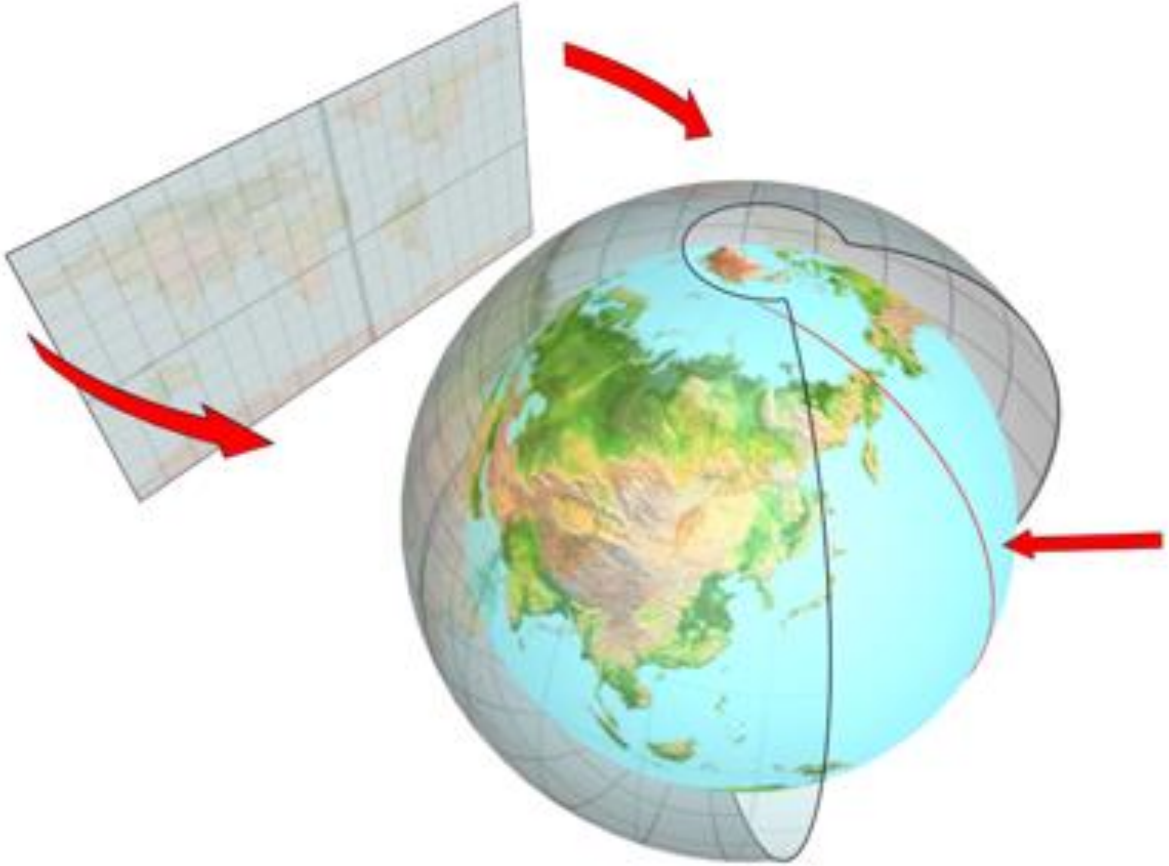
(a) Mesh on the unit sphere

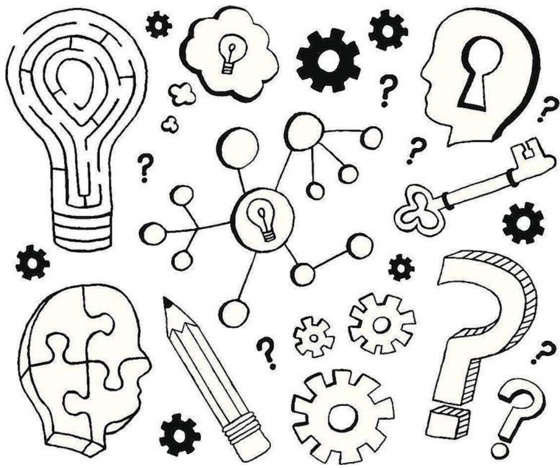


(b) Equal area projection on 2D plane

- The **linear relationship** between **variables** is **NOT** preserved

Example of Nonlinear Transformation - Mercator Projection





Reflection (Individual, 10’)

1) What do you understand from transformation?
2) Have you used transformation before? What was the case?
3) In your opinion, where is linear transformation used in computer science?
4) In your opinion, where non-linear transformation is used in computer science?

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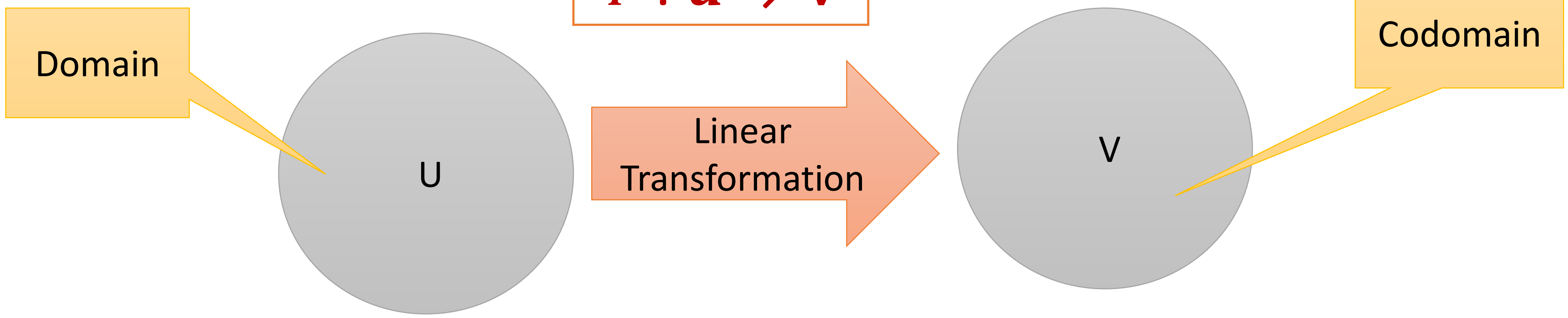
Linear Transformation as an Operator

Linear Transformation **:**

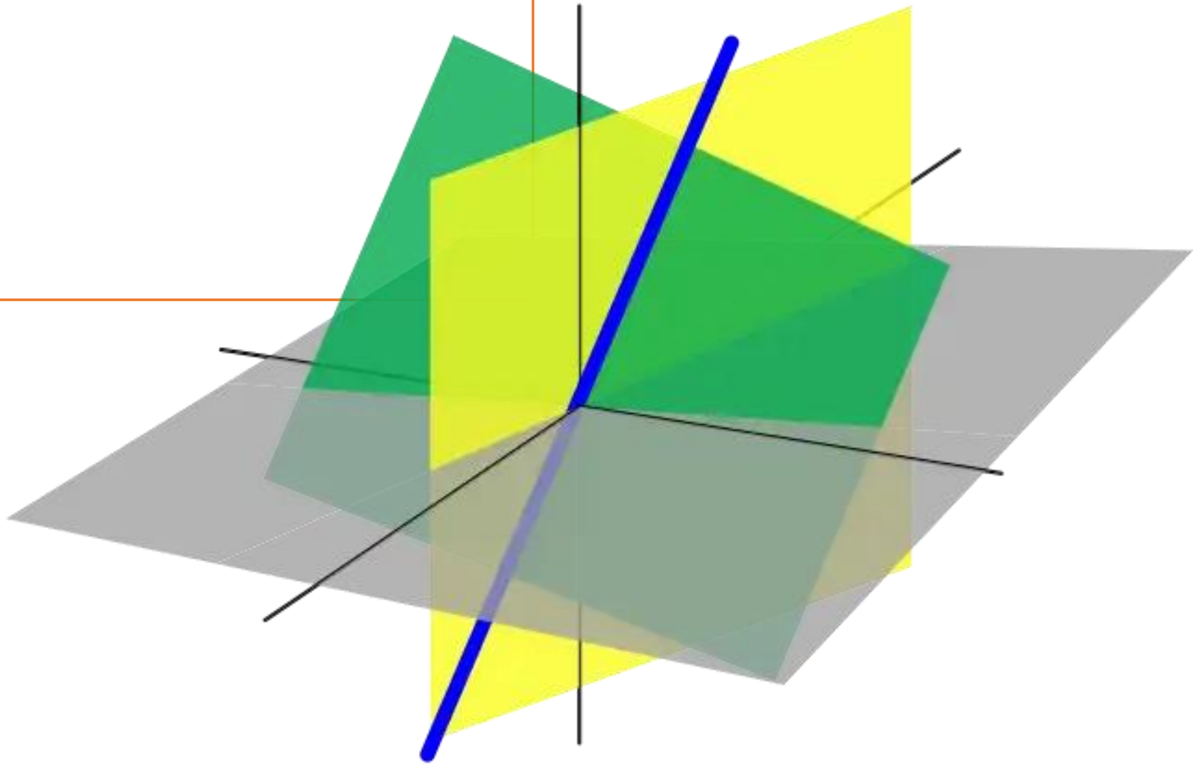
- Is an Operator T
- That **maps** the points from one manifold (U) to points in another manifold (V),
- *while preserving its linear structure.*

$$T : \mathbf{u} \rightarrow \mathbf{v}$$



Theorem

- If the variables (x1, x2, x3, …, xn) of a manifold M
- Have a linear relationship [xn=f(x(n-1),…x2,x1,C)| f:: Linear]
- Then, the manifold has a linear structure
- Corollary: And M it is a vector space

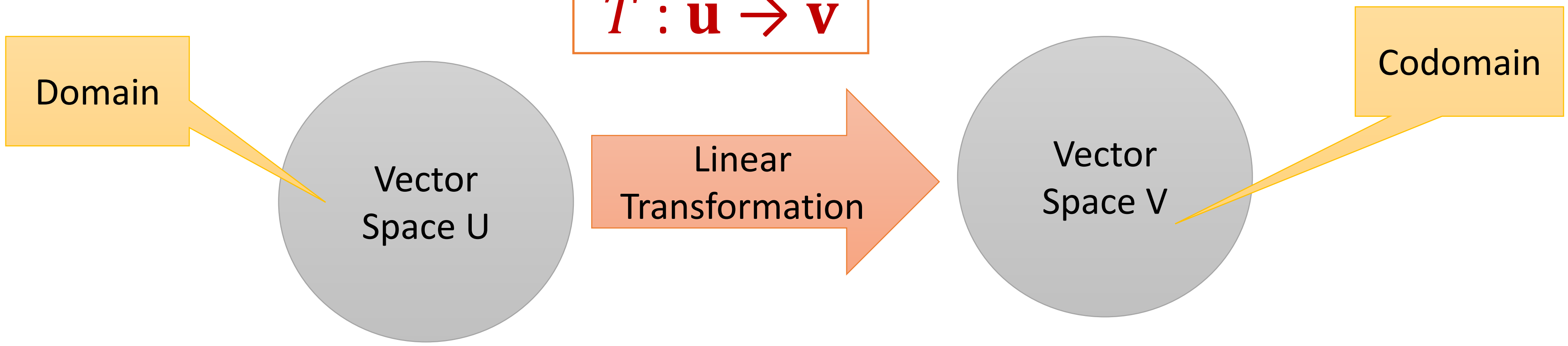


Linear Transformation- Redefined

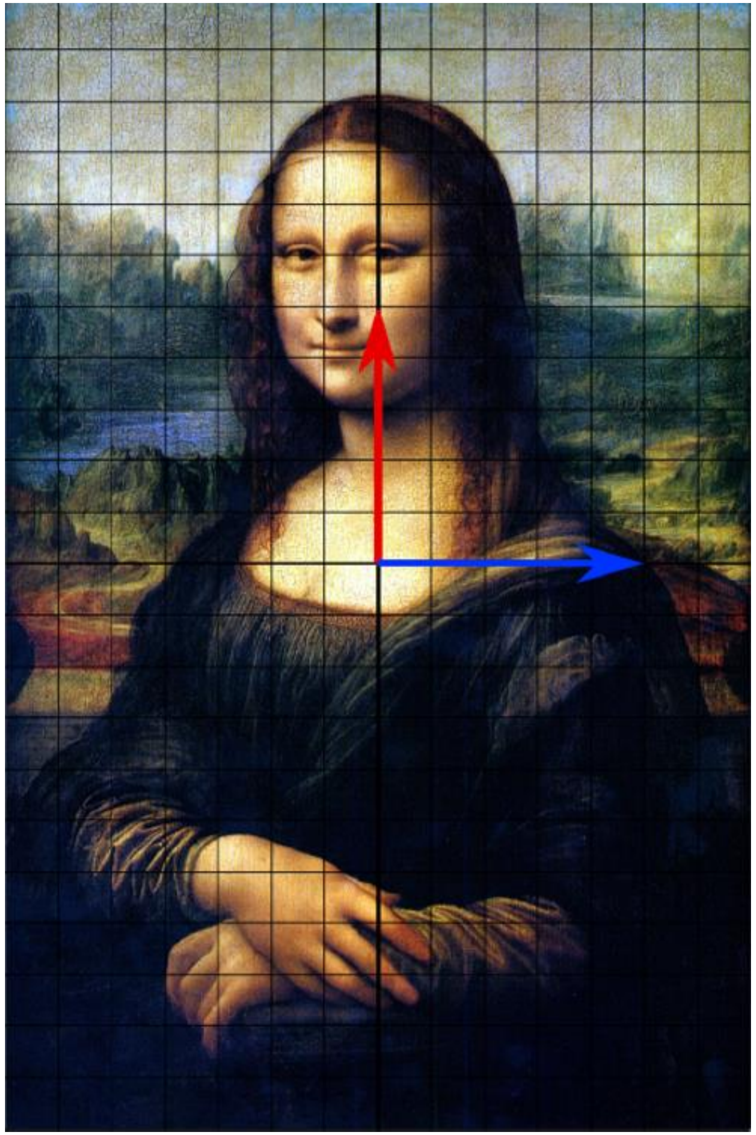
Linear Transformation :

- Is a function (T) / Operator T
- That maps one vector space(U) to another vector space (V),
- *while preserving its linear structure.*

$$T : \mathbf{u} \rightarrow \mathbf{v}$$

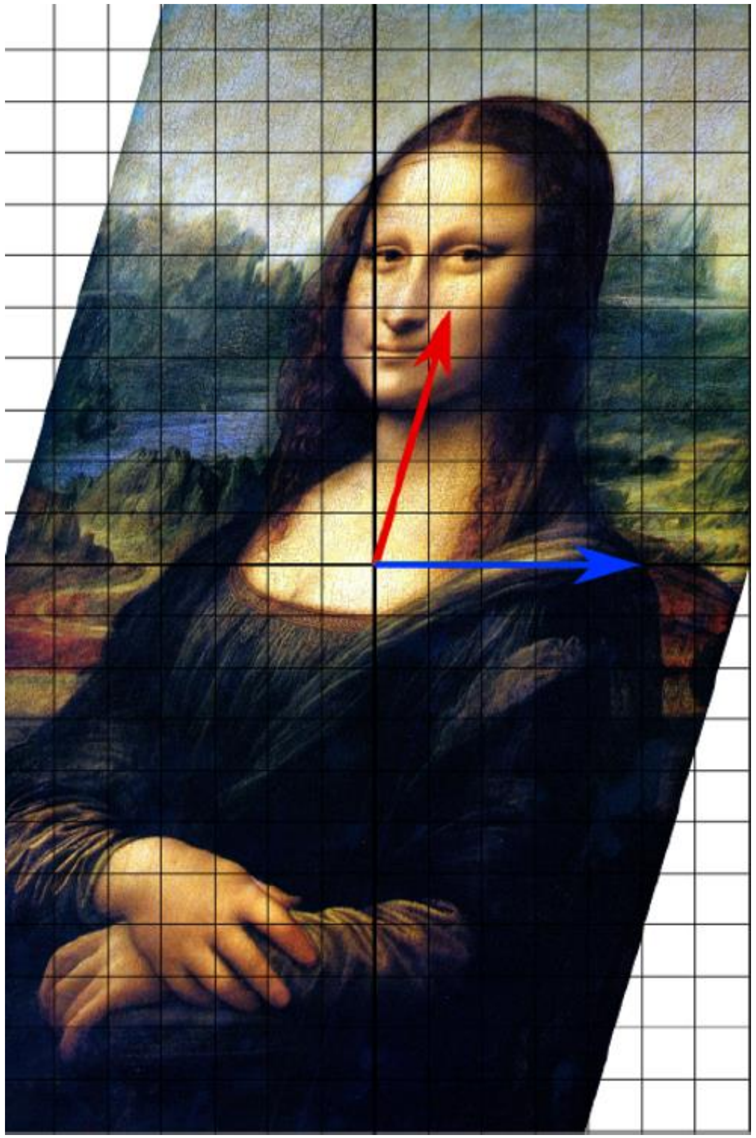


Vector space U



Linear Transformation **T**

Vector space V



Linear Transformation- Properties

- Transformation T is linear **if & Only if** it has the following properties:

$$\boxed{T : \mathbf{u} \to \mathbf{v}}$$

$$\boxed{\begin{array}{ll} I. & T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \\ II. & T(c\mathbf{u}) = cT(\mathbf{u}) \qquad (c \in R) \end{array}}$$

Example) Is T(x) a Linear Transformation?

1) $T: R \rightarrow R \quad and \; T(x) = 5x$

I) $T(x_1 + x_2) = 5\,(x_1 + x_2) = 5x_1 + 5x_2$

$T(x_1) + T(x_2) = 5x_1 + 5x_2$

$T(x_1 + x_2) = T(x_1) + T(x_2)$

II) $T(cx) = (c5x) = 5cx$

$cT(x) = c(5x) = 5cx$

$T(cx) = cT(x)$

$T(x) = 5x$ is a linear transformation

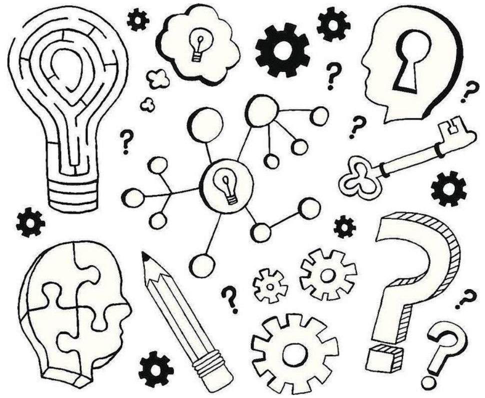
2) $T: R \rightarrow R \quad and \quad T(x) = Ln\,x$

I) $T(x_1 + x_2) = Ln\,(x_1 + x_2) \neq Ln\,x_1 + Ln\,x_2 \neq T(x_1) + T(x_2)$

II) $T(cx) = Ln(cx) \neq c\,ln\,x \neq c\,T(x)$

$T(x) = Lnx$ is not a linear transformation

Exercise 1(Individual, 10’)



1) Which of these is a linear transformation:.
   a) T= 3X
   b) T= Tan (x)
   c) T= Exp (x)

2) Let T= ax + b.
   a) Show that T is NOT a linear Transformation?
   b) Show that if b=0, then T is linear.

Exercise 1

a) T = 3X is a linear transformation:

I) $T(x_1 + x_2) = 3\,(x_1 + x_2) = 3x_1 + 3x_2$

$T(x_1) + T(x_2) = 3x_1 + 3x_2$

$$\left.\begin{array}{l} T(x_1 + x_2) = 3\,(x_1 + x_2) = 3x_1 + 3x_2 \\ T(x_1) + T(x_2) = 3x_1 + 3x_2 \end{array}\right\} \quad T(x_1 + x_2) = T(x_1) + T(x_2)$$

II) $T(cx) = (c3x) = 3cx$

$cT(x) = c(3x) = 3cx$

$$\left.\begin{array}{l} T(cx) = (c3x) = 3cx \\ cT(x) = c(3x) = 3cx \end{array}\right\} \quad T(cx) = cT(x)$$

$T(x) = 3x$
is a linear transformation

Exercise 1

b) T = Tan (x) is NOT a linear transformation:

I) $T(x_1 + x_2) = tan\ (x_1 + x_2)$

$T(x_1) + T(x_2) = tanx_1 + tanx_2$

$$T(x_1 + x_2) \neq T(x_1) + T(x_2)$$

II) $T(cx) = (\tan(c\,x)\ )$

$cT(x) = c(tanx)$

$$T(cx) \neq cT(x)$$

$T(x) = tanx$
is NOT a linear transformation

$$\tan\ (a+b) = \frac{\tan a + \tan b}{1 - \tan a \cdot \tan b}$$

Exercise 1

c) T = Exp (x) is NOT a linear transformation:

I) $T(x_1 + x_2) = exp\ (x_1 + x_2)$ = $\exp(x_1) * \exp(x_2)$

$T(x_1) + T(x_2) = \exp(\ x_1) + \exp(\ x_2)$

$$T(x_1 + x_2) \neq \quad T(x_1) + T(x_2)$$

II) $T(cx) = (\exp(c\,x))$

$cT(x) = c(\exp(\ x))$

$$T(cx) \neq cT(x)$$

$T(x) = \exp(x)$
is NOT a linear transformation

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Matrices in Geometry

Matrices have geometrical interpretations.

- Any Point on 1-D Line, can be represented by a matrix → [x]
- Any Point in 2-D plane, can be represented by a matrix → [x,y]
- Any Point in 3-D Space, can be represented by a matrix → [x,y,z]
- Any Point in 4-D Space, can be represented by a matrix→ [x,y,z,t]

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- Any point in N-D space, can be represented by a matrix→ $[x_1, x_2, x_3, \ldots x_n]$

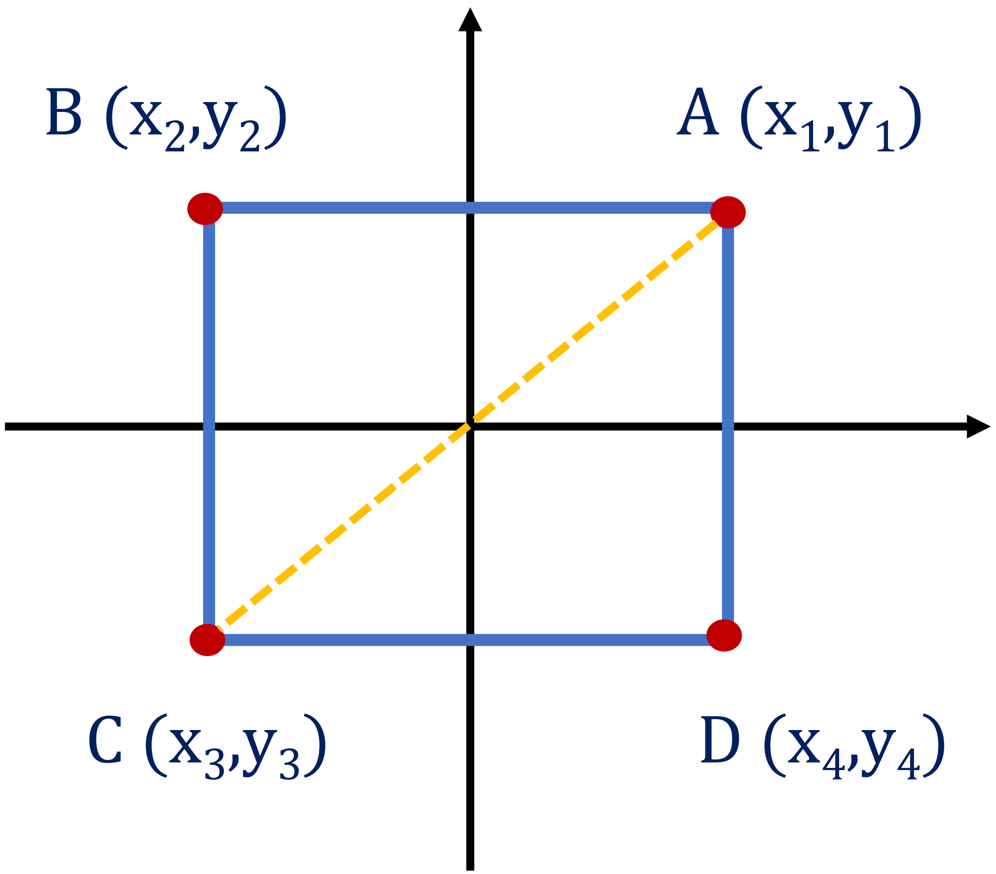
Geometrical Interpretation of Matrices- Plane

a) $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ Point A

b) $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ Line segment AB

c) $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$ Triangle ABC

d) $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$ Square ABCD

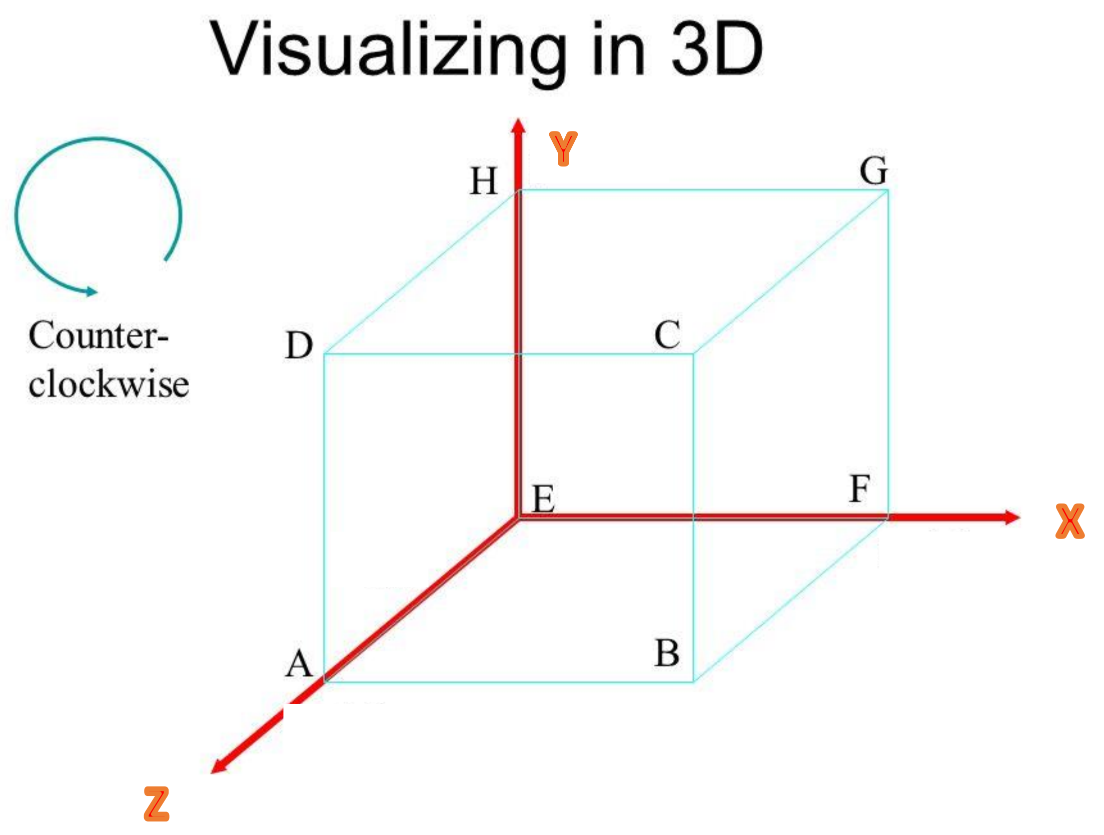


Geometrical Interpretation of Matrices- Space

- Every point P in space can be represented as a matrix:

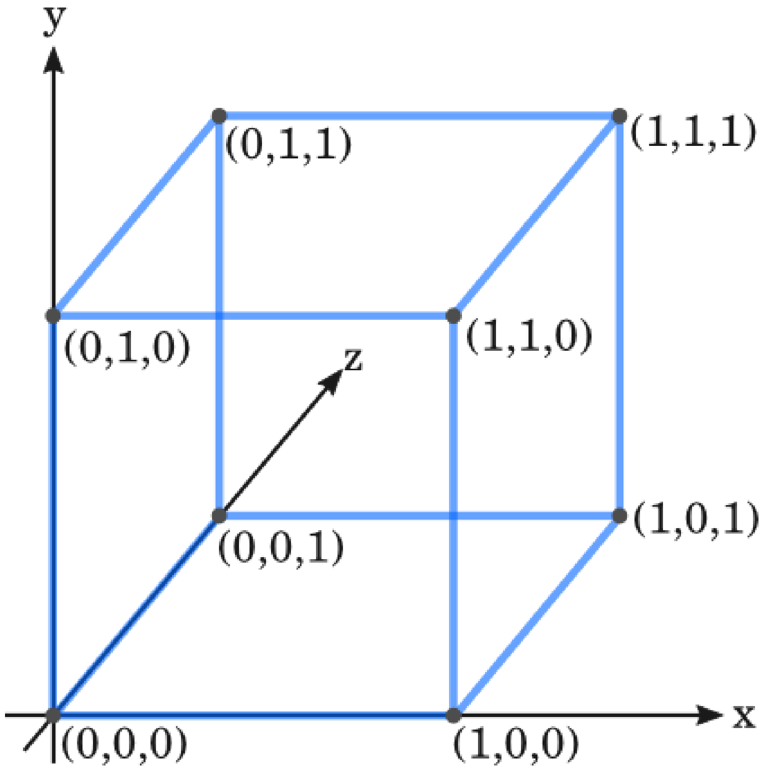
- $A\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ Point A

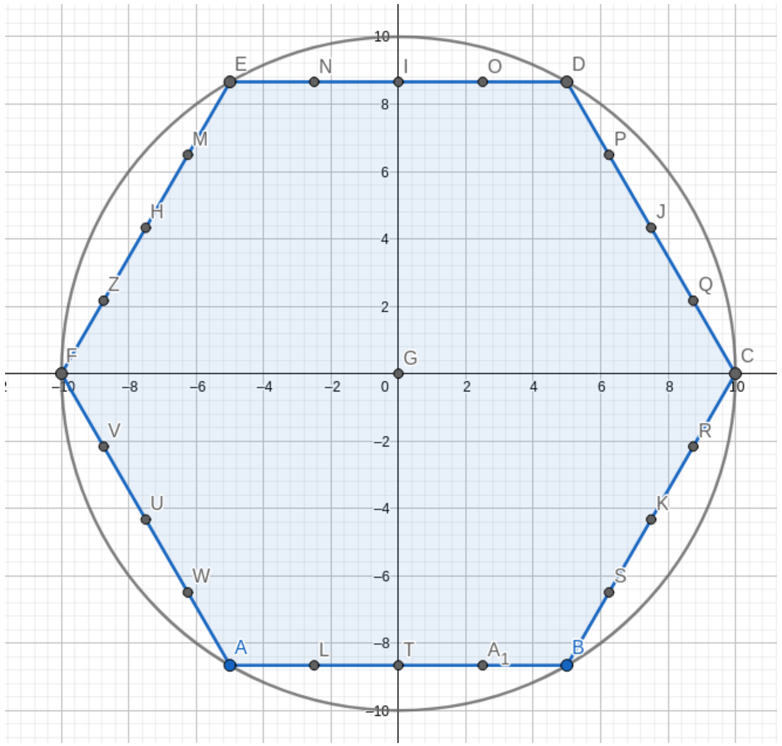
- $AB\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$ Line segment AB

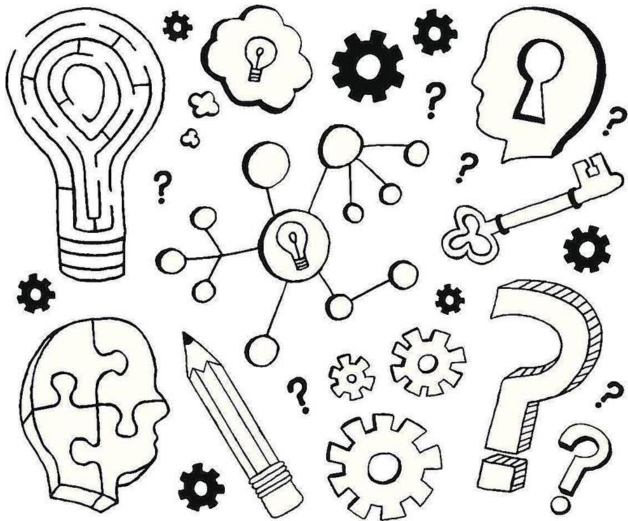


Exercise 2
(Individual, 10’)

1) Which Matrices represents the bellow cube?
2) Which matrix represents the below hexagon?
3) What is the matrix for a generic cube in 3D cartesian space.







Break- 20’

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Conceptual Example

Show **AX = b**

A= $\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$ X = $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

1) What is dimension of:

   i. Matrix A?

   ii. Vector X?

   iii. Vector b?

2) What does AX=b mean?

Conceptual Example cont

- The matrix A multiplied in a vector $\boldsymbol{x}$:

$$\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Matrix A (2x4)

Vector X (4x1)

Vector b (2x1)

Matrix A **Transforms** the vector $\boldsymbol{x}$ to another vector $\boldsymbol{b} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$$\vec{x}_{\boldsymbol{n}\times 1} \xrightarrow{A_{m\times n}} \vec{b}_{\boldsymbol{m}\times 1}$$

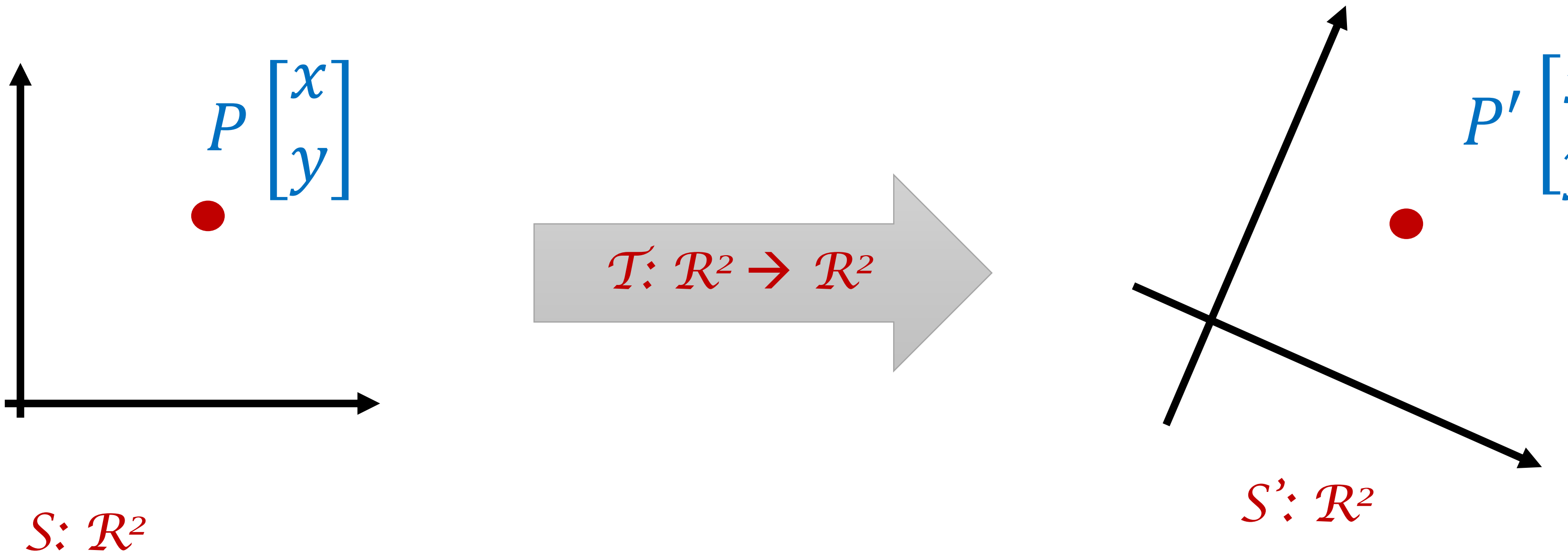
Matrix Transformation

Linear transformations - represented by **Matrices**

- This facilitates algebraic operations to a great extent.
- A *linear transformation defined by a matrix* → **matrix transformation**.

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\text{Transformation Matrix}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Initial Vector}} = \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\text{Transformed Vector}}$$

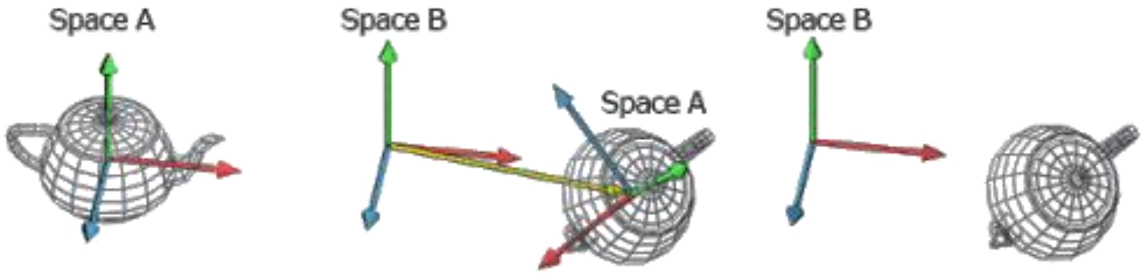
Matrix Transformation cont.



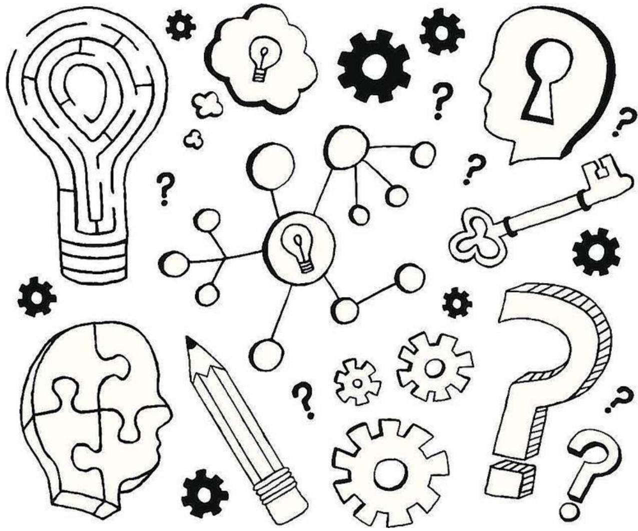
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad \left\{ \begin{array}{l} \boxed{x' = ax + by} \\ \boxed{y' = cx + dy} \end{array} \right.$$

Matrix Transformation cont

- Matrix transformation is a linear transformation
- It may have different sizes.
- Matrix transformation is widely used in image processing.
- There are various types of matrix transformation.



Exercise 3(Individual, 10’)



1) Prove that a 2x2 matrix is a linear transformation.
2) Prove that a 3x3 matrix is a linear transformation.
3) Can you conclude a nxn matrix is also a linear transformation?

Hint: You should prove, under a matrix transformation:

i) $T(x1 + x2) = T(x1) + T(x2)$

ii) $T(cx) = cT(x)$

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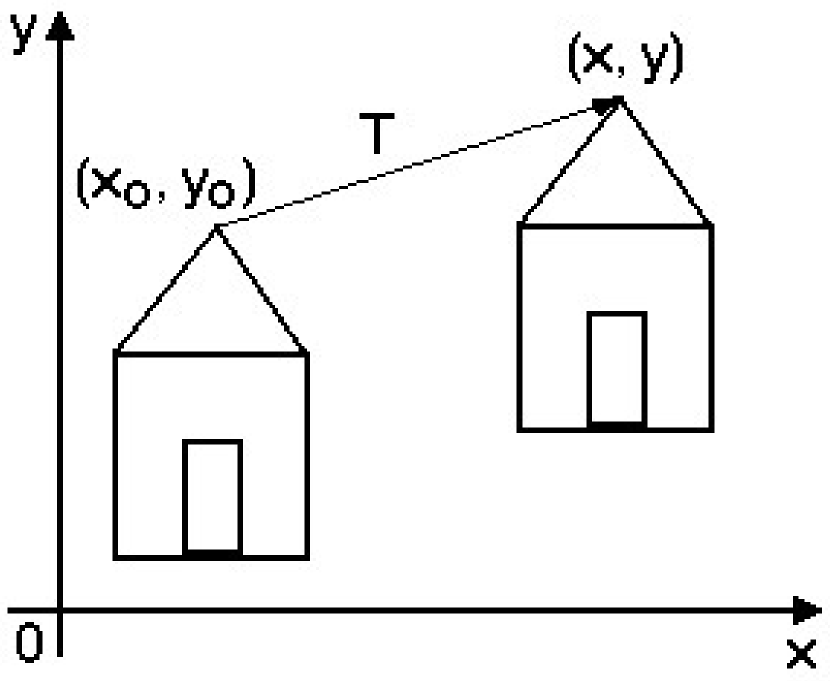
Different Types of Matrix Transformation

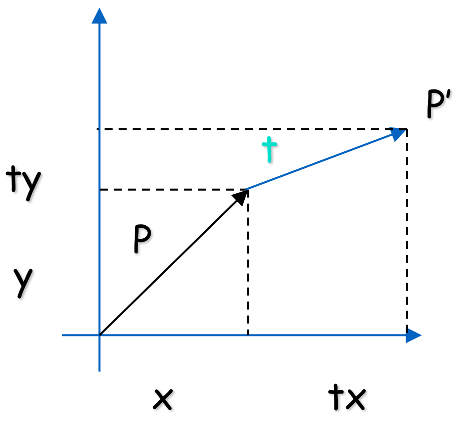
- Matrix Transformation has different types, including:
  - 2D Translation
  - Scaling
  - Rotation
  - Shears
  - Reflection
  - Projections

2D Translation

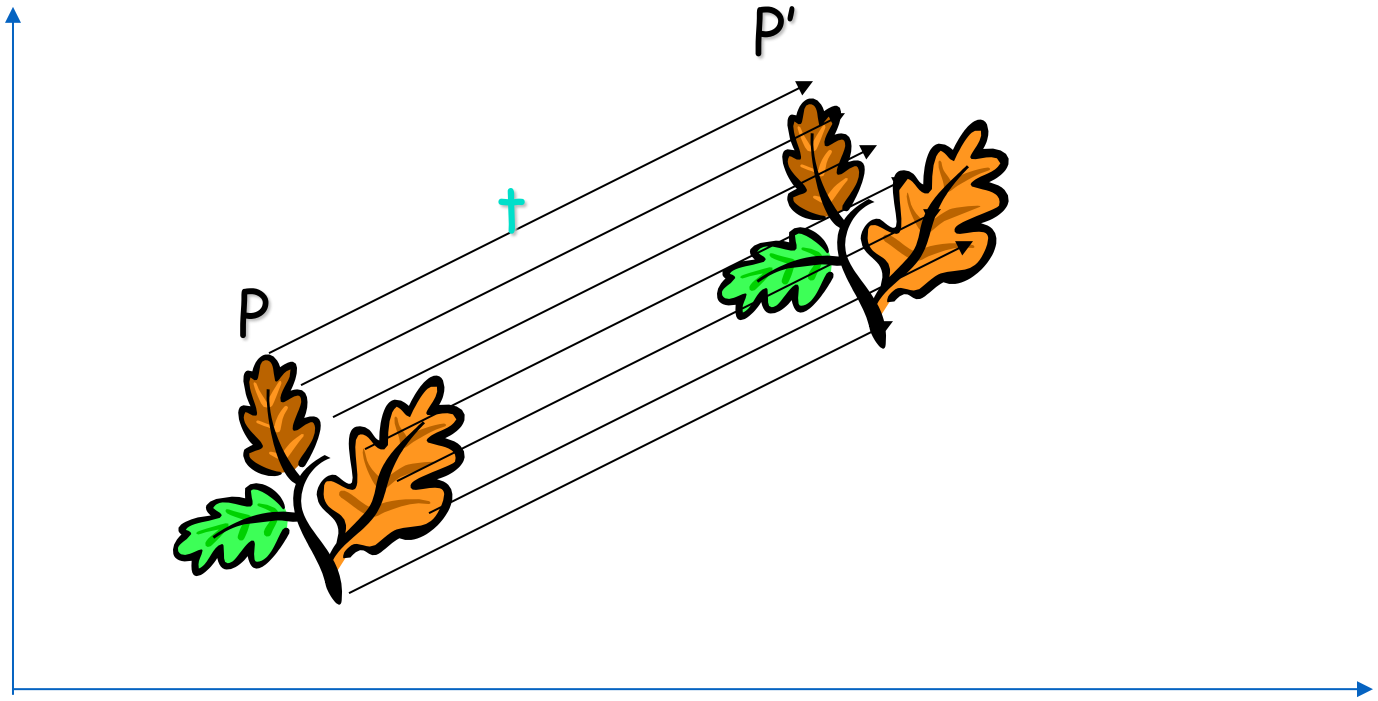
- 2D Translation is a process of moving an object from one position to another in a 2D plane.



2D Translation Matrix



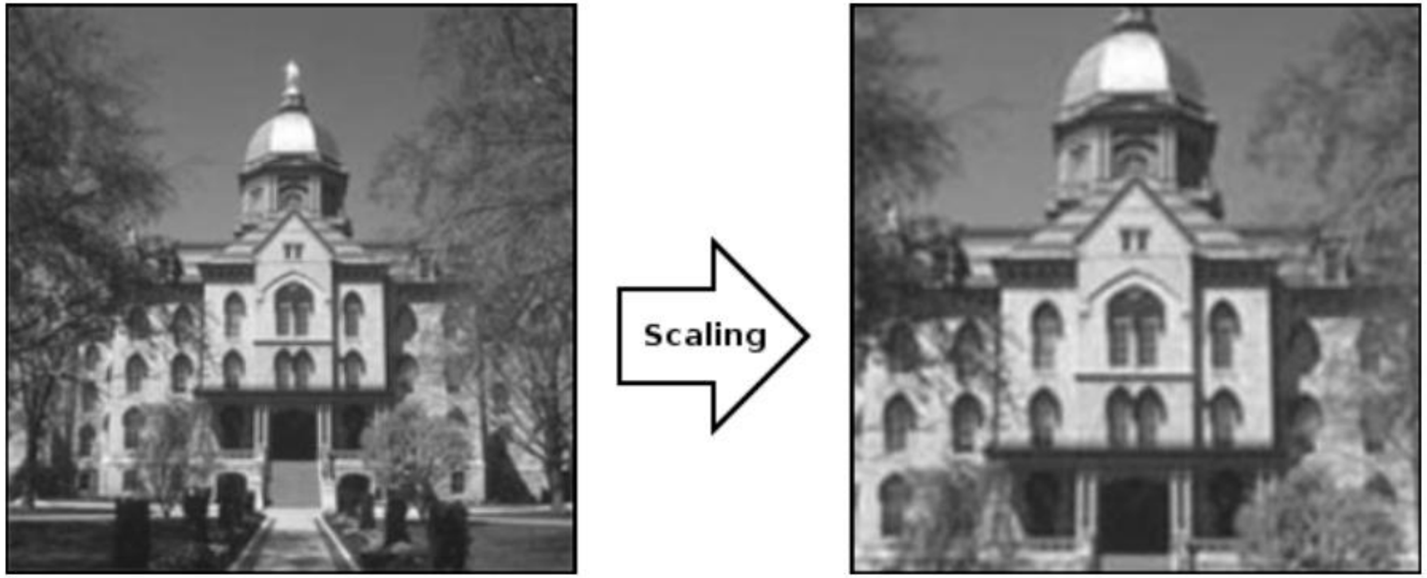
$$\mathbf{P}' = (x + t_x, y + t_y) = \mathbf{P} + \mathbf{t}$$



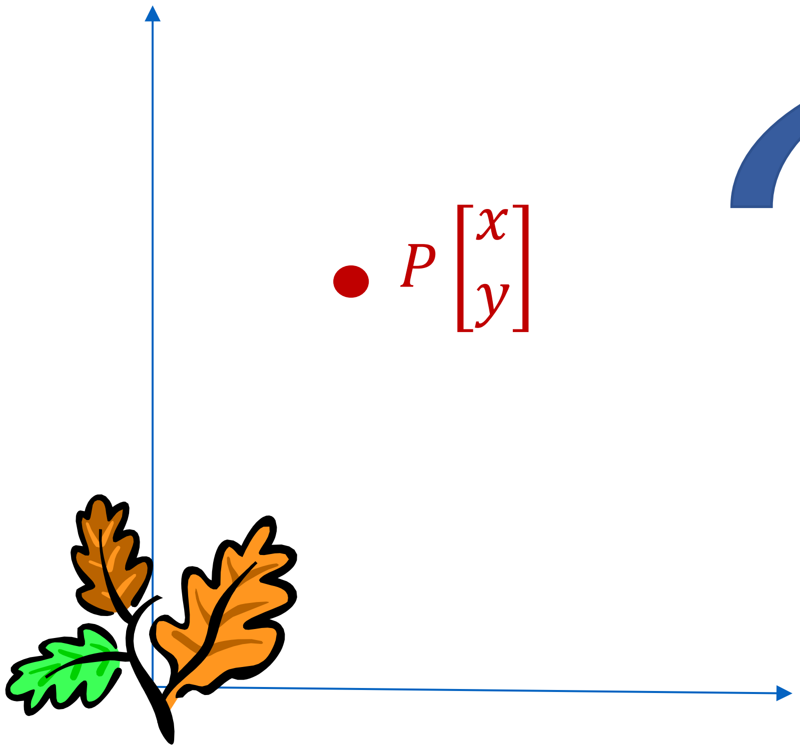
$$P' \begin{bmatrix} x' \\ y' \end{bmatrix} = P \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t(x) \\ t(y) \end{bmatrix}$$

Scaling

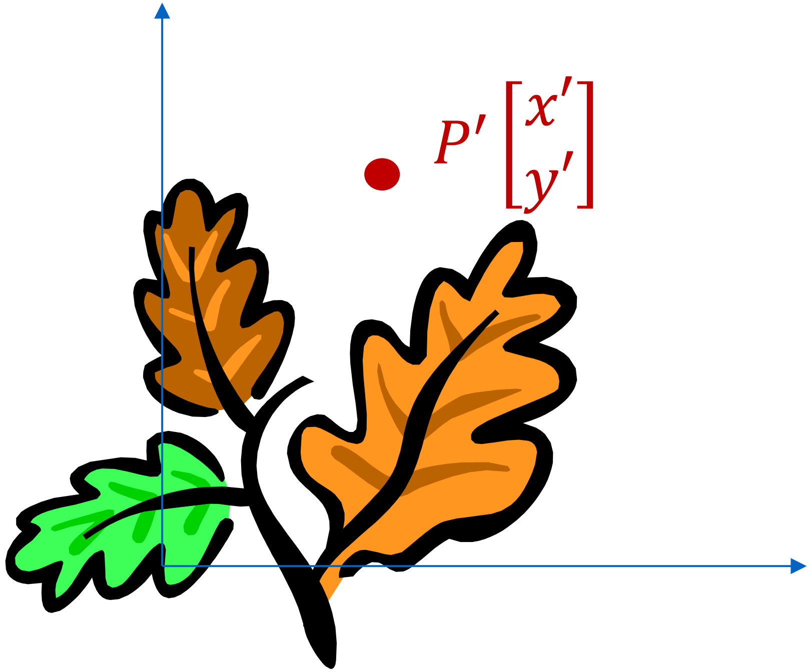
- Scaling is a linear transformation that enlarges (increases) or shrinks (diminishes) objects by a scale factor that is the *same in all directions*.



Scaling Matrix



$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

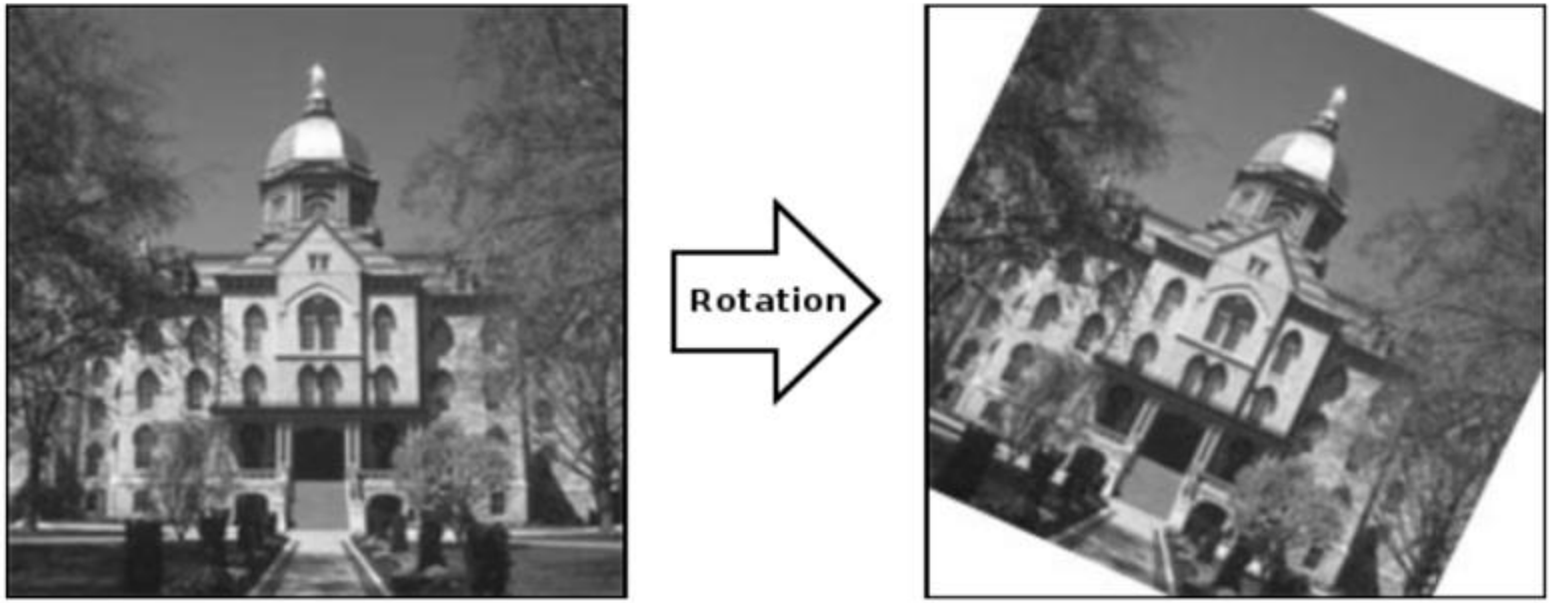


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation

- Rotation is a circular movement of an object around a centre of rotation.
- In this example, X' makes an angle *t* with X
- Z axis is the centre of rotation.

$$P' = R_z(t)P$$



Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

WHY?

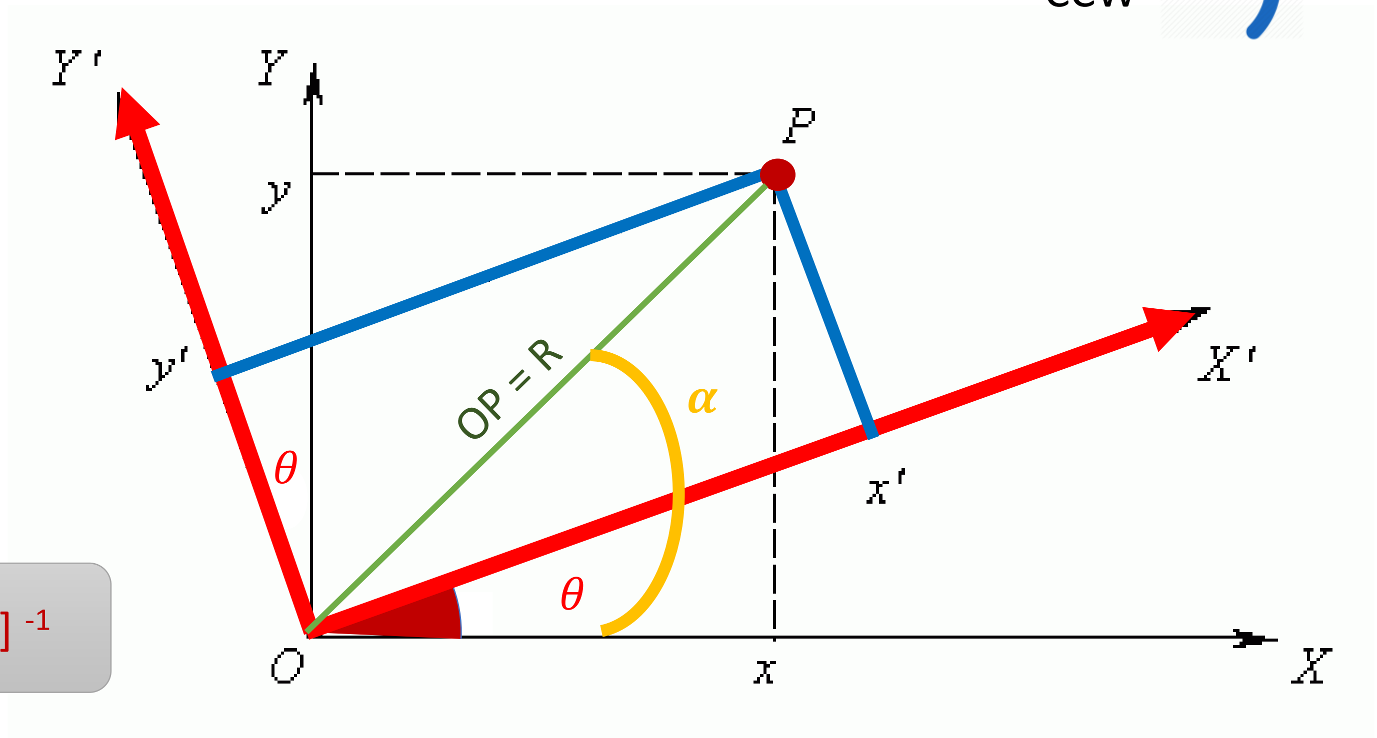
$$R'(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

WHY?

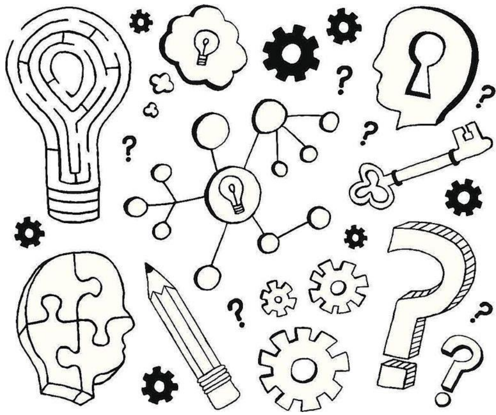
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$R(\theta) = [R'(\theta)]^{-1}$

Rotation CCW



Exercise 4(Individual, 10’)



The point A (10, 5) is in XY plane.

1. It is transformed by a vector V<1,1> to point A’. What is the matrix of transformation? What is A’?
2. The position of A is scaled by 2 to A’ . What is the scaling matrix? What is the new position?
3. The position of A is scaled by 0.5 to A’ . What is the scaling matrix? What is the new position?
4. The point A is rotated in XY plane, CCW, an angle of 30 degrees to A’. What is the matrix of rotation? What is A’?

Today’s Outline

I. Image Processing

II. Transformation

III. Two types of Transformation

IV. Linear Transformation

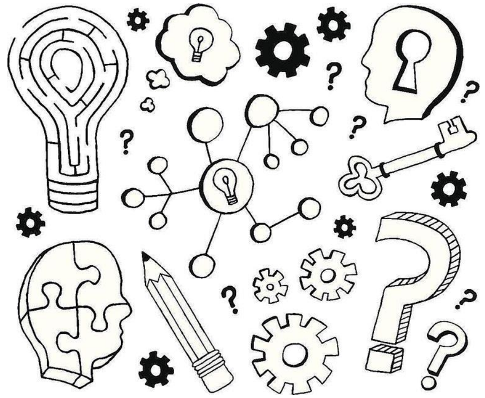
V. Geometrical Interpretation of Matrices

VI. Matrix Transformation

VII. Different types of Matrix Transformation

VIII. Tutorial

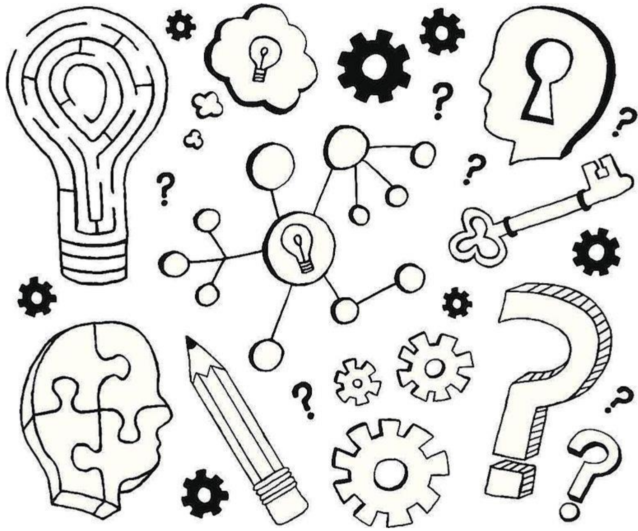
Reflection (Individual- 40')



1. What is Transformation?
2. How can T be interpreted as an operator?
3. What is linear Transformation? Give an example
4. What is nonlinear Transformation? Give an example
5. What are some application of L.T. ?
6. What are some application of N.L.T?
7. Why Matrices are useful in L.T?
8. Which of the following is a linear transformation?

   i) $Y=X^2$ ii) $Y=\mathrm{Sec}\ x$ iii) $Y= X/2$
9. Find the matrix transformation for other types of linear transformations that are important in computer science (e.g. projection).
10. Derive the matrix transformation for rotation.

Research



Read the bellow article and write a 500-words essay on the importance of Matrix transformation in computer science?

https://towardsdatascience.com/understanding-transformations-in-computer-vision-b001f49a9e61

Any Questions or Concerns?

Sources for the slides:

https://fdocuments.in/

And

https://www.xpowerpoint.com/