Mathematical Foundations of AI

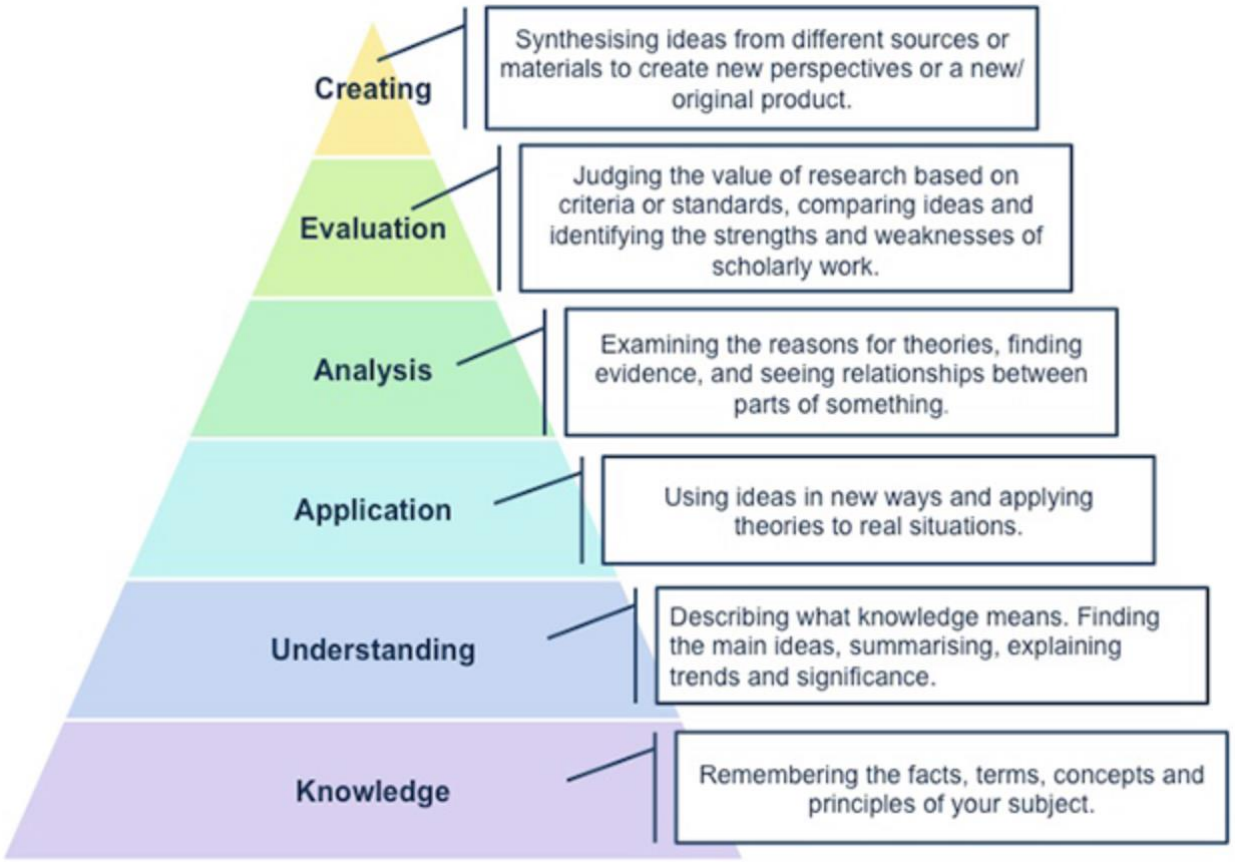
MFA501

Lecturer: Dr. James Vakilian

Welcome to MFA501

Introduction

Academic Expectations



Refer to "**Critical Thinking Guide**".

Academic Integrity

- **Plagiarism**
  - Copy from internet, book, article, video or presentation.
  - Copying another students work.
  - Simply not acknowledging the source of your research.
  - Copying your own work from a previous assignment
- **Collusion**
  - Inappropriately assisting other students in the production
    of an assessment task.
  - Submitting work that is the same, or substantially the same, as another student's piece of work for the same assessment task.
  - Assisting another student to plagiarize material
- **Ghostwriting**
  - Occurs where a person other than the student has authored, either wholly or in part, a piece of assessment.
- **Other forms**
  - Fabricated data; inventing references, quotes or sources; same work for different subjects.

Avoid plagiarism by doing extensive research and referencing where you sourced the information!!

Torrens Policy Academic Integrity Policy (and other policies): https://www.torrens.edu.au/policies-and-forms

Rules of the Class

- Punctuality
- Mobile Free Zone
- Proper learning environment : *requires your participation*

SUBJECT OVERVIEW

Intelligent Machines- Objectives

Throughout history, efforts has been made to:

1. Replace manual labor → Robots
2. Solve tedious repetitive mathematical → Calculators
3. Solve complex problems → Numerical Analysis
4. Making decisions → Predictive Analysis
5. Attempts to define consciousness → Philosophy

Intelligent Machines

What is Intelligence?

What is Intelligence?

- Ability to :

    a) Solve Complex Problems

    b) Decision making

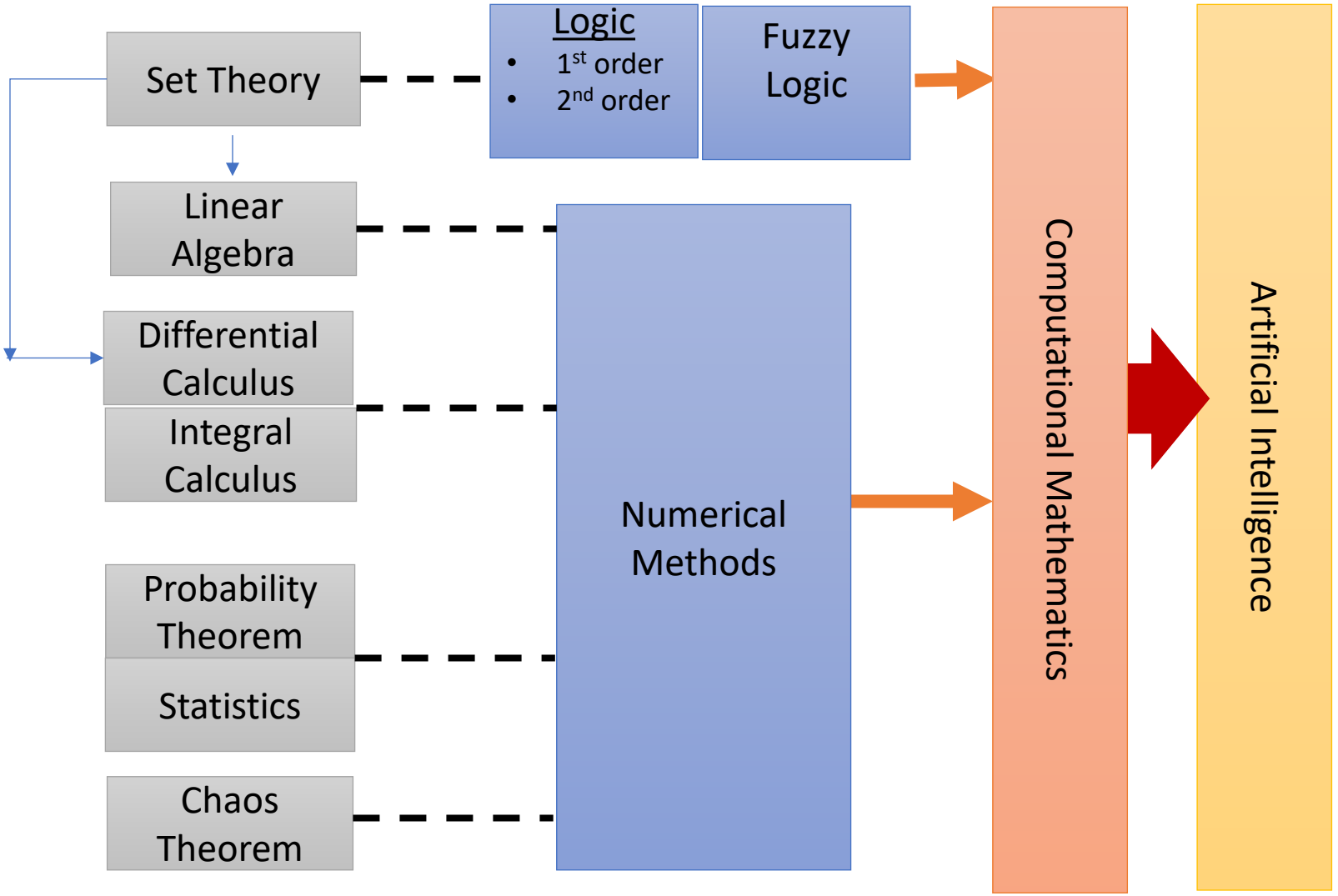
How can we make intelligent Machines?

How can we make intelligent Machines?

1. Adequate sophisticated body → Semiconductor Technology /Hardware
2. Principles of teaching machines intelligence → Logic → Mathematics →

   Mathematical foundations of AI
3. How to communicate with machine → Programing

AI & Mathematics- A Glance



Subject Overview

M.1) Introduction to Sets, Functions and Vectors

M.2) Vector Spaces, Subspaces, and Linear Transformation

M.3) Matrices

M.4) Eigenvalues and Eigenvectors

M.5) Derivatives and Integrals

M.6) Gradient and Optimisation

M.7) Introduction to Probabilities

M.8) Random Variables and Distributions

M.9) Descriptive Statistics for AI

M.10) Inferential Statistics for AI

M.11) Propositional Logic in AI

M.12) First Order Logic in AI

Assessment Overview

Assessment		Due Date	Points
Assessment 1	Online Test	Week 3	20%
Assessment 2-A	Problem Set 2A	Week 6	20%
Assessment 2-B	Problem Set 2B	Week 10	20%
Assessment 3	Solve an AI Problem set (Project)	Week 12	40%

Module 1:

Introduction to sets, functions and vectors

Today's Outline

I. Introduction to Set Theory

II. Subsets

III. Some Operations in Set Theory

IV. Undefined Concepts and Program Syntax

V. Tutorial

Logical Fallacies in Computer Science

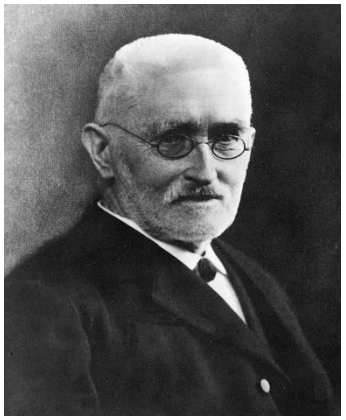
One type of Logical Fallacy(s)

Undefined concepts

Set Theory

Origins of set theory

George Cantor



Dedikind

Gödel

- **Set theory:** a branch of mathematical logic that studies sets.
- Sets are building blocks of symbolic logic.
- The origins of set theory → ancient Greece *(such as Zeno).*
- The modern study of set theory → by Georg Cantor and Richard Dedekind (1870s)
- Sets play an important role in *advanced courses*, such as groups, rings, fields, vector spaces, topologies; etc.

What is a Set?

- Set → A collection of objects.
  a. A carpenter's tool box → set.
  b. All presidents of USA → set.
  c. Your ten best friends → set.
  d. All the programming languages → Set.
- We understand sets intuitively.
- The objects in a set may be similar or different.
- A set may contain finite number of objects → Finite Sets
- A set might contain infinite number of objects → Infinite Sets

What is a Set? cont

A ***set*** is a collection of items which have a common trait.

Sets are always written using

- a Capital Letter or
- the { } (braces) notation with the elements or their description in the braces.

- Example 1) A = {1,3,5,7,9,…} → Set of Odd numbers
- Example 2) B = {2,4,6,8,10,…} → Set of Even numbers
- Example 3) C = {1,2,3,5,7,11,…} → Set of Prime numbers
- Example 4) D = {0,1} → Set of Binary Numbers

Elements of a Set

- An element (of a set): any one of the distinct objects that make up that set.

- **The elements** in a particular set must be well-defined.
  - The collection of tall boys → Not a set.
  - The collection of good movies → Not a set.

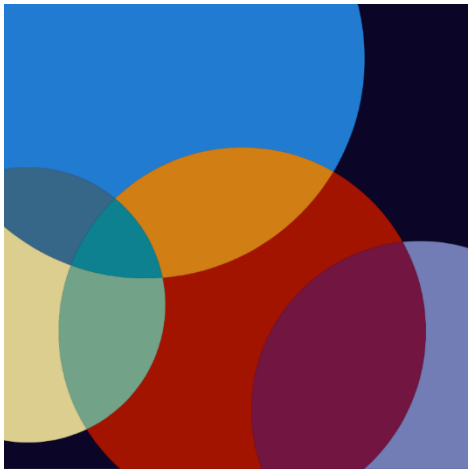
Elements of a Set cont

- If a is an element of a set A, we write: $a \in A$
- If a is not an element of a set A, we write: $a \notin A$
- ∈: Is read belongs to / is a member of set …
- Example: if A={a,b,c} → a, b & c are elements of A →

✓ a ∈ A

✓ b ∈ A

✓ c ∈ A



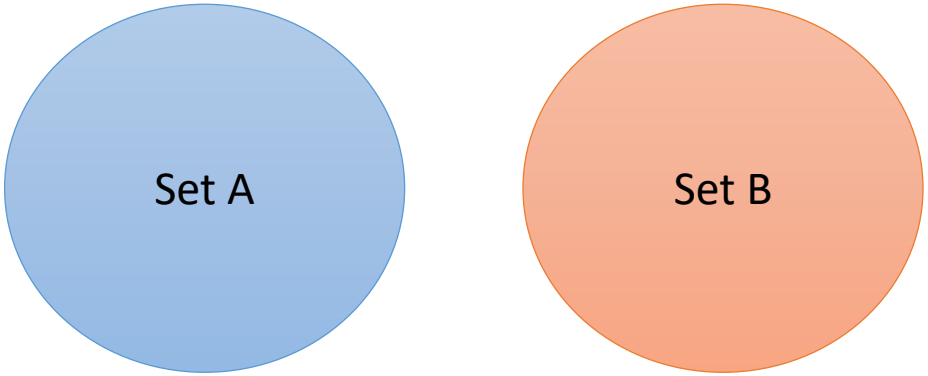
Number of Elements of a Set

n(A) : Number of elements in the set A.

- The elements of A = {1,2,3,4,5} are 1, 2, 3, 4 and 5 → n(A) = 5
- The elements of B = {1,2,3,5,7,11,13,17,19,…} are prime numbers. → Infinite Set
- {x | x=2k (K is an integer); and $1 \leq x \leq 100$} is the set of even numbers less then 100. → This set has a finite number of elements. n(A) = 50

Venn Diagram

- Venn diagram: A Tool
- Venn diagram: represent relationships between sets.
- They use overlapping circles *(or other shapes)* to **illustrate** the logical relationships between two or more sets of items.
- Often, they serve to graphically organize things, highlighting how the items are similar and different.



Null Sets

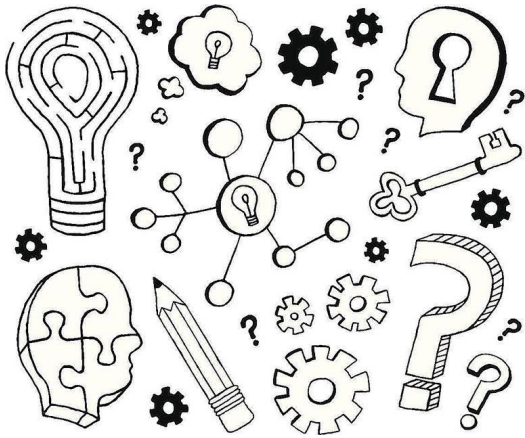
Null Set:

- A set that does not contain any element whatsoever.
- Notation: symbol ϕ or { }
- It contains no member at all.
- Even 0 or ϕ are not elements of a null set.
- *e.g. the set of integers between 4 and 5 is a null set.*

Universal Sets

Universal Sets

- A set which **contains** all objects.
- It even includes itself.
- Everything belongs to this set
- Notation: symbol U or E (usually)
- *e.g. Complex real numbers Z= ai+b, are the universal set for every existing number.*



Activity 1), 10’

Identify five different sets.

1. How many elements does each have? Are they well-defined?
2. What are its elements?
3. Give an example of a null set.
4. Give an example of a universal set.

Today's Outline

I. ~~Introduction to Set Theory~~

II. Subsets

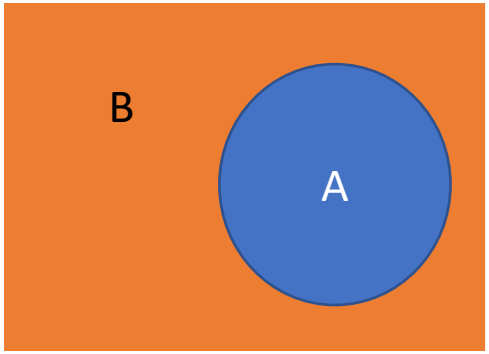
III. Some Operations in Set Theory

IV. Undefined Concepts and Program Syntax

V. Tutorial

What are Subsets?

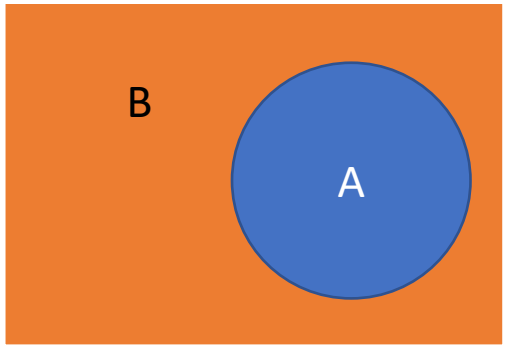
- set A is a subset of another set B → all elements of the set A are elements of the set B.
- In other words, the set A is contained inside the set B.
- The subset relationship is denoted as $A \subset B$.
- $A \subset B$ → every member of A is a member of B.



What are Subsets?

Example)

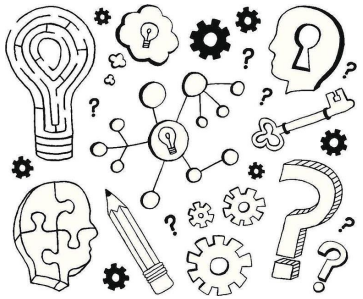
- If B = {1,2,3,4,5}
- A = {1,2,3} is a subset of B → $A \subset B$
- Other subsets of B include {2,3} or {1,4,5} or {4}, etc...
- But {1,2,6} is NOT a subset of B as it has 6 (which is not in B)



Important Notes

- Every set is a **subset** of itself. $\forall S \subset S$
- Empty set is a subset of any set. $\emptyset \subset \forall S$
- Empty set is a subset of itself. $\emptyset \subset \emptyset$
- Every set is a **subset** of the universal set. $\forall S \subset U$

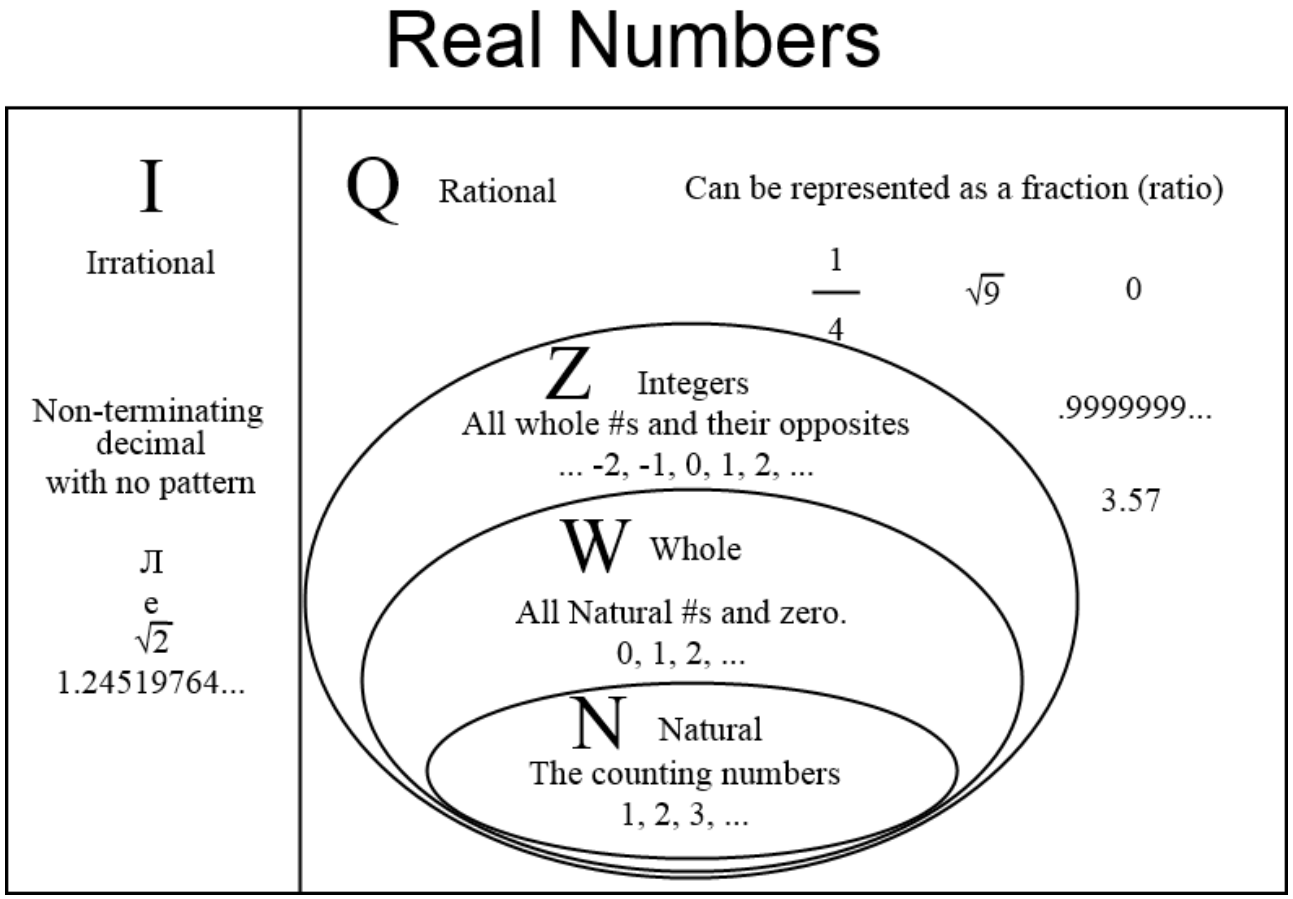
Activity 2(Individual, 10')



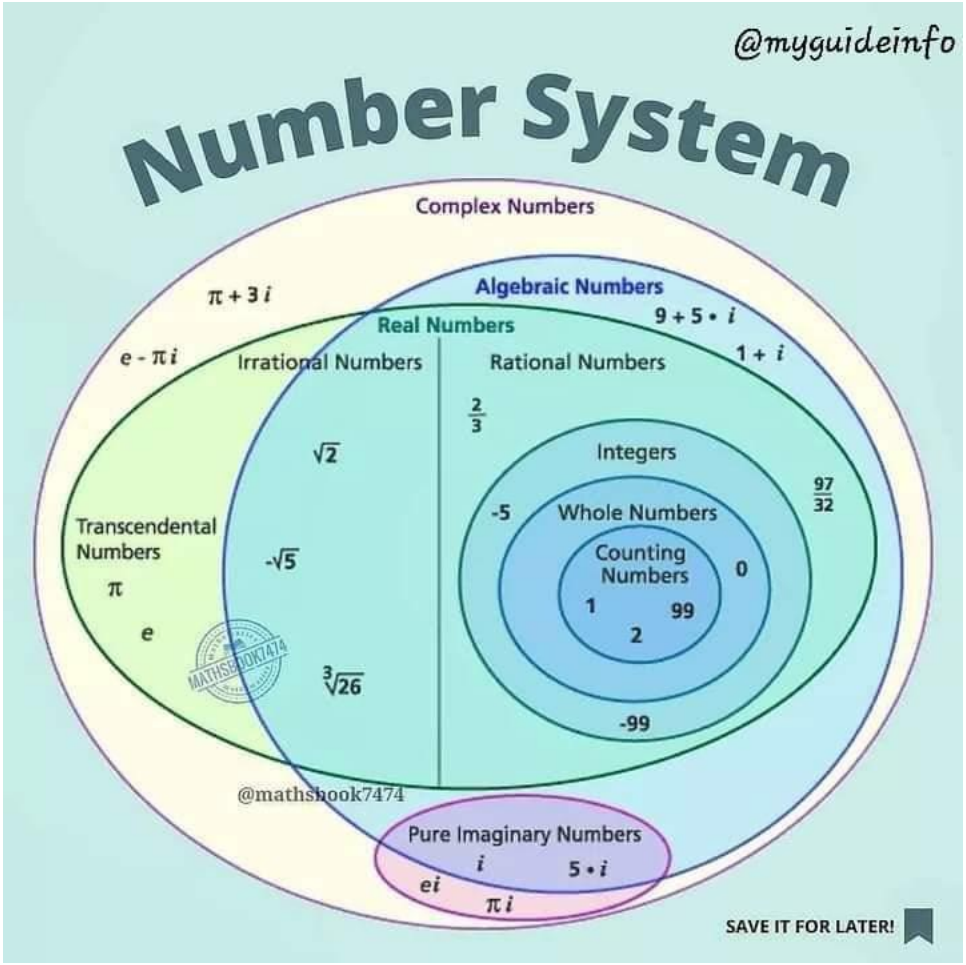
- Demonstrate the relations between the following sets in one Venn diagram.

1. Natural Numbers → N
2. Whole Numbers → W
3. Integers → Z
4. Rational Numbers → Q
5. Irrational Numbers → I
6. Real Numbers → R

Conceptual Activity



Conceptual Activity



Break- 20’

Today's Outline

I. ~~Introduction to Set Theory~~

II. ~~Subsets~~

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V. Tutorial

New Sets from Old

- Power Set
- Complement of a set
- Union
- Intersection
- Difference

Power Set

- Power set of any set A, is the set of all subsets of A.
- Power sets always include the empty set and itself.

If A has n members, then pow(A) has 2^n members

$$\text{pow}(A) ::= \{S | S \subseteq A\}$$

Example.

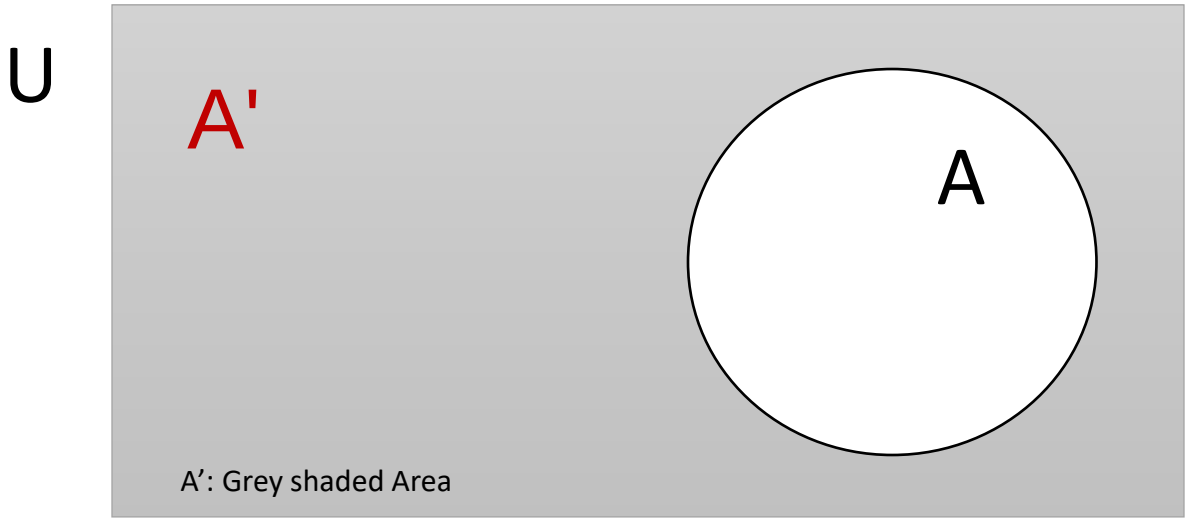
- If *A={a,b},*
- Then pow (A) has (2^2)=4 members
- That is:

$$\text{pow}(\{a, b\}) = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$$

Complement of a Set

Complement of a Set: A': is the set of all elements in the given universal set U that are not in A.

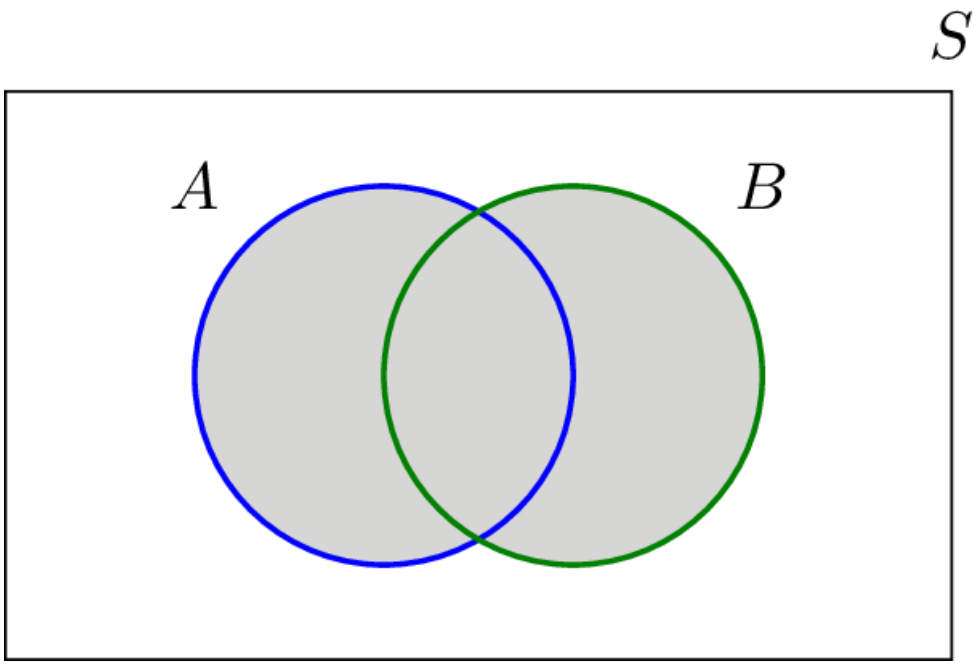
$$A' = \{x \in U \mid x \notin A\}$$



Union of Sets

Union of two sets is the set of elements which are in A and in B, or in both A and B

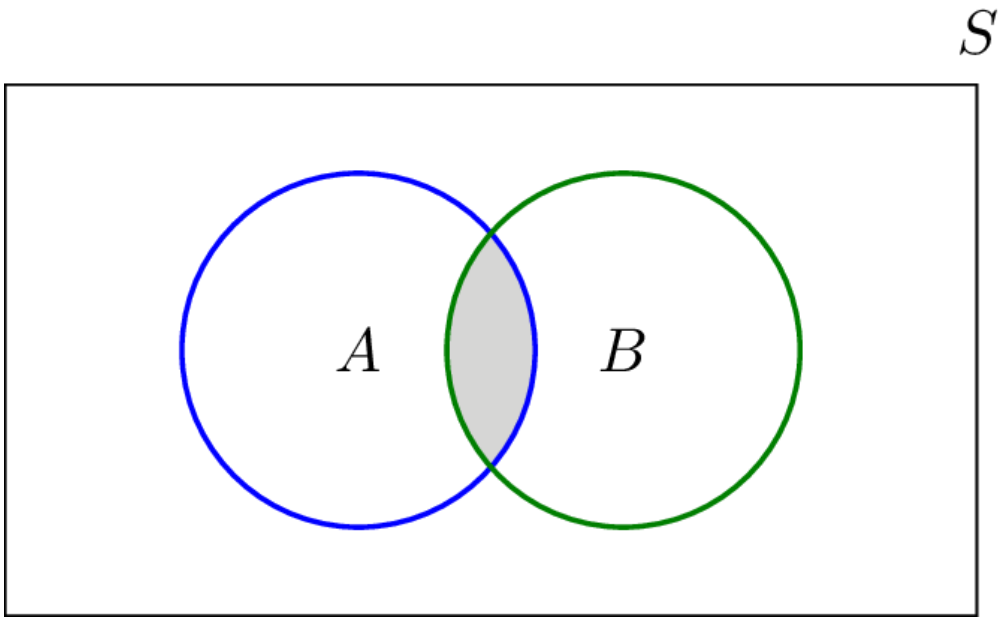
$$A \cup B ::= \{x | (x \in A) \vee (x \in B)\}$$



Intersection of Sets

Intersection of sets, is the set containing all elements of A that also belong to B (or equivalently, all elements of B that also belong to A).

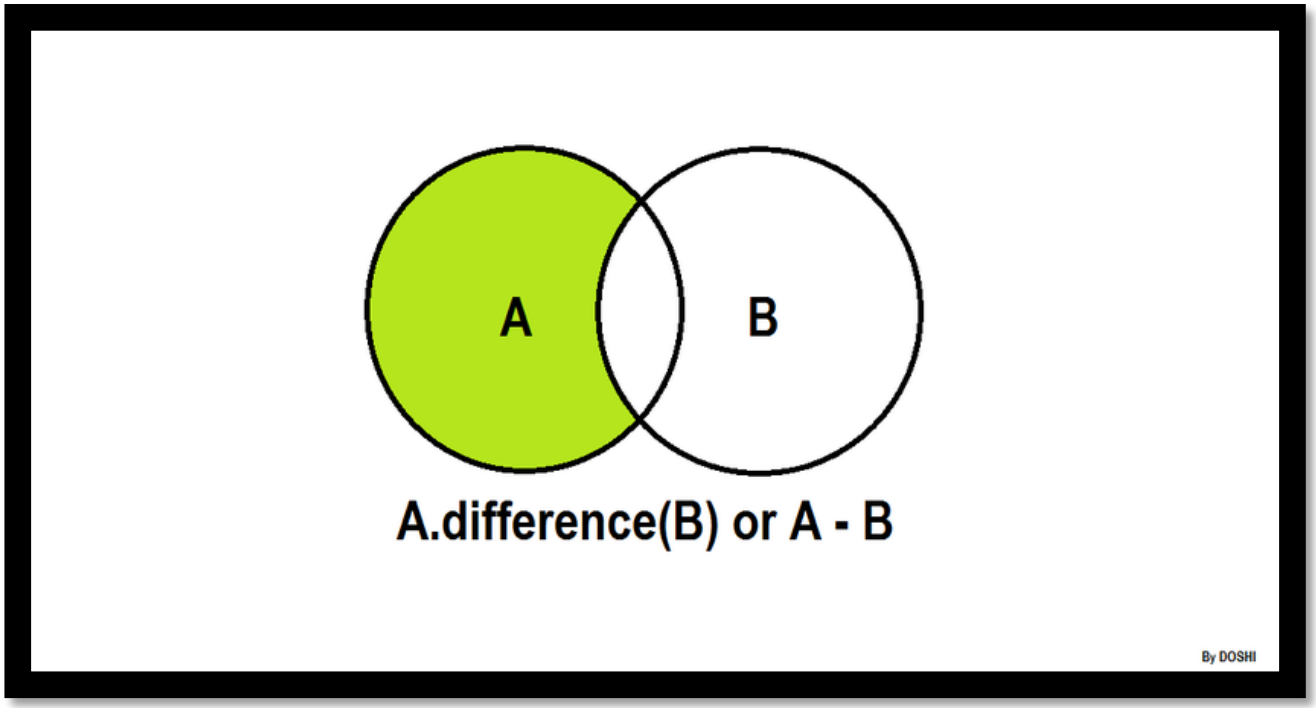
$$A \cap B::=\{x|x \in A \wedge x \in B\}$$



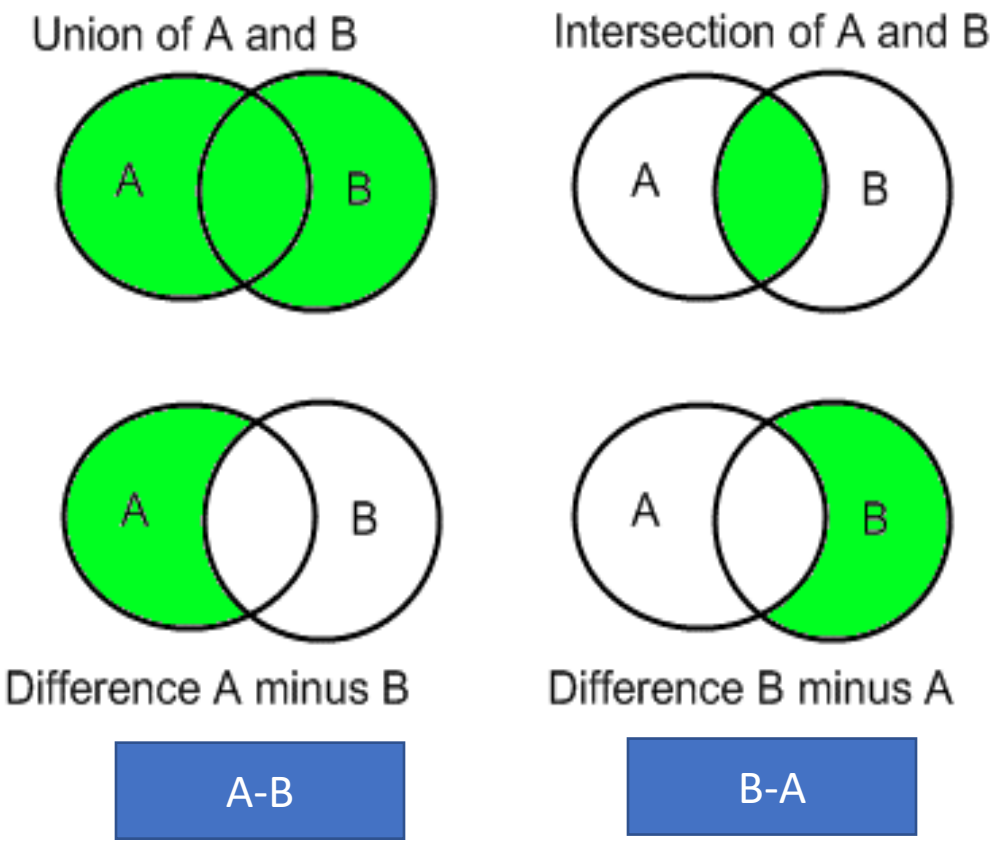
Difference of Sets

Difference of two sets, written A - B is the set of all elements of A that are not elements of B.

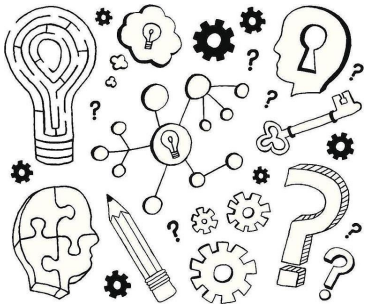
$$A - B ::= \{x | (x \in A) \wedge (x \notin B)\}$$



In a Glance


Union of A and B
A
B
Intersection of A and B
A
B
A
B
Difference A minus B
A
B
Difference B minus A
A-B
B-A

Activity 3), 10’

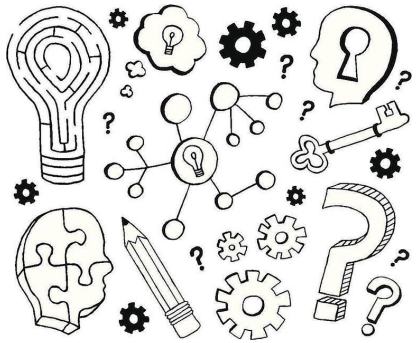


a) Using Venn diagrams, show that **Union** and **intersection** operations are i) commutative , ii) associative and iii) distributive.

i. $A \cup B = B \cup A$,

ii. $A \cup (B \cup C) = (A \cup B) \cup C$,

iii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



b) Using Venn diagrams, show that Union and intersection operations are i) commutative , ii) associative and iii) distributive.

i. $A \cap B = B \cap A$,

ii. $A \cap (B \cap C) = (A \cap B) \cap C$,

iii. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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Example of Undefined concepts

Example1:

- *Your house is in front of mine*
- *My house is in front of yours*
  - *Does that make a valid reference?*

Undefined concepts

Example 2:

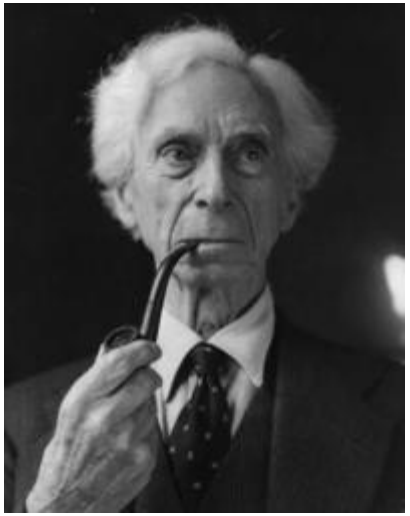
- *If you are tall*
- *And I am tall*
  - *who is taller?*

Undefined concepts:

- Every concept in mathematics → should be well defined.
- However, most disciplines in mathematics have undefined concepts → well understood term and fundamental, but undefined.
- The concept of 'set' is quite fundamental and commonly understood but they are undefined.
- Attempt to define the concept 'set' ends in 'Russell's paradox' explained next.
- Never-the-less, members of a set must be well defined.

Russell's Paradox

- There is a village in which every one should be shaved.
  - There was only one barber in a village
  - who shaved those who did not shave themselves
  - and only those.
    - Can you tell whether the barber shaved 'himself' or not ?



I. Suppose he shaved himself. → But he shaved those who did not shave themselves.
Suppose he did not shave himself → But he shaved those who did not shave themselves. So he would shave himself !

- Either way we have reached a logical contradiction → Paradox
- In short, there should have been no statement like this at all.

This is NOT a self-Consistent statement

Today's Outline

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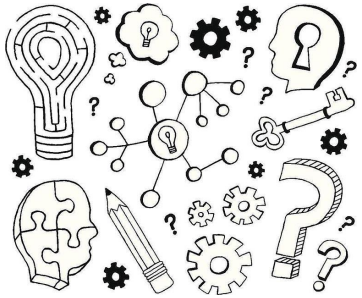
Reflection (Individual, 40')



1. What is a set ?
2. What is an element of a set?
3. Give 3 examples of sets. What are their elements?
4. Is the order of elements of a set important? (i.e. {a,b}={b,a})
5. What are venn diagrams?
6. Give an example of a null set.
7. Is 0 an element of ϕ?
8. Is ϕ an element of ϕ?
9. What is a universal set? Give an example of a universal set.
10. Is the collection of good people a set? How many elements does it have?
11. A= {x|x=2k+1 $(1 < k < 1000)$ },
    - Is "A" a set? What is n(A)?

8. A= {x| x= $\ln \alpha (\alpha \epsilon \mathbb{N})$ }
    - Is "A" a set?, What is n(A)?

9. What is a Power set ?
10. If A={1,2,3,4} what is the power set of A?
11. What is the complement of a set?
12. If A={1,2,3,4} & B={1,3,5,7}, identify the following sets & draw its Venn diagram.
    - $A \cup B$
    - $A \cap B$
    - A-B
    - B-A

9. If O is the set of odd numbers & E is the set of Even numbers what are:
    - $A \cup B$
    - $A \cap B$
    - A-B
    - B-A
10. Investigate if A-B is equal to B-A, using Vann diagram?
11. What is Russell paradox?
12. Give an example of undefined concepts.
13. If a program syntax is not well-defined, what would be the outcome of its execution?

Problem 1)

For each of the following sets, shade the named region.

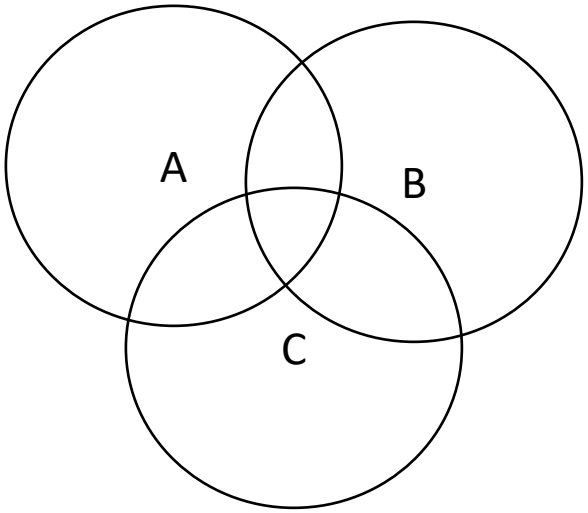
a) $A - (B \cap C)$

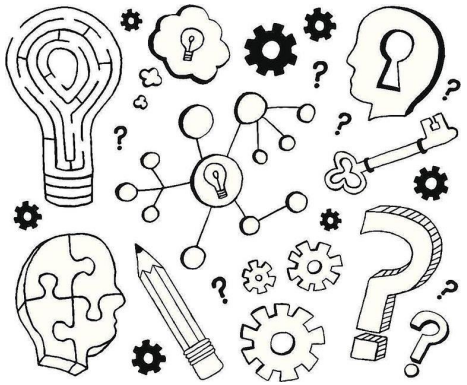
b) $A\text{- }(B\text{-}C)$

c) $B\text{- }(A\text{-}C)$

d) $(B \cap C) \cap (B \cup A)$

e) $(A\text{-}B) \cup (A - C)$

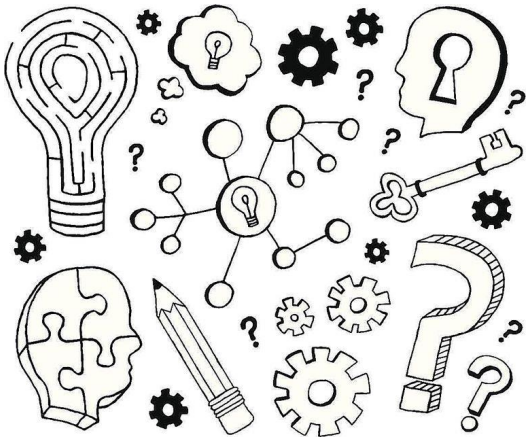




Problem 2)

G = (A – B) ∪ (B – A) = A Δ B is called symmetric difference **of A and B**. Using Venn diagrams show that:

$$A \Delta B = (A \cup B) - (A \cap B)$$

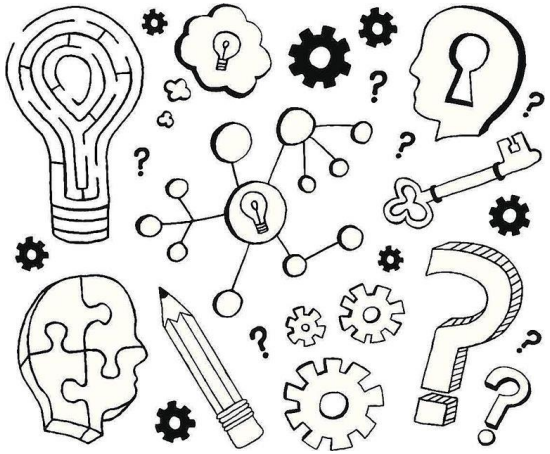


Problem 3)

Find some universal set $\mathbf{U}$ and proper subsets $S,T \subseteq \mathbf{U}$, (i.e., what are the elements of S, T, U)

Such that the following holds for these sets.

- $|S-T|=3$
- $|T-S|=1$
- $|S \cap T| = 6$
- $|\bar{s}|= 1,2$



Next Class

Vectors, Vector Spaces and Subspaces

Come prepared!

The slides resource:

https://www.wiziq.com/tutorial/350845-Chapter-2-set-theory-relations-and-functions-ppt

https://saintmaurigcsemath.files.wordpress.com/2015/01/igcsemath9a1ch8setsvectorsfunctions-notes2.pdf

https://www.slideserve.com/

Thomas Calculus (11th Edition)

Any Questions or Concerns?