CHAPTER 4

LINEAR TRANSFORMATIONS

4.1 Introduction

In this chapter, we shall consider certain mappings between two vector spaces called linear transformation. If we consider the vector spaces R^2 and R^3 and the mapping

$f: R^2 \to R^3$ defined by $f(x, y) = (x - y,\ y - x,\ -x)$, then we observe that

(i) $$\begin{aligned} f((a, b) + (c, d)) &= f(a + c, b + d) \\ &= (a + c - b - d,\ b + d - a - c, - a - c) \\ &= (a - b + c - d,\ b - a + d - c, - a - c) \\ &= (a - b,\ b - a,\ - a) + (c - d, d - c, - c) \\ &= f(a, b) + f(c, d), \end{aligned}$$

where $(a, b), (c, d) \in R^2$

(ii) For any $\alpha \in R$ and $(a,b) \in R^2$,

$$\begin{aligned} f(\alpha(a, b)) &= f(\alpha a, \alpha b) \\ &= (\alpha a - \alpha b, \alpha b - \alpha a, - \alpha a) \\ &= \alpha (a - b, b - a, - a) \\ &= \alpha\, f(a, b) \end{aligned}$$

Thus, we have defined a function f between two vector spaces such that (i) and (ii) holds good.

In other words, $f: R^2 \to R^3$ is a linear map if it preserves the two basic operations of a vector space i.e., vector addition and scalar multiplication.

Taking the above facts into consideration, we define linear transformation betwen two vector spaces.

4.2 Linear Transformation :

4.2.1 Definition : Let U and V be two vector spaces. A mapping $T : U \to V$ is called a linear mapping (or linear transformation) if it satisfies the following two conditions :

(1) For any $u_1, u_2 \in U$, $T(u_1 + u_2) = T(u_1) + T(u_2)$

and (2) For any $u \in U$, and α being a scalar, $T(\alpha u) = \alpha T(u)$.

Note : (i) A mapping $T : U \to U$ is called a linear mapping on U.

(ii) Conditions (1) and (2) can be replaced by the single condition

$$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2), \ \alpha, \beta \text{ being scalars.}$$

Proof : $u_1 \in U$, α being scalar

$\Rightarrow \alpha u_1 \in U$

$u_2 \in U, \beta$ being scalar

$\Rightarrow \beta u_2 \in U$

Now, $T(\alpha u_1 + \beta u_2)$

$= T(\alpha u_1) + T(\beta u_2)$ [by (1)]

$= \alpha T(u_1) + \beta T(u_2)$ [by (2)]

Example 4.2. 1. Let $T : V_1 \to V_3$ be defined by T (a) = (a, 2a, 3a), for all $a \in V_1$. Show that T is a linear transformation.

Solution : Let $x, y \in V_1$ and α be a scalar.

$\therefore T(x) = (x, 2x, 3x)$

$T(y) = (y, 2y, 3y)$

(i) $T(x + y)$

$= ((x + y), 2(x + y), 3(x + y))$

$= (x, 2x, 3x) + (y, 2y, 3y)$

$= T(x) + T(y)$

(ii) $T(\alpha x) = (\alpha x, 2\alpha x, 3\alpha x)$

$= \alpha (x, 2x, 3x)$

$= \alpha T(x)$.

Since T satisfies (1) and (2), therefore T is linear.

Example 4.2.2. Let $T : V_2 \to V_2$ be defined by T (a,b) = (2a + 3b, 3a – 4b)

Show that T is linear.

Solution : $x, y \in V_2$, α be a scalar.

Suppose : $x = (a_1, b_1)$ and $y = (a_2, b_2)$

$\therefore x + y = (a_1 + a_2, \ b_1 + b_2)$

$\alpha x = (\alpha a_1, \alpha b_1)$

(i) $T(x+y) = T(a_1+a_2,\ b_1+b_2)$
$= (2(a_1+a_2)+3(b_1+b_2), 3(a_1+a_2)-4(b_1+b_2))$
$= (2a_1+2a_2+3b_1+3b_2, 3a_1+3a_2-4b_1-4b_2)$
$= (2a_1+3b_1,\ 3a_1-4b_1)+(2a_2+3b_2, 3a_2-4b_2)$
$= T(x)+T(y)$

(ii) $T(\alpha x) = T(\alpha a_1, \alpha b_1)$
$= (2\alpha a_1+3\alpha b_1, 3\alpha a_1-4\alpha b_1)$
$= \alpha(2a_1+3b_1,\ 3a_1-4b_1)$
$= \alpha T(x)$

$\therefore$ T is linear.

Example 4.2.3. Let $T: V_3 \to V_3$ be defined by $T(a,b,c)=(a^2+ab, ab, bc)$. Is T linear ?

Solution : Let $x, y \in V_3$.

Suppose $x=(1,0,0)$
and $y=(2,0,0)$

$\therefore$ $x+y=(3,0,0)$

$T(x+y)=T(3,0,0)$
$=(3^2+3.0, 3.0, 0.0)$
$=(9,0,0)$

and $T(x)+T(y)=T(1,0,0)+T(2,0,0)$
$=(1^2+1.0, 1.0, 0.0)+(2^2+2.0, 2.0, 0.0)$
$=(1,0,0)+(4,0,0)$
$=(5,0,0)$

$\therefore$ $T(x+y)\neq T(x)+T(y)$

Hence T is not linear.

Example 4.2.4 : Let $T: P \to P$ be defined by $T(P)=p'$ where P is the set of polynomial function. Show that T is linear.

Solution : Let $p, q \in P$ and α be a scalar.

(i) $T(p+q)=(p+q)'=p'+q'=T(p)+T(q)$

(ii) $T(\alpha p)=(\alpha p)'=\alpha p'=\alpha T(p)$

Hence T is linear.

Example 4.2.5 : Let U and V be vector spaces defined over the same field of scalars and T maps from U to V. Then prove that T is linear if and only if $T(\alpha u_1+u_2)=\alpha T(u_1)+T(u_2)$ for all $u_1, u_2 \in U$ and scalar α.

Solution : Suppose

$T(\alpha u_1+u_2)=\alpha T(u_1)+T(u_2)$ for all $u_1, u_2 \in U$ and scalar α.

Now $T(u_1+u_2)=T(1.u_1+u_2)$
$= 1.T(u_1)+T(u_2)$
$= T(u_1)+T(u_2)$

for $u \in U$,

$$\begin{aligned} T(\alpha u) &= T(\alpha u + 0_U) \\ &= \alpha T(u) + T(0_U) \\ &= \alpha T(u) + 0_V \\ &= \alpha T(u) \qquad [\because v + 0 = v] \end{aligned}$$

$\therefore$ T is a linear transformation

Again, Let T be a linear transformation.

$$\begin{aligned} \therefore \quad & T(\alpha u_1 + u_2) \\ &= T(\alpha u_1) + T(u_2) \qquad [\text{By (1)}] \\ &= \alpha T(u_1) + T(u_2) \qquad [\text{By (2)}] \end{aligned}$$

Thus T is linear iff $T(\alpha u_1 + u_2) = \alpha T(u_1) + T(u_2)$

4.3 Some Important Results of Linear Operator :

We shall establish the following operator results :

Let $T : U \to V$ be a linear map. Then

(i) $T(\mathbf{0}_U) = \mathbf{0}_V$ where $\mathbf{0}_U$ is the zero vector of U and $\mathbf{0}_V$ is the zero vector of V.

(ii) $T(-u) = -T(u),\ u \in U$

(iii) $T(u - v) = T(u) - T(v),\ u, v \in U$

(iv) $T(\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \ldots + \alpha_n T(u_n)$, where $\alpha_i\ (i = 1, 2, \ldots, n)$ are scalars.

Proof :

(i) Let $x \in U$ so that $T(x) = x' \in V$

Now $x + \mathbf{0}_U = x$, where $\mathbf{0}_U \in U$ and $x' + \mathbf{0}_V = x'$, where $\mathbf{0}_V \in V$

$$T(x + \mathbf{0}_U) = T(x) + T(\mathbf{0}_U) \qquad [\because T \text{ is linear}]$$

$$\begin{aligned} &\Rightarrow T(x) = T(x) + T(\mathbf{0}_U) \qquad [\because x + \mathbf{0}_U = x] \\ &\Rightarrow x' = x' + T(\mathbf{0}_U) \\ &\Rightarrow x' + \mathbf{0}_V = x' + T(\mathbf{0}_U) \qquad [\because x' + \mathbf{0}_V = x'] \\ &\Rightarrow T(\mathbf{0}_U) = \mathbf{0}_V. \end{aligned}$$

(ii)
$$\begin{aligned} T(-u) &= T((-1)u) \\ &= (-1)T(u) \qquad [\because T \text{ is linear}] \\ &= -T(u) \end{aligned}$$

(iii)
$$\begin{aligned} T(u - v) &= T(u + (-v)) \\ &= T(u) + T(-v) \qquad [\because T \text{ is linear}] \\ &= T(u) - T(v) \qquad [\text{By (ii)}] \end{aligned}$$

(iv) $T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n)$, where α_i's are scalars and $u_i \in U$, $i = 1, 2, ..., n$.

To prove by method of induction

For $n = 1$, $T(\alpha_1 u_1) = \alpha_1 T(u_1)$, is true.

For $n = 2$, $T(\alpha_1 u_1 + \alpha_2 u_2) = T(\alpha_1 u_1) + T(\alpha_2 u_2) = \alpha_1 T(u_1) + \alpha_2 T(u_2)$

Assume that the statment is true for $n = k$ i.e.,

$$T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_k u_k) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_k T(u_k)$$

Now $T(\alpha_1 u_1 + ... + \alpha_k u_k + \alpha_{k+1} u_{k+1})$

$= T(u + \alpha_{k+1} u_{k+1})$ where $u = \alpha_1 u_1 + ... + \alpha_k u_k$

$= T(u) + T(\alpha_{k+1} u_{k+1})$ [$\because$ the statement is true for $n = 2$]

$= T(\alpha_1 u_1 + ... + \alpha_k u_k) + \alpha_{k+1} T(u_{k+1})$

$= \alpha_1 T(u_1) + ... + \alpha_k T(u_k) + \alpha_{k+1} T(u_{k+1})$

[$\because$ the statement is true for $n = k$]

$\therefore$ The statement in true for $n = k+1$.

Hence, by axiom of induction, the given statment is true for all $n \in N$..

Thorem 4.3.1. Let S and T be two linear transformations from U to V.

Let $B = \{u_1, u_2, ..., u_n\}$ be a basis for U.

If $S(u_i) = T(u_i)$ for $i = 1, 2, ..., n$,

then $S(u) = T(u)$, for all $u \in U$.

Proof : Let $u \in U$.

Since B is a basis for U, therefore u can be expressed uniquely as a linear combination of the elements of B i.e.,

$$u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n,$$

where α_i's are scalars, for i = 1, 2, ...n.

Now, $S(u) = S(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$

$= \alpha_1 S(u_1) + \alpha_2 S(u_2) + ... + \alpha_n S(u_n)$ [$\because$ S is linear]

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n)$ [Hypothesis]

$= T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$ [$\because$ T is linear]

$= T(u)$

Theorem 4.3.2. Let Let $\{u_1, u_2, ..., u_n\}$ be a basis for U and let $v_1, v_2, ..., v_n$ be any n vectors in V, then there exists a unique linear transformation. $T : U \to V$ such that

$$T(u_i) = v_i, \; i = 1, 2, ..., n \qquad ...(1)$$

Proof : Let $u \in U$.

Since $\{u_1, u_2, ..., u_n\}$ is a basis for U, therefore

$u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$, where α_i 's are scalars, and this expression is unique.

Define $T(u) = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$ ...(2)

Inorder to complete the proof, we must show that :

(i) T satisfies (1)

(ii) T is unique

(iii) T is linear.

We have, $u_i = 0.u_1 + 0.u_2 + ... + 0.u_{i-1} + 1.u_i + 0.u_{i+1} + ... + 0.u_n$

$$\Rightarrow T(u_i) = 0.v_1 + 0.v_2 + ... + 0.v_{i-1} + 1.v_i + 0.v_{i+1} + ... + 0.v_n$$

$$= 1.v_i = v_i$$

This Proves (i)

Let $S : U \to V$ be another linear transformation such that $S(u_i) = v_i$ for all i.

$$\therefore S(u) = S(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$$

$$= \alpha_1 S(u_1) + \alpha_2 S(u_2) + ... + \alpha_n S(u_n)$$

$$= \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$$

$$= T(u), \text{ for all } u \in U \qquad \text{[by (2)]}$$

$$\Rightarrow S = T$$

Hence T is unique. This proves (ii)

Let $u, v \in U$

$$\therefore u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$$

and $v = \beta_1 u_1 + \beta_2 u_2 + ... + \beta_n u_n$

$$u + v = (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + ... + (\alpha_n + \beta_n) u_n$$

$$\alpha u = (\alpha\alpha_1) u_1 + (\alpha\alpha_2) u_2 + ... + (\alpha\alpha_n) u_n$$

Now $T(u + v) = (\alpha_1 + \beta_1) v_1 + (\alpha_2 + \beta_2) v_2 + ... + (\alpha_n + \beta_n) v_n$ [By (2)]

$$= (\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n) + (\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n)$$

$$= T(u) + T(v)$$

Again $T(\alpha u) = (\alpha\alpha_1) v_1 + (\alpha\alpha_2) v_2 + ... + (\alpha\alpha_n) v_n$

$$= \alpha(\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n)$$

$$= \alpha T(u) \qquad \text{[By (2)]}$$

Since $T : U \to V$ satisfies

$$T(u + v) = T(u) + T(v)$$

and $T(\alpha u) = \alpha T(u)$, therefore T is linear. This proves (iii)

Hence the theorem is completely proved.

Note : Any linear transformation $T : U \to V$ is completely determined by its values on a basis for U. i.e., T exists if U has a basis.

The following examples will be useful for better understanding of the above theorem.

Example 4.3.1 Find a linear transformation

$$T : V_2 \to V_2 \text{ (if exists)}$$

defined by $T(2, 3) = (4, 5)$ and $T(1, 0) = (0, 0)$

Solution : Let $B = \{u_1, u_2\}$ where $u_1 = (2, 3)$ and $u_2 = (1, 0)$

To show that B is a basis for V_2 and $[B] = V_2$.

$$\begin{aligned}
&\text{Let } \alpha u_1 + \beta u_2 = \mathbf{0} \\
&\Rightarrow \alpha(2, 3) + \beta(1, 0) = \mathbf{0} = (0, 0) \\
&\Rightarrow (2\alpha + \beta,\ 3\alpha + 0) = (0, 0) \\
&\Rightarrow 2\alpha + \beta = 0 \text{ and } 3\alpha = 0 \\
&\Rightarrow \alpha = 0,\ \beta = 0
\end{aligned}$$

$\therefore$ (2, 3) and (1, 0) are L.I i.e., $B = \{(2, 3), (1, 0)\}$ is a basis for V_2.

Further, Let $(x, y) \in V_2$

$$\begin{aligned}
&\therefore\ (x, y) = a(2, 3) + b(1, 0) \\
&\Rightarrow (x, y) = (2a + b,\ 3a) \\
&\Rightarrow 2a + b = x,\ 3a = y \\
&\Rightarrow a = \frac{y}{3},\ b = x - \frac{2y}{3} = \frac{3x - 2y}{3} \\
&\therefore\ (x, y) = \frac{y}{3}(2, 3) + \frac{3x - 2y}{3}(1, 0) \\
&\Rightarrow [B] = V_2
\end{aligned}$$

$\therefore$ every element of V_2 can be expressed uniquely as a linear combination of the elements of B.

Hence

$$\begin{aligned}
T(x, y) &= T\left(\frac{y}{3}(2, 3) + \frac{3x - 2y}{3}(1, 0)\right) \\
&= \frac{y}{3} T(2, 3) + \frac{3x - 2y}{3} T(1, 0) \\
&= \frac{y}{3}(4, 5) + \frac{3x - 2y}{3}(0, 0) \\
&= \left(\frac{4y}{3}, \frac{5y}{3}\right)
\end{aligned}$$

Example 4.3.2 Find a linear map $T : V_3 \to V_2$ defined by

$T(e_1) = (1, 2),\ T(e_2) = (2, 3)$ and $T(e_3) = (3, 4)$

where $e_1 = (1, 0, 0),\ e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

Solution : We know that

$$B = \{e_1, e_2, e_3\} \text{ is a basis for } V_3$$

Therefore any vector $(x, y, z) \in V_3$ can be expressed as a linear canbination of the element of B.

$$\therefore (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$
$$= x\,e_1 + y\,e_2 + z\,e_3$$
$$\therefore \; T(x, y, z)$$
$$= T(x\,e_1 + y\,e_2 + z\,e_3)$$
$$= x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$$
$$= x(1, 2) + y(2, 3) + z(3, 4)$$
$$= (x + 2y + 3z,\; 2x + 3y + 4z)$$

is the required linear transformation.

Example 4.3.3 Prove that a linear transformation on one dimensional vector space is the multiplication by a fixed scalar.

Solution : Let V be a vector space.

Let $T : R \to V$ be a linear transformation.

Take $\{1\}$ as a basis for R.

Let $T(1) = v \in V$

For $\alpha \in R$,

$$T(\alpha) = \alpha T(1) = \alpha v$$

Alternatively : Let : $T : V \to V$ be a linear transformation on a 1-dimensional vector space V. Let $\{e\}$ be a basis for V and T (e) be a basis for V and $T(e) = \alpha e$, where α is a scalar.

If $v \in V$ then

$v = \beta e$, where β is a scalar.

Hence $T(v) = T(\beta e) = \beta\,T(e)$
$$= \beta(\alpha e)$$
$$= (\beta\alpha)\,e$$
$$= \gamma\,e, \quad \text{where } \gamma = \beta\alpha \text{ is a scalar.}$$

$\therefore$ A linear transformation on a 1-dimensional vector space V is nothing but multiplication by a fixed scalar.

Example 4.3.4. : Find a linear (if exists) map $T : V_3 \to V_3$, Such that

$$T(0, 1, 2) = (3, 1, 2) \text{ and}$$
$$T(1, 1, 1) = (2, 2, 2)$$
$$T(1, 0, 1) = (1, 0, 4)$$

Solution : Let u = (0, 1, 2)

v = (1, 1, 1)

w = (1, 0, 1)

To prove that u, v and w are L.I.

Let $a(o, 1, 2) + b(1, 1, 1) + c(1, 0, 1) = (0, 0, 0)$

$\Rightarrow (b + c, a + b, 2a + b + c) = (0, 0, 0)$

$\Rightarrow b + c = 0, a + b = 0, 2a + b + c = 0$

$\Rightarrow a = 0, b = 0, c = 0$

Thus (0, 1, 2) (1, 1, 1) and (1, 0, 1) are L.I

Again to show that

$\{(0, 1, 2)(1, 1, 1), (1, 0, 1)\}$ spans V_3.

Let $(x_1, x_2, x_3) \in V_3$

Let $(x_1, x_2, x_3) = a(0, 1, 2) + b(1, 1, 1) + c(1, 0, 1)$

$\Rightarrow b + c = x_1, a + b = x_2, 2a + b + c = x_3$

$$\Rightarrow a = \frac{x_3 - x_1}{2}, \quad b = \frac{2x_2 + x_1 - x_3}{2}, \quad c = \frac{x_1 - 2x_2 + x_3}{2}$$

$\therefore T(x_1, x_2, x_3) = a\,T(0, 1, 2) + b\,T(1, 1, 1) + c\,T(1, 0, 1)$

$$= \frac{x_3 - x_1}{2}(3, 1, 2) + \frac{2x_2 + x_1 - x_3}{2}(2, 2, 2) + \frac{x_1 - 2x_2 + x_3}{2}(1, 0, 4)$$

$$= \left(x_2 + x_3, \frac{x_1 + 4x_2 - x_3}{2}, 2x_1 - 2x_2 + 2x_3\right)$$

is the required linear map.

Example 4.3.5. Let $f \in P_n$ where P_n is the real vector space of all polynomials of degree less than or equal to n, definition by

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n.$$

Let $(Df)(x) = a_1 + 2a_2 x + ... + n\, a_n x^{n-1}$

Prove that $D : P_n \to P_n$ is a linear transformation.

Solution : Let f and g be two polynomials such that $f \in P_n, g \in P_n$

$\therefore \quad f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$

$g(x) = b_0 + b_1 x + b_2 x^2 + ... + b_n x^n$

Let $\alpha, \beta \in R$

$\therefore (\alpha f + \beta g)(x)$

$= \alpha(a_0 + a_1 x + a_2 x^2 + ... + a_n x^n) + \beta(b_0 + b_1 x + b_2 x^2 + ... + b_n x^n)$

$= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + (\alpha a_2 + \beta b_2)x^2 + ... + (\alpha a_n + \beta b_n)x^n$

Now $[D(\alpha f+\beta g)(x)]$

$= (\alpha a_1+\beta b_1)+2(\alpha a_2+\beta b_2)x+\ldots+n(\alpha a_n+\beta b_n)x^{n-1}$

$=\alpha(a_1+2a_2x+\ldots+n a_n x^{n-1})+\beta(b_1+2b_2x+\ldots+n b_n x^{n-1})$

$=\alpha(Df)(x)+\beta(Dg)(x)=(\alpha Df+\beta Dg)(x)$

$\therefore\ D(\alpha f+\beta g)=\alpha Df+\beta Dg$ [by 4.2 Note (ii)]

Hence D is a linear transformation.

Example 4.3.6 : Find a non-zero linear transformation $T : V_2 \to V_2$ which maps all vectors on the line $x = y$ onto the origin.

Solution : Let T (0, 1) = (0, 1)

(1, 1) is a point on the line x = y.

$\therefore T(1, 1) = (0, 0)$

To show that the vectors (1, 1) and (0, 1) are L.I

Let $a(1, 1) + b(0, 1) = (0, 0)$

$\Rightarrow (a, a + b) = (0, 0)$

$\Rightarrow a = 0, a + b = 0$

$\Rightarrow a = 0, b = 0$

$\therefore$ The vectors (1, 1) and (0, 1) are L.I.

Let $(x, y) \in V_2$

$\therefore (x, y) = a(1, 1) + b(0, 1)$

$\Rightarrow a = x, b = y - x$

$(x, y) = x(1, 1) + (y - x)(0, 1)$

$\therefore T(x, y) = x\,T(1, 1) + (y - x)\,T(0, 1)$

$= x(0, 0) + (y - x)(0, 1)$

$= (0, y - x)$

Example 4.3.7 : Determine a linear transformation $T: V_2 \to V_2$ which maps all the vectors on the line $x + y = 0$ onto themselves $(T \neq I)$. Does there exists another linear transformation ? If yes, find so.

Solution : (2, –2) is a point on the line x + y = 0. Since T maps all the vectors on the line x + y = 0 onto themselves, therefore, we can take T (2, –2) = (2, –2).

Let T (2, 0) = (3, 0).

The vectors (2, 0) and (2, –2) are L.I and form a basis of V_2.

Let $(x, y) \in V_2$

$\therefore\ (x, y) = a(2, -2) + b(2, 0)$

$\Rightarrow (x, y) = (2a + 2b, -2a)$

$\Rightarrow 2a + 2b = x,\ -2a = y$

$\Rightarrow a = -\frac{y}{2},\ b = \frac{x+y}{2}$

$\therefore\ (x, y) = -\frac{y}{2}(2, -2) + \frac{x+y}{2}(2, 0)$

$$\Rightarrow T(x,y) = -\frac{y}{2}T(2,-2) + \frac{x+y}{2}T(2,0)$$

$$= -\frac{y}{2}(2,-2) + \frac{x+y}{2}(3,0)$$

$$= (-y, y) + \left(\frac{3x+3y}{2}, 0\right)$$

$$= \left(\frac{3x+3y}{2} - y,\ y+0\right)$$

$$= \left(\frac{3x+y}{2}, y\right) \text{ is a desired linear transformation.}$$

Further,

Since $x + y = 0$, therefore we can take $T(3, -3) = (-3, 3)$.

We can consider another map defined by $T(2, 0) = (0, 2)$

The vector $(3, -3)$ and $(2, 0)$ are L.I.

Let $(x, y) \in V_2$

$$\therefore (x,y) = \alpha(3,-3) + \beta(2,0)$$

$$\Rightarrow (x,y) = (3\alpha + 2\beta, -3\alpha)$$

$$\Rightarrow -3\alpha = y,\ 3\alpha + 2\beta = x$$

$$\Rightarrow \alpha = -\frac{y}{3},\ \beta = \frac{x+y}{2}$$

$$\therefore (x,y) = -\frac{y}{3}(3,-3) + \frac{x+y}{2}(2,0)$$

$$T(x,y) = T\left(-\frac{y}{3}(3,-3) + \frac{x+y}{2}(2,0)\right)$$

$$= -\frac{y}{3}T(3,-3) + \frac{x+y}{2}T(2,0)$$

$$= -\frac{y}{3}(-3,3) + \frac{x+y}{2}(0,2)$$

$$= (y, -y) + (0, x+y) = (y, x)$$

$\therefore T(x,y) = (y,x)$ is another linear transformation.

Example 4.3.8 : Let $T : V_3 \to V_2$ be defined by $T(e_1) = (0, 1)$, $T(e_2) = (1, 2)$ and $T(e_3) = (2, 3)$, where e_1, e_2, e_3 are the standard basis of V_3. Find a formula for T.

Solution : Let Let $u = (x, y, z) \in V_3$

$$\therefore u = xe_1 + ye_2 + ze_3$$

$$T(u) = T(x,y,z)$$

$$= T(xe_1 + ye_2 + ze_3)$$

$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$= x(0,1) + y(1,2) + z(2,3)$$

$$= (y+2z,\ x+2y+3z)$$

$\therefore T(x,y,z) = (y+2z, x+2y+3z)$

is the required linear transformation.

Problem Set 4 (A)

1. Let $T : U \rightarrow V$ be a linear transformation where U and V are vector spaces over the same field of scalars. If $u_1, u_2, ..., u_n \in U$ such that $T(u_1), T(u_2), ..., T(u_n)$ are linearly independent, there show that $u_1, u_2, ..., u_n$ are linearly independent.

2. If $T : V \rightarrow V$ be defined by $T(x) = x,\ x \in V,$ show that T is linear.

3. Let R be the field of real numbers and V be the vector space of all functions from R into R, which are continuous.

   Define $T : V \rightarrow V$ by

   $(Tf)(x) = \int_0^x f(t)\,dt$. Show that T is a linear transformation.

   [Hints : Let $f, g \in V,\ \alpha, \beta$ are scalars

   $$\therefore\ \alpha f + \beta g \in V \quad [\because V \text{ is a vector space}]$$
   $$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$$
   $$\Rightarrow (T(\alpha f + \beta g))(x) = \int_0^x (\alpha f + \beta g)(t)\,dt$$
   $$= \int_0^x \alpha f(t)\,dt + \int_0^x \beta g(t)\,dt$$
   $$= \alpha \int_0^x f(t)\,dt + \beta \int_0^x g(t)\,dt$$
   $$= \alpha(Tf)(x) + \beta(tg)(x) \qquad \therefore \text{ T is linear}]$$

4. Find a non zero linear transformation $T : V_2 \rightarrow V_2$ which maps all the vectors on the line $y + 2x = 0$ onto the origin.

5. Find out which of the following are linear transformations :

   (i) $T : V_2 \rightarrow V_2$ defined by
   $T(x, y) = (y, x)$

   (ii) $T : V_2 \rightarrow V_2$ defined by
   $T(x, y) = (x - y, 0)$

   (iii) $T : V_3 \rightarrow V_3$ defined by
   $T(x, y, z) = (x - y + 2z, 2x + y, -x - 2y + 2z)$

   (iv) $T : V_3 \rightarrow V_3$ defined by
   $T(x, y, z) = (3x - 2y + z, x - 3y - 2z)$

   (v) $T : V_3 \rightarrow V_3$ defined by
   $T(x, y, z) = (x+1, y, z)$

   (vi) $T : V_2 \rightarrow V_2$ defined by
   $T(x, y) = (x + y, x)$

   (vii) $T : V_2 \rightarrow V_3$ defined by
   $T(x, y) = (x + y, x - y, y)$

(viii) $T : P \to P$ defined by
$T(p)(x) = x\,p(x) + p(1)$

(ix) $T : P \to P$ defined by

$$T(p)(x) = p(1) + x\,p'(1) + \frac{x^2}{2}p''(1)$$

(x) $T : C^2[a, b] \to C[a, b]$ defined by
$T(f) = (2x + 3)f + (3x + 4)f' + (5x^2 + 7)f''$,
where $c^n[a, b]$ denotes the set of all real valued functions defined on $[a, b]$ & differentiable n-times and nth derivatives are continuous on $[a, b]$

(xi) $T : P \to P$ be defined by $T(p) = p'$

(xii) $T : V_2 \to V_2$ be defined by $T(x, y) = (1 + x, y)$

(xiii) $T : V_3 \to V_2$ be defined by $T(x, y, z) = (x + y, y + z)$

(xiv) $T : V_1 \to V_3$ be defined by $T(x) = (x, 2x, 3x)$

(xv) $T : V_2^C \to V_2^C$ be defined by $T(x + iy, p + iq) = (x, p)$

6. Find out the linear transformations (if exists) in the following cases :

(i) $T : V_2 \to V_2$ such that
$T(1, 0) = (a, b)$ and $T(0, 1) = (c, d)$

(ii) $T : V_2 \to V_2$ such that
$T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$

(iii) $T : V_2 \to V_3$ Such that
$T(1, 2) = (3, -1, 5)$ and $T(0, 1) = (2, 1, -1)$

(iv) $T : V_3 \to V_2$ such that $T(e_1) = (1, 2)$, $T(e_2) = (2, 3)$, $T(e_3) = (3, 4)$

(v) $T : V_3 \to V_3$ such that
$T(1, 2, 3) = (5, 4, 1)$, $T(1, 0, 0) = (1, 2, -1)$, $T(0, 1, 0) = (-1, 1, -2)$

(vi) $T : P_3 \to P_3$ such that
$T(1+x) = 1+x$, $T(2+x) = x + 3x^2$ and $T(x^2) = 0$

(vii) $T : V_3 \to V_1$ such that $T(1, 1, 1) = 3$, $T(0, 1, -2) = 1$, $T(0, 0, 1) = -2$.

7. Is there a linear transformation

(i) $T : V_3 \to V_2$ such that
$T(1, -1, 1) = (1, 0)$ and $T(1, 1, 1) = (0, 1)$?

(ii) $T : V_2 \to V_2$ such that
$T(2, 2) = (8, -6), T(5, 5) = (3, -2)$?

[**Hints :** Vectors (2, 2) and (5, 5) are L.D

$$\text{i e } (5,5) = \frac{5}{2}(2, 2)$$

Hence they do not form a basis for V_2. If T is linear, then

$$T(5,5) = T\left(\frac{5}{2}(2,2)\right) = \frac{5}{2}T(2, 2)$$

$$= \frac{5}{2}(8,-6) = (20,-15) \neq (3,-2)$$

$\therefore$ T does not exist.]

8. If $u_1 = (1,-1)$, $u_2 = (2,-1)$, $u_3 = (-3,2)$ and $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (1,1)$, does there exists a linear map $T : V_2 \to V_2$ such that $T(u_i) = v_i$ for i = 1, 2, 3 ?

9. Describe explicitly the linear transformation
   (i) $T : V_2 \to V_2$ such that $T(2, 3) = (4, 5)$, $T(1, 0) = (0, 0)$
   (ii) $T : V_2 \to V_2$ such that $T(e_1) = (a, b)$, $T(e_2) = (c, d)$ where e_1 and e_2 are unit vectors.

10. Find a linear transformation $T : V_2 \to V_2$ such that $T(1, 0) = (1, 1)$ and $T(0, 1) = (-1, 2)$. Prove that T maps the square with vertices (0, 0), (1, 0), (1, 1) and (0, 1) into a parallelogram.

[**Hints :** $T(x_1, x_2) = (x_1 - x_2, x_1 + 2x_2)$

Take $A'_i = T(A_i)$, i = 1, 2, 3, 4

where A_i (i = 1, 2, 3, 4) are vertices of square and A'_i be their images. $|A'_1 A'_2| = |A'_3 A'_4|$ and slope of $A'_1 A'_2$ = slope of $A'_3 A'_4 = 1$

$\therefore$ T maps into a parallelogram].

11. Determine all elements $v \in V$ such that $T(v) = w$, where $T : V \to V$ is linear.

12. State true or false :
   (a) Let $D : V \to V$ be the differential mapping $D(v) = \frac{dv}{dt}$. Then D is linear.
   (b) Let $I : V \to R$ be the integral mapping $I(v) = \int_0^1 v(t)\,dt$. Then I is linear.
   (c) The map $T : V_2 \to V_2$ defined by $T(x, y) = (x+y, x)$ is not linear.
   (d) The map $T : V_2 \to V_3$ defined by $T(x, y) = (x+1, 2y, x+y)$ is not linear.
   (e) The map T is linear if C is viewed as a vector space over itself.

[**Hints :** Take $u = 2 + 5i$, $\alpha = 1 - i$
$T(\alpha u) = T(7 + 3i) = 7 - 3i$
$\neq \alpha T(u)$]

(f) The map T is linear if C is viewed as a vector space over the real field R.

[**Hints :** Take u = a + ib, v = c + id, $\alpha \in R$.

$\therefore\ T(u+v) = T(u) + T(v)$

$T(\alpha u) = \alpha T(u)$]

(g) There exists a linear transformation $T : V_2 \to V_2$, such that T (0,0) = (1, 0)

(h) Scalar multiplication is the only linear transformation T from V_1 to V_1.

4.4 Different types of Transformations :

(1) Zero transformations : Let $T : U \to V$ be defined by

$T(u) = 0_V$, for all $u \in U$. T is called zero transformation.

(2) Projection transformation :

Let $P : V_3 \to V$ be defined by P (x, y, z) = 'x'. P is called projection map or projection transformation.

Similarly, P(x, y, z) = y , P (x, y, z) = z are projection map from V_3 to V.

Further $P : V_3 \to V_2$ be defined by P (x, y, z) = (x, y) is a projection map. Other projection maps are P (x, y, z) = (x, z) and P (x, y, z) = (y, z) from V_3 to V_2.

(3) Reflexion transformation :

Let $T : V_2 \to V_2$ be defined by

T (x, y) = (x, – y) T is called reflexion map on x - axis.

(4) Identity Transformation :

Let $T : U \to V$ be defined $T(u) = u$, for all $u \in U$. T is called identity transformation. It is denoted by I_U or I.

(5) Quotient Map :

Let V be a subspace of a vector space U.

Let $T : U \to U/W$ be defined by $T(u) = u + W$ for all $u \in U$. T is called quotient space.

(6) Negation transformation :

Let U and V be two vectors spaces. Let $T : U \to V$ be a linear transformation. Then the map (–T) defined by (–T) (x) = – T(x), $x \in U$ is a linear transformation from U into V, which is called negation of a linear transformation.

For all $u, u_2 \in U$,

$$\begin{aligned}
&(-T)(u_1 + u_2) \\
&= -[T(u_1 + u_2)] && \text{(by definition)} \\
&= -[T(u_1) + T(u_2)] && (\because T \text{ is linear}) \\
&= (-T(u_1)) + (-T(u_2)) \\
&= (-T)(u_1) + (-T)(u_2) && \text{(by definition)}
\end{aligned}$$

and for $u \in U$, α being a scalar,

$$\begin{aligned}
&(-T)(\alpha u) \\
&= -[T(\alpha u)] && \text{(by definition)} \\
&= -[\alpha T(u)] && (\because T \text{ is linear}) \\
&= -\alpha T(u) \\
&= \alpha(-T(u)) \\
&= \alpha(-T)(u)
\end{aligned}$$

$\therefore$ $T : U \to V$ is linear.

(7) Idempotent transformation :

A linear transformation T on a vector space V is said to be idempotent if $T^2 = T$.

Example : The zero transformation and the identity transformation are idempotent. (For Proof Refer example 4.10.9)

(8) Nilpotent transformation :

A linear transformation T on a vector space V is called nilpotent transformation on V if $T^n = \mathbf{0}$ for some integer $n > 1$ and the smallest integral value of n is called the degree of nilpotence of T.

Example (1) The differential operator D is nilpotent on P_n.

(2) $T : V_3 \to V_3$ be defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$ is nilpotent.
(For Proof, Refer example 4.10.10)

Exercise : Check that the transformations, from (1) to (5) are linear.

4.5 : Range and Kernel

4.5.1 Definition : Let U and V be two vector spaces. Let $T : U \to V$ be a linear transformation.

The **Range** of T is the set of all images of U in V i.e, if $\beta \in U$ then there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

We denote range of T by R (T) or T (U).
Symbolically we write $R(T) = \{T(x) \in V \mid x \in U\}$

Note : $R(T) \subseteq V$

Example 4.5.1 : Let $T : V_3 \to V_3$ be a linear mapping defined by $T(x, y, z) = (x, y, 0)$. Find range of T.

Solution : R(T) consists of those points in xy-plane.

Let $(a, b, c) \in V_3$

$\therefore$ $(x, y, 0) = (a, b, c)$

$\Rightarrow x = a, y = b, c = 0$

$\therefore T(x, y, z)$

$= T(a, b, z) = (a, b, 0)$

This shows that every vector (a, b, 0) of V_3 is in R(T)

$\therefore R(T) = \{ (a, b, 0) : a, b \in R \}$

Example 4.5.2 : Let $T : V_2 \to V_2$ be a linear map defined by $T(x, y) = (x+y, x)$. Find R (T).

Solution : Let $(a, b) \in R(T)$

and $T(x, y) = (a, b)$

$\therefore x + y = a, x = b$

$y = a - x = a - b$

$\therefore T(x, y)$

$= T(b, a - b) = (b + a - b, b) = (a, b)$

This shows that every point of V_2 is in R(T)

$\therefore R(T) = V_2$

Example 4.5.3 : Let $T : V_3 \to V_4$ be defined by

$T(x, y, z) = (x, x+y, x+y+z, z)$. Find R(T).

Solution : Let $(p, q, r, s) \in R(T)$

$$T(x, y, z) = (p, q, r, s)$$
$$= (x, x + y, x + y + z, z)$$
$$= x(1, 1, 1, 0) + y(0, 1, 1, 0) + z(0, 0, 1, 1)$$
$$\therefore R(T) = [(1, 1, 1, 0), (0, 1, 1, 0), (0, 0, 1, 1)]$$

Example 4.5.4 : Let $T : P \to P$ be a linar map defined by $T(P)(x) = x\,p(x)$. Find R (T).

Solution : Let $q(x) \in R(T)$ There exists $p(x) \in P$ such that

$$T(p(x)) = q(x)$$
$$\Rightarrow x\,p(x) = q(x)$$
$$\Rightarrow q(0) = 0$$

Conversely, if $q(o) = o$ then

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1{}^x$$
$$= x(a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_1)$$
$$= x\,p(x)$$

$\therefore T(p(x) = x\,p(x) = q(x) \in R(T)$

Hence $q(x) \in R(T)$ iff $q(0) = 0$

$\therefore R(T) = \{q(x) \in P \mid q(0) = 0\}$

4.5.2 Definition : Let $T : U \to V$ be a linear transformation. The **kernel** of T (or null space of T) is the set of all those vectors in U, whose image by T is $\mathbf{0} \in V$. Symbolically, it is denoted by N (T) or ker T. Thus $N(T) = \{x \in U \mid T(x) = \mathbf{0}, \in V\}$

Note : $N(T) \subseteq U$

Example 4.5.5 : Referring Example 4.5.1., find N (T) .

Solution : Let $u = (x, y, z) \in N(T)$

$\therefore \; T(u) = \mathbf{0} \in V_3$

$\Rightarrow T(x, y, z) = (0, 0, 0)$

$\Rightarrow (x, y, 0) = (0, 0, 0)$

$\Rightarrow x = 0, y = 0$

$\therefore N(T) = \{(0, 0, c) \mid c \in R\}$

i.e. kernel of T is the set of all points on z-axis.

Example 4.5.6 : Find N (T), referring Example 4.5.2

Solution : Let $(x, y) \in N(T)$

$\therefore \; T(x, y) = \mathbf{0} \in V_2$

$\Rightarrow (x + y, x) = (0, 0)$

$\Rightarrow x + y = 0, x = 0$

$\Rightarrow x = 0, y = 0$

$\therefore N(T) = \{(0, 0)\} = V_0$

Example 4.5.7. Let $T : V_3 \to V_3$ be defined by

$T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$. Find N (T).

Solution : Let $v = (x, y, z) \in N(T)$

$\therefore \; T(v) = \mathbf{0} \in V_3$

$\Rightarrow (x + 2y - z, y + z, x + y - 2z) = (0, 0, 0)$

$\Rightarrow x + 2y - z = 0, y + z = 0, x + y - 2z = 0$

$\Rightarrow x + 2y - z = 0, y + z = 0, - y - z = 0$

(Putting $x = -2y + z$ in the last equation)

$\Rightarrow x + 2y - z = 0, y + z = 0$

$\Rightarrow y = -z, x = 3z, z = z$

$N(T) = \{(3z, -z, z)\}$

$= \{(3, -1, 1)\}$, taking $z = 1$

Example 4.5.8. Let $T : P \to P$ be defined by $T(p)(x) = p''(x) - p(x)$. Find N (T).

Solution : Let $p(x) \in N(T)$

$\Rightarrow T(p(x)) = 0 \in P$

$\Rightarrow p''(x) - p(x) = 0$

Suppose $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \quad a_n \neq 0$

$\therefore \; p'(x) = a_1 + 2a_2x + 3a_3x^2 + ... + n\,a_nx^{n-1}$

$p''(x) = 2a_2 + 6a_3x + ... + n(n-1)a_nx^{n-2}$

$p''(x) - p(x) = 0$

$\Rightarrow (2a_2 - a_0) + (6a_3 - a_1)x + ... + (n(n-1)a_n - a_{n-2}x^{n-2} - a_{n-1}x^{n-1} - a_nx^n = 0$

$\Rightarrow a_n = a_{n-1} = ... = a_0 = 0$

$\therefore p(x) = 0$

$\therefore N(T) = V_0$

Theorem 4.5.1 Let $T : U \to V$ be a linear transformation. Then

(a) R (T) is a subspace of V.

(b) N (T) is a subspace of U.

Proof : (a) Obviously R(T) is a non-empty subset of V.

Let $v_1, v_2 \in R(T)$. Then there exists vectors $u_1, u_2 \in U$ such that

$T(u_1) = v_1, \quad T(u_2) = v_2$.

Let α, β be scalars.

$\therefore \quad \alpha v_1 + \beta v_2 = \alpha T(u_1) + \beta T(u_2)$

$= T(\alpha u_1 + \beta u_2)$ [$\because$ T is linear]

Since U is a vector space, therefore $u_1, u_2 \in U$

$\Rightarrow \quad \alpha u_1 + \beta u_2 \in U$

Consequently

$T(\alpha u_1 + \beta u_2) = \alpha v_1 + \beta v_2 \in R(T)$

Thus, $v_1, v_2 \in R(T)$, α and β being scalars

$\Rightarrow \alpha v_1 + \beta v_2 \in R(T)$

$\therefore$ R (T) is a subspace of V.

(b) $N(T) = \{x \in U \mid T(x) = \mathbf{0} \in V\}$

Since $T(0) = \mathbf{0} \in V$

$\therefore \ 0 \in N(T)$

$\Rightarrow$ N (T) is a non - empty subset of U.

Let $u_1, u_2 \in N(T)$

$\therefore \quad T(u_1) = \mathbf{0} \in V$ and $T(u_2) = \mathbf{0} \in V$

Let α and β be scalars.

$\therefore \quad \alpha u_1 + \beta u_2 \in U$ and

$T(\alpha u_1 + \beta u_2) = \alpha T(u_1) + \beta T(u_2)$ [$\because$ T is linear]

$= \alpha . \mathbf{0} + \beta . \mathbf{0} = \mathbf{0} \in V$

$\therefore \quad \alpha u_1 + \beta u_2 \in N(T)$

$\Rightarrow$ N (T) is a subspace of U.

4.5.3 Definition : Let $T : U \to V$ be a linear transformation. T is **one-one** if

$T(u) = T(v) \Rightarrow u = v$,

where $u, v \in U$.

Example 4.5.9 : Prove that $T : V_2 \to V_3$ be defined by $T(a, b) = (a+b, a-b, b)$ is one-one.

Solution : Let $u = (p, q) \in V_2$, $v = (r, s) \in V_2$

Suppose $T(u) = T(v)$

$\Rightarrow T(p, q) = T(r, s)$

$\Rightarrow (p+q, p-q, q) = (r+s, r-s, s)$

$\Rightarrow p+q = r+s, p-q = r-s, q = s$

$\Rightarrow p = r,\ q = s$

$\Rightarrow (p, q) = (r, s) \Rightarrow u = v.$

$\therefore$ T is one-one.

In the above example, we observe that

$N(T) = \{(0, 0)\} = \{0_U\}$.

Hence, if T is one-one, then $0_U \in N(T)$ i.e., N(T) is a zero subspace of U.

We shall prove this as a theorem where the converse is also true.

4.5.4 Definition : Let : Let $T : U \to V$ be a linear transformation. T is onto if for every $v \in V$, there exists a vector $u \in U$ such that $T(u) = v$ i.e., $R(T) = V$

Example 4.5.10 : $T : V_3 \to V_3$ be defined by $T(x, y, z) = (x, z, y)$ is onto since every point of V_3 is a point of R(T) i.e., $R(T) = V_3$

Theorem 4.5.2 : Let $T : U \to V$ be a linear transformation. Then

(a) T is one - one iff $N(T) = \{\mathbf{0}_U\}$

(b) If $u_i \in U$, $i = 1, 2, ..., n$ and $[u_1, u_2, ..., u_n] = U$, then

$R(T) = [T(u_1), T(u_2), ..., T(u_n)]$

Proof : (a) Suppose T is one-one. To show that $\mathbf{0}_U \in N(T)$

Let $u \in N(T)$.

By definition of kernel of T,

$T(u) = \mathbf{0}_V$

$\Rightarrow T(u) = T(\mathbf{0}_U)$

$\Rightarrow u = \mathbf{0}_U$ [$\because$ T is one – one]

$\therefore \mathbf{0}_U \in N(T)$

Conversely, Let $\mathbf{0}_U \in N(T)$.

To show that T is one-one.

Suppose $T(u) = T(v)$

Now, $T(u - v) = T(u) + T(-v)$

$= T(u) - T(v)$

$= \mathbf{0}_V$ [$\because T(u) = T(v)$]

$\Rightarrow u - v \in N(T)$ [By definition of Kernel of T]

But $u - v = \mathbf{0}_U \Rightarrow u = v + \mathbf{0}_U = v$

$\therefore$ T is one-one

(b) Suppose $[u_1, u_2, ..., u_n] = U$. To prove that $R(T) = [T(u_1), T(u_2), ..., T(u_n)]$

Let $u \in U$. Therefore u can be written as $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$, for scalars $\alpha_i, i = 1, 2, ..., n$.

Let $v \in [T(u_1), T(u_2), ..., T(u_n)]$

$$\Rightarrow v = \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n)$$

$$\Rightarrow v = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) \quad [\because T \text{ is linear}]$$

$$\Rightarrow v \in R(T) \quad [\because u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n \in U]$$

$$\therefore [T(u_1), T(u_2), ..., T(u_n)] \subset R(T) \quad ... (1)$$

Now to prove that $R(T) \subset [T(u_1), T(u_2), ..., T(u_n)]$

Let $v \in R(T)$. Then there exists a vector $u \in U$ such that $T(u) = v$

$$\Rightarrow v = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$$

$$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n)$$

$$\Rightarrow v \in [T(u_1), T(u_2), ..., T(u_n)]$$

$$\therefore R(T) \subset [T(u_1), T(u_2), ..., T(u_n)] \quad ... (2)$$

From (1) and (2), we have $R(T) = [T(u_1), T(u_2), ..., T(u_n)]$

Example 4.5.11 Let T be a linear map on a finite-dimensional vector space V. Then prove that the following two statements are equivalent.

(a) $R(T) \cap N(T) = \{0\}$

(b) $T(T(x)) = \mathbf{0} \Rightarrow T(x) = \mathbf{0}$

Solution : First we shall show that $(a) \Rightarrow (b)$

We have $T(T(x)) = \mathbf{0} \Rightarrow T(x) \in N(T)$

$\Rightarrow T(x) \in R(T) \cap N(T) \quad [\because x \in V \Rightarrow T(x) \in R(T)]$

$\Rightarrow T(x) = \mathbf{0} \quad [\because R(T) \cap N(T) = \{0\}]$

Now we shall show that $(b) \Rightarrow (a)$

Let $x \neq \mathbf{0}$ and $x \in R(T) \cap N(T)$

$\therefore x \in R(T)$ and $x \in N(T)$

Since $x \in N(T)$, therefore $T(x) = \mathbf{0}$ ... (1)

Also $x \in R(T) \Rightarrow \exists\ y \in V$ such that $T(y) = x$

Now $T(y) = x \Rightarrow T(T(y)) = T(x) = \mathbf{0}$ [from (1)]

Thus $\exists\ y \in V$ such that $(T(T(y)) = \mathbf{0}$ but $T(y) = x \neq \mathbf{0}$

This is a contradiction to the hypothesis (b).

Therefore, the exists no $x \in R(T) \cap N(T)$ such that $x \neq 0$.

Hence $R(T) \cap N(T) = \{0\}$.

Example 4.5.11 : Find a linear transformation $T : V_3 \to V_3$ such that the set of all vectors (x_1, x_2, x_3) satisfying the equation $x_1 - 2x_2 + x_3 = 0$ is the kernel of T.

Solution : Let $T : V_3 \to V_3$ be defined by

$$T(x_1, x_2, x_3) = (x_1 - 2x_2 + x_3, 2x_1 - 4x_2 + 2x_3, 0)$$

Claim : T is linear

$$\begin{aligned}
& T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] \\
&= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= (x_1 + y_1 - 2x_2 - 2y_2 + x_3 + y_3, 2x_1 + 2y_1 - 4x_2 - 4y_2 + 2x_3 + 2y_3, 0) \\
&= (x_1 - 2x_2 + x_3, 2x_1 - 4x_2 + 2x_3, 0) + (y_1 - 2y_2 + y_3, 2y_1 - 4y_2 + 2y_3, 0) \\
&= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\
& T(\alpha(x_1, x_2, x_3)) \\
&= T(\alpha x_1, \alpha x_2, \alpha x_3) \\
&= (\alpha x_1 - 2\alpha x_2 + \alpha x_3, 2\alpha x_1 - 4\alpha x_2 + 2\alpha x_3, 0) \\
&= \alpha\ T(x_1, x_2, x_3)
\end{aligned}$$

$\therefore$ T is linear.

$\therefore$ $\ker T = N(T)$

$$= \{(x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = (0, 0, 0)\}$$
$$= \{(x_1, x_2, x_3) \mid x_1 - 2x_2 + x_3 = 0\}$$

Note : $T : V_3 \to V_3$ may be defined in various ways. One of them is

$$T(x_1, x_2, x_3) = \left(x_1 + 2x_2, x_1 + \frac{1}{2}x_3, x_1 - 2x_2 + x_3\right).$$

Problem Set 4 (B)

1. **For each of the following linear transformations T, find the range and kernel.**

(a) $T : V_2 \to V_3$ defined by
$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$.

(b) $T : V_3 \to V_2$ defined by
$T(x_1, x_2, x_3) = (x_1, x_2)$.

(c) $T : V_3 \to V_2$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1 + x_2 + x_3, x_1 + x_2 + x_3)$.

(d) $T : V_3 \to V_2$ be defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.

(e) $T : V_3 \to V_3$ be defined by
$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$.

(f) $T : V_2 \to V_2$ be defined by
$T(x_1, x_2) = (x_1, -x_2)$.

(g) $T : P \to P$ be defined by
$T(p)(x) = xp(x)$.

(h) $T : P \to P$ be defined by
$T(p)(x) = p''(x) - 2p(x)$.

(i) $T : V_2 \to V_2$ be defined by
$T(x_1, x_2) = (x_2, 0)$.

(j) $T : V_4 \to V_3$ be defined by $T(x_1, x_2, x_3, x_4)$
$= (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4)$

2. Find a linear transformation $T : V_3 \to V_3$ such that the set of all vectors (x_1, x_2, x_3) satisfying the equation $2x_1 - x_2 + x_3 = 0$ is the kernel of T.

4.6. Rank and Nullity :

4.6.1 Definition : Let $T : U \to V$ be a linear transfomation. The **rank** of T is the dimension of the range of T if R (T) is finite-dimeninal. It is denoted by $\rho(T)$ or r(T).

The **nullity** of T is the dimension of the kernel of T if N(T) is finite dimensional. It is denoted by n (T).

Theorm 4.6.1 : Let $T : U \to V$ be linear. Then.

(a) If $u_1, u_2, ..., u_n \in U$ and $T(u_i)$, $i = 1, 2, ...n$ are linearly independent, then $u_1, u_2, ..., u_n$ are linearly independent.

(b) If T is one-one and $u_1, u_2, ..., u_n$ are linearly independent vectors of U, then $T(u_1), T(u_2), ..., T(u_n)$ are L.I.

(c) If $v_1, v_2, ..., v_n$ are linearly independent vectors of R(T) and $u_1, u_2, ..., u_n$ are vectors of U such that $T(u_i) = v_i$, $i = 1, 2,...n$ then $u_1, u_2,...,u_n$ are L.I.

Proof : (a) Suppose, there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$,
such that $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = \mathbf{0}_V$
Then $\mathbf{0}_V = T(\mathbf{0}_U) = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$
$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n)$. [$\because$ T is linear]

Since $T(u_i)$, $i = 1, 2, ...$ n are linearly independent, therefore $\alpha_i = 0$ for each $i = 1, 2,...$ n.
Thus $u_1, u_2, ..., u_n$ are linearly independent.

(b) Suppose T is one-one and $u_1, u_2, ..., u_n$ are linearly independent vectors of U.

In order to prove that $T(u_1), T(u_2), ..., T(u_n)$ are linearly independent, let us assume that for scalars $\alpha_1, \alpha_2, ..., \alpha_n$,

$\alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n) = \mathbf{0}_V$

$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) = \mathbf{0}_V$ [$\because$ T is linear]

$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) = T(\mathbf{0}_U)$

$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = \mathbf{0}_U$ [$\because$ T is one-one]

Since $u_1, u_2, ..., u_n$ are L.I, therefore $\alpha_i = 0$, $i = 1, 2, ..., n$.
Hence $T(u_1), T(u_2), ..., T(u_n)$ are L.I.

(c) Suppose $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = \mathbf{0}_U$ ... (1)

Since T is linear, therefore

$$\mathbf{0}_V = T(\mathbf{0}_U) = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n)$$

$$\Rightarrow \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n) = \mathbf{0}_V$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = \mathbf{0}_V \qquad (\because T(u_i) = v_i)$$

Since $v_1, v_2, ..., v_n$ are L.I., therefore $\alpha_1 = 0 = \alpha_2 = ... = \alpha_n$.

Hence from (1), we have $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = \mathbf{0}_U$

$$\Rightarrow \alpha_1 = 0 = \alpha_2 = ... = \alpha_n$$

Hence $u_1, u_2, ..., u_n$ are L.I.

Theorem 4.6.2 : Let U and V be two vector spaces and T : U $\to$ V be linear, with U as finite dimensional. Then dim R(T) + dim N (T) = dim U.

Proof : We have seen that N (T) is a subspace of U.

Since U is finite dimensional, therefore N (T) is finite dimensional.

Let dim N (T) = k.

Let $B_1 = \{u_1, u_2, ..., u_k\}$ be a basis for N (T)

Since $\{u_1, u_2, ..., u_k\}$ is a lineary independent subset of U, therefore we can extend it to form a basis for U.

Let dim U = n, $n \geq k$.

Let $B_2 = \{u_1, u_2, ..., u_k, u_{k+1}, ..., u_n\}$ be a basis for U.

Therefore $R(T) = [T(u_1), T(u_2), T(u_k), T(u_{k+1}), ..., T(u_n)]$.

Consider the set

$$B = \{T(u_{k+1}), T(u_{k+2}), ..., T(u_n)\}.$$

We claim that B is a basis for R (T).

(i) First we shall prove that [B] = R(T).

Since, $[B_2]$ = U, therefore by theorem 4.5.2 (b)

$$R(T) = [T(u_1), T(u_2), ..., T(u_k), T(u_{k+1}), ..., T(u_n)]$$

But $u_i \in N(T)$ for i = 1, 2, k

$$\Rightarrow T(u_i) = 0$$

$$\Rightarrow T(u_1) = 0 = T(u_2) = ... = T(u_k)$$

$$\therefore R(T) = [T(u_{k+1}), T(u_{k+2}), ..., T(u_n)] = [B]$$

(ii) Now to show that the set B is L.I.

Let $\alpha_{k+1}, \alpha_{k+2}, ..., \alpha_n$ be scalars such that

$$\alpha_{k+1} T(u_{k+1}) + \alpha_{k+2} T(u_{k+2}) + ... + \alpha_n T(u_n) = \mathbf{0}_V \qquad ... (1)$$

$$\Rightarrow T(\alpha_{k+1} u_{k+1} + \alpha_{k+2} u_{k+2} + ... + \alpha_n u_n) = \mathbf{0}_V \qquad (\because T \text{ is linear})$$

$$\Rightarrow \alpha_{k+1} u_{k+1} + \alpha_{k+2} u_{k+2} + ... + \alpha_n u_n \in N(T)$$

$$\Rightarrow \alpha_{k+1} u_{k+1} + \alpha_{k+2} u_{k+2} + ... + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + ... + \beta_k u_k$$

($\because$ each element of N(T) is a linear combination of the elements of B_1)

$$\Rightarrow \beta_1 u_1 + \beta_2 u_2 + ... + \beta_k u_k + (-\alpha_{k+1}) u_{k+1} + (-\alpha_{k+2}) u_{k+2} + ... + (-\alpha_n) u_n = 0$$

Since B_2 is a basis for U, therefore B_2 is L.I.

$\Rightarrow \beta_1 = \beta_2 = ... = \beta_k = (-\alpha_{k+1}) = (-\alpha_{k+2}) = ... = (-\alpha_n) = 0$

$\Rightarrow \beta_i = 0 \quad (i = 1, 2, ..., k)$

and $\alpha_i = 0 \quad (i = k+1, k+2, ..., n)$

From (1), $\alpha_{k+1} T(u_{k+1}) + \alpha_{k+2} T(u_{k+2}) + ... + \alpha_n T(u_n) = \mathbf{0}_V$

$\Rightarrow \alpha_{k+1} = 0, \ \alpha_{k+2} = 0, ..., \alpha_n = 0$

$\Rightarrow$ B is L.I.

Hence B is a basis for R(T) and as number of elements in B = n − k, therefore dim R(T) = n − k

Since B_1 is a basis for N (T), and dim N (T) = k,

Thus dim R(T) = n − dim N (T)

$\Rightarrow$ dim R(T) + dim N (T) = n = dim U.

Hence the theorem is completely established.

(This theorem is known as **Rank-Nullity Theorem**)

Theorem 4.6.3 : Suppose U is finite dimensional and T : U $\to$ V is linear. Then R (T) is finite dimensional and dim R(T) $\leq$ dim U.

Proof : Suppose dim U = n and dim R(T) > dim U. Then there exists vectors $v_1, v_2, ..., v_{n+1} \in R(T)$, which are L.I.

Let $u_1, u_2, ..., u_{n+1}$ be vectors in U such that $T(u_i) = v_i$, i = 1, 2,..., (n + 1)

Suppose $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_{n+1} u_{n+1} = 0_U$

Then $\mathbf{0}_V = T(\mathbf{0}_U) = T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_{n+1} u_{n+1})$

$= \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_{n+1} T(u_{n+1})$ [$\because$ T is linear]

$= \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_{n+1} v_{n+1}$.

Since $\{v_1, v_2, ..., v_{n+1}\}$ is L.I.,

therefore $\alpha_1 = 0, \ \alpha_2 = 0, ..., \alpha_{n+1} = 0$.

Thus $u_1, u_2, ..., u_{n+1}$ are L.I. This contradicts the fact that dim U = n.

Hence dim R(T) $\leq$ dim U.

Example 4.6.1 : Let Z be a subspace of a finite-dimensional vector space U and V be a finite dimensional vector space. Then prove that Z will be the kernel of a linear map $T : U \to V$ iff dim Z $\geq$ dim U − dim V.

Solution : Since $T : U \to V$ is linear and U is of finite dimension, therefore by Rank-Nullity Theorem,

$$\dim R(T) + \dim N(T) = \dim U$$

But, we have given Z = N(T)

$\therefore$ dim R (T) + dim Z = dim U

$\Rightarrow$ dim R(T) = dim U − dim Z ... (1)

we know that R(T) is a subspace of V.

$\therefore$ dim R(T) $\leq$ dim V

From (1), dim U − dim Z $\leq$ dim V

$\Rightarrow$ dim Z $\geq$ dim U − dim V.

Example 4.6.2 Let U be a finite dimensional vector space having dimension n and a linear map $T : U \to U$ is such that the Range of T and kernel of T are identical. Prove that n is even.

Solution : Let N be the kernel of T. Then N is also the range of T.

By Rank-Nullity Theorem,

$$r(T) + n(T) = \dim U$$
$$\Rightarrow \dim R(T) + \dim N(T) = \dim U = n$$
$$\Rightarrow 2 \dim N = n \qquad [\because R(T) = N(T) = N]$$
$$\Rightarrow n \text{ is even.}$$

Example 4.6.3 : If $T : U \to V$ is a linear map, where U is finite dimensional, Prove that

(a) $n(T) \le \dim U$

(b) $r(T) \le \min(\dim U, \dim V)$

Solution : (a) Since $T : U \to V$ is a linear map and U is of finite dimension, by Rank-Nullity Theorem.

$$\dim R(T) + \dim N(T) = \dim U$$
$$\Rightarrow r(T) + n(T) = \dim U$$
$$\Rightarrow n(T) \le \dim U \qquad [\because r(T) \ge 0]$$

(b) Since R(T) is a subspace of V, therefore

$$\dim R(T) \le \dim V$$
$$\Rightarrow r(T) \le \dim V$$

Also, $r(T) + n(T) = \dim U$

$$\Rightarrow r(T) \le \dim U \qquad [\because n(T) \ge 0]$$

$$\therefore r(T) \le \min(\dim U, \dim V)$$

Example 4.6.4 : Let U be a vector space of dimension n and $T : U \to V$ be a linear and onto map. Then prove that T is one-one iff dim V = n.

Solution : Suppose $T : U \to V$ is one-one.

Therefore N(T) is the zero subspace of U i.e., $N(T) = \{\mathbf{0}_U\}$

$$\Rightarrow \dim N(T) = 0 \qquad \ldots (1)$$

By Rank-Nullity Theorem,

$$\dim U = \dim R(T) + \dim N(T)$$
$$= \dim R(T) + 0 = \dim R(T) \qquad [\text{Using } (1)]$$

Again $T : U \to V$ is onto

$$\Rightarrow R(T) = V$$

$\therefore \dim U = \dim V$

$$\Rightarrow \dim V = n. \qquad [\because \dim U = n]$$

$\therefore$ T is one-one $\Rightarrow \dim V = n$

If U is of dimension n and $T : U \to V$ is linear and onto and dim V = n then to show that T is one-one.

Now, By Rank-Nullity Theorem,

$$n = \dim U = \dim R(T) + \dim N(T)$$
$$= \dim V + \dim N(T) \qquad [\because T \text{ is onto} \Rightarrow R(T) = V]$$

$\Rightarrow$ $n = n + \dim N(T)$
$\Rightarrow$ $\dim N(T) = 0$
$\Rightarrow$ $N(T) = \{\mathbf{0}_U\}$
$\Rightarrow$ T is one-one
$\therefore$ T is onto and $\dim U = \dim V = n$
$\Rightarrow$ T is one-one.

Example 4.6.5 : For the following linear mapping T, find a basis and the dimension of (a) its range, (b) its null space.

Also verify $r(T) + n(T) = \dim V_4$.

$$T : V_4 \to V_3 \text{ defined by}$$

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4,\ x_1 + 2x_3 - x_4,\ x_1 + x_2 + 3x_3 - 3x_4)$$

Solution : We know that the set $A = \{e_1, e_2, e_3, e_4\}$ is a basis set for V_4.

$$e_1 = (1, 0, 0, 0),\ e_2 = (0, 1, 0, 0),\ e_3 = (0, 0, 1, 0),\ e_4 = (0, 0, 0, 1),$$

By definition, $T(e_1) = T(1, 0, 0, 0)$
$= (1 - 0 + 0 + 0,\ 1 + 2(0) - 0,\ 1 + 0 + 3(0) - 3(0)$
$= (1, 1, 1).$

Similarly $T(e_2) = (-1, 0, 1),\ T(e_3) = (1, 2, 3)$
and $T(e_4) = (1, -1, -3)$ [Verify]

Now for any $x \in V_4$ $x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4$, as A is a basis.

$\therefore$ $y \in R(T) = T(x) = T(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4)$
$= a_1 T(e_1) + a_2T(e_2) + a_3T(e_3) + a_4T(e_4)$. [$\because$ T is linear]
$= a_1(1, 1, 1) + a_2(-1, 0, 1) + a_3(1, 2, 3) + a_4(1, -1, -3).$

To verify whether $y \in R(T)$ expressed as linear combination of four vectors $\in V_3$ can be expressed as a linear combination of less number of vectors or not.

For this, we compute a matrix whose rows are these four vectors

$$B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \qquad [\text{Operating } R_2 + R_1, R_3 - R_1, R_4 - R_1]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [\text{Operating } R_4 + 2R_3, R_3 - R_2]$$

which is echelon form of matrix.

Thus the non-zero vectors {(1, 1, 1), (0, 1, 2)} is the basis for R(T).
Hence dim. R(T) i.e., rank (T) = 2.
To find the basis and dimension for N(T).

$$x \in N(T) \quad \text{if } T(x) = \mathbf{0}.$$

Now $T(x_1, x_2, x_3, x_4) = \mathbf{0}$

$$\Rightarrow (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4) = (0, 0, 0)$$

$$\Rightarrow \left.\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 0 \\ x_1 + 2x_3 - x_4 &= 0 \\ x_1 + x_2 + 3x_3 - 3x_4 &= 0 \end{aligned}\right\} \quad \dots (1)$$

$$\text{Co-efficient matrix} = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix} \quad [\text{Operating } R_2 - R_1, R_3 - R_1]$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [\text{Operating } R_3 - 2R_2]$$

which is echelon form of matrix.
Thus the system (1) is equivalent to

$$x_1 - x_2 + x_3 + x_4 = 0 \quad \dots (2)$$

$$x_2 + x_3 - 2x_4 = 0 \quad \dots (3)$$

From (3), $x_2 = -x_3 + 2x_4$

From (2), $x_1 + x_3 - 2x_4 + x_3 + x_4 = 0$

$$\Rightarrow x_1 = -2x_3 + x_4$$

Thus $x_1 = -2x_3 + x_4$

$x_2 = -x_3 + 2x_4.$

Here x_3 and x_4 can take any real value.
Hence nullity T = dim. N(T) = 2.

Choosing $x_3 = 1,\ x_4 = 0,\ x_1 = -2,\ x_2 = -1.$

Choosing $x_3 = 0,\ x_4 = 1,\ x_1 = 1,\ x_2 = 2.$

{(–2, –1, 1, 0), (1, 2, 0, 1)} constitutes a basis for N(T), because these vectors are L.I.

Now dim. $V_4 = 4$, dim. R(T) = 2, dim. N(T) = 2

$\therefore$ Rank (T) + Nullity (T) = dim V_4

$\therefore$ $r(t) + n(t) = \dim V_4$.

Example 4.6.6 : Determine the Range, rank, kernel and nullity of the linear transformation $T : C(0,1) \to C(0,1)$ defined by $T(f)(x) = f(x) \sin x$.

Solution : $T : C(0,1) \to C(0,1)$ is defined by $T(f)(x) = f(x) \sin x$

Let $f(x) \in C(0,1)$

$$\therefore \frac{f(x)}{\sin x} \in C(0,1)$$

$$T\left(\frac{f(x)}{\sin x}\right) = \frac{f(x)}{\sin x} . \sin x = f(x)$$

$\Rightarrow f(x) \in R(T)$

$\therefore \quad R(T) = C(0, 1)$

$\Rightarrow r(T) = \dim C(0, 1)$

To find N(T) and n(T).

Let $f(x) \in N(T)$

$\therefore \quad T(f)(x) = 0 \Rightarrow f(x) \sin x = 0 \Rightarrow f(x) = 0$

$\therefore \quad N(T) = V_0$ and $n(T) = \dim V_0 = 0$.

Problem Set 4 (C)

1. Determine the rank and nullity of Q.No. 1 of Problem Set 4 (B)

2. If $T : V_4 \to V_3$ be a linear map defined by $T(e_1) = (1, 1, 1)$, $T(e_2) = (1, -1, 1)$, $T(e_3) = (1, 0, 0)$, $T(e_4) = (1, 0, 1)$, then verify that $r(T) + n(T) = 4$

4.7. Singular and Non-singular transformations :

4.7.1. Definition : Let $T : U \to V$ be a linear map T is non-singular if $\mathbf{0}_U \in N(T)$ i.e., if $x \in U$ and $T(x) = \mathbf{0}$ then $x = \mathbf{0}_U$.

Hence T is non-singular if T is one-one by theorem 4.5.2

We shall prove this result in the form a theorem. Thus

Theorem 4.7.1 Let $T : U \to V$ be a linear map. Then T is non-singular iff T is one-one.

Proof : Suppose T is non-singular. To prove that T is one-one.

Let $u, v \in U$. Then $T(u) = T(v)$

$\Rightarrow T(u) - T(v) = \mathbf{0}$

$\Rightarrow T(u - v) = \mathbf{0}$

$\Rightarrow u - v = \mathbf{0}_U$ ($\because$ T is non-singular)

$\Rightarrow u = v$

$\therefore$ T is one-one.

Conversely, let T be one-one.

We know that

$T(0_U) = 0_V$

i.e., $T(0) = \mathbf{0}$

Since T is one-one, therefore $u \in U$ and $T(u) = \mathbf{0}_V = T(\mathbf{0}_U)$

$\Rightarrow u = \mathbf{0}_U$ [$\because$ T is one-one; $T(x) = T(y) \Rightarrow x = y$]

$\therefore N(T) = \{0_U\}$

$\Rightarrow$ T is non-singular.

Example 4.7.1 : Let $T : V_3 \to V_2$ be defined by $T(x, y, z) = (x, y)$

Then show that T is onto but not one-one.

Solution : $R(T) = \{T(x, y, z) \mid (x, y, z) \in V_3\}$

$= \{(x, y) \mid (x, y, z) \in V_3\}$

$= V_2$

$\therefore$ T is onto.

$N(T) = \{(x, y, z) \mid T(x, y, z) = \mathbf{0}\}$

$= \{(x, y, z) \mid (x, y) = (0, 0)\}$

$\Rightarrow \{(0, 0, z)\} \neq \{0_U\}$

$\therefore$ T is not one-one.

Example 4.7.2 : $T : V_3 \to V_3$ be defined by

$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ x_2 + x_3, x_3)$

Then show that T is non-singular.

Solution : Suppose $T(x_1, x_2, x_3) = T(y_1, y_2, y_3)$

$\Rightarrow (x_1 + x_2 + x_3,\ x_2 + x_3,\ x_3) = (y_1 + y_2 + y_3,\ y_2 + y_3, y_3)$

$\Rightarrow x_1 + x_2 + x_3 = y_1 + y_2 + y_3, x_2 + x_3 = y_2 + y_3, x_3 = y_3$

$\Rightarrow x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3$

$\Rightarrow (x_1, x_2, x_3) = (y_1, y_2, y_3)$

$\therefore$ T is one-one.

Linear map $T : V_3 \to V_3$ is one-one iff T is onto.

Hence T is non-singular.

Example 4.7.3 : Prove that the linear map $T : V_3 \to V_3$ defined by

$T(1,2,3) = (3, -1, 7)$, $T(1, -2, 3) = (3, 3, 3)$ and

$T(1, 2, -3) = (3, -1, 1)$ is one-one and onto.

Solution : Let $(x, y, z) \in V_3$.

Let $u = (1, 2, 3)$, $v = (1, -2, 3)$, $w = (1, 2, -3)$

Suppose for scalars α, β, γ

$\alpha u + \beta v + \gamma w = \mathbf{0}$

$\Rightarrow \alpha(1, 2, 3) + \beta(1, -2, 3) + \gamma(1, 2, -3) = \mathbf{0}$

$\Rightarrow (\alpha + \beta + \gamma, 2\alpha - 2\beta + 2\gamma, 3\alpha + 3\beta - 3\gamma) = (0, 0, 0)$

$\Rightarrow \alpha + \beta + \gamma = 0,\ 2\alpha - 2\beta + 2\gamma = 0,\ 3\alpha + 3\beta - 3\gamma = 0$

$\Rightarrow \alpha+\beta+\gamma=0,\ \alpha-\beta+\gamma=0,\ \alpha+\beta-\gamma=0$

$\Rightarrow \alpha=0,\ \beta=0,\ \gamma=0$

$\therefore$ The vectors u, v, w are L.I.

Now (x, y, z) = a (1, 2, 3) + b (1, –2, 3) + c (1, 2, –3)

$\Rightarrow a + b + c = x,\ 2a - 2b + 2c = y,\ 3a + 3b - 3c = z$

$$\Rightarrow a = \frac{3y+2z}{12},\ b = \frac{2x-y}{4}, c = \frac{3x-z}{6}$$

$$\Rightarrow T(x,y,z) = \frac{3y+2z}{12}(3,-1,7) + \frac{2x-y}{4}(3,3,3) + \frac{3x-z}{6}(3,-1,1)$$

$= (3x, x - y, 2x + y + z)$

To show that T is one-one.

Suppose $(x, y, z) \in N(T)$

$\Rightarrow T(x, y, z) = \mathbf{0}$

$\Rightarrow (3x, x-y, 2x+y+z) = (0, 0, 0)$

$\Rightarrow 3x = 0, x - y = 0, 2x + y + z = 0$

$\Rightarrow x = 0, y = 0, z = 0$

$\Rightarrow N(T) = \{(0, 0, 0)\}$

$\Rightarrow$ T is one-one.

Again to show that T is onto. We have seen that $[u, v, w] = V_3$

$\Rightarrow [T(u), T(v), T(w)] = R(T)$

$\Rightarrow [(3, -1, 7), (3, 3, 3) (3, -1, 1)] = R(T)$

But (3, –1, 7), (3, 3, 3), (3, –1, 1) are L.D.

$\therefore\ R(T) = V_3$

$\Rightarrow$ T is onto

Hence T is one-one and onto.

Example 4.7.4 Let $T : V_2 \to V_2$ be defined by T (x, y) = (y, 2x – y). Show that T is non-singular.

Solution : Suppose T (x, y) = T (r, s)

$\Rightarrow (y, 2x - y) = (s, 2r - s)$

$\Rightarrow y = s,\ 2x - y = 2r - s$

$\Rightarrow y = s,\ x = r.$

$\Rightarrow (x, y) = (r, s)$

$\therefore$ T is one-one.

Let $(a, b) \in V_2$

Suppose (y, 2x – y) = (a, b)

$\Rightarrow y = a,\ 2x - y = b$

$$\Rightarrow y = a,\ x = \frac{a+b}{2}$$

$$\therefore\ T(x, y) = T\left(\frac{a+b}{2}, a\right)$$

$$= \left(a, 2\left(\frac{a+b}{2}\right) - a\right)$$

$$= (a, b)$$

This shows that every vector $(a, b) \in V_2$ is in R(T) i.e., $R(T) = V_2$.

$\Rightarrow$ T is onto.

Hence T is non-singular.

Example 4.7.5 : Let $T : P_2 \to P_2$ be defined by

$T(a_0 + a_1x + a_2x^2) = (a_0 + a_1) + (a_1 + 2a_2)x + (a_0 + a_1 + 3a_2)x^2$.

Show that T is non-singular.

Solution : Suppose $T(a_0 + a_1x + a_2x^2) = T(b_0 + b_1x + b_2x^2)$

$$\Rightarrow (a_0 + a_1) + (a_1 + 2a_2)x + (a_0 + a_1 + 3a_2)x^2$$
$$= (b_0 + b_1) + (b_1 + 2b_2)x + (b_0 + b_1 + 3b_2)x^2$$

$$\Rightarrow a_0 + a_1 = b_0 + b_1,\ a_1 + 2a_2 = b_1 + 2b_2,$$
$$a_0 + a_1 + 3a_2 = b_0 + b_1 + 3b_2$$

$$\Rightarrow a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2$$

$$\Rightarrow a_0 + a_1x + a_2x^2 = b_0 + b_1x + b_2x^2$$

$\therefore$ T is one-one.

Let $\alpha_0 + \alpha_1x + \alpha_2x^2 \in P_2$

Suppose $(a_0 + a_1) + (a_1 + 2a_2)x + (a_0 + a_1 + 3a_2)x^2 = \alpha_0 + \alpha_1x + \alpha_2x^2$

$$\Rightarrow a_0 + a_1 = \alpha_0,\ a_1 + 2a_2 = \alpha_1,\ a_0 + a_1 + 3a_2 = \alpha_2$$

$$\Rightarrow a_0 = \frac{\alpha_0 - 3\alpha_1 + 2\alpha_2}{3},\ a_1 = \frac{2\alpha_0 + 3\alpha_1 - 2\alpha_2}{3},\ a_2 = \frac{\alpha_2 - \alpha_0}{3}$$

$\therefore\ T(a_0 + a_1x + a_2x^2)$

$$= T\left(\frac{\alpha_0 - 3\alpha_1 + 2\alpha_2}{3} + \frac{2\alpha_0 + 3\alpha_1 - 2\alpha_2}{3}x + \frac{\alpha_2 - \alpha_0}{3}x^2\right)$$

$$= \left(\frac{\alpha_0 - 3\alpha_1 + 2\alpha_2}{3} + \frac{2\alpha_0 + 3\alpha_1 - 2\alpha_2}{3}\right) + \left(\frac{2\alpha_0 + 3\alpha_1 - 2\alpha_2}{3} + \frac{2\alpha_2 - 2\alpha_0}{3}\right)x$$
$$+ \left(\frac{\alpha_0 - 3\alpha_1 + 2\alpha_2}{3} + \frac{2\alpha_0 + 3\alpha_1 - 2\alpha_2}{3} + \frac{3\alpha_2 - 3\alpha_0}{3}\right)x^2$$

$$= \alpha_0 + \alpha_1x + \alpha_2x^2$$

$\therefore\ R(T) = P_2$

$\Rightarrow$ T is onto.

Hence T is non-singular.

Example 4.7.6 Show that the linear map $T : V_3 \to V_3$ defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_1 - e_2 + e_3$, $T(e_3) = 3e_1 + 4e_3$ is non-singular.

Solution : First of all to find a general formula for T.

$$\begin{aligned} T(x_1, x_2, x_3) &= T(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) \\ &= x_1 (e_1 + e_2) + x_2 (e_1 - e_2 + e_3) + x_3 (3e_1 + 4e_3) \\ &= (x_1 + x_2 + 3x_3)e_1 + (x_1 - x_2) e_2 + (x_2 + 4x_3) e_3 \\ &= (x_1 + x_2 + 3x_3,\ x_1 - x_2,\ x_2 + 4x_3) \end{aligned}$$

If $T(x_1, x_2, x_3) = \mathbf{0}$ then $x_1 + x_2 + 3x_3 = 0, x_1 - x_2 = 0, x_2 + 4x_3 = 0$

$\Rightarrow x_1 = 0,\ \ x_2 = 0,\ \ x_3 = 0$

$\therefore\ N(T) = \{(0, 0, 0)\}$

$\Rightarrow$ T is one-one.

Hence T is onto.

Therefore, T is non-singular.

4.7.2 Definition : Inverse linear Transformation :

Let $T : U \to V$ be a linear transformation such that

(i) $u \neq v \Rightarrow T(u) \neq T(v)$

or $T(u) = T(v) \Rightarrow u = v$ for all $u, v \in U$ i.e., T is one-one.

(ii) To every $y \in V$ there corresponds a vector $x \in U$ such that $T(x) = y$ i.e., T is onto.

Then we say that T is invertible.

If T is invertible, then we define a mapping called inverse of T and denoted by T^{-1}.

Thus if $T : U \to V$ is invertible then $T^{-1} : V \to U$ is defined by $T^{-1}(y) = x$ iff $T(x) = y$.

Theorem 4.7.2 : Let $T : U \to V$ be a one-one and onto linear transformation. Then $T^{-1} : V \to U$ is a linear transformation which is one-one and onto.

Proof : Let $v_1, v_2 \in V$.

Let $T^{-1}(v_1) = u_1$ and $T^{-1}(v_2) = u_2$, where $u_1, u_2 \in U$.

Then by definition, $T(u_1) = v_1$ and $T(u_2) = v_2$.

Now, $v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2)$ [$\because$ T is linear]

$\Rightarrow T^{-1}(v_1 + v_2) = u_1 + u_2 = T^{-1}(v_1) + T^{-1}(v_2)$

Further, Let α be a scalar.

$\therefore\ \alpha v_1 = \alpha T(u_1) = T(\alpha u_1)$ [$\because$ T is linear]

$\Rightarrow T^{-1}(\alpha v_1) = \alpha u_1 = \alpha T^{-1}(v_1)$

$\therefore\ T^{-1}$ is linear.

We will now show that T^{-1} is one-one.

For this, let $v_1, v_2 \in V$

such that $T^{-1}(v_1) = T^{-1}(v_2)$

Let $T^{-1}(v_1) = u_1$, $T^{-1}(v_2) = u_2$

Then $T(u_1) = v_1$, $T(u_2) = v_2$

Since T is one-one, therefore $u_1 = u_2$

$\Rightarrow \quad T(u_1) = T(u_2)$

$\Rightarrow \quad v_1 = v_2$

Thus, $T^{-1}(v_1) = T^{-1}(v_2)$

$\Rightarrow \quad v_1 = v_2$

This proves that T^{-1} is one-one.

Now to show that T^{-1} is onto.

T^{-1} is onto, because for any $u \in U$, we have $T(u) = v \in V$ such that $T^{-1}(v) = u$. The proof is complete.

Note : A linear transformation $T : U \to V$ possesses an inverse if it satisfies the following two conditions:

(i) T is one-one.

(ii) T is onto.

If any one of the above two conditions or both fails, then T^{-1} does not exist.

Theorem 4.7.3 Let U and V be finite dir.-nsional vector spaces such that dim U = dim V = n. If $T : U \to V$ be a linear transformation then the following statements are equivalent :

(i) T is non-singular

(ii) T is one-one

(iii) T maps any linearly independent subset of U into a linearly independent subset of V.

(iv) T maps some basis for U into a basis for V.

(v) T is onto.

(vi) R(T) = V i.e., r(T) = n

(vii) n(T) = 0.

(viii) T^{-1} exists.

Proof: (i) $\Rightarrow$ (ii) by definition 4.7.1

(ii) $\Rightarrow$ (iii) by theorem 4.6.1(a)

(iii) $\Rightarrow$ (iv) by theorem

(iv) $\Rightarrow$ (v).

Let $\{u_1, u_2, ..., u_n\}$ be a basis for U. Then $\{T(u_1), T(u_2), ..., T(u_n)\}$ is a basis for V. Therefore by Theorem 4.5.2 (b), R(T) = V

$\Rightarrow$ T is onto.

(v) $\Rightarrow$ (vi)

Since T is onto

$R(T) = V$

$\Rightarrow$ $r(T) = \dim V = n$

(vi) $\Rightarrow$ (vii)

By rank-Nullity theorem, $r(T) + n(T) = \dim V = n$

$\Rightarrow n + n(T) = n$

$\Rightarrow n(T) = 0$

(vii) $\Rightarrow$ (viii)

$n(T) = 0$

$\Rightarrow 0_U \in N(T)$.

$\Rightarrow$ T is one-one.

Hence T is onto

$\Rightarrow T^{-1}$ exists.

(viii) $\Rightarrow$(i)

It is obvious from the definition.

Example 4.7.7 : Referring Example 4.7.4, find T^{-1}.

Solution : Since T is non-singular, T^{-1} exists.

Let $T^{-1}(x, y) = (r, s)$

Then $T(r, s) = (x, y)$

$\Rightarrow (s, 2r - s) = (x, y)$

$\Rightarrow x = s,\ 2r - s = y$

$$\Rightarrow s = x,\ r = \frac{x+y}{2}$$

$$\therefore T^{-1}(x, y) = \left(\frac{x+y}{2},\ x\right)$$

Example 4.7.8 : Referring Example 4.7.5, find T^{-1}.

Solution : Since T is non-singular, T^{-1} exists.

$$\text{Let } T^{-1}(a_0 + a_1x + a_2x^2) = \beta_0 + \beta_1 x + \beta_2 x^2$$

$$\Rightarrow T(\beta_0 + \beta_1 x + \beta_2 x^2) = a_0 + a_1x + a_2x^2$$

$$\Rightarrow (\beta_0 + \beta_1) + (\beta_1 + 2\beta_2)x + (\beta_0 + \beta_1 + 3\beta_2)x^2 = a_0 + a_1x + a_2x^2$$

$$\Rightarrow \beta_0 + \beta_1 = a_0, \beta_1 + 2\beta_2 = a_1, \beta_0 + \beta_1 + 3\beta_2 = a_2$$

$$\Rightarrow \beta_0 = \frac{a_0 - 3a_1 + 2a_2}{3},\ \beta_1 = \frac{2a_0 + 3a_1 - 2a_2}{3} \text{ and } \beta_2 = \frac{a_2 - a_0}{3}$$

$$\therefore T^{-1}(a_0 + a_1x + a_2x^2) = \frac{a_0 - 3a_1 + 2a_2}{3} + \frac{2a_0 + 3a_1 - 2a_2}{3}x + \frac{a_2 - a_0}{3}x^2$$

Example 4.7.9 : Referring Example 4.7.6, find T^{-1}.

Solution : Since T is non-singular, T^{-1} exists.

$$\text{Let } T^{-1}(x_1, x_2, x_3) = (\alpha_1, \alpha_2, \alpha_3)$$

$$\Rightarrow T(\alpha_1, \alpha_2, \alpha_3) = (x_1, x_2, x_3)$$

$$\Rightarrow (\alpha_1 + \alpha_2 + 3\alpha_3,\ \alpha_1 - \alpha_2, \alpha_2 + 4\alpha_3) = (x_1, x_2, x_3)$$

$$\Rightarrow \alpha_1 + \alpha_2 + 3\alpha_3 = x_1,\ \alpha_1 - \alpha_2 = x_2,\ \alpha_2 + 4\alpha_3 = x_3$$

$$\Rightarrow \alpha_1 = \frac{4x_1 + x_2 - 3x_3}{5},\ \alpha_2 = \frac{4x_1 - 4x_2 - 3x_3}{5},\ \alpha_3 = \frac{2x_3 - x_1 + x_2}{5}$$

$$\therefore\ T^{-1}(x_1, x_2, x_3) = \frac{1}{5}(4x_1 + x_2 - 3x_3, 4x_1 - 4x_2 - 3x_3, -x_1 + x_2 + 2x_3).$$

Problem Set 4 (D)

1. Find out the linear maps of Q.No. 1 of Problem set 4 (B) which are
   (a) one-one
   (b) onto
   (c) one-one and onto.
2. Show that the following linear transformes are non-singular and find its inverse :
   (a) $T : V_2 \to V_2$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$
   (b) $T : V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = \left(\frac{1}{2}x_1 + x_2 + x_3, x_1 - \frac{1}{3}x_2, x_3\right)$
   (c) $T : V_3 \to V_3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$
   (d) $T : V_3 \to V_3$ defined by
   $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$.
   (e) $T : P_2 \to P_2$ defined by $T(a + bx + cx^2) = (a + b) + (a + 2c)x + (a+b+3c)x^2$.
3. Let $T : U \to V$ be a non-singular linear transformation. Then prove that $(T^{-1})^{-1} = T$.
4. Prove that the linear map $T : V_3 \to V_3$ defined by
   $T(e_1) = e_1 - e_2, T(e_2) = 2e_2 + e_3, T(e_3) = e_1 + e_2 + e_3$ is neither one-one nor onto.
5. Show that $I_U : U \to U$ is non-singular and $I_U^{-1} = I_U$.

4.8 Isomorphism :

We have seen that a function has an inverse iff it is one-one and onto. Hence a linear transformation is non-singular iff it has an inverse. This linear map is called an **isomorphism.**

If there exists a vector $\mathbf{0} \neq u \in U$ such that $T(u) = \mathbf{0}$, then T is called singular.

Now we shall prove a theorem, showing that T is one-one $\Rightarrow$ T is onto

$\Rightarrow$ T is an isomorphism

$\Rightarrow$ T is one-one

4.8.1 Defintion : Two vector spaces U and V are said to be **isomorphic** to each other if there exists an isomorphism T from U to V. If U and V are isomorphic, then we write $U \cong V$.

We know that if $T : U \to V$ is an isomorphism, then T transforms a basis of U into a basis of V. Therefore dim U = dim V. i.e., if $U \cong V$ then dim U = dim V. The converse is also true. We shall prove this as a theorem.

Note : The linear map $T : U \to V$ is also known as homomorphism i.e., if U and V be two vector spaces, then the map $T : U \to V$ is called a homomorphism from U to V if

(i) $T(u_1 + u_2) = T(u_1) + T(u_2)$, $u_1, u_2 \in U$

(ii) $T(\alpha u_1) = \alpha T(u_1)$, α is a scalar.

We denote the set of homomorphism from U to V by Hom (U, V).

Theorem 4.8.1 : Let $T : U \to V$ be a linear transformation where dim U = dim V. Then the following statements are equivalent :

(a) T is one-one

(b) T is onto

(c) T is an isomorphism

Proof : In order to prove the theorem, we have to show that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

Let dim U = dim V = n

(a) $\Rightarrow$ (b)

T is one-one

$\Rightarrow$ $N(T) = \{\mathbf{0}_U\}$

$\Rightarrow$ $n(T) = 0$

$\Rightarrow$ $r(T) = n$ [Rank Nullity theorem i.e., r(T) + n(T) = dim U = n]

$\Rightarrow$ $R(T) = V$ [$\because$ R(T) is a subspace of V]

$\Rightarrow$ T is onto

$\therefore$ (a) $\Rightarrow$ (b)

(b) $\Rightarrow$ (c)

T is onto

$\Rightarrow$ $r(T) = n$

$\Rightarrow$ $n(T) = 0$

$\Rightarrow$ $N(T) = \{\mathbf{0}_U\}$

$\Rightarrow$ T is one-one.

Hence T is one-one and onto i.e., T is an isomorphism $\therefore$ (b) $\Rightarrow$ (c)

(c) $\Rightarrow$ (a)

Since T is an isomorphism, therefore T is one-one (by definition of isomorphism)

$\therefore$ (c) $\Rightarrow$ (a)

Hence, the theorem is proved.

Example 4.8.1 : Let $D : P_n \to P_n$ be defined by

$$D(a_0 + a_1x + a_2x^2 + ... + a_nx^n)$$
$$= a_1 + 2a_2x + ... + n\,a_n\,x^{n-1},$$

where P_n is the real vector space of all polynomials, then show that D is not an isomorphism.

Solution : Here P_n has finite dimension, a basis being $\{1, x, x^2, ...\}$
D is onto because any element of P_n is of the form

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n$$
$$= D\left(a_0x + a_1\frac{x^2}{2} + ... + a_n\frac{x^{n+1}}{n+1}\right)$$

D is not one-one, because D (1) = 0 = D (0), but $1 \neq 0$.

Theorem 4.8.2 : Let $T : U \to V$ be an isomorphism. Suppose $\{e_1, e_2, ..., e_n\}$ is a basis for U. Then $\{T(e_1), T(e_2), ..., T(e_n)\}$ is a basis for V.

Proof : Since $T : U \to V$ is an isomorphism,

$\therefore$ T is onto

$\Rightarrow$ R (T) = V

$\Rightarrow$ $[T(e_1), T(e_2), ..., T(e_n)] = V$

Now to show that $\{T(e_1), T(e_2), ..., T(e_n)\}$ is L.I.

Suppose there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$\alpha_1 T(e_1) + \alpha_2 T(e_2) + ... + \alpha_n T(e_n) = \mathbf{0} \qquad ... (1)$$
$$\Rightarrow T(\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n) = \mathbf{0} = T(0) \quad [\because T \text{ is linear}]$$
$$\Rightarrow \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = 0 \qquad (\because T \text{ is one-one})$$

But $\{e_1, e_2, ..., e_n\}$ is L.I.

Therefore $\alpha_1 = 0 = \alpha_2 = ... = \alpha_n$

Thus $\{T(e_1), T(e_2), ..., T(e_n)\}$ is a basis for V.

Theorem 4.8.3 : Any linear transformation $T : U \to V$ maps any linearly dependent set of vectors onto a linearly dependent set of vectors.

Proof : Let $T : U \to V$ be a linear transformation.

Let $\{u_1, u_2, ..., u_n\}$ be a linearly dependent set of vectors in U.

To show that $\{T(u_1), T(u_2), ..., T(u_n)\}$ is L.D.

Since $\{u_1, u_2, ..., u_n\}$ is L.D., there exists scalars

$\alpha_1, \alpha_2, ..., \alpha_n$ not all zero such that

$$\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n = 0$$
$$\Rightarrow \alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n) = T(0) = \mathbf{0}$$

$\Rightarrow \{T(u_1), T(u_2), ..., T(u_n)\}$ is linearly dependent.

Theorem 4.8.4 : If U is finite dimensional real vector space of dimension n, then U is isomorphic to V_n.

Proof : Let U be a real vector space of dimension n.

Let $B = \{u_1, u_2, ..., u_n\}$ be a basis for U.

Let $e_1 = (1, 0, 0, ..., 0)$

$e_2 = (0, 1, 0, ..., 0), ...,$

$e_n = (0, 0, 0, ..., 1).$

1 being the identity in V. We know that $\{e_1, e_2, ..., e_n\}$ is a basis for V_n.

Let $u \in U$.

Since $\{u_1, u_2, ..., u_n\}$ is a basis for U, there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n$ such that $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$.

Define $T : U \to V_n$ by

$$T(u) = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = (\alpha_1, \alpha_2, ..., \alpha_n) \qquad ... (1)$$

The map T is well defined, for $u = 0 \Rightarrow T(u) = (0, 0, ..., 0)$

Now to show that $U \cong V_n$ i.e.,

(i) T is linear (homomorphism)

(ii) T is one-one

(iii) T is onto.

To prove (i), we proceed as follows :

$u, v \in V$.

Let $u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$

$v = \beta_1 u_1 + \beta_2 u_2 + ... + \beta_n u_n,$

where α_i and β_i are scalars, $i = 1, 2, ..., n$.

$\therefore\ u + v = (\alpha_1 + \beta_1) u_1 + (\alpha_2 + \beta_2) u_2 + ... + (\alpha_n + \beta_n) u_n$.

$$T(u + v) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_n + \beta_n) \qquad \text{[By (1)]}$$

$$= (\alpha_1, \alpha_2, ..., \alpha_n) + (\beta_1, \beta_2, ..., \beta_n)$$

$$= T(u) + T(v).$$

and $T(\alpha u) = T(\alpha\alpha_1 u_1 + \alpha\alpha_2 u_2 + ... + \alpha\alpha_n u_n)$

$$= (\alpha\alpha_1, \alpha\alpha_2, ..., \alpha\alpha_n)$$

$$= \alpha(\alpha_1, \alpha_2, ..., \alpha_n)$$

$$= \alpha T(u).$$

$\therefore$ T is linear.

To prove (ii), T is one-one.

$$T(u) = \mathbf{0}_V$$
$$\Rightarrow (\alpha_1, \alpha_2, ..., \alpha_n) = \mathbf{0}_V = (0, 0, ..., 0)$$
$$\Rightarrow \alpha_1 = 0 = \alpha_2 = ... = \alpha_n$$
$$\Rightarrow u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$$
$$= 0\, u_1 + 0\, u_2 + ... + 0\, u_n = \mathbf{0}_U$$
$$\therefore\ T(u) = \mathbf{0}_V$$
$$\Rightarrow u = 0_U$$
$$\text{i.e., } N(T) = \{\mathbf{0}_U\}$$

$\therefore$ T is one-one.

To prove (iii) $\{u_1, u_2, ..., u_n\}$ is a basis for U.

$\Rightarrow \{T(u_1), T(u_2), ..., T(u_n)\}$ is a basis for V_n.

$\Rightarrow R(T) = V_n$ i.e., T is onto.

Hence T is an isomorophism. i.e., there exists an isomorphism from U to V_n and consequently $U \cong V_n$.

Note : Every complex vector space of dimension n is isomorphic to V_n^C.

Theorem 4.8.5 : The relation of isomorphism in any set of vector spaces is an equivalence relation.

Proof : In order to show that isomorphism is an equivalence relation, we consider any three vector spaces U, V and W and to establish the following :

(i) Reflexivity : $U \cong U$

(ii) Symmetry : If $U \cong V$ then $V \cong U$.

(iii) Transitivity : If $U \cong V$, $V \cong W$ then $U \cong W$.

Proof of (i) the identity map $I_U : U \to U$ is defined by $I_U(u) = u$, $u \in U$.

The map I_U is linear since

$$I_U(u+v) = u + v = I_U(u) + I_U(v),\ u, v \in U$$

and $I_U(\alpha u) = \alpha u = \alpha\, I_U(u)$, α is a scalar.

Let $u \in N(T)$

$\Rightarrow I_U(u) = \mathbf{0}_U$ [by definition of kernel]

$\Rightarrow I_U(u) = I_U(\mathbf{0}_U)$ [by definition of identity map]

$\Rightarrow u = \mathbf{0}_U$

$N(I_U) = \{\mathbf{0}_U\}$

$\Rightarrow I_U$ is one-one.

and Range of $I_U = U$

$\Rightarrow I_U$ is onto

Hence $I_U : U \to U$ is an isomorphism, i.e., $U \cong U$

Proof of (ii) Let $U \cong V$.

So there exists an isomorphism

$$T : U \to V.$$

Hence by Theorem 4.7.2,

$T^{-1} : V \to U$ is an isomorphism i.e., $V \cong U$.

Proof of (iii) Let $U \cong V$, $V \cong W$.

So we have isomorphisms

$T : U \to V$ and $S : V \to W$.

By Theorem (A), $ST : U \to W$ is a linear map.

Since S and T are both one-one and onto, then ST is also one-one and onto, by Theorem 4.10.4.

Consequently $ST : U \to W$ is an isomorphism

Hence $U \cong W$.

Theorem 4.8.6 : Isomorphic finite dimensional vector spaces have same dimension.

Proof : To prove that two finite dimensional vector spaces U and V are isomorphic iff dim U = dim V.

Given U and V be isomorphic. Therefore a linear map $T : U \to V$ is an isomorphism.

So T is one-one and onto.

Since U is finite dimensional, let $B_1 = \{u_1, u_2, ..., u_n\}$ be a basis for U, where dim U = n.

Consider $B_2 = \{T(u_1), T(u_2), ..., T(u_n)\}$

$\therefore$ B_2 is a subset of V, consisting n elements.

If we prove that B_2 is a basis for V then dim V = n.

For this we have to prove that B_2 is L.I. and $[B_2] = V$.

To Prove B_2 is L.I.

Suppose $\alpha_1 T(u_1) + \alpha_2 T(u_2) + ... + \alpha_n T(u_n) = \mathbf{0}_V$

$$\Rightarrow T(\alpha_1 u_1) + T(\alpha_2 u_2) + ... + T(\alpha_n u_n) = \mathbf{0}_V \quad [\because \text{T is linear}]$$

$$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) = \mathbf{0}_V \quad [\because \text{T is linear}]$$

$$\Rightarrow T(\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) = T(\mathbf{0}_U)$$

$$\Rightarrow \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n) = \mathbf{0}_U \quad [\because \text{T is one-one}]$$

$$\Rightarrow \alpha_1 = 0,\ \alpha_2 = 0, ..., \alpha_n = 0 \quad [\because B_1 \text{ is a basis}]$$

$\therefore$ B_2 is L.I.

To prove $[B_2] = V$

Let $v \in V$

Since T is onto, therefore, there exists $u \in U$ such that $T(u) = v$

But $u = \beta_1 u_1 + \beta_2 u_2 + ... + \beta_n u_n$ $\quad [\because B_1$ is a basis for U]

$$\therefore v = T(u)$$

$$= T(\beta_1 u_1 + \beta_2 u_2 + ... + \beta u_n)$$

$$= \beta_1 T(u_1) + \beta_2 T(u_2) + ... + \beta_n T(u_n) \quad [\because \text{T is linear}]$$

$\therefore$ v is a linear combination of the elements of B_2.

Thus, $[B_2] = V$.

Hence $\dim V = n \Rightarrow \dim U = \dim V$.

Conversely,

Given dim U = dim V = n

To prove $U \cong V$. i.e., there exists an isomorphism between U and V.

Since dim U = dim V = n

$\therefore$ there exists basis sets U and V, each having n elements.

Let $B_1 = \{u_1, u_2, ..., u_n\}$ and $B_2 = \{v_1, v_2, ..., v_n\}$ be the basis sets for U and V respectively. Therefore, every element of U and V can be expressed as a linear combination of the elements of B_1 and B_2 respectively.

$\therefore$ $u \in U \Rightarrow u = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$, for scalars α_i, $i = 1, 2, ..., n$.

Let us define $T : U \to V$ by

$T(u) = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$

To show that T is an isomorphism.

i.e. (i) T is linear (ii) T is one-one (iii) T is onto.

for (i) Let $x, y, \in U$, α and β be scalars.

$$\therefore \; x = \sum_{i=1}^{n} \gamma_i u_i, \quad y = \sum_{i=1}^{n} \delta_i u_i$$

$$\alpha x + \beta y = \sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i) u_i$$

Now, $T(\alpha x + \beta y)$

$$= T\left(\sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i) u_i\right)$$

$$= \sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i) T(u_i)$$

$$= \sum_{i=1}^{n} (\alpha\gamma_i + \beta\delta_i) v_i \qquad \text{[by definition of T]}$$

$$= \sum_{i=1}^{n} \alpha \gamma_i v_i + \sum_{i=1}^{n} \beta \delta_i v_i$$

$$= \alpha \sum_{i=1}^{n} \gamma_i v_i + \beta \sum_{i=1}^{n} \delta_i v_i$$

$$= \alpha T(x) + \beta T(y)$$

$\therefore$ T is linear.

For (ii) Let T(x) = T(y)

$$\Rightarrow \sum_{i=1}^{n} \gamma_i v_i = \sum_{i=1}^{n} \delta_i v_i$$

$$\Rightarrow \sum_{i=1}^{n} (\gamma_i - \delta_i) v_i = 0$$

$$\Rightarrow \gamma_i - \delta_i = 0, \quad \text{for } i = 1, 2, \ldots, n$$
$$\Rightarrow \gamma_i = \delta_i, \quad \text{for } i = 1, 2., \ldots, n$$
$$\Rightarrow \sum_{i=1}^{n} \gamma_i u_i = \sum_{i=1}^{n} \delta_i u_i$$
$$\Rightarrow x = y \qquad \therefore \text{ T is one-one.}$$

For (iii) Let $v = \sum_{i=1}^{n} \alpha_i v_i \in V$

By definition, $u = \sum_{i=1}^{n} \alpha_i u_i \in U$ such that

$$T(u) = T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = \sum_{i=1}^{n} \alpha_i v_i = v$$

$\therefore$ T is onto.

Hence $U \cong V$. The theorem is completely established.

Alternatively, we can prove the above theorem as follows :

Let $U \cong V$. So there exists an isomorphism $T : U \to V$

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis for U, n = dim U; If $V = \{0\}$, $B = \phi$

$\therefore$ T(B) is a basis for T(U) = V.

As T(B) has n elements,

$\therefore$ dim V = n = dim U

Conversely, let dim U = dim V = n

If $n = 0$, $U = V = \{0\}$,

then obviously $U \cong V$.

If $n \geq 1$, then $U \cong F^{n \times 1}$ and $F^{n \times 1} \cong V$, where $F^{n \times 1}$ is the set of (n × 1) matrix over field F.

By Theorem 4.8.5, $U \cong V$

Example 4.8.2 : Let A be the subspace of V_4 defined by

$$A = \{(x_1, x_2, x_3, x_4) \mid x_2 = 0\}$$

Prove that $A \cong V_3$.

Solution : Let $T : A \to V_3$ be defined by

$$T(x_1, x_2, x_3, 0) = (x_1, x_2, x_3)$$

Now to show that $A \cong V_3$

i.e., (i) T is one-one

(ii) T is onto

(iii) T is linear.

Let $T(x_1, x_2, x_3, 0) = T(y_1, y_2, y_3, 0)$

$$\Rightarrow (x_1, x_2, x_3) = (y_1, y_2, y_3)$$
$$\Rightarrow (x_1, x_2, x_3, 0) = (y_1, y_2, y_3, 0)$$

$\therefore$ T is one-one.

For $(x_1, x_2, x_3) \in V_3$, we have

$(x_1, x_2, x_3, 0) \in A$ and $T(x_1, x_2, x_3, 0) = (x_1, x_2, x_3)$

$\therefore$ T is onto.

Further,

$$\begin{aligned} \text{Further } & T\{(x_1, x_2, x_3, 0) + (y_1, y_2, y_3, 0)\} \\ &= T(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3, 0) \\ &= (x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) \\ &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\ &= T(x_1, x_2, x_3, 0) + T(y_1, y_2, y_3, 0) \end{aligned}$$

and $T\{\alpha(x_1, x_2, x_3, 0)\}$, α is a scalar

$$\begin{aligned} &= T(\alpha x_1, \alpha x_2, \alpha x_3, 0) \\ &= (\alpha x_1, \alpha x_2, \alpha x_3) \\ &= \alpha(x_1, x_2, x_3) \\ &= \alpha T(x_1, x_2, x_3, 0) \end{aligned}$$

$\therefore$ T is linear.

Hence T is an isomorphism i.e., $A \cong V_3$.

Problem Set 4 (E)

1. Let the map $T : P_2 \to V_3$ be defined by $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$. Prove that $P_2 \cong V_3$..
2. Let B be the subspace of P_4 defined by $B = \{p \mid p'(1) = 0,\ p''(1) = 0\}$. Prove that $B \cong V_3$.
3. Let the map $T : P_3 \to P_4$ be defined by
   $T(a_0 + a_1 x + a_2 x^2 + a_3 x^3)$
   $= a_0 + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \frac{a_3}{4} x^4$
   Prove that T is not an isomorphism.
4. Let V be the set of n-times continuously differentiable function f on [a, b]. Let a map $T : V \to V_n$ be defined by
   $T(f) = (f(a), f'(a), \ldots, f^{n-1}(a))$.
   Prove that T is an isomorphism.
5. Let A be the subspace of V_4 defined by
   $A = \{(x_1, x_2, x_3, x_4) \mid x_2 = 0\}$ and B be the subspace of P_4 defined by
   $B = \{p \mid p'(1) = 0,\ p''(1) = 0\}$.
   Show that $A \cong B$.

   [**Hints :** Let $T : A \to B$ be defined by
   $T(x_1, x_2, x_3, x_4) = x_1 + (3x_3 + 8x_4)\,x - (3x_3 + 6x_4)\,x^2 + x_3\,x^3 + x_4\,x^4$]

6. Prove that $T : P_3 \to P_4$ be defined by $T\{f(x)\} = f(x+1)$, $f(x) \in P_3$ is an isomorphism

7. Let $T : R^{3\times 1} \to R^{3\times 1}$ be given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Prove that T is an isomorphism.

4.9 The Vector Space L (U, V) :

Let U and V be vector spaces. Consider the set of all linear transformations from U to V and denote it by L (U, V).

Now our aim is to define addition and scalar multiplication in L (U, V), as a result L (U, V) becomes a vector space.

Sum of two linear Transformations :

Let $T, S \in L(U, V)$

i.e., $T : U \to V$ and $S : U \to V$ are linear transformations.

We define

$(S+T) : U \to V$ by $(S+T)(u) = S(u) + T(u)$, for all $u \in U$.

To show that sum of two linear transformations is linear.

Let $u_1, u_2 \in U$ and α be a scalar.

$(S + T)(u_1 + u_2)$

$= S(u_1 + u_2) + T(u_1 + u_2)$

$= S(u_1) + S(u_2) + T(u_1) + T(u_2)$ [$\because$ S and T are linear]

$= \{S(u_1) + T(u_1)\} + \{S(u_2) + T(u_2)\}$

$= (S + T)(u_1) + (S + T)(u_2)$

For all $u \in U$,

$(S+T)(\alpha u)$

$= S(\alpha u) + T(\alpha u)$

$= \alpha S(u) + \alpha T(u)$ [$\because$ S and T are linear]

$= \alpha (S(u) + T(u))$

$= \alpha (S+T)(u)$

$\therefore$ $S+T \in L(U, V)$.

$\therefore$ $S+T : U \to V$ is linear.

Scalar Multiple of a linear transformation :

Let $S \in L(U, V)$ and α being a scalar. Here U and V are vector spaces over same field of scalars.

We define $\alpha S : U \to V$ as $(\alpha S)(u) = \alpha S(u)$, for all $u \in U$.

To show that αS is a linear transformation.

Let $u_1, u_2 \in U$

$$
\begin{aligned}
&(\alpha S)(u_1 + u_2) \\
&= \alpha(S(u_1 + u_2)) && \text{[by definition]} \\
&= \alpha(S(u_1) + S(u_2)) && [\because \text{ S is linear}] \\
&= \alpha(S(u_1)) + \alpha(S(u_2)) \\
&= (\alpha S)(u_1) + (\alpha S)(u_2) && \text{[by definition]}
\end{aligned}
$$

Again, if $u \in U$ and β is a scalar then $(\alpha S)(\beta u)$

$$
\begin{aligned}
&= \alpha(S(\beta u)) && \text{[by definition]} \\
&= \alpha(\beta(S(u))) && [\because \text{ S is linear}] \\
&= \beta(\alpha(S(u))) \\
&= \beta((\alpha S)(u))
\end{aligned}
$$

$\therefore \alpha S \in L(U, V)$

i.e., $\alpha S : U \to V$ is a linear transformation.

Theorem 4.9.1 : Let L (U, V) be the set of all linear transformations from U to V. The operations of addition and scalar multiplications are defined by

$$(S + T)(u) = S(u) + T(u)$$

and $(\alpha S)(u) = \alpha S(u)$ for all $S, T \in L(U, V)$ and α being a scalar.

Then L (U, V) is a vector space.

Proof : In order to prove that L (U, V) is a vector space, we have to verify all the properties of a vector space.

I. (V_1) Closure Property

Let $S, T \in L(U, V)$

i.e., $S : U \to V$ and $T : U \to V$ be two linear transformations.

To prove that S + T, defined by

$(S + T)(u) = S(u) + T(u)$ for all $u \in U$ is a linear transformation i.e.,

$S + T \in L(U, V)$.

Now for $u_1, u_2 \in U$,

$$
\begin{aligned}
&(S+T)(u_1 + u_2) \\
&= S(u_1 + u_2) + T(u_1 + u_2) && \text{[By definition]} \\
&= S(u_1) + S(u_2) + T(u_1) + T(u_2) && [\because \text{ S and T are linear}] \\
&= (S(u_1) + T(u_1)) + (S(u_2) + T(u_2)) \\
&= (S+T)(u_1) + (S+T)(u_2) && \text{[By definition]}
\end{aligned}
$$

Again,

$$(S+T)(\alpha u_1)$$
$$= S(\alpha u_1) + T(\alpha u_1) \qquad \text{[By definition]}$$
$$= \alpha S(u_1) + \alpha T(u_1) \qquad [\because \text{ S and T are linear}]$$
$$= \alpha \{S(u_1) + T(u_1)\}$$
$$= \alpha (S+T)(u_1)$$

$\therefore$ $S+T : U \to V$ is linear i.e., $S+T \in L(U, V)$

(V_2) Commutative Property :

Let $u \in U$

$$\therefore (S+T)(u) = S(u) + T(u)$$
$$= T(u) + S(u)$$
$$= (T+S)(u)$$
$$\Rightarrow S+T = T+S$$

(V_3) Associative Property :

Let $S, T, R \in L(U, V)$

Let $u \in U$

$$\therefore ((S+T)+R)(u)$$
$$= (S+T)(u) + R(u) \qquad \text{[by definition]}$$
$$= \{S(u) + T(u)\} + R(u) \qquad \text{[by definition]}$$
$$= S(u) + \{T(u) + R(u)\} \qquad \text{[by associativity in V]}$$
$$= S(u) + (T+R)(u) \qquad \text{[by definition]}$$
$$= (S+(T+R))(u) \qquad \text{[by definition]}$$

$\therefore$ $(S+T)+R = S+(T+R)$

(V_4) (Existence of additive Identity in L (U, V).

First of all, we define a zero map $O : U \to V$ by $O(u) = O_V$, for all $u \in U$.

For all $u_1, u_2 \in U$, α, β be scalar.

$$O(\alpha u_1 + \beta u_2) = O_V \qquad \text{[by definition]}$$
$$= \alpha O_V + \beta O_V$$
$$= \alpha O(u_1) + \beta O(u_2)$$

$\Rightarrow O : U \to V$ is a linear transformation and consequently $O \in L(U, V)$.

Now to prove that

$$O + T = T + O = T, \text{ For all } T \in L(U, V)$$

For all $u \in U$,

$$(O+T)(u) = O(u) + T(u)$$
$$= O_V + T(u) \qquad \text{[by definition]}$$
$$= T(u)$$
$$(T+O)(u) = T(u) + O(u)$$
$$= T(u) + O_V$$
$$= T(u)$$

Thus

$$O + T = T + O = T$$

Hence O (zero map) is the additive identity for L (U, V).

(V_5) (Existence of additive Inverse in L (U, V) :

Let $T \in L(U, V)$

To prove that $-T \in L(U, V)$

$$(-T)(u_1 + u_2) = (-T)(u_1) + (-T)(u_2)$$

$$\text{and } (-T)(\alpha u) = \alpha(-T)(u)$$

for all $u_1, u_2 \in U$ and α being a scalar. [Refer 4.4. (6)]

For this, we have defined

$$-T \text{ by } (-T)(u) = -T(u) \quad \text{for all } u \in U.$$

$\therefore$ $(-T) \in L(U, V)$.

Again to prove that

$$T + (-T) = (-T) + T = O_V$$

Now, if $u \in U$, then

$$(-T + T)(u) = (-T)(u) + T(u)$$

[by definition of addition in L(U, V)]

$$= -T(u) + T(u) \qquad \text{[by definition of} -T]$$

$$= O_V.$$

$$= O(u) \qquad \text{[by definition of zero map]}$$

$$\Rightarrow (-T) + T = O \text{ for every } T \in L(U, V)$$

Similarly, it can be proved that $T + (-T) = O$.

Thus each element in L (U, V) possesses additive inverse.

Therefore, L (U, V) is an **abelian group** with respect to addition defined in it.

Further we make the following observations :

Now to show that $\alpha T \in L(U, V)$. For all $u_1, u_2 \in U$,

$$(\alpha T)(u_1 + u_2)$$

$$= \alpha T(u_1 + u_2) \qquad [\because \text{ By definition of scalar multiplication}]$$

$$= \alpha(T(u_1) + T(u_2)) \qquad [\because T \text{ is linear}]$$

$$= \alpha T(u_1) + \alpha T(u_2)$$

$$= (\alpha T)(u_1) + (\alpha T)(u_2) \quad [\because \text{ By definition of scalar multiplication}]$$

and for $u \in U$,

$$(\alpha T)(\beta u)$$

$$= \alpha T(\beta u) \qquad [\because \text{ By definition of scalar multiplication}]$$

$$= \alpha(\beta T(u)) \qquad [\because T \text{ is linear}]$$

$$= \beta(\alpha T(u)) \qquad [\because \alpha(\beta v) = \beta(\alpha v)]$$

$$= \beta(\alpha T)(u) \qquad [\because \text{ By definition of scalar multiplication}]$$

$\therefore$ $\alpha T \in L(U, V)$.

II. (i) Let α be a scalar and $S, T \in L(U, V)$.

If $u \in U$ then we have,

$[\alpha (S+T)](u)$

$= \alpha (S+T)(u)$ [by definition of scalar multiplication in L(U, V)]

$= \alpha [S(u)+T(u)]$ [$\because$ T is linear]

$= \alpha S(u) + \alpha T(u)$

$= (\alpha S)(u) + (\alpha T)(u)$ [by definition of scalar multiplication]

$= (\alpha S + \alpha T)(u)$

$\because\ \alpha (S+T) = \alpha S + \alpha T$

(ii) Let α, β be scalars and $T \in L(U, V)$.

If $u \in U$, we have

$[(\alpha+\beta) T](u)$

$= (\alpha+\beta) T(u)$

$= \alpha T(u) + \beta T(u)$

$= (\alpha T)(u) + (\beta T)(u)$

$= (\alpha T + \beta T)(u)$

$\therefore\ (\alpha+\beta) T = \alpha T + \beta T.$

(iii) For all $u \in U$,

$[\alpha(\beta T)](u)$

$= \alpha [(\beta T)(u)]$

$= \alpha [\beta T(u)]$

$= \alpha\beta T(u)$

$= ((\alpha \beta) T)(u)$

$\therefore\ \alpha (\beta T) = (\alpha\beta) T.$

Similarly, it can be proved that $\beta (\alpha T) = (\alpha\beta) T$

(iv) Let $u \in U,\ T \in L(U, V)$.

$(1\ T)(u) = 1\ T(u)$ [By definition of scalar multiplication]

$= T(u)$ [$1\ v = v$]

$\Rightarrow 1\ T = T$

Hence L (U, V) is a vector space.

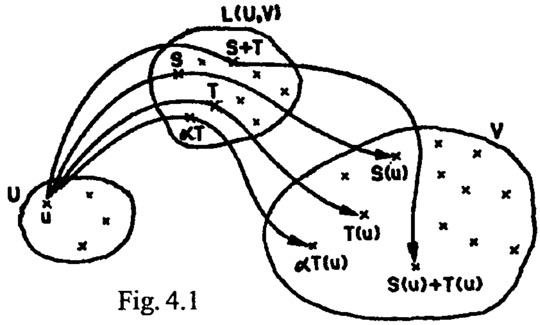

Fig. 4.1

Example 4.9.1 : Let two linear transformations $T : V_3 \to V_2$ and $S : V_3 \to V_2$ be defined by

$T(x, y, z) = (x, y)$

$S(x, y, z) = (x + y + z, 0)$.

Determine the linear transformations

(a) $S + T$ (b) $2S - 3T$ (c) $2S$ (d) $3T$

Solution :

(a) We have $(S + T) : V_3 \to V_2$ and is given by $(S + T)(x, y, z)$

$= S(x, y, z) + T(x, y, z)$

$= (x + y + z, 0) + (x, y)$

$= (2x + y + z, y)$

(b) $(2S - 3T) : V_3 \to V_2$ is given by $(2S - 3T)(x, y, z)$

$= (2S)(x, y, z) - (3T)(x, y, z)$

$= 2S(x, y, z) - 3T(x, y, z)$

$= 2(x + y + z, 0) - 3(x, y)$

$= (2x + 2y + 2z - 3x, 0 - 3y)$

$= (-x + 2y + 2z, -3y)$

(c) $(2S) : V_3 \to V_2$ is given by $(2S)(x, y, z)$

$= 2(S(x, y, z))$

$= 2(x + y + z, 0)$

$= (2x + 2y + 2z, 0)$

(d) $(3T) : V_3 \to V_2$ is given by $(3T)(x, y, z)$

$= 3(T(x, y, z))$

$= 3(x, y) = (3x, 3y)$

Example 4.9.2 : Let $T : V_3 \to V_3$ and $S : V_3 \to V_3$ be two linear transformations defined by

$T(x_1, x_2, x_3) = (x_1 + x_3, x_2 + x_3, x_1 + x_2)$

and $S(e_1) = e_1$,

$S(e_2) = e_1 + e_2$

$S(e_3) = e_3$

Determine the linear transformation : $2S + T$

Solution : $(2S + T) : V_3 \to V_3$ is given by

$(2S + T)(x_1, x_2, x_3)$

$= (2S)(x_1, x_2, x_3) + T(x_1, x_2, x_3)$

$= 2(S(x_1, x_2, x_3)) + T(x_1, x_2, x_3)$

$= 2S(x_1 e_1 + x_2 e_2 + x_3 e_3) + T(x_1, x_2, x_3)$

$= 2\{x_1 S(e_1) + x_2 S(e_2) + x_3 S(e_3)\} + T(x_1, x_2, x_3)$

$= (2x_1 e_1 + 2x_2 (e_1 + e_2) + x_3 e_3) + (x_1 + x_3, x_2 + x_3, x_1 + x_2)$

$= (2x_1 + 2x_2, 2x_2, x_3) + (x_1 + x_3, x_2 + x_3, x_1 + x_2)$

$= (3x_1 + 2x_2 + x_3, 3x_2 + x_3, x_1 + x_2 + x_3)$

Example 4.9.3 : Determine two linear transformations T and S of rank 4 from V_4 to V_4 such that (a) r(T – S) = 2 (b) r(T + 2S) = 1

Solution : (a) Let $T: V_4 \to V_4$ be defined by

$$T(e_1) = e_1, T(e_2) = -e_2, T(e_3) = e_1, T(e_4) = e_2$$

and $S: V_4 \to V_4$ be defined by

$$S(e_1) = e_2,\ S(e_2) = -e_1,\ S(e_3) = -e_2,\ S(e_4) = -e_1$$

$\therefore$ $(T-S)(e_1) = e_1 - e_2,\ (T-S)(e_2) = e_1 - e_2,\ (T-S)(e_3) = e_1 + e_2,\ (T-S)(e_4) = e_1 + e_2$

$\therefore$ $\{(T-S)(e_1), (T-S)(e_2), (T-S)(e_3), (T-S)(e_4)\} = \{e_1 - e_2, e_1 + e_2\}$

is a basis of Range of (T – S)

$\therefore$ $r(T-S) = 2$

(b) Let $T: V_4 \to V_4$ be defined by

$$T(e_1) = e_1,\ T(e_2) = -e_2,\ T(e_3) = e_1,\ T(e_4) = -e_2$$

and $S: V_4 \to V_4$ be defined by

$$S(e_1) = -\frac{1}{2}e_2,\ S(e_2) = \frac{1}{2}e_1,\ S(e_3) = -\frac{1}{2}e_2,\ S(e_4) = \frac{1}{2}e_1$$

$\therefore$ $(T+2S)(e_1) = e_1 - e_2,\ (T+2S)(e_2) = e_1 - e_2,$

$(T+2S)(e_3) = e_1 - e_2,\ (T+2S)(e_4) = e_1 - e_2$

$\because$ $\{(T+2S)e_4\} = \{e_1 - e_2\}$ is a basis of R (T + 2S)

$\therefore$ $r(T+2S) = 1.$

Problem Set 4 (F)

1. Let the linear maps $T: V_2 \to V_2$ and $S: V_2 \to V_2$ be defined by
   $T(x_1, x_2) = (x_1 + x_2, 0)$
   $S(x_1, x_2) = (2x_1, 3x_1 + 4x_2)$
   Determine the linear maps
   (a) 2S + 3T
   (b) 3S – 7T

2. Let the linear maps $T: V_3 \to V_3$ and $S: V_3 \to V_3$ be defined by
   $T(x_1, x_2, x_3) = (2x_1 - 3x_2, 4x_1 + 6x_2, x_3)$
   $S(e_1) = e_2 - e_3,\ S(e_2) = e_1, S(e_3) = e_1 + e_2 + e_3$
   Determine the linear maps
   (a) S + T
   (b) 3S – 2T.

3. Determine two linear transformations T and S from V_4 to V_4 of rank 4 such that
   (a) r(T + S) = 3
   (b) r(T – S) = 0.

4.10 Composition of Linear Maps :

Let $T : U \to V$ and $S : V \to W$ be two linear maps. The composition of S and T is a function $So\,T : U \to W$, defined by

$(So\,T)(u) = S(T(u))$, for all $u \in U$.

S 0 T can also be written as ST, which we may call the product of S and T.

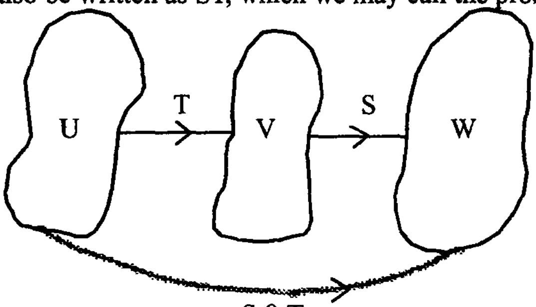

Fig. 4.2

Theorem 4.10.1 : The composition of two linear map is a linear map.

Proof : Let $T : U \to V$ and $S : V \to W$ be two linear maps.

To prove that $So\,T : U \to W$ is linear.

For this, all we need to prove that

(i) $(S0\,T)(u_1 + u_2) = (S0\,T)(u_1) + (S0\,T)(u_2)$,

$u_1, u_2 \in U$

(ii) $(S0\,T)(\alpha u) = \alpha\,(S0\,T)(u)$, $u \in U$ and α being a scalar.

Let $u_1, u_2 \in U$.

$(So\,T)(u_1 + u_2)$

$= S(T(u_1 + u_2))$ [by definition of composition]

$= S(T(u_1) + T(u_2))$ [$\because$ T is linear]

$= S(T(u_1)) + S(T(u_2))$ [$\because$ S is linear]

$= (S\,0\,T)(u_1) + (S\,0\,T)(u_2)$ [by definition of composition]

Let $u \in U$, α being a scalar.

$\therefore$ $(S\,0\,T)(\alpha u) = S(T(\alpha u))$ [by definition of composition]

$= S(\alpha\,T(u))$ [$\because$ T is linear]

$= \alpha\,(S(T(u)))$ [$\because$ S is linear]

$= \alpha\,(S\,0\,T)(u)$ [by definition of composition]

Hence $S\,0\,T : U \to W$ is a linear map.

Note : (1) If S and T are linear maps and both S 0 T and T 0 S exists and are linear maps then $S0T \neq T0S$ (in general)

Note : (2) For any linear operator T on V, T^K for $k \geq 0$, is defined as follows :

$$T^0 = I \quad \text{(identity operator on V)}$$
$$T^2 = T\,T$$
$$T^{k+1} = T^k\,T$$

It can be easily proved that $T^m\,T^n = T^{m+n}$, $m \geq 0$, $n \geq 0$.

(3) Let $T : U \to V$ be a non-singular linear map.

$\therefore$ T is linear, one-one and onto.

$\Rightarrow T^{-1} : V \to U$ exists and is linear.

Hence $TT^{-1} : V \to V$

$\Rightarrow TT^{-1} = I_V$

and $T^{-1}T : U \to U$

$\Rightarrow T^{-1}T = I_U$.

Example 4.10.1 : Let $T : V_2 \to V_3$ be defined by $T(x_1, x_2) = (0, x_1, x_2)$ and $S : V_3 \to V_2$ be defined by $S(x_1, x_2, x_3) = (x_1 + x_2,\ x_2 + x_3)$. Find S 0 T and T 0 S. Are they equal ?

Solution : We have $S0T : V_2 \to V_2$ and $ToS : V_3 \to V_3$. Both are defined.

$$\begin{aligned}\text{Now } (SoT)(x_1, x_2) &= S(T(x_1, x_2))\\ &= S(0, x_1, x_2)\\ &= (x_1, x_1 + x_2)\end{aligned}$$

$$\begin{aligned}(T0S)(x_1, x_2, x_3) &= T(S(x_1, x_2, x_3))\\ &= T(x_1 + x_2, x_2 + x_3)\\ &= (0, x_1 + x_2, x_2 + x_3)\end{aligned}$$

$\therefore S0T \neq T0S.$

Example 4.10.2 : Let D and T be two linear maps on P, the vector space of all polynomials in x defined by

$$D(f(x)) = \frac{d}{dx} f(x)$$

and $T(f(x)) = x\,f(x)$, for every $f(x) \in P$.

Then show that $D0T \neq T0D$ and $D0T - T0D = I$.

Solution : We have

$$\begin{aligned}(D0T)(f(x)) &= D(T(f(x)))\\ &= D(x f(x))\\ &= \frac{d}{dx}(x\,f(x))\\ &= f(x) + x\frac{d}{dx} f(x) \qquad ...(1)\end{aligned}$$

and $(T 0 D)(f(x)) = T(D(f(x)))$

$$= T\left(\frac{d}{dx} f(x)\right) = x \frac{d}{dx} f(x) \qquad ...(2)$$

From (1) and (2), we see that

for every $f(x) \in P$,

$(D 0 T)(f(x)) \neq (T 0 D)(f(x))$

$\Rightarrow D 0 T \neq T 0 D.$

Also, $(D 0 T - T 0 D)(f(x))$

$= (D 0 T)(f(x)) - (T 0 D)(f(x))$

$$= f(x) + x\frac{d}{dx} f(x) - x\frac{d}{dx} f(x)$$

$= f(x) = I(f(x))$

$\therefore D 0 T - T 0 D = I.$

Example 4.10.3 : Let the linear maps $T : V_2 \to V_2$ and $S : V_2 \to V_2$ be defined by

$T(x_1, x_2) = (x_1 + x_2, 0)$ and $S(x_1, x_2) = (2x_1, 3x_1 + 4x_2)$.

Determine (i) $T^2 S$ (ii) $S T S$.

Solution : We have

$T^2S : V_2 \to V_2$ and $STS : V_2 \to V_2$

(i) $(T^2S)(x_1, x_2)$

$= T^2(S(x_1, x_2))$

$= T^2(2x_1, 3x_1 + 4x_2)$

$= T(5x_1 + 4x_2, 0)$

$= (5x_1 + 4x_2, 0)$

(ii) $(STS)(x_1, x_2)$

$= (ST)(S(x_1, x_2))$

$= (ST)(2x_1, 3x_1 + 4x_2)$

$= S(T(2x_1, 3x_1 + 4x_2))$

$= S(5x_1 + 4x_2, 0)$

$= (2(5x_1 + 4x_2),\ 3(5x_1 + 4x_2) + 4.0)$

$= (10x_1 + 8x_2, 15x_1 + 12x_2)$.

Example 4.10.4 : Let V be a 1-dimensional vector space and S, T be two linear maps on V. Then prove that S 0 T = T 0 S.

Solution : Suppose $S(1) = \alpha$ and and $T(1) = \beta$, where α and β are scalars.

Let $x \in V$

$$\therefore\ (S\,0\,T)(x) = S(T(x)) = S(x\,T(1))$$
$$= S(x\beta)$$
$$= x\beta\,S(1)$$
$$= x\,\beta\,\alpha$$

Again $(T\,0\,S)(x) = T\,(S(x))$
$$= T\,(x\,S(1))$$
$$= T\,(x\,\alpha)$$
$$= x\,\alpha\,T(1)$$
$$= x\,\alpha\,\beta$$

$$\therefore\ (S\,0\,T)(x) = (T\,0\,S)(x)$$
$$\Rightarrow\ S\,0\,T = T\,0\,S, \forall\, x \in V.$$

Example 4.10.5 : Let T be an idempotent linear map on a vector space V. Then

$$V = R(T) \oplus N(T)$$

Solution : By definition, $T^2 = T$

To show that

$$V = R(T) \oplus N(T)$$

i.e., (i) $V = R\,(T) + N\,(T)$

(ii) $R(T) \cap N(T) = \{0\}$

Now $T^2 = T$
$$\Rightarrow\ T^2\alpha = T\alpha, \quad \alpha \in R(T)$$
$$\Rightarrow\ T(\alpha - T\alpha) = 0$$
$$\Rightarrow\ \alpha - T\alpha \in N(T)$$

Also $T\alpha \in R(T)$

Hence $\alpha = T\alpha + (\alpha - T\alpha) \in R(T) + N(T)$

Hence $V = R(T) + N(T)$

Again, Let $\alpha \in R(T) \cap N(T)$
$$\Rightarrow\ \alpha \in R(T) \text{ and } \alpha \in N(T)$$
$$\Rightarrow\ T\beta = \alpha, \text{ for } \beta \in V \text{ and } T\alpha = 0$$
$$\therefore\ \mathbf{0} = T\alpha = T\,T\,\beta = 0 \text{ i.e., } \alpha = 0$$

Hence, $R(T) \cap N(T) = \{0\}$

$$\therefore\ V = R(T) \oplus N(T).$$

Theorem 4.10.2 : Let T_1, T_2 be linear maps from U to V. Let S_1, S_2 be linear maps from V to W. Let P be a linear map from W to Z, where U, V, W and Z are vector spaces over the same field of scalars. Then

(a) $S_1(T_1 + T_2) = S_1 T_1 + S_1 T_2$

(b) $(S_1 + S_2) T_1 = S_1 T_1 + S_2 T_1$

(c) $P(S_1 T_1) = (PS_1) T_1$

(d) $(\alpha S_1) T_1 = \alpha (S_1 T_1) = S_1(\alpha T_1)$, where α is a scalar.

(e) $I_V T_1 = T_1$ and $T_1 I_U = T_1$

Proof : (a) We have

$$T_1 + T_2 : U \to V \text{ and } S_1 : V \to W$$
$$\Rightarrow S_1(T_1 + T_2) : U \to W$$

Again $T_1 : U \to V,\ S_1 : V \to W$

$$\Rightarrow S_1 T_1 : U \to W$$

and $T_2 : U \to V,\ S_1 : V \to W \Rightarrow S_1 T_2 : U \to W$

Hence, the products

$S_1(T_1 + T_2)$ and $S_1 T_1, S_1 T_2$ are well defined.

Now to prove that

$$S_1(T_1 + T_2) = S_1 T_1 + S_1 T_2$$

Let $u \in U$.

$\therefore (S_1(T_1 + T_2))(u) = S_1((T_1 + T_2)(u))$ [By definition of product]

$= S_1(T_1(u) + T_2(u))$ [By definition of addition in L (U, V)]

$= S_1(T_1(u)) + S_1(T_2(u))$ [$\because S_1$ is linear]

$= (S_1 T_1)(u) + (S_1 T_2)(u)$ [By definition of product]

$= (S_1 T_1 + S_1 T_2)(u)$ [By definition of addition in L (U, V)]

Hence $S_1(T_1 + T_2) = S_1 T_1 + S_1 T_2$

(b) To prove that $(S_1 + S_2) T_1 = S_1 T_1 + S_2 T_1$.

We have

$$T_1 : U \to V,\ S_1 + S_2 : V \to W$$
$$\Rightarrow (S_1 + S_2) T_1 : U \to W$$

Again $S_1 T_1 : U \to W,\ S_2 T_1 : U \to W$

$$\Rightarrow S_1 T_1 + S_2 T_1 : U \to W.$$

The products $(S_1 + S_2) T_1, S_1 T_1, S_2 T_1$ are well defined.

Let $u \in U$

$((S_1 + S_2) T_1)(u)$

$= (S_1 + S_2)(T_1(u))$ [definition of product]

$= S_1(T_1(u)) + S_2(T_1(u))$ [definition of addition in L (U, W)]

$= (S_1 T_1)(u) + (S_2 T_1)(u)$ [definition of product]

$= (S_1 T_1 + S_2 T_1)(u)$ [definition of addition L (U, W)]

Hence $(S_1 + S_2) T_1 = S_1 T_1 + S_2 T_1$

(c) To prove that $P(S_1 T_1) = (P S_1) T_1$

We have

$$S_1 T_1 : U \to W,\ P : W \to Z$$
$$\Rightarrow P(S_1 T_1) : U \to Z$$
$$PS_1 : V \to Z,\ T_1 : U \to V$$
$$\Rightarrow (P S_1) T_1 : U \to Z$$

Hence, the products $P(S_1 T_1)$ and $(P S_1) T_1$ are defined.

Now, let $u \in U$

$$\begin{aligned} \therefore (P(S_1 T_1))(u) &= P((S_1 T_1)(u)) \\ &= P(S_1 (T_1(u))) \\ &= (P S_1)(T_1(u)) \\ &= ((P S_1) T_1)(u) \end{aligned}$$

$$\Rightarrow P(S_1 T_1) = (P S_1) T_1$$

(d) To prove that $(\alpha S_1) T_1 = \alpha (S_1 T_1) = S_1 (\alpha T_1)$

We have

$$T_1 : U \to V,\ \alpha S_1 : V \to W$$
$$\Rightarrow (\alpha S_1) T_1 : U \to W$$

Again, $\alpha (S_1 T_1) : U \to W$

and $S_1 : V \to W,\ \alpha T_1 : U \to V$

$$\Rightarrow S_1 (\alpha T_1) : U \to W.$$

Let $u \in U$

$$\begin{aligned} ((\alpha S_1) T_1)(u) &= (\alpha S_1)(T_1(u)) && \text{[by definition of product]} \\ &= \alpha (S_1 (T_1(u))) && \text{[by definition of scalar multiplication]} \\ &= \alpha ((S_1 T_1)(u)) && \text{[by definition of product]} \\ &= (\alpha (S_1 T_1))(u) && \text{[by definition of scalar multiplication]} \end{aligned}$$

$$\Rightarrow (\alpha S_1) T_1 = \alpha (S_1 T_1)$$

$$\begin{aligned} \text{Again } (\alpha (S_1 T_1))(u) &= \alpha (S_1 (T_1(u))) \\ &= S_1 ((\alpha T_1)(u)) \\ &= (S_1 (\alpha T_1))(u) \end{aligned}$$

$$\Rightarrow \alpha (S_1 T_1) = S_1 (\alpha T_1)$$

Hence $(\alpha S_1) T_1 = \alpha (S_1 T_1) = S_1 (\alpha T_1)$.

(e) To prove that $I_V T_1 = T_1$ and $T_1 I_U = T_1$

We have

$$T_1 : U \to V,\ I_V : V \to V$$
$$\Rightarrow I_V T_1 : U \to V$$

Hence $I_V T_1$ and T_1, both maps from U to V.

For all $u \in U$,

$$(I_V\, T_1)(u) = I_V\,(T_1(u)) \qquad \text{[definition of product]}$$
$$= T_1(u) \qquad \text{[definition of identity map]}$$
$$\Rightarrow I_V\, T_1 = T_1$$

Further $I_U : U \to U,\ T_1 : U \to V$

$$\Rightarrow T_1\, I_U : U \to V$$

$\therefore$ both products $T_1.I_U$ and T_1 are defined.

For all $u \in U$,

$$(T_1\, I_U)(u) = T_1\,(I_U(u)) \qquad \text{[definition of product]}$$
$$= T_1(u) \qquad \text{[definition of identity map]}$$
$$\Rightarrow T_1\, I_U = T_1$$

Example 4.10.6: Let $R : V_2 \to V_2$ be defined by $R(x, y) = (y, 2x)$, $S : V_3 \to V_2$ be defined by $S(x, y, z) = (y, x + z)$ and $T : V_3 \to V_2$ be defined by $T(x, y, z) = (2z, x - y)$.

Verify that $R(S + T) = RS + RT$.

Solution : We have

$$S + T : V_3 \to V_2 \text{ and } R : V_2 \to V_2$$
$$\Rightarrow R(S+T) : V_3 \to V_2.$$

Again $R : V_2 \to V_2,\ S : V_3 \to V_2 \ \Rightarrow\ RS : V_3 \to V_2$

and $R : V_2 \to V_2,\ T : V_3 \to V_2$

$$\Rightarrow RT : V_3 \to V_2$$

$\therefore$ $RS + RT : V_3 \to V_2$

$\therefore$ Products $R(S + T)$ and RS, RT are defined. So both sides makes sense

Let $u = (x, y, z) \in V_3$

$\therefore$ $(R(S + T))(x, y, z)$

$$= R((S + T)(x, y, z))$$
$$= R(S(x, y, z) + T(x, y, z))$$
$$= R((y, x + z) + (2z, x - y))$$
$$= R(y + 2z, 2x - y + z)$$
$$= (2x - y + z, 2y + 4z)$$

and $(RS + RT)(x, y, z)$

$$= (RS)(x, y, z) + (RT)(x, y, z)$$
$$= R(S(x, y, z)) + R(T(x, y, z))$$
$$= R(y, x + z) + R(2z, x - y)$$
$$= (x + z, 2y) + (x - y, 4z)$$
$$= (2x - y + z, 2y + 4z)$$

$\therefore$ $(R(S + T))(u) = (RS + RT)(u)$

$$\Rightarrow R(S + T) = RS + RT.$$

Hence verified.

Theorem 4.10.3 : A linear map $T : U \to V$ is non-singular iff there exists a linear map $S : V \to U$ such that $TS = I_V$ and $ST = I_U$.

Then $S = T^{-1}$ and $T = S^{-1}$

Proof : Let T be non-singular.

$\therefore$ T is onto, therefore $\beta \in V \Rightarrow$ there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

Then $T^{-1}(\beta) = \alpha$

We have $T(\alpha) = \beta$

$$\Rightarrow T^{-1}(T(\alpha)) = T^{-1}(\beta)$$
$$\Rightarrow (T^{-1}T)(\alpha) = T^{-1}(\beta)$$
$$\Rightarrow (T^{-1}T)(\alpha) = \alpha = I_U(\alpha) \qquad [\because T^{-1}T : U \to U]$$
$$\Rightarrow T^{-1}T = I_U$$
$$\Rightarrow ST = I_U \qquad [\text{Taking } S = T^{-1}]$$

Again, Let $\beta \in V$. Since T is onto, therefore

$\beta \in V \Rightarrow$ there exists $\alpha \in U$ such that $T(\alpha) = \beta$

Then $T^{-1}(\beta) = \alpha$.

Now $T^{-1}(\beta) = \alpha$

$$\Rightarrow T(T^{-1}(\beta)) = T(\alpha)$$
$$\Rightarrow (TT^{-1})(\beta) = \beta$$
$$\Rightarrow (TT^{-1})(\beta) = \beta = I_V(\beta) \qquad [\because TT^{-1} : V \to V]$$
$$\Rightarrow TT^{-1} = I_V$$
$$\Rightarrow TS = I_V \qquad [\text{Taking } S = T^{-1}]$$

Conversely, Let S and T exists and $TS = I_V$ and $ST = I_U$.

To prove that T is non-singular i.e., to prove that (i) T is one-one and (ii) T is onto.

(i) T is one-one.

Let $\alpha, \beta \in U$

Then $T(\alpha) = T(\beta)$

$$\Rightarrow S(T(\alpha)) = S(T(\beta))$$
$$\Rightarrow (ST)(\alpha) = (ST)(\beta)$$
$$\Rightarrow I_U(\alpha) = I_U(\beta)$$
$$\Rightarrow \alpha = \beta$$

$\therefore$ T is one-one.

(ii) T is onto.

Let $\beta \in V$.

$T(S(\beta)) = (TS)(\beta) = I_V(\beta) = \beta$

Therefore there exists an element $\alpha = S(\beta) \in U$ such that

$T(\alpha) = \beta$

Thus $\beta \in V \Rightarrow$ there exists $\alpha \in U$ such that $T(\alpha) = \beta$.

$\therefore$ T is onto.

Since T is one-one and onto, therefore T is non-singular.

$\therefore$ T^{-1} exists.

Further to show that

$$S = T^{-1} \text{ and } T = S^{-1}$$

We have $TS = I_V$

$$\Rightarrow T^{-1}(TS) = T^{-1}(I_V)$$
$$\Rightarrow (T^{-1}T)S = T^{-1}$$
$$\Rightarrow I_U S = T^{-1}$$
$$\Rightarrow S = T^{-1}$$

Again

$$T = T I_U$$
$$= T(SS^{-1})$$
$$= (TS)S^{-1}$$
$$= I_V S^{-1}$$
$$= S^{-1}$$

Theorem 4.10.4 : Let $T : U \to V$ and $S : V \to W$ be two linear maps. Then

(a) If S and T are non-singular then ST is non-singular and

$$(ST)^{-1} = T^{-1} S^{-1}$$

(b) If ST is one-one, then T is one-one.

(c) If ST is onto, the S is onto.

(d) If ST is non-singular, then T is one-one and S is onto.

Proof : (a) Since S is non-singular, S^{-1} exists and

$$SS^{-1} = I_W \text{ and } S^{-1}S = I_V$$

Since T is non-singular, T^{-1} exists and

$$TT^{-1} = I_V \text{ and } T^{-1}T = I_U.$$

Now, we have

$$(ST)(T^{-1}S^{-1})$$
$$= S((TT^{-1})S^{-1})$$
$$= S(I_V S^{-1})$$
$$= SS^{-1} = I_W$$

and

$$(T^{-1}S^{-1})(ST)$$
$$= T^{-1}((S^{-1}S)T)$$
$$= T^{-1}(I_V T)$$
$$= T^{-1}T = I_U$$

$\therefore (ST)(T^{-1}S^{-1}) = I_W$ and $(T^{-1}S^{-1})(ST) = I_U$.

ST is non-singular with inverse $T^{-1}S^{-1}$ i.e.,

$$(ST)^{-1} = T^{-1}S^{-1}$$

(b) Given ST is one-one. To prove that T is one-one.

Let $u \in N(T)$

$\Rightarrow T(u) = 0_V$

$\Rightarrow S(T(u)) = S(0_V)$

$\Rightarrow (ST)(u) = \mathbf{0}_W$

$\Rightarrow u = 0_U$ [$\because$ ST is one-one]

$\Rightarrow N(T) = \{0_U\}$

$\therefore$ T is one-one.

(c) Given ST is onto. To prove that T is onto.

Let $w \in W$.

Since ST is onto, there exists $u \in U$ such that

$(ST)(u) = w$

$\Rightarrow S(T(u)) = w$

Hence, there exists a vector $v = T(u) \in V$

such that $S(v) = w$

$\therefore$ S is onto.

(d) Given that ST is non-singular i.e., ST is one-one and onto.

Now, ST is one-one $\Rightarrow$ T is one-one (Proved in (b))

ST is onto $\Rightarrow$ S is onto (Proved in (c))

Example 4.10.7 : If a linear transformation T on V satisfies the condition $T^2 + I = T$, then prove that T^{-1} exists and find T^{-1}.

Solution : If $T^2 + I = T$, then $T^2 - T = -I$

First we shall prove that T is one-one.

Let $u, v \in V$. Then

$T(u) = T(v)$ ... (1)

$\Rightarrow T(T(u)) = T(T(v))$

$\Rightarrow T^2(u) = T^2(v)$ ... (2)

$\Rightarrow T^2(u) - T(u) = T^2(v) - T(v)$ [From (1) and (2)]

$\Rightarrow (-I)(u) = (-I)(v)$

$\Rightarrow -(I(u)) = -(I(v))$

$\Rightarrow -u = -v$

$\Rightarrow u = v$

$\therefore$ T is one-one.

Now to prove that T is onto.

Let $v \in V$. Then $v - T(v) \in V$

We have

$$\begin{aligned} T(v - T(v)) &= T(v) - T(T(v)) \\ &= T(v) - T^2(v) \\ &= (T - T^2)(v) \\ &= I(v) \qquad [\because T^2 - T = -I \Rightarrow T - T^2 = I] \\ &= v \end{aligned}$$

Thus $v \in V \Rightarrow$ there exists $v - T(v) \in V$ such that $T(v - T(v)) = v$.

$\therefore$ T is onto.

Hence T is non-singular i.e., T^{-1} exists. Now to find T^{-1}.

$$\begin{aligned} \text{Now,} \quad & T - T^2 = I \\ \Rightarrow\ & T(I - T) = I \\ \Rightarrow\ & T^{-1} = I - T. \end{aligned}$$

Example 4.10.8 : Let T be a linear map on V_3 defined by

$T(e_1) = e_3,\ T(e_2) = e_1, T(e_3) = e_2$. Prove that $T^2 = T^{-1}$.

Solution : Given $T(e_1) = e_3,\ T(e_2) = e_1, T(e_3) = e_2$.

$$\begin{aligned} \therefore\ T^2(e_1) &= T(e_3) = e_2 \\ T^2(e_2) &= T(e_1) = e_3 \\ T^2(e_3) &= T(e_2) = e_1 \end{aligned}$$

Let $x \in V_3$

$$\begin{aligned} \therefore\ x &= (x_1, x_2, x_3) \\ &= x_1 e_1 + x_2 e_2 + x_3 e_3 \end{aligned}$$

$$\begin{aligned} T^2(x) &= T^2(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= x_1 T^2(e_1) + x_2 T^2(e_2) + x_3 T^2(e_3) \\ &= x_1 e_2 + x_2 e_3 + x_3 e_1 \\ &= (x_3, x_1, x_2) \qquad \ldots (1) \end{aligned}$$

Further,

$$T^{-1}(e_3) = e_1,\ T^{-1}(e_1) = e_2, T^{-1}(e_2) = e_3$$

$$\begin{aligned} \therefore\ T^{-1}(x) &= T^{-1}(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= x_1 T^{-1}(e_1) + x_2 T^{-1}(e_2) + x_3 T^{-1}(e_3) \\ &= x_1 e_2 + x_2 e_3 + x_3 e_1 \\ &= (x_3, x_1, x_2) \qquad \ldots (2) \end{aligned}$$

From (1) and (2),

$$T^2(x) = T^{-1}(x)$$

$$\Rightarrow T^2 = T^{-1}$$

Example 4.10.9 : The zero transformation and the identity transformation are idempotent.

Solution : Let $T: V \to V$ be a linear map.

By definition of zero map, $T(u) = \mathbf{0}_V$, $u \in V$

$$\therefore \quad T^2(u) = T(T(u))$$
$$= T(O_V) = T(u)$$
$$\therefore \quad T^2 = T$$
$$\Rightarrow T \text{ is idempotent.}$$

Again, by definition of identity map : $I_U(u) = u$, $u \in V$

$$\therefore \quad I_U^2(u) = I_U\,(I_U(u))$$
$$= I_U(u)$$
$$\therefore \quad I_U^2 = I_U$$
$$\Rightarrow I \text{ is iempotent.}$$

Example 4.10.10 : (1) Let $f \in P_n$ where

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_n\,x^n$$
$$\Rightarrow \frac{d^{n+1}}{d\,x^{n+1}}(f(x)) = 0$$

Further, $\quad x^n \in P_n$ and $\frac{d^n}{d\,x^n}(x^n) = n\,! \neq 0$

Let $D = \frac{d}{dx}$.

$\therefore \quad D^n(f) \neq 0$, but $D^{n+1}(f) = 0$

$\Rightarrow$ D is a nilpotent transformation of degree (n + 1).

(2) $\quad T(x_1, x_2, x_3) = (0, x_1, x_2)$

$\Rightarrow T^2\,(x_1, x_2, x_3) = T(0, x_1, x_2) = (0, 0, x_1)$

$\Rightarrow T^3\,(x_1, x_2, x_3) = T(0, 0, x_1) = (0, 0, 0)$

$\therefore$ T is nilpotent of degree 3.

Problem Set 4 (G)

1. Let $T : V_2 \to V_2$ be defined by $T(x, y) = (0, x)$ and $S : V_2 \to V_2$ be defined by $S(x, y) = (y, x)$. Then determine.

   (a) S T (b) T S (c) S^2 (d) T^2

2. Let $T : V_3 \to V_2$ be defined $T(x_1, x_2, x_3) = (x_2, x_1+x_3)$, $S : V_3 \to V_2$ be defined by $S(x_1, x_2, x_3) = (2x_3, x_1 - x_2)$ and $R : V_2 \to V_2$ be defined by $R(x_1, x_2) = (x_2, 2x_1)$.

   Then determine

   (a) RT (b) RS (c) TS (d) SR (e) R(S + T) (f) RS + RT.

3. Let R, S and T be three linear maps from V_3 to V_3 defined by
$R(e_1) = e_1 + e_2$, $R(e_2) = e_1 - e_2 + e_3$, $R(e_3) = 3e_1 + 4e_3$; $S(e_1) = e_1 - e_2$, $S(e_2) = e_2$, $S(e_3) = e_1 + e_2 - 7e_3$ and $T(e_1) = e_1 - e_2 + e_3$, $T(e_2) = 3e_1 - 5e_3$, $T(e_3) = 3e_1 - 2e_3$. Determine

(a) RST (b) T^2 (c) $T^2 ST$

4. Let a linear map $T : V_3 \to V_4$ be defined by
$T(e_1) = (1, 1, 0, 0)$, $T(e_2) = (1, -1, 1, 0)$, $T(e_3) = (0, -1, 1, 1)$ where $\{e_1, e_2, e_3\}$ is the standard basis for V_3 and a linear map $R : V_4 \to V_2$ be defined by
$R(f_1) = (1, 0)$, $R(f_2) = (1, 1)$, $R(f_3) = (1, -1)$, $R(f_4) = (0, 1)$,
where $\{f_1, f_2, f_3, f_4\}$ is the standard basis for V_4. Then find
(a) (RT) (e_1) (b) (RT) (e_2) (c) (RT) (e_3)

5. Let T be a linear map on a finite-dimensional vector space V. Prove that if $r(T^2) = r(T)$ then $R(T) \cap N(T) = \{0\}$

6. Let S and T be two linear maps on U, such that ST = TS. Then prove that
$(S+T)^n = S^n + {}^nC_1\ S^{n-1} + {}^nC_2\ S^{n-2}\ T^2 + \ldots + {}^nC_n\ T^n$
Hence or otherwise, Prove that
$(S+T)^2 = S^2 + 2\,ST + T^2$.

7. Find two linear maps T and S on V_2 such that
$TS = \mathbf{0}$ but $ST \neq \mathbf{0}$.
[**Hints :** $T : V_2 \to V_2$ be defined by $T(x, y) = (x, 0)$ and $S : V_2 \to V_2$ be defined by $S(x, y) = (0, x)$]

8. Let $T : V_3 \to V_3$ be defined by $T(x_1, x_2, x_3) = (0, x_1, x_2)$.
Show that $T \neq \mathbf{0}$, $T^2 \neq \mathbf{0}$ but $T^3 = \mathbf{0}$.

9. Let $T : U \to V$ and $S : V \to W$ be two linear maps. Then prove that
(a) If T is onto then r(ST) = r(S)
(b) If S is one-one then r(ST) = r(T)

10. Let $T : V_3 \to V_3$ be defined by
$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Prove that T is invertible and find T^{-1}. Also, prove that $(T^2 - I)(T - 3I) = \mathbf{0}$

11. Let $T : V_3 \to V_2$ and $S : V_2 \to V_3$ be linear maps defined by
$T(x_1, x_2, x_3) = (x_1 - 3x_2 - 2x_3, x_2 - 4x_3)$
and $S(x_1, x_2) = (2x_1, 4x_1 - x_2, 2x_1 + 3x_2)$. Find ST and TS. Is product commutative ?

12. Let T be a linear map defined on V such that $T^2 = \mathbf{0}$. Show that I – T is invertible.

13. Let T be a linear map on V such that $T^2 + 2T + I = \mathbf{0}$. Show that T is invertible.

14. Let a linear map T on V_3 be defined by $T(x_1, x_2, x_3) = (x_1, 0, 0)$.

    If T is idempotent on a vector space V_3 then show that I – T is also idempotent on V_3.

15. Let S and T be two linear maps on V_3 defined by $S(x_1, x_2, x_3) = (x_1, 0, 0)$ and $T(x_1, x_2, x_3) = (0, x_2, x_3)$. Show that S and T are idempotent on a vector space V. Also find the condition under which ST and S + T are idempotent.

16.(a) Show that the differential operator $D : P_n \to P_n$ is nilpotent.

(b) Show that $T : V_4 \to V_4$ be defined by $T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3)$ is nilpotent.

17. Let $T : V_4 \to V_4$ be defined by

    $T(x_1, x_2, x_3, x_4) = (0, 2x_1, 3x_1 + 2x_2, x_2 + 4x_3)$.

    Prove that T is nilpotent. Find the degree of nilpotence. Also show that (I + T) is non singular and $(I + T)^{-1} = I - T + T^2 - T^3$.

18. If U, V, W are of same finite dimension and ST is non-singular, then show that both S and T are non-singular.

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