Module 5:

Derivatives and Integrals

Today’s Outline

I. Introduction to Calculus

II. Mapping & Relations

III. Functions

IV. Limits

V. Continuity

VI. Tutorial

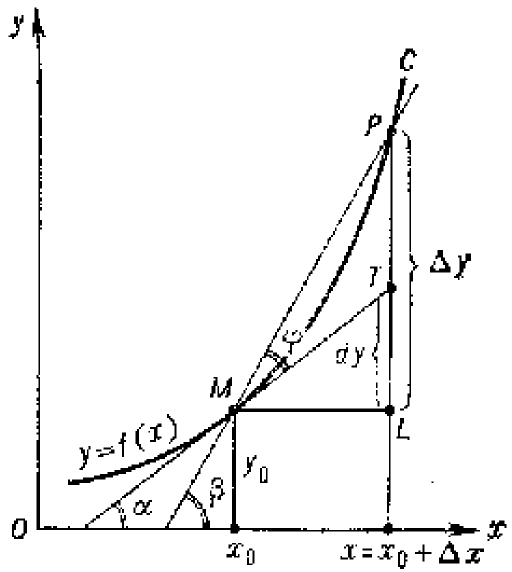
What is Calculus?

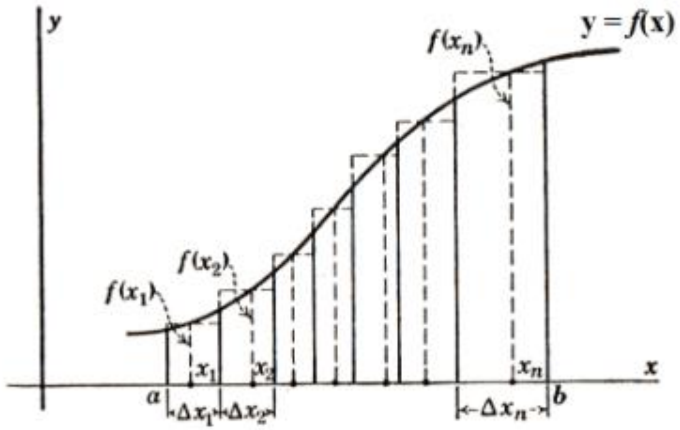
- Calculus is a branch of mathematics
- Originally :“The calculus of infinitesimals”
- It was developed first in 17th century by Newton & Leibniz (independently)
- Calculus is concerned with the study of rates of change.

Subfields of Calculus

Traditionally it is divided to:

- **Differential calculus:** A subfield of calculus that studies the rates at which quantities change.
- **Integral Calculus**: A subfield of calculus that studies the area under a curve.





Today's Outline

I. Introduction to Calculus

II. Mapping & Relations

III. Functions

IV. Limits

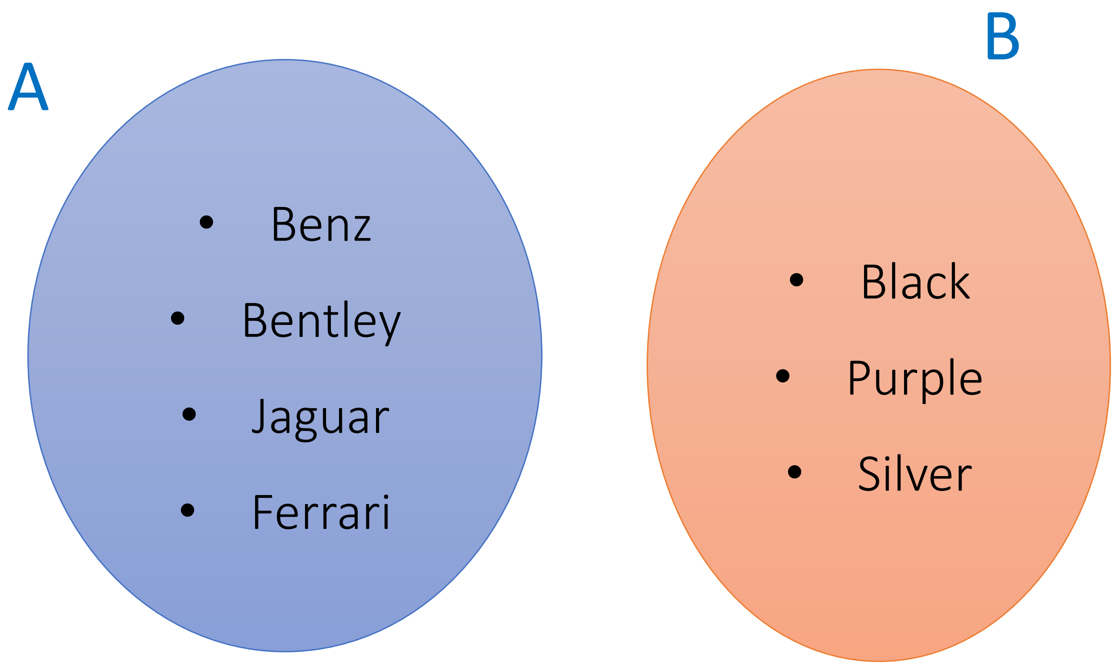
V. Continuity

VI. Tutorial

Conceptual Example

- Let A be the set of the 4 cars you like the most.
- Let B be your 3 favorite colors.
- What are the possible relations?
- What are your favorite combinations?

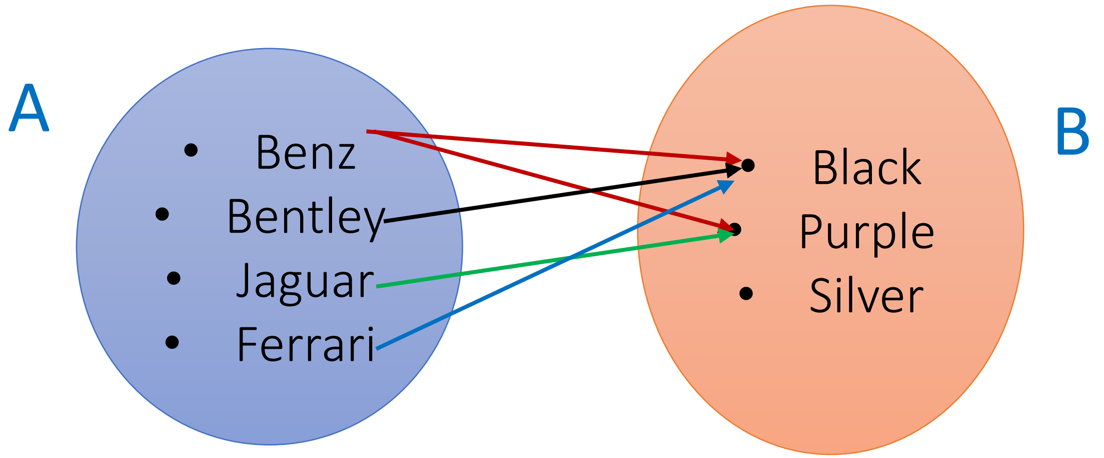
* Use this notation: (car, color)



Conceptual Example cont

- We are mapping A to B: A → B
- We have 12 relations (4x3) in this mapping → Each combination (car, color) → is an ordered pair.
- Out of the possible 12 relations, we have chosen 5 ordered pairs,

R = { (Benz, Black}, (Benz, Purple), (Bentley, Black), (Jaguar, Purple), (Ferrari, Black) }



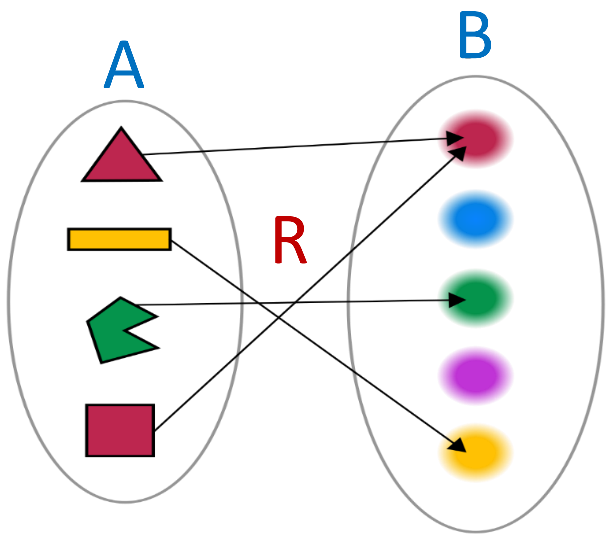
Mapping

Mapping

- Correlating members of one set A →with/to members of another set B.
- Any **prescribed way** of **assigning** to **each object** in one set a **particular object** in another set.
- The **relation R**, maps elements of set A to elements of set B

R: A → B

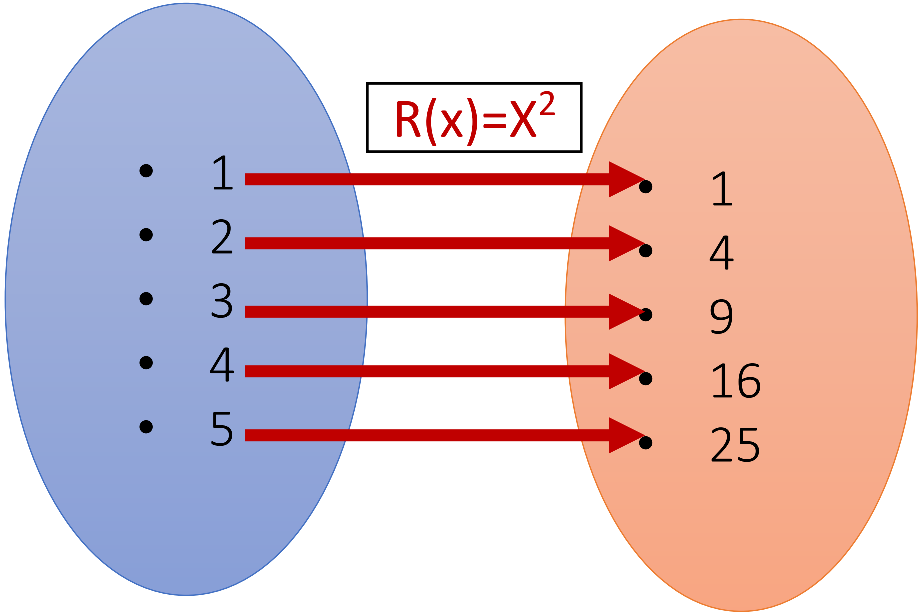
In this figure, R maps elements of A to elements of B, if they have the same color.



Mapping

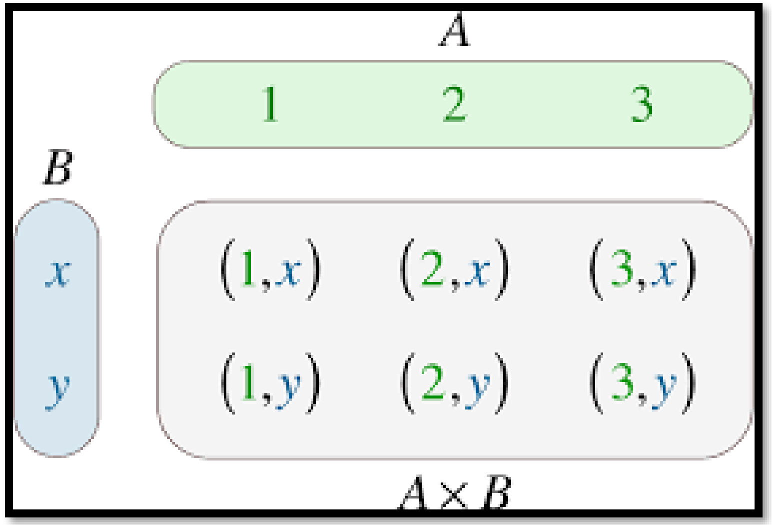
Example) If A = {1,2,3,4,5}, then the relation $R(x)=X^2$ will map every element of A to the set B:

- A = {1,2,3,4,5} → B = {1,4,9,16,25}
- R= {(1,1), (2,4), (3,9), (4,16),(5,25)} → R is a set of 5 ordered pairs



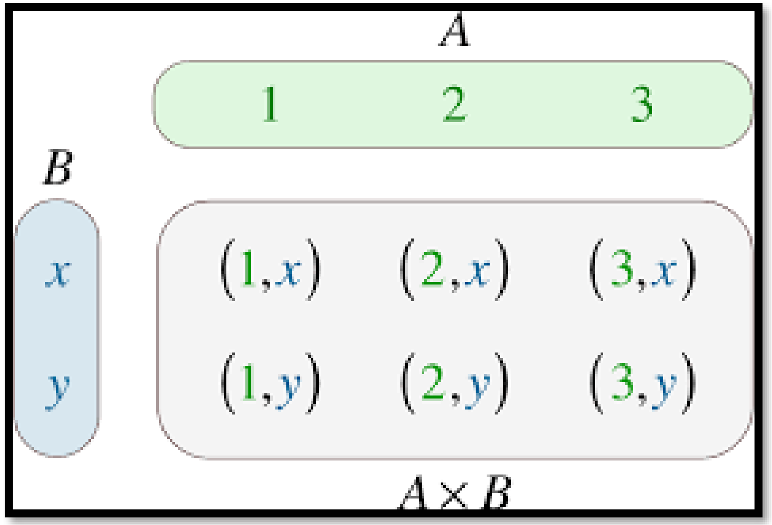
Ordered Pair

- The Cartesian product of two sets A and B
- Denoted $A \times B$, is the set of all ordered pairs where a is in A and b is in B.
- $A \times B = \{(x,y) \mid x \in A \;\&\; y \in B\}$



Cartesian Multiplication

- Note 1: (X,Y) $\neq$ (Y,X)
- Note 2: R x R = $R^2 \rightarrow$ Plane
- Note 3: R x R x R= $R^3 \rightarrow$ Space
- Note 4: R x R x R x …= $R^n \rightarrow$ n-D space



- A x B = {(x,y) | x$\in$ A & y $\in$ B }

Relation Between Two Sets

Between two sets- Mapped to one another,there may be different relations:

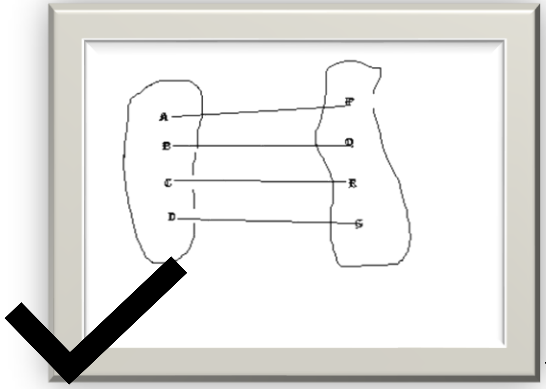
a) one-to-one

b) many-to-one

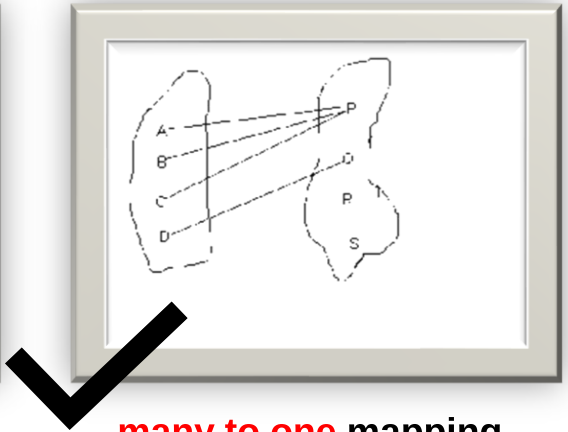
c) one-to-many

d) many-to-many

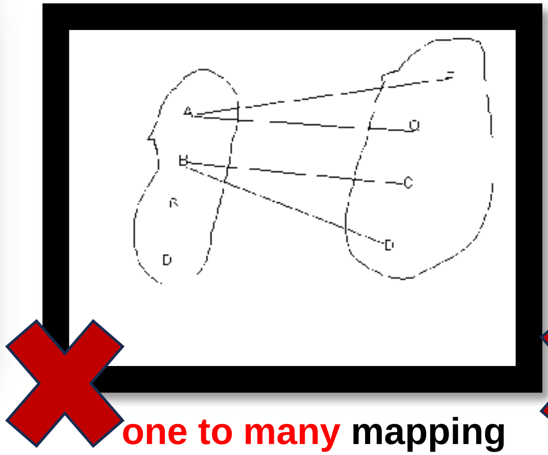
In calculus, we are **interested** in one-to-one & many-to-one relations.



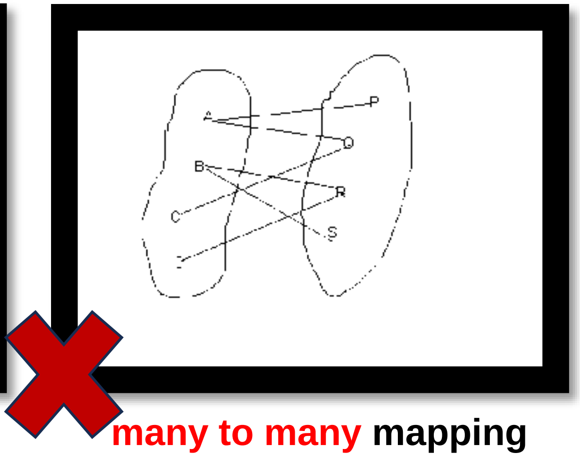
one to one **mapping**



many to one **mapping**

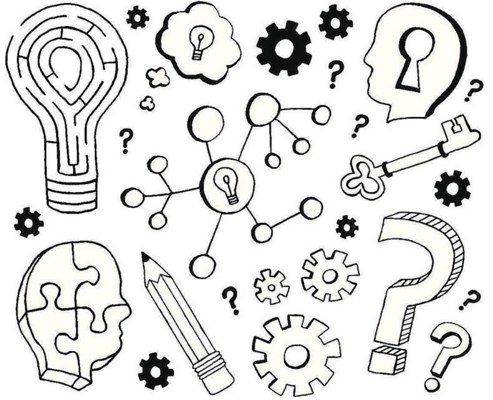


one to many **mapping**



many to many **mapping**

Activity (Individual, 10’)



1) Reflect on mapping. When have you used mapping in your coding?

2) What is the significance of a function?

3) If set A has n elements and set B has m elements, how many would A X B and B X A have?

4) If A is the set of prime numbers and B is the set of odd numbers, what are:

   a) A X A

   b) B X B

   c) A X B

   d) B X A

5) Is A X B = B X A

6) What is R x R?

7) What is R X RX R?

Today's Outline

I. Introduction to Calculus

II. Mapping & Relations

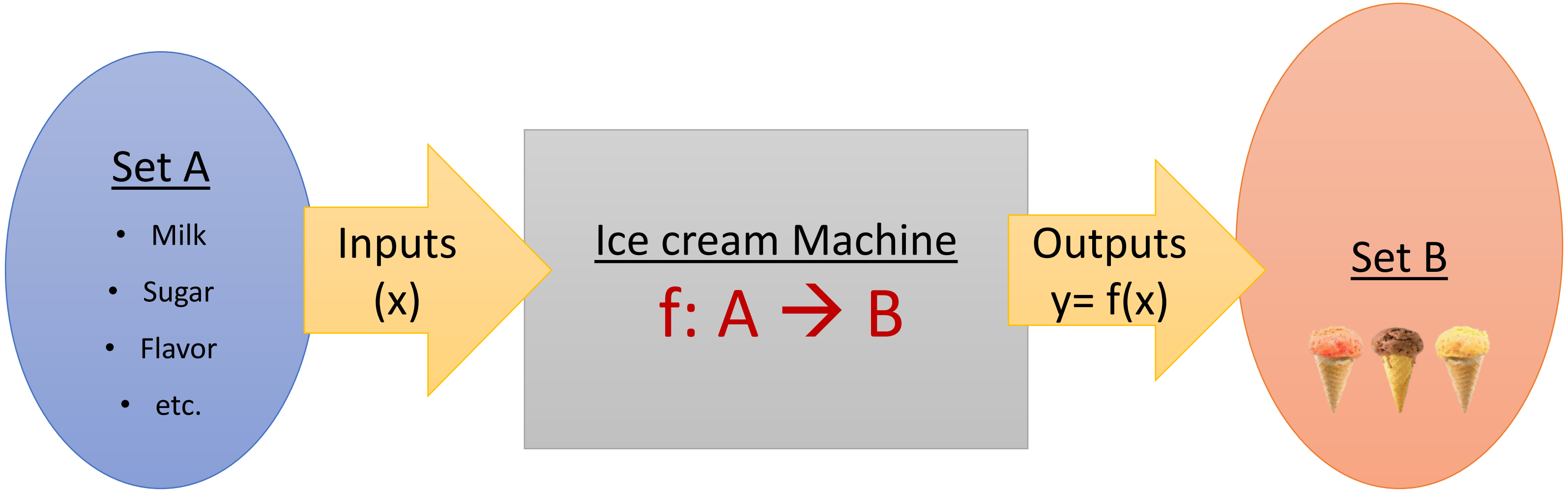
III. Functions

IV. Limits

V. Continuity

VI. Tutorial

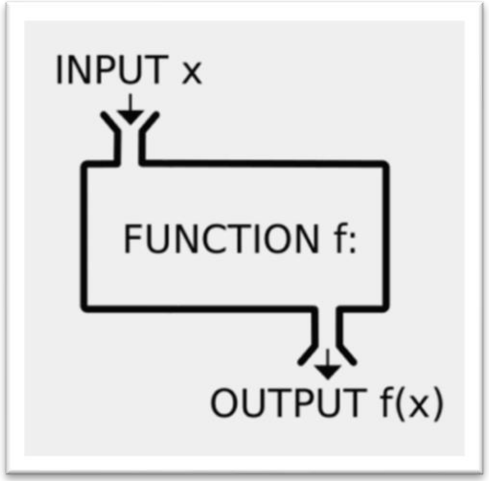
Conceptual Example



- How many inputs can we have?
- What are these inputs?
- How many outputs can we have?
- What are the outputs?
- Can the **same inputs** → **Two different ice creams**?

Functions

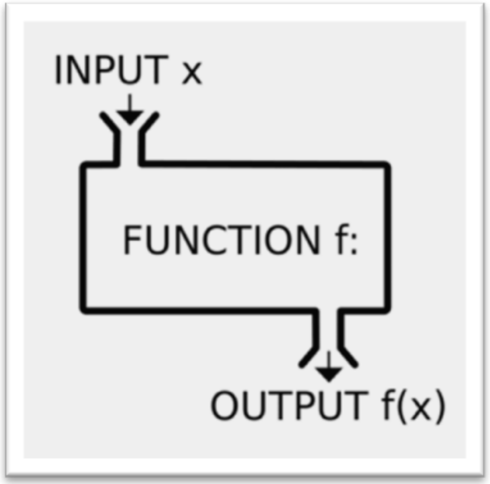
- f: A→B: Is a function from A to B
  - x is the input → independent Variable
  - y is the output → dependent variable
- Function: is a **relation** which associates to every x in set A, an element of set B
- Notation for a function: $y = f(x)$



Functions- Variables

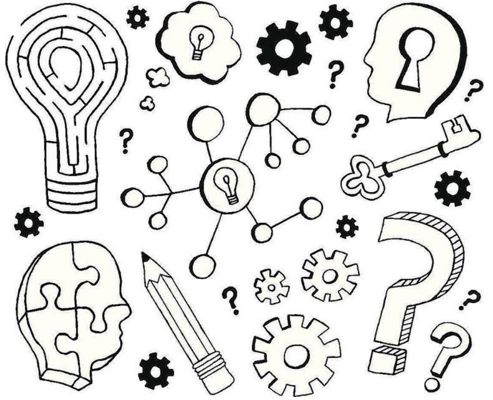
f: A→B: Is a function from A to B

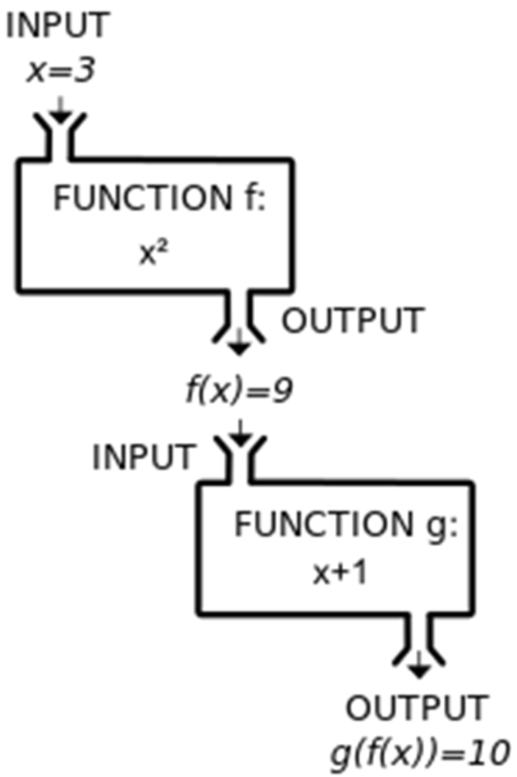
- A **function** may have several independent Variable → Set of all inputs = A
- A **function** may have several dependent Variable → Set of all outputs = B



Activity (Individual, 10’)

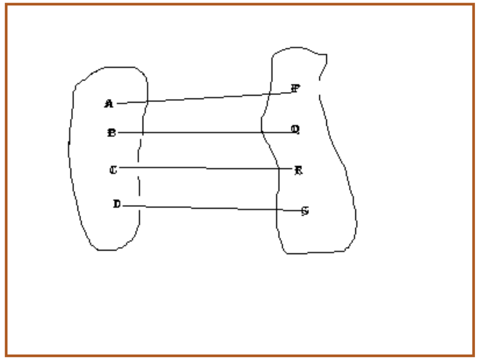
1. What is the story of this diagram?
2. Identify f(x), g(x) & g(f(x))
3. Try it with X=5 & X=-1.
4. Can X be any real number?
5. Can f(x) be any real number? [Hint: Can f(x) be negative]
6. Can g(x) be any real number?



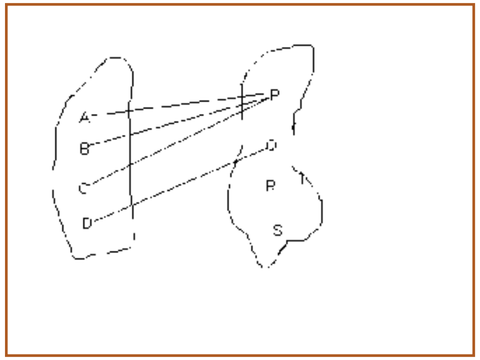


Function vs. Relations

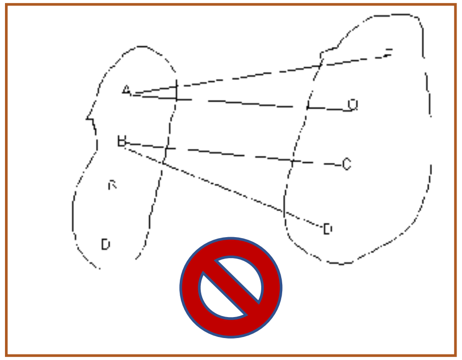
- Function: A one-to-one or many-to-one Relation
- Function : well-behaved relation.



A Function



A Function



Not A Function

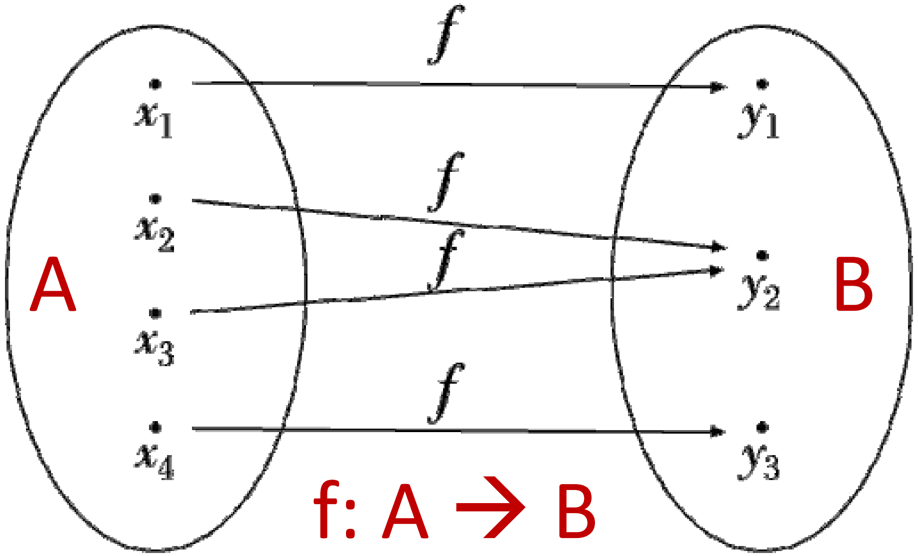
Functions- Characteristics

Input	Output	Function
One	One	Yes
One	Many	No
Many	One	Yes
Many	Many	No

- One input can not have more than one output.

Functions

- $A = \{x_1, x_2, x_3, x_4\}$
- $B = \{y_1, y_2, y_3\}$



$f: A \rightarrow B$

$y = f(x)$

- $y_1 = f(x_1)$
- $y_2 = f(x_2)$
- $y_2 = f(x_3)$
- $y_3 = f(x_4)$

Functions-Definition

f: A→B: Is a function from A to B

$$function: \{(x, y_1) \epsilon f \wedge (x, y_2) \in f \Rightarrow (y_1 = y_2)\}$$

Activity (Individual, 10’)

1) The following relations are R → R. Using Ven diagram, draw them.

   i. A = {(1,1),(2,4),(3,9),(4,16),(5,25)}

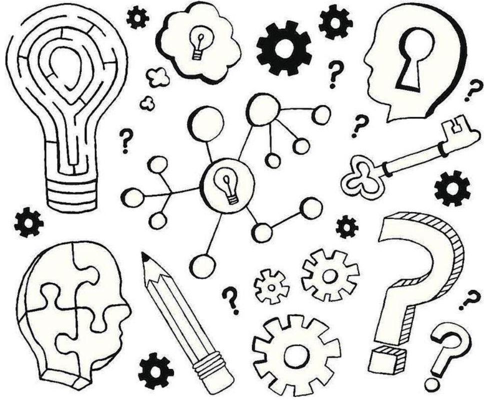
   ii. B= {(1,1),(1,2),(2,3)}

   iii. C= {(1,2),(2,2),(3,2)}

2) What are the inputs & outputs of A, B & C?

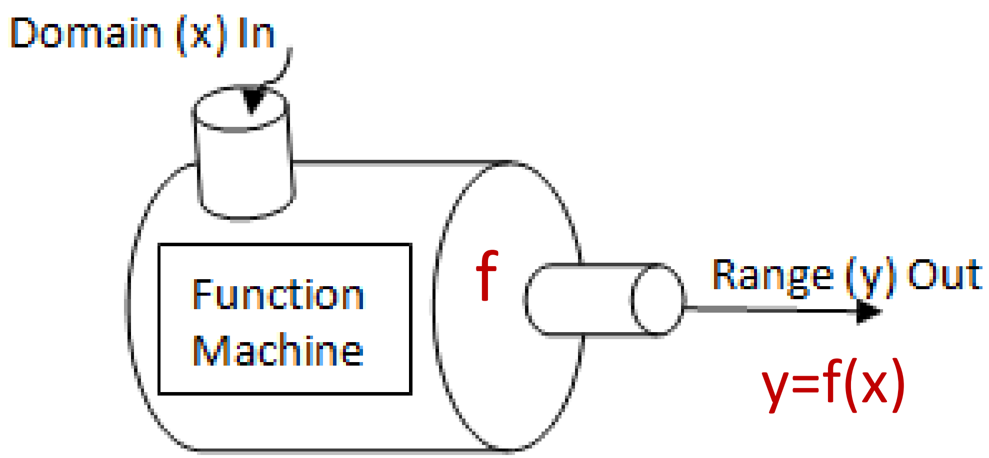
3) What kind of a relation are A, B & C?

4) Why are all these functions, but B?



Domain & Range

- Domain of a function D_f is the **complete set of** possible values of the inputs
- Range of a function R_f is the **complete set of** possible values of the outputs



Domain & Range - Definition

For a function y=f(x):

- Domain of a function D_f :

$$D = \{x \mid x \in (x, y)\}$$

- Range of a function R_f :

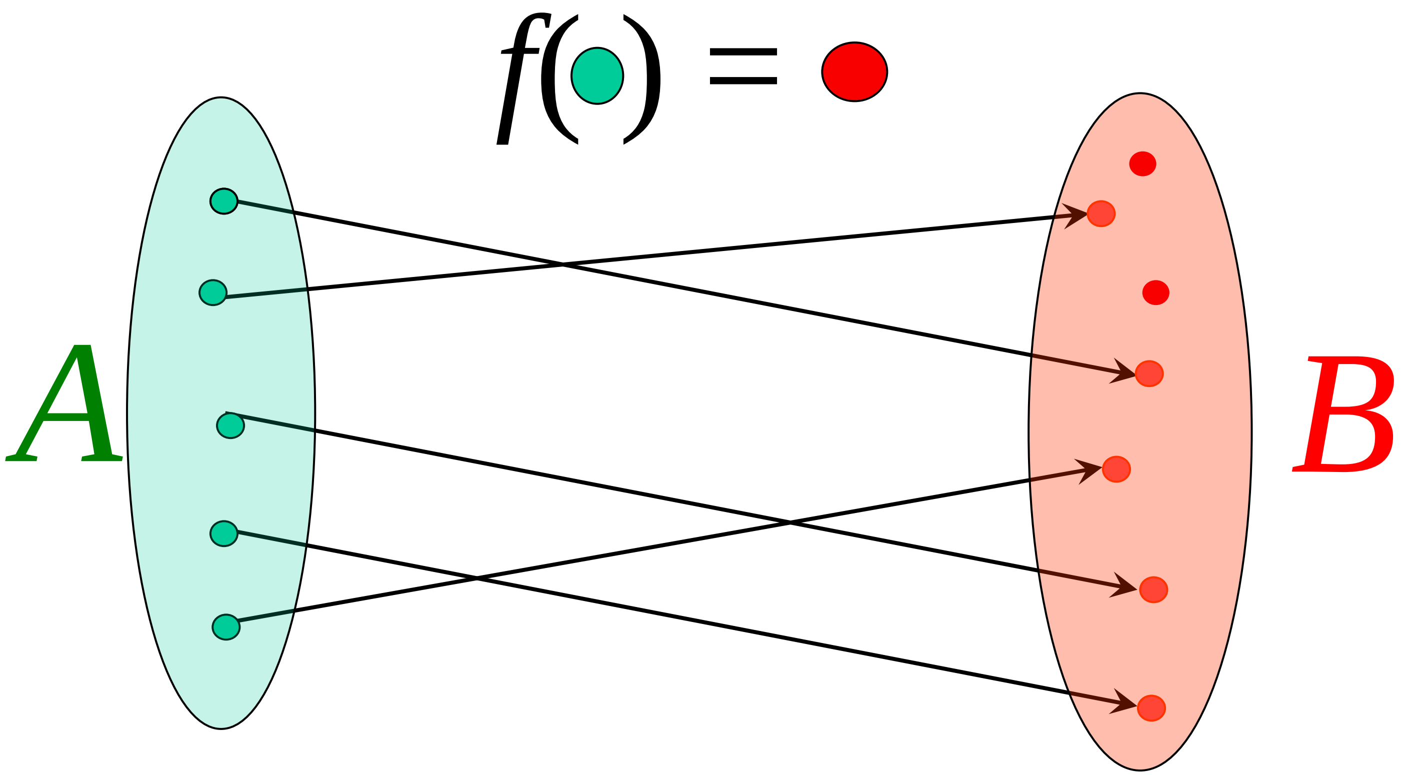
$$R = \{ y \mid y \in (x, y)\}$$

- For every function $f: R \rightarrow R$; domain & range might not be defined for several real numbers

e.g.: Divide by 0, Square root of a negative number, Logarithm of a negative number, ...

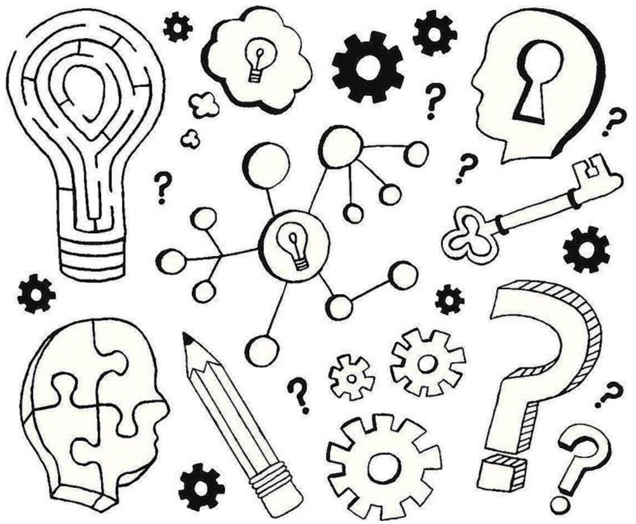
Functions & Sets

$\underline{f : A \to B}$



$$\boxed{\forall x \in A, !\exists y \in B \,\middle|\, f(x) = y}$$

Activity (Individual, 10’)



1. For the following functions, find
   - The domain of the function
   - The range of the function

- $y = x^2$
- $y = 1/x$
- $y = \sqrt{x}$
- $y = \sqrt{4 - x}$
- $y = \sqrt{1 - x^2}$

Activity - Solution

Function	**Domain (x)**	**Range (y)**
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Break- 20’

Today's Outline

I. Introduction to Calculus

II. Mapping & Relations

III. Functions

IV. Limits

V. Continuity

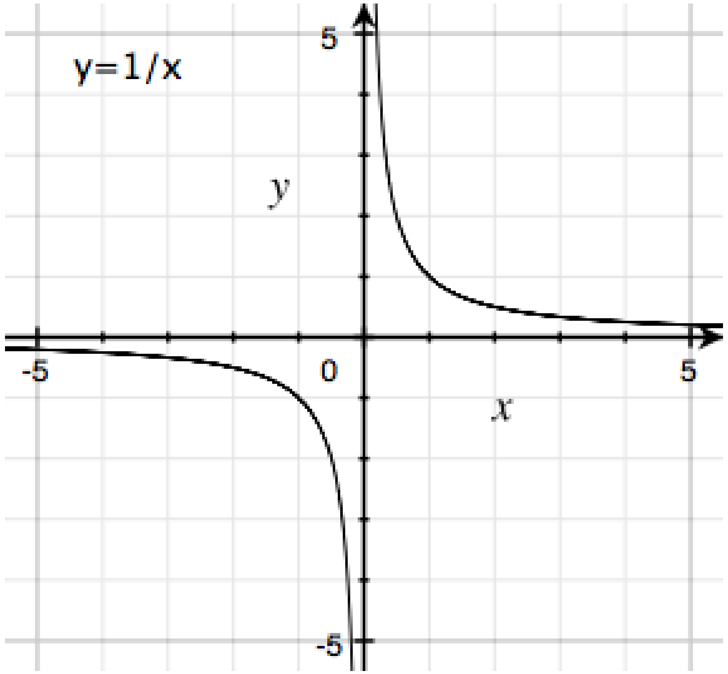
VI. Tutorial

Conceptual Example

- Graph the function $y = \frac{1}{x}$

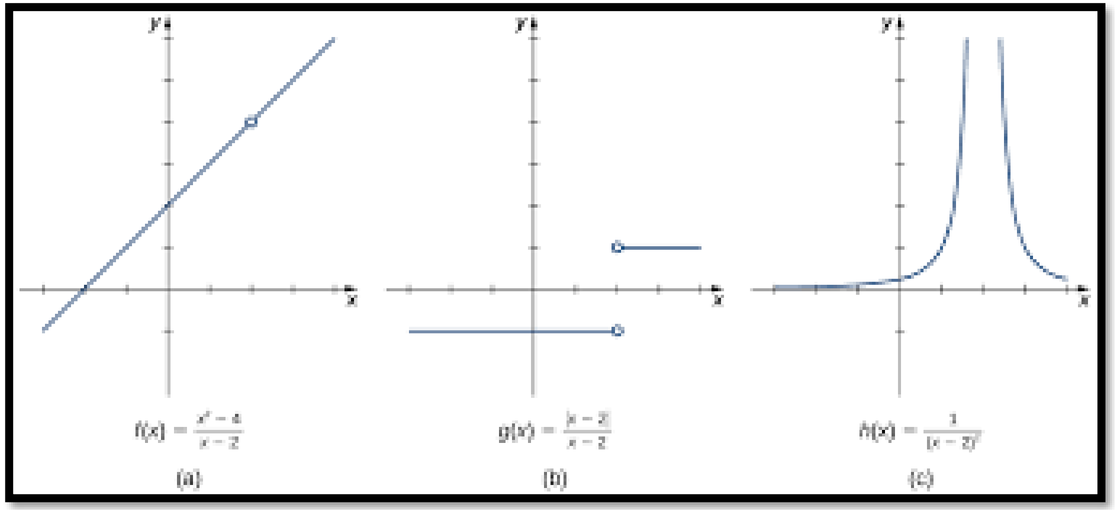
1. $D_f = \{\forall x \in \mathbb{R} - \{0\}\}$
2. $R_f = \mathbb{R}$

How to study the **behavior** of y=f(x) at x=0 ?



Concept of Limit- Informal

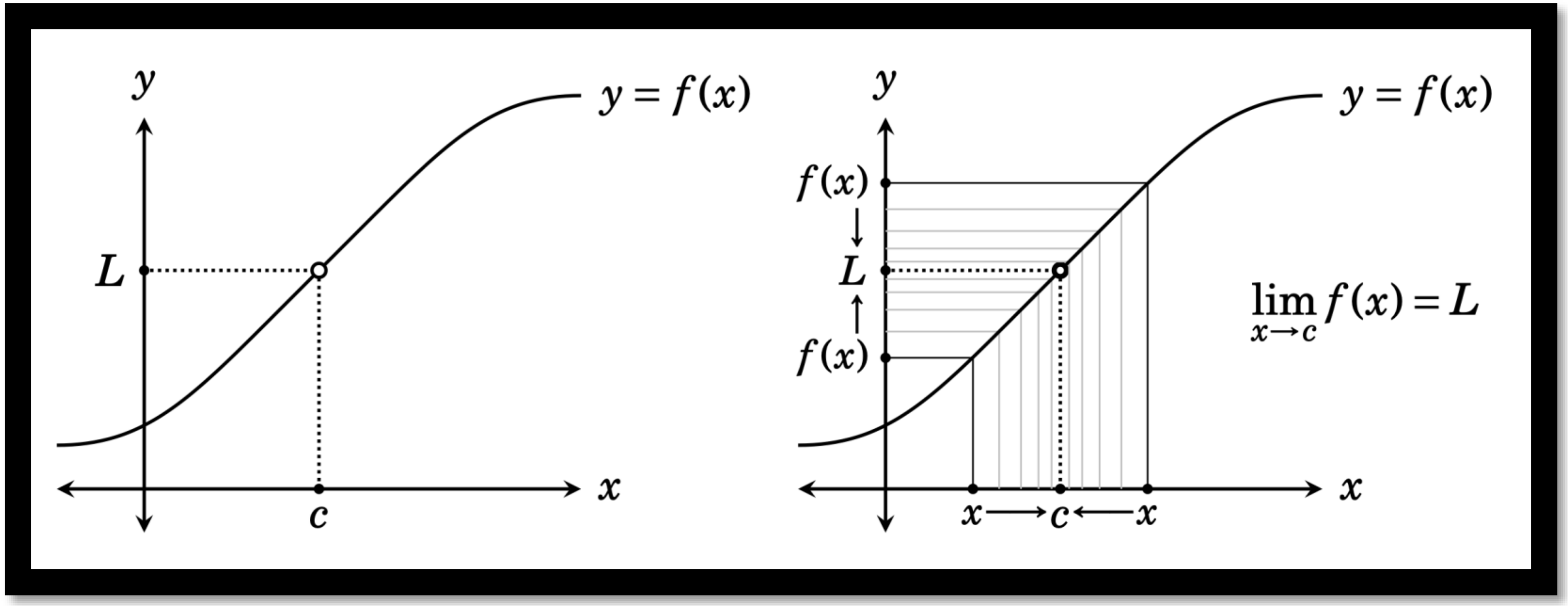
- How should we graph/study functions in the values NOT in the Domain?
- study the behavior of a function as x approaches *values in which the function is not defined* → Limit of a function
- Concept Limit → Concept continuity → derivative & integral.



Informal Definition of Limit

If x → value c *from either sides* → the value of *f(x) approaches a real number L, then:*

Lim f(x) = L *(as X → C)*



$$\lim_{x \to c} f(x) = L$$

Formal Definition of Limit

LIMIT DEFINITION

Let f be a function defined at each point on an open interval containing a, except possibly at a itself.

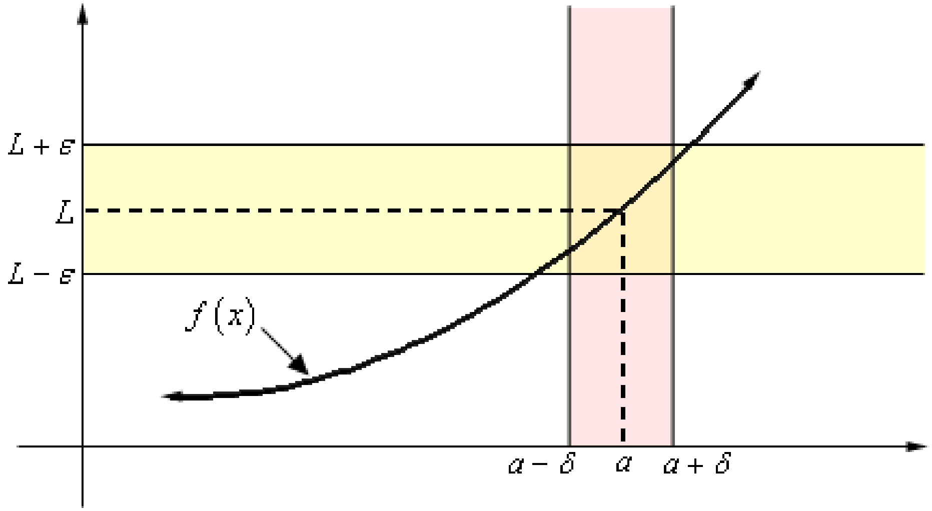
Then a number L is the **limit of f at a** if for every number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$\text{if} \quad a - \delta < x < a + \delta, \quad \text{then} \quad L - \epsilon < f(x) < L + \epsilon.$$

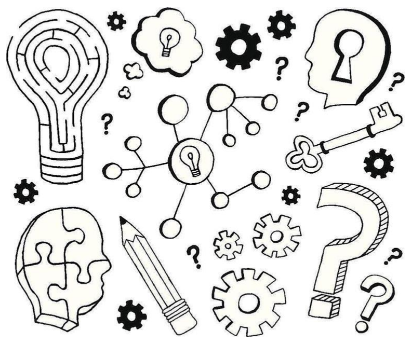
We write this limit as $\lim_{x \to a} f(x) = L$.

Lim f(x) = L
X → a

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t } |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$



Activity (Individual, 10’)



1. What are the following limits?

- $y = \sin(x)$ @ $x = \frac{\Pi}{2}$
- $y = \ln(x + 1)$ @ $x = 0$
- $y = \mathrm{Arctan}\,(x)$ @ $x \to +\infty$
- $y = \exp\,(x + \ln(x))$ @ $x = 1$

2. Find the limit of:

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x^2 - 3x + 2}.$$

Activity-Solution

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-3x+2}. = \frac{0}{0}$$ → Not Defined

- **Solution** $$\frac{x^2+2x-3}{x^2-3x+2} = \frac{(x+3)\cancel{(x-1)}}{(x-2)\cancel{(x-1)}} = \frac{x+3}{x-2}$$

$$\lim_{x\to 1} \frac{x^2+2x-3}{x^2-3x+2}. = \lim_{x\to 1} \frac{x+3}{x-2} = -4$$

Today’s Outline

I. Introduction to Calculus

II. Mapping & Relations

III. Functions

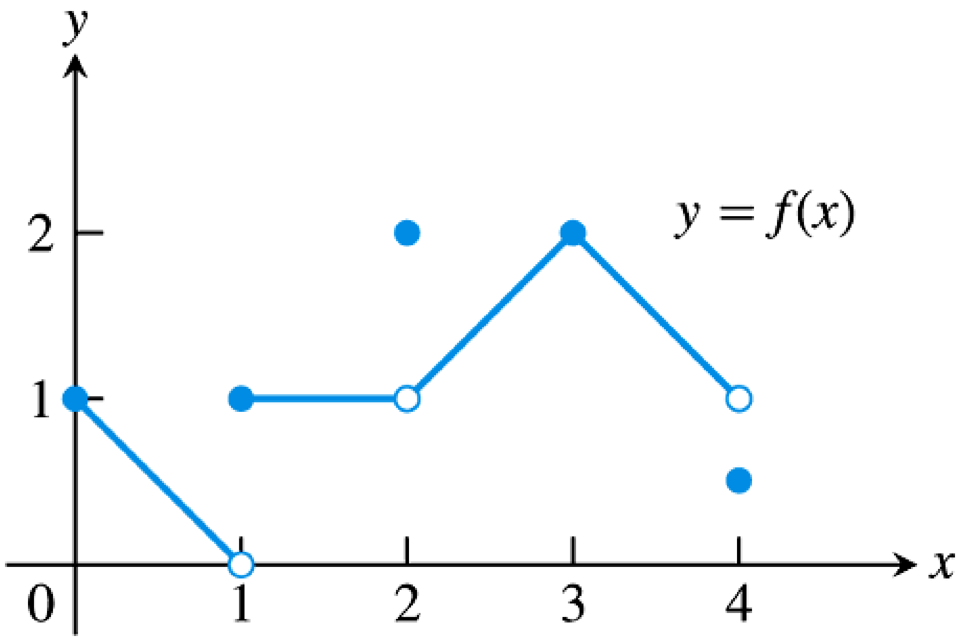
IV. Limits

V. Continuity

VI. Tutorial

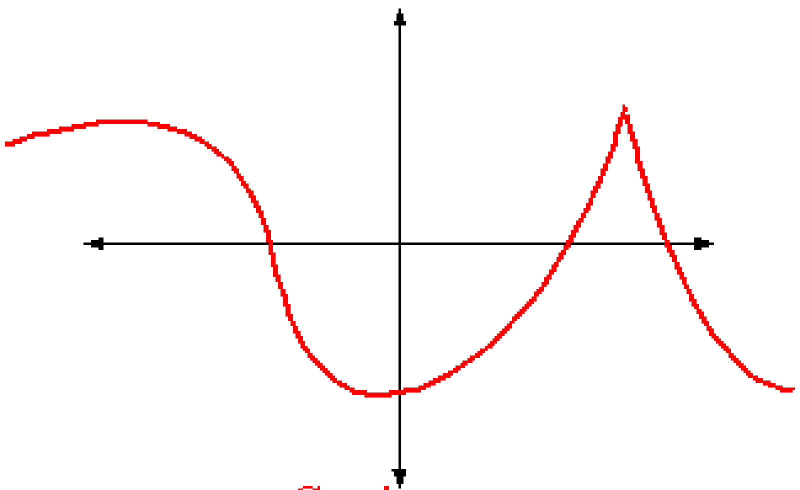
Concept of Continuity

- A continuous function is a function that does not have any abrupt changes in value, known as discontinuities.

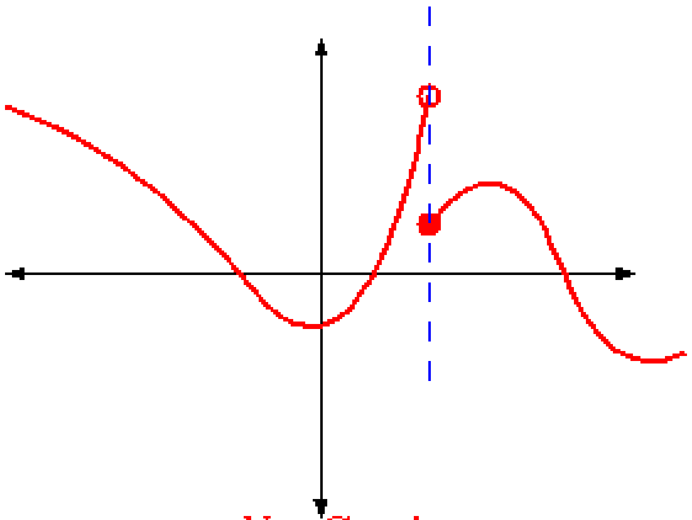


In this Figure, The function is continuous on [0, 4] except at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

Continuity vs Discontinuity

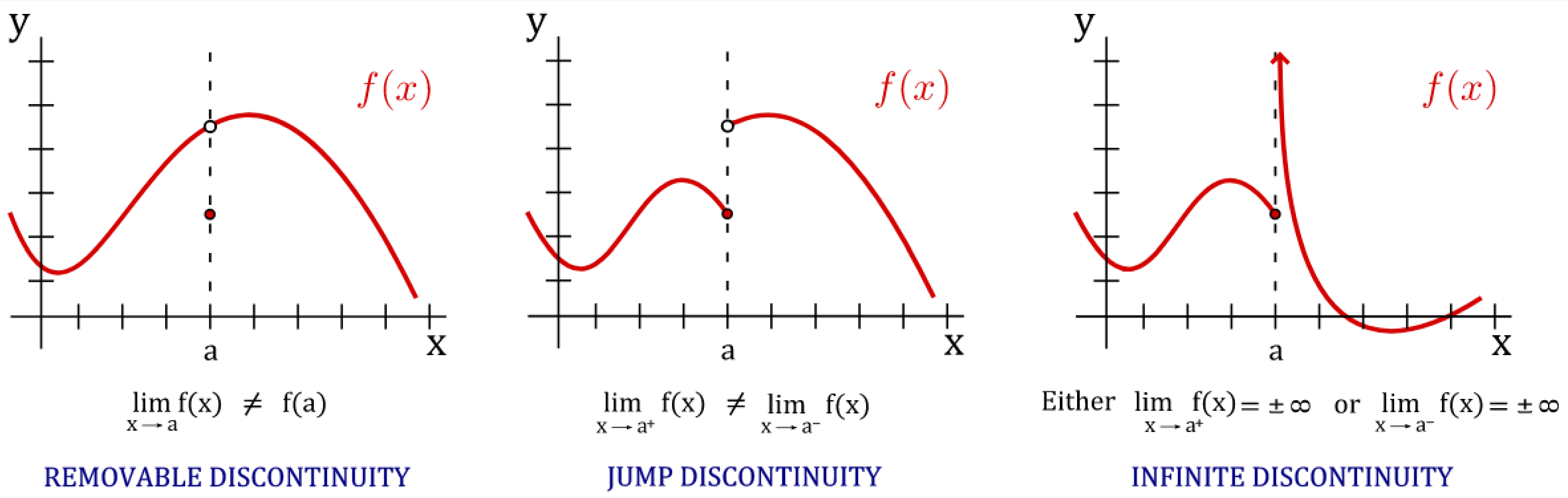


Continuous



Not Continuous

Different Types of Discontinuty



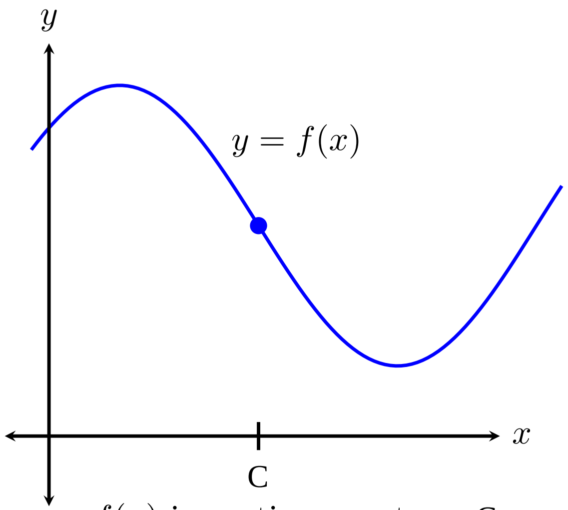
Continuity Test

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

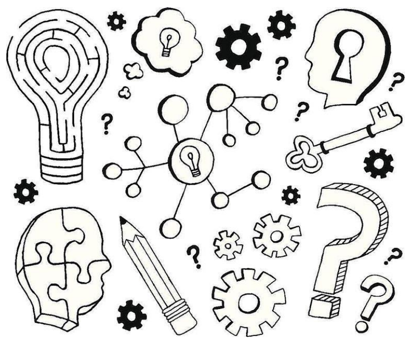
1.	$f(c)$ exists	(c lies in the domain of f)
2.	$\lim_{x \to c} f(x)$ exists	(f has a limit as $x \to c$)
3.	$\lim_{x \to c} f(x) = f(c)$	(the limit equals the function value)

A Continuous Function



$f(x)$ is continuous at x = C

Activity (Individual, 10’)

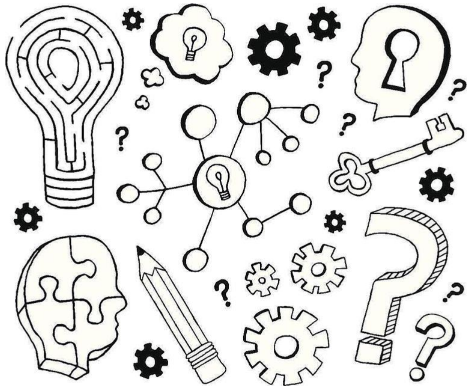


1. Are the bellow function continues?
   a. Y = x @x=0
   b. Y= cos (x) @x=pi/2
   c. Y=Ln (x) @x= 0

Today’s Outline

1. ~~I. Introduction to Calculus~~
2. ~~II. Mapping & Relations~~
3. ~~III. Functions~~
4. ~~IV. Limits~~
5. ~~V. Continuity~~
6. VI. Tutorial

Reflection (Individual, 30')



1. What is mapping? Give an example
2. What is a Relation ? Give an example
3. How are relations and mapping related? How are relations & mapping used in computer science?
4. What are different types of relations? What are Domain & Range of a relation?
5. What are ordered pairs? What are ordered triplets?
6. If A is the set of odd numbers & B is the set of even numbers, find i) A x B, ii) B x A, iii) are they equal?
7. If A is the set of odd numbers, B is the set of even numbers and C is the set of Natural numbers, What is A x B x C?
8. In your opinion, why are we only interested in one-to-one & many-to-one relations? How does this relate to coding?
9. What is a function ? What is domain? What is independent variable? What is range? What is dependent variable? How is domain a range of a function related?
10. If f(x)=Sin(x)+Cos(x); & if x is in degrees; evaluate: f(0) ; f(45) ; f(90) ; f(180)
11. What is the limit of a function?
12. What does the following mean?
    a) y=f(x) has a limit at x=L from both sides.
    b)y=f(x) has a limit at x=L from left.
    c) y=f(x) has a limit at x=L from right.
13. Reflect on the precise definition of limit. (Fig. bellow) What does it mean? Why are limits important?
14. What is continuity from left? What is continuity from right? What is continuity? Why is it important?

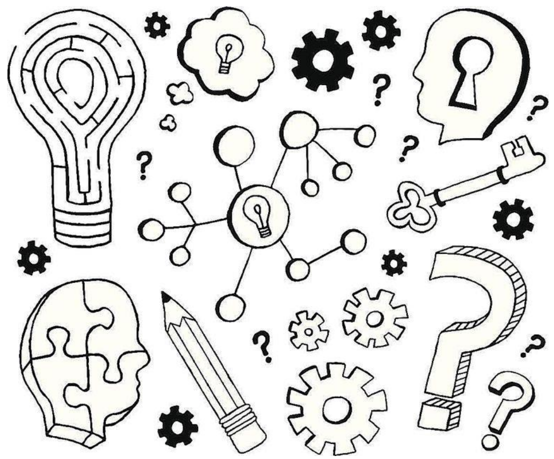
Ex.1) Why all the following sets NOT a function **but C**?

i. A = {(1,1),(2,4),(1,9),(4,16),(5,25)}

ii. B= {(1,1),(2,2),(2,3)}

iii. C= {(1,2),(2,2),(3,2)}

iv. D= {(X,1),(X,2),(2,3)}



Ex.2) In the following figures:

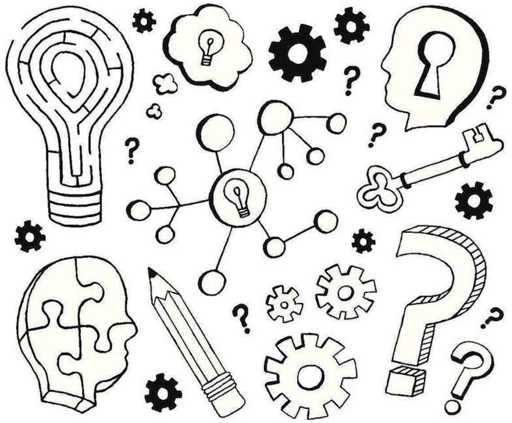
a. What are the Domain & Range?

b. What is f? Can you identify a formula for f?

c. How many order pairs are there?

d. Why Fig.1 & 3 are functions? Why Fig.2 is not a function?

e. Which is a relation?



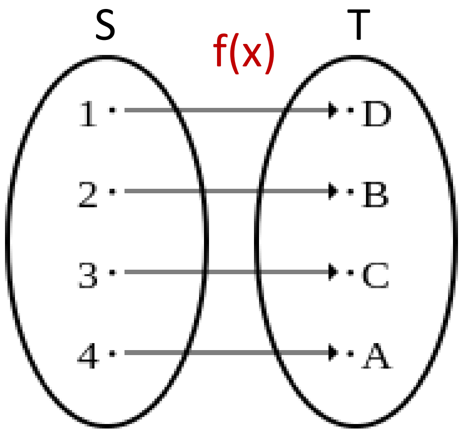


Fig.1

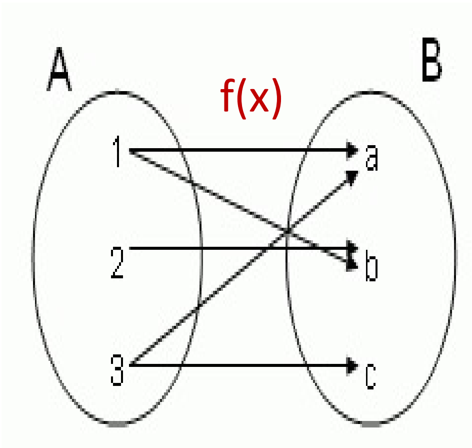


Fig.2

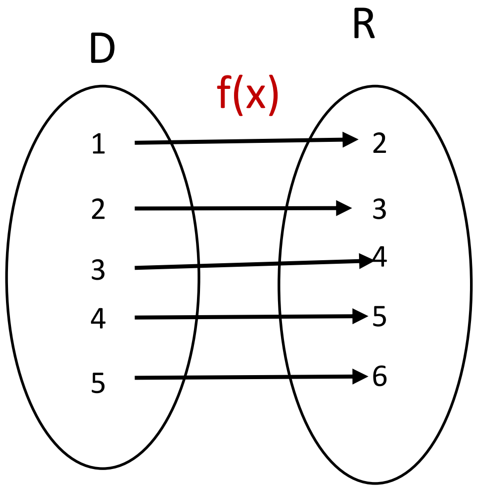


Fig.3

Ex.3) In regard to this diagram:

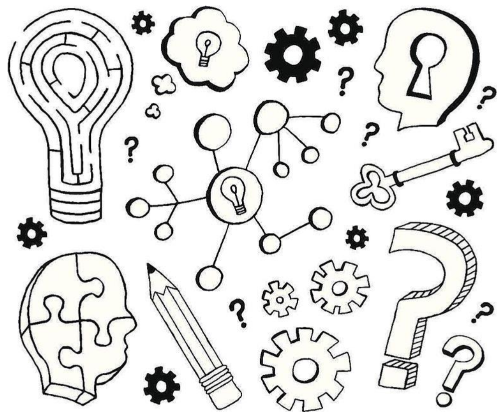
i. What is the story of this diagram?

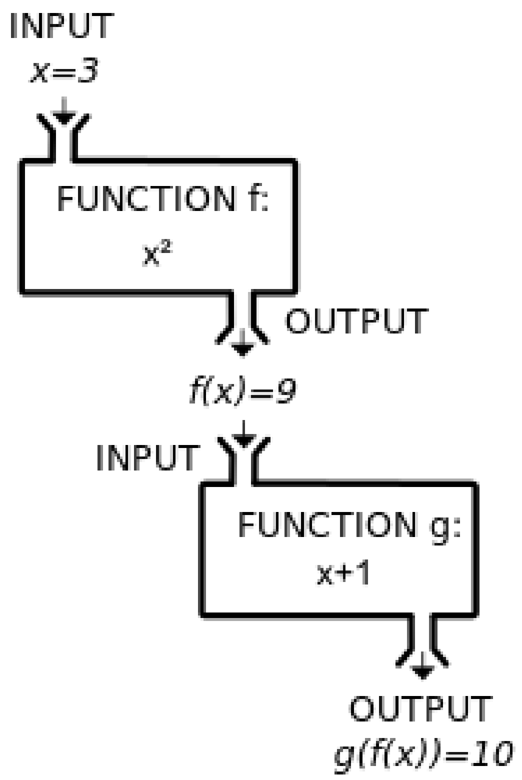
ii. Identify f(x)

iii. Identify g(x)

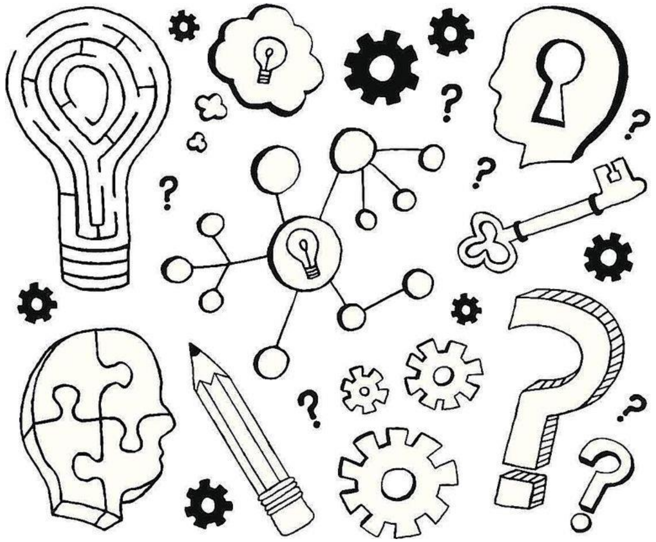
iv. Identify g(f(x))

v. Try it with X=5





EX.4) If $y = 5x$, what are the following?



i. $f(u)$

ii. $f(v)$

iii. $f(Cv)$ *(C is a constant number)*

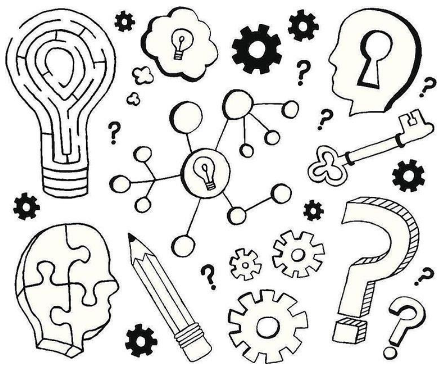
iv. $Cf(v)$

v. $f(u)+f(v)$

vi. $f(u+v)$

vii. Is $f(u)+f(v)$ equal to $f(u+v)$

viii. Is $f(Cv)$ equal to $Cf(v)$



EX.5) If $y = x^2$, what are the following?

i. f (u)

ii. f(v)

iii. f(Cv) *(C is a constant number)*

iv. Cf(v)

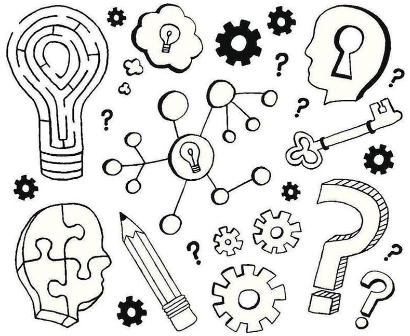
v. f(u)+f(v)

vi. f (u+v)

vii. Is f(u)+f(v) equal to f (u+v)

viii. Is f(Cv) equal to Cf(v)

Exercise 6



1. What are the application of limit in computer science?

2. What is the difference between approaching a value and equality to a value?

3. What does the precise definition of limits mean?

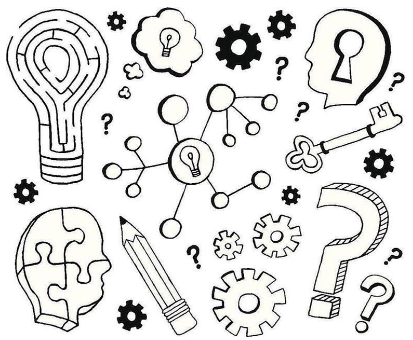
2 Precise Definition of a Limit Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if} \quad 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \varepsilon$$

Exercise 7



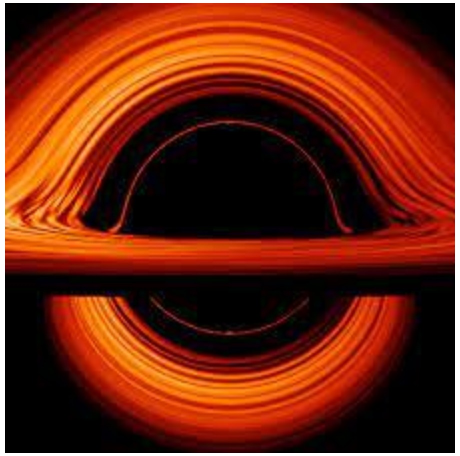
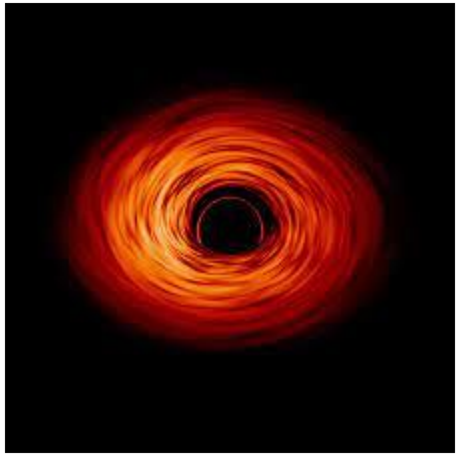
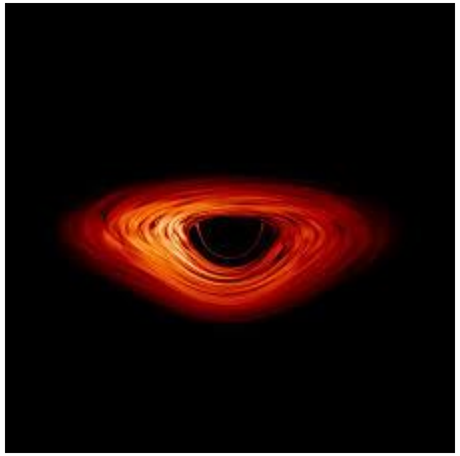
1. Are the bellow function continues?
   a. $Y = \ln x$ @$x=0$
   b. $Y=\sin(x)$ @$x=pi/2$
   c. $Y=\tan(x)$ @$x=pi/2$
   d. $Y=\exp(x)$ @R
   e. $Y=1/x^2$ @R
   f. $Y=x^2$ @R

Research Activity (Individual, 60’)

- Blackholes are known to be the nature’s largest hard drives. Read the bellow articles and conduct further research and answer the bellow questions:

1) Find the function required to evaluate the amount of data that can be stored in a blackhole?
2) How can the concept of limit help us to define the event horizon of a black hole?
3) Are black holes continues in space & time geometry? How does singularity in the center of a black hole relate to the concept of continuity?
4) In your opinion, can we harness the technology to use black hole as a hard drive?

  - https://arstechnica.com/science/2020/06/natures-cosmic-hard-drive-black-holes-could-store-information-like-holograms/
  - https://www.scientificamerican.com/article/black-hole-computers-2007-04/
  - A brief History of Time- Stephen Hawking- Chapters 6 & 7
  - Documentary: Blackholes- The edge of all we know (Netflix)

Any Questions or Concerns?

Source of the slides:

Thomas Calculus – 11e

Stewart Calculus

https://www.slideserve.com/search/presentations/derivatives-and-integrals