take simple transformations and derive more complicated transformations through matrix concatenation; more on this in Section 5.6.

Before we move on, let's review the key concepts of Section 4.2.

- The rows of a square matrix can be interpreted as the basis vectors of a coordinate space.
- To transform a vector from the original coordinate space to the new coordinate space, we multiply the vector by the matrix.
- The transformation from the original coordinate space to the coordinate space defined by these basis vectors is a linear transformation. A linear transformation preserves straight lines, and parallel lines remain parallel. However, angles, lengths, areas, and volumes may be altered after transformation.
- Multiplying the zero vector by any square matrix results in the zero vector. Therefore, the linear transformation represented by a square matrix has the same origin as the original coordinate space—the transformation does not contain translation.
- We can visualize a matrix by visualizing the basis vectors of the coordinate space after transformation. These basis vectors form an 'L' in 2D, and a tripod in 3D. Using a box or auxiliary object also helps in visualization.

4.3 The Bigger Picture of Linear Algebra

At the start of Chapter 2, we warned you that in this book we are focusing on just one small corner of the field of linear algebra—the geometric applications of vectors and matrices. Now that we've introduced the nuts and bolts, we'd like to say something about the bigger picture and how our part relates to it.

Linear algebra was invented to manipulate and solve systems of linear equations. For example, a typical introductory problem in a traditional course on linear algebra is to solve a system of equations such as

$$\begin{aligned} -5x_1 + x_2 + x_3 &= -10, \\ 2x_1 + 2x_2 + 4x_3 &= 12, \\ x_1 - 3x_3 &= 9, \end{aligned}$$

which has the solution

$$\begin{aligned} x_1 &= 3, \\ x_2 &= 7, \\ x_3 &= -2. \end{aligned}$$

Matrix notation was invented to avoid the tedium involved in duplicating every x and $=$. For example, the system above can be more quickly written as

$$\begin{bmatrix} -5 & 1 & 1 \\ 2 & 2 & 4 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -10 \\ 12 \\ 9 \end{bmatrix}.$$

Perhaps the most direct and obvious place in a video game where a large system of equations must be solved is in the physics engine. The constraints to enforce nonpenetration and satisfy user-requested joints become a system of equations relating the velocities of the dynamic bodies. This large system[3] is then solved each and every simulation frame. Another common place for traditional linear algebra methods to appear is in least squares approximation and other data-fitting applications.

Systems of equations can appear where you don't expect them. Indeed, linear algebra has exploded in importance with the vast increase in computing power in the last half century because many difficult problems that were previously neither discrete nor linear are being approximated through methods that are both, such as the finite element method. The challenge begins with knowing how to transform the original problem into a matrix problem in the first place, but the resulting systems are often very large and can be difficult to solve quickly and accurately. Numeric stability becomes a factor in the choice of algorithms. The matrices that arise in practice are not boxes full of random numbers; rather, they express organized relationships and have a great deal of structure. Exploiting this structure artfully is the key to achieving speed and accuracy. The diversity of the types of structure that appear in applications explains why there is so very much to know about linear algebra, especially numerical linear algebra.

This book is intended to fill a gap by providing the geometric intuition that is the bread and butter of video game programming but is left out of

[3]It's a system of *inequalities*, but similar principles apply.

most linear algebra textbooks. However, we certainly know there is a larger world out there for you. Although traditional linear algebra and systems of equations do not play a prominent role for basic video game programming, they are essential for many advanced areas. Consider some of the technologies that are generating buzz today: fluid, cloth, and hair simulations (and rendering); more robust procedural animation of characters; real-time global illumination; machine vision; gesture recognition; and many more. What these seemingly diverse technologies all have in common is that they involve difficult linear algebra problems.

One excellent resource for learning the bigger picture of linear algebra and scientific computing is Professor Gilbert Strang's series of lectures, which can be downloaded free from MIT OpenCourseWare at ocw.mit.edu. He offers a basic undergraduate linear algebra course as well as graduate courses on computational science and engineering. The companion textbooks he writes for his classes [67, 68] are enjoyable books aimed at engineers (rather than math sticklers) and are recommended, but be warned that his writing style is a sort of shorthand that you might have trouble understanding without the lectures.

4.4 Exercises

(Answers on page 759.)

Use the following matrices for questions 1–3:

$$\mathbf{A} = \begin{bmatrix} 13 & 4 & -8 \\ 12 & 0 & 6 \\ -3 & -1 & 5 \\ 10 & -2 & 5 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 15 & 8 \\ -7 & 3 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} a & g \\ b & h \\ c & i \\ d & j \\ f & k \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 10 & 20 & 30 & 1 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

1. For each matrix, give the dimensions of the matrix and identify whether it is square and/or diagonal.

2. Transpose each matrix.

3. Find all the possible pairs of matrices that can be legally multiplied, and give the dimensions of the resulting product. Include "pairs" in which a matrix is multiplied by itself. (Hint: there are 14 pairs.)

4. Compute the following matrix products. If the product is not possible, just say so.

   (a) $\begin{bmatrix} 1 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 7 \\ 4 & 1/3 \end{bmatrix}$

   (b) $\begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 3 & 3 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \\ 5 & 7 & -6 \\ 1 & -4 & 2 \end{bmatrix}$

   (d) $\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (e) $\begin{bmatrix} 7 & -2 & 7 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

   (f) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$

   (g) $\begin{bmatrix} 3 & 3 \end{bmatrix} \begin{bmatrix} 6 & -7 \\ -4 & 5 \end{bmatrix}$

   (h) $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

5. For each of the following matrices, multiply on the left by the row vector $[5, -1, 2]$. Then consider whether multiplication on the right by the column vector $[5, -1, 2]^{\mathrm{T}}$ will give the same or a different result. Finally, perform this multiplication to confirm or correct your expectation.

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2 & 5 & -3 \\ 1 & 7 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 7 & 2 \\ 7 & 0 & -3 \\ 2 & -3 & -1 \end{bmatrix}$

This is an example of a *symmetric* matrix. A square matrix is symmetric if $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$.

(d) $\begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix}$

This is an example of a *skew symmetric* or *antisymmetric* matrix. A square matrix is skew symmetric if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. This implies that the diagonal elements of a skew symmetric matrix must be 0.

6. Manipulate the following matrix expressions to remove the parentheses.

   (a) $\left(\left(\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

   (b) $\left(\mathbf{B}\mathbf{A}^{\mathrm{T}}\right)^{\mathrm{T}}\left(\mathbf{C}\mathbf{D}^{\mathrm{T}}\right)$

   (c) $\left(\left(\mathbf{D}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\right)\left(\mathbf{A}\mathbf{B}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

   (d) $\left(\left(\mathbf{A}\mathbf{B}\right)^{\mathrm{T}}\left(\mathbf{C}\mathbf{D}\mathbf{E}\right)^{\mathrm{T}}\right)^{\mathrm{T}}$

7. Describe the transformation $\mathbf{aM} = \mathbf{b}$ represented by each of the following matrices.

   (a) $\mathbf{M} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   (b) $\mathbf{M} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

   (c) $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

   (d) $\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$

   (e) $\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

   (f) $\mathbf{M} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

8. For 3D row vectors $\mathbf{a}$ and $\mathbf{b}$, construct a 3×3 matrix $\mathbf{M}$ such that $\mathbf{a} \times \mathbf{b} = \mathbf{aM}$. That is, show that the cross product of $\mathbf{a}$ and $\mathbf{b}$ can be represented as the matrix product $\mathbf{aM}$, for some matrix $\mathbf{M}$. (Hint: the matrix will be skew-symmetric.)

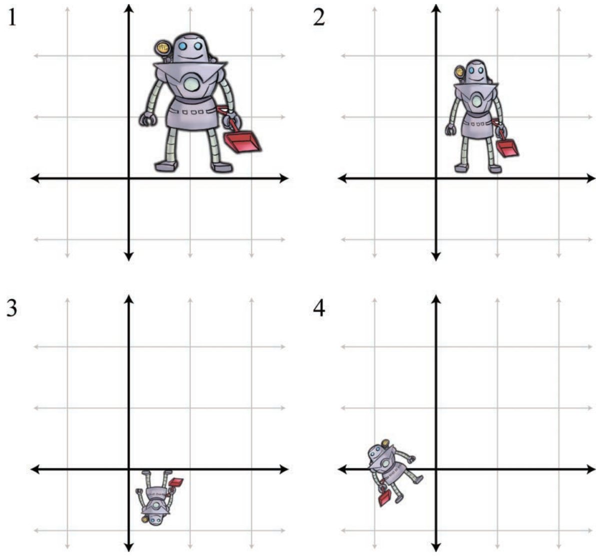
9. Match each of the following figures (1–4) with their corresponding transformations.

   (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 2.5 & 0 \\ 0 & 2.5 \end{bmatrix}$

   (c) $\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

   (d) $\begin{bmatrix} 1.5 & 0 \\ 0 & 2.0 \end{bmatrix}$



10. Given the 10×1 column vector $\mathbf{v}$, create a matrix $\mathbf{M}$ that, when multiplied by $\mathbf{v}$, produces a 10×1 column vector $\mathbf{w}$ such that

$$w_i = \begin{cases} v_1 & \text{if } i = 1, \\ v_i - v_{i-1} & \text{if } i > 1. \end{cases}$$

Matrices of this form arise when some continuous function is discretized. Multiplication by this *first difference* matrix is the discrete equivalent of continuous differentiation. (We'll learn about differentiation in Chapter 11 if you haven't already had calculus.)

11. Given the 10×1 column vector $\mathbf{v}$, create a matrix $\mathbf{N}$ that, when multiplied by $\mathbf{v}$, produces a 10×1 column vector $\mathbf{w}$ such that

$$w_i = \sum_{j=1}^{i} v_j.$$

In other words, each element becomes the sum of that element and all previous elements.

This matrix performs the discrete equivalent of *integration*, which as you might already know (but you certainly will know after reading Chapter 11) is the inverse operation of differentiation.

12. Consider $\mathbf{M}$ and $\mathbf{N}$, the matrices from Exercises 10 and 11.

    (a) Discuss your expectations of the product $\mathbf{MN}$.

    (b) Discuss your expectations of the product $\mathbf{NM}$.

    (c) Calculate both $\mathbf{MN}$ and $\mathbf{NM}$. Were your expectations correct?

To be civilized is to be potentially master of all possible ideas, and that means that one has got beyond being shocked, although one preserves one's own moral aesthetic preferences.

— Oliver Wendell Holmes (1809–1894)