Week 7

Differential Calculus

Differentiation and the Derivative

Two questions in **Differential Calculus**

1. What is the slope of a Tangent to a curve y=f(x)?

2. What is the velocity of an object with its position vector as: $\overrightarrow{R_{(t)}}$

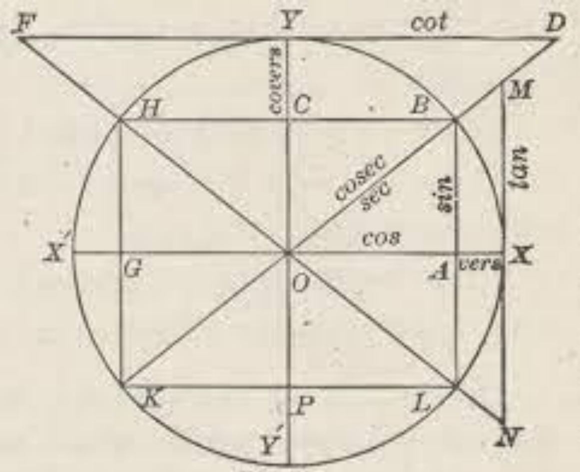


FIG. 31.

Today's Outline

I. Derivatives

II. Derivative of functions

III. Differentiability of a function

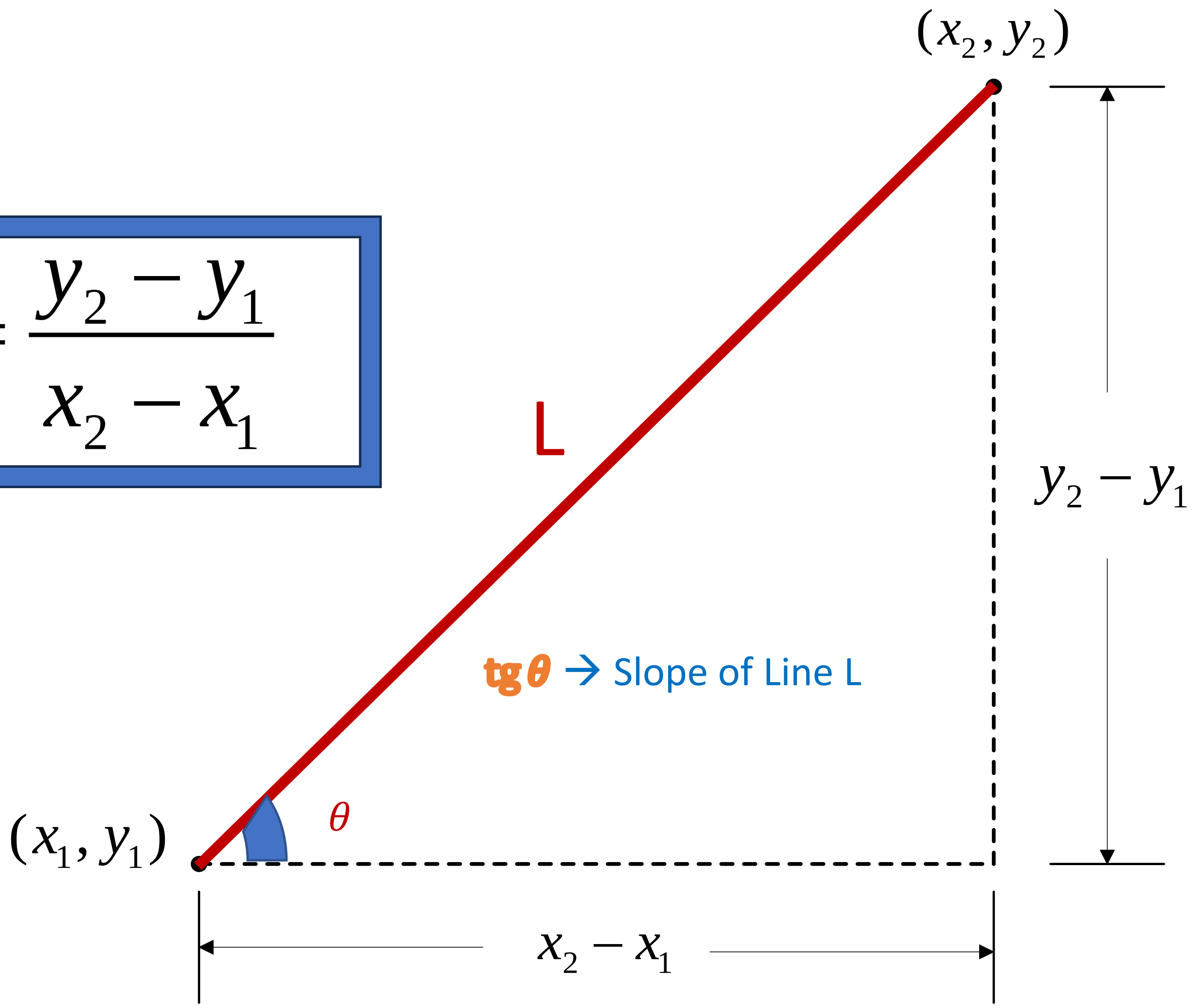
IV. Some Applications of Derivatives

V. Exercises

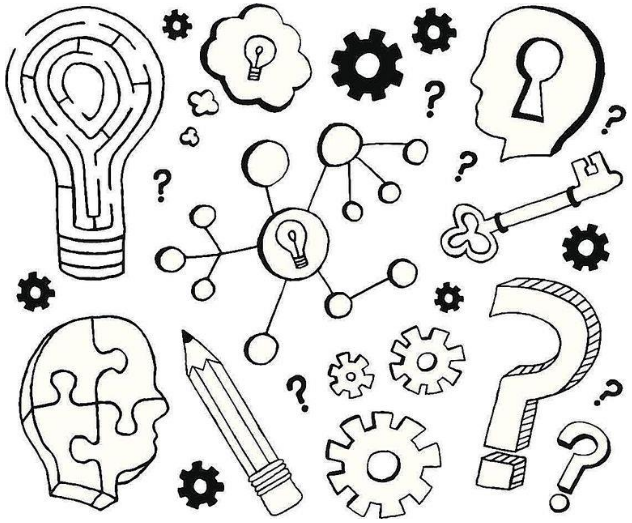
The Slope of a line

First, let's define the slope of a line.

$$\boxed{\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}}$$



Activity (Individual, 10')



1. A line passes from two points A(5,1) and B(15,6) . What is the slope of this line.
2. The general equation of a line is y=mx+b, where m & b are two constants. using the below definition, show that m is the slope of the line and that the line cuts the y axis at b.

$$Slope\ T = \tan\theta = \frac{\Delta y}{\Delta x}$$

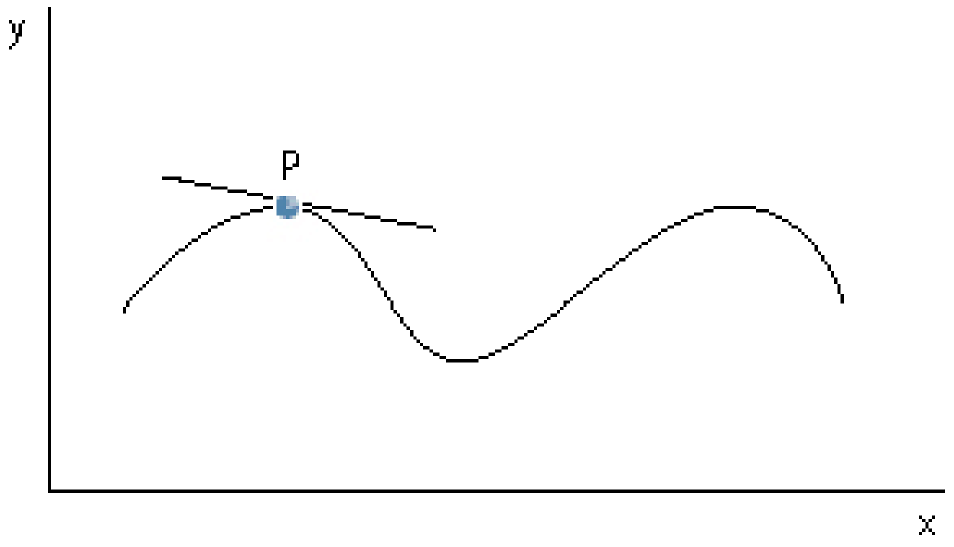
Slope of a Curve

- A curve C is given.
- What is the slope of this function at a point P?

Definition:

Slope of y=f(x) at P → The slope of the Tangent passing from P

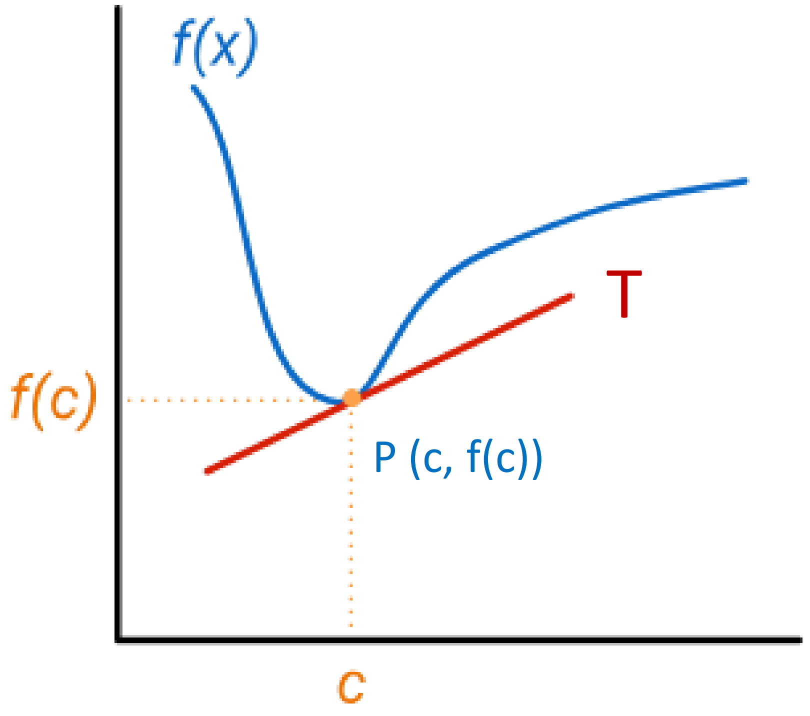
- Finding the slope of the Tangent → intrigued mathematicians from ancient times.



Tangent Line- Definition

What is the Tangent line to y= f(x) at P (x= c, y= f(c))?

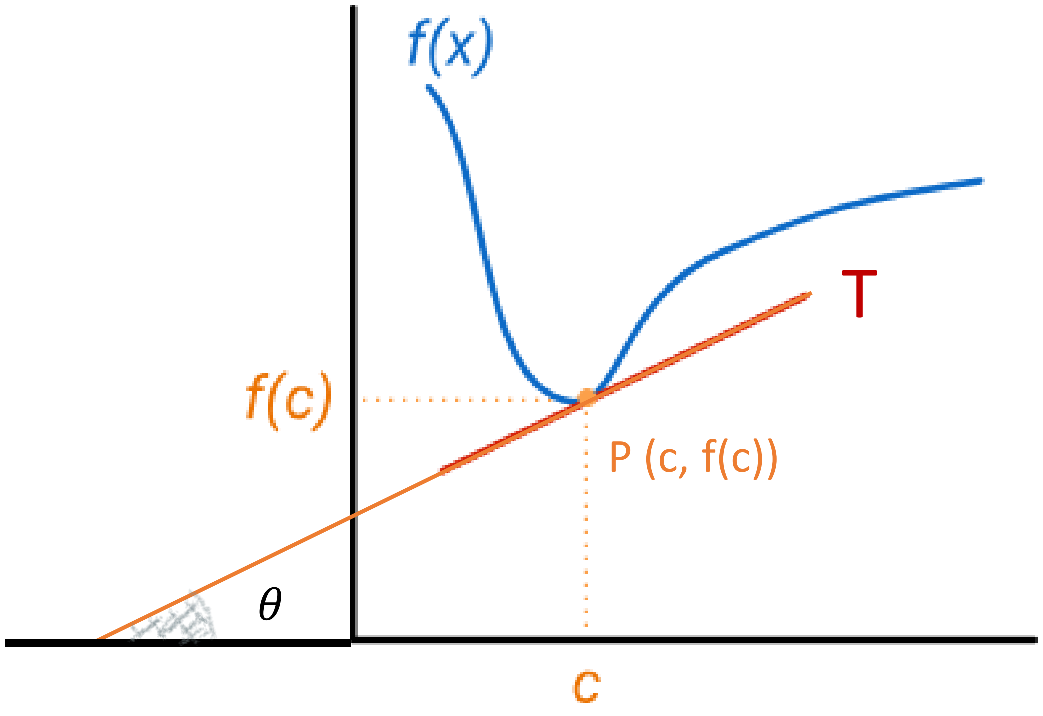
- Tangent line T to a curve/function y=f(x) at a given point P [i.e. P (c, f(c))]
- Intuitively, is the straight line that "just touches" the curve at that point.
- Leibniz defined it as the line through a pair of infinitely close points on the curve



The Slope of a Curve

How can we define the slope of a curve?

- Curve C is the plot of function y=f(x)
- The slope of the curve at a point P
- Is defined as the slope of Tangent T



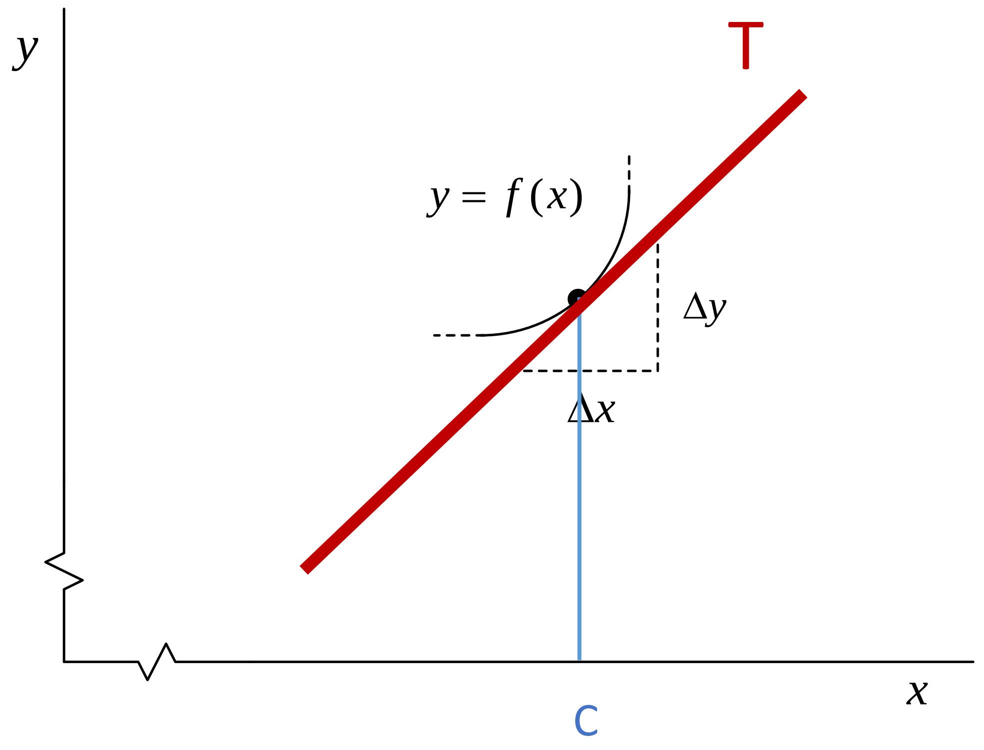
Slope of T → tgθ

The Slope of a Curve cont

- Slope of y = f(x) @ P(c, f(c)) = Slope of T
- Slope of T $\rightarrow$ tg θ

$$Slope\ T = \tan\theta = \frac{\Delta y}{\Delta x}$$

- We have $y = f(x)$
- How to find $\frac{\Delta y}{\Delta x}$ → Derivative



Derivative

- The derivative of f(x)
- With respect to x, is **defied** as:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

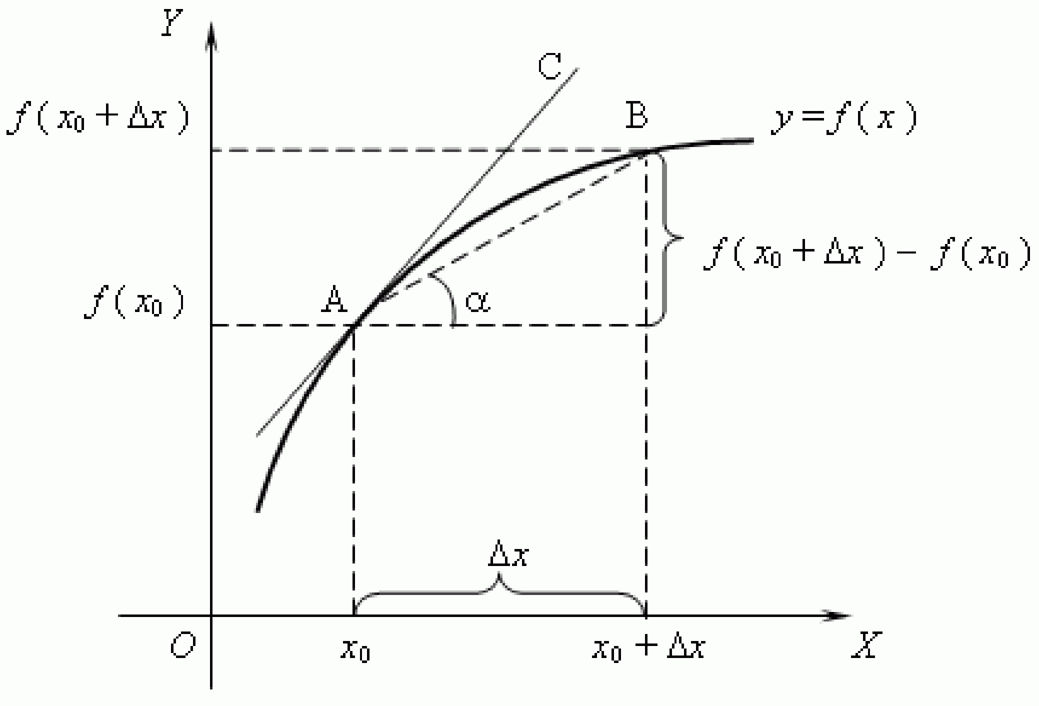


Fig. 1

$$Slope\ AB = \tan\alpha = \frac{\Delta y}{\Delta x}$$

Derivative- Definition

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

Differentiation and the Derivative

- A **derivative** is obtained through the process of *differentiation.*
- **Differential Calculus:** The study of all forms of differentiation.
- Various **notations** for derivative:

$$\boxed{\frac{dy}{dx}\ ,\ \frac{df(x)}{dx},\ f'(x),\ \dot{f}(x)}$$

Differentiation and the Derivative

$f(x)$

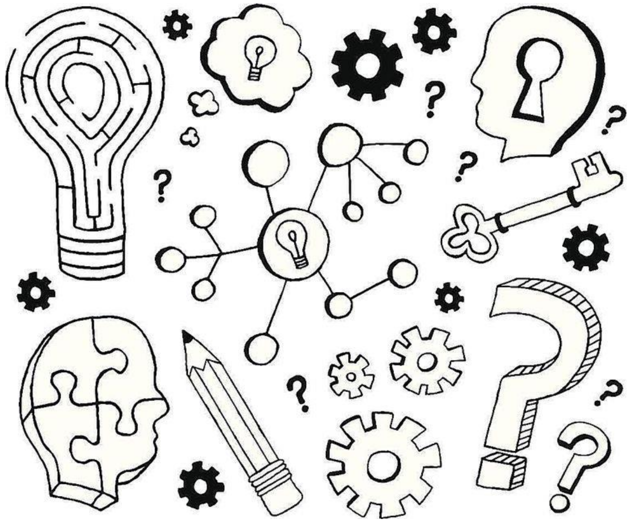
$f'(x)$

$f''(x)$

$f'''(x)$

- If we begin with a function
- To determine its derivative, we arrive at a new function called the *first derivative*.
- If we differentiate the *first derivative*, we arrive at a new function called the *second derivative*,
- ….

Activity (Individual, 15’)



Using the definition of derivative, find the derivative of the following function @ x=1:

- $f(x) = x$
- $f(x) = x^2$
- $f(x) = 1/x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

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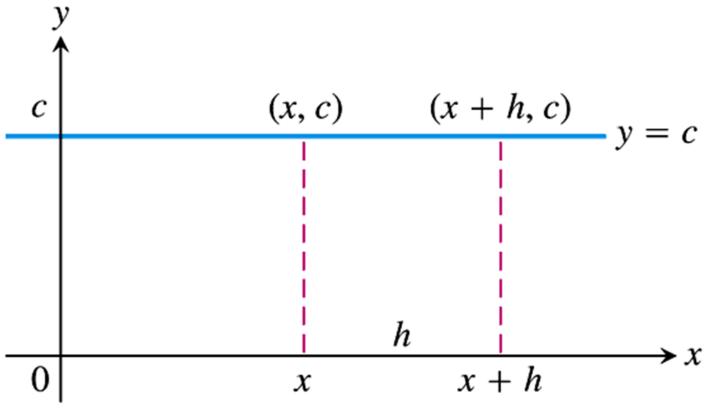
Derivative of a Function y=f(x)

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$



This Figure. he rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Derivative of a Function y=f(x)- Continued

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

Derivative of a Function y=f(x)- Continued

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Derivative of a Function y=f(x)- Continued

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Derivative of a Function y=f(x)- Continued

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Derivative of a Function y=f(x)- Continued

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Derivative of a Function y=f(x)- Continued

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

Derivative of a Function y=f(x)- Continued

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Derivative of a Function y=f(x)- Continued

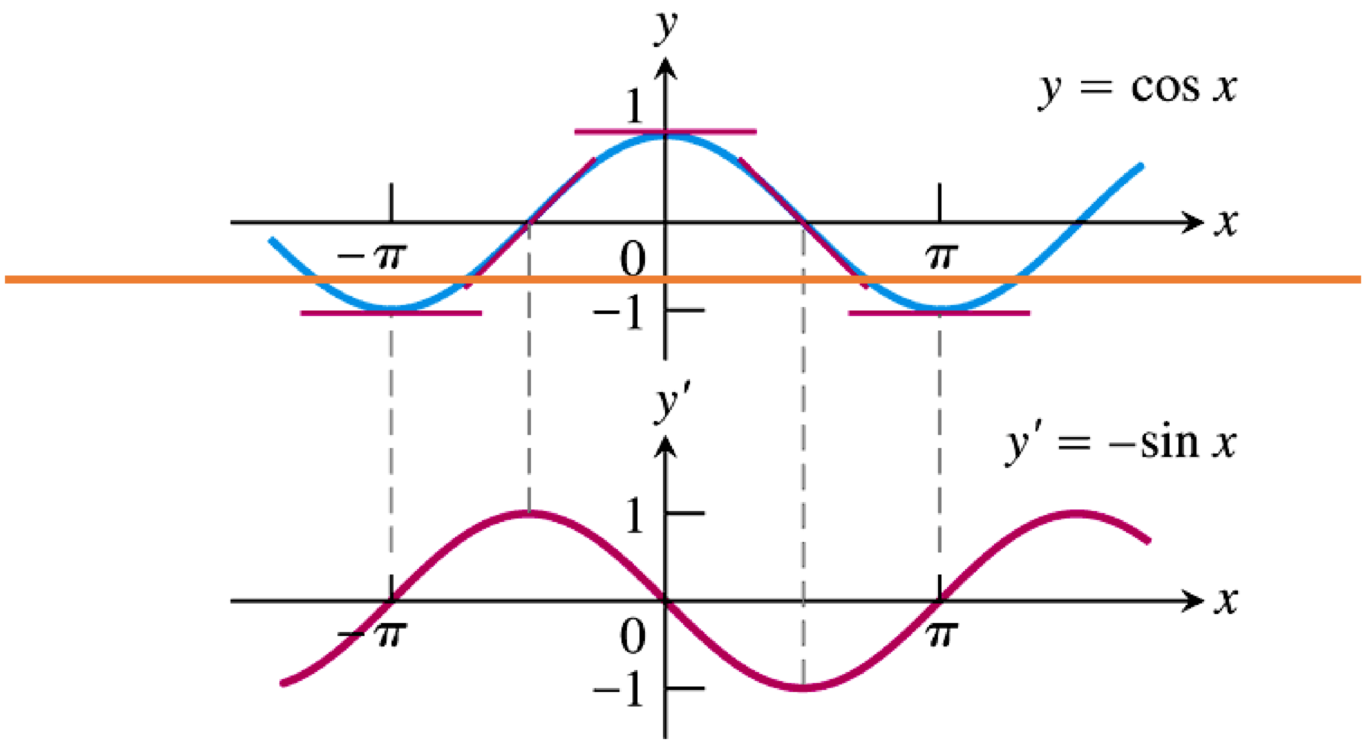


Figure. The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

Derivative of a Function y=f(x)- Continued

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

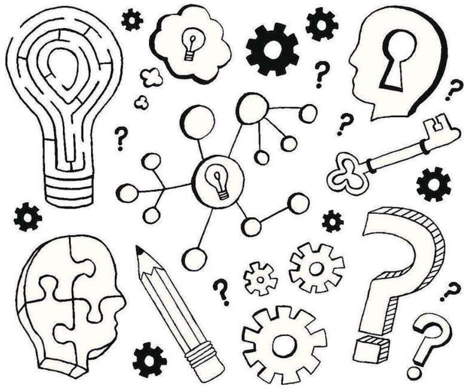
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Activity (Individual, 15’)

- Reflect on this Table. Try to come up with an example for each rule.



DIFFERENTIATION RULES

General Formulas

Assume u and v are differentiable functions of x.

Constant:	$\frac{d}{dx}(c) = 0$
Sum:	$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$
Difference:	$\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$
Constant Multiple:	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
Product:	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
Quotient:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Power:	$\frac{d}{dx}x^n = nx^{n-1}$
Chain Rule:	$\frac{d}{dx}(f(g(x)) = f'(g(x))\cdot g'(x)$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Exponential and Logarithmic Functions

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

$$\frac{d}{dx}a^x = a^x \ln a \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1 + x^2} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}} \qquad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1 + x^2}}$$

Parametric Equations

If $x = f(t)$ and $y = g(t)$ are differentiable, then

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}.$$

Chain Rule

- If y is a function of u → e.g. y=Ln u
- If u is a function of x, → e.g. $u=x^2$
- Then y is a function of x → $y = f(u(x))$ → e.g. $y = \ln(x^2)$ → $y' = \frac{2x}{x^2} = 2x^{-1}$
- Then the derivative of y with respect to x will be calculated from chain rule:

If $\begin{cases} y = f(u) \\ u = u(x) \end{cases}$ ⇒

$$\boxed{\frac{dy}{dx} = \frac{df(u)}{du}\frac{du}{dx} = f'(u)\frac{du}{dx}}$$

$$\boxed{\frac{dy}{dx} = \left[\frac{d(\text{Ln } u)}{du}\frac{dx^2}{dx}\right] = \frac{1}{u}2x = \frac{2x}{x^2} = 2x^{-1}}$$

Derivative of a Parametric Equation

- If y & x are parametric equations of t, then:

$$\begin{cases} y = f(t) \\ x = g(t) \end{cases} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\left.\begin{aligned} y &= t^2 \\ x &= t + 5 \end{aligned}\right\} \frac{dy}{dx} = \frac{2t}{1} = 2(x - 1)$$

Higher-Order Derivatives

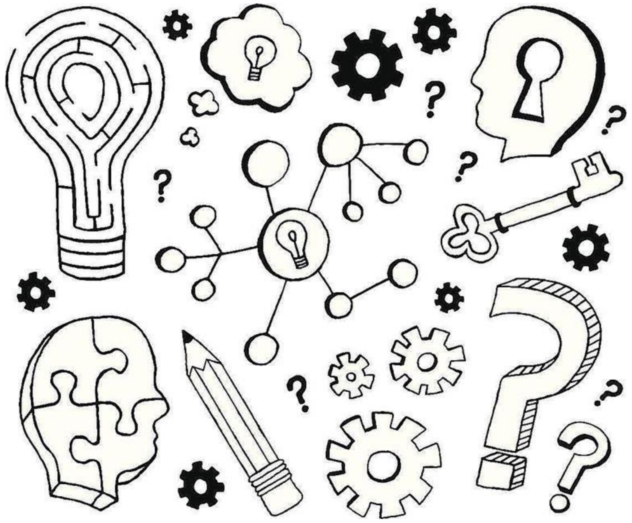
$$y = f(x)$$

$$\frac{dy}{dx} = f'(x) = \frac{df(x)}{dx}$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{d^3y}{dx^3} = f^{(3)}(x) = \frac{d^3 f(x)}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

Activity (Individual, 15')



1) Find the derivative of the following functions:

a. $y = \sum_{n=0}^{m} cx^n$

b. $y = \sqrt[n]{x^m}$

c. $y = \frac{\sin x}{x}$

d. $y = \operatorname{tg} x^m$

e. $y = sin(ln\, x)$

(m & n are constants)

2) Determine the 2nd derivative with respect to *x* of the function: $y = 5\sin 4x$

3) For the function:

- $y = sin(t)$
- $x = cos(t)$

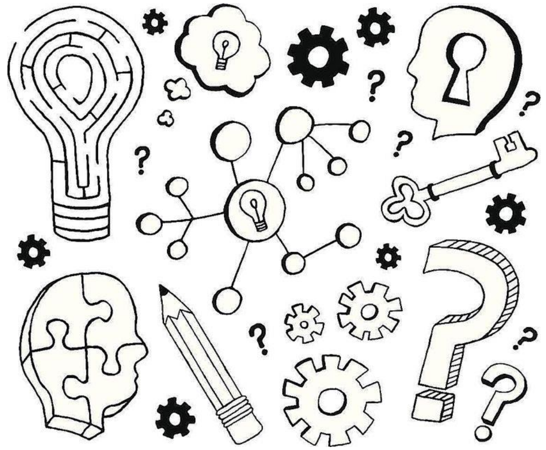
a) Find y(x)

b) Find y'(x)

c) Find x'(y)

d) What is the relation between y'(x) & x'(y)

Solutions to Q2 & Q3



Solution Q.2) Determine the 2nd derivative with respect to *x* of the function below.

$$y = 5\sin 4x$$

$$\frac{dy}{dx} = 5(\cos 4x)\cdot\frac{d}{dx}(4x) = 20\cos 4x$$

$$\frac{d^2y}{dx^2} = 20\left(-\sin 4x\right)\cdot\frac{d}{dx}(4x) = -80\sin 4x$$

Solution Q.3)

a) $y = \pm\sqrt{1-x^2}$

b) Y’(x)= - cot (t)

c) X’(y)= - tan (t)

d) Y’(x) X’(y)= 1

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III. Differentiability of a function

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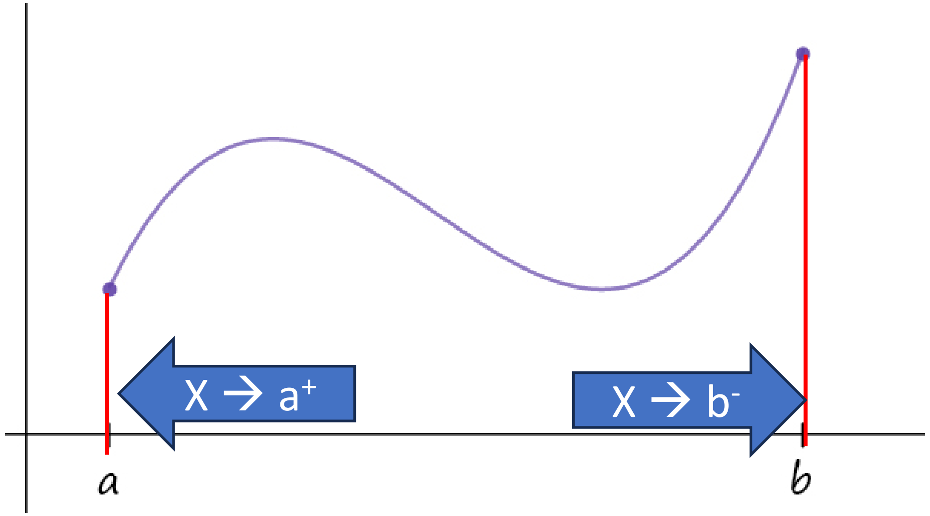
Differentiability

A function f(x) is differentiable in the interval [a,b] if :

i. The derivative exist for every point in domain (a,b)

ii. $f'(x)$ exist, for $x \rightarrow a^+$

iii. $f'(x)$ exist, for $x \rightarrow b^-$

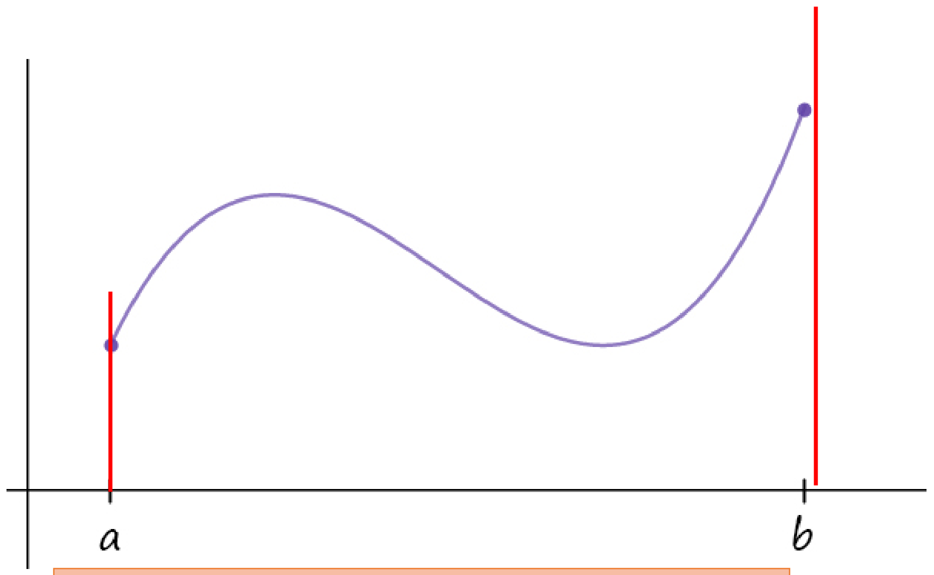


$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Differentiability cont...

If f(x) is differentiable → Graph of a f(x):

1. Must have a non-vertical tangent line at each point in its domain
2. Be relatively smooth,
3. cannot contain any breaks, bends, or cusps.

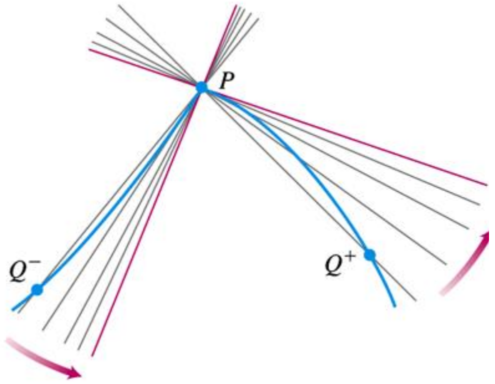


$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

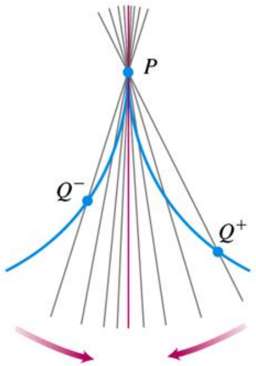
Non differentiability of a function

- The function y=f(x) is not differentiable at point P in the following graphs:

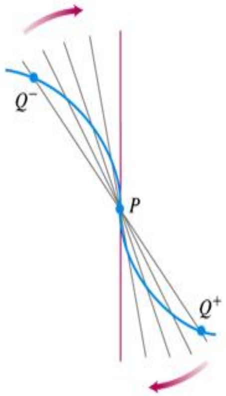
1. a *corner*, where the one-sided derivatives differ.



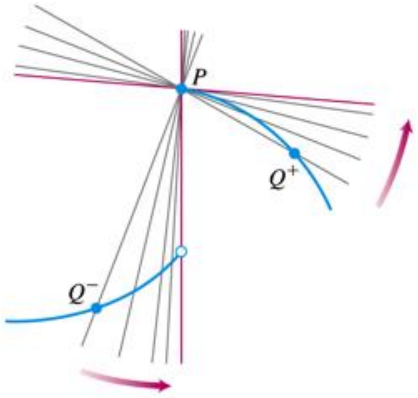
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.



Time for a break – 20’

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Some Application of Derivatives

- Absolute & Local Maximum & Minimum
- Monotonic Functions & first derivative test
- Concavity & Inflection Point
- Applied Optimization Problem
- More & More….

Absolute Maximum & Minimum

DEFINITIONS **Absolute Maximum, Absolute Minimum**

M

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

M

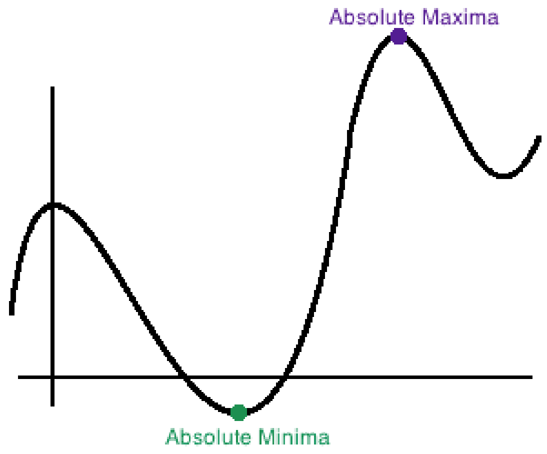
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

m

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

m

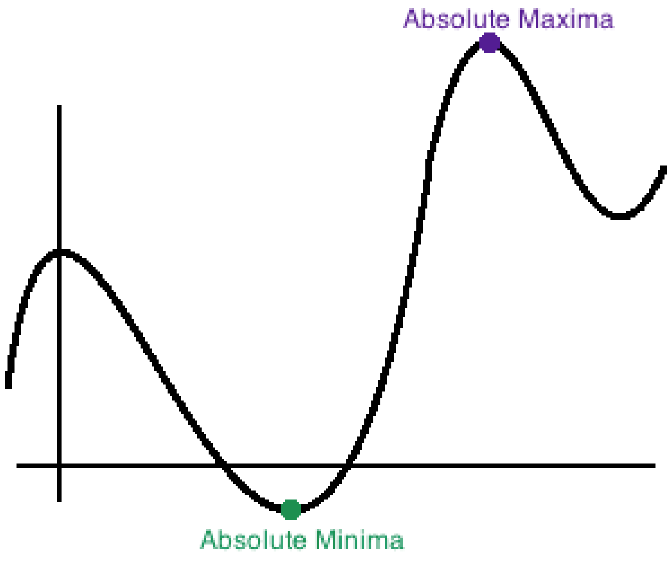


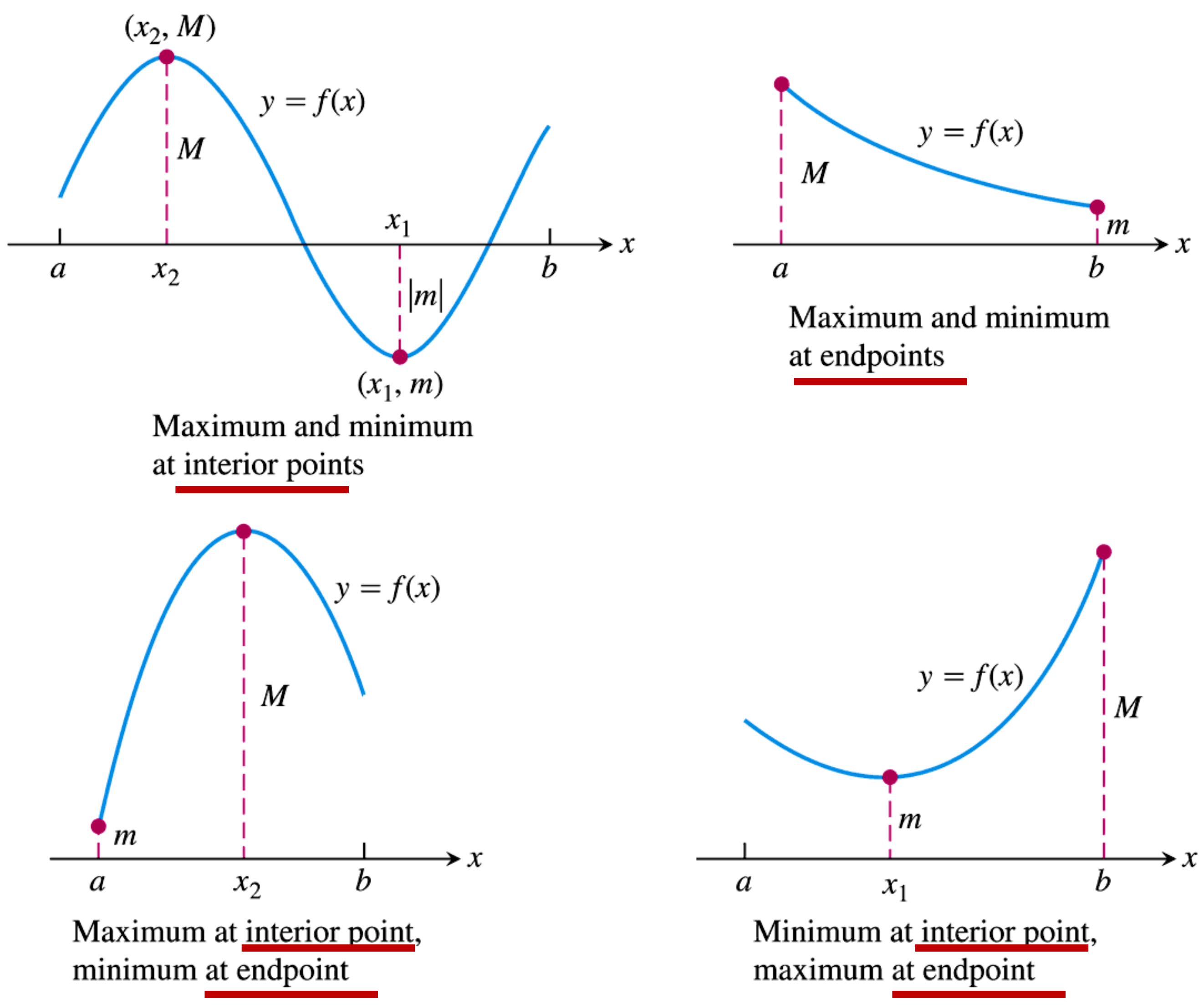
Absolute Maximum & Minimum cont

If y= f(x) is continues on [a,b]

1. Then f has an ABS. Max on [a,b] → M
2. Then f attains an ABS. Min (m) on [a,b] → m

- Then: For every x in [a,b] → $m \leq f(x) \leq M$





(This Figure): possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

Local Maximum & Minimum

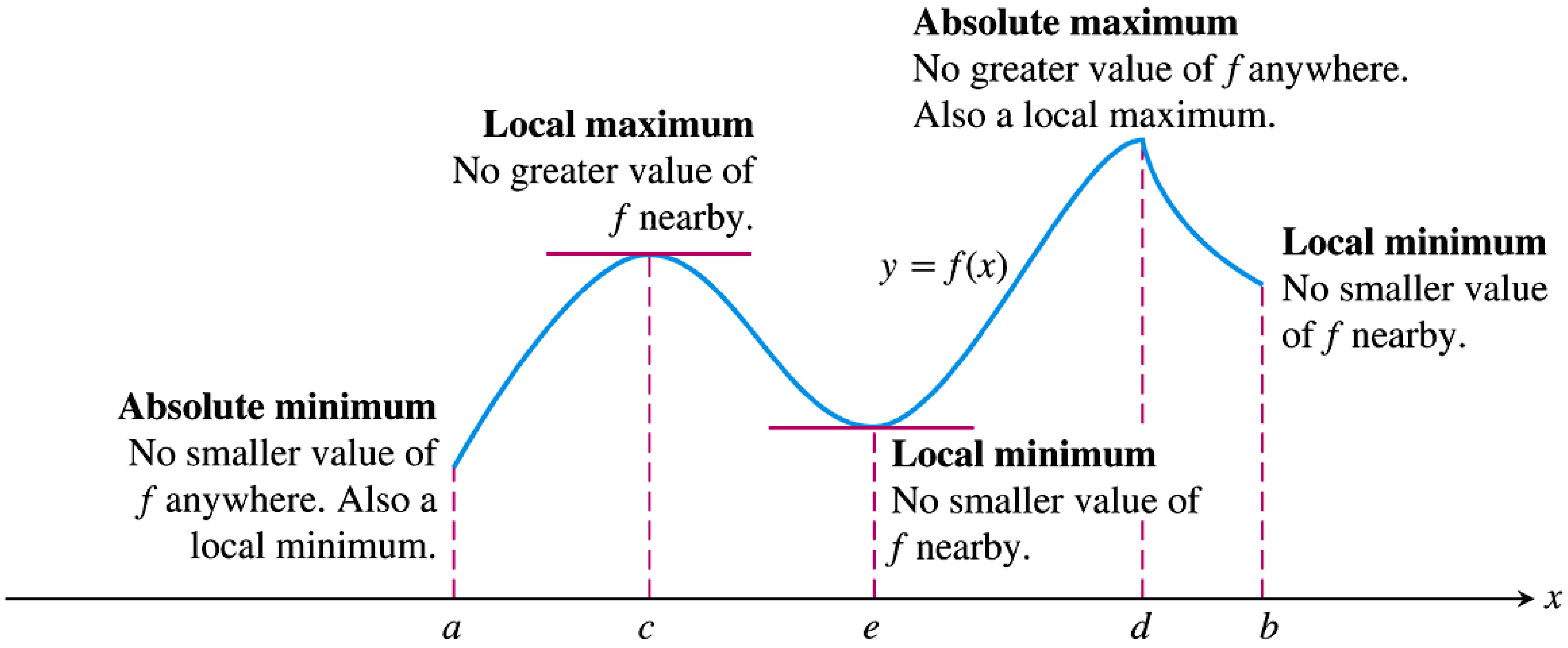
DEFINITIONS **Local Maximum, Local Minimum**

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

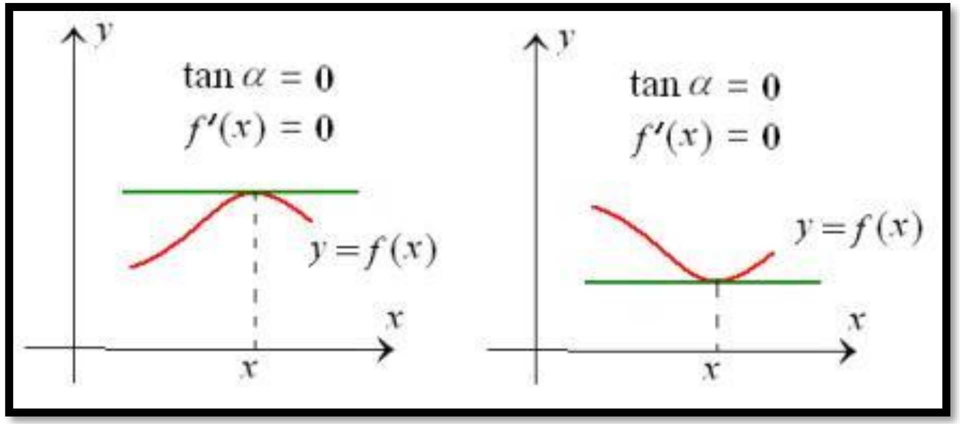
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$



(This Figure): How to classify maxima and minima.

Maximum, Minimum & Slope

- When
  - $f'(x) = 0 \rightarrow \mathrm{I}$
  - $f'(x) > 0 \rightarrow \mathrm{II}$
  - $f'(x) < 0 \rightarrow \mathrm{III}$



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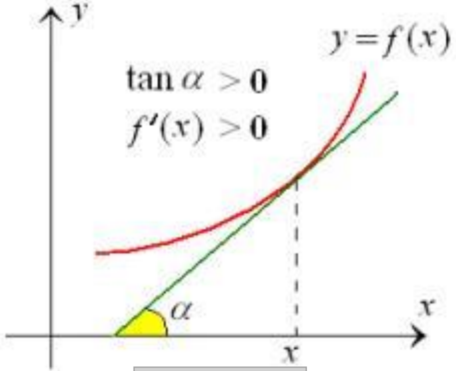

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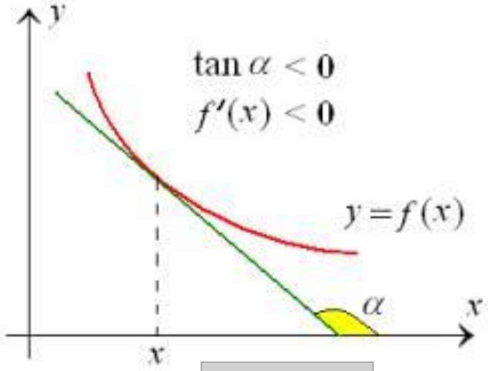


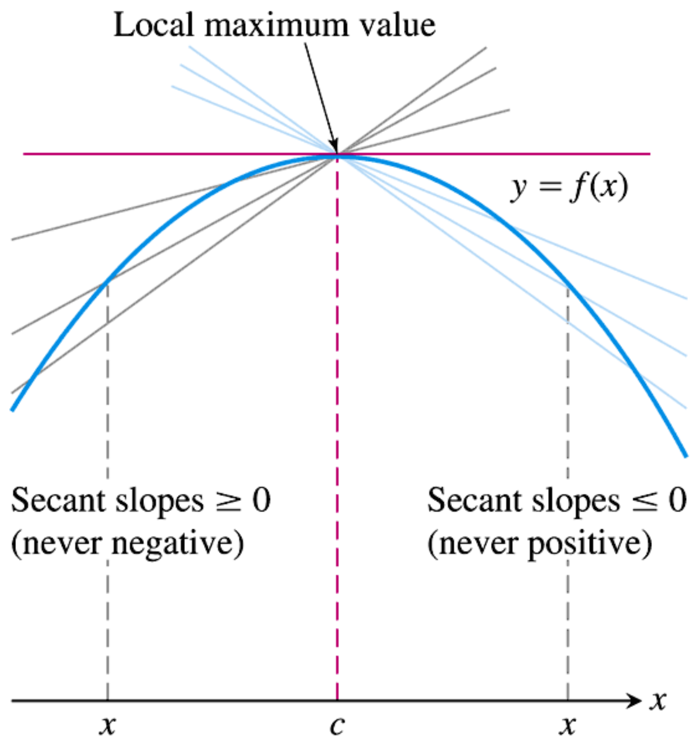
III

Derivative of Max/Min

THEOREM 2 The First Derivative Theorem for Local Extreme Values
If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

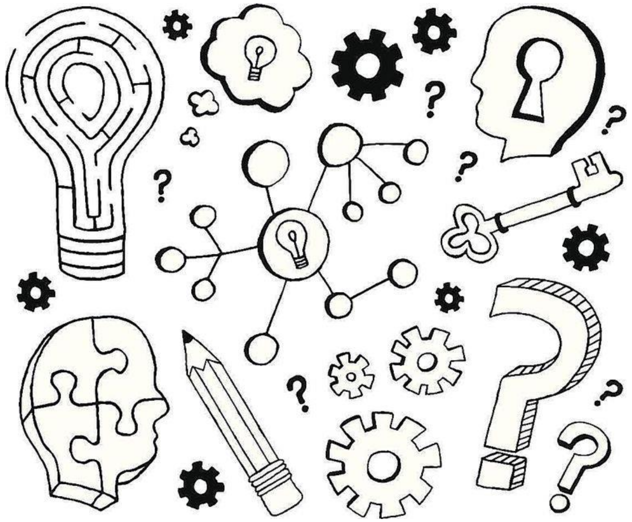
$$f'(c) = 0.$$

1. C = ABS. Max → f'(x @c) = 0
2. C = ABS. Min → f'(x @c) = 0
3. C = Local Max → f'(x @c) = 0
4. C = Local Min → f'(x @c) = 0



(This Figure): urve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Activity (Individual, 10’)



1. Determine local maxima or minima of function below

$$y = x^3 - 6x^2 + 9x + 2$$

Solution

1. Determine local maxima or minima of function below.

$$y = f(x) = x^3 - 6x^2 + 9x + 2 \rightarrow \text{Domain} = \text{R}$$

2. First Derivative:

$$\frac{dy}{dx} = 3x^2 - 12x + 9 \Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow x = 1 \text{ and } x = 3$$

3. Second Derivative:

$$\frac{d^2y}{dx^2} = 6x - 12 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$$

Solution- continue

x	$-\infty$		1		2		3		$+\infty$
Y’		+	0	-		-	0	+	
Y’’		-		-	0	+		+	
Behavior of y									
Y=f(x)			6				2		

Max: (1,6)

Min: (3,2)

Some Application of Derivatives

- Absolute & Local Maximum & Minimum
- Monotonic Functions & first derivative test
- Concavity & Inflection Point
- Applied Optimization Problem
- More & More….

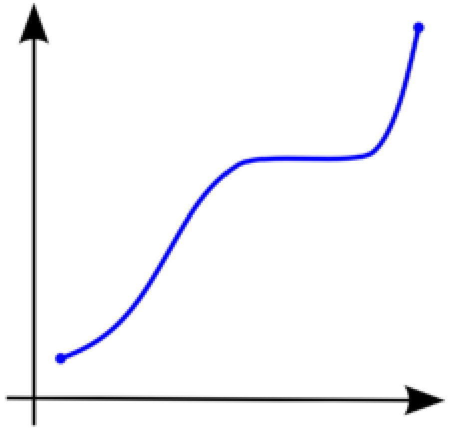
Monotonic Functions

DEFINITIONS **Increasing, Decreasing Function**

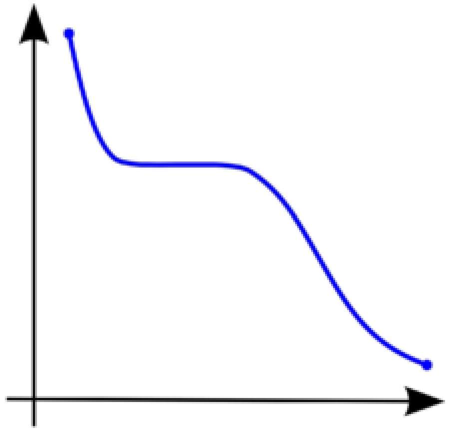
Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.



$Increasing: f'(x) > 0$



$Decreasing: f'(x) < 0$

Monotonic Functions cont

COROLLARY First Derivative Test for Monotonic Functions

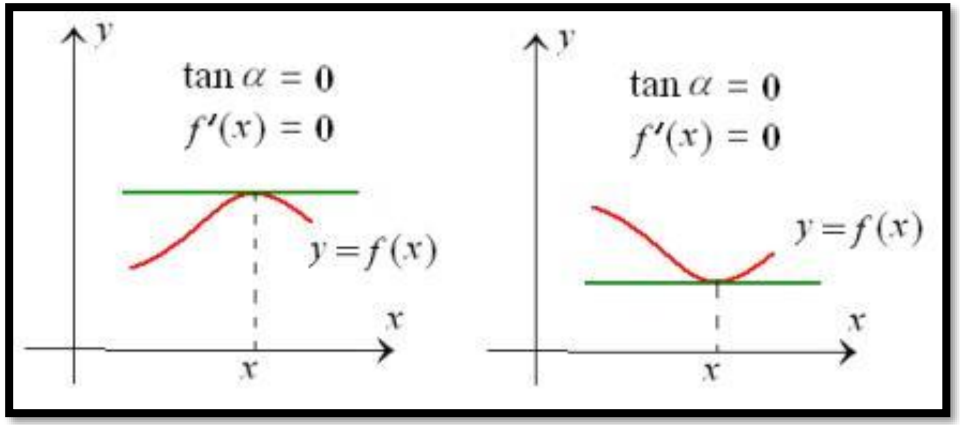
Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

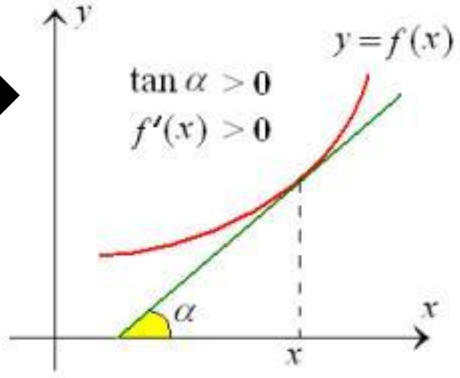
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Maximum, Minimum & Slope

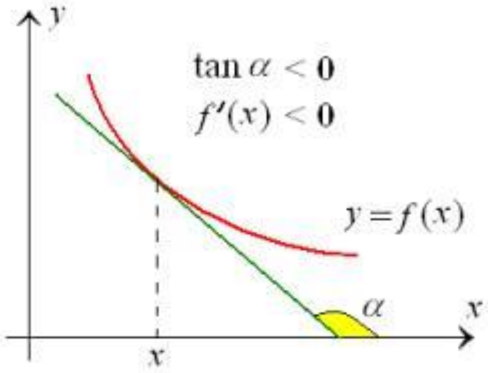
- When
  - $f'(x) = 0 \ \rightarrow \mathrm{I}$
  - $f'(x) > 0 \ \rightarrow \mathrm{II}$
  - $f'(x) < 0 \ \rightarrow \mathrm{III}$



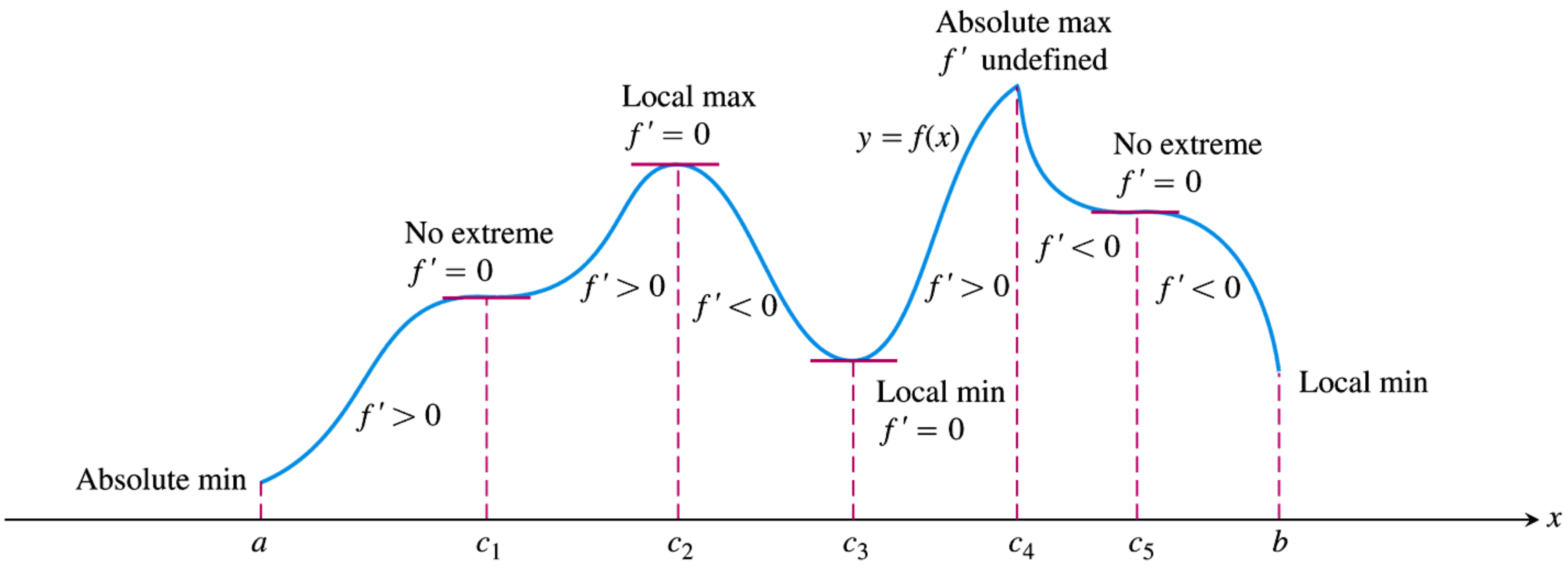
I



II

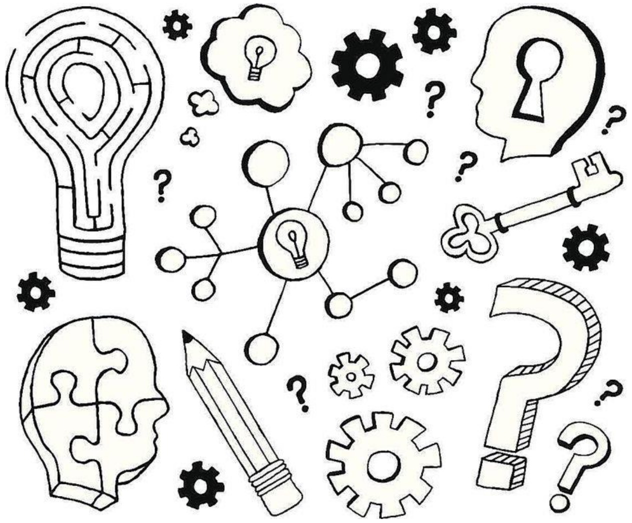


III



(This Figure): A function's first derivative tells how the graph rises and falls.

Activity (Individual, 15’)



1) Find the Max/Min of the following functions.
2) Which of them is monotonic?
3) Which is increasing and which is decreasing?

   a) $f(x) = x$

   b) $f(x) = x^2$

   c) $f(x) = 1/x$

   d) $f(x) = \sin x$

Some Application of Derivatives

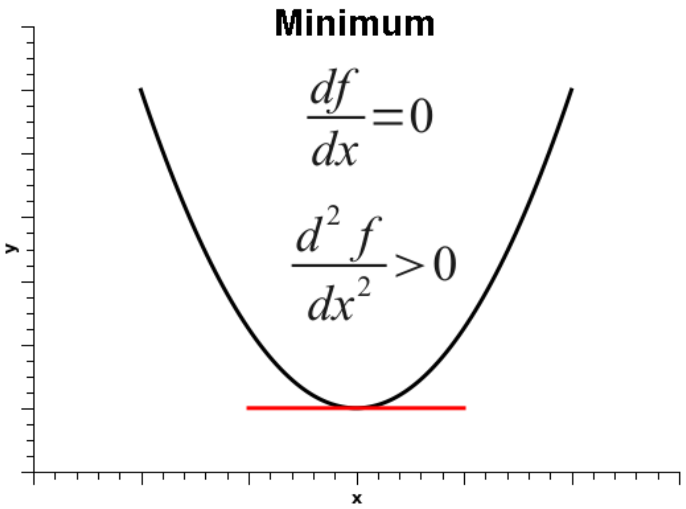
- Absolute & Local Maximum & Minimum
- Monotonic Functions & first derivative test
- Concavity & Inflection Point
- Applied Optimization Problem
- More & More….

Concavity

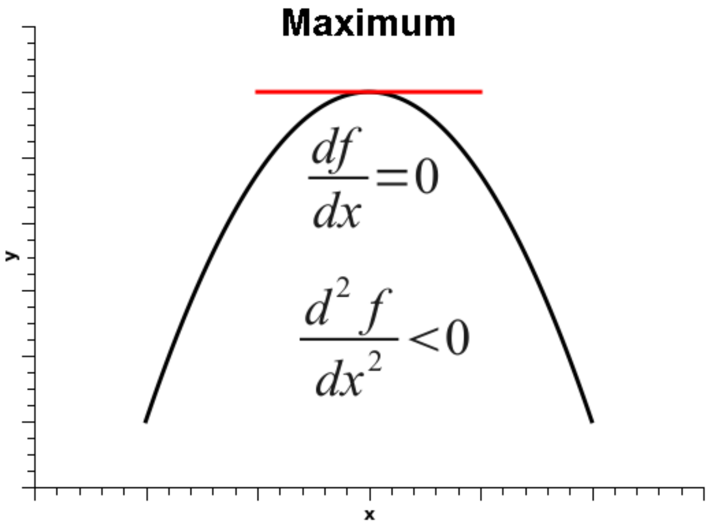
The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.



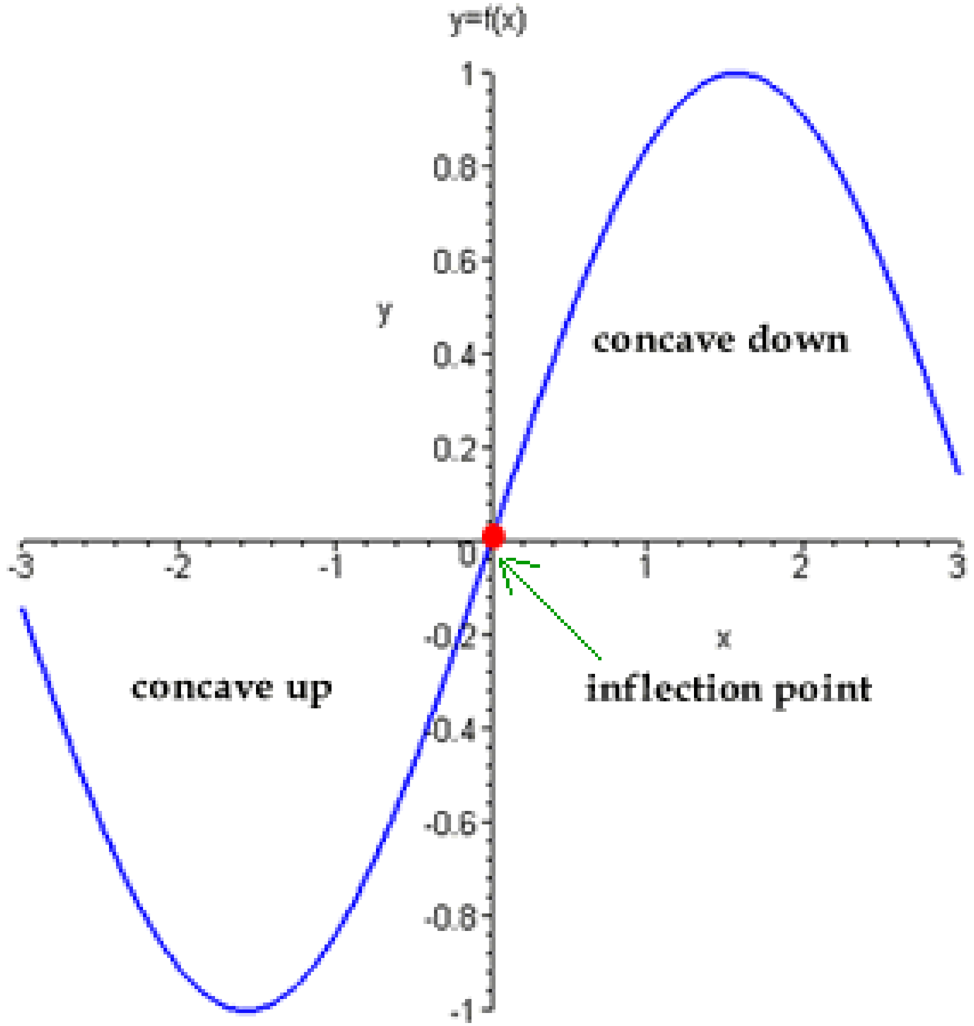
Concavity: Up



Concavity: Down

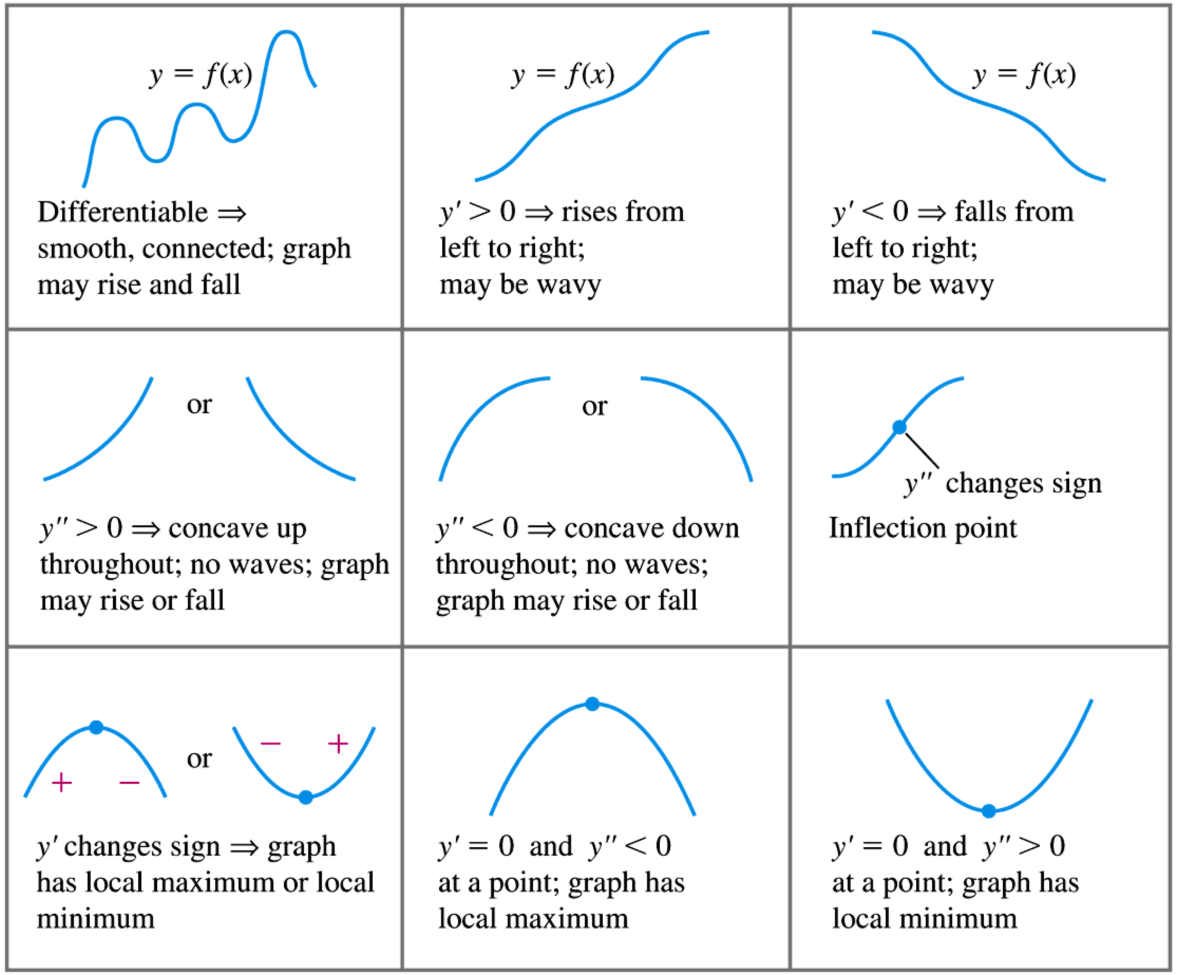
Concavity & Inflection Point

- Inflection Point → Curvature changes
- At inflection point P → $f''(x) = 0$



Activity (Individual, 10’)

- Reflect on each graph.



Graphing a function

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Some Application of Derivatives

- Absolute & Local Maximum & Minimum
- Monotonic Functions & first derivative test
- Concavity & Inflection Point
- Applied Optimization Problem
- Graphing a function
- ODE & PDE
- More & more

Today's Outline

I. Derivatives

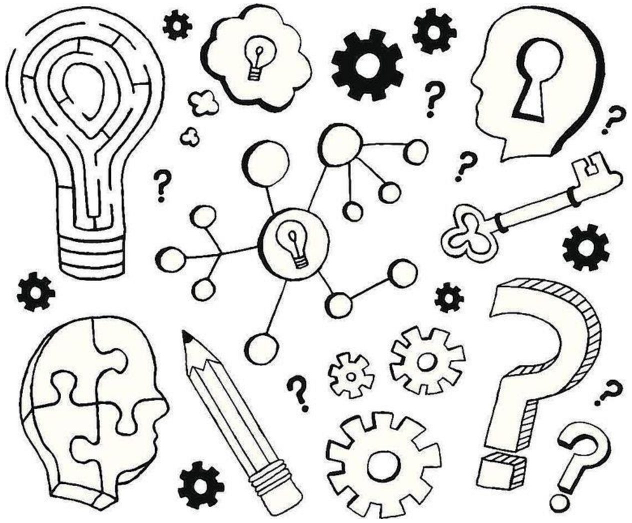
II. Derivative of functions

III. Differentiability of a function

IV. Some Applications of Derivatives

V. Tutorial

Reflection, Individual – 40’



Define the following:

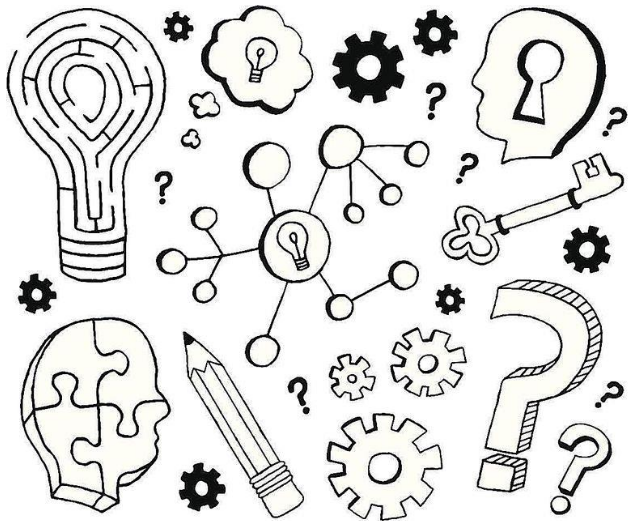
1. Absolute Maximum
2. Absolute Minimum
3. Local Maximum
4. Local Minimum
5. Critical points
6. First Derivative Test for local extreme values
7. Increasing functions
8. Decreasing functions
9. Monotonic functions
10. First Derivative Test for monotonic functions
11. Determine local maxima or minima of function below.

$$y = f(x) = x^5 - 8x^3 + x + 2$$

Research

What is the procedure for graphing a function y = f(x),

using what you have learned in calculus?



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Source of the slides:

Thomas Calculus – 11e

Stewart Calculus

https://www.slideserve.com/search/presentations/derivatives-and-integrals