Learning Goals:

- How **imprecision** in concepts can be discussed using the basics of fuzzy sets
- The basic principles of **organizing a fuzzy logic system**
- What is inside the **rule-base of a fuzzy control system**
- Methods of **building a fuzzy control system**

Course Content:

- **Terminology:**
  - Uncertainty
  - Approximations
  - Vagueness
- **Fuzzy Sets**
- **Fuzzy logic and Fuzzy systems**
- **Fuzzy control**
- **Neuro-Fuzzy systems**

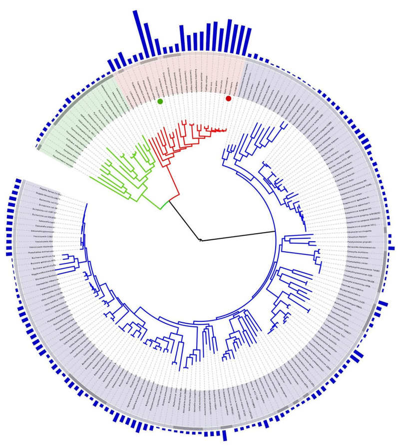
Soft computing - umbrella term for sub-disciplines of computing including:

- Fuzzy logic, and control
- Neural network based systems and ML
- Genetic Algorithms
- Chaos Theory in Mathematics

Zadeh: “More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership.”

- E.g. set of animals: includes horses, dogs, cats, but not rocks, fluids, plants
  - Starfish and bacteria are then ambiguous in the context of an animal classification
- E.g. numbers much greater than one, does it include 10? Hard to say yes or no

Zadeh: **“Imprecisely defined ‘classes’ play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction.”**



Example: Tree of life in evolutionary theory.

- The interconnection between species **blurs or fuzzifies the boundaries**
- The notion of this fuzzification has allowed evolutionary biologists to talk about **clades (term for organisms with common ancestry due to branches)**

Fuzziness is not Randomness

- “Much of the decision making in the real world takes place in an environment in which the goals, the constraints and the consequences of possible actions are not known precisely.”

Fuzziness is not Randomness

- By dealing with imprecision using probability theory, the premise of imprecision equating with randomness is accepted. This is wrong according to Bellman & Zadeh.

Given a fuzzy goal G and a fuzzy constraint C in a space of alternatives X: The decision achieving G and satisfying C, is itself a fuzzy set, D. This occurs at the intersection of the goal and the constraint. Symbolically:

$$D = G \cap C$$

Fuzzy is:

- **(Computing and logic)** Defined so as to allow for imprecise membership criteria and for the gradations of membership; pertaining or belonging to such a set
- **(Fuzzy logic)** the logic of fuzzy sets and concepts; fuzzy matching, the matching of items that are similar but not identical.
- A deviant or alternative logic

What are fuzzy sets and systems?

- The theory of fuzzy sets now encompasses a corpus of basic notions including aggregation operations, a generalised theory of relations, specific measures of information content, a calculus of fuzzy numbers.
- Fuzzy sets have led to:
    - Non-additive uncertainty theory
    - A tool for both linguistic and numerical modelling: fuzzy rule-based system
- **Used to mathematically represent uncertainty and vagueness, providing formalized tools for dealing with imprecision**

Fuzzy logic is being developed as a discipline to meet 2 objectives

- A professional subject dedicated to the building of systems of high utility - for example fuzzy control
- A theoretical subject - a symbolic logic with a comparative notion of truth

Zadeh introduced the theory of fuzzy sets

- A collection of objects that might belong to the set to a degree
    - **1 for full belongingness**
    - **0 for full non-belongingness**
- **Membership functions** designed to assign each element a number **indicating its intensity of belongingness**

Fuzzy Logic provides a **formal methodology for representing, manipulating, and implementing a human's heuristic knowledge about control**.

- Generated rules of thumb - created by engineers out of mathematical analysis, and operators running complex control systems

Zadeh also devised fuzzy logic: devised model human reasoning processes comprising of

- **Vague predicates** - large,beautiful,small

- **Partial truths** - not very true, more or less false
- **Linguistic quantifiers** - most, almost all, a few
- **Linguistic hedges** - very, more or less

Vagueness can facilitate certainty

- Jones/Jill is tall - **“Confidence in the truth of a vague assertion, just because of its vagueness”**
- Jones/Jill is 1.8297m tall - **“The laws of physics can acquire this minuteness of detail only by sacrificing some of the fixed absolute certainty of common sense laws.”** - there exists a trade-off between certainty and precision.

Fuzzy logic is not a vague logic system, it is a system of logic for dealing with vague concepts.

Distance	Speed	
	Stationary	Fast
Very Near	Very Angry, Not surprised, No Fear	Not Angry, Not surprised, Very Fearful
Far	Very Angry, Not surprised, No Fear	Not Angry, Very Surprised, No Fear

Distance	Speed		
	Stationary	Slow	Fast
Very Near	VA, NS, NF	A, NS, F	NA, NS, VF
Near	A, NS, NF	NA, NS, NF	NA, S, F
Far	VA, NS, NF	A, S, NF	NA, VS, NF

Perception Linguistic Variable	Term Set
SPEED:	{F → Fast; SL → Slow; ST →Stationary
DISTANCE:	{VN → Very Near; N → Near; F→ Far Away}

Emotional Linguistic Variable	Term Set
ANGER:	{VA → Very Angry; A → Angry; NA → Not Angry}
SURPRISE:	{VS → Very Surprised; S →Surprised; NS→ Not Surprised}
FEAR:	{VF → Very Fearful; F → Fearful; NF → No Fear}

Getting computers to do what we want (what we naturally do)

- **“Soft computing differs from conventional (hard) computing ... unlike hard computing, it is tolerant of imprecision, uncertainty, partial truth, and approximation. ... The guiding principle of soft computing is: Exploit the tolerance for imprecision, uncertainty, partial truth, and approximation to achieve tractability, robustness and low solution cost.”**

Crisp Set: Membership of classical set theory is described according to a bivalent condition - all members of the set definitely belong to the set while all non members definitely do not.

Estimative Probability: a statement of indisputable fact, describing something knowable, that is known with a high degree of certainty

Estimative Certainty: a judgement or estimate, something that is knowable in the realm of human understanding but not precisely known by the person saying it (inferable)

Estimative Certainty: A weaker judgement or estimate, made without evidence direct or indirect, possibly an estimate that is not knowable

“The notions of inclusion, union, intersection, complement, relation, convexity, … can be extended to such sets, and various properties of these notions in the context of fuzzy sets have been established”

System	Variable	Relationships	
		Simple	Complex
Conventional	**Quantitative, e.g. numerical**	**Conditional and Relational Statements between domain objects *A*, *B*: IF *A* THEN B; *A* is-a-part-of *B* *A* weighs 5KG**	**Ordered sequences of instructions comprising *A*=5; IF *A* < *5* THEN *B*=*A*+*5* ………**
Fuzzy	**Quantitative (e.g. numerical) and *linguistic* variables**	**Conditional and Relational Statements between domain objects *A*, *B*: IF *A* (Ψ_A) THEN *B* (Ψ_B) *A* weighs *about* 5KG**	**Ordered sequences of instructions comprising *A* IS-SMALL; IF *A* IS_SMALL THEN *B* IS LARGE**

Fuzzy System vs Crisp System

1. **Linguistic Variables**: defined as a variable whose values are sentences in a natural or artificial language (e.g. tall, not tall, very tall etc are values of HEIGHT - height is a linguistic variable)
2. **Fuzzy Conditional Statement**: expressions of the form IF A THEN B where A and B have some ‘fuzzy meaning’ (e.g. IF x is small THEN y is large, where small and large are labels of fuzzy sets)
3. **Fuzzy Algorithm**: ordered sequence of instructions which may contain fuzzy assignment and conditional statements (e.g. x = very_small IF x is small THEN y is large … The execution of such instructions is **governed by the compositional rule of inference and the rule of the preponderant alternative**)

Fuzzy Restriction:

- A fuzzy relation which acts as an **elastic constraint** on the values that may be assigned to a variable

Fuzzy Logic is not Fuzzy Set Theory

- Fuzzy logic is a branch of fuzzy set theory which deals with the representation and inference from knowledge. Fuzzy logic handles **imprecise or uncertain knowledge.**

Elements of a fuzzy set may **belong to a set, not belong to a set, or may belong to the set to a degree.**

Fuzzy set notation

'Large Number'	Comment	'Degree of membership'
10	'Surely'	1
9	'Surely'	1
8	'Quite poss.'	0.8
7	'Maybe'	0.5
6	'In some cases, not usually'	0.2
5, 4, 3, 2, 1	'Definitely Not'	0

We can denote Johnny's notion of 'large number' by the fuzzy set

$$A = 0/1 + 0/2 + 0/3 + 0/4 + 0/5 + 0.2/6 + 0.5/7 + 0.8/8 + 1/9 + 1/10$$

For convenience, a fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots + \mu_A(x_n)/x_n$$

That belongs to a finite universe of discourse:

$$A \subseteq \{x_1, x_2, \ldots, x_n\} \equiv X$$

Can define a small number membership function set B using **NOT (** μ_A **(.))**

Fuzzy (sub-)sets: Membership Functions

Let X = { x } be a universe of discourse i.e., a set of all possible, e.g., feasible or relevant, elements with regard to a fuzzy (vague) concept (property). Then

$$A \underset{\sim}{\subset} X \text{ (A of X)}$$

denotes a fuzzy subset, or loosely fuzzy set, a set of ordered pairs $\{(x, \mu_A(x))\}$ where $X \in x$.

$\mu_A : X \rightarrow [0, 1]$ the membership function of A

$\mu_A(x) \in [0, 1]$ is grade of membership of x in A

Properties of fuzzy sets

- **Equality**
  - Fuzzy set A = Fuzzy set B iff $\mu_A(x) = \mu_B(x)$
- **Inclusion**
  - Fuzzy set A ⊂ X is included in (is a subset of) fuzzy set B ⊂ X where $\mu_A(x) \leq \mu_B(x) \, \forall x \in X$

- Consider X = {1, 2, 3} and
  A = 0.3/1 + 0.5/2 + 1/3;
  B = 0.5/1 + 0.55/2 + 1/3
  Then A is a subset of B

- **Cardinality**
  - The cardinality of a fuzzy set is the **sigma count**, the sum of the values of the membership function A
- **Emptiness**
  - A fuzzy set if empty iff $\mu_A(x) = 0, \forall x \in X$
- **Alpha level cuts**
  - An alpha level cut/set of a fuzzy set A ⊂ X is an ordinary set $A_\alpha \subset X$ such that $A_\alpha = \{x \in X; \mu_A(x) \geq \alpha\}$
    - E.g. A=0.3/1 + 0.5/2 + 1.0/3 ; X = {1,2,3}
    - $A_{0.5} = \{2, 3\}$; $A_{0.1} = \{1, 2, 3\}$ etc.
- **Normality**
  - A fuzzy set is considered to be normal if at least one element has a membership of 1 to the set
  - A fuzzy set is said to be subnormal if the above is false
- **Height**
  - The height of a fuzzy subset A is the largest membership grade of an element in A
- **Support**
  - The support of fuzzy subset A of X is the crisp subset of X whose elements all have non-zero membership grades in A
    - supp(A) = {x| $\mu_A(x) > 0$ *and* $x \in X$}
- **Core**
  - The core of a fuzzy subset A of X is the crisp subset of X whose elements have membership grade 1
    - Core(A) = {x| $\mu_A(x) = 1$ *and* $x \in X$}
  - **Obviously a subnormal set has a null core, and a normal set does not**

Operations on fuzzy sets

- **Complementation**
  - a.k.a (A of X)' = NOT (A of X)
  - The membership function to the complementary set of any fuzzy set is 1 less the set membership function
  - The intersection of a fuzzy set and its complement is not a null set
- **Bounded Sum**
  - The bounded sum of two fuzzy subsets A and B of fuzzy set X is the fuzzy subset C, where the membership function of C: $\mu_C = min[1, (\mu_A(x) + \mu_B(x))]$
- **Bounded Difference**
  - The bounded difference of two fuzzy subsets A and B of fuzzy set X is the fuzzy set D where the membership function of D: $\mu_D = max[0, (\mu_A(x) - \mu_B(x))]$

- **Intersection**
    - The intersection of fuzzy subsets A and B of X is denoted as the fuzzy subset D of X: $\mu_D(x) = min[\mu_A(x), \mu_B(x)]$
- **Union**
    - The union of fuzzy subsets A and B of the set X, is denoted as the fuzzy subset C of x. The membership function for C is:
      $\mu_C(x) = max[\mu_A(x), \mu_B(x)] = \mu_A(x) \vee \mu_B(x)$
- **Concentration**
    - To concentrate is to reduce the membership
    - Some fuzzy set A is concentrated by a factor of α where $\alpha > 1$: $A^{\alpha} \subset A$
- **Dilation**
    - To dilate if to increase membership
    - Some fuzzy set B is dilated by a factor of α where $\alpha < 1$: $A^{\alpha} \supset A$
- **Level Set**
    - If A is a fuzzy subset of X and $0 \le \alpha \le 1$ then we can define another fuzzy subset F such that: $F = \alpha A$; $\mu_F(x) = \alpha\mu_A(x) \quad x \varepsilon X$

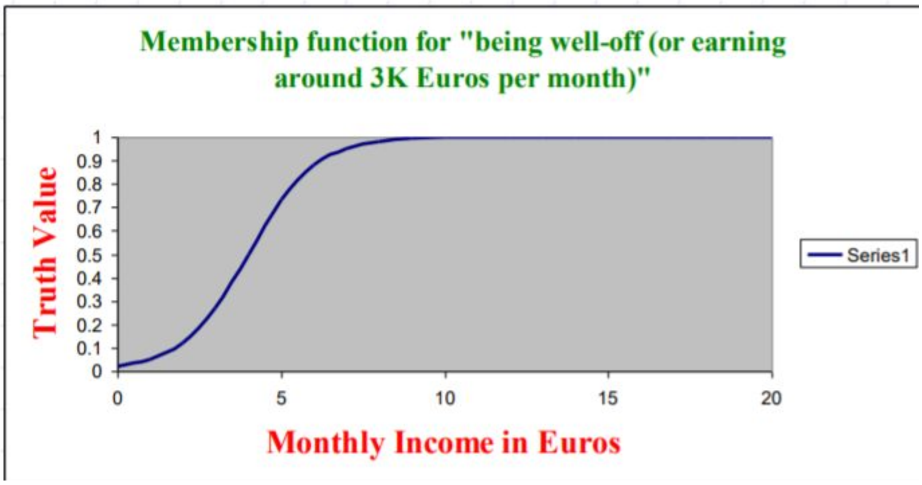
Fuzzy membership functions shapes

Fuzzy Relationships
- **Cartesian/Cross Product**
    - Of fuzzy subsets A and B of sets X and Y respectively is denoted by A x B
    - This cross product relationship T on the set X x Y is denoted as:
      $\mu_T(x,y) = min[((x), \mu_B(y))]$
    - E.g. A={1/a1, 0.6/a2} ; B={0.6/b1,0.9/b2}
    - A x B = {0.6/(a1,b1) , 0.9/(a1,b2),
    - 0.6/(a2,b1) , 0.6/(a2,b2) }
    - **Cross products facilitate the mapping of fuzzy subsets that belong to disparate quantities or observations. This mapping is crucial for fuzzy rule based systems in general and fuzzy control systems in particular.**
- **Membership functions**

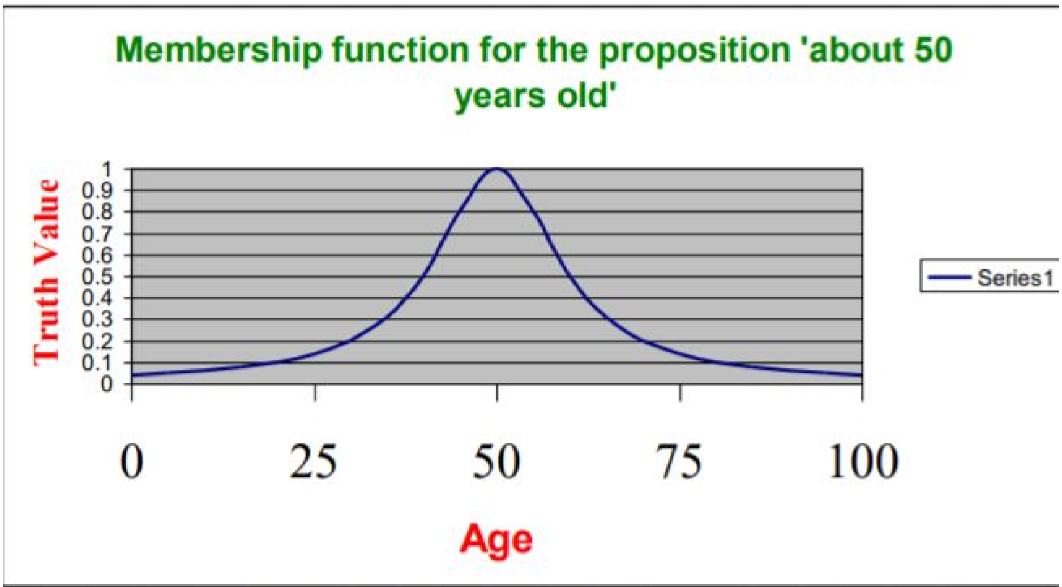
Membership Function	Mathematical Formulation
Triangular	$trimf(x;a,b,c) = max\left(min\left(\frac{x-a}{b-a},\frac{c-x}{c-b}\right),0\right)$
Trapezoidal	$trapmf(x;a,b,c,d) = max\left(min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),0\right)$
Gaussian	$gaussmf\ (x;a,b,c) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$
Generalized Gaussian	$gbellmf\ (x;a,b,c) = \frac{1}{1+\left\lvert\frac{x-c}{\sigma}\right\rvert^{2b}}$

- **Sigmoid Functions map the concept of ‘around x or more’**

$$\mu_{wealthy}(x) = \frac{1}{1+e^{(-a\times(x-c))}}$$



- **Peaked membership function for ‘about’-ness**



$$\mu_{50\ or\ so\ old}(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^2}$$

- **Triangular membership functions**

$$f(x;a,b,c) = \max(\min(\frac{x-a}{b-a},\frac{c-x}{c-b}),0)$$

- **Fuzzy Patches**

    - Mapping of two membership functions, a patch is an area that represents the causal association between the cause (input) and the effects (output)
    - Size of the patch is indicative of the vagueness - big patch caused by mapping of fuzzy-fuzzy. Point occurs on crisp to crisp mapping.
- **Primary terms**
    - The terms of a fuzzy set which must be defined a priori - they may be defined by their own membership functions which are in turn used to define other terms:
        - E.g. T(age) has terms young, old, very young, very old, not young, not old
        - Young and old are primary terms, other terms are defined based off them.
- **Linguistic Variables** can be interpreted quantitatively using a corresponding membership function, and qualitatively using an expression in linguistic terms.