Advanced Algorithms, Fall 2014

Prof. Bernard Moret

Homework Assignment #3

due Sunday night, Oct. 12

There are five questions of equal value—but perhaps not equal difficulty.
Write your solutions in LaTeX using the template provided on the Moodle and web sites and upload your PDF file through Moodle by 4:00 am Monday morning, Oct. 13.

Question 1.
Prove Corollary 1 (page 2 of the notes for Oct. 6).

Question 2.
Let us return to our randomized mincut algorithm. It should be clear that its probability of picking an edge of the cut (and thus failing to return a mincut) increases with the number of steps taken. This observation motivates a different strategy: instead of taking all $n-2$ steps in this manner, we are going to start with k steps of the regular algorithm, thereby obtaining a graph with $n-k$ vertices; we will then make m copies of this graph and run our randomized algorithm for $n-k-2$ steps independently on each copy, thus obtaining m different cut solutions, of which we retain the smallest. The rationale is that the original algorithm can be more or less trusted for the first k edge contraction steps, but that beyond that point it makes too many mistakes and so we need to generate alternatives; the overall algorithm then should obtain a level of performance not far from the level we would obtain by running the original algorithm m times, yet use fewer contractions, since the first k contractions are done only once.

Find nearly optimal values for k and m such that the total number of edges contracted by the algorithm is $2(n-2)$, the same number of edges contractions performed by running the original algorithm twice, and the probability that the new algorithm finds a mincut is maximized. Compare the two probabilities.

Question 3.
Alice and Bob are famous characters in Computer Science, engaging in any number of fascinating algorithmic endeavors. Here is the latest one from Alice. She is eating out with Bob at a fancy restaurant and brought with her 3 different dice with which to play for who will pay for the meals. Each die is a regular cube and is fair—the probability for each face to be the up face after a throw is $\frac{1}{6}$. However, the values on the faces are different from the usual: die A has values 1, 1, 6, 6, 8, and 8; die B has values 2, 2, 4, 4, 9, and 9; and die C has values 3, 3, 5, 5, 7, and 7. Alice remarks to Bob that the expected value for a single roll is the same for all three dice, namely 5. Alice explains to Bob that he and she will each choose one die, make a throw, and the one who rolls the lower number will pay for the meals. She graciously (!) tells Bob that he gets to choose his die first.

Alice never lies to Bob, but neither does she volunteer all the truth... Verify that, no matter which die Bob picks, Alice can pick one of the remaining two so as to give herself a probability of winning the throw larger than $\frac{1}{2}$.

Question 4.
You start with a bin containing one black ball and one white ball; you also have two boxes of extra balls, one containing $n-1$ black balls and the other containing $n-1$ white balls. At each step, you draw one ball uniformly at random from the bin, pick up a second ball of the same color from the appropriate box, and put the two balls into the bin. You continue until the bin contains n balls. For each $1 \le i \le n-1$, what is the probability that the bin now contains i balls?

Question 5.
Your company has developed a blood test for the Ebola virus and has been given a contract to test all n people living in Liberia. The test is too expensive to run separately on each of the n individuals, so you have come up with the idea of pooling blood: blood will be drawn from k individuals and pooled, and the resulting single sample will be tested. (The test runs just as well on this larger sample as on blood from a single person.) If the test on the pooled sample comes out negative, nothing more needs be done and you have saved your company $k-1$ tests; if the test comes out positive, you will test each of the k individuals in the pool separately, and you will have cost your company one extra test. Assume that each individual has the same probability p of testing positive; further assume, to simplify the calculations, that n is divisible by k.

For a given value of k, what is the expected number of tests? Assuming that you could choose k, how would you go about computing the value that yields the smallest expected number of tests? Prove that, as along as p is not 1, pooling is always better than testing each individual separately.