Compatible Connectivity-Augmentation of Disconnected Graphs

Second Bellairs Workshop on Geometry and Graphs

Abstract

1 Introduction

We consider the following problem, which will be more carefully formalized below. We are given several different planar straight-line drawings of the same graph, $\mathcal{G}$, which is not connected. We wish to make $\mathcal{G}$ connected by adding vertices and edges in such a way that these vertices and edges can also be added to the planar straight-line drawings of $\mathcal{G}$ while preserving the planarity of the drawings. The objective is to do this while minimizing the number of edges and vertices added. As the example in Figure 1 shows, it is not always possible to just add edges to $\mathcal{G}$; sometimes additional vertices are necessary.

The motivation for studying this problem comes the problem of morphing planar graphs, which has many applications [6, 7, 8, 11, 12] including computer animation. Imagine an animator who wishes to animate a scene in which a character's expression goes from neutral, to surprised, to happy. The animator can draw these three faces, but does not want to hand-draw the 30–60 frames required to animate the change of expression. The strokes used to draw the character's features can be converted into paths and these can be merged into components corresponding to the character's eyes, nose,

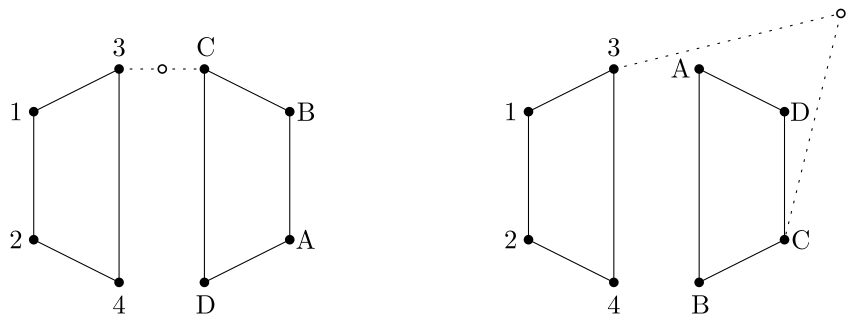


Figure 1: Two drawings of the same graph, $\mathcal{G}$, where making $\mathcal{G}$ connected requires the addition both of edges and vertices. In this case, $\mathcal{G}$ is made connected by adding the hollow vertex and two dashed edges.

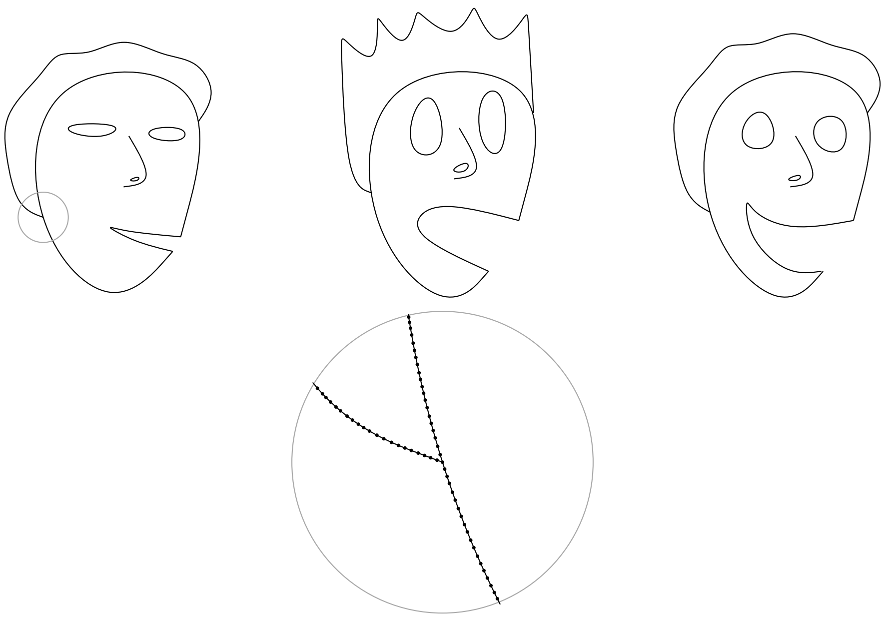


Figure 2: Computer-assisted animation frequently involves morphing between a sequence of embeddings of the same planar graph. Zooming in on a section of the image reveals that the artist's strokes are approximated by polygonal paths

mouth and so on. A correspondence between the same elements in different pictures is also given.[1]

In this setting, animating the face becomes a problem of *morphing* (i.e., continuously deforming) one drawing of a planar graph into another drawing of the same planar graph while maintaining planarity of the drawing throughout the deformation. This morphing problem has been studied since 1944, when Cairns [5] showed such a transformation always exists between any two drawings of the same graph. Since then, a sequence of results has shown that such transformations can be done efficiently, so that the motion can be described concisely [13, 9, 1, 2]. The most recent such result [2] shows that any planar drawing of a n-vertex connected planar graph can be morphed into any compatible drawing[2] of the same planar using a sequence of $O(n)$ *linear morphs*, in which vertices move along linear trajectories at constant speed.

These morphing algorithms require that the input graph, $\mathcal{G}$, be connected. In many applications of morphing (for example in Figure 2) the input graph is not connected. Before these morphing algorithms can be used, $\mathcal{G}$ must be augmented into a connected graph, $\mathcal{H}$, but this augmentation must be compatible with the drawings of $\mathcal{G}$. At the same time, the complexity of the morph produced by a morphing algorithm depends

[1]In many cases, the correspondence is a byproduct of the creation process. For example, in the Figure 2, the second two faces were obtained by copying and then editing the first one.

[2]Two drawings of the same graph are compatible if they have the same ordering of edges around vertices and the same structure of faces; see Section **??**.

on the number of vertices of $\mathcal{H}$. Therefore, we want to find an augmentation with the fewest number of vertices.

This raises the question studied in the current paper: How can we add vertices and edges to $\mathcal{G}$ to obtain a supergraph $\mathcal{H} \supset \mathcal{G}$ that is connected and such that the vertices and edges of $\mathcal{H}$ can also be added to the drawings of $\mathcal{G}$ while maintaining the planarity of each drawing? In this paper, we obtain a tight bound for the extremal question: In the worst-case (over all n vertex planar graphs, $\mathcal{G}$, and over all sets of k planar drawings of $\mathcal{G}$), how many edges have to be added to $\mathcal{G}$ to obtain connectivity while preserving planarity of the drawings?

1.1 Formal Problem Statement and Main Result

Let $\mathcal{G}$ be a planar graph with n vertices and r connected components. Given k geometric planar isomorphic embeddings $G_1, \ldots, G_k$ of $\mathcal{G}$, a *compatible augmentation*, $\mathcal{H}$, of $\mathcal{G}$ is a supergraph of $\mathcal{G}$ such that (1) $\mathcal{H}$ is connected, and (2) there exist geometric planar isomorphic embeddings, $H_1, \ldots, H_k$, of $\mathcal{H}$ such that $H_i \supset G_i$ for every $1 \leq i \leq k$. In this paper, we show that G always has a compatible embedding of size $O(nr^{1-1/k})$ and that this bound is tight; there exists a graph $\mathcal{G}$ with r components and having embeddings $G_1, \ldots, G_k$ for which any compatible augmentation has size $\Omega(nr^{1-1/k})$.

This needs more details. In particular, we need to properly define planar isomorphic embeddings. Yuck!

1.2 Related Work

To the best of our knowledge, there is very little work on compatible connectivity-augmentation of planar graphs, though there is work on isomorphic triangulations of polygons. Refer to Figure 3. In this setting, the graph $\mathcal{G}$ is a cycle and one has two non-crossing drawings, P and Q, of $\mathcal{G}$ The goal is to augment $\mathcal{G}$ (and the two drawings P and Q) so that G becomes a near-triangulation, and P and Q become (geometric) triangulations of the polygons whose boundaries are P and Q. Aronov *et al* [3] showed that this can always be accomplished with the addition of $O(n^2)$ vertices and that $O(n^2)$ vertices are sometimes necessary. Kranakis and Urrutia [10] showed that this result can be made sensitive to the number of reflex vertices of P and Q, so that the number of triangles required is $O(n + pq)$ where p and q are the number of reflex vertices of p and q, respectively.

Babikov *et al* [4] showed that the result of Aronov *et al* can be extended to polygons with holes. This work is the most closely related to ours because it encounters our problem as a subproblem. In their setting, the graph $\mathcal{G}$ is a collection of cycles and the embeddings P and Q are such that one cycle of $\mathcal{G}$ contains all the others in its interior. In the first stage of their algorithm, they build a connected supergraph $\mathcal{H}'$ of $\mathcal{G}$, but their supergraph has size $\Theta(n^2)$ in the worst-case. A byproduct of our main theorem is that this step of their algorithm could be done with a graph $\mathcal{H}'$ having only $O(n^{3/2})$ edges (but completing this graph to a triangulation may still require $\Omega(n^2)$ edges).

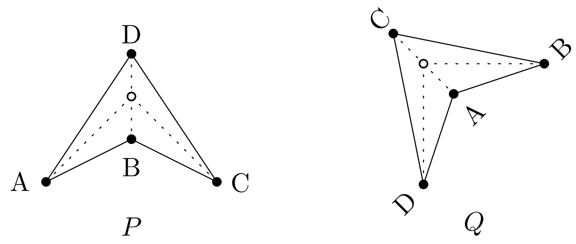


Figure 3: Compatible triangulations of two polygons P and Q.

We should mention, maybe later on, that they have to solve similar routing problems as us; they have two paths along the boundaries of P and Q that visit the components in the wrong order.

2 Upper bounds for trivial components

To provide some intuition, we start by showing the result for the case when $\mathcal{G}$ is a graph containing n vertices and no edges, i.e., $\mathcal{G}$ contains n (trivial) connected components. Before constructing the compatible augmentations, we provide a subroutine that constructs a "short" plane spanning path of a point set. Later, we use this subroutine on each of the geometric drawings of $\mathcal{G}$ to obtain isomorphic compatible augmentations of bounded length.

2.1 Spanning paths of point sets

Let S be a set of n points in the plane such that no two points of S have the same x-coordinate. Given a point $v \in S$, let $rank(v)$ denote the number of points of S that lie to the left of v.

Given an arbitrary order $(v_1, v_2, \ldots, v_n)$ of the points of S, we want to construct a path R that connects them in the given order such that:

$$|R| = O\left(\sum_{i=1}^{n-1} |rank(v_i) - rank(v_{i+1})|\right).$$

Consider a horizontal line such that each point of S lies above it and let π be the closed halfspace supported by this line that contains S. After each round of the algorithm, we maintain the invariant that the boundary of π does not intersect R.

Throughout, we maintain the *escape invariant* which states that for each point $v_j \in S \setminus R$, there is a cone Δ_j with apex u_j such that (1) u_j lies above v_j (or on v_j) and has the same x-coordinate as v_j, (2) Δ_j contains v_j and no other point of S, (3) Δ_j contains the ray shooting from v_j in the direction of the negative y-axis, (4) Δ_j does not intersect R, and (5) Δ_i and Δ_j are disjoint inside π. Before starting the construction, we establish the escape invariant. To do this, for each $0 \leq j \leq n$ let u_j be an arbitrary

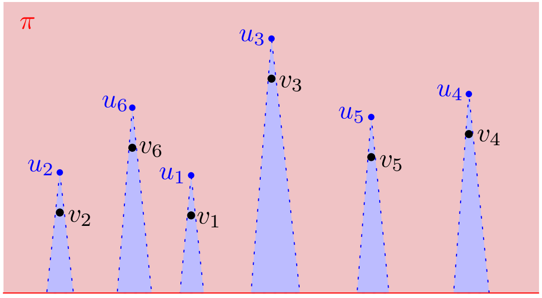


Figure 4:

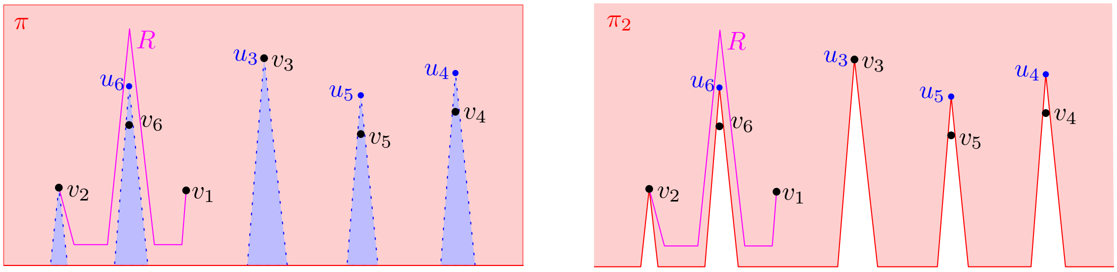


Figure 5:

translation of v_j in the direction of the positive y-axis and let Δ_j be a cone with apex on u_j sufficiently narrow such that these cones doe not intersect inside π; see Figure 4.

To construct R, we add the points of S, one by one, according to the given order while maintaining the escape invariant. Assume that R is a path connecting v_1 with v_j (initially, $j = 1$ and R consists of the single vertex v_1). We extend R by appending a path that connects v_j with v_{j+1}. As a first step, we translate down the cones Δ_j and Δ_{j+1} until their apexes u_j and u_{j+1} coincide with v_j and v_{j+1}, respectively. Moreover, we narrow the cone Δ_j while keeping the ray shooting downwards from v_j contained in Δ_j.

Let π_j be the closure of the set obtained from π by removing every cone Δ_h, where $1 \le h \le n$ and v_h is not an internal vertex of R; see Figure 5. That is, π_j is a halfspace with dents made by the remotion of $n - j + 1$ cones. Therefore, the length of the boundary of π_j is $O(n-j)$. Moreover, the distance between two apexes u_i and u_h along the boundary of π_j is proportional to the number of apexes with x-coordinate between those of u_i and u_h. That is, the length of the path that joins u_i with u_h along π_j is $O(|rank(v_i) - rank(v_h)|)$.

Because the boundary of π does not intersect R and by property (4) of the escape invariant, the boundary of π_j intersects the portion of R constructed so far only at v_j. Moreover, by property (2) of the escape invariant, each point of $S \setminus V(R)$ lies outside of π_j except for v_{j+1} that lies on its boundary. Because both v_j and v_{j+1} lie on the boundary of π_j and since this boundary does not intersect the interior of R, we can connect v_j with v_{j+1} with a path contained in the boundary of π_j. Recall that this path has length $O(|rank(v_j) - rank(v_{j+1})|)$. In this way, we extend R to a planar path that connects v_1 with v_{j+1}.

After connecting v_j with v_{j+1}, for each $v_h \notin R$, either Δ_h is disjoint from R, or it shares some portion of its boundary with R. However, the interior of Δ_h does not intersect R. To preserve the escape invariant, for each $1 \leq h \leq n$ such that $v_h \notin R$, translate the cone Δ_h downwards a distance $\varepsilon > 0$. Because the translated cone is contained in the previous one and since its apex u_h lies above v_h by property (1) of the escape invariant, by choosing ε sufficiently small, we guarantee that the escape invariant is maintained. Similarly, we move ℓ a distance ε in the direction of the negative y-axis. Using this algorithm repeatedly until all points of S are connected, we obtain the following result.

Lemma 2.1. *Given an arbitrary order $(v_1, \dots, v_n)$ of the vertices of S, the previous algorithm computes a plane path R that connects every point of S in the given order such that $|R| = O\left(\sum_{i=1}^{n-1} |rank(v_i) - rank(v_{i+1})|\right)$.*

Proof. Recall that in each iteration, the algorithms computes a path connecting v_j with v_{j+1} that does not cross the portion of the path already constructed. Because this invariant is maintained throughout, the resulting path is planar.

Since the path that connects v_j with v_{j+1} follows the boundary of π_j and since this boundary has length $O(|rank(v_j) - rank(v_{j+1})|)$ between v_j and v_{j+1}, the path that connects v_j with v_{j+1} has length $O(|rank(v_j) - rank(v_{j+1})|)$. Consequently, the total length of R is given by $O\left(\sum_{i=1}^{n-1} |rank(v_i) - rank(v_{i+1})|\right)$. □

2.2 Compatible drawings of point sets

Recall that in this section, $\mathcal{G}$ is a graph with n trivial components. Let $G_1, \dots, G_k$ be k isomorphic drawings of $\mathcal{G}$, i.e., G_i is a set of n points in the plane. Assume without loss of generality that no two points of G_i share the same x-coordinate. Otherwise, rotate the coordinate system slightly. Given a vertex v of $\mathcal{G}$, let $rank_{G_i}(v)$ denote the number of points of G_i to the left of v.

For each v of $\mathcal{G}$, let $x_v = (rank_{G_1}(v), rank_{G_2}(v), \dots, rank_{G_k}(v))$ be a point in the integer grid of side-length n contained in $\mathbb{R}^k$. Let $X = \{x_v : v \in V(\mathcal{G})\}$ and let P be the shortest hamiltonian path of X, i.e., the shortest path that visits every point of X. It is known that the length of P is $O(n^{2-1/k})$ [].

Note that the order of the points of P induces an order on the vertices of $\mathcal{G}$ and hence, an order on the points of each G_i.

Theorem 2.2. *For each $1 \leq i \leq n$, we can construct a path R_i of length $O(n^{2-1/k})$ that connects every point of G_i such that $G_i \cup R_i$ is plane. Moreover, for each $1 \leq i < j \leq n$, $G_i \cup R_i$ is isomorphic to $G_j \cup R_j$.*

Proof. For each G_i, we use Lemma 2.1 to construct a plane path R_i that connects the points of G_i in the order induced by P. Assume that $(v_1, \dots, v_n)$ is the order of the points of G_i induced by P. Therefore, $|R_i| = O\left(\sum_{j=1}^{n-1} |rank_{G_i}(v_j) - rank_{G_i}(v_{j+1})|\right)$. Note that the if d_j denotes the distance between x_{v_j} and $x_{v_{j+1}}$ along P, then $|rank_{G_i}(v_j) - rank_{G_i}(v_{j+1})| \leq d_j$. Therefore, the path contained in R_i that connects v_j with v_{j+1} has length $O(d_j)$. Because $\sum_{j=1}^{n} d_j = |P| = O(n^{2-1/k})$, the total length R_i is $O(n^{2-1/k})$.

Since the vertices of each G_i are connected in the same order, $G_i \cup R_i$ is isomorphic to $G_j \cup R_j$ for each $1 \leq i < j \leq n$. □

3 The general problem

In this section, we extend the result presented in Section 2 to graphs with non-trivial components. As in Section 2, we provide a subroutine that constructs a path connecting all components of a plane graph in any given order.

3.1 Preliminaries

Let C be a connected geometric plane graph. Let $v_0, v_1, \ldots, v_k, v_0$ be the sequence of vertices of C visited by a counterclockwise Eulerian tour along the boundary of C. Note that v_i may be equal to v_j for some $i \neq j$. A vertex of v_i in this sequence is called a *corner* of C. In this paper, we consider the boundary of C, denoted by ∂C, to be the boundary of the weakly-simple polygon $(v_0, \ldots, v_k, v_0)$ whose vertex set is the set of corners of C.

Let $\varepsilon > 0$. For each cornet v_i of ∂C, let ℓ_i be the line passing through v_i that bisects the angle between the edges $v_{i-1}v_i$ and v_iv_{i+1}. Let z_i be the point at distance ε from v_i along ℓ_i such that $v_{i-1}v_iz_i$ either makes a right turn or defines three collinear points such that $v_i \in [v_{i-1}, z_i]$. We call z_i the *ε-copy* of v_i. Let $\partial_\varepsilon C$ be the polygon defined by the sequence $(z_0, z_1, \ldots, z_k, z_0)$, i.e., $\partial_\varepsilon C$ is isomorphic to ∂C. We call $\partial_\varepsilon C$ the *ε-fattening* of C. An ε-fattening $\partial_\varepsilon C$ is *simple* if $\partial_\varepsilon C$ is a simple polygon that contains C. Note that $\partial_\varepsilon C$ is simple, provided that ε is sufficiently small. In this paper, we consider only simple ε-fattenings; see Figure 6. Note that the (graph) distance between two corners of ∂C along the boundary of C is the same as the distance between their ε-copies along $\partial_\varepsilon C$.

3.2 Connected augmentations

Let G be a geometric plane graph with r connected components such that each component is adjacent to the outer face. Consider the visibility graph of G where two vertices are visible if the open segment joining them does not intersect G. Let T_G be a smallest set of edges of the visibility graph of G that need to be added to G to make it connected. Because there are always two components visible from each other, we can connect them and repeat recursively with one less component. Therefore, $r-1$ edges are always sufficient to connect all the components of G. Moreover, because G has r connected components, T_G consists of exactly $r-1$ edges. Let $G^* = G \cup T_G$. Note that ∂G^* consists of at most $n+r-1$ edges. We say that G^* is a *connected augmentation* of G; see Figure 6.

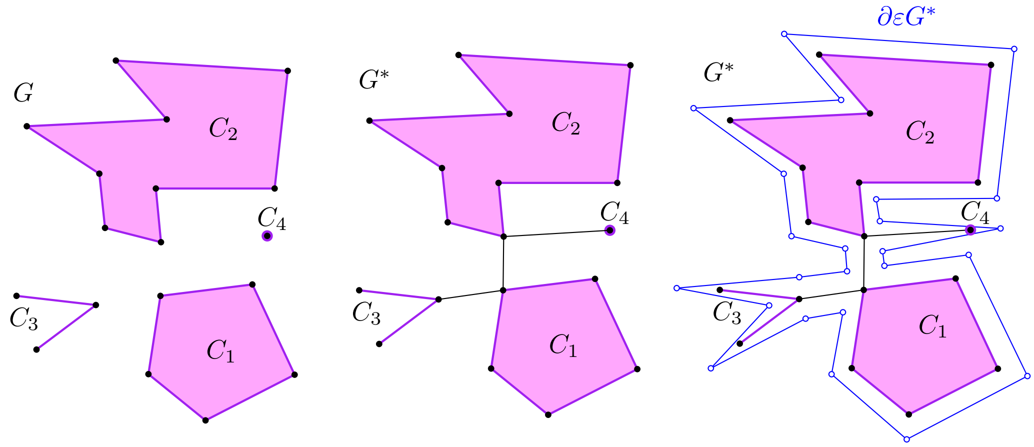


Figure 6:

Let $C_1, \ldots, C_r$ be the components of G. Recall that we consider ∂C_i to be the boundary of a weakly-simple polygon. Therefore, even though a vertex of C_i can appear multiple times along ∂C_i, we consider them as different corners of ∂C_i. For each $1 \leq i \leq r$, let $a_i \in C_i$ be an arbitrary corner of ∂C_i (note that a_i is adjacent to the outer face). We call a_i the *attachment corner* of C_i.

Let φ be the path obtained by splitting ∂G^* at the corner a_1. That is, φ is a path with both endpoints equal to a_1. Moreover, we assume that φ is oriented in counterclockwise order along ∂G^*. Note that φ visits every corner of ∂G^* exactly once except for a_1.

Given two corners u and v in ∂G^*, let $\varphi(u, v)$ denote the shortest path contained in φ that connects u with v. Let $A(u, v)$ be the set of attachment corners of G visited by $\varphi(u, v)$. Let $\sigma_G(u, v) = |\varphi(u, v)| + \sum_{a_i \in A(u,v)} |C_i|$ be the *cost* of going from u to v.

Lemma 3.1. *If a is an attachment corner of G, then $\sigma_G(a_1, a) \leq 5n$. Moreover, if b is another attachment corner of G, then $\sigma_G(a, b) = |\sigma_G(a_1, a) - \sigma_G(a_1, b)|$.*

Proof. Because $|\partial G^*| \leq n + r - 1$ and from the fact that $|\varphi(a_1, a)| \leq |\varphi| = |\partial G^*|$, we know that $|\varphi(a_1, a)| \leq n+r-1$. Therefore, since $\sum_{i=1}^{r} |\partial C_i| \leq |E(G)| \leq 3n$, we conclude that $\sigma_G(a_1, a) = |\varphi(a_1, a)| + \sum_{a_i \in A(a_1,a)} |C_i| \leq 4n + r - 1 \leq 5n$.

Assume without loss of generality that a is visited before b by φ. Because each attachment corner of G visited by $\varphi(a_1, a)$ is also visited by $\varphi(a_1, b)$ and from the fact that $|\varphi(a_1, b)| - |\varphi(a_1, a)| = |\varphi(a, b)|$, we get that $\sigma_G(a_1, b) - \sigma_G(a_1, a) = |\sigma_G(a_1, a) - \sigma_G(a_1, b)| = |\varphi(a, b)| + \sum_{a_i \in A(a,b)} |C_i| = \sigma_G(a, b)$. □

3.3 Spanning paths for connected augmentations

Let $a_1, \ldots, a_r$ be an arbitrary order of the attachment corners of G (we can get the incremental indexing by relabeling the components). Given a path $R = (\rho_1, \rho_2, \ldots, \rho_t)$ that passes through all attachment corners of G, we say that C_i lies to the right of R if (1) a_i is the only vertex of C_i that belongs to $V(R)$, and (2) if $a_i = \rho_j$ for some $1 \leq j \leq t$,

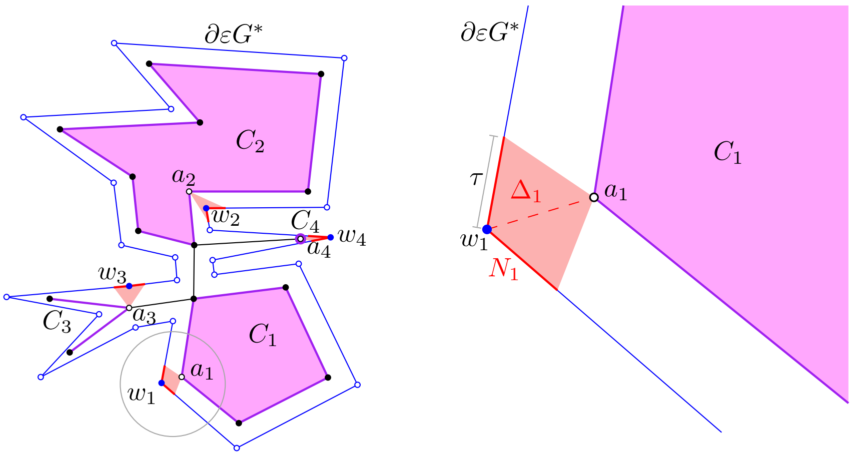


Figure 7:

then ρ_{j-1} and ρ_{j+1} appear as consecutive vertices when sorting the neighbors of a_i in clockwise order around a_i.

We want to construct a path R (possibly containing Steiner vertices) that connects the attachment corners of G in the given order, i.e., if $i < j$, then a_i is visited before a_j by R. We want to construct R in such a way that each component C_i of G lies to the right of R. Moreover, we want that $|R| = O(\sum_{j=1}^{n-1} \sigma_G(a_j, a_{j+1}))$. We construct such a path incrementally, starting with the trivial path that contains only a_1.

Recall that for any given $\varepsilon > 0$, $\partial_\varepsilon G^*$ denotes the ε-fattening of G^* (see Section 3.1). Let $\mu > 0$ be a small constant to be specified later. Initially, let $\varepsilon = 2\mu$ and let $\delta = \mu/2$. Let $\lambda < \mu/2^{r+1}$ be a constant sufficiently small so that $\partial_\lambda C_i \cap \partial_\lambda C_j = \emptyset$ for each $1 \le i < j \le r$. Throughout, λ remains constant while ε and δ are redefined on each round. However, as an invariant we maintain $\lambda < \delta < \varepsilon$.

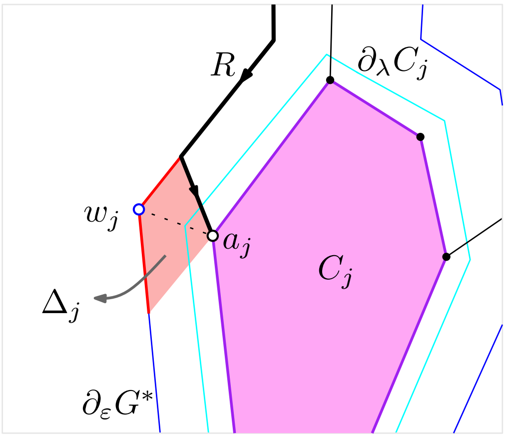
For each $1 \le i \le r$, let w_i be ε-copy of a_i. Split $\partial_\varepsilon G^*$ at w_1, i.e., $\partial_\varepsilon G^*$ is a path with both endpoints equal to w_1. By choosing ε sufficiently small, we guarantee that $\partial_\varepsilon G^*$ is simple, i.e., $\partial_\varepsilon G^*$ has the same combinatorial structure than φ.

We say that two points in the plane are *R-visible* if the open segment joining them does not intersect R. Let $\tau > 0$. For each $1 \le i \le r$ such that a_i is not an interior point of R, consider the set of points $N_i \subset \partial_\varepsilon G^*$ that are at distance at most τ from w_i. Let $\Delta_i = \textsc{ch}(N_i \cup \{a_i\})$, i.e., Δ_i is a "cone" with apex at a_i; see Figure 7.

Throughout, we maintain also the *escape invariant* which states that (1) R intersects neither $\partial_\varepsilon G^*$ nor its unbounded face, (2) for each $a_i \notin V(R)$, R intersects neither the simple polygon bounded by $\partial_\delta C_i$ nor the cone Δ_i, and (3) for each $1 \le i < j \le r$, $\Delta_i \cap \Delta_j = \emptyset$.

In particular, the escape invariant implies that every point in N_i is R-visible from a_i (including w_i). Note that the escape invariant holds at the beginning, provided that the initial choice of τ is sufficiently small.

Assume that we have constructed a path R that connects a_1 with a_j for some $1 \le$



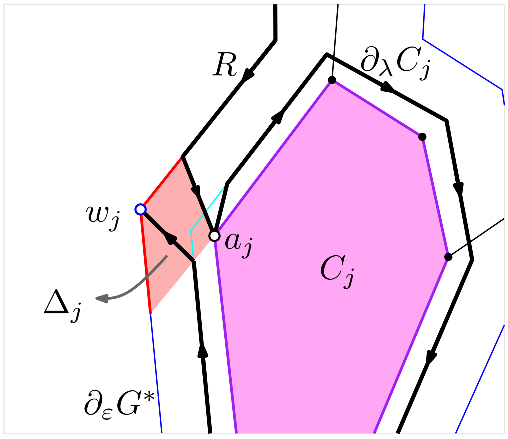


Figure 8:

$j \leq r$ (initially $j = 1$). Moreover, assume that the escape invariant holds. To extend R, we create a new path that connects a_j with a_{j+1} without crossing R while maintaining the invariant. Recall that we consider $\partial_\varepsilon G_i^*$ to be a path with both endpoints on w_1.

If $j > 1$, then we need to be careful in our way out of a_j as we want R to leave C_j to its right. If the path R together with the edge $a_j w_j$ leaves C_j to its right, then walk from a_j to w_j in straight line. Because the escape invariant holds, we know that w_j is R-visible form a_i and hence, this edge does not cross R. If R together with $a_j w_j$ does not leave C_j to its right, then walk from a_j to $\partial_\lambda C_j$ instead and traverse $\partial_\lambda C_j$ in clockwise order without crossing R before moving to w_j on $\partial_\varepsilon G^*$. In this way, we guarantee that C_j lies to the right of the constructed path; see Figure 8 for an illustration. Because $\lambda < \delta < \varepsilon$ and since $a_i \in V(R)$, the escape invariant is preserved.

After reaching w_j, we follow $\partial_\varepsilon G^*$ along the shortest walk that connects w_j with w_{j+1}. However, whenever we reach an endpoint of N_i for some $1 \leq i \leq r$ such that $a_i \notin V(R)$, we take a detour to the other endpoint of N_i while avoiding its interior. In this way, the points in the interior of N_i remain R-visible from a_i (including w_i). Formally, we walk from the reached endpoint of N_i to $\partial_\delta C_i \setminus \Delta_i$ along the boundary of Δ_i. Then, we traverse the path $\partial_\delta C_i \setminus \Delta_i$ before moving to the other endpoint of N_i from the endpoint of $\partial_\delta C_i \setminus \Delta_i$. In this way, we avoid crossing the cone Δ_i; see Figure 9 for an illustration. Note that R does not intersect the interior of the simple polygon bounded by $\partial_\delta C_i$ nor the interior of Δ_i. Moreover, R never goes out of the simple polygon bounded by $\partial_\varepsilon G^*$.

Once we go around C_i, we are back on $\partial_\varepsilon G^*$ on the other endpoint of N_i. In this way, we continue going towards w_{j+1} along $\partial_\varepsilon G^*$ until reaching an endpoint of N_{j+1}. Once we reach an endpoint of N_{j+1}, we move directly from this endpoint to a_{j+1}.

Because $\partial_\varepsilon G^*$ is isomorphic to φ, the constructed a_j-a_{j+1}-path has length at most $|\varphi(a_j, a_{j+1})|$ plus the length of the boundaries of the components that we walked around. Because each avoided component has its attachment corner on the path $\varphi(a_j, a_{j+1})$, the

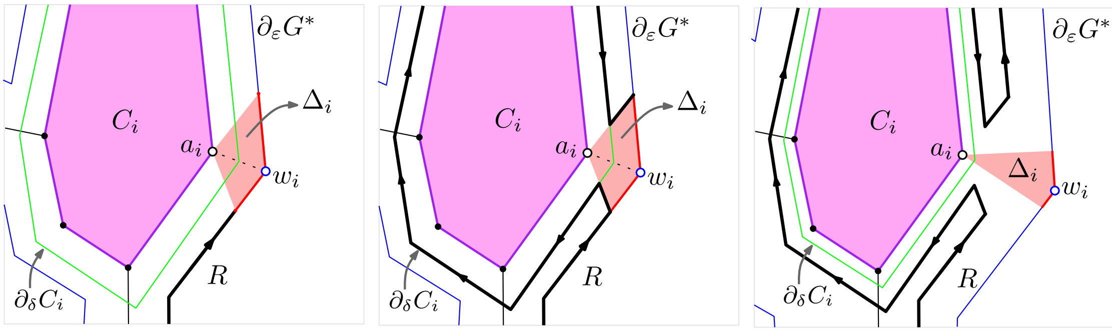


Figure 9:

length of the constructed a_j-a_{j+1}-path is

$$O(|\varphi(a_j, a_{j+1})| + \sum_{a_i \in A(a_j, a_{j+1})} |C_i|) = O(\sigma_G(a_j, a_{j+1})) \ .$$

After reaching a_{j+1}, we increase ε by a factor of two. Similarly, we decrease de value of δ by a factor of two. That is, after reaching a_{j+1}, $\varepsilon = \mu 2^{j+1}$ while $\delta = \mu/2^{j+1}$ and hence, we guarantee that $\lambda < \delta < \varepsilon$. We can guarantee that $\partial_\varepsilon G^*$ remains simple, provided that the original choice of μ is sufficiently small. Finally, for each $1 \leq i \leq n$, we reduce the size of the neighborhood N_i by reducing τ by a factor of two and updating the cone Δ_i accordingly, i.e., we narrow the cone Δ_i; see Figure 9 (c).

Recall that for each $a_i \notin R$, R intersected neither the interior of Δ_i nor the interior of the polygon bounded by $\partial_\delta C_i$. Moreover, R never went out of $\partial_\varepsilon G^*$. Therefore, after increasing (*resp.* reducing) ε (*resp.* δ), we preserve the correctness invariant should there be a subsequent round of the algorithm. We repeat this algorithm until all attachment corners of G are visited by R.

Lemma 3.2. *Given an arbitrary order $a_1, \ldots, a_r$ of the attachment corners of G, the previous algorithm computes a path R connecting all attachment corners of G in the given order such that $R \cup G$ is plane, $|R| = O(\sum_{j=1}^{n-1} \sigma_G(a_j, a_{j+1}))$, and for each $1 \leq i \leq r$, C_i lies to the right of R when oriented from a_1 to a_r.*

Proof. By construction, the attachment corners are visited by R in the given order. For each component C_i, a_i is the only vertex of C_i visited by R. Moreover, the construction guarantees that C_i lies to the right of R when oriented from a_1 to a_r.

To prove that R is plane, recall that in each round we extend R by constructing a path γ_j that connects a_j with a_{j+1}. We claim that at this point, no edge of γ_j crosses the portion of R constructed so far. Indeed, because the value of ε (*resp.* δ) increases (*resp.* decreases) by a factor of two in each round, the edges of γ_j that lie on the boundaries of some $\partial_\delta C_i$ or on $\partial_\varepsilon G^*$ cannot cross R by the escape invariant. Moreover, this invariant states that for each $a_i \notin R$, R does not intersect Δ_i. Because each cone Δ_i is narrowed in each round, the edges of γ_j that lie on the boundary of this cone cannot cross R.

Finally, because $\lambda < \delta$, the edges of γ_j that lie on $\partial_\lambda C_j$ do not cross R. Therefore, we conclude that by concatenating γ_j and R, we obtain a plane path.

Because the length of γ_j is $O(\sigma_G(a_j, a_{j+1}))$, the total length of R after connecting every attachment corner is given by $|R| = O(\sum_{j=1}^{n-1} \sigma_G(a_j, a_{j+1}))$. $\square$

3.4 Compatible drawings of planar graphs

Let $\mathcal{G}$ be a planar graph with n vertices and r connected components. Let $G_1, \ldots, G_k$ be k plane isomorphic drawings of $\mathcal{G}$. We show how to construct a compatible augmentation of $\mathcal{G}$ of size $O(nr^{1-1/k})$.

Let $C_1, \ldots, C_r$ be the connected components of $\mathcal{G}$. Because all the plane drawings of $\mathcal{G}$ that we consider are isomorphic, the boundaries of corresponding components in different drawings are also isomorphic. Thus, for each $1 \leq j \leq r$, we can choose an attachment corner a_j in ∂C_j such that a_j is adjacent to the outer face. Note that this corner has isomorphic copies in each of the embeddings of G.

For each $1 \leq j \leq r$, let G_j^* be a connected augmentation of G_j (see Section 3.2 for the construction of such a graph). For each $1 \leq i \leq k$ and for each $1 \leq j \leq r$, let $rank_i(j) = \sigma_{G_i}(a_1, a_j)$. For each $1 \leq j \leq r$, we define a point $x_j \in \mathbb{R}^k$ that correspond to the component C_j such that $x_j = (rank_1(a_j), rank_2(a_j), \ldots, rank_k(a_j))$. Let $X = \{x_1, \ldots, x_r\}$ be a set of points in $\mathbb{R}^k$. Note that Lemma 3.1 implies that X is contained in an integer grid of side length $5n$ of dimension k.

Let P be the shortest hamiltonian path of X, i.e., the shortest path that visits every point of X. Because X is contained in the k-dimensional integer grid of side-length $5n$ and from the fact that $|X| = r$, we known that the length of P is $O(nr^{1-1/k})$ [].

Note that the order of the points of P induces an order on the components of $\mathcal{G}$ and hence, an order on the attachment corners of each G_i.

Theorem 3.3. *For each $1 \leq i \leq k$, we can construct a path R_i of length $O(nr^{1-1/k})$ that connects every component of G_i such that $G_i \cup R_i$ is plane. Moreover, for each $1 \leq i < j \leq k$, $G_i \cup R_i$ is isomorphic to $G_j \cup R_j$.*

Proof. For each G_i, we use Lemma 3.2 to construct a plane path R_i that connects the attachment corners of G_i in the order induced by P. Assume that $(a_1, \ldots, a_r)$ is the order of the attachment corners of G_i induced by P. Lemma 3.2 implies that $|R_i| = O(\sum_{j=1}^{n-1} \sigma_{G_i}(a_j, a_{j+1}))$.

Because $\sigma_{G_i}(a_j, a_{j+1}) = |\sigma_{G_i}(a_1, a_{j+1}) - \sigma_{G_i}(a_1, a_j)|$ by Lemma 3.1 and from the fact that $rank_i(j) = \sigma_{G_i}(a_1, a_j)$, we get that $\sigma_{G_i}(a_j, a_{j+1}) = |rank_i(j+1) - rank_i(j)|$. Therefore, $|R_i| = O\left(\sum_{j=1}^{n-1} |rank_i(j+1) - rank_i(j)|\right)$.

Let d_j denote the distance between x_j and x_{j+1} in P. Because $|rank_i(j+1) - rank_i(j)|$ represents the difference in the i-th coordinates of x_j and x_{j+1}, by the triangle inequality we infer that $|rank_i(j+1) - rank_i(j)| \leq d_j$. Therefore, the path contained in R_i has length $O\left(\sum_{j=1}^{n-1} |rank_i(j+1) - rank_i(j)|\right) = O(\sum_{j=1}^{n-1} d_j)$. Because $\sum_{j=1}^{n-1} d_j = |P| = O(nr^{1-1/k})$, the length R_i is $O(nr^{1-1/k})$.

Since each R_i visits each component only at its attachment corner and from the fact that R_i leaves every component to the right when oriented from a_1 to a_r, we conclude that $G_i \cup R_i$ is isomorphic to $G_j \cup R_j$ for each $1 \le i < j \le k$. □

4 Lower Bounds

Our lower bounds are based on the following lemma. In words, this lemma says that we can find k permutations of $\{1,\ldots,r\}$ such that, for half the indices $i \in \{1,\ldots,r\}$, and every $j \in \{1,\ldots,r\} \setminus \{i\}$, there is a permutation in which i and j are at distance $\Omega(r^{1-1/k})$.

Lemma 4.1. *Let $t = (1/2)^{1+1/k} \cdot (r-1)^{1-1/k}$. There exists permutations $\pi^{(1)},\ldots,\pi^{(k)}$ of $\{1,\ldots,r\}$ such that, for at least half the values of $i \in \{1,\ldots,r\}$ and for every $j \in \{1,\ldots,r\} \setminus \{i\}$,*

$$\max\left\{\left|\pi_i^{(s)} - \pi_j^{(s)}\right| : s \in \{1,\ldots,k\}\right\} \ge t \ . \tag{1}$$

Proof. This proof is an application of the probabilistic method. Select each of $\pi^{(1)},\ldots,\pi^{(k)}$ independently and uniformly from among all $r!$ permutations of $\{1,\ldots,r\}$. Fix a particular index i and a particular index j. For a particular $s \in \{1,\ldots,k\}$, the probability that $|\pi_i^{(s)} - \pi_j^{(s)}| \le t$ is at most $2t/(r-1)$ since the set $\{\hat{\jmath} \in \{1,\ldots,r\} \colon |\pi_i^{(s)} - \pi_{\hat{\jmath}}^{(s)}| \le t\}$ is a random subset of $2t$ elements drawn without replacement from $\{1,\ldots,r\} \setminus \{i\}$.

Therefore, since $\pi^{(1)},\ldots,\pi^{(k)}$ are chosen independently,

$$\Pr\left\{\max\left\{\left|\pi_i^{(s)} - \pi_j^{(s)}\right| : s \in \{1,\ldots,k\}\right\} \le t\right\} \le (2t/(r-1))^k = \frac{1}{2(r-1)} \ .$$

In particular, the expected number of such $j \in \{1,\ldots,r\} \setminus \{i\}$ is at most $1/2$ so, by Markov's Inequality, the probability that there exists at least one such j is at most $1/2$. Thus, with probability at least $1/2$, the index i satisfies (1) and therefore the expected number of indices $i \in \{1,\ldots,r\}$ that satisfy (1) is $r/2$. We conclude that there must exist some permutations $\pi^{(1)},\ldots,\pi^{(k)}$ that satisfy (1) for at least half the indices $i \in \{1,\ldots,r\}$. □

Using Lemma 4.1, we can prove a lower bound that matches the upper bound obtained in our general construction.

Theorem 4.2. *For every integer n and every $r \in \{1,\ldots,\lfloor n/4 \rfloor\}$, there exists a graph $\mathcal{G}$ having n vertices, r connected components, and k isomorphic drawings $G_1,\ldots,G_k$ such that any compatible augmentation of G has size $\Omega(nr^{1-1/k})$.*

Proof. Since the lemma only claims an asymptotic result, we may assume without loss of generality that r is even and that $2r$ divides n.

The graph $\mathcal{G}$ consists of r disjoint paths, $\mathcal{C}_1,\ldots,\mathcal{C}_r$, each of length n/r. Each of the drawings G_1,...,G_k embeds the vertices of $\mathcal{G}$ on the same point and segment set. The point set consists of the vertices of r nested regular n/r-gons, $P_1,\ldots,P_r$, each

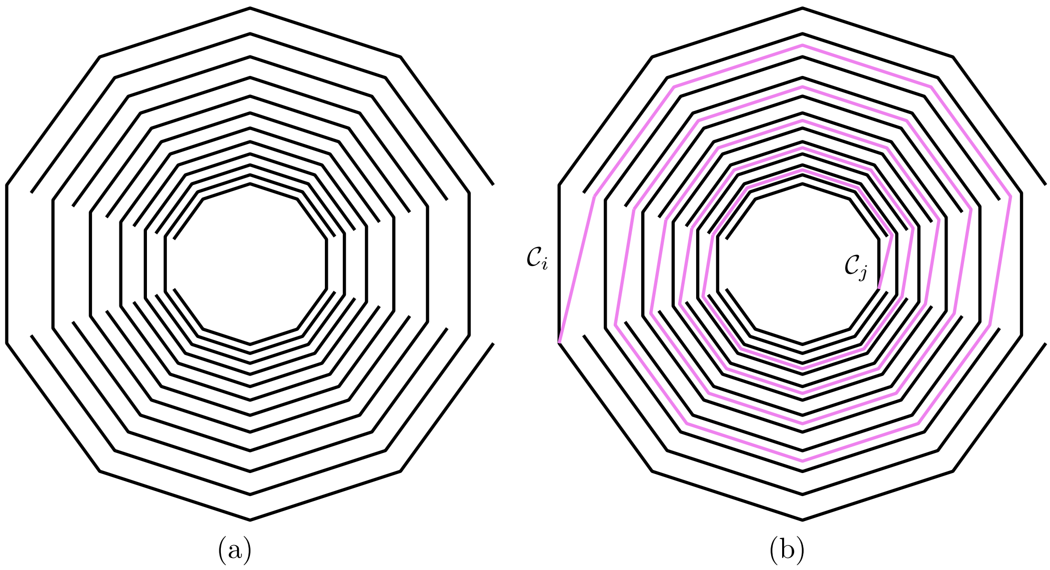


Figure 10: In the lower bound of Theorem 4.2, (a) all drawings use the same set of points for vertices and segments for edges and (b) the drawing of a path that joins $\mathcal{C}_i$ to $\mathcal{C}_j$ must travel around all the paths embedded between the drawing of $\mathcal{C}_i$ and the drawing of $\mathcal{C}_j$.

centered at the origin and having nearly the same size. Refer to Figure 10.a. More precisely, $P_1 \subset P_2 \subset \cdots \subset P_r$ and the sizes are chosen so that any segment joining two non-consecutive vertices of P_i intersects the interior of P_{i-1}.

The drawings $G_1, \ldots, G_k$ are obtained from the permutations $\pi^{(1)}, \ldots, \pi^{(k)}$ given by Lemma 4.1. In the drawing G_x, the path $\mathcal{C}_i$ is embedded on the vertices of $P_{\pi_i^{(x)}}$. If $y = \pi_i^{(x)}$ is even, the drawing uses all the edges of P_y except the left-most edge. If y is odd, the drawing uses all the edges of P_y except the right-most edge.

Now, without loss of generality, consider some edge-minimal compatible augmentation $\mathcal{H}$ of $\mathcal{G}$. For each component $\mathcal{C}_i$ of G, let T_i be any path in $\mathcal{H}$ that has one endpoint on $\mathcal{C}_i$, one endpoint on some other component $\mathcal{C}_j$, $j \neq i$, and no vertices of $\mathcal{G}$ in its interior.

Now, for each of the $r/2$ indices $i \in \{1, \ldots, r\}$ that satisfy (1), the path T_i joins a vertex of $P_{\pi_i^{(s)}}$ to a vertex of $P_{\pi_j^{(s)}}$, $j \neq i$, and $|\pi_i^{(s)} - \pi_i^{(s)}| \geq t$. This path must have length $\Omega(tn/r)$ since it has to "go around" the paths between $P_{\pi_i^{(s)}}$ and $P_{\pi_j^{(s)}}$; see Figure 10.b.

Thus far, we have shown that for at least $r/2$ values of $i \in \{1, \ldots, r\}$, the component C_i is the endpoint of a path, T_i, of length at least $\Omega(tn/r) = \Omega(nr^{-1/k})$. It is tempting to claim the result at this point, since $(r/2) \cdot \Omega(nr^{-1/k}) = \Omega(nr^{1-1/k})$. Unfortunately, there is a little more work that needs to be done, since two such paths T_i and T_j may not be disjoint, so summing their lengths double-counts the contribution of the shared portion.

To finish up we note that, since the augmentation $\mathcal{H}$ is minimal, it is a tree; $\mathcal{G}$ contains no cycles, so any cycle in $\mathcal{H}$ contains an edge not in $\mathcal{G}$ that could be removed. Now, observe that if we traverse the outer face of (any planar drawing of) $\mathcal{H}$ then we obtain a non-simple path, P, that traverses each edge of $\mathcal{H}$ exactly twice. If we consider the set of maximal subpaths of P with no vertex of $\mathcal{G}$ in their interior, we obtain a set of r paths, $Q_1, \ldots, Q_r$ and, for every component $\mathcal{C}_i$ of $\mathcal{G}$, there is a vertex of $\mathcal{C}_i$ that is an endpoint of at least one such path. Therefore, from the preceding discussion, the total length of $Q_1 \ldots, Q_r$ is $\Omega(nr^{1-1/k})$. But since each edge of $\mathcal{H}$ appears at most twice in these subpaths, we conclude that $\mathcal{H}$ has $\Omega(nr^{1-1/k})$ edges. Since $\mathcal{H}$ is a tree, it has $\Omega(nr^{1-1/k})$ vertices. □

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