

# A linear-time algorithm for the geodesic center of a simple polygon

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## Abstract

Let  $P$  be a simple polygon with  $n$  vertices. Given two points in  $P$ , the geodesic distance between them is the length of the shortest path that connects them among all paths contained in  $P$ . The geodesic center of  $P$  is the unique point in  $P$  that minimizes the largest geodesic distance to all other points of  $P$ . In 1989, Pollack, Sharir and Rote [Disc. & Comput. Geom. 89] showed an  $O(n \log n)$ -time algorithm that computes the geodesic center of  $P$ . Since then, a longstanding question has been whether this running time can be improved (explicitly posed by Mitchell [Handbook of Computational Geometry, 2000]). In this paper we affirmatively answer this question and present a linear time algorithm to solve this problem.

## 1 Introduction

Let  $P$  be a simple polygon with  $n$  vertices. Given two points  $x, y$  in  $P$ , the *geodesic path*  $\pi(x, y)$  is the shortest path contained in  $P$  connecting  $x$  with  $y$ . If the straight-line segment connecting  $x$  with  $y$  is contained in  $P$ , then  $\pi(x, y)$  is a straight-line segment. Otherwise,  $\pi(x, y)$  is a polygonal chain whose vertices (other than its endpoints) are reflex vertices of  $P$ . We refer the reader to [22] for more information on geodesic paths.

The *geodesic distance* between  $x$  and  $y$ , denoted by  $|\pi(x, y)|$ , is the sum of the Euclidean lengths of each segment in  $\pi(x, y)$ . Throughout this paper, when referring to the distance between two points in  $P$ , we mean the geodesic distance between them. Given a point  $x \in P$ , a (geodesic) *farthest neighbor* of  $x$ , is a point  $f_P(x)$  (or simply  $f(x)$ ) of  $P$  whose geodesic distance to  $x$  is maximized. To ease the description, we assume that each vertex of  $P$  has a unique farthest neighbor. This *general position* condition was also assumed by Aronov et al. [2] and can be obtained by applying a slight perturbation to the positions of the vertices [10].

Let  $F_P(x)$  be the function that maps each  $x \in P$  to the distance to a farthest neighbor of  $x$  (i.e.,  $F_P(x) = |\pi(x, f(x))|$ ). A point  $c_P \in P$  that minimizes  $F_P(x)$  is called the *geodesic center* of  $P$ . Similarly, a point  $s \in P$  that maximizes  $F_P(x)$  (together with its farthest neighbor) is called a *geodesic diametral pair* and their distance is known as the *geodesic*

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42 *diameter*. Asano and Toussaint [3] showed that the geodesic center is unique (whereas it is  
 43 easy to see that several geodesic diametrical pairs may exist).

44 In this paper, we show how to compute the geodesic center of  $P$  in  $O(n)$  time.

## 45 1.1 Previous Work

46 Since the early 1980s the problem of computing the geodesic center (and its counterpart,  
 47 the geodesic diameter) has received a lot of attention from the computational geometry  
 48 community. Chazelle [7] gave the first algorithm for computing the geodesic diameter (which  
 49 runs in  $O(n^2)$  time using linear space). Afterwards, Suri [27] reduced it to  $O(n \log n)$ -time  
 50 without increasing the space constraints. Finally, Hershberger and Suri [15] presented a fast  
 51 matrix search technique, one application of which is a linear-time algorithm for computing  
 52 the diameter.

53 The first algorithm for computing the geodesic center was given by Asano and Toussaint [3],  
 54 and runs in  $O(n^4 \log n)$ -time. In 1989, Pollack, Sharir, and Rote [25] improved it to  $O(n \log n)$   
 55 time. Since then, it has been an open problem whether the geodesic center can be computed  
 56 in linear time (indeed, this problem was explicitly posed by Pollack et al. [25] and later by  
 57 Mitchell [22, Chapter 27]).

58 Several other variations of these two problems have been considered. Indeed, the same  
 59 problem has been studied under different metrics. For example, the  $L_1$  geodesic distance [6],  
 60 the link distance [26, 16, 9] (where we look for the path with the minimum possible number  
 61 of bends or *links*), or even rectilinear link distance [23, 24] (a variation of the link distance in  
 62 which only isothetic segments are allowed). The diameter and center of a simple polygon for  
 63 both the  $L_1$  and rectilinear link metrics can be computed in linear time (whereas  $O(n \log n)$   
 64 time is needed for the link distance).

65 Another natural extension is the computation of the diameter and center in polygonal  
 66 domains (i.e., polygons with one or more holes). Polynomial time algorithms are known for  
 67 both the diameter [4] and center [5], although the running times are significantly larger (i.e.,  
 68  $O(n^{7.73})$  and  $O(n^{12+\varepsilon})$ , respectively).

## 69 1.2 Outline

70 In order to compute the geodesic center, Pollack *et al.* [25] introduce a linear time *chord-*  
 71 *oracle*. Given a chord  $C$  that splits  $P$  into two sub-polygons, the oracle determines which  
 72 sub-polygon contains  $c_P$ . Combining this operation with an efficient search on a triangulation  
 73 of  $P$ , Pollack *et al.* narrow the search of  $c_P$  within a triangle (and find the center using  
 74 optimization techniques). Their approach however, does not allow them to reduce the  
 75 complexity of the problem in each iteration, and hence it runs in  $\Theta(n \log n)$  time.

76 The general approach of our algorithm described in Section 6 is similar: partition  $P$   
 77 into  $O(1)$  cells, use an oracle to determine which cell contains  $c_P$ , and recurse within the  
 78 cell. Our approach differs however in two important aspects that allows us to speed-up the  
 79 algorithm. First, we do not use the chords of a triangulation of  $P$  to partition the problem  
 80 into cells. We use instead a cutting of a suitable set of chords. Secondly, we compute a set  $\Phi$   
 81 of  $O(n)$  functions, each defined in a triangular domain contained in  $P$ , such that their upper  
 82 envelope,  $\phi(x)$ , coincides with  $F_P(x)$ . Thus, we can “ignore” the polygon  $P$  and focus only  
 83 on finding the minimum of the function  $\phi(x)$ .

84 The search itself uses  $\varepsilon$ -nets and cutting techniques, which guarantee that both the size of  
 85 the cell containing  $c_P$  and the number of functions of  $\Phi$  defined in it decrease by a constant  
 86 fraction (and thus leads to an overall linear time algorithm). This search has however two

stopping conditions, (1) reach a subproblem of constant size, or (2) find a triangle containing  $c_P$ . In the latter case, we show that  $\phi(x)$  is a convex function when restricted to this triangle. Thus, finding its minimum becomes an optimization problem that we solve in Section 7 using cuttings in  $\mathbb{R}^3$ .

The key of this approach lies in the computation of the functions of  $\Phi$  and their triangular domains. Each function  $g(x)$  of  $\Phi$  is defined in a triangular domain  $\Delta$  contained in  $P$  and is associated to a particular vertex  $w$  of  $P$ . Intuitively speaking,  $g(x)$  maps points in  $\Delta$  to their (geodesic) distance to  $w$ . We guarantee that, for each point  $x \in P$ , there is one function  $g$  defined in a triangle containing  $x$ , such that  $g(x) = F_P(x)$ . To compute these triangles and their corresponding functions, we proceed as follows.

In Section 3, we use the matrix search technique introduced by Hershberger and Suri [15] to decompose the boundary of  $P$ , denoted by  $\partial P$ , into connected edge disjoint chains. Each chain is defined by either (1) a consecutive list of vertices that have the same farthest neighbor  $v$  (we say that  $v$  is *marked* if it has such a chain associated to it), or (2) an edge whose endpoints have different farthest neighbors (such edge is called a *transition edge*).

In Section 4, we consider each transition edge  $ab$  of  $\partial P$  independently and compute its *hourglass*. Intuitively, the hourglass of  $ab$ ,  $H_{ab}$ , is the region of  $P$  between two chains, the edge  $ab$  and the chain of  $\partial P$  that contains the farthest neighbors of all points in  $ab$ . Inspired by a result of Suri [27], we show that the sum of the complexities of all hourglasses defined on a transition edge is  $O(n)$ . In addition, we provide a new technique to compute all these hourglasses in linear time.

In Section 5 we show how to compute the functions in  $\Phi$  and their respective triangles. We distinguish two cases: (1) Inside each hourglass  $H_{ab}$  of a transition edge, we use a technique introduced by Aronov et al. [2] that uses the shortest-path trees of  $a$  and  $b$  in  $H_{ab}$  to decompose  $H_{ab}$  into  $O(|H_{ab}|)$  triangles with their respective functions (for more information on shortest-path trees refer to [11]). (2) For each marked vertex  $v$  we compute triangles that encode the distance from  $v$ . Moreover, we guarantee that these triangles cover every point of  $P$  whose farthest neighbor is  $v$ . Overall, we compute the  $O(n)$  functions of  $\Phi$  in linear time.

## 2 Hourglasses and Funnels

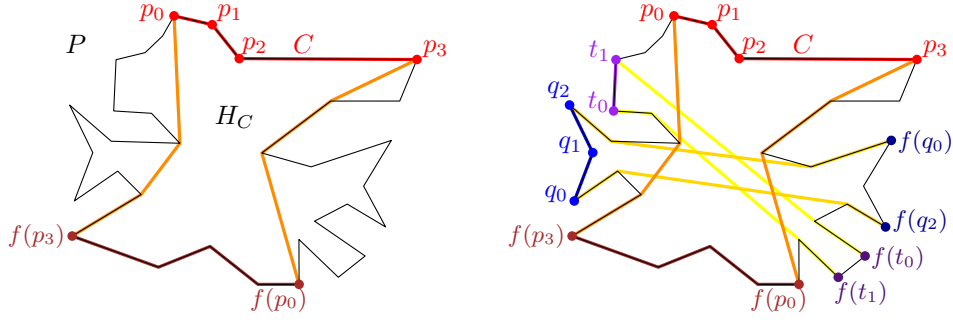
In this section, we introduce the main tools that are going to be used by the algorithm. Some of the results presented in this section have been shown before in different papers.

### 2.1 Hourglasses

Given two points  $x$  and  $y$  on  $\partial P$ , let  $\partial P(x, y)$  be the polygonal chain that starts at  $x$  and follows the boundary of  $P$  clockwise until reaching  $y$ .

For any polygonal chain  $C = \partial P(p_0, p_k)$ , the *hourglass* of  $C$ , denoted by  $H_C$ , is the simple polygon contained in  $P$  bounded by  $C$ ,  $\pi(p_k, f(p_0))$ ,  $\partial P(f(p_0), f(p_k))$  and  $\pi(f(p_k), p_0)$ ; see Figure 1. We call  $C$  and  $\partial P(f(p_0), f(p_k))$  the *top* and *bottom* chains of  $H_C$ , respectively, while  $\pi(p_k, f(p_0))$  and  $\pi(f(p_k), p_0)$  are referred to as the *walls* of  $H_C$ .

We say that the hourglass  $H_C$  is *open* if its walls are vertex-disjoint. We say  $C$  is a *transition chain* if  $f(p_0) \neq f(p_k)$  and neither  $f(p_0)$  nor  $f(p_k)$  are interior vertices of  $C$ . In particular, if an edge  $ab$  of  $\partial P$  is a transition chain, we say that it is a *transition edge* (see Figure 1).



■ **Figure 1** Given two edge disjoint transition chains, their hourglasses are open and the bottom chains of their hourglasses are also edge disjoint. Moreover, these bottom chains appear in the same cyclic order as the top chains along  $\partial P$ .

130 ► **Lemma 1.** [A restatement of Lemma 3.1.3 of [2]] If  $C$  is a transition chain of  $\partial P$ , then  
 131 the hourglass  $H_C$  is an open hourglass.

132 Note that by Lemma 1, the hourglass of each transition chain is open. In the remainder  
 133 of the paper, all the hourglasses considered are defined by a transition chain, i.e., they are  
 134 open and their top and bottom chains are edge disjoint.

135 The following lemma is depicted in Figure 1 and is a direct consequence of the Ordering  
 136 Lemma proved by Aronov et al. [2, Corollary 2.7.4].

137 ► **Lemma 2.** Let  $C_1, C_2, C_3$  be three edge disjoint transition chains of  $\partial P$  that appear in this  
 138 order when traversing clockwise the boundary of  $P$ . Then, the bottom chains of  $H_{C_1}, H_{C_2}$  and  
 139  $H_{C_3}$  are also edge disjoint and appear in this order when traversing clockwise the boundary  
 140 of  $P$ .

141 Let  $\gamma$  be a geodesic path joining two points on the boundary of  $P$ . We say that  $\gamma$  separates  
 142 two points  $x_1$  and  $x_2$  of  $\partial P$  if the points of  $X = \{x_1, x_2\}$  and the endpoints of  $\gamma$  alternate  
 143 along the boundary of  $P$  ( $x_1$  and  $x_2$  could coincide with the endpoints of  $\gamma$  in degenerate  
 144 cases). We say that a geodesic path  $\gamma$  separates an hourglass  $H$  if it separates the points of  
 145 its top chain from those of its bottom chain.

146 ► **Lemma 3.** Let  $C_1, \dots, C_r$  be edge disjoint transition chains of  $\partial P$ . Then, there is a set  
 147 of  $t \leq 10$  geodesic paths  $\gamma_1, \dots, \gamma_t$  with endpoints on  $\partial P$  such that for each  $1 \leq i \leq r$  there  
 148 exists  $1 \leq j \leq t$  such that  $\gamma_j$  separates  $H_{C_i}$ . Moreover, this set can be computed in  $O(n)$   
 149 time.

150 **Proof.** Aronov et al. showed that there exist four vertices  $v_1, \dots, v_4$  of  $P$  and geodesic paths  
 151  $\pi(v_1, v_2), \pi(v_2, v_3), \pi(v_3, v_4)$  such that for any point  $x \in \partial P$ , one of these paths separates  $x$   
 152 from  $f(x)$  [2, Lemma 2.7.6]. Moreover, they show how to compute this set in  $O(n)$  time.

153 Let  $\Gamma = \{\pi(v_i, v_j) : 1 \leq i < j \leq 4\}$  and note that  $v_1, \dots, v_4$  split the boundary of  $P$  into  
 154 at most four connected components. If a chain  $C_i$  is completely contained in one of these  
 155 components, then one path of  $\Gamma$  separates the top and bottom chain of  $H_{C_i}$ . Otherwise,  
 156 some vertex  $v_j$  is an interior vertex of  $C_i$ . However, because the chains  $C_1, \dots, C_r$  are edge  
 157 disjoint, there are at most four chains in this situation. For each chain  $C_i$  containing a vertex  
 158  $v_j$ , we add the geodesic path connecting the endpoints of  $C_i$  to  $\Gamma$ . Therefore,  $\Gamma$  consists of at  
 159 most 10 geodesic paths and each hourglass  $H_{C_i}$  has its top and bottom chain separated by  
 160 some path of  $\Gamma$ . Since only  $O(1)$  additional paths are computed, this can be done in linear  
 161 time. ◀

162 A *chord* of  $P$  is an edge joining two non-adjacent vertices  $a$  and  $b$  of  $P$  such that  $ab \subseteq P$ .  
 163 Therefore, a chord splits  $P$  into two sub-polygons.

164 ► **Lemma 4.** [A restatement of Lemma 3.4.3 of [2]] Let  $C_1, \dots, C_r$  be a set of edge disjoint  
 165 transition chains of  $\partial P$  that appear in this order when traversing clockwise the boundary of  
 166  $P$ . Then each chord of  $P$  appears in  $O(1)$  hourglasses among  $H_{C_1}, \dots, H_{C_r}$ .

167 **Proof.** Note that chords can only appear on walls of hourglasses. Because hourglasses are  
 168 open, any chord must be an edge on exactly one wall of each of these hourglasses. Assume,  
 169 for the sake of contradiction, that there exist two points  $s, t \in P$  whose chord  $st$  is in three  
 170 hourglasses  $H_{C_i}, H_{C_j}$  and  $H_{C_k}$  (for some  $1 \leq i < j < k \leq r$ ) such that  $s$  visited before  $t$   
 171 when going from the top chain to the bottom one along the walls of the three hourglasses.  
 172 Let  $s_i$  and  $t_i$  be the points in the in the top and bottom chains of  $H_{C_i}$ , respectively, such  
 173 that  $\pi(s_i, t_i)$  is the wall of  $H_{C_i}$  that contains  $st$  (analogously, we define  $s_k$  and  $t_k$ )

174 Because  $C_j$  lies in between  $C_i$  and  $C_k$ , Lemma 2 implies that the bottom chain of  $C_j$   
 175 appears between the bottom chains of  $C_i$  and  $C_k$ . Therefore,  $C_j$  lies between  $s_i$  and  $s_k$  and  
 176 the bottom chain of  $H_{C_j}$  lies between  $t_i$  and  $t_k$ . That is, for each  $x \in C_j$  and each  $y$  in the  
 177 bottom chain of  $H_{C_j}$ , the geodesic path  $\pi(x, y)$  is “sandwiched” by the paths  $\pi(s_i, t_i)$  and  
 178  $\pi(s_k, t_k)$ . In particular,  $\pi(x, y)$  contains  $st$  for each pair of points in the top and bottom  
 179 chain of  $H_{C_j}$ . However, this implies that the hourglass  $H_{C_j}$  is not open—a contradiction  
 180 that comes from assuming that  $st$  lies in the wall of three open hourglasses, when this wall  
 181 is traversed from the top chain to the bottom chain. Analogous arguments can be used to  
 182 bound the total number of walls that contain the edge  $st$  (when traversed in any direction)  
 183 to  $O(1)$ . ◀

184 ► **Lemma 5.** Let  $x, u, y, v$  be four vertices of  $P$  that appear in this cyclic order in a clockwise  
 185 traversal of  $\partial P$ . Given the shortest-path trees  $T_x$  and  $T_y$  of  $x$  and  $y$  in  $P$ , respectively, such  
 186 that  $T_x$  and  $T_y$  can answer lowest common ancestor (LCA) queries in  $O(1)$  time, we can  
 187 compute the path  $\pi(u, v)$  in  $O(|\pi(u, v)|)$  time. Moreover, all edges of  $\pi(u, v)$ , except perhaps  
 188 one, belong to  $T_x \cup T_y$ .

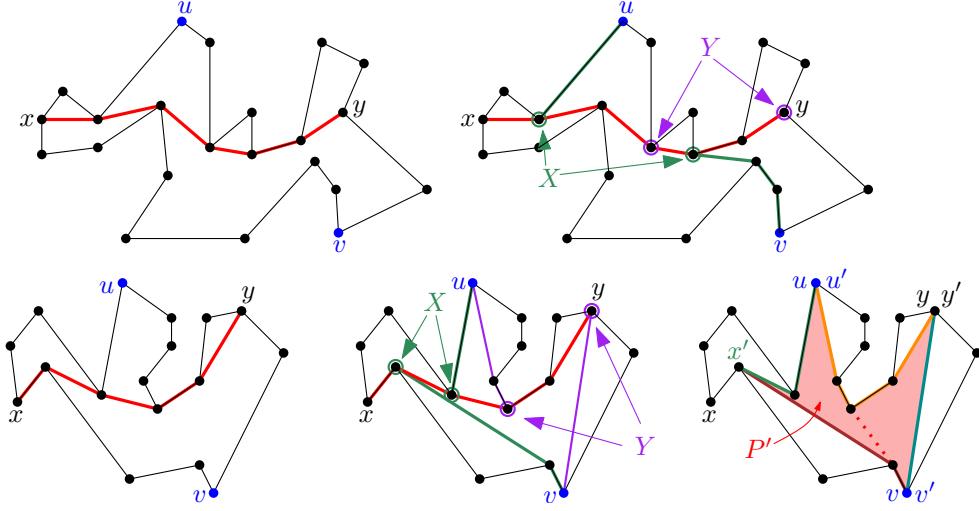
189 **Proof.** Let  $X$  (resp.  $Y$ ) be the set containing the LCA in  $T_x$  (resp.  $T_y$ ) of  $u, y$ , and of  $v, y$   
 190 (resp.  $u, x$  and  $x, y$ ). Note that the points of  $X \cup Y$  lie on the path  $\pi(x, y)$  and can be  
 191 computed in  $O(1)$  time by hypothesis. Moreover, using LCA queries, we can decide their  
 192 order along the path  $\pi(x, y)$  when traversing it from  $x$  to  $y$ . (Both  $X$  and  $Y$  could consist of  
 193 a single vertex in some degenerate situations.) Two cases arise:

194 **Case 1.** If there is a vertex  $x^* \in X$  lying after a vertex  $y^* \in Y$  along  $\pi(x, y)$ , then the  
 195 path  $\pi(u, v)$  contains the path  $\pi(y^*, x^*)$ . In this case, the path  $\pi(u, v)$  is the concatenation of  
 196 the paths  $\pi(u, y^*)$ ,  $\pi(y^*, x^*)$ , and  $\pi(x^*, v)$  and that the three paths are contained in  $T_x \cup T_y$ .  
 197 Moreover,  $\pi(u, v)$  can be computed in time proportional to its length by traversing along the  
 198 corresponding tree; see Figure 2 (top).

199 **Case 2.** In this case the vertices of  $X$  appear before the vertices of  $Y$  along  $\pi(x, y)$ . Let  
 200  $x'$  (resp.  $y'$ ) be the vertex of  $X$  (resp.  $Y$ ) closest to  $x$  (resp.  $y$ ).

201 Let  $u'$  be the last vertex of  $\pi(u, x)$  that is also in  $\pi(u, y)$ . Note that  $u'$  can be constructed  
 202 by walking from  $u'$  towards  $x$  until the path towards  $y$  diverges. Thus,  $u'$  can be computed  
 203 in  $O(|\pi(u, u')|)$  time. Define  $v'$  analogously and compute it in  $O(|\pi(v, v')|)$  time.

204 Let  $P'$  be the polygon bounded by the geodesic paths  $\pi(x', u')$ ,  $\pi(u', y')$ ,  $\pi(y', v')$  and  
 205  $\pi(v', x')$ . Because the vertices of  $X$  appear before those of  $Y$  along  $\pi(x, y)$ ,  $P'$  is a simple  
 206 polygon; see Figure 2 (bottom).



■ **Figure 2** (top) Case 1 of the proof of Lemma 5 where the path  $\pi(u, v)$  contains a portion of the path  $\pi(x, y)$ . (bottom) Case 2 of the proof of Lemma 5 where the path  $\pi(u, v)$  has exactly one edge being the tangent of the paths  $\pi(u', y')$  and  $\pi(v', x')$ .

207 In this case the path  $\pi(u, y)$  is the union of  $\pi(u, u')$ ,  $\pi(u', v')$  and  $\pi(v', v)$ . Because  $\pi(u, u')$   
 208 and  $\pi(v', v)$  can be computed in time proportional to their length, it suffices to compute  
 209  $\pi(u', v')$  in  $O(|\pi(u', v')|)$  time.

210 Note that  $P'$  is a simple polygon with only four convex vertices  $x', u', y'$  and  $v'$ , which  
 211 are connected by chains of reflex vertices. Thus, the shortest path from  $x'$  to  $y'$  can have at  
 212 most one diagonal edge connecting distinct reflex chains of  $P'$ . Since the rest of the points  
 213 in  $\pi(u', v')$  lie on the boundary of  $P'$  and from the fact that each edge of  $P'$  is an edge of  
 214  $T_x \cup T_y$ , we conclude all edges of  $\pi(u, v)$ , except perhaps one, belong to  $T_x \cup T_y$ .

215 We want to find the common tangent between the reflex paths  $\pi(u', x')$  and  $\pi(v', y')$ , or  
 216 the common tangent of  $\pi(u', y')$  and  $\pi(v', x')$  as one of them belongs to the shortest path  
 217  $\pi(u', v')$ . Assume that the desired tangent lies between the paths  $\pi(u', x')$  and  $\pi(v', y')$ .  
 218 Since these paths consist only of reflex vertices, the problem can be reduced to finding the  
 219 common tangent of two convex polygons. By slightly modifying the linear time algorithm to  
 220 compute this tangents, we can make it run in  $O(|\pi(u', v')|)$  time.

221 Since we do not know if the tangent lies between the paths  $\pi(u', x')$  and  $\pi(v', y')$ , we  
 222 process the chains  $\pi(u', y')$  and  $\pi(v', x')$  in parallel and stop when finding the desired tangent.  
 223 Consequently, we can compute the path  $\pi(u, v)$  in time proportional to its length. ◀

► **Lemma 6.** Let  $P$  be a simple polygon with  $n$  vertices. Given  $k$  disjoint transition chains  $C_1, \dots, C_k$  of  $\partial P$ , it holds that

$$\sum_{i=1}^k |H_{C_i}| = O(n).$$

224 **Proof.** Because the given transition chains are disjoint, Lemma 2 implies that the bottom  
 225 chains of their respective hourglasses are also disjoint. Therefore, the sum of the complexities  
 226 of all the top and bottom chains of these hourglasses is  $O(n)$ . To bound the complexity of  
 227 their walls we use Lemma 4. Since no chord is used more than a constant number of times,  
 228 it suffices to show that the total number of chords used by all these hourglasses is  $O(n)$ .

To prove this, we use Lemma 3 to construct  $O(1)$  *splitting chains*  $\gamma_1, \dots, \gamma_t$  such that for each  $1 \leq i \leq k$ , there is a splitting chain  $\gamma_j$  that separates the top and bottom chains of  $H_{C_i}$ . For each  $1 \leq j \leq t$ , let

$$\mathcal{H}^j = \{H_{C_i} : \text{the top and bottom chain of } H_{C_i} \text{ are separated by } \gamma_j\}.$$

Since the complexity of the shortest-path trees of the endpoints of  $\gamma_j$  is  $O(n)$  [11], and from the fact that the chains  $C_1, \dots, C_k$  are disjoint, Lemma 5 implies that the total number of edges in all the hourglasses of  $\mathcal{H}^j$  is  $O(n)$ . Moreover, because each of these edges appears in  $O(1)$  hourglasses among  $C_1, \dots, C_k$ , we conclude that

$$\sum_{H \in \mathcal{H}^j} |H| = O(n).$$

229 Since we have only  $O(1)$  splitting chains, our result follows. ◀

## 2.2 Funnels

231 Let  $C = (p_0, \dots, p_k)$  be a chain of  $\partial P$  and let  $v$  be a vertex of  $P$  not in  $C$ . The *funnel* of  $v$  to  
232  $C$ , denoted by  $S_v(C)$ , is the simple polygon bounded by  $C$ ,  $\pi(p_k, v)$  and  $\pi(v, p_0)$ ; see Figure 3  
233 (a). Note that the paths  $\pi(v, p_k)$  and  $\pi(v, p_0)$  may coincide for a while before splitting into  
234 disjoint chains. See Lee and Preparata [17] or Guibas et al. [11] for more details on funnels.

235 A subset  $R \subset P$  is *geodesically convex* if for every  $x, y \in R$ , the path  $\pi(x, y)$  is contained  
236 in  $R$ . This funnel  $S_v(C)$  is also known as the geodesic convex hull of  $C$  and  $v$ , i.e., the  
237 minimum geodesically convex set that contains  $v$  and  $C$ .

238 Given two points  $x, y \in P$ , the (geodesic) *bisector* of  $x$  and  $y$  is the set of points contained  
239 in  $P$  that are equidistant from  $x$  and  $y$ . This bisector is a curve, contained in  $P$ , that consists  
240 of straight-line segments and hyperbolic arcs. Moreover, this curve intersects  $\partial P$  only at its  
241 endpoints [1, Lemma 3.22].

242 The (farthest) *Voronoi region* of a vertex  $v$  of  $P$  is the set of points  $R(v) = \{x \in P : F_P(x) = |\pi(x, v)|\}$  (including boundary points).

244 ► **Lemma 7.** *Let  $v$  be a vertex of  $P$  and let  $C$  be a transition chain such  $R(v) \cap \partial P \subseteq C$   
245 and  $v \notin C$ . Then,  $R(v)$  is contained in the funnel  $S_v(C)$*

246 **Proof.** Let  $a$  and  $b$  be the endpoints of  $C$  such that  $a, b, f(a)$  and  $f(b)$  appear in this order  
247 in a clockwise traversal of  $\partial P$ . Because  $R(v) \cap \partial P \subseteq C$ , we know that  $v$  lies between  $f(a)$   
248 and  $f(b)$ .

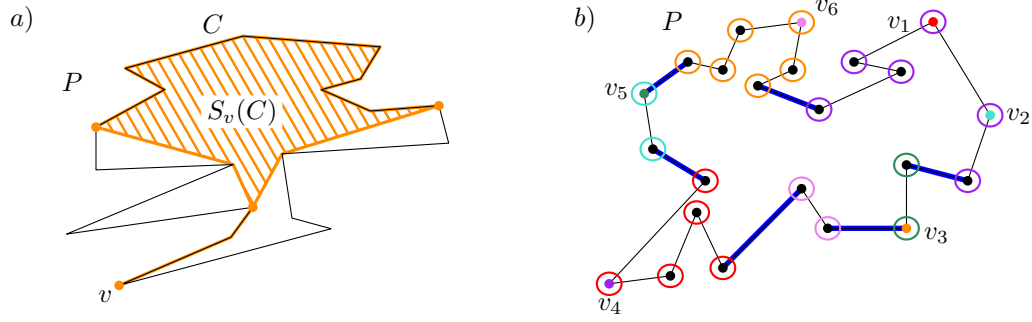
249 Let  $\alpha$  (resp.  $\beta$ ) be the bisector of  $v$  and  $f(a)$  (resp.  $f(b)$ ). Let  $h_a$  (resp.  $h_b$ ) be the set of  
250 points of  $P$  that are farther from  $v$  than from  $f(a)$  (resp.  $f(b)$ ). Note that  $\alpha$  is the boundary  
251 of  $h_a$  while  $\beta$  bounds  $h_b$ .

252 By definition, we know that  $R(v) \subseteq h_a \cap h_b$ . Therefore, it suffices to show that  $h_a \cap h_b \subseteq S_v(C)$ .  
253 Assume for a contradiction that there is a point of  $h_a \cap h_b$  lying outside of  $S_v(C)$ .  
254 By continuity of the geodesic distance, the boundaries of  $h_a \cap h_b$  and  $S_v(C)$  must intersect.  
255 Because  $a \notin h_a$  and  $b \notin h_b$ , both bisectors  $\alpha$  and  $\beta$  must have an endpoint on the edge  
256  $ab$ . Since the boundaries of  $h_a \cap h_b$  and  $S_v(C)$  intersect, we infer that  $\beta \cap \pi(v, b) \neq \emptyset$  or  
257  $\alpha \cap \pi(v, a) \neq \emptyset$ . Without loss of generality, assume that there is a point  $w \in \beta \cap \pi(v, b)$ , the  
258 case where  $w$  lies in  $\alpha \cap \pi(v, a)$  is analogous.

Since  $w \in \beta$ , we know that  $|\pi(w, v)| = |\pi(w, f(b))|$ . By the triangle inequality and since  $w$  cannot be a vertex of  $P$  as  $w$  intersects  $\partial P$  only at its endpoints, we get that

$$|\pi(b, f(b))| < |\pi(b, w)| + |\pi(w, f(b))| = |\pi(b, w)| + |\pi(w, v)| = |\pi(b, v)|.$$





**Figure 3** a) The funnel  $S_v(C)$  of a vertex  $v$  and a chain  $C$  contained in  $\partial P$  are depicted. b) Each vertex of the boundary of  $P$  is assigned with a farthest neighbor which is then marked. The boundary is then decomposed into vertex-disjoint chains, each associated with a marked vertex, joined by transition edges (blue) whose endpoints have different farthest neighbors.

Which implies that  $b$  is farther from  $v$  than from  $f(b)$ —a contradiction that comes from assuming that  $h_a \cap h_b$  is not contained in  $S_v(C)$ . ◀

### 3 Decomposing the boundary

In this section, we decompose the boundary of  $P$  into chains of consecutive vertices that share the same farthest neighbor and edges of  $P$  whose endpoints have distinct farthest neighbors.

Using a result from Hershberger and Suri [15], in  $O(n)$  time we can compute the farthest neighbor of each vertex of  $P$ . Recall that the farthest neighbor of each vertex of  $P$  is always a convex vertex of  $P$  [3] and is unique by our general position assumption.

We mark the vertices of  $P$  that are farthest neighbors of at least one vertex of  $P$ . Let  $M$  denote the set of marked vertices of  $P$  (clearly this set can be computed in  $O(n)$  time after applying the result of Hershberger and Suri). In other words,  $M$  contains all vertices of  $P$  whose Voronoi region contains at least one vertex of  $P$ .

Given a vertex  $v$  of  $P$ , the vertices of  $P$  whose farthest neighbor is  $v$  appear contiguously along  $\partial P$  [2]. Therefore, after computing all these farthest neighbors, we effectively split the boundary into subchains, each associated with a different vertex of  $M$ ; see Figure 3 (b).

Let  $a$  and  $b$  be the endpoints of a transition edge of  $\partial P$  such that  $a$  appears before  $b$  in the clockwise order along  $\partial P$ . Because  $ab$  is a transition edge, we know that  $f(a) \neq f(b)$ . Recall that we have computed  $f(a)$  and  $f(b)$  in the previous step and note that  $f(a)$  appears also before  $f(b)$  along this clockwise order. For every vertex  $v$  that lies between  $f(a)$  and  $f(b)$  in the bottom chain of  $H_{ab}$ , we know that there cannot be a vertex  $u$  of  $P$  such that  $f(u) = v$ . As proved by Aronov et al. [2, Corollary 2.7.4], if there is a point  $x$  on  $\partial P$  whose farthest neighbor is  $v$ , then  $x$  must lie on the open segment  $(a, b)$ . In other words, the Voronoi region  $R(v)$  restricted to  $\partial P$  is contained in  $(a, b)$ .

### 4 Building hourglasses

Let  $E$  be the set of transition edges of  $\partial P$ . Given a transition edge  $ab \in E$ , we say that  $H_{ab}$  is a *transition hourglass*. In order to construct the triangle cover of  $P$ , we construct the transition hourglass of each transition edge of  $E$ . By Lemma 6, we know that  $\sum_{ab \in E} |H_{ab}| = O(n)$ . Therefore, our aim is to compute the cover in time proportional to the size of  $H_{ab}$ .



By Lemma 3 we can compute a set of  $O(1)$  separating paths such that for each transition edge  $ab$ , the transition hourglass  $H_{ab}$  is separated by one (or more) paths in this set. For each endpoint of the  $O(1)$  separating paths we compute its shortest-path tree in linear time [8, 11]. In addition, we preprocess these trees in linear time to support LCA queries [14]. Both computations need linear time per endpoint and use  $O(n)$  space. Since we do this process for a constant number of endpoints, overall this preprocessing takes  $O(n)$  time.

Let  $\gamma$  be a separating path. Note that  $\gamma$  separates the boundary of  $P$  into two chains  $S$  and  $S'$  such that  $S \cup S' = \partial P$ . Let  $\mathcal{H}(\gamma)$  be the set of transition hourglasses separated by  $\gamma$  whose transition edge is contained in  $S$  (whenever an hourglass is separated by more than one path, we pick one arbitrarily). Note that we can classify all transition hourglasses into the sets  $\mathcal{H}(\gamma)$  in  $O(n)$  time (since  $O(1)$  separating paths are considered).

We claim that we can compute all transition hourglasses of  $\mathcal{H}(\gamma)$  in  $O(n)$  time. By construction, the wall of each of these hourglasses consists of a (geodesic) path that connects a point in  $S$  with a point in  $S'$ . Let  $u \in S$  and  $v \in S'$  be two vertices such that  $\pi(u, v)$  is the wall of a hourglass in  $\mathcal{H}(\gamma)$ . Because LCA queries can be answered in  $O(1)$  time [14], Lemma 5 allows us to compute this path in  $O(|\pi(u, v)|)$  time. Therefore, we can compute all hourglasses of  $\mathcal{H}(\gamma)$  in  $O(\sum_{H \in \mathcal{H}(\gamma)} |H| + n) = O(n)$  time by Lemma 6. Because only  $O(1)$  separating paths are considered, we obtain the following result.

► **Lemma 8.** *We can construct the transition hourglass of all transition edges of  $P$  in  $O(n)$  time.*

## 5 Covering the polygon with apexed triangles

An *apexed triangle*  $\Delta = (a, b, c)$  with *apex*  $a$  is a triangle contained in  $P$  with an associated distance function  $g_\Delta(x)$ , called the *apex function* of  $\Delta$ , such that (1)  $a$  is a vertex of  $P$ , (2)  $b, c \in \partial P$ , and (3) there is a vertex  $w$  of  $P$ , called the *definer* of  $\Delta$ , such that

$$g_\Delta(x) = \begin{cases} -\infty & \text{if } x \notin \Delta \\ |xa| + |\pi(a, w)| = |\pi(x, w)| & \text{if } x \in \Delta \end{cases}$$

In this section, we show how to find a set of  $O(n)$  apexed triangles of  $P$  such that the upper envelope of their apex functions coincides with  $F_P(x)$ . To this end, we first decompose the transition hourglasses into apexed triangles that encode all the geodesic distance information inside them. For each marked vertex  $v \in M$  we construct a funnel that contains the Voronoi region of  $v$ . We then decompose this funnel into apexed triangles that encode the distance from  $v$ .

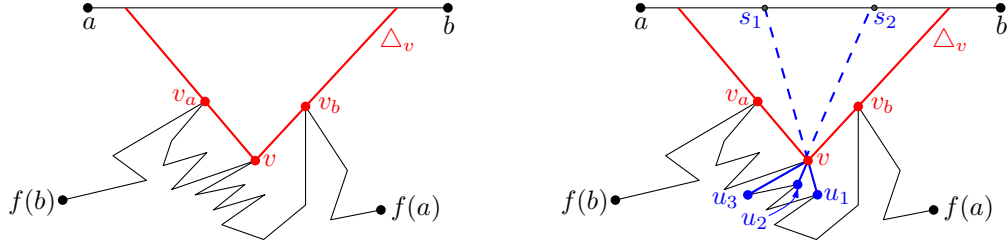
### 5.1 Inside the transition hourglass

Let  $ab$  be a transition edge of  $P$  such that  $b$  is the clockwise neighbor of  $a$  along  $\partial P$ . Let  $B_{ab}$  denote the bottom chain of  $H_{ab}$  after removing its endpoints. As noticed above, a point on  $\partial P$  can be farthest from a vertex in  $B_{ab}$  only if it lies in the open segment  $ab$ . That is, if  $v$  is a vertex of  $B_{ab}$  such that  $R(v) \neq \emptyset$ , then  $R(v) \cap \partial P \subset ab$ .

In fact, not only this Voronoi region is inside  $H_{ab}$  when restricted to the boundary of  $P$ , but also  $R(v) \subset H_{ab}$ . The next result follows trivially from Lemma 7.

► **Corollary 9.** *Let  $v$  be a vertex of  $B_{ab}$ . If  $R(v) \neq \emptyset$ , then  $R(v) \subset H_{ab}$ .*

Our objective is to compute  $O(|H_{ab}|)$  apexed triangles that cover  $H_{ab}$ , each with its distance function, such that the upper envelope of these apex functions coincides with  $F_P(x)$  restricted to  $H_{ab}$  where it “matters”.



■ **Figure 4** (left) A vertex  $v$  visible from the segment  $ab$  lying on the bottom chain of  $H_{ab}$ , and the triangle  $\Delta_v$  which contains the portion of  $ab$  visible from  $v$ . (right) The children  $u_1$  and  $u_2$  of  $v$  are visible from  $ab$  while  $u_3$  is not. The triangle  $\Delta_v$  is split into apexed triangles by the rays going from  $u_1$  and  $u_2$  to  $v$ .

326 The same approach was already used by Pollack et al. in [25, Section 3]. Given a segment  
 327 contained in the interior of  $P$ , they show how to compute a linear number of apexed triangles  
 328 such that  $F_P(x)$  coincides with the upper envelope of the corresponding apex functions in  
 329 the given segment.

330 While the construction we follow is analogous, we use it in the transition hourglass  $H_{ab}$   
 331 instead of the full polygon  $P$ . Therefore, we have to specify what is the relation between the  
 332 upper envelope of the computed functions and  $F_P(x)$ . We will show that the upper envelope  
 333 of the apex functions computed in  $H_{ab}$  coincides with  $F_P(x)$  inside the Voronoi region  $R(v)$   
 334 of every vertex  $v \in B_{ab}$ .

335 Let  $T_a$  and  $T_b$  be the shortest-path trees in  $H_{ab}$  from  $a$  and  $b$  rooted at  $a$  and  $b$ , respectively.  
 336 We can compute these trees in  $O(|H_{ab}|)$  time [11]. For each vertex  $v$  between  $f(a)$  and  $f(b)$ ,  
 337 let  $v_a$  and  $v_b$  be the neighbors of  $v$  in the paths  $\pi(v, a)$  and  $\pi(v, b)$ , respectively. We say that  
 338 a vertex  $v$  is *visible* from  $ab$  if  $v_a \neq v_b$ . Note that if a vertex is visible, then the extension of  
 339 these segments must intersect the top segment  $ab$ . Therefore, for each visible vertex  $v$ , we  
 340 obtain a triangle  $\Delta_v$  as shown in Figure 4.

341 We further split  $\Delta_v$  into a series of triangles with apex at  $v$  as follows: Let  $u$  be a child  
 342 of  $v$  in either  $T_a$  or  $T_b$ . As noted by Pollack et al.,  $v$  can be of three types, either (1)  $u$  is not  
 343 visible from  $ab$  (and is hence a child of  $v$  in both  $T_a$  and  $T_b$ ); or (2)  $u$  is visible from  $ab$ , is a  
 344 child of  $v$  only in  $T_b$ , and  $v_bvu$  is a left turn; or (3)  $u$  is visible from  $ab$ , is a child of  $v$  only in  
 345  $T_a$ , and  $v_avu$  is a right turn.

346 Let  $u_1, \dots, u_{k-1}$  be the children of  $v$  of type (2) sorted in clockwise order around  $v$ . Let  
 347  $c(v)$  be the maximum distance from  $v$  to any invisible vertex in the subtrees of  $T_a$  and  $T_b$   
 348 rooted at  $v$ ; if no such vertex exists, then  $c(v) = 0$ . Define a function  $d_l(v)$  on each vertex  $v$   
 349 of  $H_{ab}$  in a recursive fashion as follows: If  $v$  is invisible from  $ab$ , then  $d_l(v) = c(v)$ . Otherwise,  
 350 let  $d_l(v)$  be the maximum of  $c(v)$  and  $\max\{d_l(u_i) + |u_iv| : u_i \text{ is a child of } v \text{ of type (2)}\}$ .  
 351 Symmetrically, we define a function  $d_r(v)$  using the children of type (3) of  $v$ .

For each  $1 \leq i \leq k-1$ , extend the segment  $u_iv$  past  $v$  until it intersects  $ab$  at a point  $s_i$ .  
 Let  $s_0$  and  $s_k$  be the intersections of the extensions of  $vv_a$  and  $vv_b$  with the segment  $ab$ . We  
 define then  $k$  apexed triangles contained in  $\Delta_v$  as follows. For each  $0 \leq i \leq k-1$ , consider  
 the triangle  $\Delta(s_i, v, s_{i+1})$  whose associated apexed (left) function is

$$f_i(x) = \begin{cases} |xv| + \max_{j>i}\{c(v), |vu_j| + d_l(u_j)\} & \text{if } x \in \Delta(s_i, v, s_{i+1}) \\ -\infty & \text{otherwise} \end{cases}$$

352 In a symmetric manner, we define a set of apexed triangles induced by the type (3) children  
 353 of  $v$  and their respective apexed (right) functions.

Let  $g_1, \dots, g_r$  and  $\Delta_1, \dots, \Delta_r$  respectively be an enumeration of all the generated apex functions and apexed triangles such that  $g_i$  is defined in the triangle  $\Delta_i$ . Because each function is determined uniquely by a pair of adjacent vertices in  $T_a$  or in  $T_b$ , and since these trees have  $O(|H_{ab}|)$  vertices, we conclude that  $r = O(|H_{ab}|)$ .

Note that for each  $1 \leq i \leq r$ , the apexed triangle  $\Delta_i$  has two vertices on the segment  $ab$  and a third vertex, say  $a_i$ , being its apex such that for each  $x \in \Delta_i$ ,  $g_i(x) = |\pi(x, w_i)|$  for some vertex  $w_i$  of  $H_{ab}$ . Recall that  $w_i$  is called the definer of  $\Delta_i$ . Intuitively,  $\Delta_i$  defines a portion of the geodesic distance function from  $w_i$  in a constant complexity region.

► **Lemma 10.** *Given a transition edge  $ab$  of  $P$ , we can compute a set  $\mathcal{A}_{ab}$  of  $O(|H_{ab}|)$  apexed triangles in  $O(|H_{ab}|)$  time with the property that for any point  $p \in P$  such that  $f(p) \in B_{ab}$ , there is an apexed triangle  $\Delta \in \mathcal{A}_{ab}$  with apex function  $g$  and definer equal to  $f(p)$  such that*

1.  $p \in \Delta$  and
2.  $g(p) = F_P(p)$ .

**Proof.** Because  $p \in R(f(p))$ , Lemma 9 implies that  $p \in H_{ab}$ . Consider the path  $\pi(p, f(p))$  and let  $v$  be the neighbor of  $p$  along this path. By construction of  $\mathcal{A}_{ab}$ , there is a triangle  $\Delta \in \mathcal{A}_{ab}$  apexed at  $v$  with definer  $w$  that contains  $p$ . The apex function  $g(x)$  of  $\Delta$  encodes the geodesic distance from  $x$  to  $w$ . Because  $F_P(x)$  is the upper envelope of all the geodesic functions, we know that  $g(p) \leq F_P(p)$ .

To prove the other inequality, note that if  $v = f(p)$ , then trivially  $g(p) = |pv| + |\pi(v, w)| \geq |pv| = |\pi(p, f(p))| = F_P(p)$ . Otherwise, let  $z$  be the next vertex after  $v$  in the path  $\pi(p, f(p))$ . Three cases arise:

(a) If  $z$  is invisible from  $ab$ , then so is  $f(p)$  and hence,

$$|\pi(p, f(p))| = |pv| + |\pi(v, f(p))| \leq |pv| + c(v) \leq g(p).$$

(b) If  $z$  is a child of type (2), then  $z$  plays the role of some child  $u_j$  of  $v$  in the notation used during the construction. In this case:

$$|\pi(p, f(p))| = |pv| + |vz| + |\pi(z, f(p))| \leq |pv| + |vu_j| + d_l(u_j) \leq g(p).$$

(c) If  $z$  is a child of type (3), then analogous arguments hold using the (right) distance  $d_r$ .

Therefore, regardless of the case  $F_P(p) = |\pi(p, f(p))| \leq g(p)$ .

To bound the running time, note that the recursive functions  $d_l, d_r$  and  $c$  can be computed in  $O(|T_a| + |T_b|)$  time. Then, for each vertex visible from  $ab$ , we can process it in time proportional to its degree in  $T_a$  and  $T_b$ . Because the sum of the degrees of all vertices in  $T_a$  and  $T_b$  is  $O(|T_a| + |T_b|)$  and from the fact that both  $|T_a|$  and  $|T_b|$  are  $O(|H_{ab}|)$ , we conclude that the total running time to construct  $\mathcal{A}_{ab}$  is  $O(|H_{ab}|)$ . ◀

In other words, Lemma 10 says that no information on farthest neighbors is lost if we only consider the functions of  $\mathcal{A}_{ab}$  within  $H_{ab}$ . In the next section we use a similar approach to construct a set of apexed triangles (and their corresponding apex functions), so as to encode the distance from the vertices of  $M$ .

## 5.2 Inside the funnels of marked vertices

Recall that for each marked vertex  $v \in M$ , we know at least of one vertex on  $\partial P$  such that  $v$  is its farthest neighbor. For any marked vertex  $v$ , let  $u_1, \dots, u_{k-1}$  be the vertices of  $P$  such that  $v = f(u_i)$  and assume that they appear in this order when traversing  $\partial P$  clockwise. Let  $u_0$  and  $u_k$  be the neighbors of  $u_1$  and  $u_{k-1}$  other than  $u_2$  and  $u_{k-2}$ , respectively. Note that

both  $u_0u_1$  and  $u_{k-1}u_k$  are transition edges of  $P$ . Thus, we can assume that their transition hourglasses have been computed.

Let  $C_v = (u_0, \dots, u_k)$  and consider the funnel  $S_v(C_v)$ . We call  $C_v$  the *main chain* of  $S_v(C_v)$  while  $\pi(u_k, v)$  and  $\pi(v, u_0)$  are referred to as the *walls* of the funnel. Because  $v = f(u_1) = f(u_{k-1})$ , we know that  $v$  is a vertex of both  $H_{u_0u_1}$  and  $H_{u_{k-1}u_k}$ . By definition, we have  $\pi(v, u_0) \subset H_{u_0u_1}$  and  $\pi(v, u_k) \subset H_{u_{k-1}u_k}$ . Thus, we can explicitly compute both paths  $\pi(v, u_0)$  and  $\pi(v, u_k)$  in  $O(|H_{u_0u_1}| + |H_{u_{k-1}u_k}|)$  time. So, overall, the funnel  $S_v(C_v)$  can be constructed in  $O(k + |H_{u_0u_1}| + |H_{u_{k-1}u_k}|)$  time. Recall that, by Lemma 6, the total sum of the complexities of the transition hourglasses is  $O(n)$ . In particular, we can bound the total time needed to construct the funnels of all marked vertices by  $O(n)$ .

Since the complexity of the walls of these funnels is bounded by the complexity of the transition hourglasses used to compute them, we get that

$$\sum_{v \in M} |S_v(C_v)| = O\left(n + \sum_{ab \in E} |H_{ab}|\right) = O(n).$$

► **Lemma 11.** *Let  $x$  be a point in  $P$ . If  $v = f(x)$  is a vertex of  $M$ , then  $x \in S_v(C_v)$ .*

**Proof.** Since  $f(u_0) \neq f(u_k)$ ,  $C_v$  is a transition chain. Moreover,  $C_v$  contains  $R(v) \cap \partial P$  by definition. Therefore, Lemma 7 implies that  $R(v) \subset S_v(C_v)$ . Since  $v = f(x)$ , we know that  $x \in R(v)$  and hence that  $x \in S_v(C_v)$ . ◀

We now proceed to split a given funnel into  $O(|S_v(C_v)|)$  apexed triangles that encode the distance function from  $v$ . To this end, we use the algorithm described by Guibas et al. [12, Section 2] to compute the shortest-map of  $v$  in  $S_v(C_v)$  in  $O(|S_v(C_v)|)$  time. This algorithm produces a partition of  $S_v(C_v)$  into  $O(|S_v(C_v)|)$  interior disjoint triangles, such that each triangle consists of all points in  $S_v(C_v)$  whose shortest path to  $v$  consists of the same sequence of vertices. Let  $\Delta$  be a triangle in this partition and let  $a$  be its apex, i.e., the first vertex found along each path  $\pi(x, v)$ , where  $x \in \Delta$ . We define the apex function  $g_\Delta(x)$  of  $\Delta$  as follows:

$$g_\Delta(x) = \begin{cases} |xa| + |\pi(a, v)| & \text{if } x \in \Delta \\ -\infty & \text{otherwise} \end{cases}$$

Therefore, for each  $x \in \Delta$ ,  $g_\Delta(x) = |\pi(x, v)|$ .

► **Lemma 12.** *The shortest-path map of  $v$  in  $S_v(C_v)$  can be computed in  $O(|S_v(C_v)|)$  time and produces  $O(|S_v(C_v)|)$  interior disjoint apexed triangles such that their union covers  $S_v(C_v)$ . Moreover, for each point  $x \in R(v)$ , there is an apexed triangle  $\Delta$  with apex function  $g(x)$  such that (1)  $x \in \Delta$  and (2)  $g(x) = F_P(x)$ .*

**Proof.** The above procedure splits  $S_v(C_v)$  into  $O(|S_v(C_v)|)$  apexed triangles, such that the apex function in each of them is defined as the geodesic distance to  $v$ . By Lemma 11, if  $x \in R(v)$ , then  $x \in S_v(C_v)$ . Therefore, there is an apexed triangle  $\Delta$  with apex function  $g(x)$  such that  $x \in \Delta$  and  $g(x) = |\pi(x, v)| = F_P(x)$ . Thus, we obtain properties (1) and (2). ◀

## 6 Prune and search

With the tools introduced in the previous sections, we can proceed to give the prune and search algorithm to compute the geodesic center. The idea of the algorithm is to partition  $P$  into  $O(1)$  cells, determine in which cell of  $P$  the center lies and recurse on that cell as a new subproblem with smaller complexity.

Naturally, we can discard all apexed triangles that do not intersect the new cell containing the center. Using cuttings to produce this partition of  $P$ , we can show that both the complexity of the cell containing the center, and the number of apexed triangles that intersect it decrease by a constant fraction in each iteration of the algorithm. This process is then repeated until either of the two objects has constant descriptive size.

Let  $\tau$  be the set of all apexed triangles computed in previous sections. Lemmas 6 and 12 directly provide a bound on the complexity of  $\tau$ .

► **Corollary 13.** *The set  $\tau$  consists of  $O(n)$  apexed triangles.*

Let  $\phi(x)$  be the upper envelope of the apex functions of the triangles in  $\tau$  (i.e.,  $\phi(x) = \max\{g(x) : \Delta \in \tau \text{ and } g(x) \text{ is the apex function of } \Delta\}$ ). The following result is a direct consequence of Lemmas 10 and 12, and shows that the  $O(n)$  apexed triangles of  $\tau$  not only cover  $P$ , but their apex functions suffice to reconstruct the function  $F_P(x)$ .

► **Lemma 14.** *The functions  $\phi(x)$  and  $F_P(x)$  coincide in the domain of points of  $P$ , i.e., for each  $p \in P$ ,  $\phi(p) = F_P(p)$ .*

Given a chord  $C$  of  $P$  a *half-polygon* of  $P$  is one of the two simple polygons in which  $C$  splits  $P$ . A  $k$ -cell of  $P$  is a simple polygon obtained as the intersection of at most  $k$  half-polygons. Because a  $k$ -cell is the intersection of geodesically convex sets, it is also geodesically convex.

The recursive algorithm described in this section takes as input a 4-cell (initially equal to  $P$ ) containing the geodesic center of  $P$  and the set of apexed triangles of  $\tau$  that intersect  $R$ . In each iteration, it produces a new 4-cell of smaller complexity that intersects just a fraction of the apexed triangles and contains the geodesic center of  $P$ . By recursing on this new cell, the complexity of the problem is reduced in each iteration.

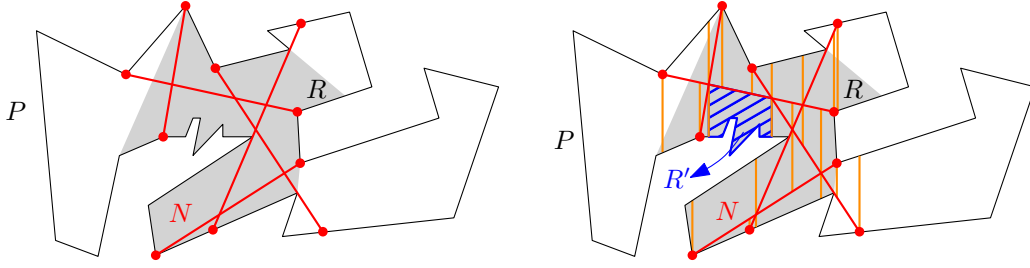
Let  $R$  be a 4-cell of  $P$  containing the geodesic center of  $P$  and let  $\tau_R$  be the set of apexed triangles of  $\tau$  that intersect  $R$ . Let  $m_R = \max\{|R|, |\tau_R|\}$ , where  $|R|$  denotes the combinatorial complexity of  $R$ , i.e., the number of vertices and edges of  $R$ . Recall that, by construction of the apexed triangles, for each triangle of  $\tau_R$  at least one and at most two of its boundary segments is a chord of  $P$ . Let  $\mathcal{C}$  be the set containing all chords that belong to the boundary of a triangle of  $\tau_R$ . Therefore,  $|\mathcal{C}| \leq 2|\tau_R| \leq 2m_R$ .

To construct  $\varepsilon$ -nets, we need some definitions (for more information on  $\varepsilon$ -nets refer to [20]). Let  $\varphi$  be the set of all open 4-cells of  $P$ . For each  $t \in \varphi$ , let  $\mathcal{C}_t = \{C \in \mathcal{C} : C \cap t \neq \emptyset\}$  be the set of chords of  $\mathcal{C}$  induced by  $t$ . Finally, let  $\varphi_{\mathcal{C}} = \{\mathcal{C}_t : t \in \varphi\}$  be the family of subsets of  $\mathcal{C}$  induced by  $\varphi$ . Consider the set system  $(\mathcal{C}, \varphi_{\mathcal{C}})$  (denoted by  $(\mathcal{C}, \varphi)$  for simplicity) defined by  $\mathcal{C}$  and  $\varphi$ . The proof of the next lemma is deferred to the Appendix for ease of readability.

► **Lemma 15.** *The set system  $(\mathcal{C}, \varphi)$  has constant VC-dimension.*

Let  $\varepsilon > 0$  (the exact value of  $\varepsilon$  will be specified later). Because the VC-dimension of the set system  $(\mathcal{C}, \varphi)$  is finite by Lemma 15, we can compute an  $\varepsilon$ -net  $N$  of  $(\mathcal{C}, \varphi)$  in  $O(|\mathcal{C}|/\varepsilon) = O(m_R)$  time [20]. The size of  $N$  is  $O(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}) = O(1)$  and its main property is that any 4-cell that does not intersect a chord of  $N$  will intersect at most  $\varepsilon|\mathcal{C}|$  chords of  $\mathcal{C}$ .

Observe that  $N$  partitions  $R$  into  $O(1)$  sub-polygons (not necessarily 4-cells). We further refine this partition by performing a 4-cell decomposition. That is, we shoot vertical rays up and down from each endpoint of  $N$ , and from the intersection point of any two segments of  $N$ , see Figure 5. Overall, this partitions  $R$  into  $O(1)$  4-cells such that each either (i) is a convex polygon contained in  $P$  of at most four vertices, or otherwise (ii) contains some chain of  $\partial P$ . Since  $|N| = O(1)$ , the whole decomposition can be computed in  $O(m_R)$  time



■ **Figure 5** The  $\epsilon$ -net  $N$  splits  $R$  into  $O(1)$  sub-polygons that are further refined into a 4-cell decomposition using  $O(1)$  ray-shooting queries from the vertices of the arrangement defined by  $N$ .

464 (the intersections between segments of  $N$  are done in constant time, and for the ray shooting  
 465 operations we walk along the boundary of  $R$  once).

466 In order to determine which 4-cell contains the geodesic center of  $P$ , we extend each edge  
 467 of a 4-cell to a chord  $C$ . This can be done with two ray-shooting queries (each of which  
 468 takes  $O(m_R)$  time). We then use the chord-oracle from Pollack et al. [25, Section 3] to decide  
 469 which side of  $C$  contains  $c_P$ . The only requirement of this technique is that the function  
 470  $F_P(x)$  coincides with the upper envelope of the apex functions when restricted to  $C$ . Which  
 471 is true by Lemma 14 and from the fact that  $\tau_R$  consists of all the apexed triangles of  $\tau$  that  
 472 intersect  $R$ .

473 Because the chord-oracle described by Pollack et al. [25, Section 3] runs in time linear in  
 474 the number of functions defined on  $C$ , we can decide in total  $O(m_R)$  time in which side of  $C$   
 475 the geodesic center of  $P$  lies. Since our decomposition into 4-cells has constant complexity,  
 476 we need to perform  $O(1)$  calls to the oracle before determining the 4-cell  $R'$  that contains  
 477 the geodesic center of  $P$ .

478 The chord-oracle computes the minimum of  $F_P(x)$  restricted to the chord before determ-  
 479 ining the side containing the minimum. In particular, if  $c_P$  lies on any chord bounding  $R'$ ,  
 480 then the chord-oracle will find it. Therefore, we can assume that  $c_P$  lies in the interior of  $R'$ .  
 481 Moreover, since  $N$  is a  $\epsilon$ -net, we know that at most  $\epsilon|C|$  chords of  $C$  intersect  $R'$ .

482 Using a similar argument, we can show that the complexity of  $R'$  also decreases: since  
 483  $|C| \leq 2|\tau_R| \leq 2m_R$ , we guarantee that at most  $2\epsilon m_R$  apexed triangles intersect  $R'$ . Because  
 484  $F_P(x)$  is defined in each point of  $R'$ , Lemma 14 implies that each vertex of  $R'$  is covered  
 485 by at least one apexed triangle of  $\tau_R$ . Since each apexed triangle can cover at most three  
 486 vertices, by the pigeonhole principle we conclude that  $R'$  can have at most  $6\epsilon m_R$  vertices.  
 487 Otherwise, an apexed triangle would contain at least four vertices of  $R'$ . Thus, if we choose  
 488  $\epsilon = 1/12$ , we guarantee that both the size of the 4-cell  $R'$  and the number of apexed triangles  
 489 in  $\tau_{R'}$  are at most  $m_R/2$ .

490 In order to proceed with the algorithm on  $R'$  recursively, we need to compute the set  
 491  $\tau_{R'}$  with the at most  $\epsilon|C|$  apexed triangles of  $\tau_R$  that intersect  $R'$  (i.e., prune the apexed  
 492 triangles that do not intersect with  $R'$ ). For each apexed triangle  $\triangle \in \tau_R$ , we can determine  
 493 in constant time if it intersects  $R'$  (either one of the endpoints is in  $R' \cap \partial P$  or the two  
 494 boundaries have non-empty intersection in the interior of  $P$ ). Overall, we need  $O(m_R)$  time  
 495 to compute the at most  $\epsilon|C|$  triangles of  $\tau_R$  that intersect  $R'$ .

496 By recursing on  $R'$ , we guarantee that after  $O(\log m_R)$  iterations, we reduce the size of  
 497 either  $\tau_R$  or  $R'$  to constant. In the former case, the minimum of  $F_P(x)$  can be found by  
 498 explicitly constructing function  $\phi$  in  $O(1)$  time. In the latter case, we triangulate  $R'$  and  
 499 apply the chord-oracle to determine which triangle will contain  $c_P$ . The details needed to  
 500 find the minimum of  $\phi(x)$  inside this triangle are given in the next section.



501 ► **Lemma 16.** *In  $O(n)$  time we can find either the geodesic center of  $P$  or a triangle*  
 502 *containing the geodesic center.*

## 503 **7 Solving the problem restricted to a triangle**

504 In order to complete the algorithm it remains to show how to find the geodesic center of  $P$   
 505 for the case in which  $R'$  is a triangle. If this triangle is in the interior of  $P$ , it may happen  
 506 that several apexed triangles of  $\tau$  fully contain  $R'$ . Thus, the pruning technique used in the  
 507 previous section cannot be further applied. We solve this case with a different approach.

508 Recall that  $\phi(x)$  denotes the upper envelope of the apex functions of the triangles in  $\tau$ ,  
 509 and the geodesic center is the point that minimizes  $\phi$ . The key observation is that, as it  
 510 happened with chords, the function  $\phi(x)$  restricted to  $R'$  is convex.

Let  $\Delta_1, \Delta_2, \dots, \Delta_m$  be the set of  $m = O(n)$  apexed triangles of  $\tau$  that intersect  $R'$ . Let  
 $a_i$  and  $w_i$  be the apex and the definer of  $\Delta_i$ , respectively. Let  $g_i(x)$  be the apex function of  
 $\Delta_i$  such that

$$g_i(x) = \begin{cases} |xa_i| + \kappa_i & \text{if } x \in \Delta_i \\ -\infty & \text{otherwise} \end{cases},$$

511 where  $\kappa_i = |\pi(a_i, w_i)|$  is a constant.

512 By Lemma 14,  $\phi(x) = F_P(x)$ . Therefore, the problem of finding the center is equivalent  
 513 to the following optimization problem in  $\mathbb{R}^3$ :

(P1). Find a point  $(x, r) \in \mathbb{R}^3$  minimizing  $r$  subject to  $x \in R'$  and

$$g_i(x) \leq r, \text{ for } 1 \leq i \leq m.$$

514 Thus, we need only to find the solution to (P1) to find the geodesic center of  $P$ . We use  
 515 some remarks described by Megiddo in order to simplify the description of (P1) [21].

To simplify the formulas, we square the equation  $|xa_i| \leq r - \kappa_i$ :

$$\|x\|^2 - 2x \cdot a_i + \|a_i\|^2 = |xa_i|^2 \leq (r - \kappa_i)^2 = r^2 - 2r\kappa_i + \kappa_i^2.$$

And finally for each  $1 \leq i \leq m$ , we define the function  $h_i(x, r)$  as follows:

$$h_i(x, r) = \begin{cases} \|x\|^2 - 2x \cdot a_i + \|a_i\|^2 - r^2 + 2r\kappa_i - \kappa_i^2 & \text{if } x \in \Delta_i \\ -\infty & \text{otherwise} \end{cases}.$$

516 Therefore, our optimization problem can be reformulated as:

(P2). Find a point  $(x, r) \in \mathbb{R}^3$  such that  $r$  is minimized subject to  $x \in R'$  and

$$h_i(x, r) \leq 0 \text{ and } r > \max\{\kappa_i\}, \text{ for } 1 \leq i \leq m.$$

517 Let  $h'_i(x, r) = \|x\|^2 - 2x \cdot a_i + \|a_i\|^2 - r^2 + 2r\kappa_i - \kappa_i^2$  be a function defined in the entire  
 518 plane and let (P2') be an optimization problem analogous to (P2) where every instance of  
 519  $h_i(x, r)$  is replaced by  $h'_i(x, r)$ . The optimization (P2') was studied by Megiddo in [21]. We  
 520 provide some of the intuition used by Megiddo to solve this problem.

Although the functions  $h'_i(x, r)$  are not linear, they all have the same non-linear terms.  
 Therefore, for  $i \neq j$ , we get that  $h'_i(x, r) = h'_j(x, r)$  defines a *separating plane*

$$\gamma_{i,j} = \{(x, r) \in \mathbb{R}^3 : 2(\kappa_i - \kappa_j)r - 2(a_i - a_j) \cdot x + \|a_i\|^2 - \|a_j\|^2 - \kappa_i^2 + \kappa_j^2 = 0\}$$

521 As noted by Megiddo [21], this separating plane has the following property: If the solution  
 522  $(x, r)$  to (P2') is known to lie to one side of  $\gamma_{i,j}$ , then we know that one of the constraints is  
 523 redundant.



Thus, to solve (P2') it sufficed to have a *side-decision oracle* to determine in which side of a plane  $\gamma_{i,j}$  the solution lies. Megiddo showed how to implement this oracle in a way that the running time is proportional to the number of constraints [21].

Once we have such an oracle, Megiddo's problem can be solved using a prune and search approach: pair the functions arbitrarily, and consider the set of  $m/2$  separating planes defined by these pairs. For some constant  $t$ , compute a  $1/t$ -cutting in  $\mathbb{R}^3$  of the separating planes. A  $1/t$ -cutting is a partition of the plane into  $O(t^3) = O(1)$  convex regions each of which is of constant complexity and intersects at most  $m/2t$  separating planes. A cutting of planes can be computed in linear time in  $\mathbb{R}^3$  for any  $t = O(1)$  [18]. After computing the cutting, determine in which of the regions the minimum lies by performing  $O(1)$  calls to the side-decision oracle. Because at least  $(t-1)m/2t$  separating planes do not intersect this constant complexity region, for each of them we can discard one of the constraints as it becomes redundant. Repeating this algorithm recursively we obtain a linear running time.

To solve (P2) we follow a similar approach, but our set of separating planes needs to be extended in order to handle apex functions as they are only defined in the same way as in (P2') in a triangular domain. Note that the vertices of each apexed triangle that intersect  $R'$  have their endpoints either outside of  $R'$  or on its boundary.

## 7.1 Optimization problem in a convex domain

In this section we describe our algorithm to solve the optimization problem (P2). To this end, we pair the apexed triangles arbitrarily to obtain  $m/2$  pairs. By identifying the plane where  $P$  lies with the plane  $Z_0 = \{(x, y, z) : z = 0\}$ , we can embed each apexed triangle in  $\mathbb{R}^3$ . A *plane-set* is a set consisting of at most five planes in  $\mathbb{R}^3$ . For each pair of apexed triangles  $(\Delta_i, \Delta_j)$  we define its plane-set as follows: For each chord of  $P$  bounding either  $\Delta_i$  or  $\Delta_j$  (at most two chords on each triangle), consider the line extending this chord and the vertical extrusion of this line in  $\mathbb{R}^3$ , i.e., the plane containing this chord orthogonal to  $Z_0$ . Moreover, consider the separating plane  $\gamma_{i,j}$ . The set containing these planes is the plane-set of the pair  $(\Delta_i, \Delta_j)$ .

Let  $\Gamma$  be the union of all the plane-sets defined by the  $m/2$  pairs of apexed triangles. Because the plane-set of each pair  $(\Delta_i, \Delta_j)$  consists of at most five planes and contains at least one plane unique to this pair, say  $\gamma_{i,j}$ , we infer that  $m/2 \leq |\Gamma| \leq 5m/2$ .

Compute a  $1/t$ -cutting of  $\Gamma$  in  $O(m)$  time for some constant  $t$  to be specified later. Because  $t$  is constant, this  $1/t$ -cutting splits the space into  $O(1)$  convex regions, each bounded by a constant number of planes [18]. Using a side-decision algorithm (to be specified later), we can determine the region  $Q$  of the cutting that contains the solution to (P2). Because  $Q$  is the region of a  $1/t$ -cutting of  $\Gamma$ , we know that at most  $|\Gamma|/t$  planes of  $\Gamma$  intersect  $Q$ . In particular, at most  $|\Gamma|/t$  plane-sets intersect  $Q$  and hence, at least  $(t-1)|\Gamma|/t$  plane-sets do not intersect  $Q$ . Since  $|\Gamma| \geq m/2$ , at least  $(t-1)m/2t$  plane-sets do not intersect  $Q$ .

Let  $(\Delta_i, \Delta_j)$  be a pair such that its plane-set does not intersect  $Q$ . Let  $Q'$  be the projection of  $Q$  on the plane  $Z_0$ . Because the plane-set of this pair does not intersect  $Q$ , we know that  $Q'$  intersects neither the boundary of  $\Delta_i$  nor that of  $\Delta_j$ . Two cases arise:

**Case 1.** If either  $\Delta_i$  or  $\Delta_j$  does not intersect  $Q'$ , then we know that their apex function is redundant and we can drop the constraint associated with this apexed triangle.

**Case 2.** If  $Q' \subset \Delta_i \cap \Delta_j$ , then we need to decide which constraint to drop. To this end, we consider the separating plane  $\gamma_{i,j}$ . Notice that inside the vertical extrusion of  $\Delta_i \cap \Delta_j$  (and hence in  $Q$ ), the plane  $\gamma_{i,j}$  has the property that if we know its side containing the solution, then one of the constraints can be dropped. Since  $\gamma_{i,j}$  does not intersect  $Q$  as  $\gamma_{i,j}$

belongs to the plane-set of  $(\Delta_i, \Delta_j)$ , we can decide which side of  $\gamma_{i,j}$  contains the solution to (P2) and drop one of the constraints.

Regardless of the case if the plane-set of a pair  $(\Delta_i, \Delta_j)$  does not intersect  $Q$ , then we can drop one of its constraints. Since at least  $(t-1)m/2t$  plane-sets do not intersect  $Q$ , we can drop at least  $(t-1)m/2t$  constraints. By choosing  $t=2$ , we are able to drop at least  $(t-1)m/2t = m/4$  constraints. Consequently, after  $O(m)$  time, we are able to drop  $m/4$  apexed triangles. By repeating this process recursively, we end up with a constant size problem in which we can compute the upper envelope of the functions explicitly and find the solution to (P2) using exhaustive search. Thus, the running time of this algorithm is bounded by the recurrence  $T(m) = T(3m/4) + O(m)$  which solves to  $O(m)$ . Because  $m = O(n)$ , we can find the solution to (P2) in  $O(n)$  time.

The last detail is the implementation of the side-decision algorithm. Given a plane  $\gamma$ , we want to decide in which side lies the solution to (P2). To this end, we solve (P2) restricted to  $\gamma$ , i.e., with the additional constraint of  $(x, r) \in \gamma$ . This approach was used by Megiddo [21], the idea is to recurse by reducing the dimension of the problem. Another approach is to use a slight modification of the chord-oracle described by Pollack et al. [25, Section 3].

Once the solution to (P2) restricted to  $\gamma$  is known, we can follow the same idea used by Megiddo [21] to find the side of  $\gamma$  containing the global solution to (P2). Intuitively, we find the apex functions that define the minimum restricted to  $\gamma$ . Since  $\phi(x) = F_P(x)$  is locally defined by this functions, we can decide in which side the minimum lies using convexity. We obtain the following result.

► **Lemma 17.** *Let  $R'$  be a convex trapezoid contained in  $P$  such that  $R'$  contains the geodesic center of  $P$ . Given the set of all apexed triangles of  $\tau$  that intersect  $R'$ , we can compute the geodesic center of  $P$  in  $O(n)$  time.*

The following theorem summarizes the result presented in this paper.

► **Theorem 18.** *We can compute the geodesic center of any simple polygon  $P$  of  $n$  vertices in  $O(n)$  time.*

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## 654 **A Bounding VC dimension**

655 In this section, we provide the proof of Lemma 15. That is, we want to prove that the set  
656 system  $(\mathcal{C}, \varphi)$  has constant VC-dimension. Recall that  $\mathcal{C}$  is a set of chords of  $P$  and  $\varphi$  is the  
657 set of all open 4-cells of  $P$ .

658 Let  $A \subseteq \mathcal{C}$  be a subset of chords. We say that  $A$  is *shattered* by  $\varphi$  if each of the subsets  
659 of  $A$  can be obtained as the intersection of some  $S \in \varphi$  with  $A$ , i.e., if for each  $\sigma \subseteq A$ , there  
660 exists  $S \in \varphi$  such that  $\sigma = S \cap A$ . The *VC-dimension* of  $(\mathcal{C}, \varphi)$  is the supremum of the sizes  
661 of all finite shattered subsets of  $\mathcal{C}$ .

Let  $\mathcal{H} = \{H : H \text{ is a half-polygon of } P\}$ . Because each 4-cell of  $P$  is the intersection of at most four half-polygons of  $P$ , the following result is a direct consequence of Proposition 10.3.3 of [19, Chapter 10].

► **Lemma 19.** *If  $(\mathcal{C}, \mathcal{H})$  has VC-dimension  $d$ , then  $(\mathcal{C}, \varphi)$  has VC-dimension  $O(d)$ .*

Let  $\kappa$  be a sufficiently large constant to be determined later. Note that if no subset of  $\mathcal{C}$  of size  $\kappa$  is shattered by  $\mathcal{H}$ , then the VC-dimension of  $(\mathcal{C}, \mathcal{H})$  is at most  $\kappa$ .

Let  $A$  be an arbitrary subset of  $\mathcal{C}$  such that  $|A| = \kappa$ . By Lemma 19, it suffices to show that  $A$  is not shattered by  $\mathcal{H}$  to bound the VC-dimension of  $(\mathcal{C}, \mathcal{H})$  and hence of  $(\mathcal{C}, \varphi)$ . We spend the remainder of this section proving that  $A$  is not shattered by  $\mathcal{H}$ .

For any  $B \subseteq A$ , notice that  $B$  partitions  $P$  into sub-polygons.

► **Lemma 20.** *Let  $B \subseteq A$ . If there is a 6-cell  $\sigma$  among the sub-polygons in which  $B$  partitions  $P$ , then  $A$  is not shattered by  $\mathcal{H}$ .*

**Proof.** Let  $C_1, \dots, C_6$  be the chords supporting the six half-polygons whose intersection defines  $\sigma$ . Assume that  $C_1, \dots, C_6$  appear in these order when walking clockwise along the boundary of  $\sigma$ . Note that any half-polygon that intersects  $C_1, C_3$  and  $C_5$  must intersect either  $C_2, C_4$  or  $C_6$ . Therefore, the set  $\{C_1, C_3, C_5\}$  cannot be obtained as the intersection of some half-polygon  $H \in \mathcal{H}$  with  $H$ . Because  $\{C_1, C_3, C_5\}$  is also a subset of  $A$ , we conclude that  $A$  is not shattered by  $\mathcal{H}$ . ◀

Given two chords  $C_1$  and  $C_2$  of  $A$ ,  $C_1$  is *separated* from  $C_2$  if there exists a chord  $S \in A$  such that  $C_1$  and  $C_2$  lie on different half-polygons supported by  $S$ .

Note that if  $A$  contains two chords  $C_1$  and  $C_2$  that are separated, then the subset  $\{C_1, C_2\}$  cannot be obtained as the intersection of a half-polygon  $H \in \mathcal{H}$  with  $A$ , i.e.,  $A$  is not shattered by  $\mathcal{H}$ . Therefore, we assume from now on that no two chords of  $A$  are separated.

Let  $G$  be the intersection graph of  $A$ , i.e., the graph with vertex set  $A$  and an edge between two chords if they intersect. An Erdős-Szekeres type result from Harborth and Möller [13] shows that every arrangement of nine pseudo-lines defines a hexagonal face. Thus, if  $G$  has a clique of size nine, then that subset of chords is a set of pseudo-lines and splits  $P$  into sub-polygons, one of which is a 6-cell. Therefore, if  $G$  has a clique of size nine, then Lemma 20 implies that  $A$  is not shattered by  $\mathcal{H}$ . Consequently, we assume from now on that  $G$  has no clique of size nine.

Turán's Theorem states that if  $G$  has no clique of size nine, then it has at most  $(7/16)\kappa^2$  edges. Let  $C_1$  be the chord in  $A$  with the smallest degree in  $G$ . Notice that  $C_1$  has degree at most  $7\kappa/16$ , as otherwise  $G$  would have more than  $(7/16)\kappa^2$  edges. Therefore, there are at least  $9\kappa/16$  chords of  $A$  do not intersect  $C_1$ . Consider the two half-polygons supported by  $C_1$  and note that one of them, say  $P'$ , contains at least  $9\kappa/32$  chords of  $A$  that do not intersect  $C_1$ . Let  $A'$  be the set containing these chords.

Let  $G'$  be subgraph of  $G$  induced by  $A'$  and let  $C_2$  be the chord of  $A'$  with smallest degree in  $G'$ . Because  $G'$  has no clique of size nine, we infer that  $C_2$  has degree at most  $7|A'|/16$ . Thus, there is a set  $A''$  of at least  $9|A'|/16$  chords of  $A'$  that do not intersect  $C_2$  (nor  $C_1$ ). Because no two chords of  $A$  are separated, all chords in  $A''$  are contained in the half-polygon supported by  $C_2$  that contains  $C_1$ . Otherwise, these chords are separated from  $C_1$  by  $C_2$ .

Let  $G''$  be the subgraph of  $G$  induced by  $A''$ . By repeating the above procedure recursively on  $G''$  and  $A''$  four times more, we can obtain a set  $C_1, \dots, C_6$  such that for each  $1 \leq i \leq 6$ ,  $C_1, \dots, C_{i-1}$  are contained in the same half-polygon supported by  $C_i$ . Consequently, the set  $\{C_1, \dots, C_6\} \subseteq A$  partitions  $P$  into sub-polygons one of which is a 6-cell bounded by the chords  $C_1, \dots, C_6$ .

Note that the above construction can be applied as long as

$$\left(\frac{9}{32}\right) \left(\frac{9}{16}\right)^4 \kappa \geq 1.$$

708 That is, as long as  $|A| \geq 36$  we can always find a subset  $B \subseteq A$  such that there is a 6-cell  
 709 among the sub-polygons in which  $B$  partitions  $P$ . Therefore, as long as  $|A| \geq 36$ ,  $A$  is not  
 710 shattered by  $\mathcal{H}$ , i.e., the set system  $(\mathcal{C}, \mathcal{H})$  has VC-dimension at most 36. By Lemma 19, we  
 711 obtain the following result.

712 ► **Lemma 15.** *The set system  $(\mathcal{C}, \varphi)$  has constant VC-dimension.*