Triangulating with many tetrahedra

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Abstract

Given a set of n points in convex position in $\mathbb{R}^3$, we show that there exists a triangulation of this point set that consists of $\Omega(n \log \log n)$ tetrahedra.

1 Outline

2 Preliminaries

For any given integer k, let $Z_k = \{(x, y, z) \in \mathbb{R}^3 : z = k\}$. Let $P = P_0 \cup P_1$ be a set of $2n$ points in $\mathbb{R}^3$ such that $|P_0| = |P_1|$ and $P_i \subset Z_i$ for $i \in \{0, 1\}$. Assume for ease of description that P is in general position: no three points lie on a line and no two parallel lines contain more than one point.

Let S^k denote the unit sphere in dimension $k+1$. Given a vector $\vec{u} \in S^1$, its rotation matrix is defined as follows:

$$M_{\vec{u}} = \begin{bmatrix} \vec{u}_x & -\vec{u}_y \\ \vec{u}_y & \vec{u}_x \end{bmatrix},$$

where $\vec{u}_x$ and $\vec{u}_y$ denote the x- and y-coordinates of $\vec{u}$.

Recall that the surface of the unit torus, $\mathcal{T}$, is equivalent to $\mathcal{T}$, i.e., for each $w \in \mathcal{T}$, $w = (\vec{v}, \vec{u})$ where $\vec{v}, \vec{u} \in S^1$.

Given a point $w = (\vec{v}, \vec{u})$ on $\mathcal{T}$, we use it to define a partial order on the points of P_i. To this end, let $\vec{v}_{\vec{u}} = M_{\vec{u}}(\vec{v})$ be the vector $\vec{v}$ rotated by the angle given by $\vec{u}$. For each point $p \in P_i$, let $\kappa_w(p) = \{p + \alpha\vec{v} + \beta\vec{v}_{\vec{u}} \in Z_i : \alpha > 0, \beta \geq 0\}$ be the cone contained in Z_i apexed at p. Note that since α is always greater than zero, the cone does not contain the ray shooting from p in the direction of $\vec{u}$; see Figure 1. (If $\vec{v} = -\vec{u}$, then $\kappa_w(p)$ is the halfplane passing through p with normal vector $M_{(0,-1)}(\vec{v})$, minus the ray shooting from p in the direction of $\vec{u}$).

Let $(P_i, \leq_w)$ be a partial order set (poset) defined as follows: Given two points $p, q \in P_i$, we say that $p \leq_w q$ if and only if $q \in \kappa_w(p)$.

We say that two partial orders $(P_i, \leq')$ and $(P_i, \leq'')$ are *complementary* if every two points of P_i are comparable in $(P_i, \leq') \cup (P_i, \leq'')$ and only symmetric pairs (of the form (p, p)) are comparable in $(P_i, \leq') \cap (P_i, \leq'')$.

Lemma 2.1. *The posets $(P_i, \leq_w)$ and $(P_i, \leq_{-w})$ are complementary.*

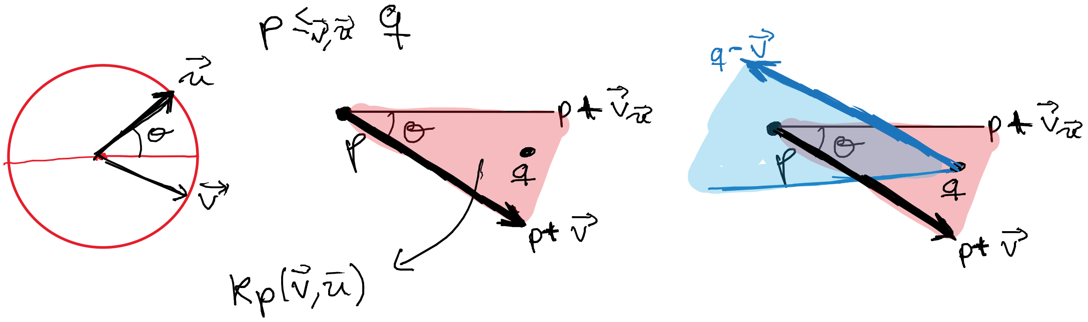

Figure 1:

Proof. Let p and q be two distinct points in P_i. Assume without loss of generality that $\vec{v} = (1, 0)$ and that p lies below q in P_i.

Because q lies above p, and since the union of $\kappa_w(p)$ and $\kappa_{-w}(p)$ contains all points with y-coordinate larger than p in P_i, q lies either in $\kappa_w(p)$ or in $\kappa_{-w}(p)$. This implies that either $p \leq_w q$ or $p \leq_{-w} q$, but not both. Since p and q are arbitrary points, we conclude that any two points are either comparable in $(P_i, \leq_w)$ or in $(P_i, \leq_{-w})$. That is, $(P_i, \leq_w)$ and $(P_i, \leq_{-w})$ are complementary. $\square$

2.1 Triangulation using chains and antichains

Let w be a point on $\mathcal{T}$. Let $C = \{p_1, \ldots, p_k\}$ be a chain of $(P_0, \leq_w)$ and let $A = \{q_1, \ldots, q_t\}$ be an antichain of $(P_1, \leq_w)$. Recall that an antichain in $(P_1, \leq_w)$ is a chain in $(P_1, \leq_{-w})$. Assume without loss of generality that the elements of C are sorted according to $\leq_w$ and that the elements of A are sorted according to $\leq_{-w}$. In this section, we show how to construct a triangulation of $C \cup A$ that has at least $|C| \cdot |A|$ tetrahedra.

To this end, we first construct a set of $|C| \cdot |A|$ tetrahedra that is a triangulation of a non necessarily convex polyhedra contained in $\text{CH}(C \cup A)$. We then show how to extend this to a triangulation of $\text{CH}(C \cup A)$ by gluing more tetrahedra.

Let $\varphi_0 = \cup\{p_j p_{j+1} : 1 \leq j < k\}$ be the curve contained in Z_0, obtained by the union of the segments connecting consecutive elements of A. Define $\varphi_1 = \cup\{q_i q_{i+1} : 1 \leq i < t\}$ analogously.

For each $x \in \varphi_0$ and each $1 \leq i < t$, let $\triangle_i(x) = \triangle(x, q_i, q_{i+1})$. Let $\Pi_i(x)$ be the plane extending $\triangle_i(x)$ and let $\Pi_i^+(x)$ be the closed halfspace supported by $\Pi_i(x)$ that contains p_k.

Lemma 2.2. *For each $x, y \in \varphi_0$ such that $x \leq_w y \leq_w p_k$, it holds that $y \in \Pi_i^+(x)$.*

Proof. Let λ be the line passing through q_i and q_{i+1} and let λ_x be the line parallel to λ passing through x. Thus, $\lambda_x \subset \Pi_i(x)$.

Because q_i and q_{i+1} are not comparable in $(P_1, \leq_w)$, both $x + \vec{v}$ and $x + \vec{v}_{\vec{u}}$ lie on the same halfplane supported by λ_x in Z_0, i.e., the cone $\kappa_w(x)$ is contained in this halfplane.

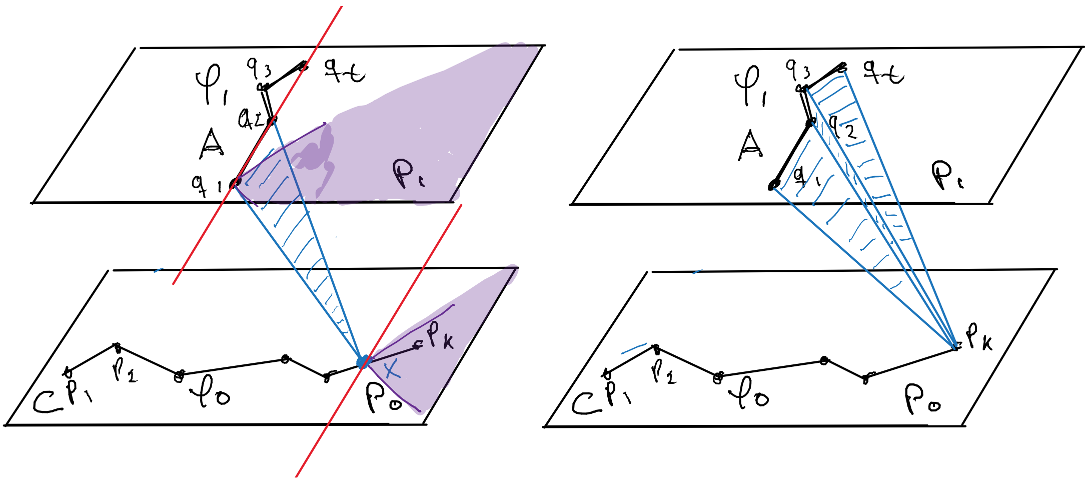

Figure 2:

Because $p_k, y \in \kappa_w(x)$ as $x \leq_w y \leq_w p_k$, and since $\Pi_i(x)$ contains λ_x, we conclude that $y \in \kappa_w(x) \subset \Pi_i^+(x)$. $\square$

Note that for any two points $x, y \in \varphi_0$ such that y lies in the path contained in φ_0 connecting x with p_k, it holds that $x \leq_w, y \leq_w p_k$.

Let $S_x = \cup\{\triangle_i(x) : 1 \leq i < t\}$ be the *wavefront* at x. Because all points in A are coplanar, S_x is a surface topologically equivalent to a disk; see Figure 2.

Lemma 2.3. *For each $x, y \in \varphi_0$ such that y lies in the path contained in φ_0 connecting x with p_k, it holds that $S_x \cap S_y = \varphi_1$.*

Proof. Assume for a contradiction that there is a point $z \in \varphi_1$ such that the open segment zy intersects S_x, i.e., S_x and S_y intersect. Let $\triangle_i(x)$ be the first triangle intersected by zy when going from z towards y. Because the segment zy crosses the plane $\Pi_i(x)$, we conclude that zy is not contained in $\Pi_i^+(x)$.

Assume that z belongs to the segment $q_j q_{j+1}$ for some $1 \leq j \leq t$. Thus, z lies on the boundary of $\triangle_j(x)$. Let $R = \Pi_i^+(x) \cap \Pi_j^+(x)$ and note that R is a convex set. Because $y \in R$ by Lemma 2.2 and since $z \in R$, we know that $zy \subset R$. Since $R \subset \Pi_i^+(x)$, we conclude that $zy \subset \Pi_i^+(x)$—a contradiction. Therefore, for each $z \in \varphi_1$, the open segment zy does not intersect S_x. Consequently, $S_x \cap S_y = \varphi_1$. $\square$

To construct the triangulation, imagining moving x continuously from p_1 to p_k along φ_0 and taking the union of all wavefronts S_x. Let T' be a set of $k \cdot t$ tetrahedra such that

$$T' = \{\text{CH}(p_i, p_{i+1}, q_j, q_{j+1}) : 1 \leq i < k, 1 \leq j < t\}.$$

Lemma 2.4. *The set T' consists of interior disjoint tetrahedra whose union defines a simple polyhedron.*

Proof. Let $\sigma_{i,j} = \text{CH}(p_i, p_{i+1}, q_j, q_{j+1})$. Assume for a contradiction that the interiors of $\sigma_{i,j}$ and $\sigma_{s,r}$ intersect for some $i, s \in \{1, \ldots, k\}$ and $j, r \in \{1, \ldots, t\}$. Therefore, the boundaries of $\sigma_{i,j}$ and $\sigma_{s,r}$ intersect at a point not on φ_1. That is, a triangle $\triangle_j(x)$ intersects a segment wy where $x \in p_i p_{i+1}, y \in p_s p_{s+1}$ and $w \in q_r q_{r+1}$. Since $\triangle_j(x) \subset S_x$ and since $wy \subset S_y$, this implies that the wavefronts S_x and S_y intersect—a contradiction with Lemma 2.3. Therefore, T' consists of interior disjoint tetrahedra.

To see that $\cup T'$ is a simple polyhedra, recall that S_x is topologically equivalent to a disk for each $x \in \varphi_0$. Notice that $\cup T' = \cup\{S_x : x \in \varphi_0\}$. Because each S_x is disjoint, $\cup T'$ is topologically equivalent to a sphere, i.e., it is a simple polyhedra. $\square$

By Lemmas 2.4, we conclude that T' is a triangulation of a simple polyhedron, say $\cup T'$, whose facets are triangles.

Our objective is to complete T' to a triangulation T of $\text{CH}(C \cup A)$. To this end, we proceed as follows. We say that two points are *T-visible* if the open segment joining them does not intersect the interior of $\cup T$. Let τ_i be a triangulation of P_i in Z_i such that γ_i is contained in the union of the edges of τ_i. Notice that for each point $z \in \text{CH}(C)$, either q_1 or q_t is T-visible from z.

In fact, the curve γ_i splits $\text{CH}(C)$ into two sets, the one containing the points visible from q_1 and the other containing those points visible from q_t. This partition of $\text{CH}(C)$ induces a partition of τ_1 into two sets of the triangles, those visible from q_1 and those visible form q_t. For each triangle $\triangle \in \tau_1$ visible from q_1 (*resp.* q_t), we add the tetrahedron $\text{CH}(\triangle, q_1)$ (*resp.* $\text{CH}(\triangle, q_t)$) to T'.

In the same way, the triangulation τ_0 can be partitioned into two sets of triangles, those visible from p_1 and those visible from p_k. We construct tetrahedra analogously for them and add them to T'. In this manner, we obtain a new triangulation T whose union is equal to $\text{CH}(C \cup A)$; see Figure 3 for an illustration. Since the triangulation T' has at least $|C| \cdot |A|$ tetrahedra and $T' \subseteq T$, T consists of at least $|C| \cdot |A|$ tetrahedra.

The following theorem summarizes the results presented in this section.

Theorem 2.5. *Give a chain C of $(P_0, \leq_w)$ and an antichain A of $(P_1, \leq_{-w})$, there exists a triangulation of $\text{CH}(C \cup A)$ that has at least $|C| \cdot |A|$ tetrahedra.*

2.2 Large chains and antichains

For each w on $\mathcal{T}$, Dilworth's theorem implies the existence of a chain C_w^i and an antichain A_w^i in $(P_i, \leq_w)$ such that $|C_w^i| \cdot |A_w^i| \geq |P_i| = n$. Assume that C_w^i and A_w^i are the lexicographically minimum subsets of P_i with this property.

Let $x(w) = |C_w^0| - |A_w^1|$ and $y(w) = |C_w^1| - |A_w^0|$.

Lemma 2.6. *For each $w \in \mathcal{T}$, it holds that $x(w) = -x(-w)$.*

Proof. By Lemma 2.1 $(P_i, \leq_w)$ and $(P_i, \leq_{-w})$ are complementary. This implies that any chain in $(P_i, \leq_w)$ is an antichain in $(P_i, \leq_{-w})$ and viceversa. Therefore, the value of $x(w) = -x(-w)$.

$\square$

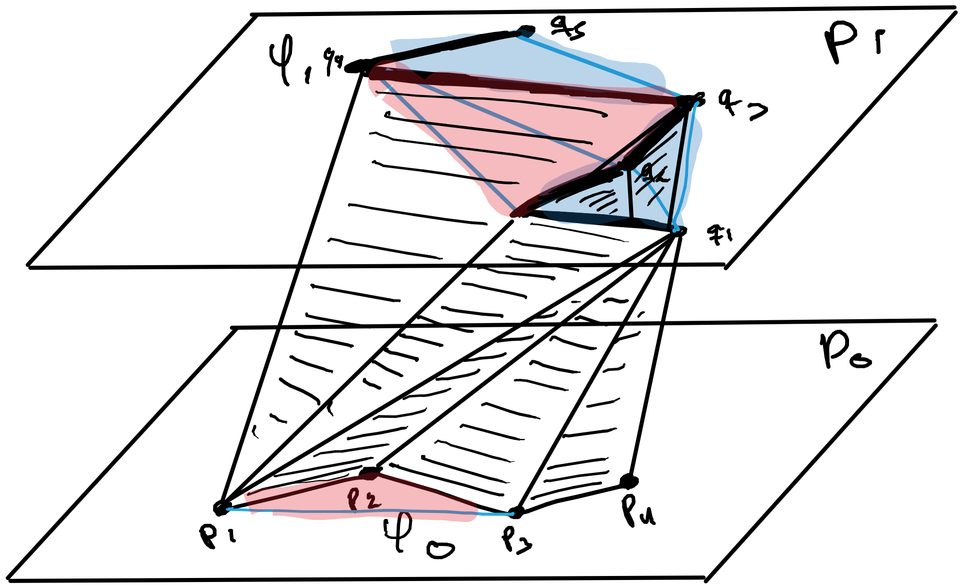

Figure 3:

We say that a function $g : \mathbb{R}^m \to \mathbb{R}$ is *almost continuous* if for any converging sequence $\{x_i\}_{i\in\mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that for any $i, j > n_0$, $|g(x_i) - g(x_j)| \leq 1$.

Lemma 2.7. *The functions x and y are almost continuous.*

Proof. Recall that $w = (\vec{v}, \vec{u})$, where $\vec{v}, \vec{u} \in S^1$. If we fix the value of $\vec{u}$, then by modifying the value of $\vec{v}$, we obtain changes in $x(w)$ only when a point stops (or starts) being inside a cone $\kappa_w(p)$ for some $p \in P_i$ as it rotates around it. By the general position assumption of P_i, this change can increase or decrease the size of a chain (or antichain) by at most one. Therefore, for a sufficiently small change in $\vec{v}$, the value of $x(w)$ changes by at most one.

If we fix the value of $\vec{v}$, then by modifying the value of $\vec{u}$, we change the aperture of the $\kappa_w(p)$. In this case however, we have to be careful when $\vec{u}$ reaches $(\cos(\pi), \sin(\pi))$. Notice that the cone will point upwards for $(\cos(\pi - \varepsilon), \sin(\pi - \varepsilon))$ and downwards for $(\cos(\pi + \varepsilon), \sin(\pi + \varepsilon))$. However, as ε approaches zero, all the points will be comparable except for those lying in lines parallel to $\vec{v}$. By our general position assumption, at most one line parallel to $\vec{v}$ contains two points. Therefore, $x(w)$ is equal to either n or $n - 1$ when ε reaches zero. An analogous argument shows that for a fixed value of $\vec{u}$, any sufficiently small change in $\vec{v}$ produces a change in $x(w)$ of at most one.

Consequently, for any converging sequence $\{w_i\}_{i\in\mathbb{N}}$ in $\mathcal{T}$, there exists n_0 such that for any $i, j > n_0$, $|x(w_i) - x(w_j)| \leq 1$.

The same argument holds for the function y. □

Lemma 2.8. *There exist continuous functions $g_x, g_y : \mathcal{T} \to \mathbb{R}$ such that for any $w \in \mathcal{T}$, it holds that $|x(w) - g_x(w)| \leq 1$ and $|y(w) - g_y(w)| \leq 1$.*

Proof. Because discontinuities in x arise only when the boundary of a cone $\kappa_w(p)$ passes through a second point, function x has a set of discontinuities of measure zero. Therefore, x is a Baire class one-function [**?**], i.e., it can be defined as the pointwise limit of continuous functions. Thus, there exist a function $g_x : \mathcal{T} \to \mathbb{R}$ such that $|x(w) - g_x(w)| \leq 1$ for all $w \in \mathcal{T}$. An analogous proof applies for y. □

Lemma 2.9. *There exists $w^* \in \mathcal{T}$ and an integer $i \in \{0, 1\}$ such that $|C^i_{w^*}| \cdot |A^{1-i}_{w^*}| \geq n$, and both $C^i_{w^*}$ and $A^{1-i}_{w^*}$ have size at least $\sqrt{n} - 1$.*

Proof. Because g_x is a continuous function from the torus $\mathcal{T}$ to $\mathbb{R}$ by Lemma 2.8, the Borsuk-Ulam theorem [**?**] implies the existence of a point w^* such that $g_x(w^*) = g_x(-w^*)$, i.e., two antipodal points with the same value. Thus, Lemma 2.8 implies that $|x(w^*) - x(-w^*)| \leq 2$. Because $x(w^*) = -x(-w^*)$ by Lemma 2.6, we get that $|x(w^*) + x(w^*)| \leq 2$, which implies that $|x(w^*)| \leq 1$. Consequently $-1 \leq x(w^*) = |C^0_{w^*}| - |A^1_{w^*}| \leq 1$, i.e., the sizes of $C^0_{w^*}$ and $A^1_{w^*}$ differ at most by one. If $\max\{|C^0_{w^*}|, |A^1_{w^*}|)\} \geq \sqrt{n}$, then our result follows. Otherwise, we can assume that

$$\max\{|C^0_{w^*}|, |A^1_{w^*}|)\} < \sqrt{n}. \tag{1}$$

Note that by definition $|C^i_{w^*}| \cdot |A^i_{w^*}| \geq n$ for $i \in \{0, 1\}$. Therefore, by (1) we know that $|C^1_{w^*}| > \sqrt{n}$ and $|A^0_{w^*}| > \sqrt{n}$. Moreover, we have

$$(|C^0_{w^*}| \cdot |A^1_{w^*}|)(|C^1_{w^*}| \cdot |A^0_{w^*}|) \geq n^2.$$

Because $\max\{|C^0_{w^*}|, |A^1_{w^*}|)\} < \sqrt{n}$ by (1), $|C^0_{w^*}| \cdot |A^1_{w^*}| < n$. Consequently, $|C^1_{w^*}| \cdot |A^0_{w^*}| > n$ proving our result. □

Corollary 2.10. *There exists a set S of $2\sqrt{n} - 2$ points of P such that $\mathrm{CH}(S)$ has a triangulation with $\Theta(n)$ tetrahedra.*

2.3 Constructing complete triangulations

Let S be the set of $2\sqrt{n} - 2$ points of P given by Corollary 2.10 having a triangulation with $\Theta(n)$ tetrahedra. To extend this triangulation of S to a triangulation of P, we proceed recursively. Let $\Sigma = \{\sigma_1, \dots, \sigma_k\}$ be the set of planes that extend all the facets of $\mathrm{CH}(S)$. Because $|S| = \Theta(\sqrt{n})$, we know that $k = O(\sqrt{n})$ (it could be that many triangles are coplanar in either P_0 or P_1). We construct k sets $P^1, \dots P^k$ as follows: Let P^i be the set of remaining points of P that lie on the halfspace supported by σ_i that does not contain $\mathrm{CH}(S)$. Remove P^i from P and continue the proceed recursively with $i+1$. As a *separation invariant*, note that σ_i separates P_i from S and from the remaining points of P.

Since the convex hulls of the P^is are pairwise disjoint, we can proceed recursively and obtain a triangulation of their convex hull. To complete the process, we merge the convex hull S with the convex hulls of the P^is in reverse order. That is, we first merge $\textsc{ch}(S)$ with $\textsc{ch}(P^k)$, then the resulting convex hull with $\textsc{ch}(P_{k-1})$ and so on. We claim that this merge is always possible.

To prove this claim, let $S_0 = S$, and let $S_j = S \cup (\cup_{i=0}^{j-1} P^{k-i})$ for each $1 \leq j \leq k$. Notice that by the separation invariant, for each $1 \leq j \leq k$, the plane σ_j separates $\textsc{ch}(S_{j-1})$ from $\textsc{ch}(P_j)$. Therefore, for each $1 \leq i < j \leq k$, $\textsc{ch}(P_i)$ cannot intersect $\textsc{ch}(S_j)$. Otherwise, σ_i does not separate $\textsc{ch}(P_i)$ from $\textsc{ch}(S_{i-1})$. Therefore, the recursive process ends with a triangulation of $\textsc{ch}(S_k) = \textsc{ch}(P)$.

It remains only to analyze the size of this triangulation. To this end, recall that the triangulation of S contains $\Theta(n)$ triangles. Moreover, since $k = O(n)$, at least one of the P^is has $\Omega(\sqrt{n})$ points. We obtain the following recurrence $T(n) = \sum_{i=1}^{k} T(n_i) + \Omega(n)$, where $k = O(\sqrt{n})$ (and hence some $n_i = \Omega(\sqrt{n})$). By induction, we can show that $T(n) = \Omega(n \log\log n)$.