Triangulating with many tetrahedra

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Abstract

For any given integer k, let $Z_k = \{(x, y, z) \in \mathbb{R}^3 : z = k\}$. Given a set P consisting of n points in Z_0 and n points in X_1, we show that there exists a triangulation of P that consists of $\Omega(n \log \log n)$ tetrahedra.

1 Introduction

1.1 Outline

2 Preliminaries

For any given integer k, let $Z_k = \{(x, y, z) \in \mathbb{R}^3 : z = k\}$. Let $P = P_0 \cup P_1$ be a set of $2n$ points in $\mathbb{R}^3$ such that $|P_0| = |P_1|$ and $P_i \subset Z_i$ for $i \in \{0, 1\}$. Throughout, we assume that P is in general position: no three points lie on a line, no two points have the same y-coordinate and no two parallel lines contain more than one point.

Given an angle $-\pi \leq \phi \leq \pi$, we use it to define a partial order on the points of P_i. To this end, let $\vec{v} = (1, 0)$ and let $\vec{v}_\phi = (\cos \phi, \sin \phi)$ be the vector that makes an angle ϕ with the positive x-axis. For each point $p \in P_i$, let $\kappa_\phi(p) = \{p + \alpha\vec{v} + \beta\vec{v}_\phi \in Z_i : \alpha > 0, \beta \geq 0\}$ be the cone contained in Z_i apexed at p (if $\phi = \pi$, $\kappa_\phi(p)$ is the halfplane containing all the points with y-coordinate larger or equal to x). Let $(P_i, \leq_\phi)$ be a partial order set (poset) defined as follows: Given two points $p, q \in P_i$, we say that $p \leq_\phi q$ if and only if $q \in \kappa_\phi(p)$; see Figure 1 for an illustration.

We say that two partial orders $(P_i, \leq')$ and $(P_i, \leq'')$ are *complementary* if every two points of P_i are comparable in $(P_i, \leq') \cup (P_i, \leq'')$ and only symmetric pairs (of the form (p, p)) are comparable in $(P_i, \leq') \cap (P_i, \leq'')$.

For ease of notation, given $0 \leq \theta \leq \pi$, we let $\overline{\theta} = \theta - \pi$.

Lemma 2.1. *Given $0 \leq \theta \leq \pi$, the posets $(P_i, \leq_\theta)$ and $(P_i, \leq_{\overline{\theta}})$ are complementary.*

Proof. Let p and q be two distinct points in P_i. Note that the union of $\kappa_\theta(p)$ and $\kappa_{\overline{\theta}}(p)$ defines a halfplane passing through p whose boundary makes an angle of θ with the positive x-axis. Moreover, since no two points share the same y-coordinate by the general position assumption, no point of P_i other than p lies in $\kappa_\theta(p) \cap \kappa_{\overline{\theta}}(p)$. That is only symmetric pairs are comparable in $(P_i, \leq_\theta) \cap (P_i, \leq_{\overline{\theta}})$.

Assume without loss of generality that $q \in \kappa_\theta(p) \cup \kappa_{\overline{\theta}}(p)$, otherwise invert the roles of p and q. Therefore, q lies either in $\kappa_\theta(p)$ or in $\kappa_{\overline{\theta}}(p)$. This implies that either $p \leq_\theta q$

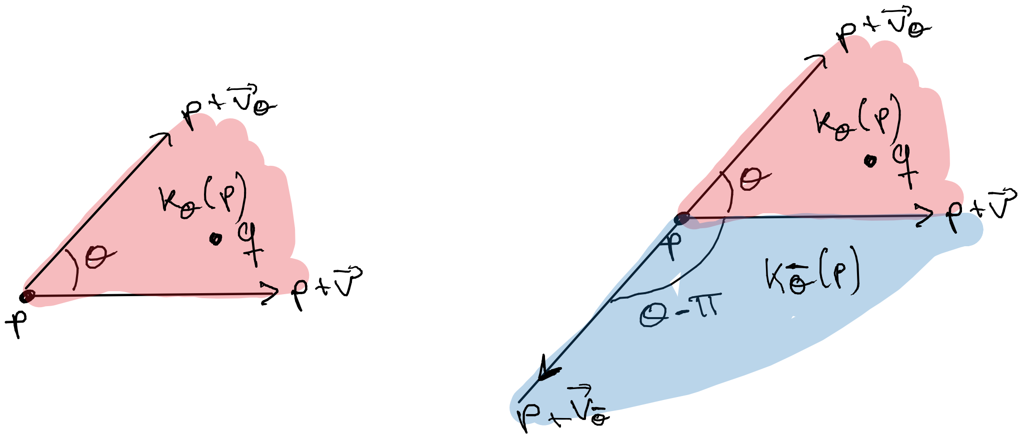

Figure 1:

or $p \leq_{\bar{\theta}} q$, but not both. Since p and q are arbitrary points, we conclude that any two points are either comparable in $(P_i, \leq_\theta)$ or in $(P_i, \leq_{\bar{\theta}})$. That is, $(P_i, \leq_\theta)$ and $(P_i, \leq_{\bar{\theta}})$ are complementary; see Figure 1. $\square$

3 Triangulation using chains and antichains

Let $0 \leq \theta \leq \pi$ be an arbitrary angle. Let $C = \{p_1, \ldots, p_k\}$ be a chain of $(P_0, \leq_\theta)$ and let $A = \{q_1, \ldots, q_t\}$ be an antichain of $(P_1, \leq_\theta)$. Recall that an antichain in $(P_1, \leq_\theta)$ is a chain in $(P_1, \leq_{\bar{\theta}})$ by Lemma 2.1. Assume without loss of generality that the elements of C are sorted according to $\leq_\theta$ and that the elements of A are sorted according to $\leq_{\bar{\theta}}$. In this section, we show how to construct a triangulation of $C \cup A$ that has at least $|C| \cdot |A|$ tetrahedra.

To this end, we first construct a set of $|C| \cdot |A|$ tetrahedra that is a triangulation of a non necessarily convex polyhedra contained in $\text{CH}(C \cup A)$. We then show how to extend this to a triangulation of $\text{CH}(C \cup A)$ by gluing more tetrahedra.

Let $\varphi_0 = \cup\{p_j p_{j+1} : 1 \leq j < k\}$ be the curve contained in Z_0, obtained by the union of the segments connecting consecutive elements of A. Define $\varphi_1 = \cup\{q_i q_{i+1} : 1 \leq i < t\}$ analogously.

For each $x \in \varphi_0$ and each $1 \leq i < t$, let $\triangle_i(x) = \triangle(x, q_i, q_{i+1})$. Let $\Pi_i(x)$ be the plane extending $\triangle_i(x)$ and let $\Pi_i^+(x)$ be the closed halfspace supported by $\Pi_i(x)$ that contains p_k.

Lemma 3.1. *For each $x, y \in \varphi_0$ such that $x \leq_\theta y \leq_\theta p_k$, it holds that $y \in \Pi_i^+(x)$.*

Proof. Let λ be the line passing through q_i and q_{i+1} and let λ_x be the line parallel to λ passing through x. Thus, $\lambda_x \subset \Pi_i(x)$.

Recall that $\vec{v} = (1, 0)$ and $\vec{v}_\phi = (\cos\theta, \sin\theta)$. Because q_i and q_{i+1} are not comparable in $(P_1, \leq_\theta)$, λ (and λ_x) makes an angle larger than θ with the positive x-axis; otherwise

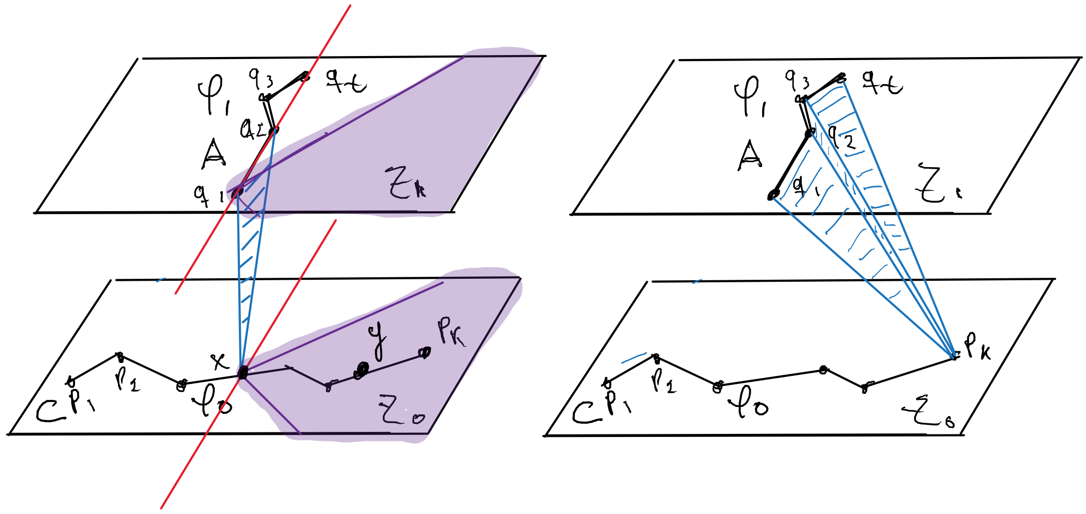

Figure 2:

$q_i \in \kappa_\theta(q_{i+1})$ or $q_{i+1} \in \kappa_\theta(q)$. Therefore, both $x + \vec{v}$ and $x + \vec{v}_\theta$ lie on the same halfplane supported by λ_x in Z_0, i.e., the cone $\kappa_\theta(x)$ is contained in this halfplane. Because $p_k, y \in \kappa_\theta(x)$ as $x \leq_\theta y \leq_\theta p_k$, and since $\Pi_i(x)$ contains λ_x, we conclude that $y \in \kappa_\theta(x) \subset \Pi_i^+(x)$; see Figure 2. $\square$

Note that for any two points $x, y \in \varphi_0$ such that y lies in the path contained in φ_0 connecting x with p_k, it holds that $x \leq_\theta, y \leq_\theta p_k$.

Let $S_x = \cup\{\triangle_i(x) : 1 \leq i < t\}$ be the *wavefront* at x. Because all points in A are coplanar, S_x is a surface topologically equivalent to a disk; see Figure 2.

Lemma 3.2. *For each $x, y \in \varphi_0$ such that y lies in the path contained in φ_0 connecting x with p_k, it holds that $S_x \cap S_y = \varphi_1$.*

Proof. Assume for a contradiction that there is a point $z \in \varphi_1$ such that the open segment zy intersects S_x, i.e., S_x and S_y intersect. Let $\triangle_i(x)$ be the first triangle intersected by zy when going from z towards y. Because the segment zy crosses the plane $\Pi_i(x)$, we conclude that zy is not contained in $\Pi_i^+(x)$.

Assume that z belongs to the segment q_jq_{j+1} for some $1 \leq j \leq t$. Thus, z lies on the boundary of $\triangle_j(x)$. Let $R = \Pi_i^+(x) \cap \Pi_j^+(x)$ and note that R is a convex set. Because $y \in R$ by Lemma 3.1 and since $z \in R$, we know that $zy \subset R$. Since $R \subset \Pi_i^+(x)$, we conclude that $zy \subset \Pi_i^+(x)$—a contradiction. Therefore, for each $z \in \varphi_1$, the open segment zy does not intersect S_x. Consequently, $S_x \cap S_y = \varphi_1$. $\square$

To construct the triangulation, imagining moving x continuously from p_1 to p_k along φ_0 and taking the union of all wavefronts S_x. Let T' be a set of $k \cdot t$ tetrahedra such that

$$T' = \{\text{CH}(p_i, p_{i+1}, q_j, q_{j+1}) : 1 \leq i < k, 1 \leq j < t\}.$$

Lemma 3.3. *The set T' consists of interior disjoint tetrahedra whose union defines a simple polyhedron.*

Proof. Let $\sigma_{i,j} = \text{CH}(p_i, p_{i+1}, q_j, q_{j+1})$. Assume for a contradiction that the interiors of $\sigma_{i,j}$ and $\sigma_{s,r}$ intersect for some $i, s \in \{1, \ldots, k\}$ and $j, r \in \{1, \ldots, t\}$. Therefore, the boundaries of $\sigma_{i,j}$ and $\sigma_{s,r}$ intersect at a point not on φ_1. That is, a triangle $\triangle_j(x)$ intersects a segment wy where $x \in p_ip_{i+1}, y \in p_sp_{s+1}$ and $w \in q_rq_{r+1}$. Since $\triangle_j(x) \subset S_x$ and since $wy \subset S_y$, this implies that the wavefronts S_x and S_y intersect—a contradiction with Lemma 3.2. Therefore, T' consists of interior disjoint tetrahedra.

To see that $\cup T'$ is a simple polyhedra, recall that S_x is topologically equivalent to a disk for each $x \in \varphi_0$. Notice that $\cup T' = \cup\{S_x : x \in \varphi_0\}$. Because each S_x is disjoint, $\cup T'$ is topologically equivalent to a sphere, i.e., it is a simple polyhedra. □

By Lemmas 3.3, we conclude that T' is a triangulation of a simple polyhedron, say $\cup T'$, whose facets are triangles.

Our objective is to complete T' to a triangulation T of $\text{CH}(C \cup A)$. To this end, we proceed as follows. We say that two points are *T-visible* if the open segment joining them does not intersect the interior of $\cup T$. Let τ_i be a triangulation of P_i in Z_i such that γ_i is contained in the union of the edges of τ_i. Notice that for each point $z \in \text{CH}(C)$, either q_1 or q_t is T-visible from z.

In fact, the curve γ_i splits $\text{CH}(C)$ into two sets, the one containing the points visible from q_1 and the other containing those points visible from q_t. This partition of $\text{CH}(C)$ induces a partition of τ_1 into two sets of the triangles, those visible from q_1 and those visible form q_t. For each triangle $\triangle \in \tau_1$ visible from q_1 (*resp.* q_t), we add the tetrahedron $\text{CH}(\triangle, q_1)$ (*resp.* $\text{CH}(\triangle, q_t)$) to T'.

In the same way, the triangulation τ_0 can be partitioned into two sets of triangles, those visible from p_1 and those visible from p_k. We construct tetrahedra analogously for them and add them to T'. In this manner, we obtain a new triangulation T whose union is equal to $\text{CH}(C \cup A)$; see Figure 3 for an illustration. Since the triangulation T' has at least $|C| \cdot |A|$ tetrahedra and $T' \subseteq T$, T consists of at least $|C| \cdot |A|$ tetrahedra.

The following theorem summarizes the results presented in this section.

Theorem 3.4. *Give a chain C of $(P_0, \leq_\theta)$ and an antichain A of $(P_1, \leq_\theta)$, there exists a triangulation of $\text{CH}(C \cup A)$ that has at least $|C| \cdot |A|$ tetrahedra.*

4 Large chains and antichains

For each $0 \leq \theta \leq \pi$, Dilworth's theorem implies the existence of a chain C^i_θ and an antichain A^i_θ of $(P_i, \leq_\theta)$ such that $|C^i_\theta| \cdot |A^i_\theta| \geq |P_i| = n$. Assume that C^i_θ and A^i_θ are the lexicographically minimum subsets of P_i with this property.

Let $x(\theta) = |C^0_\theta| - |A^1_\theta|$ and $y(\theta) = |C^1_\theta| - |A^0_\theta|$.

Lemma 4.1. *For each $0 \leq \theta \leq \pi$, it holds that $x(\theta) = -x(\overline{\theta})$. Moreover, for $x(0) = 1-n$ and $x(\pi) = n - 1$.*

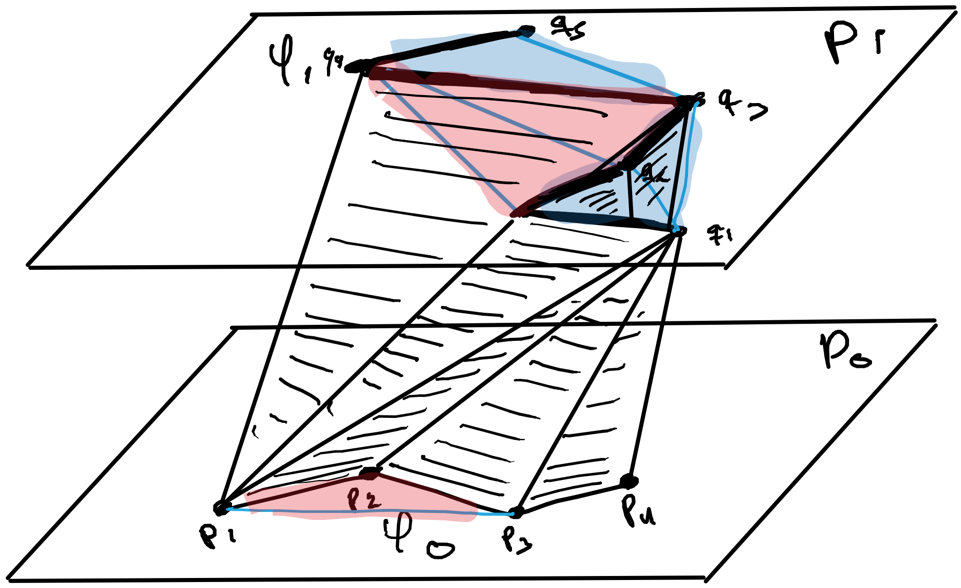

Figure 3:

Proof. By Lemma 2.1 $(P_i, \leq_\theta)$ and $(P_i, \leq_{\overline{\theta}})$ are complementary. This implies that any chain in $(P_i, \leq_\theta)$ is an antichain in $(P_i, \leq_{\overline{\theta}})$ and viceversa. Therefore, $|C^i_\theta| = |A^i_{\overline{\theta}}|$ and $|A^i_\theta| = |C^i_{\overline{\theta}}|$. Consequently, $x(\theta) = -x(\overline{\theta})$.

For the second part, note that if $\theta = \pi$, then $\kappa_\theta(p)$ consists of a halfplane containing all the points with y-coordinate larger or equal than p. Because no two points have the same y-coordinate by our general position assumption, the longest chain in $(P_i, \leq_\theta)$ contains every point of P_i. On the other hand, every chain in $(P_i, \leq_{\overline{\theta}})$ consists of at most one point. Therefore, $x(\pi) = n - 1$. The first part of the lemma implies that, $x(0) = 1 - n$. □

We say that a function $g : [0, \pi] \to \mathbb{R}$ is *almost continuous* if for any converging sequence $\{x_i\}_{i \in \mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that for any $i, j > n_0$, $|g(x_i) - g(x_j)| \leq 2$.

Lemma 4.2. *The functions x and y are almost continuous.*

Proof. By modifying the value of θ, we obtain changes in $x(\theta)$ only when a point stops (or starts) being inside a cone $\kappa_\theta(p)$ for some $p \in P_i$ as it changes its aperture. By the general position assumption of P_i, this change can only increase (*resp.* decrease) the size of a chain (*resp.* antichain) by at most two. Therefore, for a sufficiently small change in θ, the value of $x(\theta)$ changes by at most two. That is, x is almost continuous.

Consequently, for any converging sequence $\{\theta_i\}_{i\in\mathbb{N}}$ in $[0,\pi]$, there exists n_0 such that for any $i,j > n_0$, $|x(\theta_i) - x(\theta_j)| \le 1$. The same argument holds for the function y. □

Lemma 4.3. *There exist continuous functions $g_x, g_y : [0,\pi] \to \mathbb{R}$ such that for any $\theta \in [0,\pi]$, it holds that $|x(\theta) - g_x(\theta)| \le 2$ and $|y(\theta) - g_y(\theta)| \le 1$.*

Proof. Because discontinuities in x arise only when the boundary of a cone $\kappa_\theta(p)$ passes through a second point, function x has a finite set of discontinuities. Because x is almost continuous by Lemma 4.2, each of these discontinuities is a jump of at most two. Thus, if there is a discontinuity at a point θ_0, we can define g_x in the neighborhood $[\theta_0-\varepsilon, \theta_0+\varepsilon]$ as a linear function that connects $x(\theta_0-\varepsilon)$ with $x(\theta_0+\varepsilon)$; see Figure **??**. By repeating this process with every discontinuity of x, and making g_x coincide with x outside of these neighborhoods, we obtain a continuous function $g_x : [0,\pi] \to \mathbb{R}$ such that $|x(\theta) - g_x(\theta)| \le 1$ for all $0 \le \theta \le \pi$. An analogous proof applies for y. □

Lemma 4.4. *There exists $0 \le \theta^* \le \pi$ and an integer $i \in \{0,1\}$ such that $|C^i_{\theta^*}| \cdot |A^{1-i}_{\theta^*}| \ge n$, and both $C^i_{\theta^*}$ and $A^{1-i}_{\theta^*}$ have size at least $\sqrt{n}-1$.*

Proof. Because $g_x(0) = x(0) = 1-n$ and $g_x(\pi) = x(\pi) = n-1$, and since g_x is continuous, there exists a value θ^* such that $g_x(\theta^*) = 0$. Thus, by Lemma 4.3 we infer that $|x(\theta^*)| \le 1$. Consequently

$$-1 \le x(\theta^*) = |C^0_{\theta^*}| - |A^1_{\theta^*}| \le 1.$$

That is, the sizes of $C^0_{\theta^*}$ and $A^1_{\theta^*}$ differ at most by one.

If $\max\{|C^0_{\theta^*}|, |A^1_{\theta^*}|)\} \ge \sqrt{n}$, then our result follows. Otherwise, we can assume that

$$\max\{|C^0_{\theta^*}|, |A^1_{\theta^*}|)\} < \sqrt{n}. \tag{1}$$

Note that by definition $|C^i_{\theta^*}| \cdot |A^i_{\theta^*}| \ge n$ for $i \in \{0,1\}$. Therefore, by (1) we know that $|C^1_{\theta^*}| > \sqrt{n}$ and $|A^0_{\theta^*}| > \sqrt{n}$. Moreover, we have

$$(|C^0_{\theta^*}| \cdot |A^1_{\theta^*}|)(|C^1_{\theta^*}| \cdot |A^0_{\theta^*}|) \ge n^2.$$

Because $\max\{|C^0_{\theta^*}|, |A^1_{\theta^*}|)\} < \sqrt{n}$ by (1), $|C^0_{\theta^*}| \cdot |A^1_{\theta^*}| < n$. Consequently, $|C^1_{\theta^*}| \cdot |A^0_{\theta^*}| > n$ proving our result. □

Corollary 4.5. *There exists a set S of $2\sqrt{n}-2$ points of P such that* $\textsc{ch}(S)$ *has a triangulation with $\Theta(n)$ tetrahedra.*

5 Constructing complete triangulations

Let S be the set of $2\sqrt{n}-2$ points of P given by Corollary 4.5 having a triangulation with $\Theta(n)$ tetrahedra. To extend this triangulation of S to a triangulation of P, we proceed recursively. Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be the set of planes that extend all the facets of $\textsc{ch}(S)$. Because $|S| = \Theta(\sqrt{n})$, we know that $k = O(\sqrt{n})$ (several triangles are coplanar in either

P_0 or P_1). We construct k sets $P^1, \ldots, P^k$ as follows: Let P^i be the set of remaining points of P that lie on the halfspace supported by σ_i that does not contain $\textsc{ch}(S)$. Remove P^i from P and continue the proceed recursively with $i+1$. As a *separation invariant*, note that σ_i separates P_i from S and from the remaining points of P.

Since the convex hulls of the P^is are pairwise disjoint, we can proceed recursively and obtain a triangulation of their convex hull. To complete the process, we merge the convex hull S with the convex hulls of the P^is in reverse order. That is, we first merge $\textsc{ch}(S)$ with $\textsc{ch}(P^k)$, then the resulting convex hull with $\textsc{ch}(P^{k-1})$ and so on. We claim that this merge is always possible.

To prove this claim, let $S_0 = S$, and let $S_j = S \cup (\cup_{i=0}^{j-1} P^{k-i})$ for each $1 \leq j \leq k$. Notice that by the separation invariant, for each $1 \leq j \leq k$, the plane σ_j separates $\textsc{ch}(S_{j-1})$ from $\textsc{ch}(P^j)$. Therefore, for each $1 \leq i < j \leq k$, $\textsc{ch}(P^i)$ cannot intersect $\textsc{ch}(S_j)$. Otherwise, σ_i does not separate $\textsc{ch}(P^i)$ from $\textsc{ch}(S_{i-1})$. Therefore, the recursive process ends with a triangulation of $\textsc{ch}(S_k) = \textsc{ch}(P)$.

It remains only to analyze the size of this triangulation. To this end, recall that the triangulation of S contains $\Theta(n)$ triangles. Moreover, since $k = O(n)$, at least one of the P^is has $\Omega(\sqrt{n})$ points. We obtain the following recurrence $T(n) = \sum_{i=1}^{k} T(n_i) + \Omega(n)$, where $k = O(\sqrt{n})$ and $\sum_{i=1}^{k} n_i = n - \sqrt{n}$ (and hence some $n_i = \Omega(\sqrt{n})$). By induction, we can show that $T(n) = \Omega(n \log\log n)$.

$T(n) = \sum_{i=1}^{k} T(n_i) + \Omega(n)$, where $k = O(\sqrt{n})$ and $\sum_{i=1}^{k} n_i = n - \sqrt{n}$ (and hence some $n_i = \Omega(\sqrt{n})$). I want to show show that $T(n) = \Omega(n \log \log n)$.