Math 231 **Proof Portfolio** **Name: <u>Minjia Bettcher</u>**

Please write your proofs neatly in the space provided. Remember that searching the internet and discussing these questions with anyone, except your instructor or fellow students in this class through the general discussion board, is considered academic dishonesty.

(1) The unique positive real number solution to the equation $x^5 + x = 10$ is irrational.

Hints: You may use the following without justification.

- $\forall n \in \mathbb{N}$, $\forall a_0, a_1, \ldots, a_n \in \mathbb{R}$, the poly. $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ satisfies $\forall a, b, y \in \mathbb{R}$ if y is in between $p(a)$ and $p(b)$ then for some c in between a and b, $y = p(c)$.
- Any positive rational number can be represented by a fraction $\frac{p}{q}$ such that $p, q \in \mathbb{N}$ and the greatest common divisor of p, q is 1.
- The function $f(x) = x^5 + x$ is an increasing function for $x > 0$.

Proof: There exists a unique positive real number solution x to the equation $x^5 + x = 10$, where $f(x) = x^5 + x$ is an increasing function for $x > 0$, such that x must satisfy $1 < x < 2$ because for $x = 1$, $(1)^5 + (1) = 2$ which is less than 10, and for $x = 2$, $(2)^5 + (2) = 34$, which is greater than 10. Suppose that x is rational. Then x can be represented by a fraction p/q such that $p, q \in \mathbb{N}$ and the greatest common divisor of p, q is 1. Substitution into the equation yields $\frac{p^5}{q^5} + \frac{p}{q} = 10$, $p^5 + pq^4 = 10q^5$, $p(p^4 + q^4) = 10q^5$, $pr = 10q^5$ where r is the integer $p^4 + q^4$, so p divides $10q^5$. Since p and q do not share any prime divisors, p must divide 10. So p can be $1, 2, 5$, or 10. In the case $p = 1$, $1 < \frac{1}{q} < 2$, $\frac{1}{2} < q < 1$. There are no integers in $(\frac{1}{2}, 1)$. In the case $p = 2$, $1 < \frac{2}{q} < 2$, $1 < q < 2$. Again, there are no integers in $(1, 2)$. In the case $p = 5$, $1 < \frac{5}{q} < 2$, $2.5 < q < 5$, so q can be 3 or 4. If $q = 3$, $(\frac{5}{3})^5 + (\frac{5}{3}) = \frac{3530}{243} \neq 10$ and if $q = 4$, $(\frac{5}{4})^5 + (\frac{5}{4}) = \frac{4405}{1024} \neq 10$. In the case $p = 10$, $1 < \frac{10}{q} < 2$, $5 < q < 10$. The integers in $(5, 10)$ are $6, 7, 8, 9$, but q cannot be 6 or 8 because it would share the common divisor 2 with p, so that leaves 7 and 9. If $q = 7$, $(\frac{10}{7})^5 + (\frac{10}{7}) = \frac{124010}{16807} \neq 10$ and if $q = 9$, $(\frac{10}{9})^5 + (\frac{10}{9}) = \frac{165610}{59049} \neq 10$. Therefore there are no integers p and q for $\frac{p^5}{q^5} + \frac{p}{q} = 10$, so x is not rational. That completes the proof $\square$

(2) For all sets S, T, $S \subseteq T$ if and only if $\mathscr{P}(S) \subseteq \mathscr{P}(T)$.

Proof: Let S, T be sets. First let's prove the forward direction. Assume $S \subseteq T$. Let X be an arbitrary element of $\mathscr{P}(S)$, then $X \subseteq S$ so $X \subseteq T$. Since $\mathscr{P}(T)$ is the set of all subsets of T, then $X \in \mathscr{P}(T)$. Since X was arbitrary, it follows that all elements of $\mathscr{P}(S)$ are in $\mathscr{P}(T)$, so $\mathscr{P}(S) \subseteq \mathscr{P}(T)$.

Next let's prove the reverse direction. Assume $\mathscr{P}(S) \subseteq \mathscr{P}(T)$. Since $S \in \mathscr{P}(S)$, it follows that $S \in \mathscr{P}(T)$. Therefore $S \subseteq T$. That completes the proof $\square$

(3) For all integers $n \geq 0$, $5^{5n+1} + 4^{5n+2} + 3^{5n}$ is a multiple of 11.

Proof: Let's prove this by induction. First we must show that $5^{5n+1} + 4^{5n+2} + 3^{5n}$ is a multiple of 11 when $n = 0$. So $5^{5(0)+1} + 4^{5(0)+2} + 3^{5(0)} = 22$, which is a multiple of 11. That establishes the base case. Let $k \geq 0$ be an integer, and assume that $5^{5k+1} + 4^{5k+2} + 3^{5k}$ is a multiple of 11. Then $\exists j \in \mathbb{Z}$ such that $5^{5k+1} + 4^{5k+2} + 3^{5k} = 11j$. This is the inductive hypothesis.

We must show that $5^{5(k+1)+1} + 4^{5(k+1)+2} + 3^{5(k+1)} = 11z$ for some integer z. Then

$$5^{5(k+1)+1} + 4^{5(k+1)+2} + 3^{5(k+1)} = 5^{(5k+1)+5} + 4^{(5k+2)+5} + 3^{5k+5}$$

$$\begin{aligned}
&= 3125 \cdot 5^{5k+1} + 1024 \cdot 4^{5k+2} + 243 \cdot 3^{5k} \\
&= 3124 \cdot 5^{5k+1} + 1023 \cdot 4^{5k+2} + 242 \cdot 3^{5k} + (5^{5k+1} + 4^{5k+2} + 3^{5k}) \\
&= 3124 \cdot 5^{5k+1} + 1023 \cdot 4^{5k+2} + 242 \cdot 3^{5k} + 11j \text{ (by the inductive hypothesis)} \\
&= 3124 \cdot 5^{5k+1} + 1023 \cdot 4^{5k+2} + 242 \cdot 3^{5k} + 11j \text{ (by the inductive hypothesis)} \\
&= 2 \cdot 3^{k+1} + (-2)^{k+1}.
\end{aligned}$$

This completes the proof by induction $\square$

feedback: To prove the proposition, by induction. We can see that it works for n = 0. Then if k is... and if it works for k then it works with k+1. "Substituted the inductive hypothesis". Why it is an integer. Sum of products of integers. Use more words! Say conclusion.

(4) For all positive integers n, $\sum_{k=1}^{n} k(C(n,k))^2 = nC(2n-1, n-1)$.

Hint: Consider forming committees of n people (with a chairperson) that are chosen from a group of n Oregonians and n Washingtonians such that the chairperson is an Oregonian.

THIS is my proof!

From n oregonians, choose 1 oregoninans, from those k choose 1 to have be the chair