Hubert Hahn

Rigid Body Dynamics of Mechanisms

1 Theoretical Basis

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To
Mechthild and Elke

Preface

The dynamics of mechanical rigid-body mechanisms is a highly developed discipline. The model equations that apply to the tremendous variety of applications of rigid-body systems in industrial practice are based on just a few basic laws of, for example, Newton, Euler, or Lagrange. These basic laws can be written in an extremely compact, symmetrical, and esthetic form, simple enough to be easily learned and kept in mind by students and engineers, not only from the area of mechanics but also from other disciplines such as physics, or mathematics, or even control, hydraulics, or electronics. This latter aspect is of immense practical importance since mechanisms, machines, robots, and vehicles in modern industrial practice (sometimes called mechatronic systems) usually include various subsystems from the areas of hydraulics, electronics, pneumatics, informatics, and control, and are built by engineers trained in quite different disciplines.

Conventional methods of modeling rigid-body mechanisms

In contrast to the comparatively simple and easy-to-learn basic laws of rigid-body systems, the practical application of these laws to the planar or spatial motions of industrial mechanisms rapidly leads to extremely lengthy and complex equations of motion, where the form and complexity of the model equations depends critically on the choice of the model coordinates. Until recently this had the following consequences:

1. A large variety of *specialized techniques* have been developed, each suitable for efficiently modeling a *special-purpose mechanism.*
2. These techniques have usually been applied to comparatively *simple mechanisms*, as most of them were developed at universities or academic institutes, where there was no need to model complex realistic industrial systems, and no pressure to do this within a predetermined time schedule.
3. The overwhelming *majority of practicing industrial engineers have not had the opportunity to learn all these special modeling techniques.* They were usually neither capable of finding a special modeling approach suitable to a given mechanism, nor of deriving efficiently and correctly the realistic model equations, nor of estimating in advance the effort required to derive those models and to set up a time schedule for the task.

As a consequence there has been a large gap between the available basic laws of mechanics and the ability of practicing industrial engineers to apply them to large rigid-body systems.

General-purpose rigid-body analysis programs as efficient modeling tools

In the past two decades the above problems have been overcome by worldwide intensive research activity. As a result, various *general-purpose rigid-body analysis programs* have been developed that:

1. *Automatically set up the equations of motion* of rather complex kinematic and dynamic mechanisms.
2. *Provide efficient and accurate computer simulations* of most of these systems.
3. *Perform the first analysis steps*, such as static analysis, kinematic analysis, local linearization, eigenvalue analysis, and sensitivity analysis.

Examples of general-purpose rigid-body analysis programs include ADAMS ([1],[2]), DADS ([3]), NUSTAR ([4], [5]), and various other software packages discussed in ([6], [7]). Teaching computers to automatically formulating the equations of motion was equivalent to developing *systematic general methods* for *setting up* and *solving model equations of quite general mechanisms.* Using these computer programs, practicing industrial engineers can simulate and analyse complex rigid-body systems:

1. By *setting up an engineering model* of the mechanism based on their intuitive practical understanding of that system.
2. By *handling a rigid-body analysis program* without the burden of deriving complex analytical model equations, developing computer simulation code, and developing numerical solution algorithms of these equations.

Many of these rigid-body analysis programs have been equipped with graphical user interfaces that can be easily handled even by engineers who have a limited understanding both of the underlying mechanics and numerics, and of the *problems that may occur in the computer-aided modeling and solution process.* However this latter inexperience may have serious consequences: numerical results may be obtained by these programs that are far more erroneous than any results obtained in laboratory experiments.

Objectives of this monograph

Volume I of this monograph presents:

1. An introduction into the *theoretical background* of rigid-body mechanics.
2. A *systematic approach* for *deriving model equations* of mechanisms, as a first step in symbolic *differential–algebraic equations (DAE) form.*

Volume II presents:

1. Various *exercises to systematically apply this approach to examples* of *planar and spatial mechanisms.*
2. A symbolic approach for mapping the DAEs in a second step into symbolic *differential equations (DEs)*, into *nonlinear* and *linear state-space equations*, and sometimes also into *transfer function form.*

The *objectives* of both the *theoretical discussions* (Volume I) and the *practical applications* (Volume II) are:

1. *To prepare the reader for efficiently handling and application of general-purpose rigid-body analysis programs to complex mechanisms*, and
2. To *set up symbolic mathematical models of mechanisms in DAE form* for *computer simulations* and/or in *DE form*, as is often required in *dynamic analysis* and *control design.*

From the point of view of these two objectives this monograph can be considered as an introduction to basic mechanical aspects of *mechatronic systems.*

Organization of the books (Volumes I and II)

The two volumes of this monograph provide a *systematic theoretical approach for setting up model equations of planar and spatial rigid-body systems* in *DAE form* (Volume I), and present various *applications of the modeling methodology* to examples of planar and spatial mechanisms (Volume II).

Volume I includes *six chapters* and *four appendices. Chapter 1* gives a brief introduction to the subject of modeling rigid-body mechanisms, which is illustrated by several simple examples and by some more complex applications of mechanisms from industrial practice. *Chapter 2* presents a brief review of vector and matrix algebra and of multivariable calculus for the planar and spatial cases. Spatial rotations are derived in terms of Bryant angles together with the associated kinematic DEs. Due to the introductory character of this book, quaternions or Euler parameters of spatial rotations are not considered here (despite the fact that singularities may occur in the kinematic DEs of Bryant angles). Time derivatives of vector functions together with the gradient vector and the Jacobian matrix of those functions are introduced. They will be used extensively for describing constraint relations. Some useful relations of scalar products and cross products of vectors are derived in *Appendix A.1*, together with different expressions for the time derivatives of vectors and orientation matrices of planar and spatial vectors, and with a brief review of derivatives of vector functions. Relations of planar and spatial kinematic and active constraints, represented in Cartesian coordinates, are discussed in *Chapter 3* together with the associated velocity and acceleration constraint equations, including formal relationships between constraint reaction forces and torques, and with a discussion of possible singularities of the constraint equations, illustrated by an example. Kinetic equations of planar and spatial rigid-body mechanisms are developed in *Chapter 4* and in *Appendix A.2.* Starting with the concepts of linear momentum and angular momentum in *Section 4.1*, the Newton–Euler equations of the planar and spatial motion of a single unconstrained rigid body are derived in *Section 4.2*, together with the model equations of planar and spatial mechanisms in *Section 4.3.* A brief discussion of the numerical solution of DAEs is presented in *Section 4.4.* Parallel to the Newton–Euler approach, the Lagrange formalism is briefly discussed in *Appendix A.2.* Basic differences between the theoretical constituents of planar and spatial mechanisms are collected in *Appendix A.3.* In *Chapter 5* a systematic approach for deriving the constraint equations of planar and spatial joints is presented based on suitable representations and projections of vector and orientation loop equations. The constraint equations of various joint types in common use are derived there. Theoretical models of joints of planar mechanisms are presented in *Section 5.1.* Model equations of joints of spatial mechanisms are derived in *Section 5.2* and in *Appendix A.4.* Constitutive relations of applied forces and torques of planar and spatial mechanisms are discussed in *Chapter 6.* Among those, theoretical models of translational and torsional springs and dampers as well as models of actuators and motors are briefly presented.

Various simple and some more complex applications of rigid-body mechanisms are modeled in symbolic *DAE form* and in *DE form*, and for selected mechanisms also in *nonlinear* and *linear state-space form* and using the *transfer function matrix* representation in *Volume II.* They include various combinations of theoretical models of joints, and of active and passive force elements. In *Chapter 1* of Volume II, the modeling methodology is summarized, and a software package is briefly discussed ([8]) that maps symbolic model equations from DAE form into DE form (in most cases where this is feasible). Two applications of *planar models* of an unconstrained rigid body are discussed in *Chapter 2.* Several applications of a planar rigid body under constrained motion are presented in *Chapter 3.* Various applications of planar mechanisms that include two rigid bodies under constraints are discussed in *Chapter 4.* Applications of a rigid body under unconstrained *spatial motion* are collected in *Chapter 5*, followed by several applications of a constrained spatial rigid body in *Chapter 6*, and by several applications of spatial mechanisms including between two and thirteen constrained rigid bodies in *Chapter 7.*

Use of the text

The text of the books is intended for use and self-study by practicing industrial engineers that have a bachelor's degree, and by students of undergraduate university courses. The contents of the books have been used in lectures and courses held over many years:

1. In several *industrial companies* (like BMW and IABG) for practicing engineers from the areas of mechanics, vibration techniques, vehicle simulation, control, hydraulics, pneumatics, measurement, testing, electromagnetics, and electronics.
2. In the undergraduate courses of several *universities* (Universities of Munich, Tübingen, and Kassel) for students from the areas of mechanical engineering, control engineering, electrical engineering, civil engineering, physics, and mathematics.

The practicing engineers who attended these courses have influenced both the contents and the direction of this monograph, resulting in more emphasis being placed on:

1. A *systematic choice of notation* (with indices of the variables that uniquely identify the frames of their representations and time derivatives).
2. An *algebraic formulation* of all expressions in a form suitable for direct *implementation in a computer.*
3. *Applying these methods to both simple and complex mechanisms.*

The engineers and students that attended these lectures had the opportunity to apply these methods to practical examples of mechanisms using general-purpose rigid-body analysis programs like NUSTAR, ADAMS, and DADS.

Spatial mechanics is conceptually more complex and its theoretical modeling provides much lengthier and more unwieldy formal expressions than *planar* mechanics. To enable the beginner reader to successfully master his or her study of rigid-body dynamics and to keep the amount of notation and formal expressions of the applications presented within acceptable limits, only *planar* rigid-body systems are considered in the first parts of *Chapters 2, 3, 5* and *6 of Volume I.* They present vectors, matrices, kinematics, forces and torques of *planar* geometry and *planar* mechanics. The equations of motion of rigid bodies under *planar* motion are collected in *Chapter 4 of Volume I.* Various *planar* mechanisms are discussed in *Chapters 2, 3* and *4 of Volume II.* Teaching experience shows that the methodology of modeling rigid-body systems can be basically understood by considering *planar* systems only. Having developed confidence and enough intuition in the basic methods of theoretical modeling of *planar* mechanisms, the reader is encouraged to study *spatial* mechanisms in the second parts of *Chapters 2, 3, 5, 6 and in all of Chapter 4 of Volume I*, and the applications of *spatial* mechanisms of *Chapters 5, 6*, and *7 of Volume II.* Basic differences between the model equations of *planar* and *spatial* mechanisms are summarized in *Appendix A.3* of *Volume I.*

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Sporke/Westfalen
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April 2001

Contents

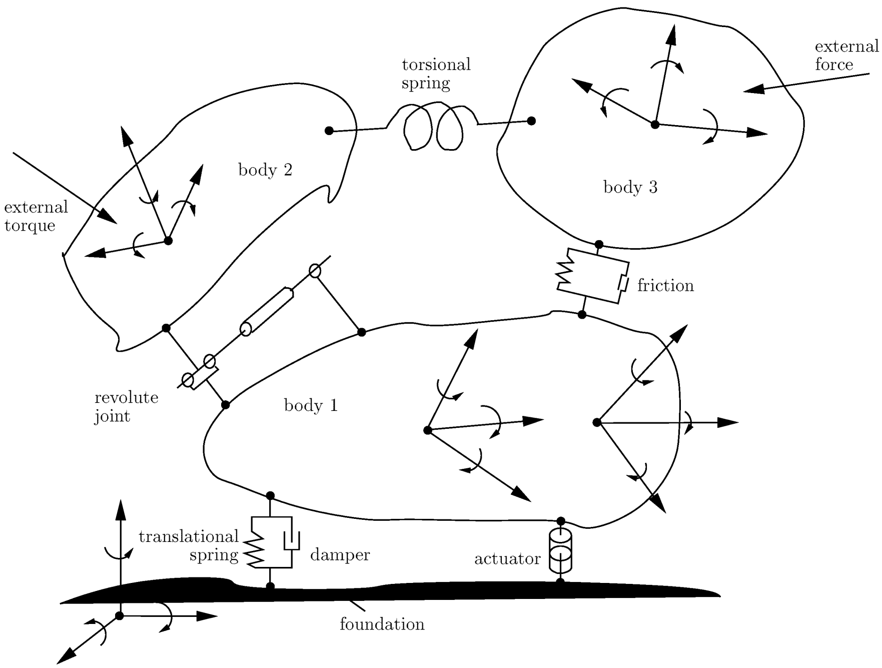
1. Introduction

The mechanical systems discussed in this book (e.g., Figure 1.1) are collections of *rigid-bodies* connected by translational and torsional *spring*, *damper* and *friction* elements, and by *joints*, *links*, *bearings* and *gear boxes*, in which some or all of the bodies can move relative to each other. They may be driven by *external forces* or *torques* to achieve specified performance requirements as well as desired loading and operation conditions. They are called *rigid-body systems* or *mechanisms.* A *rigid body* is defined as an assembly of particles that do not move relative to each other. This means in reality that "deformations of rigid bodies" have no significant influence on the gross body motion. Rigid bodies of mechanisms move relative to each other consistent with the joints that limit their relative motion. Simultaneous large *displacements* and *rotations* of those bodies lead to *nonlinear model equations* with geometric nonlinearities that in most cases must be solved numerically.

1.1 Tasks in multibody simulation, analysis, and control

Multibody systems are commonly investigated under different aspects, depending on the task to be solved. In *kinematic analysis*, the *motion* of a system (positions, orientations, velocities, and accelerations) is considered *without taking into account forces* that cause this motion. *Usually time histories of some position coordinates (independent variables) of the rigid bodies are prescribed*, and *time histories of the remainder position, velocity, and acceleration coordinates* (dependent variables) *are determined* by solving *nonlinear algebraic equations* for the position, and *linear algebraic equations* for the velocities and accelerations. In kinematic analysis, the number of degrees of freedom of the mechanism must be equal to the number of independent driver constraint equations. All required model *parameters* are assumed to be known in kinematic analysis.

In *kinetic (dynamic) analysis* and *computer simulation*, the *motion* (dependent variables) *of a system is determined from given time histories of forces and torques* (independent variables) applied to the system, by solving a set of *nonlinear differential equations (DEs)* or *differential–algebraic equations (DAEs).* In dynamic analysis the number of unknown variables in the constraint equations is larger than the number of constraint equations. Therefore a unique solution is only obtained by specifying a proper set of initial conditions. The model *parameters* are assumed to be known here, too.



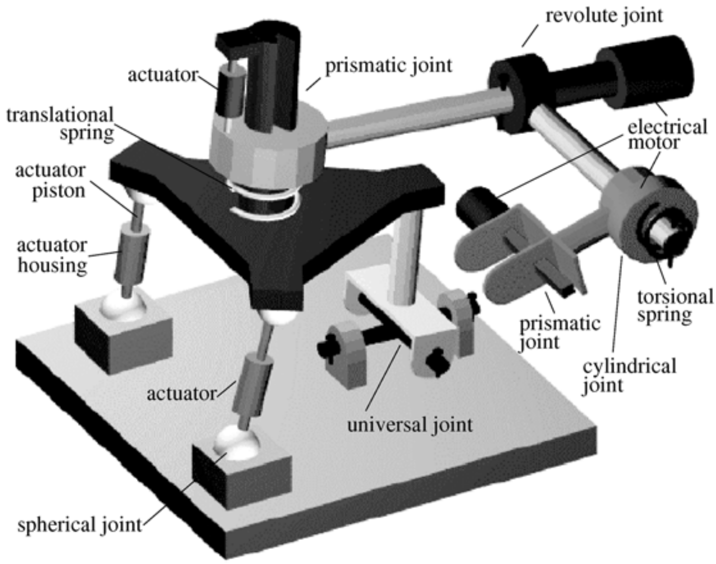


Fig. 1.1: Examples of multibody systems

In *inverse kinetic (dynamic) analysis* the time histories of the minimal coordinates of the mechanism are prescribed. Solving the *nonlinear algebraic constraint equations* of the kinematics provides the time histories of the position, velocity, and acceleration of the remainder coordinates. Solving the dynamic equations as *nonlinear algebraic equations* with respect to the forces and torques provides their time histories, associated with the prescribed motions. Again the model *parameters* are assumed to be known.

In *parameter identification*, the time histories of the motion (position, velocity, acceleration) of each rigid body together with the time histories of the associated forces and torques are measured. The kinematic and dynamic model equations are solved as *nonlinear algebraic equations* with respect to the unknown model parameters, taking into account measurement errors.

In *control synthesis*, desired motions (position, velocity and/or acceleration) of selected bodies are chosen. Assuming that the dynamic and kinematic model equations of the system together with the model parameters are exactly or approximately known, dynamic or static *control algorithms are computed* by specific design techniques that tend to minimize deviations of the actual motions from the desired motions, and simultaneously guarantee the stability and sometimes certain robustness properties of the closed-loop system.

Each of the preceding investigations and tasks is based on *analytical and numerical models* of the *dynamics of the mechanism considered*, where rigid *body dynamics* includes *kinematics* and *kinetics.*

1.2 Coordinates and frames

Model equations (equations of motion) of rigid-body systems may be formulated in quite *different* (moving and/or inertial) *frames* and *coordinates.* As a general result, depending on the *coordinates* chosen, some model equations of a system will be more involved than others. Any set of variables (coordinates) that *uniquely specifies* the *position* and *orientation of all bodies in a mechanism*, that is, the configuration of the mechanism, is referred to as set of *generalized coordinates*[1] ($\boldsymbol{p}$) (coordinates in *general*, regardless of their nature). *Generalized coordinates* may be *independent* (each free to vary arbitrarily) or *dependent* (required to satisfy constraint equations).

Independent generalized coordinates are called *minimal coordinates.* The *minimal number* of *independent* coordinates required to specify uniquely and completely the *position* and *orientation* of each "component part" of a rigid-body mechanism is called set of *degrees of freedom (DOFs)* of the system, where the term *"component part"* used in this context refers to any part of the system such as a platform, wheel, motor, disk, or lever, which must be treated

[1] This definition of generalized coordinates in rigid-body mechanisms differs from the traditional definition of *generalized coordinates in mechanics.*

as a *rigid body*. Rigid-body configurations may be specified by introducing an *inertial (global, absolute) frame* R and *body-fixed (local, relative) reference frames* L_{ij}, with j as index of the local frame and i as index of the body (if only a single local frame is defined on body i, it will be called L_i). Then each rigid body may be located by specifying *global (absolute, inertial) coordinates* of the *position* of the origin of a frame L_i, and its *orientation* with respect to a global frame R (Figure 1.2). The relative location and orientation of a frame L_i fixed on a body i with respect to a frame L_j, fixed on a body j, is specified by *local (relative) coordinates.*

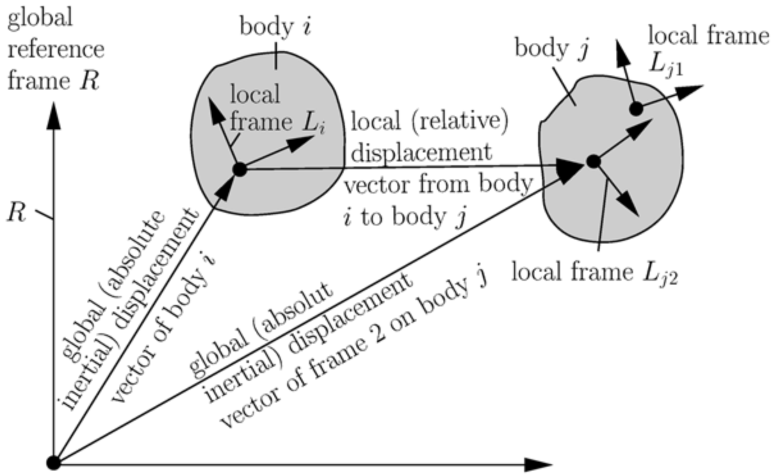


Fig. 1.2: Global (inertial) and local (relative) vectors and frames

1.3 Formulation of the model equations

A lot of research work into rigid-body dynamics has been devoted to the selection of system coordinates and DOFs that provide a trade-off between the *generality* and *efficiency* of dynamic formulation and simulation. The modeling methods of multibody systems may in general be devided into *two* main approaches:

In the *first* approach, a *minimum number of relative (local) or joint coordinates* are used to formulate a *minimum number of DEs* that are expressed in terms of the system DOFs. In many applications, this approach leads to a complex recursive formulation based on loop closure equations. The incorporation of general forcing functions, constraint equations (e.g. model equations of joints, compare Chapter 5) and/or specified trajectories in the recursive formulation is difficult. This approach, however, may be desirable in several applications (e.g. serial robots).

In a *second* approach, the configuration of the system is identified by using a set of *Cartesian (global) coordinates* that describe the location and orientation of the bodies in the mechanism. This approach leads to model equations in *DAE form.* It has the advantage that the *dynamic formulation* of the equations that govern the motion of the system is *straight forward*, and that it allows *easy and flexible addition and removal of rigid bodies, of complex force functions, and constraint equations.* For each spatial rigid body in the system, six coordinates are sufficient to describe the body configuration.

The second approach to set up model equations in DAE form and *absolute coordinates* from the *Newton–Euler equations* will primarily be used in this monograph, where:

1. The position and orientation of a rigid body under *planar motion* will be specified by *planar* Cartesian coordinates $\boldsymbol{p} := (x\,,\,y\,,\,\psi)^{\mathrm{T}}$.
2. The position and orientation of a rigid body under *spatial motion* will be specified by *spatial* Cartesian coordinates $\boldsymbol{p} := (x, y, z, \varphi, \theta, \psi)^{\mathrm{T}}$.

Formulation of model equations and system constraints in *global coordinates* is flexible with practically no limitation on the type of multibody system. This easier modeling is obtained at the expense of a *larger system of model equations (maximum number of coordinates).* Moreover, various *analysis* and *control design* techniques, developed for systems in minimal coordinates and in *state-space form*, are not yet available or may become much more complicated for system equations written in *DAE form* ([9] and [10]).

As a consequence, the model equations of the applications of Volume II will be mapped from symbolic *DAE form* to symbolic *DE form* by suitable elimination or projection techniques. It should be mentioned that this symbolic elimination process is in general restricted to mechanisms that are not too complex, or to mechanisms with a particular structure (like a tree structure).

Newton's and Euler's laws, together with the concept of *virtual work*, may be regarded as a foundation on which all considerations of rigid-body mechanics rest. However, it should be realized that the basic laws of mechanics can be formulated (mathematically written) in several ways other than that given by Newton, such as *D'Alembert's principle*, *Lagrange's equations*, *Hamilton's principle* and *Hamilton's equations*, all of which are basically equivalent to Newtons's laws and the principle of virtual work.

The *"basic laws" of dynamics* are merely statements of a wide range of experience. They cannot be obtained by logic or mathematical manipulations alone but are founded on careful experimentation. We cannot "explain" why these laws are valid. We can only say that they represent a compact statement of past experience regarding the behavior of a wide variety of mechanical systems.

Here the equations of *kinetics* (dynamics) of rigid bodies will be mainly derived from *Newton's* and *Euler's* basic laws, and sometimes also from *Lagrange's equations* (cf. *Appendix A.2*). *Newton's and Euler's basic laws* of unconstrained motions of a rigid body are:

1. "Force ($\boldsymbol{F}$) equals the product of the mass (m) times the acceleration ($\dot{\boldsymbol{v}}$) of a rigid body" *(Newton's second axiom)*;

$$m \cdot \dot{\boldsymbol{v}} = \boldsymbol{F} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{P}) = \boldsymbol{F} \;, \tag{1.1}$$

   with

   $\boldsymbol{P} = m \cdot \boldsymbol{v}$ as the *linear momentum* of the rigid body.

2. "Torque ($\boldsymbol{M}$) equals the product of the moment of inertia (J) times the angular acceleration ($\dot{\boldsymbol{\omega}}$) of a rigid body" *(Euler's law)*;

$$J \cdot \dot{\boldsymbol{\omega}} = \boldsymbol{M} \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{D}) = \boldsymbol{M} \;, \tag{1.2}$$

   with

   $\boldsymbol{D} = J \cdot \boldsymbol{\omega}$ as the *angular momentum* of the rigid body.

3. "Reaction forces ($\boldsymbol{F}_{ij}$, $\boldsymbol{F}_{ji}$) (or torques $\boldsymbol{M}_{ij}$, $\boldsymbol{M}_{ji}$) between two bodies i and j are equal in magnitude and opposite in direction" *(Newton's third axiom)*;

$$\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji} \;. \tag{1.3}$$

The above equations (1.1) and (1.2) are only valid in this *simple form* if the *time derivatives* of $\boldsymbol{P}$, $\boldsymbol{v}$, and $\boldsymbol{D}$ are measured relative to an *"inertial frame"*, and if all vectors are represented in *"inertial coordinates"*.

Choosing local (noninertial) coordinates of the accelerations, velocities, and force and torque vectors may provide quite complex representations of Newton's and Euler's equations, as will be shown in *Chapter 4*. As a consequence, the treatment of every theoretical problem in rigid-body mechanics begins with a consideration of an *inertial frame*. The concept of "inertial frame" is of fundamental theoretical and practical importance though it is only a *hypothesis* that is never really satisfied in applications; it is a fictious concept introduced for formal convenience. Due to rotations and other motions of the earth, a coordinate frame attached to its surface is obviously noninertial. Nevertheless, the acceleration of this frame is so "low" that for most technical purposes it may be regarded as inertial.

Lagrange's equations of the second type of an unconstrained rigid body are (cf. *Appendix A.2*)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) + \frac{\partial L}{\partial \boldsymbol{p}} = \boldsymbol{Q} \tag{1.4}$$

with L as the Lagrange function, $\boldsymbol{p} = (\boldsymbol{r}^{\mathrm{T}}, \boldsymbol{\eta}^{\mathrm{T}})^{\mathrm{T}}$ as vector of a minimum set of generalized coordinates, $\boldsymbol{v} = (\dot{\boldsymbol{r}}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}})^{\mathrm{T}}$ as velocity vector, and $\boldsymbol{Q}$ as vector of generalized forces, associated to $\boldsymbol{p}$ and $\boldsymbol{v}$.

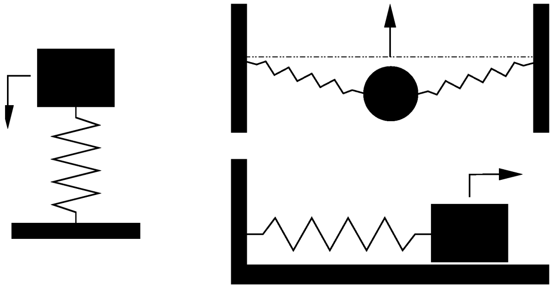
1.4 Prototype applications of rigid-body mechanisms

Rigid-body systems may range from very simple to very complex mechanisms. *Simple rigid-body mechanisms* (Figures 1.3 and 1.4) are traditionally modeled by writing down the Newton–Euler equations by direct inspection of associated free-body diagrams or by using the Lagrange equations after having defined the Lagrange function. This modeling approach can still be applied by engineers with some experience in this field to *slightly more complex systems* such as those of Figure 1.5. The complexity of a model of a technical system depends on its purpose. A model for *vibration analysis* of the *steering mechanism* of Figure 1.6 may be obtained by direct application of the Lagrange equations (compare Figure 1.6d and [11], [12], [13]). Another model for simultaneously studying both, the *spatial kinematics and dynamics of the steering mechanism* of Figure 1.6a to 1.6c, may be already quite complex when it includes large spatial motions of the wheels, the steering gear, and the steering wheel.

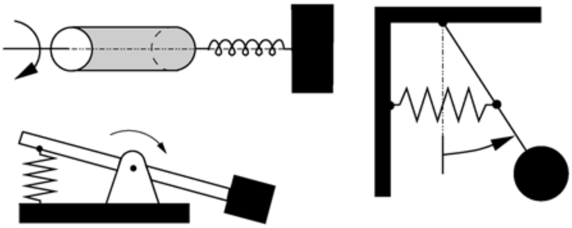
Mathematical *models of kinematic and dynamic systems with several DOFs* have traditionally been modeled in terms of *"clever formulations"* that take advantage of specific properties of the system considered to obtain simplified forms of model equations. *Ingenious selection of independent position and orientation coordinates* occasionally may lead to a formulation that allows manual derivation of the equations of motion. This *"clever formulation approach"* is nevertheless limited to relative simple rigid-body systems and can only be performed by specialists that have quite a deal of experience in this field.

More complex rigid-body systems like, for example:

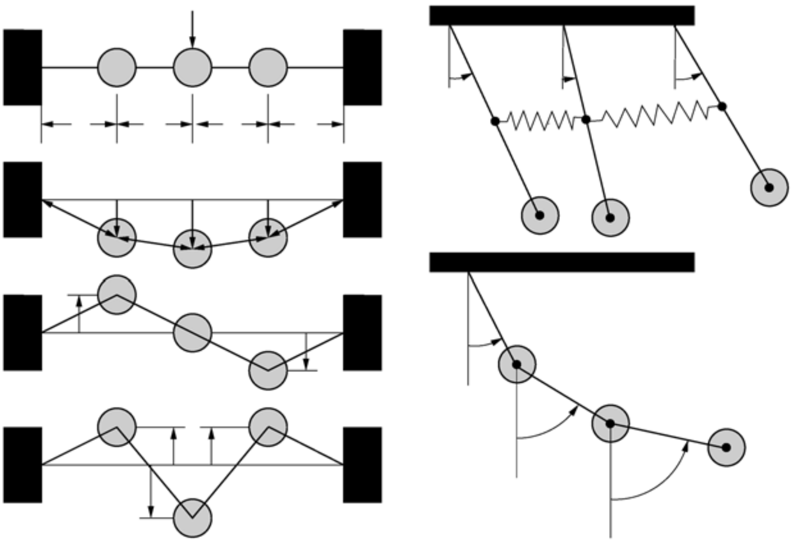
1. the *press model* of Figure 1.7 that includes three rigid-bodies subject to large spatial motion (ram, pitman, and eccentric drive), additional 16 rigid bodies that model small spatial deflections of the frame, and several revolute and universal joints as well as various springs and dampers ([14], [15], [16]); or
2. the model of a loaded *roller rig* of Figures 1.8, and 1.9, 1.10, 1.11 and 1.12 that includes more than 60 rigid bodies (most subject to small spatial motion), connected by various revolute, prismatic, and universal joints, and driven by several servo-hydraulic actuators (compare the technical drawing of the roller stand of Figure 1.10), and that includes models of the rolling contact of elastic bodies in the presence of dry friction; ([17], [18], [19]);



(a) Systems with one translational degree of freedom



(b) Systems with one rotational degree of freedom



(c) Discrete model of a string (d) Coupled pendulum

Fig. 1.3: Examples of simple unconstrained and constrained mechanical systems

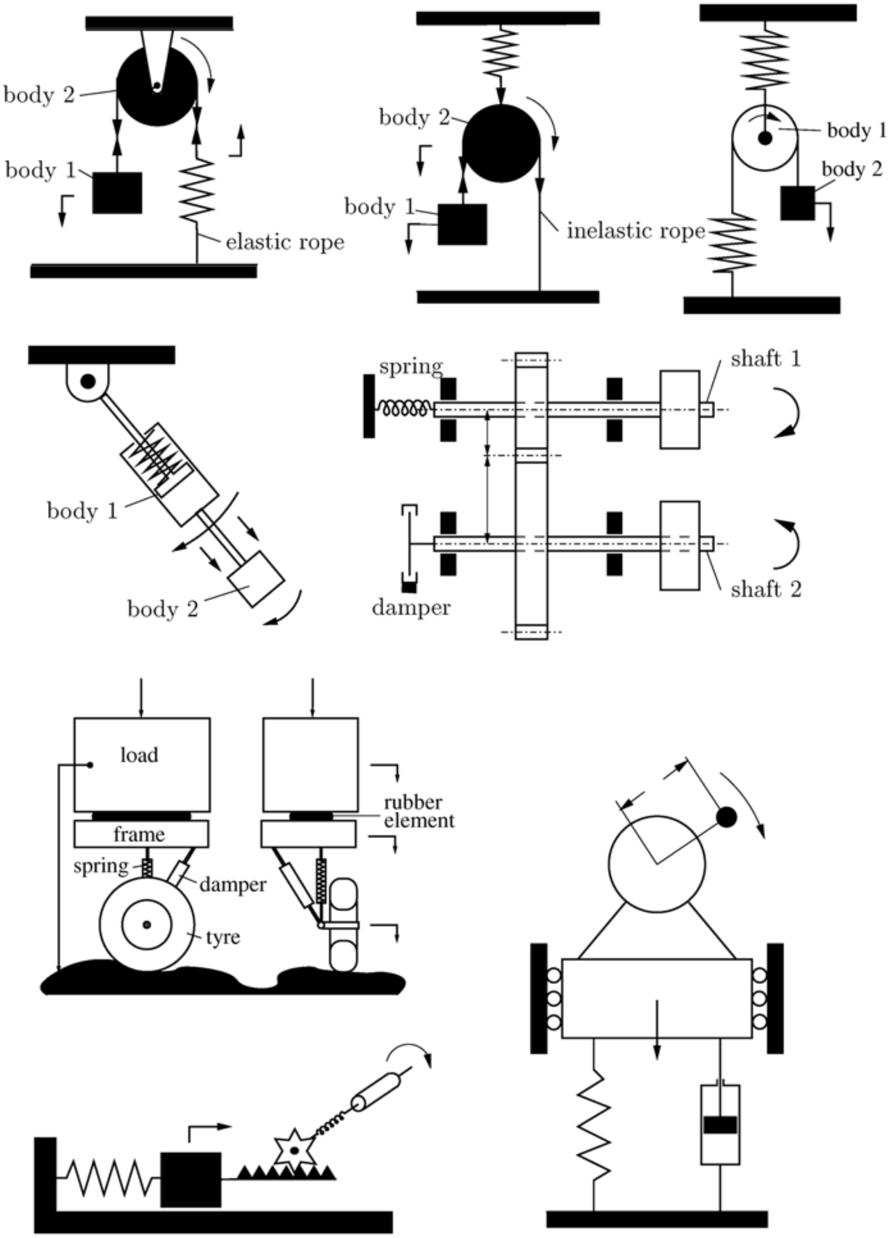
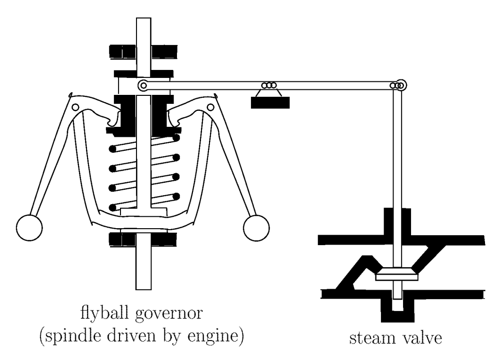
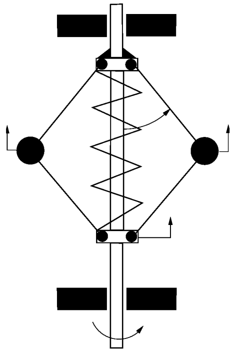


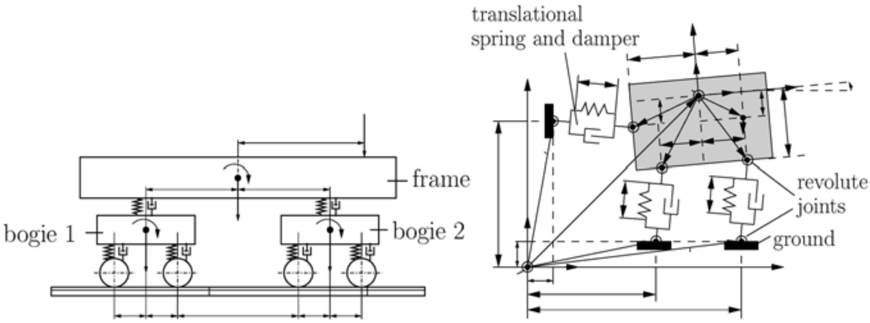
Fig. 1.4: Examples of simple mechanisms including joints



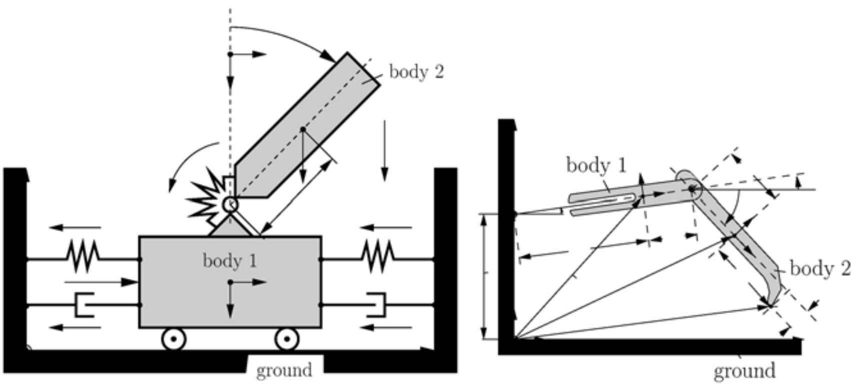
(a) Watt's steam engine governor



(b) Schematic drawing of a flyball governor



(c) Planar models of a simple locomotive and of a planar test facility



(d) Planar models of an inverted pendulum and of an excavator

Fig. 1.5: Slightly more complex mechanisms

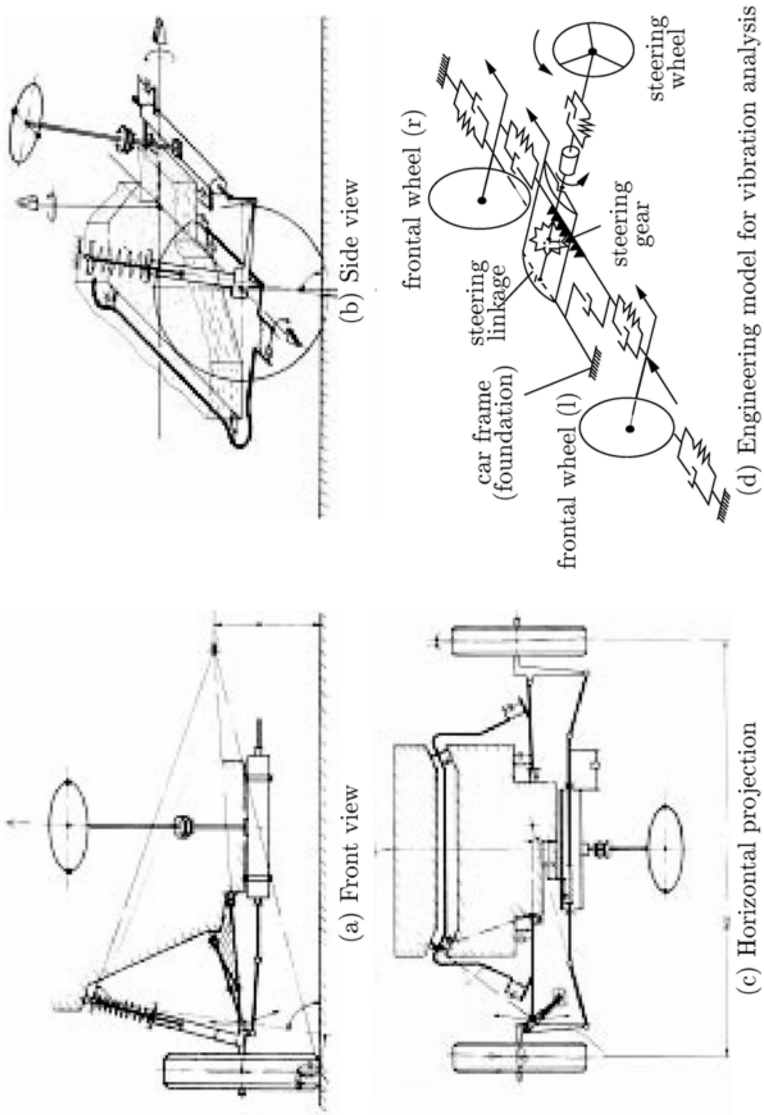


Fig. 1.6: Steering mechanism of an automobile

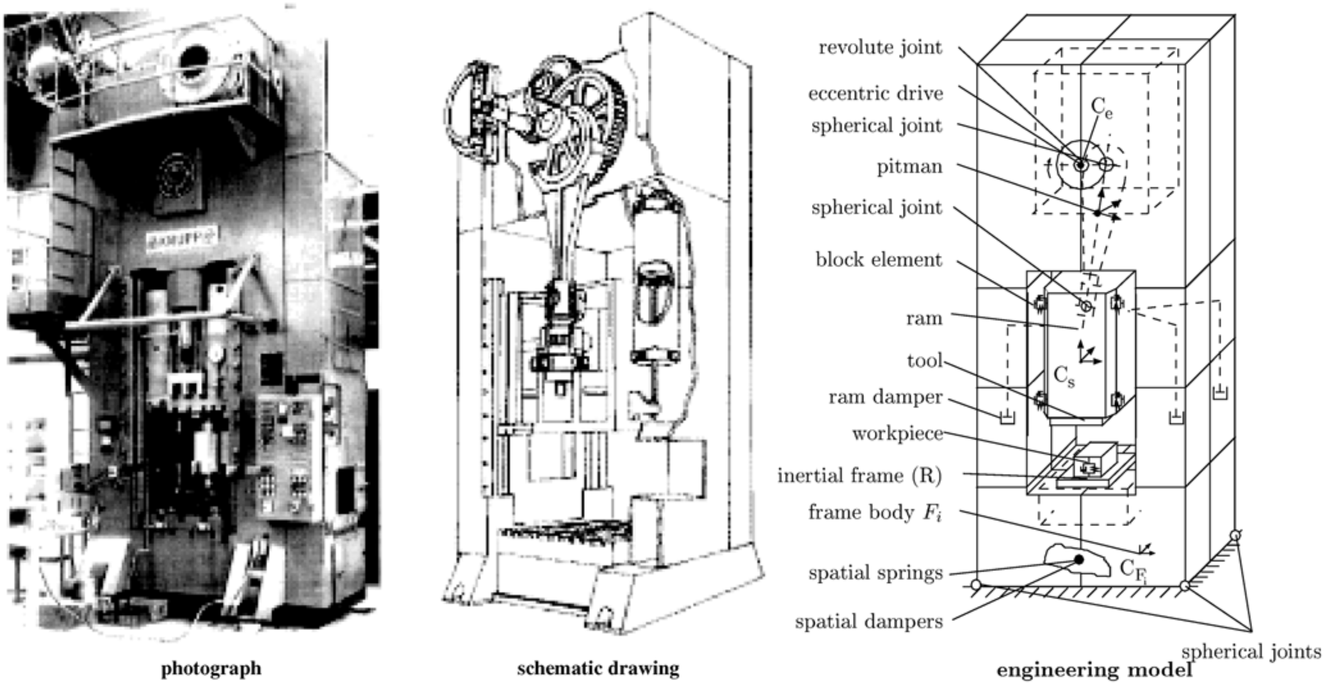
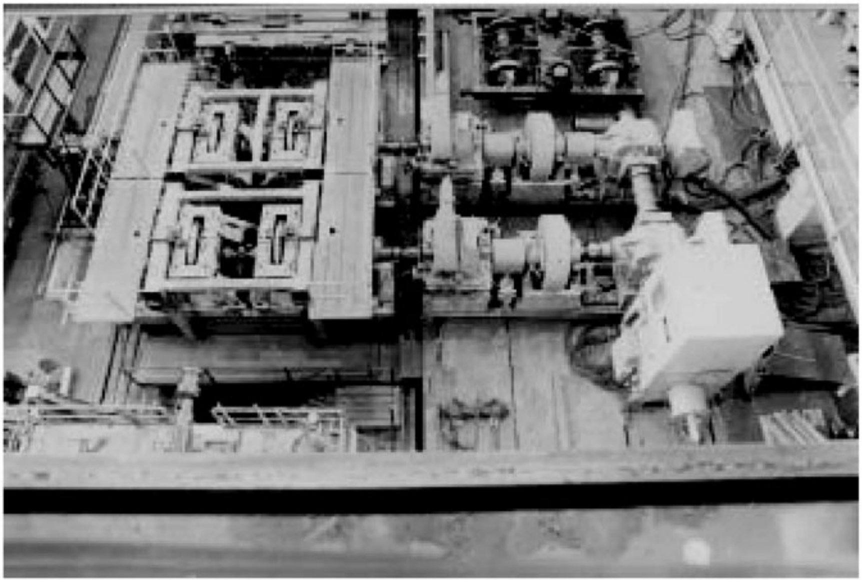
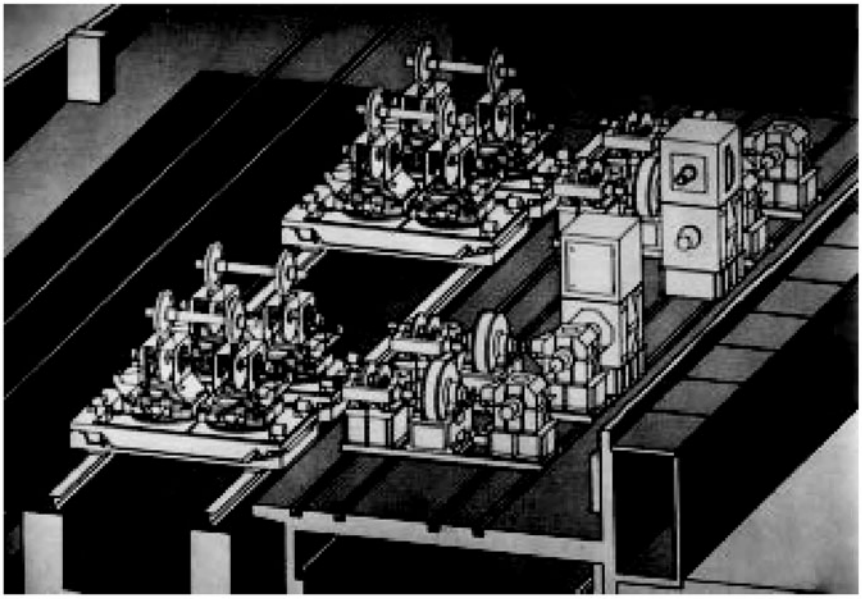


Fig. 1.7: Photograph, schematic drawing, and engineering model of a single-point-drive eccentric press (LVWU laboratory, University of Kassel)

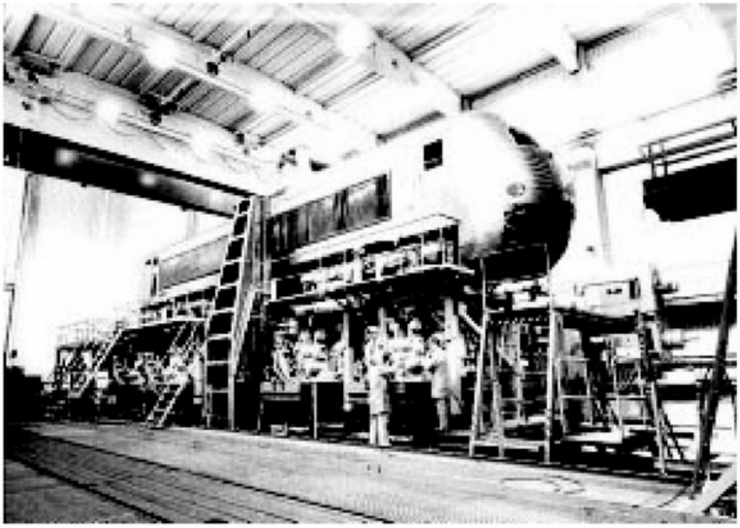


(a) Photograph of the roller rig

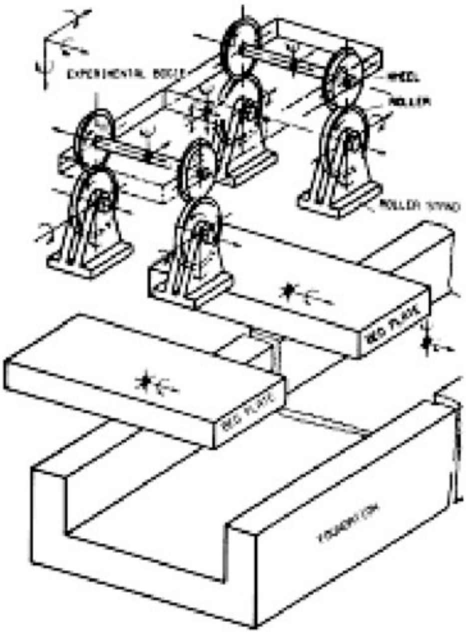


(b) Technical drawing of the roller rig

Fig. 1.8: Roller rig of the German railway company (DB AG)

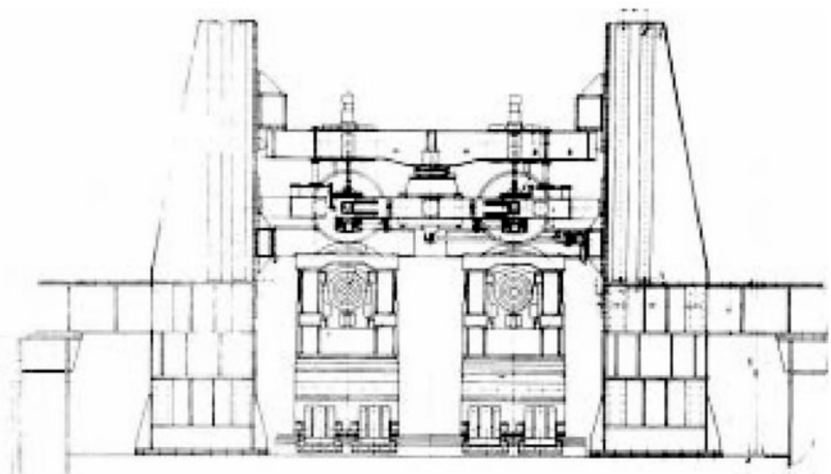


(a) Roller rig loaded by an ICE high-speed locomotive

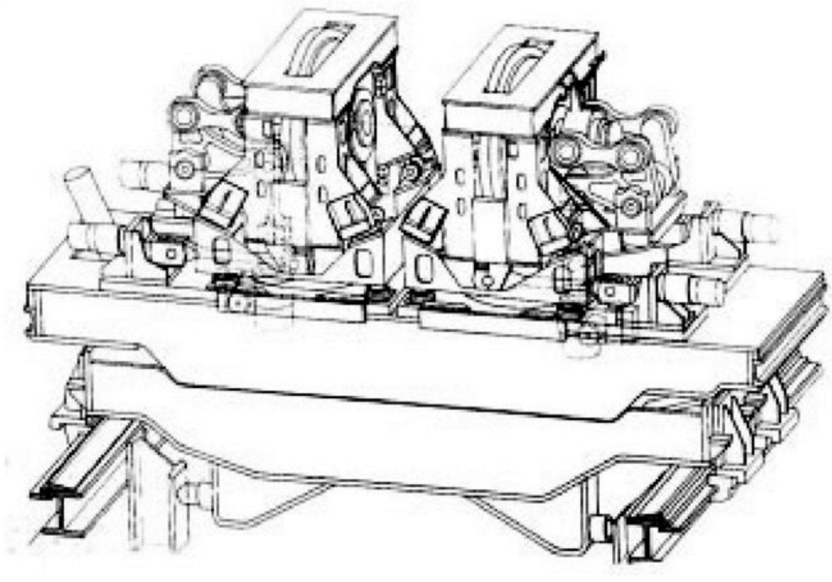


(b) Engineering model of a roller stand loaded by a wheel set

Fig. 1.9: Roller rig of the German railway company (DB AG)

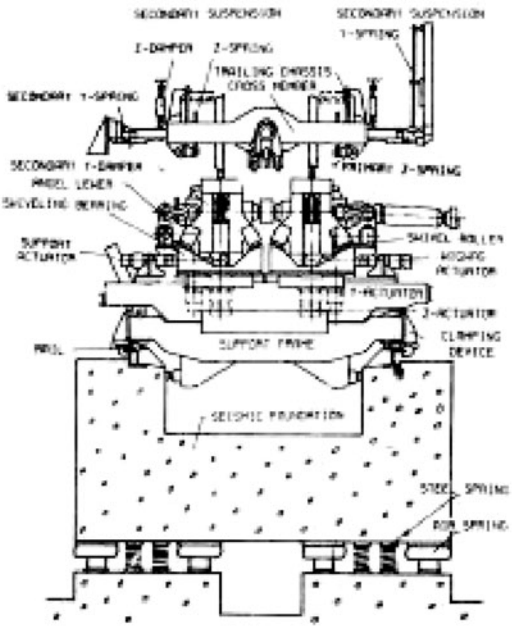


(a) Roller rig with bogie (technical drawing)

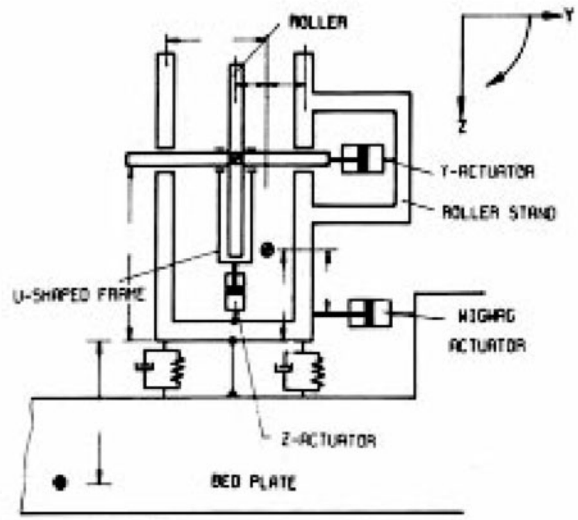


(b) Bed plate and roller stands (technical drawing)

Fig. 1.10: Roller rig and bogie

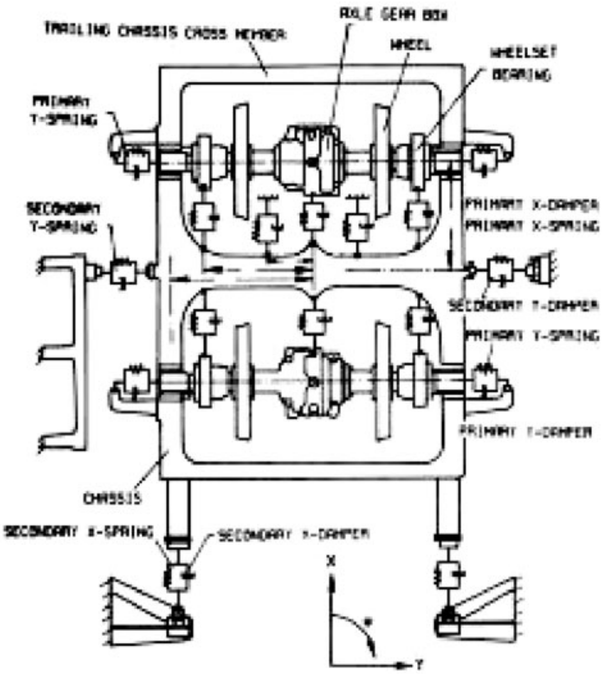


(a) Roller rig with bogie (technical drawing)

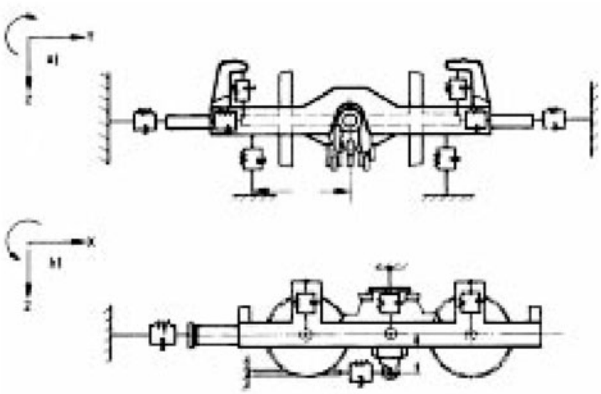


(b) Rollerstand (engineering model)

Fig. 1.11: Roller rig and bogie



(horizontal projection)



(front and side views)

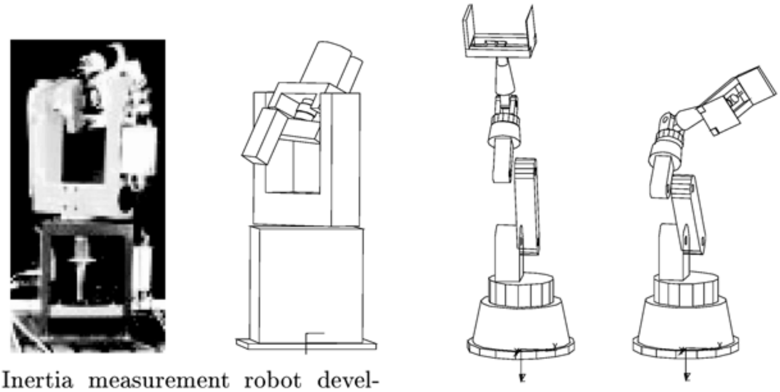
Fig. 1.12: Drawing of elastic and dissipative coupling of a bogie

may be modeled by *special-purpose rigid-body programs*, obtained for example by *symbolic computation*, where the computer is used to differentiate the Lagrange function of a mechanism according to the Lagrange formalism, to substitute variables, and to perform algebraic manipulation. The chance (probability) of *deriving correct model equations* of the roller rig (that cover more than 300 pages) by hand without using symbolic computation tends towards zero, even for engineers that are well trained in providing "clever formulations".

A *special purpose simulation program* deals with only a single type of applications. Such a program can be well adapted to the particular structure of the application, taking into account, for example, specific different motions of subsets of rigid bodies (like spatial, planar, or single axis motions) and typical kinematic behavior. Such a tailor-made program for a specific single application can be made computationally quite efficient. The major drawback of such a special-purpose program is its lack of flexibility for handling other types of applications.

Practicing engineers must usually be capable of theoretically modeling and simulating complex mechanisms of quite different types in a short time:

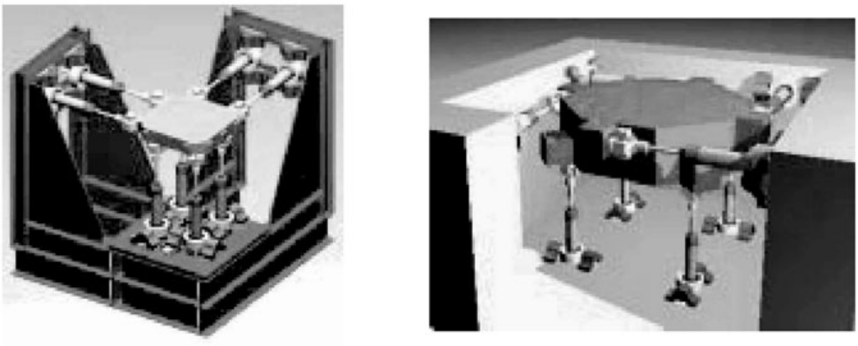
1. Like *serial robots* with various degrees of freedom that include many rigid bodies subject to large spatial motion, various joints and actuators (e.g. Figure 1.13a, [20], [21], [22], [23], [24]).
2. Like *parallel robots*
   2.1 constructed as *multi-axis test facilities* including up to 17 rigid bodies subject to large spatial motion together with 8 universal, 8 spherical, and 8 prismatic joints (e.g. Figure 1.13b, [25], [26], [27], [28], [29], [30], [31]); or
   2.2 constructed as *hexapods* including from 1 to 13 rigid bodies subject to large spatial motion, with 6 universal, 6 spherical, and 6 prismatic joints (e.g. Figure 1.13c, [32], [33], [34]).
3. Like *off-road vehicles* including various rigid bodies subject to large spatial motion and a large number of dissipative and elastic connection elements as well as revolute and universal joints. Compare the following two examples:
   3.1 The *truck* of Figure 1.14 that has been modeled by the *general purpose rigid-body analysis program* NUSTAR. The model includes 17 rigid bodies subject to large spatial motion, 8 universal joints, 5 revolute joints, 4 tire models, an engine model and more than 32 spatial spring and damper elements. More than 96 spatial frames were needed to specify the geometry of this vehicle ([35], [36], [37]).
   3.2 The *tank* model of Figure 1.15 that has been constructed from a large number of rigid bodies subject to large spatial motion and from a large number of elements connecting these bodies ([38], [39]). It has been modeled and simulated by the general-purpose rigid-body analysis program NUSTAR.



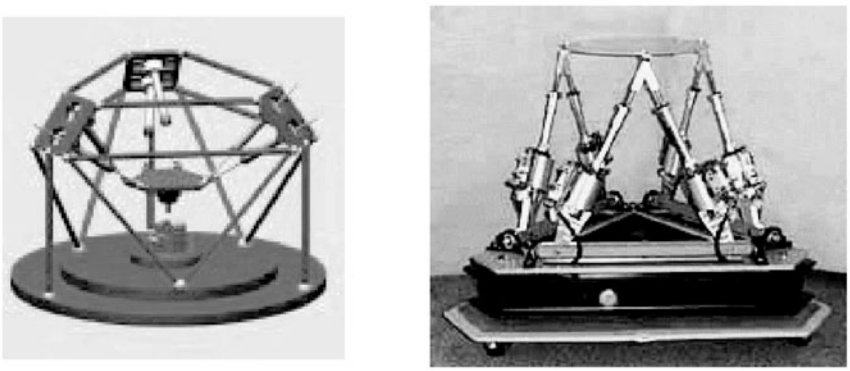
Inertia measurement robot developed by RTS, University of Kassel

Serial robot simulated by NUSTAR

(a) Serial robots



(b) Parallel robots (left-hand side robot built as multi-axis test facility by RTS, University of Kassel)



(c) Parallel robots built as hexapods

Fig. 1.13: Drawings and photographs of serial and parallel robots

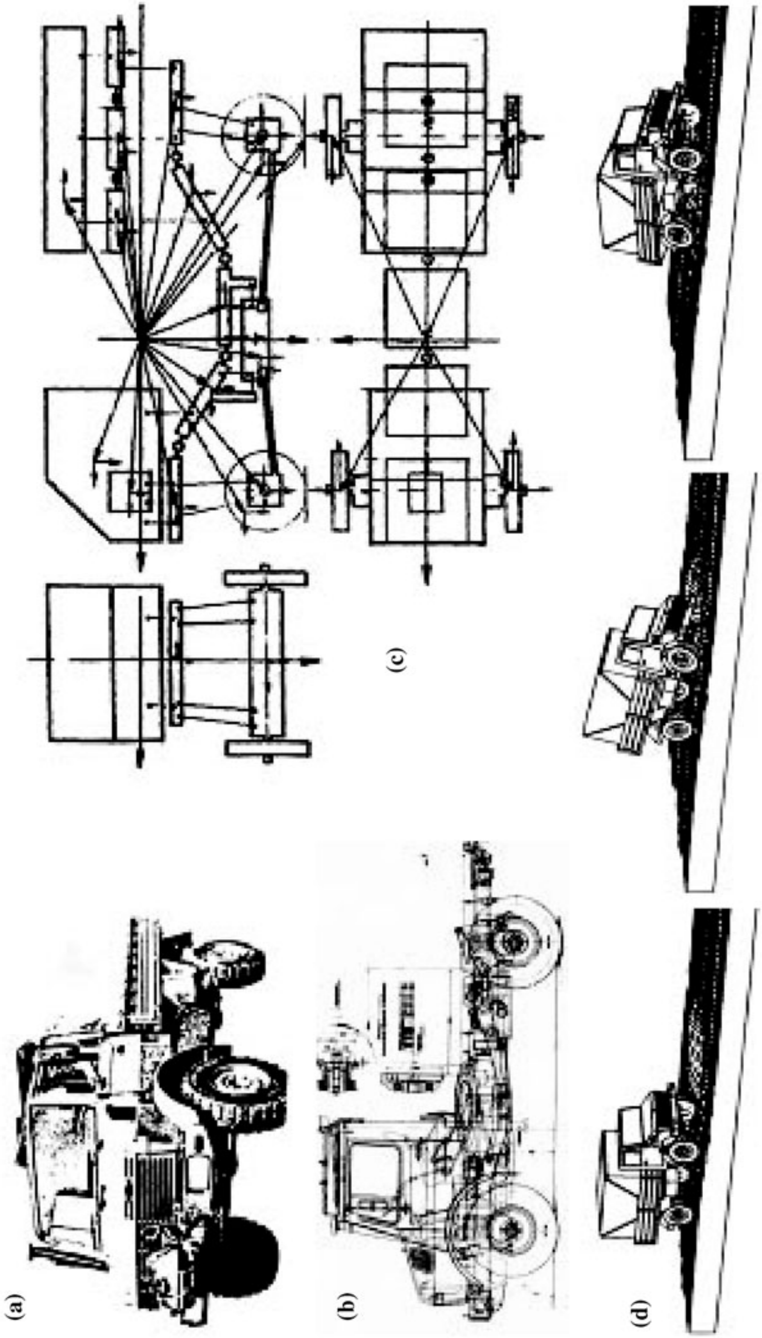


Fig. 1.14: Photograph (a), technical drawing (b), engineering model (c), and animation graphics (d) of a truck obtained using the program NUSTAR at IABG, Ottobrunn

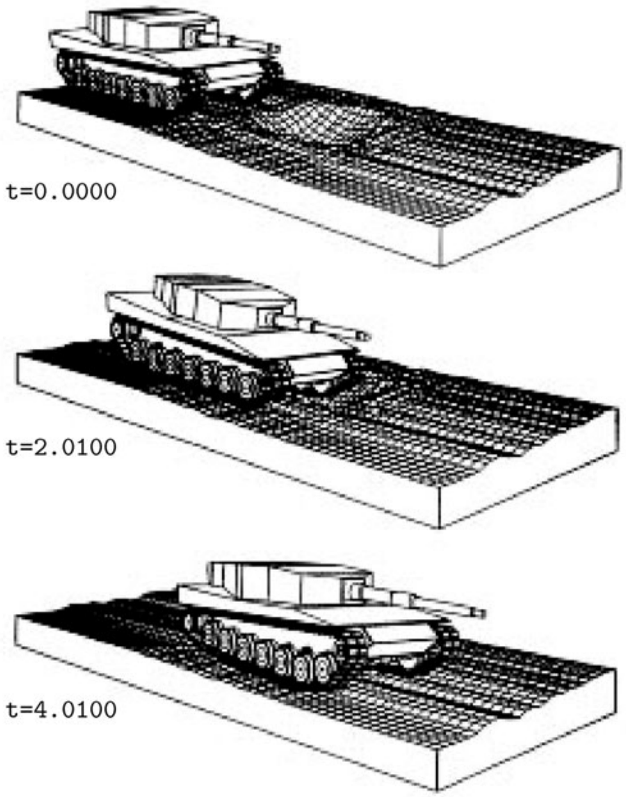


Fig. 1.15: Animation graphics of the german tank Leopard II, obtained using the program NUSTAR at IABG, Ottobrunn

Programs that can handle a large variety of different complex rigid-body mechanisms are called *general-purpose rigid-body analysis programs.*

1.5 General-purpose rigid-body analysis programs

Rather than relying on *"clever formulations"* and on *special-purpose programs* for simulating mechanisms, *general-purpose rigid-body analysis programs* have been developed in the past two decades to automatically *set up model equations* of those systems (usually in numerical form), and *solve* them *numerically* ([1], [3], [4], [5], [6], [7], [40], [41]). While the number and type of elements and the kinematics of the above applications may differ significantly, the modeling concept remains the same. *General purpose programs* are based on methods for *systematically deriving the model equations.* As there are mechanisms for which dependent coordinates cannot be eliminated symbolically (e.g. for special classes of mechanisms with loop structure), symbolic computer algebra computations are usually replaced by pure numerical computations. Due to the efficient and accurate numerical algorithms used in these programs, they are capable of providing *reliable numerical solutions* of the model equations. Due to user-friendly graphical interfaces and efficient pre- and postprocessors, many of these programs enable users with comparatively poor knowledge of theoretical mechanics to rapidely model, simulate and analyze complex rigid-body systems numerically. As a result, *general-purpose rigid-body analysis programs* are very flexible. They can be easily handled and applied to a large variety of different industrial mechanisms.

A *general purpose rigid-body analysis program* performs four basic tasks:

1. Accepts the model and control data from the user (preprocessor, input phase).
2. Generates the model equations (usually in numerical form as DAEs).
3. Solves the model equations.
4. Delivers the desired results to the user (postprocessor, output phase).

Steps 2 and 3 are essentially performed by the *program.* The *user* of a general purpose analysis program has to:

1. *Set up an engineering model* of the mechanism taking into account the *purpose* of the model *(pre-input phase).*
2. *Choose*, *collect* and *enter control data of the program*, and *data of the engineering model (input phase).*
3. Select desired *output data (output phase).*
4. Judge their quality *(post-output phase).*

1.5.1 Design of an engineering model

An engineering model is a schematic drawing of a mechanism that includes all components and data needed to fulfill its purpose. Therefore an engineering model of a mechanism is very dependent on the *purpose of the model.* Depending on the intended application of the model, *simplifying assumptions* of the mechanism are made to reduce the expenditure of modeling and simulation effort. Consider, for example, the vehicle of Figure 1.14. If investigations focus on the *driving behavior* and *driving stability* of the vehicle, high-frequency vibrational motions of the vehicle components play only a minor role; they will be excluded from the engineering model. Instead, sophisticated tire models will be included in this engineering model. As a consequence the design of a suitable engineering model is of crucial importance for the efficient and successful application of the model. The user has to decide: (1) which *components of the mechanism* must be included in the model, and (2) which type of *component models* and *characteristics* must be chosen, in order to efficiently achieve the objectives of the intended use of the model. Choosing inadequate component models prevents the user from achieving satisfactory model validation results even when the model parameters are carefully identified in laboratory experiments. Figure 1.16a shows a comparison between field experiments and computer simulation results obtained by a truck model that includes a simple engine model ([38]). Various model parameter identification and model validation experiments (that took months of intensive work) could not provide a satisfactory agreement between the computer simulations and the field experiments. Replacing the simplified engine model of the truck by a more sophisticated engine model, and introducing a simple driver model,provided in a single step excellent agreement between the time histories of all simulated and measured variables of the system (Figure 1.16b).

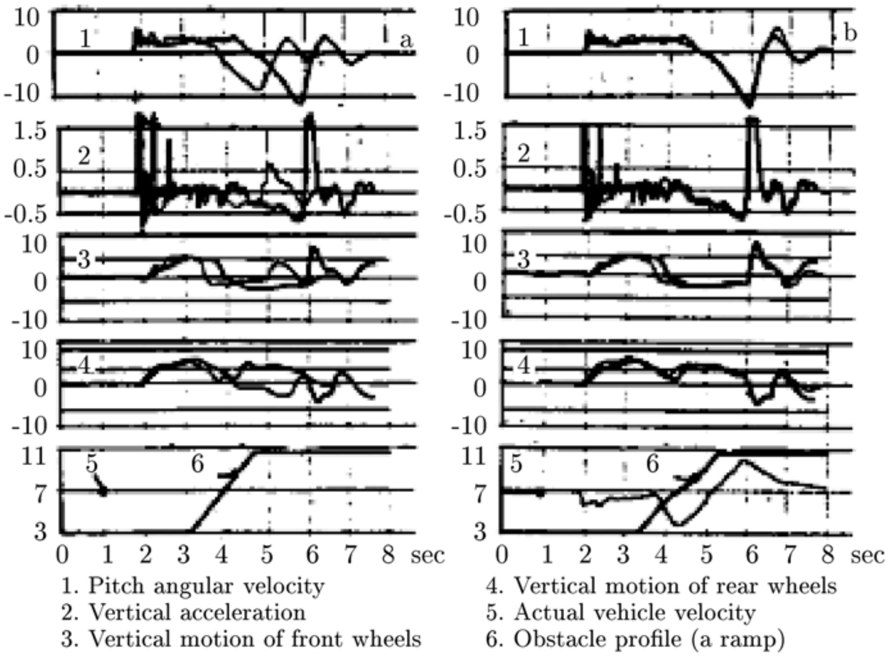


Fig. 1.16: Measured and simulated vertical transient motions of a truck crossing an obstacle (a ramp): without sophisticated engine and driver modules (a) and including sophisticated engine and driver modules (b)

The success in setting up an *engineering model* depends crucially on the *practical experience* of the user, on his intuitive *understanding of the components and properties* of the mechanism, and on his *understanding of possible* *critical situations* that may occur in the practical behavior of a mechanism as well as, to a certain extent, in the modeling and numerical solution process. In detail, the design of an *engineering model* includes the following steps (compare the truck model of Figure 1.14):

1. *Simplification of the system* by isolating elements or components of primary importance (for the *purpose* of the model) and *construction of a schematic drawing of the mechanism.* The actual shape or outline of a body may not be of immediate concern to the modeling process.
2. *Choice of rigid-bodies* as models of components with significant inertial properties and collection of *inertial parameters* such as the *mass*, the *coordinates of the center of mass*, and the *moments and products of inertia.*
3. *Choice* of an *inertial frame* R of the mechanism, of a *local reference frame* L_i for each rigid body i, fixed on that body, and of various other *local (body-fixed) frames* L_{ij} used to identify attachment points, locations, and orientations of connection elements between the bodies, and of forces, torques, actuators, and sensing elements.
4. Choice of the *joint types* of the connection elements between the bodies.
5. Choice of the *spring, damper, and friction elements* together with the associated characteristics.
6. Choice of the types of the *external forces/torques* acting on the bodies together with the force/torque *characteristics*, lines of action of forces, and rotation axes of torques.
7. Refinement of the *diagram or network* of the engineering model that includes all components (e.g. rigid-bodies, connection elements and drivers), all frames, all displacement vectors, rotation angles, all force and torque vectors, all lines of action of forces, and all rotation axes of torques of the mechanism.

1.5.2 Input and output data

Based on the engineering model (network) of a chosen mechanism, all *physical* and *geometric model data* needed to setup the simulation program are collected and entered into the model data file. They include:

1. Data that control the *size* and *complexity* of the mechanism such as the *number* (n_b) *of rigid-bodies*, and the *number* (n_c) *and type of joints.*
2. Data of the *reference points* and *local frames* on the bodies with respect to a chosen *inertial frame.*
3. Data of the *inertia parameters* of the rigid bodies such as the *mass* (m), the *center of mass* ($\boldsymbol{r}_{\mathrm{PC}}$), the *moments and products of inertia* ($J_{ij\mathrm{C}}$).
4. Data that specify the *connectivity* of the mechanism such as the *attachment points* of the joints, springs, dampers, actuators and the *orientation* of the external forces and torques.
5. Data about the *characteristics* of the springs, dampers, tires, etc.

In addition to these model data, *control data* are entered that select and control desired *options of the intended simulation runs* and analysis steps. Other control data select *desired output data* and control their *representation.*

The collection and input of model and control data of theoretical models of complex mechanisms is a cumberful and tedious task that must be prepared very carefully in order to efficiently achieve correct simulation results, represented in a form that can be easily interpreted and efficiently judged.

1.6 Purpose of this monograph

General purpose rigid-body analysis programs are widely and successfully used in industry to automatically *set up model equations* of quite complex rigid-body mechanisms in numerical form, *solve* these *equations*, and hence *simulate* these mechanisms on a *computer.* As well as usually not being as computationally efficient as *special-purpose programs*, *general-purpose rigid-body analysis programs* may have two further *drawbacks*:

1. Due to the user-friendly interface of these programs, users with minor knowledge and understanding of the underlying laws of mechanics and of the problems and fallacies that may occur in the modeling and solution process can apply these programs to complex applications, with the consequence that they may provide simulation results that are much more erroneous than any results measured in laboratory experiments. This is an increasingly observed phenomenon in industry (similar observations are made for laboratory experiments based on digital measurement equipment, performed by engineers with minor knowledge of digital signal analysis).
2. These programs usually do not provide model equations of rigid-body mechanisms in *symbolic form* that are often needed in the design of various nonlinear *control algorithms,* nonlinear *prefilter algorithms, disturbance compensation algorithms*, *signal-* or *image-processing algorithms*, and in algorithms used as model hypotheses in *model parameter identification.*

This monograph is addressed to all engineers that whish to *model*, *simulate*, *control*, or/and *experimentally identify* rigid-body mechanisms. These engineers may be from the areas of *mechanics*, *robotics*, and *mechatronics*, as well as from other areas such as *control*, *electronics*, *hydraulics*, and *signal processing*, or even from disciplines such as *physics*, *informatics*, and *applied mathematics.*

Volume I presents:

1. An introduction into the *foundations* of rigid-body mechanics of mechanisms.

2. A *systematic approach* for *deriving, as a first step, symbolic model equations* of mechanisms in *DAE form.*

Volume II presents:

1. Various *exercises to systematically apply this modeling approach to examples of planar and spatial mechanisms.*
2. A systematic approach for *mapping the DAEs* in a second step *into symbolic DEs*, into *nonlinear* and *linear state-space equations*, and sometimes also into *transfer function form.*

The *objectives* of both the *theoretical discussions* (Volume I), and the *practical applications* (Volume II) are (Table 1.1) to overcome some of the above mentioned drawbacks of general-purpose rigid-body analysis programs, by:

1. *Preparing the reader for efficiently handling and application of general-purpose computer programs to complex mechanisms*:
   1.1 To obtain a deeper understanding of the basic mechanical relations behind the software packages.
   1.2 To set up adequate engineering models of mechanisms and to choose suitable component models, coordinates and frames.
   1.3 To become more sensitive and confident with the possibilities, restrictions, and fallacies when applying rigid-body programs to practical applications.
   1.4 To find adequate interpretations of the simulation results obtained by these programs.
   1.5 To gain enough intuitive understanding for reasonably and critically judging and evaluating the modeling approach.
2. *Systematically deriving analytical mathematical models of mechanisms in DAE and/or DE form*, as is often needed in:
   2.1 The *design of sophisticated linear and nonlinear control, disturbance compensation, and signal-processing algorithms.*
   2.2 The *theoretical analysis* of model equations like symbolic linearization, eigenvalue analysis, and stability analysis, sensitivity analysis or frequency response analysis.
   2.3 *Experimental identification* of model parameters and *model validation* of mechanisms.

Various simple and more advanced *examples* of planar and spatial mechanisms of Figures 1.17, 1.18, 1.19, and 1.20 will be theoretically modeled in detail and discussed in *Volume II.*

The *objectives* of these books are to provide:	Applications		
	Computer simulations	Analysis	Control and identi-fication
background *understanding* of the *theoretical foundations* of rigid body mechanisms	X	X	X
the ability of *setting up engineering models* that serve certain purposes (choice of component models, coordinates, and frames)	X	X	X
the ability of systematically *deriving model equations* of mechanisms in *symbolic DAE form*		X	X
sufficient *sensitivity* with respect to possible *singular situations and fallacies* in the model equations	X	X	X
some routine in *mapping and in simplifying model equations from DAE to DE form* (linearization, state-space and frequency response representations) for special applications and purposes		X	X

Table 1.1: Objectives of this monograph (Volumes I and II)

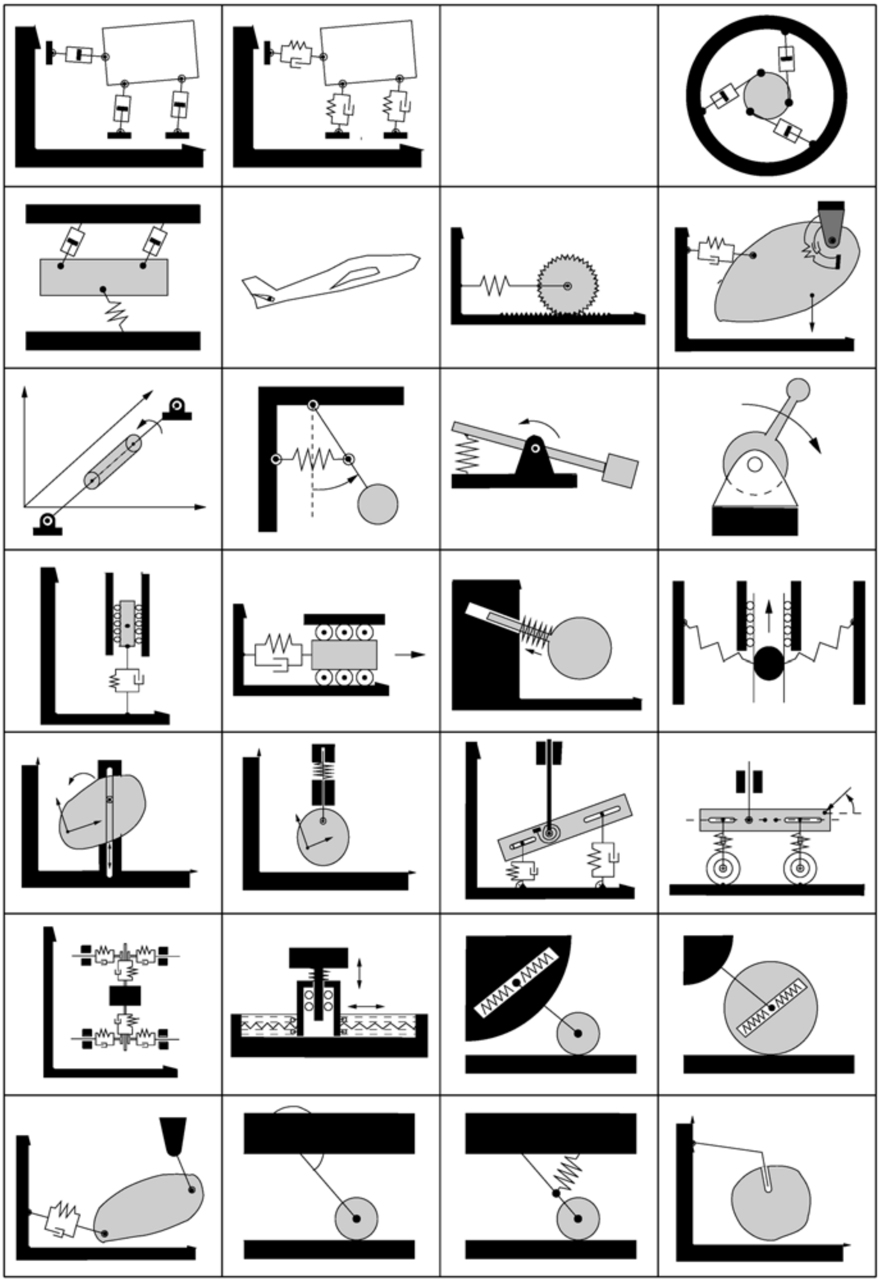


Fig. 1.17: Planar mechanisms including a single rigid body

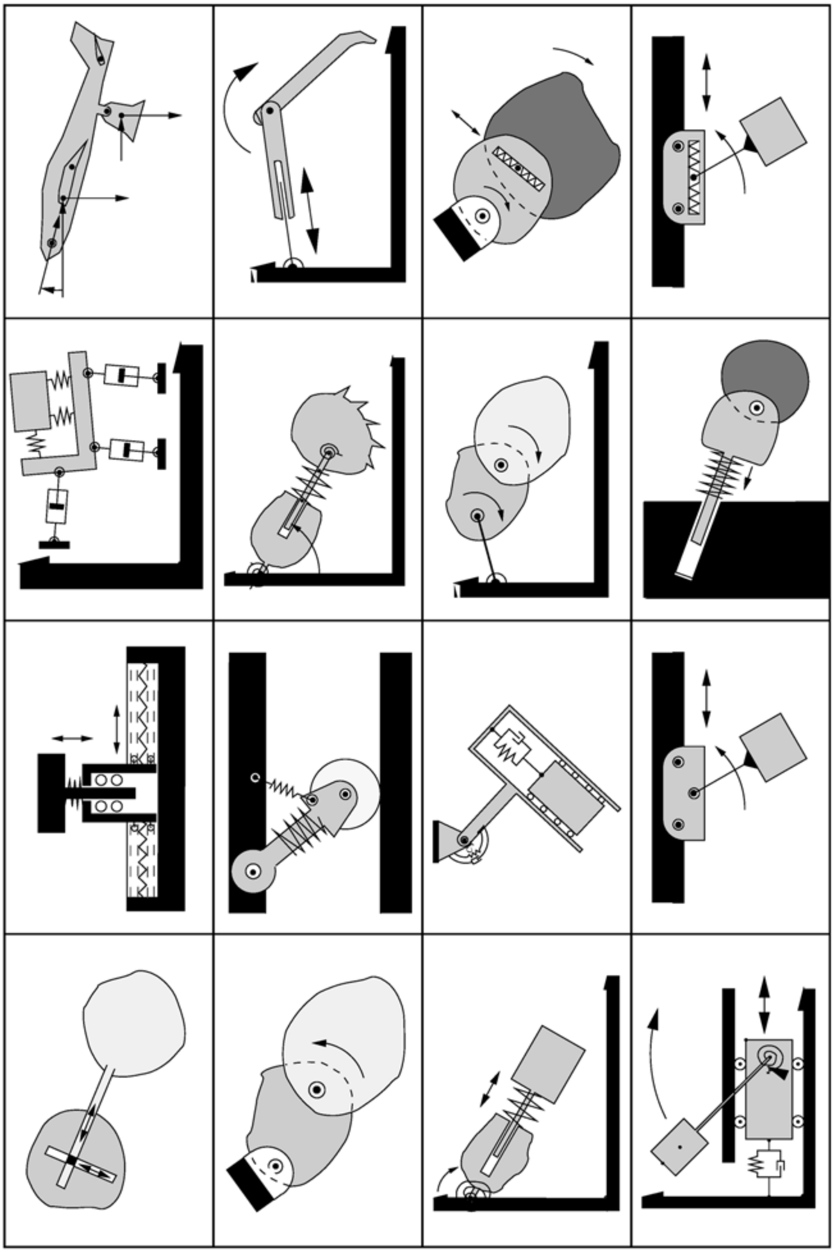


Fig. 1.18: Planar mechanisms including two rigid-bodies

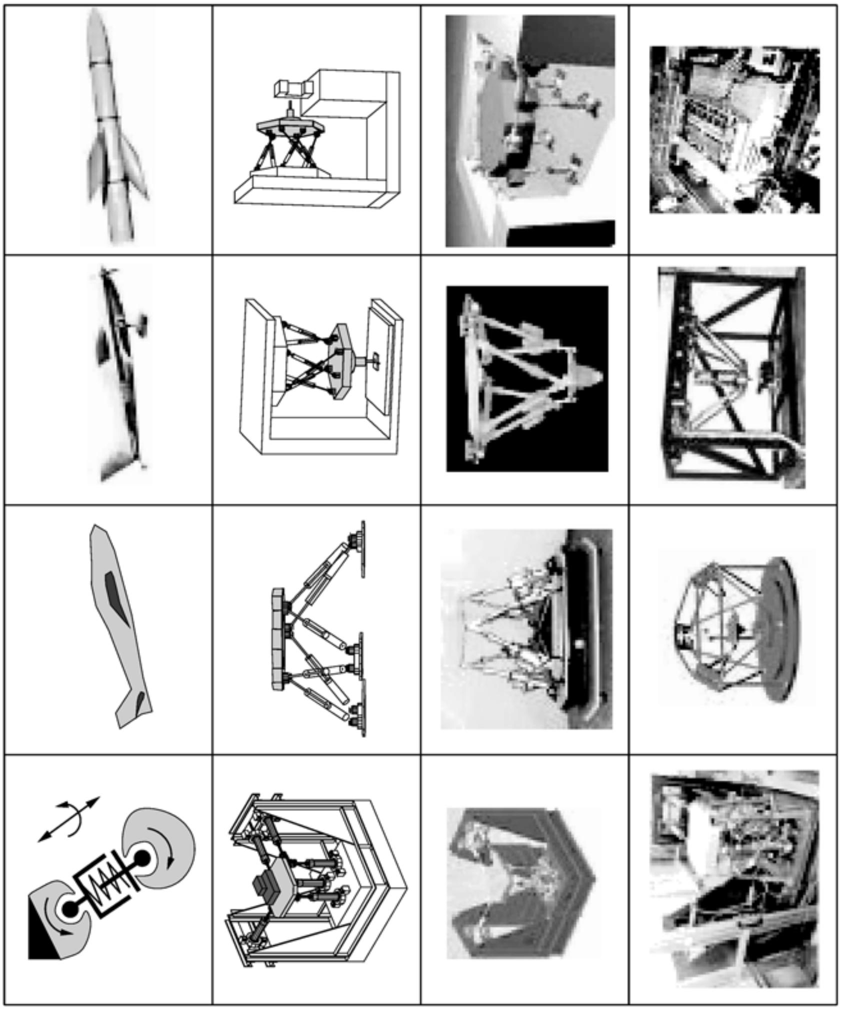


Fig. 1.19: Mechanisms including a single rigid body subject to large spatial motion

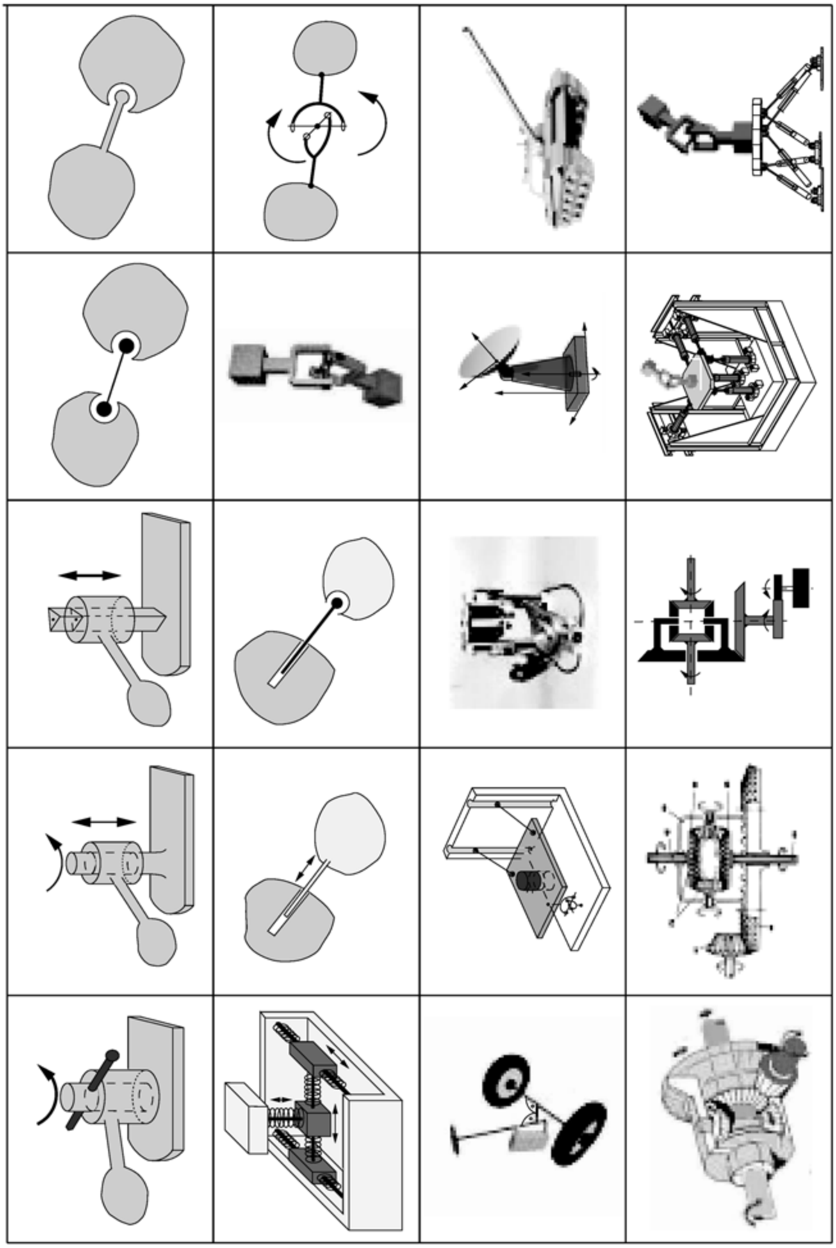


Fig. 1.20: Mechanisms including several rigid bodies subject to large spatial motion

2. Planar and spatial vectors, matrices, and vector functions

A treatise of vector algebra and vector analysis is a standard ingredient in basic engineering education, and the reader will undoubtedly have some knowledge of this topic. However, in this chapter, we shall provide a refresher on some basic concepts that will be instrumental for the further development in this book.

Vectors are basic entities of analytic geometry. They are defined by their geometric properties (invariants). These so-called *geometric vectors* are entities in their own right without referring to any special basis of a vector space. They allow compact formulations of physical laws and theoretical relationships. In particular, geometric vector notations provide an adequate tool for efficiently formulating the kinematics and dynamics of rigid-body systems. Alternative representations of vectors (with respect to a basis of an underlying vector space), referred to as *algebraic vectors*, are better suited to engineering applications of rigid-body dynamics including formula manipulation and computer implementation. *Differential calculus* of *vector functions* provides an approved tool for formulating and analysing *kinematic relations*. The development in this chapter is for *planar (Section 2.1)* and *spatial (Section 2.2)* vectors. *Spatial rotations* will be expressed in terms of *Bryant angles*. They may include singular situations that might be avoided by Euler parameter or quaternion formulations of rotations. Due to the introductory character of this book, quaternion formulation will not be discussed here. Multivariable calculus is written in a form that will be directly used in the formulation of constraint equations in *Section 3*. Some elementary results from Euclidean vector spaces, elementary vector algebra, and geometry, together with a discussion of time derivatives of vectors, orientation matrices, vector functions, and some results from multivariable calculus are collected in *Appendix A.1*.

2.1 Planar vectors and matrices

In this section vector representations, operations and transformations in the *plane* will be briefly discussed together with their time derivatives. Vectors and matrices will be written in bold faced letters.

2.1.1 Elementary vector and matrix operations

In this section planar *geometric* and *algebraic* vectors will be considered together with elementary vector operations and mappings of vectors by means of orientation matrices.

2.1.1.1 Geometric vectors. A vector (displacement, velocity, acceleration, angular velocity, angular acceleration, force, torque, linear momentum, or angular momentum) is an entity in its own right. Its basic properties (invariants) are independent of any reference frame and special coordinate representation. Those properties of vectors are theoretically studied and analysed in mathematics (analytical geometry), theoretical physics and analytical mechanics without referring to special coordinates. These vectors are sometimes called *geometric vectors.*

Consider the *geometric displacement vector* $\boldsymbol{r}_{PO}$ in Figure 2.1, with start point O and end point P. It is defined as the *directed straight line* from O to P and is represented by an arrow pointing from O to P. This vector has the *length* $|\boldsymbol{r}_{PO}|$ and a *direction* with respect to another vector $\boldsymbol{r}_{QO}$ (from point O to point Q), described by an angle ψ_{QP}, measured from $\boldsymbol{r}_{PO}$ to $\boldsymbol{r}_{QO}$. The *sum of two vectors* $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QP}$ is defined as the vector $\boldsymbol{r}_{QO}$ from point O to point Q, written as (Figure 2.1)

$$\boldsymbol{r}_{QO} = \boldsymbol{r}_{QP} + \boldsymbol{r}_{PO} \ . \tag{2.1}$$

The *scalar product* (or dot product) of two vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$ is defined as product of the magnitudes of the vectors times the cosine of the angle between them,

$$\boldsymbol{r}_{PO} \bullet \boldsymbol{r}_{QO} := |\boldsymbol{r}_{PO}| \cdot |\boldsymbol{r}_{QO}| \cdot \cos\psi_{QP} \in \mathbb{R}^1 , \tag{2.2a}$$

where $|\boldsymbol{r}_{QO}| \cdot \cos\psi_{QP}$ is the *projection of* $\boldsymbol{r}_{QO}$ *onto* $\boldsymbol{r}_{PO}$ (Figure 2.1). For $\boldsymbol{r}_{PO} \neq \boldsymbol{0}$ and $\boldsymbol{r}_{QO} \neq \boldsymbol{0}$ the scalar product is only zero if $\cos\psi_{QP} = 0$. Two vectors are said to be *orthogonal* to each other if their scalar product is zero; i.e.,

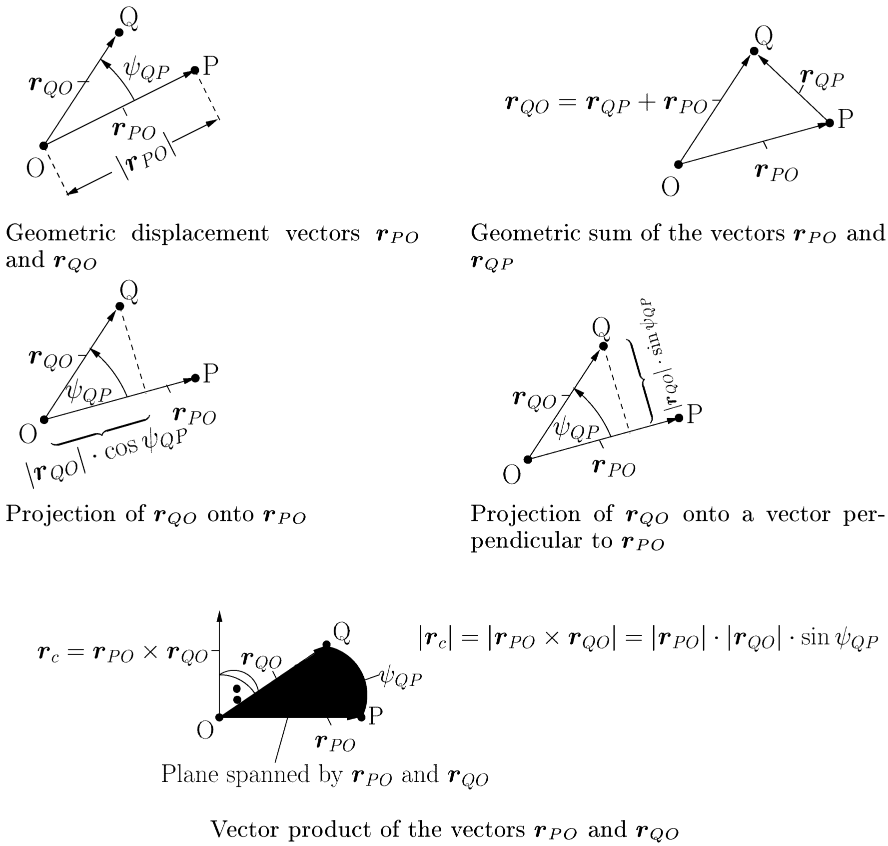
$$\boldsymbol{r}_{PO} \bullet \boldsymbol{r}_{QO} = 0 \quad \textit{(orthogonal vectors } \boldsymbol{r}_{PO} \textit{ and } \boldsymbol{r}_{QO}\textit{)} \ . \tag{2.2b}$$

Since $\psi_{QP} = 2\pi - \psi_{QP}$, the order of the factors of a scalar product is immaterial. For each vector $\boldsymbol{r}_{PO}$

$$\boldsymbol{r}_{PO} \bullet \boldsymbol{r}_{PO} = |\boldsymbol{r}_{PO}|^2 \in \mathbb{R}^1 \quad \text{(square of the length of } \boldsymbol{r}_{PO}\text{)} , \tag{2.3}$$

due to $\cos\psi_{PP} = \cos 0 = 1$. The *vector product (or cross product)* of two vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$ is defined as the *vector* (Figure 2.1)

$$\boldsymbol{r}_{\mathrm{c}} := \boldsymbol{r}_{PO} \times \boldsymbol{r}_{QO} = \underbrace{(|\boldsymbol{r}_{PO}| \cdot |\boldsymbol{r}_{QO}| \cdot \sin\psi_{QP})}_{\in \mathbb{R}^1} \cdot \boldsymbol{e}_{\mathrm{c}} , \tag{2.4}$$



Geometric displacement vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$

Geometric sum of the vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QP}$

Projection of $\boldsymbol{r}_{QO}$ onto $\boldsymbol{r}_{PO}$

Projection of $\boldsymbol{r}_{QO}$ onto a vector perpendicular to $\boldsymbol{r}_{PO}$

Vector product of the vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$

Fig. 2.1: Geometric vector, vector length, vector sum, projections, and vector product

where $\boldsymbol{e}_\mathrm{c}$ is a unit vector (vector of length 1) that is orthogonal to the plane spanned by $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$, taken in the positive right-hand direction. Since reversal of the order of vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$ in (2.4) yields an opposite direction of $\boldsymbol{e}_\mathrm{c}$,

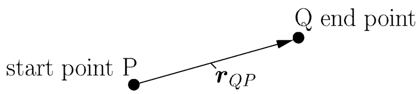
$$\boldsymbol{r}_{QP} \times \boldsymbol{r}_{PO} = -\boldsymbol{r}_{PQ} \times \boldsymbol{r}_{QP}. \tag{2.5}$$

Comment 2.1.1 (Cross product of planar vectors): The definition of the cross product of two vectors implies for the planar case ($\mathbb{R}^2$) that the vector generated by this cross product is orthogonal to this plane ($\mathbb{R}^2$). As a consequence the *plane is not closed under cross product operations.* This implies that cross product operations of vectors in $\mathbb{R}^2$ can only be formulated in the space $\mathbb{R}^3$, considered as an extension of $\mathbb{R}^2$. This will be done subsequently in order to describe *planar rotations* by expressions that are formal identical to *spatial rotations.*

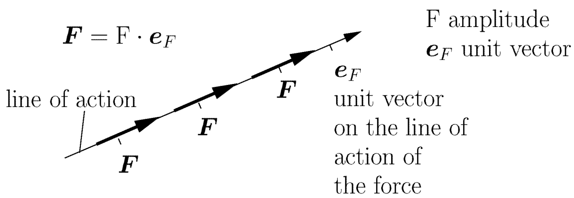
Comment 2.1.2 (Cross product vector): The above vector $\boldsymbol{r}_c$, introduced as the result of the cross product of the displacement vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$, is no longer a displacement vector. It has no start point and no target point, but only a direction and a length (compare *Comment 2.1.3* and the notion of a moment or torque vector, defined later).

Comment 2.1.3 (Different geometric vectors in mechanics): In rigid-body dynamics, different types of geometric vectors occur:

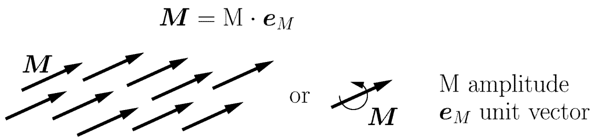
1. A *displacement vector* ($\boldsymbol{r}_{QP}$) defined by a fixed *start point* and by a fixed *end point* (Figure 2.2a).



(a) *Displacement vector* $\boldsymbol{r}_{QP}$ with fixed start point and fixed end point



(b) *Force vector* $\boldsymbol{F}$ as an element of a set of vectors with common line of action and equal length



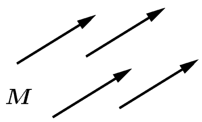
(c) *Moment (torque) vector* $\boldsymbol{M}$ as an element of a set of vectors of identical orientation, direction, and length; these vectors are not restricted to a common line of action

Fig. 2.2: Different types of geometric vectors in rigid-body dynamics

2. A *force vector* ($\boldsymbol{F} = \mathrm{F} \cdot \boldsymbol{e}_F$) as an element of a set of geometric vectors on a common *line of action*, with equal *length* $|\mathrm{F}|$ and with the amplitude F, where F is counted positive if the force acts in the direction of the arrow $\boldsymbol{F}$, and negative otherwise (Figure 2.2b) (vectors ($\boldsymbol{F}$, $\boldsymbol{M}$) are drawn by bold faced arrows).
3. A *moment (torque) vector* ($\boldsymbol{M} = \mathrm{M} \cdot \boldsymbol{e}_M$) as an element of a set of geometric vectors of *identical orientation*, *identical direction*, equal *length* $|M|$, and with an amplitude M. They are *not restricted to a common line of action* (Figure 2.2c). M is counted positive if the torque acts in the direction of the arrow $\boldsymbol{M}$.

Comment 2.1.4 (Vectors in rigid-body dynamics:):
Torque vectors of

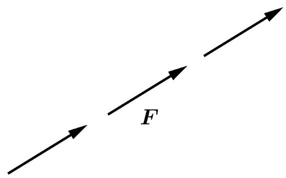
identical orientation,
identical direction, and
equal length



have an *equal action on a rigid body.*

Force vectors of

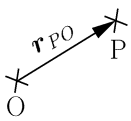
identical orientation,
identical direction,
equal length, and
placed on the same line of action



have an *equal action on a rigid body.*

Displacement vectors of

identical orientation,
identical direction,
equal length, and
with an identical start point



are equal.

These latter statements hold for *planar* and *spatial* geometric vectors.

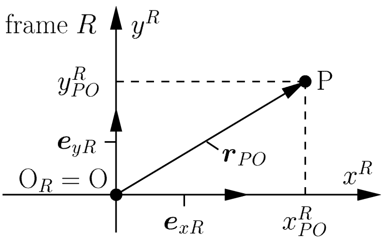
2.1.1.2 Algebraic vectors. Traditional vector analysis and algebra of *geometric vectors* are not well suited for computer implementation and formula manipulation. As a consequence, instead of using geometric vectors and vector products, etc., the equations of motion of rigid-body systems will be written in terms of *algebraic vectors* and *matrices* here, represented with respect to suitably chosen *local* and *global frames*. This *simplifies numerical computations* and *provides conceptual clearness* (as, for example, cross products of vectors are not included in the axioms of linear vector spaces and of linear algebra (*Appendix A.1.1*)).

Consider a reference frame R with the origin $O = \mathrm{O}_R$ and with the *planar orthogonal basis* $B_R = \{\boldsymbol{e}_{xR}\,,\,\boldsymbol{e}_{yR}\}$, where $\boldsymbol{e}_{iR}$ is a basis vector in $\mathbb{R}^2$, with

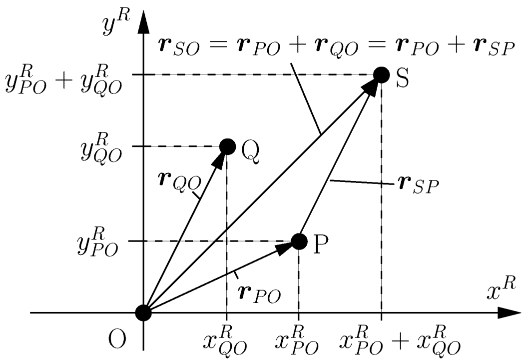
$$\boldsymbol{e}_{iR} \bullet \boldsymbol{e}_{jR} = \delta_{ij} := \begin{cases} 1 \;, \; i = j \\ 0 \;, \; i \neq j \end{cases} \;;\; i, j = x, y \;. \tag{2.6a}$$

Then (Figure 2.3a) $\boldsymbol{r}_{PO}$ can be represented with respect to frame R as

$$\boldsymbol{r}_{PO} = x^R_{PO} \cdot \boldsymbol{e}_{xR} + y^R_{PO} \cdot \boldsymbol{e}_{yR} \;, \tag{2.6b}$$



(a) Components of a vector $\boldsymbol{r}_{PO}$ in frame R



(b) Sum of two vectors

Fig. 2.3: Algebraic vectors

or in abbreviated *algebraic* form

$$\boldsymbol{r}_{PO}^{R} = \left(x_{PO}^{R}\,,\, y_{PO}^{R}\right)^{\mathrm{T}} = \begin{pmatrix} x_{PO}^{R} \\ y_{PO}^{R} \end{pmatrix} \in \mathbb{R}^2, \tag{2.6c}$$

where x_{PO}^{R} and y_{PO}^{R} are the *Cartesian coordinates* or components of $\boldsymbol{r}_{PO}$ with respect to frame R. Consider two vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$, represented in frame R (Figure 2.3b):

$$\boldsymbol{r}_{PO}^{R} = \begin{pmatrix} x_{PO}^{R} \\ y_{PO}^{R} \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_{QO}^{R} = \begin{pmatrix} x_{QO}^{R} \\ y_{QO}^{R} \end{pmatrix} . \tag{2.6d}$$

Then

$$\boldsymbol{r}_{SO}^{R} := \boldsymbol{r}_{PO}^{R} + \boldsymbol{r}_{QO}^{R} = \boldsymbol{r}_{PO}^{R} + \boldsymbol{r}_{SP}^{R} \tag{2.6e}$$

is defined as the *vector sum* of $\boldsymbol{r}_{PO}^{R}$ and $\boldsymbol{r}_{QO}^{R}$ with

$$\begin{aligned} \boldsymbol{r}_{SO}^{R} = \begin{pmatrix} x_{SO}^{R} \\ y_{SO}^{R} \end{pmatrix} &:= \begin{pmatrix} x_{PO}^{R} \\ y_{PO}^{R} \end{pmatrix} + \begin{pmatrix} x_{QO}^{R} \\ y_{QO}^{R} \end{pmatrix} \\ &= \begin{pmatrix} x_{PO}^{R} + x_{QO}^{R} \\ y_{PO}^{R} + y_{QO}^{R} \end{pmatrix} = \begin{pmatrix} x_{PO}^{R} + x_{SP}^{R} \\ y_{PO}^{R} + y_{SP}^{R} \end{pmatrix} . \end{aligned} \tag{2.6f}$$

Consider two reference frames R and L with a common origin $\mathrm{O}_L = \mathrm{O}_R = \mathrm{O}$ and with orthogonal basis vectors $\{\boldsymbol{e}_{xR}\,,\, \boldsymbol{e}_{yR}\}$ and $\{\boldsymbol{e}_{xL}\,,\, \boldsymbol{e}_{yL}\}$ (Figure 2.4). Then $\boldsymbol{r}_{PO}$ can be written (cf. Equation 2.6b) as

$$\boldsymbol{r}_{PO} = x_{PO}^{L} \cdot \boldsymbol{e}_{xL} + y_{PO}^{L} \cdot \boldsymbol{e}_{yL} \ . \tag{2.7}$$

The two algebraic representations (2.6b) and (2.7) of $\boldsymbol{r}_{PO}$ are abbreviated as

$$\boldsymbol{r}_{PO}^{R} = \begin{pmatrix} x_{PO}^{R} \\ y_{PO}^{R} \end{pmatrix} \quad \left(\begin{array}{l} \textit{representation of } \boldsymbol{r}_{PO} \textit{ by} \\ \textit{Cartesian coordinates in } R \end{array}\right) \tag{2.8a}$$

and

$$\boldsymbol{r}_{PO}^{L} = \begin{pmatrix} x_{PO}^{L} \\ y_{PO}^{L} \end{pmatrix} \quad \left(\begin{array}{l} \textit{representation of } \boldsymbol{r}_{PO} \textit{ by} \\ \textit{Cartesian coordinates in } L \end{array}\right). \tag{2.8b}$$

These representations of the *geometric vector* $\boldsymbol{r}_{PO}$ are sometimes called *algebraic representations* of $\boldsymbol{r}_{PO}$, or *algebraic vectors*. Then the following relations among the coordinates of a vector $\boldsymbol{r}_{PO}$ represented in different frames R and L with coinciding origins $\mathrm{O} = \mathrm{O}_R = \mathrm{O}_L$ are obtained by elementary geometry (Figure 2.4):

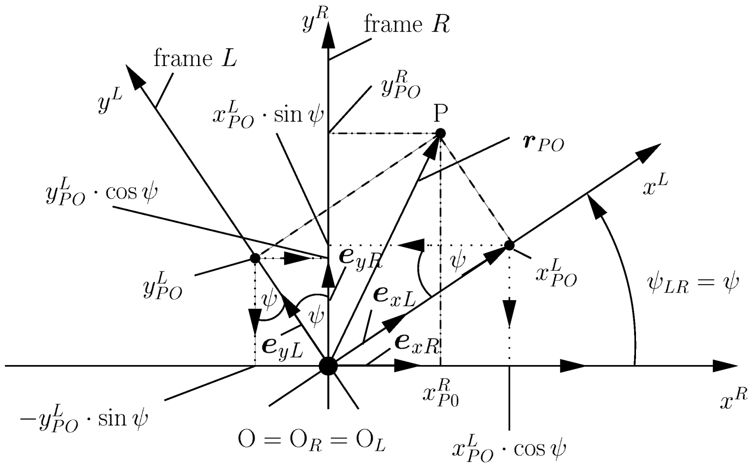


Fig. 2.4: Vector $\boldsymbol{r}_{PO}$ represented in frames R and L

$$\begin{aligned} x^R_{PO} &= x^L_{PO} \cdot \cos\psi - y^L_{PO} \cdot \sin\psi \\ y^R_{PO} &= x^L_{PO} \cdot \sin\psi + y^L_{PO} \cdot \cos\psi \end{aligned} \tag{2.9a}$$

or

$$\begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix}}_{=:\ \boldsymbol{A}^{RL}} \cdot \begin{pmatrix} x^L_{PO} \\ y^L_{PO} \end{pmatrix} \quad , \quad \psi := \psi_{LR} \ , \tag{2.9b}$$

or

$$\boldsymbol{r}^R_{PO} = \boldsymbol{A}^{RL}(\psi) \cdot \boldsymbol{r}^L_{PO} = \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO} \tag{2.9c}$$

with

$$\boldsymbol{A}^{RL} := \boldsymbol{A}^{RL}\left(\psi_{LR}\right) = \begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix} \tag{2.10}$$

as the *planar coordinate transformation matrix* or *planar orientation matrix* that maps a vector, represented in L, into a vector, represented in R, where $\psi = \psi_{LR}$ is the angle of rotation from R to L. The relations

$$\boldsymbol{A}^{RL} \cdot \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix} \cdot \begin{pmatrix} \cos\psi \ , & \sin\psi \\ -\sin\psi \ , & \cos\psi \end{pmatrix} \tag{2.11a}$$

$$= \begin{pmatrix} \cos^2\psi + \sin^2\psi & , \cos\psi\sin\psi - \sin\psi\cos\psi \\ \sin\psi\cos\psi - \cos\psi\sin\psi \ , & \sin^2\psi + \cos^2\psi \end{pmatrix} = \begin{pmatrix} 1 \ , & 0 \\ 0 \ , & 1 \end{pmatrix} = \mathbf{I}_2$$

and

$$\left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \left(\boldsymbol{A}^{RL}\right)^{-1} \ . \tag{2.11b}$$

hold for arbitrary values of ψ. As a consequence, the matrix $\boldsymbol{A}^{RL}$ is an *orthogonal matrix.* Due to

$$\psi_{LR} = -\psi_{RL} \tag{2.11c}$$

the following relations hold:

$$\boldsymbol{A}^{RL}(\psi_{LR}) = \boldsymbol{A}^{RL}(-\psi_{RL}) = \begin{pmatrix} \cos(-\psi_{RL}) \ , & -\sin(-\psi_{RL}) \\ \sin(-\psi_{RL}) \ , & \cos(-\psi_{RL}) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi_{RL} \ , & \sin\psi_{RL} \\ -\sin\psi_{RL} \ , & \cos\psi_{RL} \end{pmatrix} = \left(\boldsymbol{A}^{RL}(\psi_{RL})\right)^{\mathrm{T}} = \boldsymbol{A}^{LR}(\psi_{RL})$$

or

$$\boldsymbol{A}^{RL}(\psi_{LR}) = \boldsymbol{A}^{RL}(-\psi_{RL}) = \left(\boldsymbol{A}^{RL}(\psi_{RL})\right)^{\mathrm{T}} = \boldsymbol{A}^{LR}(\psi_{RL}) \ . \tag{2.11d}$$

As a *special example* of the above coordinate transformation, the basis vectors $\{\boldsymbol{e}_{xL} \, , \, e_{yL}\}$ of L will be represented in frame R (Figure 2.5). Due to (2.6a), $\boldsymbol{e}_{xL}$ and $\boldsymbol{e}_{yL}$, represented in L, are written as

$$\begin{aligned} &\boldsymbol{e}_{xL} = 1 \cdot \boldsymbol{e}_{xL} + 0 \cdot \boldsymbol{e}_{yL} \qquad \text{or} \qquad \boldsymbol{e}_{xL}^{L} = (1 \, , \, 0)^{\mathrm{T}} \\ &\text{and} \\ &\boldsymbol{e}_{yL} = 0 \cdot \boldsymbol{e}_{xL} + 1 \cdot \boldsymbol{e}_{yL} \qquad \text{or} \qquad \boldsymbol{e}_{yL}^{L} = (0 \, , \, 1)^{\mathrm{T}} \, . \end{aligned} \tag{2.12a}$$

Then, in agreement with Figure 2.5, the following relations hold:

$$\begin{aligned} &\boldsymbol{e}_{xL} = \cos\psi \cdot \boldsymbol{e}_{xR} + \sin\psi \cdot \boldsymbol{e}_{yR}, \quad \psi := \psi_{LR}, \quad \text{or} \\ &\boldsymbol{e}_{xL}^{R} = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix} = \begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \boldsymbol{e}_{xL}^{L} \end{aligned} \tag{2.12b}$$

(where $\boldsymbol{e}_{xL}^{R}$ is identical to the *first column vector of* $\boldsymbol{A}^{RL}$), and

$$\begin{aligned} &\boldsymbol{e}_{yL} = -\sin\psi \cdot \boldsymbol{e}_{xR} + \cos\psi \cdot \boldsymbol{e}_{yR}, \quad \text{or} \\ &\boldsymbol{e}_{yL}^{R} = \begin{pmatrix} -\sin\psi \\ \cos\psi \end{pmatrix} = \begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \boldsymbol{e}_{yL}^{L} \end{aligned} \tag{2.12c}$$

(where $\boldsymbol{e}_{yL}^{R}$ is identical to the *second column vector of* $\boldsymbol{A}^{RL}$). By analogy with (2.12b) and (2.12c), the following relations hold:

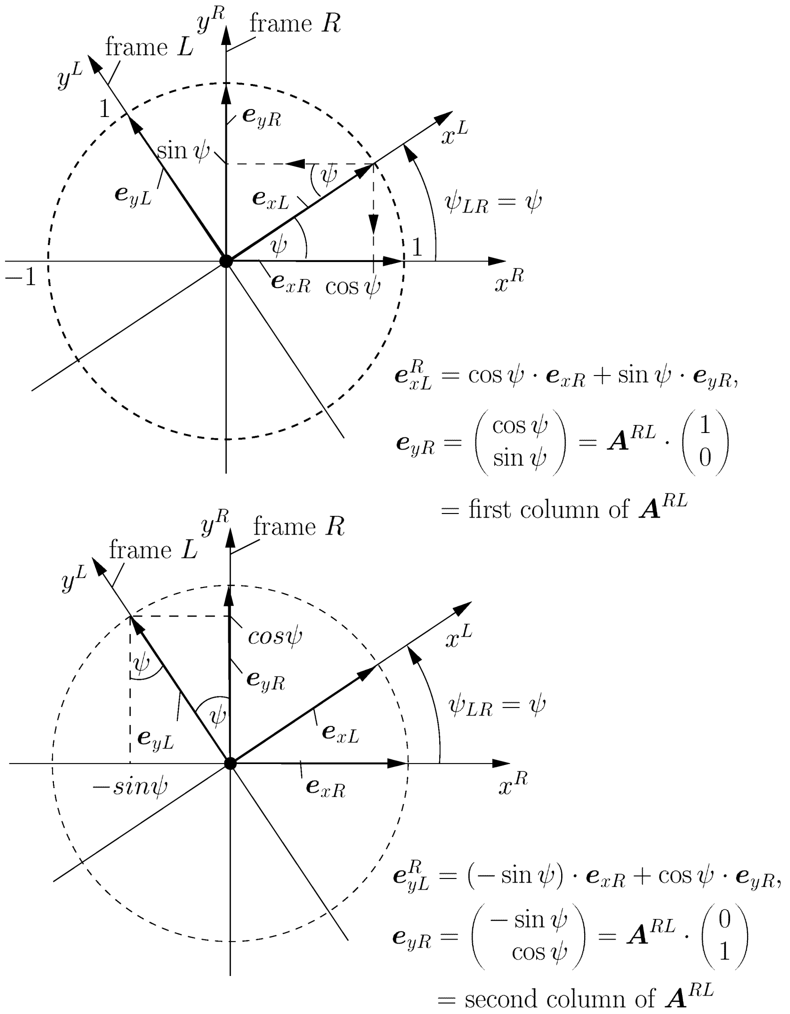


Fig. 2.5: Representation of the basis vectors $\boldsymbol{e}_{xL}$, $\boldsymbol{e}_{yL}$ of frame L in frame R

$$\boldsymbol{e}^L_{xR} = \boldsymbol{A}^{LR} \cdot \boldsymbol{e}^R_{xR} = \begin{pmatrix} \cos\psi \\ -\sin\psi \end{pmatrix} \quad \text{or} \quad \boldsymbol{e}_{xR} = \cos\psi \cdot \boldsymbol{e}_{xL} - \sin\psi \cdot \boldsymbol{e}_{yL}$$

and (2.13)

$$\boldsymbol{e}^L_{yR} = \boldsymbol{A}^{LR} \cdot \boldsymbol{e}^R_{yR} = \begin{pmatrix} \sin\psi \\ \cos\psi \end{pmatrix} \quad \text{or} \quad \boldsymbol{e}_{yR} = \sin\psi \cdot \boldsymbol{e}_{xL} + \cos\psi \cdot \boldsymbol{e}_{yL} \ .$$

Compared to the previously considered geometrical situation with *coinciding origins* O_R and O_L of frames R and L (Figure 2.4), a slightly new situation occurs when these *origins do not coincide* (Figure 2.6). Then the vector $\boldsymbol{r}_{PO}$ from $\mathrm{O}_R = \mathrm{O}$ to the point P can be written as (Figure 2.6)

$$\boldsymbol{r}_{PO} = \boldsymbol{r}_{O_LO} + \boldsymbol{r}_{PO_L} ,$$

or in coordinates of R

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_LO} + \boldsymbol{r}^R_{PO_L} \tag{2.14a}$$

with

$\boldsymbol{r}_{O_LO}$ as a *translation* (displacement vector) from $\mathrm{O}_R = \mathrm{O}$ to O_L.

Taking into account the above *rotation relation* (2.9c) of L with respect to R yields

$$\boldsymbol{r}^R_{PO_L} = \boldsymbol{A}^{RL}(\psi) \cdot \boldsymbol{r}^L_{PO_L} \quad , \quad \psi := \psi_{LR} \tag{2.14b}$$

with $\boldsymbol{r}^L_{PO_L}$ as a representation of $\boldsymbol{r}_{PO_L}$ in frame L. Combining the above two transformations implies that each coordinate transformation can be interpreted as a superposition of a *translation*

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_LO} + \boldsymbol{r}^R_{PO_L} \tag{2.15a}$$

and of a *rotation*

$$\boldsymbol{r}^R_{PO_L} = \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \ . \tag{2.15b}$$

Then

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \tag{2.15c}$$

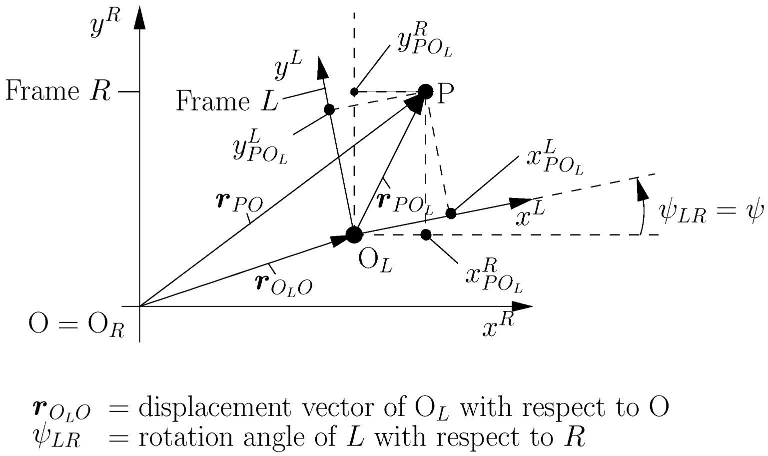


Fig. 2.6: Translation and rotation of frame L with respect to frame R

is a combined *translation rotation transformation.* Written in coordinate representation we have, with ψ_{LR} as the rotation angle of frame L from frame R

$$\begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} = \begin{pmatrix} x^R_{O_LO} \\ y^R_{O_LO} \end{pmatrix} + \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \end{pmatrix}$$

or (2.16)

$$\begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} = \begin{pmatrix} x^R_{O_LO} \\ y^R_{O_LO} \end{pmatrix} + \begin{pmatrix} x^L_{PO_L} \cdot \cos\psi_{LR} \ - \ y^L_{PO_L} \cdot \sin\psi_{LR} \\ x^L_{PO_L} \cdot \sin\psi_{LR} \ + \ y^L_{PO_L} \cdot \cos\psi_{LR} \end{pmatrix} .$$

The inverse relation of (2.15c) is

$$\boldsymbol{r}^L_{PO_L} = \boldsymbol{A}^{LR} \cdot \left(\boldsymbol{r}^R_{PO} - \boldsymbol{r}^R_{O_LO}\right) \tag{2.17a}$$

with the *special cases*:

$$\begin{array}{llll} \textit{pure rotation} & \boldsymbol{r}_{O_LO} = \mathbf{0} & (\text{for } \mathrm{O}_L \equiv \mathrm{O}), & \text{and} \\ \textit{pure translation} & \boldsymbol{A}^{RL} \ = \mathbf{I}_2 & (\text{for } \psi_{LR} \equiv \mathbf{0}). & \end{array} \tag{2.17b}$$

The relations (2.11b) and (2.15c) serve as a basis for deriving *kinematic relations* and *inverse kinematic relations* of planar rigid-body systems (based on *vector loop* and *orientation loop equations* in *Section 3*).

The *scalar product* of two vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$, introduced in (2.2), may be algebraically written as (*Appendix A.1.2*)

$$\begin{aligned} \boldsymbol{r}^R_{PO} \bullet \boldsymbol{r}^R_{QO} &:= \left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{QO}\right) \\ &= \left(x^R_{PO} \, , \, y^R_{PO}\right) \cdot \begin{pmatrix} x^R_{QO} \\ y^R_{QO} \end{pmatrix} = x_{PO} \cdot x_{QO} + y_{PO} \cdot y_{QO} \in \mathbb{R}^1 . \end{aligned} \tag{2.18}$$

The *scalar product* of two vectors is *independent of their representations:*

$$\begin{aligned} \left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{QO}\right) &= \left(\boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{QO}\right) \\ &= \left(\boldsymbol{r}^L_{PO}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{QO} = \left(\boldsymbol{r}^L_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^L_{QO}\right) . \end{aligned} \tag{2.19}$$

The *length of a vector* $\boldsymbol{r}_{PO}$ is

$$|\boldsymbol{r}_{PO}| = \left(\left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \boldsymbol{r}^R_{PO}\right)^{1/2} = \left({x^R_{PO}}^2 + {y^R_{PO}}^2\right)^{1/2} . \tag{2.20}$$

Due to (2.2b), the *scalar product* of *two orthogonal vectors* $\boldsymbol{r}^R_{PO}$ and $\boldsymbol{r}^R_{QO}$ is zero. Then

$$\left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \boldsymbol{r}^R_{QO} = \left(x^R_{PO}\,,\, y^R_{PO}\right) \cdot \begin{pmatrix} x^R_{QO} \\ y^R_{QO} \end{pmatrix} = x^R_{PO} \cdot x^R_{QO} + y^R_{PO} \cdot y^R_{QO} = 0 \tag{2.21a}$$

or

$$x^R_{PO} \cdot x^R_{QO} = -y^R_{PO} \cdot y^R_{QO}\ . \tag{2.21b}$$

Consider the vector $\left(\boldsymbol{r}^R_{PO}\right)^{\perp}$ generated from $\boldsymbol{r}^R_{PO}$ by the relation

$$\left(\boldsymbol{r}^R_{PO}\right)^{\perp} := \mathbf{R} \cdot \left(\boldsymbol{r}^R_{PO}\right) = \begin{pmatrix} 0\,, & -1 \\ 1\,, & 0 \end{pmatrix} \cdot \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} = \begin{pmatrix} -y^R_{PO} \\ x^R_{PO} \end{pmatrix}$$

with (2.22a)

$$\mathbf{R} := \begin{pmatrix} 0\,, & -1 \\ 1\,, & 0 \end{pmatrix}\ .$$

Then the relation

$$\begin{aligned} \left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{PO}\right)^{\perp} &= \left(x^R_{PO}\,,\, y^R_{PO}\right) \cdot \begin{pmatrix} -y^R_{PO} \\ x^R_{PO} \end{pmatrix} \\ &= -x^R_{PO} \cdot y^R_{PO} + x^R_{PO} \cdot y^R_{PO} = 0 \end{aligned} \tag{2.22b}$$

holds, $\left(\boldsymbol{r}^R_{PO}\right)^{\perp}$ *is orthogonal to* $\boldsymbol{r}^R_{PO}$, and the matrix $\mathbf{R}$ satisfies the *orthogonality relation*

$$\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \begin{pmatrix} 1\,, & 0 \\ 0\,, & 1 \end{pmatrix} = \mathbf{I}_2\ , \tag{2.22c}$$

and the relation

$$\mathbf{R} \cdot \mathbf{R} = \begin{pmatrix} -1\,, & 0 \\ 0\,, & -1 \end{pmatrix} = -\mathbf{I}_2 \tag{2.22d}$$

holds. In addition

$$\begin{aligned} &\left|\left(\boldsymbol{r}^R_{PO}\right)^{\perp}\right|^2 := \left(\left(\boldsymbol{r}^R_{PO}\right)^{\perp}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{PO}\right)^{\perp} \\ &= \left(-y^R_{PO}\,,\, x^R_{PO}\right) \cdot \begin{pmatrix} -y^R_{PO} \\ x^R_{PO} \end{pmatrix} = \left(x^R_{PO}\right)^2 + \left(y^R_{PO}\right)^2 \\ &= \left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{PO}\right) = \left(\boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\mathbf{R}^{\mathrm{T}} \cdot \mathbf{R}\right) \cdot \boldsymbol{r}^R_{PO} = \left(\mathbf{R} \cdot \boldsymbol{r}^R_{PO}\right)^{\mathrm{T}} \cdot \left(\mathbf{R} \cdot \boldsymbol{r}^R_{PO}\right). \end{aligned}$$

In summary, the matrix $\mathbf{R}$ maps a vector $\boldsymbol{r}^R_{PO} \in \mathbb{R}^2$ into a vector $\left(\boldsymbol{r}^R_{PO}\right)^{\perp}$ that is *orthogonal* to $\boldsymbol{r}^R_{PO}$ and has *equal length.* Then $\mathbf{R}$ *rotates* $\boldsymbol{r}^R_{PO}$ by 90 degrees around an axis perpendicular to the x–y plane. As a consequence, $\mathbf{R}$ will be called *orthogonal rotation matrix in the x–y plane.* This is in agreement with the relation

$$\mathbf{R} = \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} = \begin{pmatrix} \cos 90° \ , & -\sin 90° \\ \sin 90° \ , & \cos 90° \end{pmatrix} = \boldsymbol{A}^{RL}(\psi_{LR} = 90°), \tag{2.22e}$$

where the frame L is obtained from the frame R by a rotation of $\psi_{LR} = 90°$.

The *vector product of two vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$ in $\mathbb{R}^3$*, and represented in frame R, is (*Appendices A.1.2 and A.1.3*)

$$\begin{aligned} \boldsymbol{r}_{PO} \times \boldsymbol{r}_{QO} = & \left(y_{PO}^R \cdot z_{QO}^R - z_{PO}^R \cdot y_{QO}^R\right) \cdot \boldsymbol{e}_{xR} \\ & + \left(z_{PO}^R \cdot x_{QO}^R - x_{PO}^R \cdot z_{QO}^R\right) \cdot \boldsymbol{e}_{yR} \\ & + \left(x_{PO}^R \cdot y_{QO}^R - y_{PO}^R \cdot x_{QO}^R\right) \cdot \boldsymbol{e}_{zR} \end{aligned} \tag{2.23a}$$

or

$$\boldsymbol{r}PO^R \times \boldsymbol{r}QO^R = \begin{pmatrix} y_{PO}^R \cdot z_{QO}^R - z_{PO}^R \cdot y_{QO}^R \\ z_{PO}^R \cdot x_{QO}^R - x_{PO}^R \cdot z_{QO}^R \\ x_{PO}^R \cdot y_{QO}^R - y_{PO}^R \cdot x_{QO}^R \end{pmatrix} . \tag{2.23b}$$

Introducing the *skew-symmetric matrix*

$$\tilde{\boldsymbol{r}}_{PO}^R := \begin{pmatrix} 0 & , -z_{PO}^R \ , & y_{PO}^R \\ z_{PO}^R \ , & 0 & , -x_{PO}^R \\ -y_{PO}^R \ , & x_{PO}^R \ , & 0 \end{pmatrix}$$

constructed from $\boldsymbol{r}_{PO}^R = (x_{PO}^R\,,\, y_{PO}^R\,,\, z_{PO}^R)^{\mathrm{T}}$, shows that the product

$$\begin{aligned} \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{r}_{QO}^R &= \begin{pmatrix} 0 & , -z_{PO}^R \ , & y_{PO}^R \\ z_{PO}^R \ , & 0 & , -x_{PO}^R \\ -y_{PO}^R \ , & x_{PO}^R \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{QO}^R \\ y_{QO}^R \\ z_{QO}^R \end{pmatrix} \\ &= \begin{pmatrix} y_{PO}^R \cdot z_{QO}^R - z_{PO}^R \cdot y_{QO}^R \\ z_{PO}^R \cdot x_{QO}^R - x_{PO}^R \cdot z_{QO}^R \\ x_{PO}^R \cdot y_{QO}^R - y_{PO}^R \cdot x_{QO}^R \end{pmatrix} \end{aligned} \tag{2.24}$$

yields the same vector as the cross product (2.23b). As a consequence, the *vector product* (2.23) can be replaced by the *product of a matrix* $\tilde{\boldsymbol{r}}_{PO}^R$ times an *algebraic vector* $\boldsymbol{r}_{QO}^R$ (2.24):

$$\boldsymbol{r}_{PO}^R \times \boldsymbol{r}_{QO}^R = \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{r}_{QO}^R \in \mathbb{R}^3 \ . \tag{2.25}$$

Given two *vectors* $\boldsymbol{r}^R_{PO}$ and $\boldsymbol{r}^R_{QO}$, located inside the x–y plane in $\mathbb{R}^3$,

$$\boldsymbol{r}^R_{PO} := \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}^R_{QO} := \begin{pmatrix} x^R_{QO} \\ y^R_{QO} \\ 0 \end{pmatrix} . \tag{2.26a}$$

Then the vector

$$\tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{r}^R_{QO} = \begin{pmatrix} y^R_{PO} \cdot z^R_{QO} - z^R_{PO} \cdot y^R_{QO} \\ z^R_{PO} \cdot x^R_{QO} - x^R_{PO} \cdot z^R_{QO} \\ x^R_{PO} \cdot y^R_{QO} - y^R_{PO} \cdot x^R_{QO} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x^R_{PO} \cdot y^R_{QO} - y^R_{PO} \cdot x^R_{QO} \end{pmatrix} \tag{2.26b}$$

is perpendicular (orthogonal) to the x–y plane. As a consequence, *the x–y plane is not closed under vector-product operations.* In the *planar* case the above cross product operation will be abbreviated in the form

$$\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO} = \mathbf{R} \cdot \dot{\psi}_{LR} \cdot \boldsymbol{r}^L_{PO} \tag{2.27a}$$

with

$$\boldsymbol{\omega}^L_{LR} = \boldsymbol{\omega}^R_{LR} = \left(0\,,\,0\,,\,\dot{\psi}_{LR}\right)^{\mathrm{T}} = (0\,,\,0\,,\,1)^{\mathrm{T}} \cdot \dot{\psi}_{LR} \tag{2.27b}$$

as the angular velocity vector around the z-axis perpendicular to the x–y plane, with $\dot{\psi}_{LR}$ as the time derivative of the rotation angle ψ_{LR}, with $\mathbf{R}$ as the orthogonal rotation matrix in the x–y plane (Equation 2.22e and Section 2.1.2.1), and with

$$\tilde{\boldsymbol{\omega}}^L_{LR} := \left(\tilde{\boldsymbol{\omega}}^L_{LR} \text{ restricted to the } x\text{–}y \text{ plane in } \mathbb{R}^3\right) \tag{2.27c}$$

$$= \left[\begin{pmatrix} 0\,, & -1\,, & 0 \\ 1\,, & 0\,, & 0 \\ 0\,, & 0\,, & 0 \end{pmatrix} \cdot \dot{\psi}_{LR}\right]\Bigg|_{\mathbb{R}^2 = \mathbb{R}^2_{xy}} = \begin{pmatrix} 0\,, & -1 \\ 1\,, & 0 \end{pmatrix} \cdot \dot{\psi}_{LR} = \mathbf{R} \cdot \dot{\psi}_{LR},$$

or (2.27d)

$$\tilde{\boldsymbol{\omega}}_{LR} = \mathbf{R} \cdot \dot{\psi}_{LR} \cdot \boldsymbol{e}_{zR} = \mathbf{R} \cdot \dot{\psi}_{LR} \cdot \boldsymbol{e}_{zL} \ .$$

The above notation will be used in the planar case to obtain *identical formal relations for rotations* of *planar* and *spatial* mechanisms.

2.1.2 Time derivatives of displacement vectors and orientation matrices

Time derivatives of displacement vectors and orientation matrices provide velocity vectors, angular velocity vectors, acceleration vectors, and angular acceleration vectors as basic ingredients of rigid-body dynamics. In this section, time derivatives of *planar* vectors and orientation matrices will be considered.

2.1.2.1 Velocities and angular velocities. Consider a *planar* vector $\boldsymbol{r}_{O_L O}$ from point O of frame R to point O_L (Figure 2.6), where $\boldsymbol{r}_{O_L O}$ is assumed to change its length and orientation smoothly in time; i.e.,

$$\boldsymbol{r}_{O_L O} = \boldsymbol{r}_{O_L O}(t). \tag{2.28a}$$

The *time derivative* of $\boldsymbol{r}_{O_L O}(t)$

$$\boldsymbol{v}_{O_L O}(t) := \dot{\boldsymbol{r}}_{O_L O}(t) := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{O_L O}(t) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}_{O_L O}(t + \Delta t) - \boldsymbol{r}_{O_L O}(t)}{\Delta t}$$

can be measured with respect to frame R while $\boldsymbol{r}_{O_L O}(t)$ may be represented in frame L, and vice versa. Consider the vector $\boldsymbol{r}_{O_L O}(t)$, represented in frame R,

$$\boldsymbol{r}_{O_L O}(t) = x^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR} + y^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR}$$

or (2.28b)

$$\boldsymbol{r}^R_{O_L O}(t) = \left(x^R_{O_L O}\,,\, y^R_{O_L O}\right)^{\mathrm{T}} .$$

Introducing the time derivative operator with respect to frame R "${}^R\mathrm{d}/\mathrm{d}t$" and applying this operator to $\boldsymbol{r}_{O_L O}(t)$ yields:

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{O_L O}(t)\right) = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(x^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR}\right) + \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(y^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR}\right). \tag{2.28c}$$

The basis vectors $\boldsymbol{e}_{iR}$ $(i = x\,,\, y)$ of frame R are, by definition, constant in R. This yields together with $\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{iR} \equiv \boldsymbol{0}$ $(i = x\,,\, y)$ the relation

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{O_L O}(t)\right) = \dot{x}^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR} + \dot{y}^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR}\ , \tag{2.28d}$$

or written in abbreviated form

$${}^R\dot{\boldsymbol{r}}^R_{O_L O}(t) := \left(\dot{x}^R_{O_L O}(t)\,,\, \dot{y}^R_{O_L O}(t)\right)^{\mathrm{T}} \tag{2.28e}$$

with unambiguous real numbers $\dot{x}^R_{O_L O}(t)\,,\, \dot{y}^R_{O_L O}(t)$. The same result holds for the time derivative of another vector $\boldsymbol{r}^R_{PO_L}(t)$ with respect to frame R:

$${}^R\dot{\boldsymbol{r}}^R_{PO_L}(t) = \left(\dot{x}^R_{PO_L}(t)\,,\, \dot{y}^R_{PO_L}(t)\right)^{\mathrm{T}} . \tag{2.28f}$$

On the other hand, representing the vector $\boldsymbol{r}^R_{PO_L}(t)$ in the form

$$\boldsymbol{r}^R_{PO_L} = \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \tag{2.29}$$

yields (A.1.11f)

$$\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{R}_{PO_L}\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL}\cdot \boldsymbol{r}^{L}_{PO_L}\right) = \dot{\boldsymbol{A}}^{RL}\cdot \boldsymbol{r}^{L}_{PO_L} + \boldsymbol{A}^{RL}\cdot \frac{{}^{L}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{L}_{PO_L}\right), \quad (2.30a)$$

or written more compactly

$${}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} = \dot{\boldsymbol{A}}^{RL}\cdot \boldsymbol{r}^{L}_{PO_L} + \boldsymbol{A}^{RL}\cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L}, \quad (2.30b)$$

or (A.1.11f)

$${}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} = \boldsymbol{A}^{RL}\cdot \left(\tilde{\boldsymbol{\omega}}^{L}_{LR}\cdot \boldsymbol{r}^{L}_{PO_L} + {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L}\right) = \boldsymbol{A}^{RL}\cdot {}^{R}\dot{\boldsymbol{r}}^{L}_{PO_L}, \quad (2.31a)$$

with (A.1.11e)

$${}^{R}\dot{\boldsymbol{r}}^{L}_{PO_L} := \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{L}_{PO_L}\right) = \tilde{\boldsymbol{\omega}}^{L}_{LR}\cdot \boldsymbol{r}^{L}_{PO_L} + {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L}, \quad (2.31b)$$

with (A.1.11b)

$$\boldsymbol{A}^{LR}\cdot \dot{\boldsymbol{A}}^{RL} = \mathbf{R}\cdot \dot{\psi}_{LR} = \begin{pmatrix} 0 & , -\dot{\psi}_{LR} \\ \dot{\psi}_{LR} & , \quad 0 \end{pmatrix} =: \tilde{\boldsymbol{\omega}}^{L}_{LR}, \quad (2.31c)$$

or (A.1.11c)

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL}\cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} = \boldsymbol{A}^{RL}\cdot \mathbf{R}\cdot \dot{\psi}_{LR}, \quad (2.31d)$$

with the orthogonal *planar* rotation matrix $\mathbf{R}$, with the *angular velocity vector* (A.1.11d)

$$\boldsymbol{\omega}_{LR} := \dot{\psi}_{LR}\cdot \boldsymbol{e}_{zL} \quad \text{or} \quad \boldsymbol{\omega}^{L}_{LR} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}_{LR} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cdot \dot{\psi}_{LR}, \quad (2.31e)$$

and with $\tilde{\boldsymbol{\omega}}^{L}_{LR}$ as defined in (2.27c). Subsequently the following abbreviations will be used:

$$\dot{\boldsymbol{r}}^{L}_{PO_L} := {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} = \frac{{}^{L}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{L}_{PO_L}\right) = \left(\dot{x}^{L}_{PO_L}\,,\, \dot{y}^{L}_{PO_L}\right)^{\mathrm{T}} \quad (2.32a)$$

and

$$\dot{\boldsymbol{r}}^{R}_{PO_L} := {}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{R}_{PO_L}\right) = \left(\dot{x}^{R}_{PO_L}\,,\, \dot{y}^{R}_{PO_L}\right)^{\mathrm{T}}. \quad (2.32b)$$

Then

$${}^{R}\dot{\boldsymbol{r}}^{R}_{PO} = {}^{R}\dot{\boldsymbol{r}}^{R}_{O_LO} + \boldsymbol{A}^{RL}\cdot \dot{\boldsymbol{r}}^{L}_{PO_L} + \boldsymbol{A}^{RL}\cdot \tilde{\boldsymbol{\omega}}^{L}_{LR}\cdot \boldsymbol{r}^{L}_{PO_L} \quad (2.33)$$

or

$$\begin{pmatrix} \dot{x}^R_{PO} \\ \dot{y}^R_{PO} \end{pmatrix} = \begin{pmatrix} \dot{x}^R_{O_LO} \\ \dot{y}^R_{O_LO} \end{pmatrix} + \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix}$$
$$\cdot \left[\begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \end{pmatrix} + \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \end{pmatrix} \cdot \dot{\psi}_{LR} \right],$$

and finally

$$\begin{aligned} \dot{x}^R_{PO} &= \dot{x}^R_{O_LO} + \cos\psi \left(\dot{x}^L_{PO_L} - \dot{\psi} \cdot y^L_{PO_L} \right) - \sin\psi \left(\dot{y}^L_{PO_L} + \dot{\psi} \cdot x^L_{PO_L} \right) \\ \dot{y}^R_{PO} &= \dot{y}^R_{O_LO} + \sin\psi \left(\dot{x}^L_{PO_L} - \dot{\psi} \cdot y^L_{PO_L} \right) + \cos\psi \left(\dot{y}^L_{PO_L} + \dot{\psi} \cdot x^L_{PO_L} \right). \end{aligned}$$

2.1.2.2 Accelerations and angular accelerations. Consider the previously derived *velocity vector* (2.33)

$$\begin{aligned} \boldsymbol{v}^R_{PO} := \dot{\boldsymbol{r}}^R_{PO} &:= \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}^R_{PO} \right) \qquad (2.34) \\ &= \left({}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} \right) \\ &= {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \left({}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} \right) \\ &= {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L}. \end{aligned}$$

The associated *acceleration vector* $\boldsymbol{a}^R_{PO} := \dfrac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{v}^R_{PO} \right)$ can be written as

$$\begin{aligned} \boldsymbol{a}^R_{PO} &:= \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{v}^R_{PO} \right) =: {}^R\dot{\boldsymbol{v}}^R_{PO} \qquad (2.35a) \\ &= \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left({}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} \right), \end{aligned}$$

or as

$$\begin{aligned} \boldsymbol{a}^R_{PO} = {}^R\ddot{\boldsymbol{r}}^R_{O_LO} &+ \dot{\boldsymbol{A}}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} \\ &+ \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \dot{\boldsymbol{A}}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\ddot{\boldsymbol{r}}^L_{PO_L}. \qquad (2.35b) \end{aligned}$$

Inserting the relation (2.31d)

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \qquad (A.1.3)$$

into (2.35b) yields the *acceleration relation*

$$\begin{aligned} {}^R\ddot{\boldsymbol{r}}^R_{PO} = {}^R\ddot{\boldsymbol{r}}^R_{O_LO} &+ \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot \dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} \\ &+ 2 \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\ddot{\boldsymbol{r}}^L_{PO_L}. \qquad (2.36) \end{aligned}$$

Assuming that the vector $\boldsymbol{r}^L_{PO_L}$ is constant in the frame L (*rigid-body property* in the case that L is a frame fixed on a rigid body) provides

$$ {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} \equiv \mathbf{0} \quad \text{and} \quad {}^{L}\ddot{\boldsymbol{r}}^{L}_{PO_L} \equiv \mathbf{0}, \tag{2.37} $$

and yields the relations

$$ {}^{R}\dot{\boldsymbol{r}}^{R}_{PO} = {}^{R}\dot{\boldsymbol{r}}^{R}_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \boldsymbol{r}^{L}_{PO_L} \tag{2.38a} $$

and

$$ {}^{R}\ddot{\boldsymbol{r}}^{R}_{PO} = {}^{R}\ddot{\boldsymbol{r}}^{R}_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \boldsymbol{r}^{L}_{PO_L} + \boldsymbol{A}^{RL} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L}_{LR} \cdot \boldsymbol{r}^{L}_{PO_L} . \tag{2.38b} $$

Written in components, the acceleration relation (2.36) is

$$ \begin{aligned} \begin{pmatrix} \ddot{x}^{R}_{PO} \\ \ddot{y}^{R}_{PO} \end{pmatrix} &= \begin{pmatrix} \ddot{x}^{R}_{O_LO} \\ \ddot{y}^{R}_{O_LO} \end{pmatrix} \\ &+ \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{L}_{PO_L} \\ y^{L}_{PO_L} \end{pmatrix} \cdot \dot{\psi}^{2}_{LR} \\ &+ \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{L}_{PO_L} \\ y^{L}_{PO_L} \end{pmatrix} \cdot \ddot{\psi}_{LR} \\ &+ 2 \cdot \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{x}^{L}_{PO_L} \\ \dot{y}^{L}_{PO_L} \end{pmatrix} \cdot \dot{\psi}_{LR} \\ &+ \begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} \ddot{x}^{L}_{PO_L} \\ \ddot{y}^{L}_{PO_L} \end{pmatrix} \end{aligned} $$

or

$$ \begin{aligned} \begin{pmatrix} \ddot{x}^{R}_{PO} \\ \ddot{y}^{R}_{PO} \end{pmatrix} = \begin{pmatrix} \ddot{x}^{R}_{O_LO} \\ \ddot{y}^{R}_{O_LO} \end{pmatrix} &+ \begin{pmatrix} -\cos\psi_{LR} \ , & +\sin\psi_{LR} \\ -\sin\psi_{LR} \ , & -\cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} x^{L}_{PO_L} \\ y^{L}_{PO_L} \end{pmatrix} \cdot \dot{\psi}^{2}_{LR} \\ &+ \begin{pmatrix} -\sin\psi_{LR} \ , & -\cos\psi_{LR} \\ \cos\psi_{LR} \ , & -\sin\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} x^{L}_{PO_L} \\ y^{L}_{PO_L} \end{pmatrix} \cdot \ddot{\psi}_{LR} \\ &+ 2 \cdot \begin{pmatrix} -\sin\psi_{LR} \ , & -\cos\psi_{LR} \\ \cos\psi_{LR} \ , & -\sin\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} \dot{x}^{L}_{PO_L} \\ \dot{y}^{L}_{PO_L} \end{pmatrix} \cdot \dot{\psi}_{LR} \\ &+ \begin{pmatrix} \ddot{x}^{L}_{PO_L} \cdot \cos\psi_{LR} - \ddot{y}^{L}_{PO_L} \cdot \sin\psi_{LR} \\ \ddot{x}^{L}_{PO_L} \cdot \sin\psi_{LR} + \ddot{y}^{L}_{PO_L} \cdot \cos\psi_{LR} \end{pmatrix} , \end{aligned} \tag{2.39} $$

and finally

$$\begin{aligned}\ddot{x}^R_{PO} = \ddot{x}^R_{O_LO} &- x^L_{PO_L} \cdot \left(\dot{\psi}_{LR}\right)^2 \cdot \cos\psi_{LR} + y^L_{PO_L} \cdot \left(\dot{\psi}_{LR}\right)^2 \cdot \sin\psi_{LR} \\ &- x^L_{PO_L} \cdot \ddot{\psi}_{LR} \cdot \sin\psi_{LR} - y^L_{PO_L} \cdot \ddot{\psi}_{LR} \cdot \cos\psi_{LR} \\ &- 2 \cdot \dot{x}^L_{PO_L} \cdot \dot{\psi}_{LR} \cdot \sin\psi_{LR} - 2 \cdot \dot{y}^L_{PO_L} \cdot \dot{\psi}_{LR} \cdot \cos\psi_{LR} \\ &+ \ddot{x}^L_{PO_L} \cdot \cos\psi_{LR} - \ddot{y}^L_{PO_L} \cdot \sin\psi_{LR}\end{aligned} \tag{2.40a}$$

and

$$\begin{aligned}\ddot{y}^R_{PO} = \ddot{y}^R_{O_LO} &- x^L_{PO_L} \cdot \left(\dot{\psi}_{LR}\right)^2 \cdot \sin\psi_{LR} - y^L_{PO_L} \cdot \left(\dot{\psi}_{LR}\right)^2 \cdot \cos\psi_{LR} \\ &+ x^L_{PO_L} \cdot \ddot{\psi}_{LR} \cdot \cos\psi_{LR} - y^L_{PO_L} \cdot \ddot{\psi}_{LR} \cdot \sin\psi_{LR} \\ &+ 2 \cdot \dot{x}^L_{PO_L} \cdot \dot{\psi}_{LR} \cdot \cos\psi_{LR} - 2 \cdot \dot{y}^L_{PO_L} \cdot \dot{\psi}_{LR} \cdot \sin\psi_{LR} \\ &+ \ddot{x}^L_{PO_L} \cdot \sin\psi_{LR} + \ddot{y}^L_{PO_L} \cdot \cos\psi_{LR}.\end{aligned} \tag{2.40b}$$

In addition, the following relations hold:

1. $${}^R\dot{\boldsymbol{\omega}}^R_{LR} = \boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{\omega}}^L_{LR} \quad \text{with} \quad \dot{\boldsymbol{\omega}}^L_{LR} := {}^L\dot{\boldsymbol{\omega}}^L_{LR}, \tag{2.41a}$$

2. $$\tilde{\boldsymbol{\omega}}^R_{LR} = A^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}, \tag{2.41b}$$

3. $${}^R\dot{\tilde{\boldsymbol{\omega}}}^R_{LR} = \boldsymbol{A}^{RL} \cdot {}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{A}^{LR}. \tag{2.41c}$$

Proof of (2.41a):

The relation

$${}^R\dot{\boldsymbol{\omega}}^R_{LR} = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL} \cdot \boldsymbol{\omega}^L_{LR}\right) = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{\omega}^L_{LR} + \boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{\omega}}^L_{LR}$$

yields together with

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR}$$

the relation

$${}^R\dot{\boldsymbol{\omega}}^R_{LR} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\omega}^L_{LR} + \boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{\omega}}^L_{LR},$$

and with

$$\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\omega}^L_{LR} = \mathbf{0}$$

the relation (2.41a). □

Proof of (2.41b):

The relation

$$\tilde{\boldsymbol{\omega}}^R_{LR} \cdot \boldsymbol{r}^R = \boldsymbol{A}^{RL} \cdot \left(\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L\right)$$
$$= \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \left(\boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL}\right) \cdot \boldsymbol{r}^L = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{r}^R$$

implies

$$\tilde{\boldsymbol{\omega}}^R_{LR} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}. \qquad \square$$

Proof of (2.41c):

$$^R\dot{\tilde{\boldsymbol{\omega}}}^R_{LR} = \frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}\right)$$
$$= \dot{\boldsymbol{A}}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{A}^{LR} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \dot{\boldsymbol{A}}^{LR}$$

provides, together with

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{and} \quad \dot{\boldsymbol{A}}^{LR} = \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL} = -\boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{LR},$$

the relation

$$^R\dot{\tilde{\boldsymbol{\omega}}}^R_{LR} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}$$
$$+ \boldsymbol{A}^{RL} \cdot {}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{A}^{LR} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL}.$$

Using

$$\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL}$$
$$= -\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \underbrace{\boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{LR} \cdot \boldsymbol{A}^{RL}}_{:\,\tilde{\boldsymbol{\omega}}^L_{LR}} \cdot \boldsymbol{A}^{LR} = -\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}$$

yields

$$^R\dot{\tilde{\boldsymbol{\omega}}}^R_{LR} = \boldsymbol{A}^{RL} \cdot {}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR} \cdot \boldsymbol{A}^{LR}\,. \qquad \square$$

2.2 Spatial vectors and matrices

In this chapter, vector representations, operations and transformations in the *space* $\mathbb{R}^3$ will be discussed together with their time derivatives. The basic differences from the planar case $\mathbb{R}^2$ are:

1. That the space $\mathbb{R}^3$ is closed under vector-product operations.
2. That the angular velocity vector in $\mathbb{R}^3$ is in general no longer computed as time derivative of an angle as in case of the $\mathbb{R}^2$.
   In the spatial case angles and angular velocities are related by means of the *kinematic differential equation.* This equation may include singulari-

ties that can be avoided by formulating rotations by means of quaternions or by Euler parameters ([42], [43], [44], [45]) instead of Bryant angles. Due to the introductory character of this book, quaternions and Euler parameters will not be considered here.

2.2.1 Displacement vectors, frames, and orientation matrices

The basic definitions and statements of *geometric vectors* in the *plane* presented in Section 2.1.1.1 also hold in the *spatial* case, with slight modifications. As in the planar case, the spatial vectors and matrices will also be written in bold faced letters.

Consider a reference frame R with origin $\mathrm{O} = \mathrm{O}_R$ and with the *orthogonal basis* $B_R = \{\boldsymbol{e}_{xR}\,, \boldsymbol{e}_{yR}\,, \boldsymbol{e}_{zR}\}$, $\boldsymbol{e}_{iR} \in \mathbb{R}^3$, defined by the relation

$$\left(\boldsymbol{e}_{iR}^R\right)^{\mathrm{T}} \cdot \boldsymbol{e}_{jR}^R = \begin{cases} 1 \ , \ i=j \\ 0 \ , \ i \neq j \end{cases} \ ; \ i,j = x,y,z, \tag{2.42a}$$

with $\boldsymbol{e}_{iR}^R$ as the basis vector $\boldsymbol{e}_{iR}$ represented in R. Consider a second frame L with origin O_L and with orthogonal basis $B_L = \{\boldsymbol{e}_{xL}\,, \boldsymbol{e}_{yL}\,, \boldsymbol{e}_{zL}\}$ fixed to a rigid body (Figure 2.7a). Let $\boldsymbol{r}_{O_L O_R}$ be the (*geometrical*) displacement vector from O to O_L. Then $\boldsymbol{r}_{O_L O_R} = \boldsymbol{r}_{O_L O}$ can be represented with respect to R as

$$\boldsymbol{r}_{O_L O} = x_{O_L O}^R \cdot \boldsymbol{e}_{xR} + y_{O_L O}^R \cdot \boldsymbol{e}_{yR} + z_{O_L O}^R \cdot \boldsymbol{e}_{zR}, \tag{2.42b}$$

or in *algebraic* form

$$\boldsymbol{r}_{O_L O}^R = \left(x_{O_L O}^R\,, y_{O_L O}^R\,, z_{O_L O}^R\right)^{\mathrm{T}} \in \mathbb{R}^3 \tag{2.42c}$$

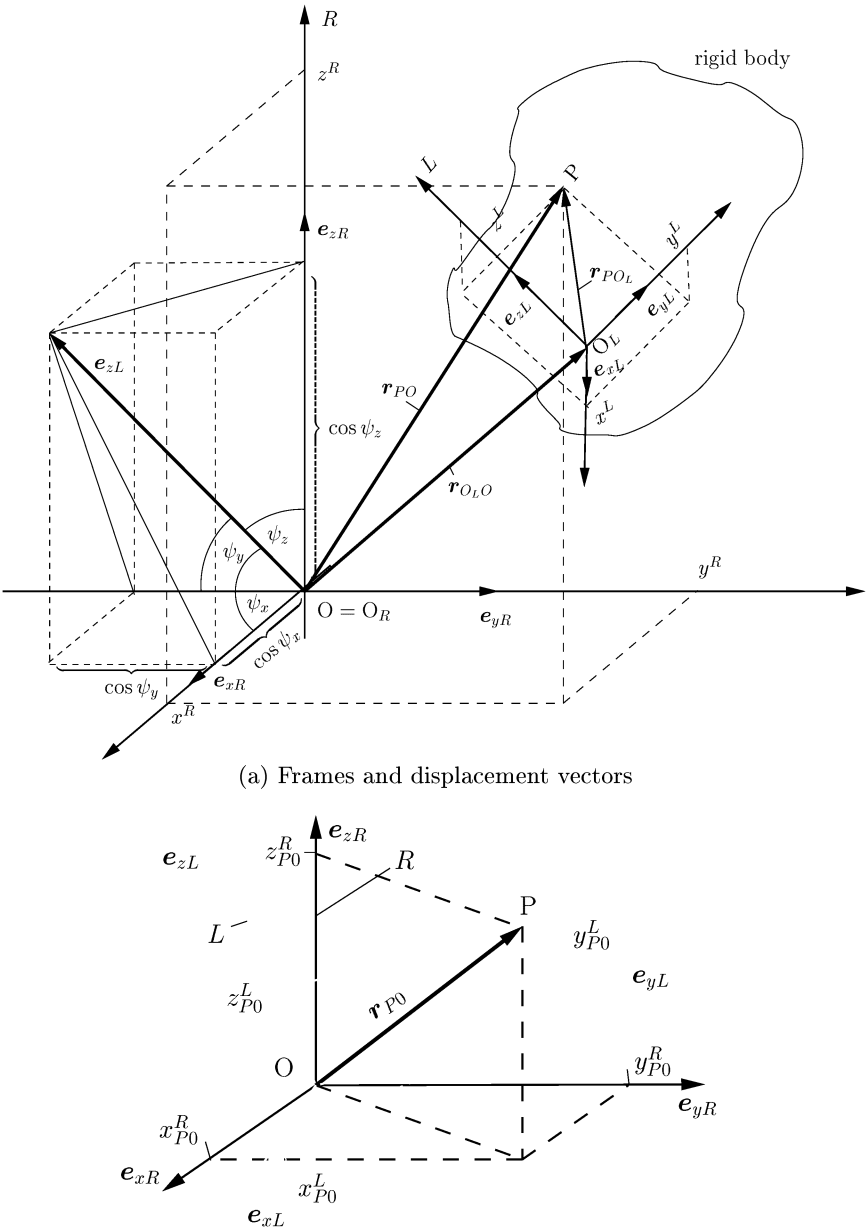
with $x_{O_L O}^R\,, y_{O_L O}^R\,, z_{O_L O}^R$ as *Cartesian coordinates* of $\boldsymbol{r}_{O_L O}$ with respect to frame R. Consider an arbitrary point P on a rigid body (Figure 2.7a) and the displacement vectors $\boldsymbol{r}_{PO_L}$ from O_L to P and $\boldsymbol{r}_{PO}$ from O to P. Then the following *translation relation* holds:

$$\boldsymbol{r}_{PO} = \boldsymbol{r}_{PO_L} + \boldsymbol{r}_{O_L O},$$

or, representing $\boldsymbol{r}_{PO}$ in the frame R, (2.43)

$$\boldsymbol{r}_{PO}^R = \boldsymbol{r}_{PO_L}^R + \boldsymbol{r}_{O_L O}^R = \left(x_{PO_L}^R\,, y_{PO_L}^R\,, z_{PO_L}^R\right)^{\mathrm{T}} + \left(x_{O_L O}^R\,, y_{O_L O}^R\,, z_{O_L O}^R\right)^{\mathrm{T}}.$$

Consider the vector $\boldsymbol{r}_{PO}$ in Figure 2.7b and two orthogonal frames R and L with a common origin O and with basis vectors $\{\boldsymbol{e}_{xR}\,, \boldsymbol{e}_{yR}\,, \boldsymbol{e}_{zR}\}$ and $\{\boldsymbol{e}_{xL}\,, \boldsymbol{e}_{yL}\,, \boldsymbol{e}_{zL}\}$, respectively. Then $\boldsymbol{r}_{PO}$ can be written as



(a) Frames and displacement vectors

(b) Representation of a vector $\boldsymbol{r}_{PO}$ in different frames R and L

Fig. 2.7: Vector representations

$$\boldsymbol{r}_{PO} = x_{PO}^{R} \cdot \boldsymbol{e}_{xR} + y_{PO}^{R} \cdot \boldsymbol{e}_{yR} + z_{PO}^{R} \cdot \boldsymbol{e}_{zR}, \tag{2.44a}$$

or

$$\boldsymbol{r}_{PO} = x_{PO}^{L} \cdot \boldsymbol{e}_{xL} + y_{PO}^{L} \cdot \boldsymbol{e}_{yL} + z_{PO}^{L} \cdot \boldsymbol{e}_{zL}, \tag{2.44b}$$

or abbreviated in algebraic form

$$\boldsymbol{r}_{PO}^{R} = \left(x_{PO}^{R}\,,\, y_{PO}^{R}\,,\, z_{PO}^{R}\right)^{\mathrm{T}} \qquad \text{(representation of } \boldsymbol{r}_{PO} \text{ in } R) \tag{2.44c}$$

and

$$\boldsymbol{r}_{PO}^{L} = \left(x_{PO}^{L}\,,\, y_{PO}^{L}\,,\, z_{PO}^{L}\right)^{\mathrm{T}} \qquad \text{(representation of } \boldsymbol{r}_{PO} \text{ in } L). \tag{2.44d}$$

2.2.1.1 Basis transformation. Given two frames R and L (Figure 2.7b) with orthogonal basis vectors $\{\boldsymbol{e}_{xR}\,,\, \boldsymbol{e}_{yR}\,,\, \boldsymbol{e}_{zR}\}$ of R and $\{\boldsymbol{e}_{xL}\,,\, \boldsymbol{e}_{yL}\,,\, \boldsymbol{e}_{zL}\}$ of L. Then

$$\begin{aligned} \boldsymbol{e}_{xR} &= 1 \cdot \boldsymbol{e}_{xR} + 0 \cdot \boldsymbol{e}_{yR} + 0 \cdot \boldsymbol{e}_{zR}, \\ \boldsymbol{e}_{yR} &= 0 \cdot \boldsymbol{e}_{xR} + 1 \cdot \boldsymbol{e}_{yR} + 0 \cdot \boldsymbol{e}_{zR}, \\ \boldsymbol{e}_{zR} &= 0 \cdot \boldsymbol{e}_{xR} + 0 \cdot \boldsymbol{e}_{yR} + 1 \cdot \boldsymbol{e}_{zR}, \end{aligned} \tag{2.45a}$$

or

$$\boldsymbol{e}_{xR}^{R} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_{yR}^{R} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{e}_{zR}^{R} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and

$$\begin{aligned} \boldsymbol{e}_{xL} &= 1 \cdot \boldsymbol{e}_{xL} + 0 \cdot \boldsymbol{e}_{yL} + 0 \cdot \boldsymbol{e}_{zL}, \\ \boldsymbol{e}_{yL} &= 0 \cdot \boldsymbol{e}_{xL} + 1 \cdot \boldsymbol{e}_{yL} + 0 \cdot \boldsymbol{e}_{zL}, \\ \boldsymbol{e}_{zL} &= 0 \cdot \boldsymbol{e}_{xL} + 0 \cdot \boldsymbol{e}_{yL} + 1 \cdot \boldsymbol{e}_{zL}, \end{aligned}$$

or

$$\boldsymbol{e}_{xL}^{L} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_{yL}^{L} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{e}_{zL}^{L} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.45b}$$

Consider the *projections of* $\boldsymbol{e}_{xL}$, $\boldsymbol{e}_{yL}$ *and* $\boldsymbol{e}_{zL}$ *onto the basis vectors of* R; i.e., the expressions

$$\begin{aligned} &\left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{xL}^{L}\right) =: \ell_x \;,\; \left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{yL}^{L}\right) =: \ell_y \;,\; \left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{zL}^{L}\right) =: \ell_z \\ &\left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{xL}^{L}\right) =: m_x \,,\, \left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{yL}^{L}\right) =: m_y \,,\, \left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{zL}^{L}\right) =: m_z \\ &\left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{xL}^{L}\right) =: n_x \;,\; \left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{yL}^{L}\right) =: n_y \;,\; \left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{e}_{zL}^{L}\right) =: n_z \end{aligned} \tag{2.45c}$$

with numbers ℓ_i, m_i, n_i, $(i = x\,,\,y\,,\,z)$, called *direction cosines* due to the relation

$$\left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{xL}^{L}\right) := \left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{e}_{xL}^{L} = |\boldsymbol{e}_{xR}^{R}| \cdot |\boldsymbol{e}_{xL}^{L}| \cdot \cos\alpha_x = \cos\alpha_x\,,$$

with α_x as the angle between the basis vectors $\boldsymbol{e}_{xR}$ and $\boldsymbol{e}_{xL}$. Then

$$\begin{aligned}
\boldsymbol{e}_{xL} &= \left(\left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{xL}^{L}\right)\right)\cdot\boldsymbol{e}_{xR} + \left(\left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{xL}^{L}\right)\right)\cdot\boldsymbol{e}_{yR} + \left(\left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{xL}^{L}\right)\right)\cdot\boldsymbol{e}_{zR},\\
\boldsymbol{e}_{yL} &= \left(\left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{yL}^{L}\right)\right)\cdot\boldsymbol{e}_{xR} + \left(\left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{yL}^{L}\right)\right)\cdot\boldsymbol{e}_{yR} + \left(\left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{yL}^{L}\right)\right)\cdot\boldsymbol{e}_{zR},\\
\boldsymbol{e}_{zL} &= \left(\left(\boldsymbol{e}_{xR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{zL}^{L}\right)\right)\cdot\boldsymbol{e}_{xR} + \left(\left(\boldsymbol{e}_{yR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{zL}^{L}\right)\right)\cdot\boldsymbol{e}_{yR} + \left(\left(\boldsymbol{e}_{zR}^{R}\right)^{\mathrm{T}}\left(\boldsymbol{e}_{zL}^{L}\right)\right)\cdot\boldsymbol{e}_{zR},
\end{aligned} \tag{2.46a}$$

or

$$\begin{aligned}
\boldsymbol{e}_{xL} &= \ell_x \cdot \boldsymbol{e}_{xR} + m_x \cdot \boldsymbol{e}_{yR} + n_x \cdot \boldsymbol{e}_{zR},\\
\boldsymbol{e}_{yL} &= \ell_y \cdot \boldsymbol{e}_{xR} + m_y \cdot \boldsymbol{e}_{yR} + n_y \cdot \boldsymbol{e}_{zR},\\
\boldsymbol{e}_{zL} &= \ell_z \cdot \boldsymbol{e}_{xR} + m_z \cdot \boldsymbol{e}_{yR} + n_z \cdot \boldsymbol{e}_{zR},
\end{aligned} \tag{2.46b}$$

and

$$\begin{aligned}
\boldsymbol{e}_{xL}^{R} &= \ell_x \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + m_x \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + n_x \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\ell_x\\m_x\\n_x\end{pmatrix}, &\qquad (2.46\text{c})\\
\boldsymbol{e}_{yL}^{R} &= \ell_y \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + m_y \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + n_y \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\ell_y\\m_y\\n_y\end{pmatrix},\\
\boldsymbol{e}_{zL}^{R} &= \ell_z \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + m_z \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + n_z \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\ell_z\\m_z\\n_z\end{pmatrix}.
\end{aligned}$$

By analogy to (2.45c) and (2.46b), the following *basis transformation relations* hold:

$$\begin{aligned}
\boldsymbol{e}_{xR} &= \ell_x \cdot \boldsymbol{e}_{xL} + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL},\\
\boldsymbol{e}_{yR} &= m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL},\\
\boldsymbol{e}_{zR} &= n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL},
\end{aligned} \tag{2.47a}$$

or

$$\boldsymbol{e}_{xR}^{L} = \ell_x \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} + \ell_y \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} + \ell_z \cdot \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\ell_x\\\ell_y\\\ell_z\end{pmatrix}, \tag{2.47b}$$

$$\boldsymbol{e}_{yR}^{L} = m_x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + m_y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + m_z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix},$$
$$\boldsymbol{e}_{zR}^{L} = n_x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + n_y \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + n_z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}.$$

Inserting (2.47a) into (2.46a) yields, together with (2.45c),

$$\begin{aligned}
\boldsymbol{e}_{xL} = {} & \ell_x \cdot (\ell_x \cdot \boldsymbol{e}_{xL} + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL}) \\
& + m_x \cdot (m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL}) \\
& + n_x \cdot (n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL}), \\
\boldsymbol{e}_{yL} = {} & \ell_y \cdot (\ell_x \cdot \boldsymbol{e}_{xL} + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL}) \\
& + m_y \cdot (m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL}) \\
& + n_y \cdot (n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL}), \\
\boldsymbol{e}_{zL} = {} & \ell_z \cdot (\ell_x \cdot \boldsymbol{e}_{xL} + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL}) \\
& + m_z \cdot (m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL}) \\
& + n_z \cdot (n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL}),
\end{aligned} \tag{2.48a}$$

or

$$\begin{aligned}
\boldsymbol{e}_{xL} = {} & \underbrace{\left(\ell_x^2 + m_x^2 + n_x^2\right)}_{=:\,1} \cdot \boldsymbol{e}_{xL} + \underbrace{(\ell_x \cdot \ell_y + m_x \cdot m_y + n_x \cdot n_y)}_{=:\,0} \cdot \boldsymbol{e}_{yL} \\
& + \underbrace{(\ell_x \cdot \ell_z + m_x \cdot m_z + n_x \cdot n_z)}_{=:\,0} \cdot \boldsymbol{e}_{zL}, \\
\boldsymbol{e}_{yL} = {} & \underbrace{(\ell_y \cdot \ell_x + m_y \cdot m_x + n_y \cdot n_x)}_{=:\,0} \cdot \boldsymbol{e}_{xL} + \underbrace{\left(\ell_y^2 + m_y^2 + n_y^2\right)}_{=:\,1} \cdot \boldsymbol{e}_{yL} \\
& + \underbrace{(\ell_y \cdot \ell_z + m_y \cdot m_z + n_y \cdot n_z)}_{=:\,0} \cdot \boldsymbol{e}_{zL}, \\
\boldsymbol{e}_{zL} = {} & \underbrace{(\ell_z \cdot \ell_x + m_z \cdot m_x + n_z \cdot n_x)}_{=:\,0} \cdot \boldsymbol{e}_{xL} + \underbrace{(\ell_z \cdot \ell_y + m_z \cdot m_y + n_z \cdot n_y)}_{=:\,0} \cdot \boldsymbol{e}_{yL} \\
& + \underbrace{\left(\ell_z^2 + m_z^2 + n_z^2\right)}_{=:\,1} \cdot \boldsymbol{e}_{zL}
\end{aligned} \tag{2.48b}$$

together with the six relations

$$\begin{aligned}
(\ell_x \cdot \ell_y + m_x \cdot m_y + n_x \cdot n_y) &= 0, \\
(\ell_x \cdot \ell_z + m_x \cdot m_z + n_x \cdot n_z) &= 0, \\
(\ell_y \cdot \ell_z + m_y \cdot m_z + n_y \cdot n_z) &= 0, \\
\left(\ell_i^2 + m_i^2 + n_i^2\right) &= 1 \quad , \quad (i = x\,,\, y\,,\, z).
\end{aligned} \tag{2.48c}$$

Introducing the *direction cosine matrix*

$$\boldsymbol{A}^{RL} := \begin{pmatrix} \ell_x \ , & \ell_y \ , & \ell_z \\ m_x \ , & m_y \ , & m_z \\ n_x \ , & n_y \ , & n_z \end{pmatrix}, \tag{2.48d}$$

the relations (2.48b) and (2.48c) prove the *orthogonality relations*

$$\boldsymbol{A}^{RL} \cdot \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \boldsymbol{A}^{RL} \cdot \boldsymbol{A}^{LR} = \mathbf{I}_3$$

or (2.49a)

$$\left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \left(\boldsymbol{A}^{RL}\right)^{-1} = \boldsymbol{A}^{LR}.$$

Then, due, to (2.46b) and (2.46c),

$$\boldsymbol{e}_{xL}^{R} = \begin{pmatrix} \ell_x \\ m_x \\ n_x \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \boldsymbol{e}_{xL}^{L}, \tag{2.49b}$$

$$\boldsymbol{e}_{yL}^{R} = \begin{pmatrix} \ell_y \\ m_y \\ n_y \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = A^{RL} \cdot \boldsymbol{e}_{yL}^{L},$$

$$\boldsymbol{e}_{zL}^{R} = \begin{pmatrix} \ell_z \\ m_z \\ n_z \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boldsymbol{A}^{RL} \cdot \boldsymbol{e}_{zL}^{L},$$

and

$$\boldsymbol{e}_{xR}^{L} = \boldsymbol{A}^{LR} \cdot \boldsymbol{e}_{xR}^{R}, \quad \boldsymbol{e}_{yR}^{L} = \boldsymbol{A}^{LR} \cdot \boldsymbol{e}_{yR}^{R}, \quad \boldsymbol{e}_{zR}^{L} = \boldsymbol{A}^{LR} \cdot \boldsymbol{e}_{zR}^{R}$$

with (2.50)

$$\boldsymbol{A}^{LR} = \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}}.$$

2.2.1.2 Coordinate transformation. Given two orthogonal frames R and L with a common origin O (Figure 2.7b) and a vector $\boldsymbol{r}_{PO}$ from O to P with representations in R and L,

$$\boldsymbol{r}_{PO} = x_{PO}^{R} \cdot \boldsymbol{e}_{xR} + y_{PO}^{R} \cdot \boldsymbol{e}_{yR} + z_{PO}^{R} \cdot \boldsymbol{e}_{zR} \tag{2.51a}$$

and

$$\boldsymbol{r}_{PO} = x_{PO}^{L} \cdot \boldsymbol{e}_{xL} + y_{PO}^{L} \cdot \boldsymbol{e}_{yL} + z_{PO}^{L} \cdot \boldsymbol{e}_{zL}, \tag{2.51b}$$

or

$$\boldsymbol{r}^R_{PO} = \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \\ z^R_{PO} \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}^L_{PO} = \begin{pmatrix} x^L_{PO} \\ y^L_{PO} \\ z^L_{PO} \end{pmatrix}.$$

Inserting the basis transformation relations (2.47a) into (2.51a) yields

$$\begin{aligned} \boldsymbol{r}_{PO} = \quad & x^R_{PO} \cdot \underbrace{(\ell_x \cdot exL + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL})}_{=:\, \boldsymbol{e}_{xR}} \\ & + y^R_{PO} \cdot \underbrace{(m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL})}_{=:\, \boldsymbol{e}_{yR}} \\ & + z^R_{PO} \cdot \underbrace{(n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL})}_{=:\, \boldsymbol{e}_{zR}}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{r}_{PO} = \quad & \left(x^R_{PO} \cdot \ell_x + y^R_{PO} \cdot m_x + z^R_{PO} \cdot n_x\right) \cdot \boldsymbol{e}_{xL} \\ & + \left(x^R_{PO} \cdot \ell_y + y^R_{PO} \cdot m_y + z^R_{PO} \cdot n_y\right) \cdot \boldsymbol{e}_{yL} \\ & + \left(x^R_{PO} \cdot \ell_z + y^R_{PO} \cdot m_z + z^R_{PO} \cdot n_z\right) \cdot \boldsymbol{e}_{zL} \end{aligned}$$

or

$$\underbrace{\begin{pmatrix} x^L_{PO} \\ y^L_{PO} \\ z^L_{PO} \end{pmatrix}}_{=:\, \boldsymbol{r}^L_{PO}} = \begin{pmatrix} \ell_x \cdot x^R_{PO} & + & m_x \cdot y^R_{PO} & + & n_x \cdot z^R_{PO} \\ \ell_y \cdot x^R_{PO} & + & m_y \cdot y^R_{PO} & + & n_y \cdot z^R_{PO} \\ \ell_z \cdot x^R_{PO} & + & m_z \cdot y^R_{PO} & + & n_z \cdot z^R_{PO} \end{pmatrix} \tag{2.52}$$

$$= \underbrace{\begin{pmatrix} \ell_x \ , & m_x \ , & n_x \\ \ell_y \ , & m_y \ , & n_y \\ \ell_z \ , & m_z \ , & n_z \end{pmatrix}}_{=:\, \boldsymbol{A}^{LR}} \cdot \underbrace{\begin{pmatrix} x^R_{PO} \\ y^R_{PO} \\ z^R_{PO} \end{pmatrix}}_{\boldsymbol{r}^R_{PO}},$$

and finally

$$\boldsymbol{r}^L_{PO} = \boldsymbol{A}^{LR} \cdot \boldsymbol{r}^R_{PO} \tag{2.53a}$$

with the transformation matrix

$$\boldsymbol{A}^{LR} = \begin{pmatrix} \ell_x \ , & m_x \ , & n_x \\ \ell_y \ , & m_y \ , & n_y \\ \ell_z \ , & m_z \ , & n_z \end{pmatrix} = \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \left(\boldsymbol{A}^{RL}\right)^{-1} \tag{2.53b}$$

that maps the coordinates of $\boldsymbol{r}_{PO}$ represented in frame R into coordinates represented in frame L (*coordinate transformation*).

2.2.1.3 Bryant angles. Combining the *translation* and *rotation* relations (2.43) and (2.53a) provides the following representations of the vector $\boldsymbol{r}_{PO}$ in Figure 2.7a:

$$\boldsymbol{r}_{PO} = \boldsymbol{r}_{O_L O} + \boldsymbol{r}_{PO_L} \tag{2.54a}$$

and

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_L O} + \boldsymbol{r}^R_{PO_L} \qquad (\boldsymbol{r}_{PO_L} \text{ represented in } \textit{frame } R) \tag{2.54b}$$

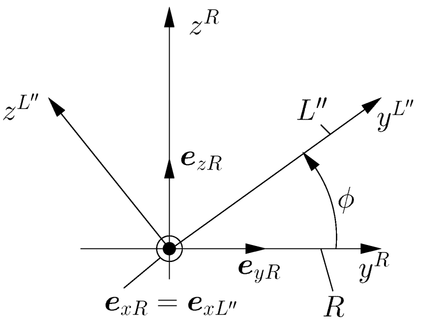
or

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_L O} + \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \qquad (\boldsymbol{r}_{PO_L} \text{ represented in } \textit{frame } L). \tag{2.54c}$$

The matrices $\boldsymbol{A}^{RL}$ and $\boldsymbol{A}^{LR}$ of (2.53a) include nine elements, called *direction cosines.* Due to the *orthogonality relation (2.49a)*, these coordinates are restricted by six additional algebraic relations (2.48c). As a consequence, only three of these nine numbers are independent. In practical applications, representations of $\boldsymbol{A}^{RL}$ usually only include *three* (independent) *coordinates.* Among those coordinate representations, the *Cardan*- or *Bryant-angle* representation is often used in mechatronic systems, due to the fact that the matrix $\boldsymbol{H}(\boldsymbol{p})$ of the associated kinematic differential equation (Section 2.2.2.3) has no singularity for small Cardan angles (contrary to the Euler angle representation). The Cardan angle representation of $\boldsymbol{A}^{LR}$ is constructed by interpreting the spatial rotation of a rigid body with a body fixed frame L with respect to frame R as result of three successive rotations. Before the first rotation, frames L and R coincide (Figure 2.8). The *first* rotation is carried about the $\boldsymbol{e}_{xR}$-axis through an angle $\phi := \phi_{L''R}$. This results in the auxiliary frame L'' with basis vectors $\{\boldsymbol{e}_{xL''}, \boldsymbol{e}_{yL''}, \boldsymbol{e}_{zL''}\}$ and $\boldsymbol{e}_{xL''} = \boldsymbol{e}_{xR}$. The *second* rotation of L'' through an angle $\theta := \theta_{L'L''}$ about the $\boldsymbol{e}_{yL''}$-axis provides a second auxiliary frame L' with basis vectors $\{\boldsymbol{e}_{xL'}, \boldsymbol{e}_{yL'}, \boldsymbol{e}_{zL'}\}$ and $\boldsymbol{e}_{yL'} = \boldsymbol{e}_{yL''}$. The *third* rotation of L' through an angle $\psi := \psi_{LL'}$ about the $\boldsymbol{e}_{zL'}$-axis provides the desired final orientation of the body and frame L. The transformation matrices of these different planar rotations are:

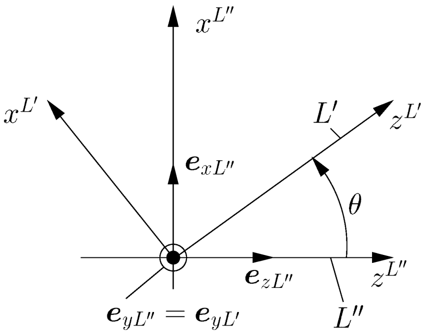
$$\boldsymbol{A}^{L''R} = \begin{pmatrix} 1\,, & 0\,, & 0 \\ 0\,, & \mathrm{c}\,\phi\,, & \mathrm{s}\,\phi \\ 0\,, & -\mathrm{s}\,\phi\,, & \mathrm{c}\,\phi \end{pmatrix} \tag{2.55a}$$

(first rotation of frame R about the $\boldsymbol{e}_{xR}$-axis into L'', $\phi := \phi_{L''R} = -\phi_{RL''}$),



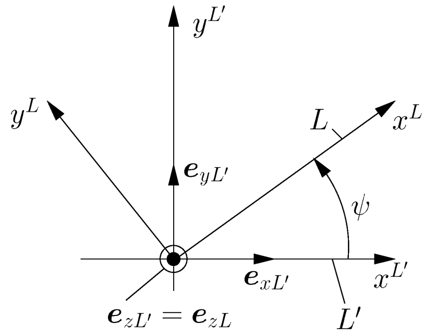
$$\boldsymbol{A}^{L'L''} = \begin{pmatrix} \mathrm{c}\,\theta & , 0 , & -\mathrm{s}\,\theta \\ 0 & , 1 , & 0 \\ \mathrm{s}\,\theta & , 0 , & \mathrm{c}\,\theta \end{pmatrix} \qquad (2.55\mathrm{b})$$

(second rotation of frame L'' about the $\boldsymbol{e}_{yL''}$-axis into L', $\theta := \theta_{L'L''} = -\theta_{L''L'}$),



$$\boldsymbol{A}^{L'L''} = \begin{pmatrix} \mathrm{c}\,\psi & , \mathrm{s}\,\psi & , 0 \\ -\mathrm{s}\,\psi & , \mathrm{c}\,\psi & , 0 \\ 0 & , \ 0 & , 1 \end{pmatrix} \qquad (2.55\mathrm{c})$$

(third rotation of frame L' about the $\boldsymbol{e}_{zL''}$-axis into L, $\psi := \psi_{LL'} = -\psi_{L'L}$) ,



with

$$\mathrm{s}\,\phi := \sin\phi \quad \text{and} \quad \mathrm{c}\,\phi := \cos\phi. \tag{2.55d}$$

The complete rotation matrix $\boldsymbol{A}^{LR}$ that describes the *rotation* from R to L is

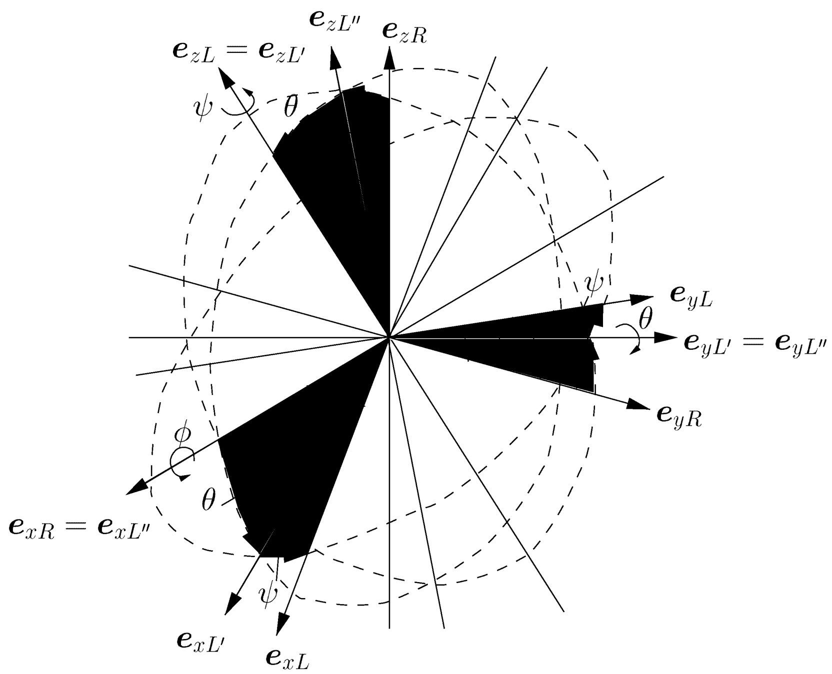
$$\boldsymbol{A}^{LR} = \boldsymbol{A}^{LL'} \cdot \boldsymbol{A}^{L'L''} \cdot \boldsymbol{A}^{L''R}$$

or

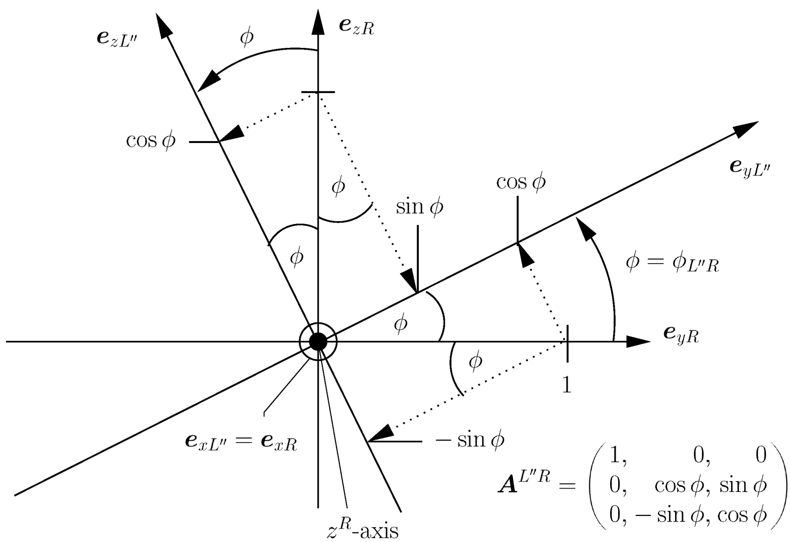
$$\boldsymbol{A}^{LR} = \begin{pmatrix} \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi + \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \\ -\mathrm{c}\,\theta \cdot \mathrm{s}\,\psi & , \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi & , \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \\ \mathrm{s}\,\theta & , \quad -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta & , \quad \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix} . \tag{2.56}$$

This matrix describes the *transformation of coordinates* of a vector represented in frame R into coordinates represented in frame L in terms of *Bryant angles.*

For sufficiently *small rotation angles* ϕ, θ, and ψ, the linear approximations



(a) Diagram of three successive rotations (around $\boldsymbol{e}_{xR}$ by ϕ, around $\boldsymbol{e}_{yL''}$ by θ and around $\boldsymbol{e}_{zL'}$ by ψ)



(b) First rotation around $\boldsymbol{e}_{xR}$ by $\phi = \phi_{L''R} = -\phi_{RL''}$

Fig. 2.8: Bryant-angle transformation

$$\sin\alpha_i \approx \alpha_i \quad \text{and} \quad \cos\alpha_i \approx 1 \quad , \quad \alpha_i \cdot \alpha_j \approx 0, \tag{2.57a}$$

provide the matrix

$$\boldsymbol{A}^{LR}_{\text{small}} \cong \begin{pmatrix} 1\,, & \psi\,, & -\theta \\ -\psi\,, & 1\,, & \phi \\ \theta\,, & -\phi\,, & 1 \end{pmatrix} = \mathbf{I}_3 - \tilde{\boldsymbol{\Phi}} \tag{2.57b}$$

with the skew-symmetric matrix

$$\tilde{\boldsymbol{\Phi}} := \begin{pmatrix} 0\,, & -\psi\,, & +\theta \\ +\psi\,, & 0\,, & -\phi \\ -\theta\,, & +\phi\,, & 0 \end{pmatrix} =: \widetilde{\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}}. \tag{2.57c}$$

that is constructed from the vector $(\phi\,,\,\theta\,,\,\psi)^{\mathrm{T}}$. Then

$$\widetilde{\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_2} = \tilde{\boldsymbol{\Phi}}_1 + \tilde{\boldsymbol{\Phi}}_1 = \widetilde{\begin{pmatrix} \phi_1 \\ \theta_1 \\ \psi_1 \end{pmatrix}} + \widetilde{\begin{pmatrix} \phi_2 \\ \theta_2 \\ \psi_2 \end{pmatrix}}. \tag{2.57d}$$

This implies that within linear approximations *small rotation angles can be added like vectors.* Due to the relation

$$\boldsymbol{\Phi}^L = \boldsymbol{A}^{LR} \cdot \boldsymbol{\Phi}^R \approx (\mathbf{I}_3 - \tilde{\boldsymbol{\Phi}}^R) \cdot \boldsymbol{\Phi}^R = \boldsymbol{\Phi}^R - \tilde{\boldsymbol{\Phi}}^R \boldsymbol{\Phi}^R = \boldsymbol{\Phi}^R, \tag{2.57e}$$

it makes no difference whether this “small angle vector” is represented in frame R or in frame L. As a consequence, the *sequence of small rotations is arbitrary.*

A general transformation of a vector can be interpreted as a superposition of a *translation* (see Equation 2.43)

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_L O} + \boldsymbol{r}^R_{PO_L} \tag{2.58a}$$

and a *rotation*

$$\boldsymbol{r}^R_{PO_L} = \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L}. \tag{2.58b}$$

This yields the combined (*translation–rotation*) transformation

$$\boldsymbol{r}^R_{PO} = \boldsymbol{r}^R_{O_L O} + \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \tag{2.59a}$$

and its inverse

$$\boldsymbol{r}^L_{PO_L} = \boldsymbol{A}^{LR} \cdot \left(\boldsymbol{r}^R_{PO} - \boldsymbol{r}^R_{O_L O}\right), \quad \left(\boldsymbol{A}^{RL}\right)^{-1} = \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} = \boldsymbol{A}^{LR} \tag{2.59b}$$

with the special cases

$$\begin{aligned}
&\textit{pure rotation} \qquad && \boldsymbol{r}_{O_L O} = \mathbf{0} \qquad \text{and} \\
&\textit{pure translation} \qquad && \boldsymbol{A}^{RL} \;= \mathbf{I}_3 .
\end{aligned} \tag{2.59c}$$

These latter two relations are the basis for deriving *kinematic relations* of rigid-body systems (vector loops and orientation loops) under spatial motion (see *Sections 3 and 5*).

2.2.2 Time derivatives of displacement vectors and orientation matrices

In this section, time derivatives of spatial displacement vectors and orientation matrices will be considered together with the kinematic differential equation that relates angular velocities to the time derivatives of angles.

2.2.2.1 Velocities and angular velocities. Consider a vector from the point O of frame R to a point O_L (Figure 2.7a), where $\boldsymbol{r}_{O_L O}L$ is assumed to move smoothly in time,

$$\boldsymbol{r}_{O_L O} = \boldsymbol{r}_{O_L O}(t). \tag{2.60a}$$

Represented in frame R, the vector $\boldsymbol{r}_{O_L O}(t)$ is written as

$$\boldsymbol{r}_{O_L O}(t) = x^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR} + y^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR} + z^R_{O_L O}(t) \cdot \boldsymbol{e}_{zR}. \tag{2.60b}$$

Taking the *time derivative of* $\boldsymbol{r}^R_{O_L O}(t)$ with respect to *frame* R yields

$$\begin{aligned}
{}^R\dot{\boldsymbol{r}}^R_{O_L O}(t) := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{O_L O}(t)\right) &:= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(x^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR}\right) \\
&+ \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(y^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR}\right) + \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(z^R_{O_L O}(t) \cdot \boldsymbol{e}_{zR}\right).
\end{aligned} \tag{2.60c}$$

By the definition of the operator "$\dfrac{{}^R\mathrm{d}}{\mathrm{d}t}$" (see Equations 2.28a to 2.28f), the *basis vectors* $\boldsymbol{e}_{iR}$ $(i = x\,,\, y\,,\, z)$ *of frame* R *do not depend on* t. This yields, together with $\dot{\boldsymbol{e}}_{iR} \equiv \mathbf{0}$, the relation

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{O_L O}(t)\right) = \dot{x}^R_{O_L O}(t) \cdot \boldsymbol{e}_{xR} + \dot{y}^R_{O_L O}(t) \cdot \boldsymbol{e}_{yR} + \dot{z}^R_{O_L O}(t) \cdot \boldsymbol{e}_{zR} \tag{2.60d}$$

or

$$\dot{\boldsymbol{r}}^R_{O_L O}(t) := {}^R\dot{\boldsymbol{r}}^R_{O_L O}(t) = \left(\dot{x}^R_{O_L O}(t)\,,\, \dot{y}^R_{O_L O}(t)\,,\, \dot{z}^R_{O_L O}(t)\right)^{\mathrm{T}} \tag{2.60e}$$

with unambiguous time derivatives $\dot{x}^R_{O_L O}(t)\,,\, \dot{y}^R_{O_L O}(t)\,,\, \dot{z}^R_{O_L O}(t)$ of the scalar functions $x^R_{O_L O}(t)\,,\, y^R_{O_L O}(t)$ and $z^R_{O_L O}(t)$. The same result holds for the time derivative of the vector $\boldsymbol{r}_{PO_L}(t)$ with respect to frame R:

$$\dot{\boldsymbol{r}}^R_{PO_L}(t) := {}^R\dot{\boldsymbol{r}}^R_{PO_L}(t) = \left(\dot{x}^R_{PO_L}(t)\,,\, \dot{y}^R_{PO_L}(t)\,,\, \dot{z}^R_{PO_L}(t)\right)^{\mathrm{T}}. \tag{2.60f}$$

Assume that the vector $\boldsymbol{r}_{PO_L}(t)$, fixed on a moving body, is represented in a body-fixed frame L, but differentiated with respect to time in frame R. Then (see Equation A.1.31f):

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^R_{PO_L}\right) = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) \tag{2.61}$$

or

$${}^R\dot{\boldsymbol{r}}^R_{PO_L} = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L}. \tag{2.62}$$

Together with (A.1.31b) and (A.1.39a)

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} = \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{A}^{LR} = \tilde{\boldsymbol{\omega}}^R_{LR},$$

and with (A.1.39b)

$$\tilde{\boldsymbol{\omega}}^L_{LR} = \begin{pmatrix} 0 & , -\boldsymbol{\omega}^L_{zLR} \,, & \boldsymbol{\omega}^L_{yLR} \\ \omega^L_{zLR}\,, & 0 & , -\boldsymbol{\omega}^L_{xLR} \\ -\omega^L_{yLR}\,, & \omega^L_{xLR}\,, & 0 \end{pmatrix},$$

and with the *formal vector* $\boldsymbol{\omega}^L_{LR} = \left(\omega^L_{xLR}\,,\, \omega^L_{yLR}\,,\, \omega^L_{zLR}\right)^{\mathrm{T}}$, this provides the expressions (see Equation A.1.31d)

$${}^R\dot{\boldsymbol{r}}^R_{PO_L} = \boldsymbol{A}^{RL} \cdot \underbrace{\left(\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + {}^L\dot{\boldsymbol{r}}^L_{PO_L}\right)}_{= {}^R\dot{\boldsymbol{r}}^L_{PO_L}} = \boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L} \tag{2.63}$$

and (see Equation A.1.31c)

$${}^R\dot{\boldsymbol{r}}^L_{PO_L} = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) = \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + {}^L\dot{\boldsymbol{r}}^L_{PO_L} = \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO_L}. \tag{2.64}$$

The relation

$$\boldsymbol{v}^R_{PO} := {}^R\dot{\boldsymbol{r}}^R_{PO} := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^R_{PO}\right) = \left({}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L}\right) \tag{2.65a}$$

$$= {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \left({}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}\right) = {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L},$$

together with

$${}^L\dot{\boldsymbol{r}}^L_{PO_L} \equiv \boldsymbol{0} \quad \textit{(rigid-body condition)} \tag{2.65b}$$

implies

$$\boldsymbol{v}^R_{PO} := {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}. \tag{2.65c}$$

2.2.2.2 Accelerations and angular accelerations. The *acceleration vector* $\boldsymbol{a}^R_{PO} := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}^R_{PO}\right)$ can be written as

$$\begin{aligned}\boldsymbol{a}^R_{PO} &:= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{v}^R_{PO}\right) = {}^R\dot{\boldsymbol{v}}^R_{PO} \\ &= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left({}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L}\right)\end{aligned} \tag{2.66a}$$

or as

$$\begin{aligned}\boldsymbol{a}^R_{PO} = {}&{}^R\ddot{\boldsymbol{r}}^R_{O_LO} + \dot{\boldsymbol{A}}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot{}^L\dot{\tilde{\boldsymbol{\omega}}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} \\ &+ \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L} + \dot{\boldsymbol{A}}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot{}^L\ddot{\boldsymbol{r}}^L_{PO_L}\,.\end{aligned} \tag{2.66b}$$

Inserting the relation (A.1.39a)

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}$$

into (2.66b) yields the *acceleration relation*

$$\begin{aligned}{}^R\ddot{\boldsymbol{r}}^R_{PO} = {}&{}^R\ddot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot\dot{\tilde{\boldsymbol{\omega}}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} \\ &+ \left(2\cdot\boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot{}^L\ddot{\boldsymbol{r}}^L_{PO_L}\right).\end{aligned} \tag{2.67}$$

Assuming that the vector $\boldsymbol{r}^L_{PO_L}$ is constant in frame L (*rigid-body property* in the case that L is a frame fixed on a rigid body) provides

$${}^L\dot{\boldsymbol{r}}^L_{PO_L} \equiv \boldsymbol{0} \quad \text{and} \quad {}^L\ddot{\boldsymbol{r}}^L_{PO_L} \equiv \boldsymbol{0} \tag{2.68}$$

and yields the relations

$${}^R\dot{\boldsymbol{r}}^R_{PO} = {}^R\dot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} \tag{2.69a}$$

and

$${}^R\ddot{\boldsymbol{r}}^R_{PO} = {}^R\ddot{\boldsymbol{r}}^R_{O_LO} + \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL}\cdot\dot{\tilde{\boldsymbol{\omega}}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L}\,. \tag{2.69b}$$

2.2.2.3 Kinematic differential equation. In this section the *formal relation (A.1.39a)* will be derived and *physically interpreted* for the special case that $\boldsymbol{A}^{RL}$ is represented in terms of *Bryant angles* (see Equation 2.56)

$$\begin{aligned}\boldsymbol{A}^{LR} &= \left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} \\ &= \begin{pmatrix} \mathrm{c}\,\theta\cdot\mathrm{c}\,\psi & , & \mathrm{c}\,\phi\cdot\mathrm{s}\,\psi+\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi & , & \mathrm{s}\,\phi\cdot\mathrm{s}\,\psi-\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \\ -\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi & , & \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi-\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi & , & \mathrm{s}\,\phi\cdot\mathrm{c}\,\psi+\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \\ \mathrm{s}\,\theta & , & -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta & , & \mathrm{c}\,\phi\cdot\mathrm{c}\,\theta \end{pmatrix}.\end{aligned}$$

This will achieve a *physical interpretation* of the vector $\boldsymbol{\omega}^{L}_{LR}$ as an *angular velocity vector*. The time derivative of $\boldsymbol{A}^{LR}$ is (using the abbreviations 2.55d)

$$\dot{\boldsymbol{A}}^{LR} \tag{2.70}$$

$$= \left(\begin{matrix} (-\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\theta - \mathrm{c}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\psi) & , \\ (\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\theta - \mathrm{c}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\psi) & , \\ \mathrm{c}\,\theta\cdot\dot\theta & , \end{matrix} \right.$$

$$\begin{matrix} -\mathrm{s}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot\phi + \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot\psi + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\phi + \mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\theta - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\psi & , \\ -\mathrm{s}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot\phi - \mathrm{c}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot\psi - \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\phi - \mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\theta - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\psi & , \\ -\mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\dot\phi + \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\dot\theta & , \end{matrix}$$

$$\left. \begin{matrix} \mathrm{c}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot\phi + \mathrm{s}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot\psi + \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\phi - \mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\theta + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\psi \\ \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot\phi - \mathrm{s}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot\psi - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\phi + \mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot\theta + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot\psi \\ -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\dot\phi - \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\dot\theta \end{matrix} \right)$$

or

$$\begin{aligned} \dot{\boldsymbol{A}}^{LR} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{LR}(\eta)\right) \\ &= \begin{pmatrix} 0 \ , & -\mathrm{s}\,\phi\cdot\mathrm{s}\,\psi + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \ , & \mathrm{c}\,\phi\cdot\mathrm{s}\,\psi + \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \\ 0 \ , & -\mathrm{s}\,\phi\cdot\mathrm{c}\,\psi - \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \ , & \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \\ 0 \ , & -\mathrm{c}\,\phi\cdot\mathrm{c}\,\theta \ , & -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta \end{pmatrix}\cdot\dot\phi \\ &+ \begin{pmatrix} -\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \ , & \mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi \ , & -\mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi \\ \mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \ , & -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi \ , & \mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi \\ \mathrm{c}\,\theta \ , & \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta \ , & -\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta \end{pmatrix}\cdot\dot\theta \\ &+ \begin{pmatrix} -\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi \ , & \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \ , & \mathrm{s}\,\phi\cdot\mathrm{c}\,\psi + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \\ -\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi \ , & -\mathrm{c}\,\phi\cdot\mathrm{s}\,\psi - \mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \ , & -\mathrm{s}\,\phi\cdot\mathrm{s}\,\psi + \mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \\ 0 \ , & 0 \ , & 0 \end{pmatrix}\cdot\dot\psi \ . \end{aligned}$$

On the other hand

$$\dot{\boldsymbol{A}}^{LR} = \frac{\partial(\boldsymbol{A}^{LR})}{\partial\phi}\cdot\dot\phi + \frac{\partial(\boldsymbol{A}^{LR})}{\partial\theta}\cdot\dot\theta + \frac{\partial(\boldsymbol{A}^{LR})}{\partial\psi}\cdot\dot\psi \tag{2.71a}$$

$$= \left(\frac{\partial(\boldsymbol{A}^{LR})}{\partial\phi} \ , \ \frac{\partial(\boldsymbol{A}^{LR})}{\partial\theta} \ , \ \frac{\partial(\boldsymbol{A}^{LR})}{\partial\psi}\right)\cdot\begin{pmatrix} \mathbf{I}_3\cdot\dot\phi \\ \mathbf{I}_3\cdot\dot\theta \\ \mathbf{I}_3\cdot\dot\psi \end{pmatrix} = \frac{\partial(\boldsymbol{A}^{LR})}{\partial\boldsymbol{\eta}}\cdot\dot{\hat{\boldsymbol{\eta}}}$$

with $\dot{\hat{\boldsymbol{\eta}}} := (\mathbf{I}_3\cdot\dot\phi, \mathbf{I}_3\cdot\dot\theta, \mathbf{I}_3\cdot\dot\psi)^{\mathrm{T}}$. The product $\dot{\boldsymbol{A}}^{RL}\cdot\boldsymbol{A}^{LR}$ is

$$\dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{A}^{LR} = (\gamma_{ij})_{\substack{i=1,2,3 \\ j=1,2,3}} \qquad \text{with} \tag{2.72}$$

$$\begin{aligned}
\gamma_{11} =& -\mathrm{s}\,\psi \cdot \mathrm{c}\,\theta \cdot \left(\dot{\theta} \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta\right) \\
& + \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta \cdot \left(-\dot{\theta} \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta\right) + \dot{\theta} \cdot \mathrm{c}\,\theta \cdot \mathrm{s}\,\theta, \\
\gamma_{12} =& \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta \cdot \Big(-\dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta \\
& + \dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& - \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta \cdot \Big(-\dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta \\
& - \dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta - \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& + \mathrm{s}\,\theta \cdot \left(\dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta - \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta\right), \\
\gamma_{13} =& -\mathrm{s}\,\psi \cdot \mathrm{c}\,\theta \cdot \Big(-\dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta \\
& + \dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& + \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta \cdot \Big(\dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta \\
& - \dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi + \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& + \mathrm{s}\,\theta \cdot \left(-\dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta\right), \\
\gamma_{21} =& \left(\dot{\theta} \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta\right) \cdot (\mathrm{c}\,\phi \cdot \mathrm{c}\,\psi - \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta) \\
& + \left(-\dot{\theta} \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta\right) \cdot (\mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi) \\
& - \dot{\theta} \cdot \mathrm{s}\,\phi\,(\mathrm{c}\,\theta)^2, \\
\gamma_{22} =& (\mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi) \cdot \Big(-\dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
& + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& + \Big(-\dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta \\
& - \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \cdot (\mathrm{c}\,\phi \cdot \mathrm{c}\,\psi - \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta) \\
& - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \left(\dot{\theta} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta - \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta\right), \\
\gamma_{23} =& (\mathrm{c}\,\phi \cdot \mathrm{c}\,\psi - \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta) \cdot \Big(-\dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
& + \dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta - \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& + (\mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi) \cdot \Big(\dot{\psi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
& + \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta + \dot{\phi} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi + \dot{\psi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
& - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \left(-\dot{\theta} \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta - \dot{\phi} \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta\right),
\end{aligned}$$

$$\begin{aligned}
\gamma_{31} = &\left(\dot\theta \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta - \dot\psi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta\right) \cdot (\mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi) \\
&+ \left(-\dot\theta \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot\psi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta\right) \cdot (\mathrm{s}\,\phi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta) \\
&+ \dot\theta \cdot \mathrm{c}\,\phi \cdot (\mathrm{c}\,\theta)^2 ,
\end{aligned}$$

$$\begin{aligned}
\gamma_{32} = &(\mathrm{s}\,\phi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta) \cdot \Big(- \dot\psi \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
&+ \dot\phi \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \dot\theta \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta - \dot\phi \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot\psi \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
&+ (\mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi) \cdot \Big(- \dot\phi \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
&- \dot\psi \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot\theta \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta - \dot\psi \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi - \dot\phi \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
&+ \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \left(\dot\theta \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta - \dot\phi \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta\right) , \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
\gamma_{33} = &(\mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta + \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi) \cdot \Big(- \dot\phi \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
&+ \dot\psi \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta + \dot\theta \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{c}\,\theta - \dot\psi \cdot \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi + \dot\phi \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
&+ (\mathrm{s}\,\phi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta) \cdot \Big(\dot\psi \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \mathrm{s}\,\theta \\
&+ \dot\phi \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{s}\,\theta - \dot\theta \cdot \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \mathrm{c}\,\theta + \dot\phi \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi + \dot\psi \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi\Big) \\
&+ \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \left(-\dot\theta \cdot \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta - \dot\phi \cdot \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta\right) .
\end{aligned}$$

This provides (after some trigonometric manipulations) the *skew-symmetric matrix*

$$\dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{A}^{LR} \tag{2.73}$$

$$= \begin{pmatrix} 0 & , -\mathrm{s}\,\phi \cdot \dot\theta - \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi & , \mathrm{c}\,\phi \cdot \dot\theta - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \\ \mathrm{s}\,\phi \cdot \dot\theta + \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \;, & 0 & , \quad -\dot\phi - \mathrm{s}\,\theta \cdot \dot\psi \\ -\mathrm{c}\,\phi \cdot \dot\theta + \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \;, & +\dot\phi + \mathrm{s}\,\theta \cdot \dot\psi & , \quad 0 \end{pmatrix} .$$

Consider, on the other hand, the *angular velocity vectors* (Section 2.2.1.2)

$$\boldsymbol{\omega}^{R}_{L''R} := \begin{pmatrix} \dot\phi_{L''R} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_{xL''R} \\ 0 \\ 0 \end{pmatrix} \quad \text{(rotation around the } x\text{-axis from } R \text{ to } L''\text{, represented in } R\text{),} \tag{2.74a}$$

$$\boldsymbol{\omega}^{L''}_{L'L''} := \begin{pmatrix} 0 \\ \dot\theta_{L'L''} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_{yL'L''} \\ 0 \end{pmatrix} \quad \text{(rotation around the } y''\text{-axis from } L'' \text{ to } L'\text{, represented in } L''\text{),} \tag{2.74b}$$

$$\boldsymbol{\omega}^{L'}_{LL'} := \begin{pmatrix} 0 \\ 0 \\ \dot\psi_{LL'} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_{xLL'} \end{pmatrix} \quad \text{(rotation around the } z'\text{-axis from } L' \text{ to } L\text{, represented in } L'\text{).} \tag{2.74c}$$

The *resulting angular velocity vector* of these three relations, represented in frame R, is

$$\boldsymbol{\omega}^R_{LR} := \begin{pmatrix} \omega^R_{xLR} \\ \omega^R_{yLR} \\ \omega^R_{zLR} \end{pmatrix} = \boldsymbol{\omega}^R_{L''R} + \boldsymbol{A}^{RL''} \cdot \boldsymbol{\omega}^{L''}_{L'L''} + \boldsymbol{A}^{RL''} \cdot \boldsymbol{A}^{L''L'} \cdot \boldsymbol{\omega}^{L'}_{LL'} . \tag{2.74d}$$

Using the abbreviations

$$\dot\phi := \dot\phi^R_{L''R} \quad , \quad \dot\theta := \dot\theta^{L''}_{L'L''} \quad , \quad \dot\psi := \dot\psi^{L'}_{LL'} \tag{2.74e}$$

and the relations (see Equation 2.55)

$$\boldsymbol{A}^{RL''} \cdot \boldsymbol{\omega}^{L''}_{L'L''} = \begin{pmatrix} 1\ , & 0\ , & 0 \\ 0\ , & \mathrm{c}\,\phi\ , & -\mathrm{s}\,\phi \\ 0\ , & \mathrm{s}\,\phi\ , & \mathrm{c}\,\phi \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{c}\,\phi \cdot \dot\theta \\ \mathrm{s}\,\phi \cdot \dot\theta \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{c}\,\phi \\ \mathrm{s}\,\phi \end{pmatrix} \cdot \dot\theta$$

and

$$\begin{aligned} \boldsymbol{A}^{RL''} \cdot \boldsymbol{A}^{L''L'} \cdot \boldsymbol{\omega}^{L'}_{LL'} &= \begin{pmatrix} 1\ , & 0\ , & 0 \\ 0\ , & \mathrm{c}\,\phi\ , & -\mathrm{s}\,\phi \\ 0\ , & \mathrm{s}\,\phi\ , & \mathrm{c}\,\phi \end{pmatrix} \cdot \begin{pmatrix} \mathrm{c}\,\theta\ , & 0\ , & \mathrm{s}\,\theta \\ 0\ , & 1\ , & 0 \\ -\mathrm{s}\,\theta\ , & 0\ , & \mathrm{c}\,\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} \\ &= \begin{pmatrix} 1\ , & 0\ , & 0 \\ 0\ , & \mathrm{c}\,\phi\ , & -\mathrm{s}\,\phi \\ 0\ , & \mathrm{s}\,\phi\ , & \mathrm{c}\,\phi \end{pmatrix} \cdot \begin{pmatrix} \mathrm{s}\,\theta \cdot \dot\psi \\ 0 \\ \mathrm{c}\,\theta \cdot \dot\psi \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{s}\,\theta \cdot \dot\psi \\ -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \\ \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \end{pmatrix} = \begin{pmatrix} \mathrm{s}\,\theta \\ -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \\ \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix} \cdot \dot\psi \end{aligned}$$

yields the resulting vector $\boldsymbol{\omega}^R_{LR}$:

$$\boldsymbol{\omega}^R_{LR} = \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathrm{c}\,\phi \\ \mathrm{s}\,\phi \end{pmatrix} \cdot \dot\theta + \begin{pmatrix} \mathrm{s}\,\theta \\ -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \\ \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix} \cdot \dot\psi = \begin{pmatrix} \dot\phi + \mathrm{s}\,\theta \cdot \dot\psi \\ \mathrm{c}\,\phi \cdot \dot\theta - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \\ \mathrm{s}\,\phi \cdot \dot\theta + \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \end{pmatrix} \tag{2.75}$$

and the associated skew-symmetric matrix

$$\tilde{\boldsymbol{\omega}}^R_{LR} = \tag{2.76}$$

$$\begin{pmatrix} 0 & , -\mathrm{s}\,\phi \cdot \dot\theta - \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi\ , & \mathrm{c}\,\phi \cdot \dot\theta - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi \\ \mathrm{s}\,\phi \cdot \dot\theta + \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot\psi\ , & 0\ , & -\dot\phi - \mathrm{s}\,\theta \cdot \dot\psi \\ -\mathrm{c}\,\phi \cdot \dot\theta + s\phi \cdot c\theta \cdot \dot\psi\ , & \dot\phi + \mathrm{s}\,\theta \cdot \dot\psi\ , & 0 \end{pmatrix} .$$

Comparing (2.73) and (2.76) yields

$$\dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{A}^{LR} = \tilde{\boldsymbol{\omega}}^R_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} = \tilde{\boldsymbol{\omega}}^R_{LR} \cdot \boldsymbol{A}^{RL}$$

or (2.77)

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR}.$$

This proves that the *formal vector*

$$\boldsymbol{\omega}^L_{LR} = \boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}^R_{LR},$$

introduced in (A.1.38), can be *physically interpreted* as the *angular velocity* of frame L, with respect to frame R, represented in L. The relation (2.75) can be written as

$$\begin{pmatrix} \omega^R_{xLR} \\ \omega^R_{yLR} \\ \omega^R_{zLR} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \,, & 0 \,, & \mathrm{s}\,\theta \\ 0 \,, & \mathrm{c}\,\phi \,, & -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \\ 0 \,, & \mathrm{s}\,\phi \,, & \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix}}_{=:\, \boldsymbol{H}^{-1}(\boldsymbol{\eta})} \cdot \underbrace{\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}}_{=:\, \dot{\boldsymbol{\eta}}} \quad , \quad \dot{\boldsymbol{\eta}} := \left(\dot{\phi}\,, \dot{\theta}\,, \dot{\psi}\right)^{\mathrm{T}}, \tag{2.78a}$$

and finally as

$$\boldsymbol{\omega}^R_{LR} = \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{\eta}}$$

or (2.78b)

$$\boldsymbol{\omega}^L_{LR} = \boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}^R_{LR} = \boldsymbol{A}^{LR} \cdot \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{\eta}}$$

with the *Bryant angles* ϕ, θ, and ψ, and with

$$\dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} \quad \text{and} \quad \boldsymbol{H}^{-1}(\boldsymbol{\eta}) = \begin{pmatrix} 1 \,, & 0 \,, & \mathrm{s}\,\theta \\ 0 \,, & \mathrm{c}\,\phi \,, & -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \\ 0 \,, & \mathrm{s}\,\phi \,, & \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix}. \tag{2.78c}$$

This yields the well-known *kinematic differential equation*

$$\dot{\boldsymbol{\eta}} = \boldsymbol{H}(\boldsymbol{\eta}) \cdot \boldsymbol{\omega}^R_{LR} = \boldsymbol{H}(\boldsymbol{\eta}) \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{\omega}^L_{LR} \tag{2.79a}$$

with

$$\boldsymbol{H}(\boldsymbol{\eta}) = \frac{1}{\mathrm{c}\,\theta} \cdot \begin{pmatrix} \mathrm{c}\,\theta \,, & \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \,, & -\mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \\ 0 \,, & \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \,, & \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \\ 0 \,, & -\mathrm{s}\,\phi \,, & \mathrm{c}\,\phi \end{pmatrix}, \tag{2.79b}$$

and (after some trigonometric manipulations)

$$\boldsymbol{H}(\boldsymbol{\eta}) \cdot \boldsymbol{A}^{RL} = \begin{pmatrix} \dfrac{\cos\psi}{\cos\theta} & , & \dfrac{-\sin\psi}{\cos\theta} & , 0 \\ \sin\psi & , & \cos\psi & , 0 \\ -\cos\psi \cdot \dfrac{\sin\theta}{\cos\theta} & , & \sin\psi \cdot \dfrac{\sin\theta}{\cos\theta} & , 1 \end{pmatrix} \tag{2.79c}$$

for *Bryant angles* ϕ, θ, and ψ. The matrix $\boldsymbol{H}(\boldsymbol{\eta})$ is *singular* for

$$\theta = (n+1) \cdot (\pi/2) \quad , \quad n \in \boldsymbol{Z} \cup \{0\}.$$

Taking into account

$$\begin{aligned} &\boldsymbol{p} = (\boldsymbol{r}^{\mathrm{T}}, \boldsymbol{\eta}^{\mathrm{T}})^{\mathrm{T}} = (x, y, z, \phi, \theta, \psi)^{\mathrm{T}}, \\ &\boldsymbol{r} = (x, y, z)^{\mathrm{T}}, \quad \boldsymbol{\eta} = (\phi, \theta, \psi)^{\mathrm{T}} \end{aligned} \tag{2.80a}$$

and

$$\dot{\boldsymbol{p}} = \left(\dot{\boldsymbol{r}}^{\mathrm{T}}, \dot{\boldsymbol{\eta}}^{\mathrm{T}}\right)^{\mathrm{T}} \tag{2.80b}$$

provides, together with

$$\begin{aligned} \boldsymbol{v} := \begin{pmatrix} \dot{\boldsymbol{r}}^R \\ \boldsymbol{\omega}_{LR}^L \end{pmatrix} &= \begin{pmatrix} \mathbf{I}_3 \cdot \dot{\boldsymbol{r}}^R \\ \boldsymbol{A}^{LR}(\boldsymbol{\eta}) \cdot \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{\eta}} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{A}^{LR}(\boldsymbol{\eta}) \cdot \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \end{pmatrix}}_{=: \boldsymbol{T}^{-1}(\boldsymbol{\eta})} \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^R \\ \dot{\boldsymbol{\eta}} \end{pmatrix}, \end{aligned}$$

the *kinematic differential equation*

$$\boldsymbol{v} = \boldsymbol{T}^{-1}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{p}} \tag{2.80c}$$

with

$$\boldsymbol{T}^{-1}(\boldsymbol{\eta}) = \begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{A}^{LR}(\boldsymbol{\eta}) \cdot \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \end{pmatrix}. \tag{2.80d}$$

Conversely,

$$\dot{\boldsymbol{p}} = \begin{pmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{\eta}} \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\eta}) \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^R \\ \boldsymbol{\omega}_{LR}^L \end{pmatrix} = \boldsymbol{T}(\boldsymbol{\eta}) \cdot \boldsymbol{v} \tag{2.81a}$$

with

$$\boldsymbol{T}(\boldsymbol{\eta}) := \begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{H}(\boldsymbol{\eta}) \cdot \boldsymbol{A}^{RL}(\boldsymbol{\eta}) \end{pmatrix}. \tag{2.81b}$$

3. Constraint equations and constraint reaction forces of mechanisms

The constraint equations of mechanisms come in many forms. They may be *kinematic constraint relations* due to passive *joints*, or *active constraint relations* that stem from reactions due to *active force elements.* Constraint equations are used to solve quite different tasks in *mechanisms* and *mechatronic systems.*

Kinematic constraint equations may serve:

1. As *theoretical models* and *numerical algorithms* of *joints.*
2. As *projection operators* of *forces.*
3. As *nonlinear and linear prefilters* and *kinematic decoupling controllers* of mechanisms and *mechatronic systems.*
4. As *algorithms* used for *transformation of signals* and for *image processing.*

Active constraint equations may be used:

1. As *theoretical and numerical models* of *active drives.*
2. As *algorithms* for *computing desired forces* acting on a mechanism.
3. As *models of disturbances* acting on a mechanism (compare Examples 3.7 and 7.1 of Volume II).
4. As *algorithms* for *disturbance rejection in control systems.*

In this chapter general representations of *position, velocity and acceleration equations* of planar and spatial *kinematic* and *active constraints* will be presented together with a discussion of the *singular behavior* of constraint equations (*lock-ups* and *bifurcation points* of a mechanism, *Sections 3.1.1.2* and *3.1.3*). *Reaction forces* and *torques* generated by these constraints will be discussed in *Section 3.2.*

3.1 Kinematics of planar and spatial rigid-body systems

In this section basic relations of holonomic kinematic and active constraints will be briefly discussed both for *planar* mechanisms *(Section 3.1.1)*, and for *spatial* mechanisms *(Section 3.1.2).* The *singular behavior* of constraints will be extensively discussed by means of an example in *Section 3.1.3.*

3.1.1 Kinematics of *planar* mechanisms

Kinematics is the *study of motion* (position, velocity, acceleration) of mechanisms *without considering the forces and torques* that generate the motion. A mechanism or configuration is considered as an assembly of rigid bodies and links that are arranged and connected by joints to allow absolute and relative motion. Joints impose constraints on the absolute and relative motion of the bodies. They will be theoretically modeled by n_c kinematic constraint equations

$$\boldsymbol{g}^k(\boldsymbol{p}(t)) \equiv \mathbf{0}, \quad \boldsymbol{g}^k(\boldsymbol{p}) \in \mathcal{C}^2(\mathbb{R}^{n_c}) \quad \text{(twice differentiable)}, \tag{3.1}$$

that reflect the geometry of the joint. In the planar case, $\boldsymbol{p}$ ($\boldsymbol{p} \in \mathbb{R}^{3n_b}$) is the vector of generalized coordinates of the mechanism. Each scalar constraint relation reduces the number of DOFs of a mechanism by one. Kinematic constraint relations of the form (3.1) do not depend explicitly on time but only on the generalized coordinates. They are called *holonomic constraint equations.* Holonomic constraint equations may also depend explicitly on time and on the velocity vector $\dot{\boldsymbol{p}}$ if $\dot{\boldsymbol{p}}$ is integrable in closed form. If the n_c kinematic constraint relations (3.1) are *consistent* and *independent*, a system of n_b rigid bodies under planar motion has $3n_b - n_c$ *DOFs.* Then either $(3n_b - n_c)$ additional *active (driving)* constraint relations are needed (*kinematic analysis*), or external forces together with initial conditions (*dynamic analysis*) are used to uniquely determine the motion $\boldsymbol{p}(t)$ of a mechanism.

Constraints imposed on a system by a drive are described by *active* or *driving constraint relations.* Active constraint relations model the time histories of some coordinates produced by ideal actuators. They usually depend explicitly on time and on some generalized coordinates. They are written in the form

$$\boldsymbol{g}^a\,(\boldsymbol{p}(t)\,,\,t) \equiv \mathbf{0} \quad , \quad \boldsymbol{g}^a\,(\boldsymbol{p}\,,\,t) \in \mathcal{C}^2(\mathbb{R}^{3n_b - n_c}) \quad \text{(twice differentiable)}.$$

More general constraint relations that include inequalities or velocity components that are not integrable in closed form are called *nonholonomic constraint relations.*

Here *holonomic kinematic constraint relations* will be considered for bodies: (1) that undergo *motions in a single plane* or in *parallel planes*, and (2) that are *connected by joints* imposing constraints on the motion of these bodies.

These constraint elements may be *joints*, *massless links*, *spur gears*, or *driving links* that connect two different bodies to each other, or a singe body to its base. To specify a configuration (the position and orientation or location of each rigid body of the configuration), an inertial (global) frame R with global Cartesian coordinates x^R and y^R, and a body-fixed (local) frame L_i for each rigid body i with local Cartesian coordinates x^{L_i}, y^{L_i} and with $\psi_{L_iR} =: \psi_i$ as the angle of rotation of L_i with respect to R are introduced (Figure 3.1).

The location of the body i in the plane is then specified by the *global coordinates* of the body-fixed origin O_i of the frame L_i with respect to the frame R:

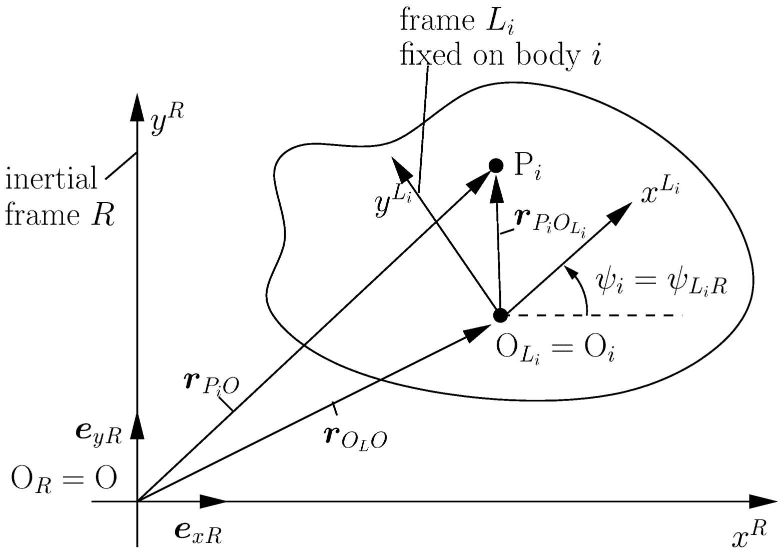


Fig. 3.1: Global and local frames in the plane

$$\boldsymbol{r}^R_{O_iO} = \left(x^R_{O_iO}\,,\,y^R_{O_iO}\right)^{\mathrm{T}}. \tag{3.2a}$$

A point P_i fixed on this body is specified by *constant local coordinates* with respect to the frame L_i:

$$\boldsymbol{r}^{L_i}_{P_iO_i} = \begin{pmatrix} x^{L_i}_{P_iO_i} \\ y^{L_i}_{P_iO_i} \end{pmatrix} \quad \text{(constant local vector).} \tag{3.2b}$$

Alternatively, the point P_i can also be specified by *global coordinates* as

$$\boldsymbol{r}^R_{P_iO} = \boldsymbol{r}^R_{O_iO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{P_iO_i}, \tag{3.3a}$$

or written in expanded form

$$\begin{pmatrix} x^R_{P_iO} \\ y^R_{P_iO} \end{pmatrix} = \begin{pmatrix} x^R_{O_iO} \\ y^R_{O_iO} \end{pmatrix} + \begin{pmatrix} \cos\psi_i\ , & -\sin\psi_i \\ \sin\psi_i\ , & \cos\psi_i \end{pmatrix} \cdot \begin{pmatrix} x^{L_i}_{P_iO_i} \\ y^{L_i}_{P_iO_i} \end{pmatrix},$$

or as

$$\begin{aligned} x^R_{P_iO} &= x^{L_i}_{O_iO} + x^{L_i}_{P_iO_i} \cdot \cos\psi_i - y^{L_i}_{P_iO_i} \cdot \sin\psi_i \\ y^R_{P_iO} &= y^{L_i}_{O_iO} + x^{L_i}_{P_iO_i} \cdot \sin\psi_i + y^{L_i}_{P_iO_i} \cdot \cos\psi_i \end{aligned} \tag{3.3b}$$

with

$$\psi_i := \psi_{L_iR}$$

as the rotation of the body (frame L_i) with respect to frame R. The *geometric vector* $\boldsymbol{r}_{O_iO}$ together with the angle ψ_i that specify the location of the body i in the inertial frame R will henceforth be written in the form of an *algebraic vector* of Cartesian coordinates

$$\boldsymbol{p}_i := \left(p^R_{xi}\,,\, p^R_{yi}\,,\, \psi_i\right)^{\mathrm{T}} := \left(\boldsymbol{r}^{R}_{O_iO}{}^{\mathrm{T}}\,,\, \psi_{L_iR}\right)^{\mathrm{T}} = \left(x^R_{O_iO}\,,\, y^R_{O_iO}\,,\, \psi_i\right)^{\mathrm{T}}. \quad (3.4\mathrm{a})$$

This vector includes two position coordinates and a rotation coordinate. It uniquely describes the position and orientation (location) of a rigid body in the x^R–y^R plane.

Then a *planar mechanism including n_b rigid bodies* is specified by a global algebraic vector of $3n_b$ generalized Cartesian coordinates

$$\begin{aligned}\boldsymbol{p} &= \left(\boldsymbol{p}_1^{\mathrm{T}}, \boldsymbol{p}_2^{\mathrm{T}}, \ldots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} &(3.4\mathrm{b})\\ &= \left(x^R_{O_1O}, y^R_{O_1O}, \psi_1, \ldots, x^R_{O_{n_b}O}, y^R_{O_{n_b}O}, \psi_{n_b}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b}\,.\end{aligned}$$

Based on these notations, holonomic constraint relations of planar configurations that include n_b bodies are written in the form:

$$\left.\begin{array}{rcl}\boldsymbol{g}^k : \mathbb{R}^{3n_b} & \longrightarrow & \mathbb{R}^{n_c}\\ \Cup & & \Cup\\ \boldsymbol{p} & \longmapsto & \boldsymbol{g}^k(\boldsymbol{p}) = \boldsymbol{0}\end{array}\right\}\ n_c \textit{ kinematic constraint equations} \quad (3.5\mathrm{a})$$

and

$$\left.\begin{array}{rcccl}\boldsymbol{g}^a : \mathbb{R}^{3n_b} & \times & \mathbb{R}^1 & \longrightarrow & \mathbb{R}^{3n_b-n_c}\\ \Cup & & \Cup & & \\ (\boldsymbol{p} & , & t) & \longmapsto & \boldsymbol{g}^a(\boldsymbol{p},t) = \boldsymbol{0}\end{array}\right\}\ \begin{array}{l}3n_b - n_c \textit{ active or driving}\\ \textit{constraint relations.}\end{array} \quad (3.5\mathrm{b})$$

In *pure kinematic analysis* of a mechanism with $(3n_b - n_c)$ DOFs exactly $(3n_b - n_c)$ *active constraint relations* are needed to completely specify the motion of the system. In *dynamic analysis*, in general, there are no driver equations to be specified. For $3n_b > n_c$ there are $(3n_b - n_c)$ more unknowns in the constraint equations then there are equations. As a consequence, there is no unique solution to these equations. A unique solution is obtained here by specifying a proper set of initial conditions and by solving the kinematic constraint equations simultaneously with the kinetic equations, either as a system of *d*ifferential *a*lgebraic *e*quations *(DAEs)*, or as a system of *DEs*, after having eliminated n_c dependent coordinates and the Lagrange multipliers. In *dynamic analysis*, the motion of the system is completely specified by the external forces and torques, by the n_c *kinematic constraint equations*, and by the initial conditions of $\boldsymbol{p}$.

3.1.1.1 Pure kinematic analysis of planar mechanisms. In *kinematic analysis, position constraint equations* and the associated *velocity* and *acceleration* equations are analyzed and solved. They will be written in the following form (*Appendix A.1.6*):

Kinematic constraint equations

Constraint position equation

$$\boldsymbol{g}^k(\boldsymbol{p}) \equiv \boldsymbol{0}. \tag{3.6a}$$

Constraint velocity equation

$$\dot{\boldsymbol{g}}^k(\boldsymbol{p}) = \boldsymbol{g}_p^k(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} \equiv \boldsymbol{0}. \tag{3.6b}$$

Constraint acceleration equation

$$\ddot{\boldsymbol{g}}^k(\boldsymbol{p}) = \left(\boldsymbol{g}_p^k(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}}\right)_p \cdot \dot{\boldsymbol{p}} + \boldsymbol{g}_p^k(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} \equiv \boldsymbol{0}$$

or

$$\boldsymbol{g}_p^k(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} = -\left(\boldsymbol{g}_p^k(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}}\right)_p \cdot \dot{\boldsymbol{p}} =: \boldsymbol{\beta}_{\mathrm{c}}^k\left(\boldsymbol{p}\,,\, \dot{\boldsymbol{p}}\right). \tag{3.6c}$$

Active constraint equations

Constraint position equation

$$\boldsymbol{g}^a(\boldsymbol{p}\,,\, t) \equiv \boldsymbol{0}. \tag{3.7a}$$

Constraint velocity equation

$$\dot{\boldsymbol{g}}^a = \boldsymbol{g}_p^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} + \boldsymbol{g}_t^a(\boldsymbol{p}) \equiv \boldsymbol{0}. \tag{3.7b}$$

Constraint acceleration equation

$$\ddot{\boldsymbol{g}}^a = \left(\boldsymbol{g}_p^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}}\right)_p \cdot \dot{\boldsymbol{p}} + \boldsymbol{g}_{pt}^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} + \boldsymbol{g}_p^a(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} + \boldsymbol{g}_{tp}^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} + \boldsymbol{g}_{tt}^a(\boldsymbol{p}) \equiv \boldsymbol{0}$$

or

$$\boldsymbol{g}_p^a(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} = -\left(\boldsymbol{g}_p^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}}\right)_p \cdot \dot{\boldsymbol{p}} - 2 \cdot \boldsymbol{g}_{pt}^a(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} - \boldsymbol{g}_{tt}^a(\boldsymbol{p}) =: \boldsymbol{\beta}_{\mathrm{c}}^a\left(\boldsymbol{p}\,,\, \dot{\boldsymbol{p}}\,,\, t\right). \tag{3.7c}$$

Complete set of kinematic and active constraint equations

Combining the two sets of *kinematic and active* constraint equations yields the total *constraint position equation*

$$\boldsymbol{g}(\boldsymbol{p}\,,\,t) = \begin{pmatrix} \boldsymbol{g}^k(\boldsymbol{p}) \\ \boldsymbol{g}^a(\boldsymbol{p}\,,\,t) \end{pmatrix} = \boldsymbol{0}, \qquad \boldsymbol{g}(\boldsymbol{p}\,,\,t) \in \mathbb{R}^{3n_b}, \tag{3.8a}$$

and, together with

$$\dot{\boldsymbol{g}}(\boldsymbol{p}\,,\,t) = \begin{pmatrix} \dot{\boldsymbol{g}}^k(\boldsymbol{p}) \\ \dot{\boldsymbol{g}}^a(\boldsymbol{p}\,,\,t) \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_p^k(\boldsymbol{p}) \\ \boldsymbol{g}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \cdot \dot{\boldsymbol{p}} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g}_t^a(\boldsymbol{p}\,,\,t) \end{pmatrix} = \boldsymbol{0},$$

the *constraint velocity equation*

$$\boldsymbol{g}_p(\boldsymbol{p}\,,\,t) \cdot \dot{\boldsymbol{p}} = -\boldsymbol{g}_t(\boldsymbol{p}\,,\,t) =: \boldsymbol{\alpha}_{\mathrm{c}}(\boldsymbol{p}\,,\,t) \tag{3.8b}$$

with

$$\boldsymbol{g}_p(\boldsymbol{p}\,,\,t) := \begin{pmatrix} \boldsymbol{g}_p^k(\boldsymbol{p}) \\ \boldsymbol{g}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \quad \text{and} \quad \boldsymbol{g}_t(\boldsymbol{p}\,,\,t) := \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g}_t^a(\boldsymbol{p}\,,\,t) \end{pmatrix} =: -\boldsymbol{\alpha}_{\mathrm{c}}(\boldsymbol{p}\,,\,t),$$

and from

$$\ddot{\boldsymbol{g}}(\boldsymbol{p}\,,\,t) = \begin{pmatrix} \ddot{\boldsymbol{g}}_p^k(\boldsymbol{p}) \\ \ddot{\boldsymbol{g}}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} = \left[\begin{pmatrix} \boldsymbol{g}_p^k(\boldsymbol{p}) \\ \boldsymbol{g}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \cdot \dot{\boldsymbol{p}} \right]_p \cdot \dot{\boldsymbol{p}} + \begin{pmatrix} \boldsymbol{g}_p^k(\boldsymbol{p}) \\ \boldsymbol{g}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \cdot \ddot{\boldsymbol{p}}$$
$$+ \begin{pmatrix} \boldsymbol{0} \\ 2 \cdot \boldsymbol{g}_{pt}^a(\boldsymbol{p}\,,\,t) \cdot \dot{\boldsymbol{p}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g}_{tt}^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \equiv \boldsymbol{0}$$

the *constraint acceleration equation*

$$\begin{aligned} \boldsymbol{g}_p(\boldsymbol{p}\,,\,t) \cdot \ddot{\boldsymbol{p}} &= -\left[\boldsymbol{g}_p(\boldsymbol{p}\,,\,t) \cdot \dot{\boldsymbol{p}}\right]_p \cdot \dot{\boldsymbol{p}} - 2 \cdot \boldsymbol{g}_{pt}(\boldsymbol{p}\,,\,t) \cdot \dot{\boldsymbol{p}} - \boldsymbol{g}_{tt}(\boldsymbol{p}\,,\,t) \\ &=: \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}\,,\,\dot{\boldsymbol{p}}\,,\,t) \end{aligned} \tag{3.8c}$$

with

$$\boldsymbol{g}_{pt}(\boldsymbol{p}\,,\,t) := \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g}_{pt}^a(\boldsymbol{p}\,,\,t) \end{pmatrix}, \qquad \boldsymbol{g}_{tt}(\boldsymbol{p}\,,\,t) := \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g}_{tt}^a(\boldsymbol{p}\,,\,t) \end{pmatrix},$$

and

$$\left(\boldsymbol{g}_p(\boldsymbol{p}\,,\,t) \cdot \dot{\boldsymbol{p}}\right)_p \cdot \dot{\boldsymbol{p}} := \left[\begin{pmatrix} \boldsymbol{g}_p^k(\boldsymbol{p}) \\ \boldsymbol{g}_p^a(\boldsymbol{p}\,,\,t) \end{pmatrix} \cdot \dot{\boldsymbol{p}} \right]_p \cdot \dot{\boldsymbol{p}}.$$

In the previous relations the following notations have been used:

$$\boldsymbol{g}_t := \frac{\partial}{\partial t}\left(\boldsymbol{g}(\boldsymbol{p}\,,\,t)\right) = \left(g_{1t}, \ldots, g_{3n_b t}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad g_{it} := \partial g_i / \partial t,$$

$$\boldsymbol{g}_{tt} := (g_{1tt}, \ldots, g_{3n_b tt})^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad g_{itt} := \partial^2 g_i / \partial t^2$$

$$\boldsymbol{g}_p := \begin{pmatrix} \partial g_1 / \partial x^R_{O_1 O} & , \ldots , & \partial g_1 / \partial \psi_{n_b} \\ \vdots & & \vdots \\ \partial g_{3n_b} / \partial x^R_{O_1 O} & , \ldots , & \partial \boldsymbol{g}_{3n_b} / \partial \psi_{n_b} \end{pmatrix} \in \mathbb{R}^{3n_b , 3n_b},$$

$$\boldsymbol{g}_{pt} := \frac{\partial}{\partial t} \left(\boldsymbol{g}_p \right) \in \mathbb{R}^{3n_b , 3n_b}.$$

3.1.1.2 Regular and singular planar kinematics. Consider the n_c kinematic constraint position equations

$$\boldsymbol{g}(p) := g^k(\boldsymbol{p}) = \mathbf{0} \in \mathbb{R}^{n_c} \quad , \quad \boldsymbol{p} = \boldsymbol{p}(t) \in \mathbb{R}^{3n_b} . \tag{3.9a}$$

Consider the partitioning of $\boldsymbol{p}$ into

$$\boldsymbol{p} = (\boldsymbol{u}^{\mathrm{T}} , \boldsymbol{w}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{3n_b} \tag{3.9b}$$

with

$\boldsymbol{u} \in \mathbb{R}^{n_c}$ as a vector of *dependent coordinates*, and (3.9c)

$\boldsymbol{w} \in \mathbb{R}^{3n_b - n_c}$ as a vector of *independent coordinates*, (3.9d)

sometimes called the *driving coordinates in kinematic analysis.* The objective of kinematic *position analysis* is to solve the n_c kinematic constraint equations

$$\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{g}(\boldsymbol{u} , \boldsymbol{w}) = \mathbf{0} \tag{3.10}$$

with respect to the vector $\boldsymbol{u}$ for a given vector $\boldsymbol{w}$ of driving coordinates. This system of algebraic equations is usually highly nonlinear in the coordinates of $\boldsymbol{u}$. As a consequence, usually it must be solved numerically.

3.1.1.2.1 Regular constraint Jacobian matrix. A *regular (nonsingular) constraint Jacobian matrix*

$$\boldsymbol{g}_{\boldsymbol{u}}(\boldsymbol{u}_0 , \boldsymbol{w}_0) \in \mathbb{R}^{n_c , n_c} \tag{3.11a}$$

with

$$\det(\boldsymbol{g}_u(\boldsymbol{u}_0 , \boldsymbol{w}_0)) \neq 0 \quad \text{or} \quad \operatorname{rank}(\boldsymbol{g}_u(\boldsymbol{u}_0 , \boldsymbol{w}_0)) = n_c \tag{3.11b}$$

guarantees, due to the *implicit function theorem* ([46], [47]):

1. The *existence* of a *unique local solution*,

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{w}(t)), \tag{3.12a}$$

   of the *constraint position equation (3.9a)* in $(\boldsymbol{u}_0 , \boldsymbol{w}_0)$.

2. The *existence* of a *unique local solution*,

$$\dot{\boldsymbol{u}} = -\left(\boldsymbol{g}_u\right)^{-1} \cdot \left(\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}\right), \tag{3.12b}$$

of the *constraint velocity equation*

$$\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}} + \boldsymbol{g}_w \cdot \dot{\boldsymbol{w}} = \boldsymbol{0}. \tag{3.12c}$$

3. The *existence* of a *unique local solution*,

$$\begin{aligned}\ddot{\boldsymbol{u}} = \left(\boldsymbol{g}_u\right)^{-1} \cdot \Big[&- \left(\boldsymbol{g}_w \cdot \ddot{\boldsymbol{w}}\right) - \left(\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}}\right)_{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \left(\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}\right)_{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} \\ &- \left(\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}\right)_{\boldsymbol{w}} \cdot \dot{\boldsymbol{w}} - \left(\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}}\right)_{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} \Big],\end{aligned} \tag{3.12d}$$

of the *constraint acceleration equation*

$$\begin{aligned}\boldsymbol{g}_u \cdot \ddot{u} = &- \left(\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}}\right)_u \cdot \dot{\boldsymbol{u}} - \left(\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}}\right)_w \cdot \dot{\boldsymbol{w}} - \left(\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}\right)_u \cdot \dot{\boldsymbol{u}} \\ &- \left(\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}\right)_w \cdot \dot{\boldsymbol{w}} - \boldsymbol{g}_w \cdot \ddot{\boldsymbol{w}}\end{aligned}$$

or (3.12e)

$$\boldsymbol{g}_u \cdot \ddot{\boldsymbol{u}} = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{u}\,, \dot{\boldsymbol{u}}\,, \boldsymbol{w}\,, \dot{\boldsymbol{w}}\,, \ddot{\boldsymbol{w}}).$$

This result already indicates that the *constraint Jacobian matrix* $\boldsymbol{g}_u(\boldsymbol{u})$ includes useful information about the theoretical and numerical (local) solvability of the model equations (cf. *Sections 3.1.3 and 4.4*).

3.1.1.2.2 Singular constraint Jacobian matrix. Constraints with a *singular constraint Jacobian matrix* with

$$\det(\boldsymbol{g}_u(\boldsymbol{u}_0\,, \boldsymbol{w}_0)) = 0 \tag{3.13a}$$

describe *singular configurations.* They may provide *pathological kinematic behavior* of the mechanism and may lead to *computational trouble* in kinematic and dynamic analysis. In those situations the constraint position equations no longer satisfy the *implicit function theorem* locally, or even globally. Then the constraint position equations are *not independent* but *redundant*, or even *not compatible* with each other. Consider, for example, the constraint acceleration equation

$$\boldsymbol{g}_u(\boldsymbol{u}_0\,, \boldsymbol{w}_0) \cdot \ddot{\boldsymbol{u}} = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{u}_0\,, \dot{\boldsymbol{u}}_0\,, \boldsymbol{w}_0\,, \dot{\boldsymbol{w}}_0\,, \ddot{\boldsymbol{w}}_0) \tag{3.12e}$$

as a linear equation in $\ddot{\boldsymbol{u}}$. Then a *singular* constraint Jacobian $\boldsymbol{g}_u(\boldsymbol{u}_0\,, \boldsymbol{w}_0)$ may have two consequences:

1. For (3.13a) and

$$\begin{aligned}\operatorname{rank}(\boldsymbol{g}_u(\boldsymbol{u}_0\,, \boldsymbol{w}_0)) \neq \operatorname{rank}\Big(&\, \boldsymbol{g}_u(\boldsymbol{u}_0\,, \boldsymbol{w}_0), \\ &\boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{u}_0\,, \dot{u}_0\,, (\boldsymbol{w}_0\,, \dot{\boldsymbol{w}}_0\,, (\ddot{\boldsymbol{w}}_0)\Big),\end{aligned} \tag{3.13b}$$

(3.12e) does not have a local or global solution $\ddot{\boldsymbol{u}}(t)$ (loss of existence of solutions). This behavior of a mechanism is often called *lock-up*. In such a singular point $(\boldsymbol{u}_0\,,\,\boldsymbol{w}_0)$ the mechanism can either *not be moved* by a chosen driving mechanism, or it *cannot even be assembled* in this form.

2. For (3.13a) and for

$$\operatorname{rank}(\boldsymbol{g}_u(\boldsymbol{u}_0\,,\,\boldsymbol{w}_0)) = \operatorname{rank}\Big(\,\boldsymbol{g}_u(\boldsymbol{u}_0\,,\,\boldsymbol{w}_0), \quad \boldsymbol{\beta}_c(\boldsymbol{u}_0\,,\,\dot{\boldsymbol{u}}_0\,,\,\boldsymbol{w}_0\,,\,\dot{\boldsymbol{w}}_0\,,\,\ddot{\boldsymbol{w}}_0)\Big) < n_c \tag{3.13c}$$

(3.12e) does not have a unique local solution $\ddot{\boldsymbol{u}}(t)$ (loss of uniqueness of solutions). Then (3.12e) may have a *finite number* of different solutions (which may branch or bifurcate, trifurcate etc., from a chosen solution), called *bifurcation solutions*, or it may even have an *infinite number* of different solutions.

A refined analysis of the *singular* behavior of the constraint equations may be obtained from the *Weierstraß preparation theorem* for analytical constraint equations ([46], [47]), from the *Malgrange preparation theorem* for $\mathcal{C}^\infty$ constraint equations ([48]), and from *Fredholm's alternative theorem* for constraint equations of quite general form ([46]). Bifurcation properties are often discussed in the framework of Newton diagram techniques ([49]), ([50]) or by an approach of Ljapunow–Schmidt and others ([51], [52]). Parametric classification schemes of singularities have been derived in an approach of Thom ([53], [54]). To convince the reader that singularities of constraint equations are not only of theoretical interest and do not only appear in pathological mechanisms, an example of the *singular behavior of a simple* and *often discussed mechanism* will be analyzed in *Section 3.1.3* (cf. [40], [41],[55]).

3.1.1.3 Kinematics in planar dynamic analysis. Consider n_c kinematic and active constraint equations in the dynamic analysis of a planar mechanism. Then

$$n_c < 3 \cdot n_b \quad \text{for} \quad \boldsymbol{g} : \mathbb{R}^{3n_b} \longrightarrow \mathbb{R}^{n_c} \quad \text{and} \quad \boldsymbol{g}_p(\boldsymbol{p}_0) \in \mathbb{R}^{n_c,3n_b}. \tag{3.14a}$$

A *regular constraint situation*, necessary for providing a unique solution of the dynamic model equations, is obtained for

$$\operatorname{rank}\big(\boldsymbol{g}_p(\boldsymbol{p}_0)\big) = n_c \quad (\boldsymbol{g}_p(\boldsymbol{p}_0)\text{has maximum rank}). \tag{3.14b}$$

Singular behavior of the mechanism is obtained for

$$\operatorname{rank}\big(\boldsymbol{g}_p(\boldsymbol{p}_0)\big) < n_c. \tag{3.14c}$$

The discussion of the singular behavior in pure kinematic analysis of Section 3.1.1.2 still holds for this situation by assuming $3n_b - n_c$ (independent) coordinates to be dynamically predetermined. In order to achieve unique solutions of the *dynamic model equations* in all $3n_b$ coordinates, (3.14b) must be satisfied (cf. *Chapter 4.4*), and proper initial conditions must be chosen.

3.1.2 Kinematics of *spatial* mechanisms

Spatial mechanisms including n_b rigid bodies are specified by a Cartesian *coordinate vector*

$$\boldsymbol{p} = \left(\boldsymbol{p}_1^{\mathrm{T}}, \cdots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \text{ with } \boldsymbol{p}_i = \left(x^R_{O_iO}, y^R_{O_iO}, z^R_{O_iO}, \varphi_i, \theta_i, \psi_i\right)^{\mathrm{T}}$$

or (3.15a)

$$\boldsymbol{p} = \Big(x^R_{O_1O}, y^R_{O_1O}, z^R_{O_1O}, \varphi_1, \theta_1, \psi_1, \cdots,$$
$$x^R_{O_{n_b}O}, y^R_{O_{n_b}O}, z^R_{O_{n_b}O}, \varphi_{n_b}, \theta_{n_b}, \psi_{n_b}\Big)^{\mathrm{T}} \in \mathbb{R}^{6n_b}.$$

If this mechanism includes constraints modeled by n_c *consistent and independent holonomic constraint equations*

$$\left.\begin{array}{rcl} \boldsymbol{g}^k : \mathbb{R}^{6n_b} & \longrightarrow & \mathbb{R}^{n_c} \\ \Downarrow & & \Downarrow \\ \boldsymbol{p} & \longmapsto & \boldsymbol{g}^k(\boldsymbol{p}) = \boldsymbol{0} \end{array}\right\} n_c \textit{ kinematic constraint equations}, \tag{3.15b}$$

it has $(6n_b - n_c)$ *DOFs*. Then either $(6n_b - n_c)$ additional *active (driving)* conditions

$$\left.\begin{array}{rcccl} \boldsymbol{g}^a : \mathbb{R}^{6n_b} & \times & \mathbb{R}^1 & \longrightarrow & \mathbb{R}^{6n_b - n_c} \\ \Downarrow & & \Downarrow & & \\ (\boldsymbol{p} & , & t) & \longmapsto & \boldsymbol{g}^a(\boldsymbol{p}, t) = \boldsymbol{0} \end{array}\right\} \begin{array}{l} 6n_b - n_c \textit{ active or driving} \\ \textit{constraint equations,} \end{array} \tag{3.15c}$$

are needed (in *kinematic analysis*) or external forces and initial conditions of $(6n_b - n_c)$ coordinates are used (in *dynamic analysis*) to uniquely determine the motion $\boldsymbol{p}(t)$ of the mechanism.

3.1.2.1 Pure kinematic analysis of spatial mechanisms. The kinematic and active constraint equations (3.10), (3.12c) and (3.12e) of pure kinematic analysis of *planar* mechanisms similarly hold for *spatial* mechanisms with modified dimensions of the functions and variables, and with $\dot{\boldsymbol{p}}$ *replaced by* $\boldsymbol{v}$ according to the kinematic differential equation (2.81a):

$$\begin{array}{rcccl} \boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}^k \\ \boldsymbol{g}^a \end{pmatrix} : \mathbb{R}^{6n_b} & \times & R^1 & \longrightarrow & \mathbb{R}^{6n_b} \\ \Downarrow & & \Downarrow & & \Downarrow \\ (\boldsymbol{p} & , & t) & \longmapsto & \boldsymbol{g}(\boldsymbol{p}, t) = \boldsymbol{0}, \end{array}$$

$$\boldsymbol{g}_t = (g_{1t}, \ldots, g_{6n_bt})^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \quad g_{it} = 0 \text{ for } i = 1, \ldots, n_c, \tag{3.15d}$$

$$\boldsymbol{g}_{tt} = (g_{1tt}, \ldots, g_{6n_b tt})^{\mathrm{T}} \in \mathbb{R}^{6n_b} \quad , \quad g_{itt} = 0 \;\; \text{for} \;\; i = 1\,, \ldots\,, n_c,$$

$$\boldsymbol{g}_p = \begin{pmatrix} \partial g_1/\partial x^R_{O_1 O} & , \ldots, & \partial g_1/\partial \psi_{n_b} \\ \vdots & & \\ \partial g_{6n_b}/\partial x^R_{O_1 O} & , \ldots, & \partial g_{6n_b}/\partial \psi_{n_b} \end{pmatrix} \in \mathbb{R}^{6n_b , 6n_b},$$

$$\boldsymbol{g}_{pt} = \frac{\partial}{\partial t}\left(\boldsymbol{g}_p\right) \in \mathbb{R}^{6n_b , 6n_b}$$

and

$$\left(\boldsymbol{g}_p \cdot \dot{\boldsymbol{p}}\right)_p \in \mathbb{R}^{6n_b , 6n_b}.$$

In order to *describe the complete set of model equations of a mechanism in consistent coordinates*, the kinematic differential equation (cf. Equation 2.81a)

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \tag{3.16a}$$

must be included in the *constraint equations* of *spatial* mechanisms with

$$\boldsymbol{p} = \left(\boldsymbol{p}_1^{\mathrm{T}}, \ldots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{p}_i = \left(\boldsymbol{r}^{\mathrm{T}}_{O_i O}, \boldsymbol{\eta}_i^{\mathrm{T}}\right)^{\mathrm{T}}, \tag{3.16b}$$

$$\boldsymbol{r}^R_{O_i O} = \left(x^R_{O_i O}, y^R_{O_i O}, z_{O_i O}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{\eta}_i^R = (\varphi_i\,, \theta_i\,, \psi_i)^{\mathrm{T}},$$

φ_i , θ_i , ψ_i as Bryant angles of the body i, measured from frame R to frame L_i,

$$\dot{\boldsymbol{p}} = \left(\dot{\boldsymbol{p}}_1^{\mathrm{T}}, \ldots, \dot{\boldsymbol{p}}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b} \quad , \quad \dot{\boldsymbol{p}}_i = \left(\dot{\boldsymbol{r}}^{\mathrm{T}}_{Q_i O}, \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^6, \tag{3.16c}$$

$$\dot{\boldsymbol{r}}^R_{Q_i O} = \left(\dot{x}^R_{O_i O}, \dot{y}^R_{O_i O}, \dot{z}_{O_i O}\right)^{\mathrm{T}} \quad , \quad \dot{\boldsymbol{\eta}}_i^R = \left(\dot{\varphi}_i\,, \dot{\theta}_i\,, \dot{\psi}_i\right)^{\mathrm{T}},$$

$$\boldsymbol{v}_i = \left((\dot{\boldsymbol{r}}^R_{Q_i O})^{\mathrm{T}}, (\boldsymbol{\omega}^{L_i}_{L_i R})^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{\omega}^{L_i}_{L_i R} = \left(\omega^{L_i}_{x L_i R}, \omega^{L_i}_{y L_i R}, \omega^{L_i}_{z L_i R}\right)^{\mathrm{T}},$$

with (2.79a)

$$\dot{\boldsymbol{\eta}}_i^R = \boldsymbol{H}_i(\boldsymbol{\eta}_i) \cdot \boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \boldsymbol{\omega}^{L_i}_{L_i R},$$

with (cf. Equation 2.79b)

$$\boldsymbol{H}_i(\boldsymbol{\eta}_i) = \frac{1}{\mathrm{c}\,\theta_i} \cdot \begin{pmatrix} \mathrm{c}\,\theta_i\,, \mathrm{s}\,\phi_i \cdot \mathrm{s}\,\theta_i\,, -\mathrm{c}\,\phi_i \cdot \mathrm{s}\,\theta_i \\ 0 \;\;, \mathrm{c}\,\phi_i \cdot \mathrm{c}\,\theta_i\,, \;\; \mathrm{s}\,\phi_i \cdot \mathrm{c}\,\theta_i \\ 0 \;\;, \;\; -\mathrm{s}\,\phi_i \;\;, \;\;\; \mathrm{c}\,\phi_i \end{pmatrix}, \tag{3.16d}$$

with (2.56)

$$\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) :=$$
$$\begin{pmatrix} \mathrm{c}\,\theta_i \cdot \mathrm{c}\,\psi_i & , & -\mathrm{c}\,\theta_i \cdot \mathrm{s}\,\psi_i & , & s\theta_i \\ \mathrm{c}\,\phi_i \cdot \mathrm{s}\,\psi_i + \mathrm{s}\,\phi_i \cdot \mathrm{s}\,\theta_i \cdot \mathrm{c}\,\psi_i & , & \mathrm{c}\,\phi_i \cdot \mathrm{c}\,\psi_i - \mathrm{s}\,\phi_i \cdot \mathrm{s}\,\theta_i \cdot \mathrm{s}\,\psi_i & , & -\mathrm{s}\,\phi_i \cdot \mathrm{c}\,\theta_i \\ \mathrm{s}\,\phi_i \cdot \mathrm{s}\,\psi_i - \mathrm{c}\,\phi_i \cdot \mathrm{s}\,\theta_i \cdot \mathrm{c}\,\psi_i & , & \mathrm{s}\,\phi_i \cdot \mathrm{c}\,\psi_i + \mathrm{c}\,\phi_i \cdot \mathrm{s}\,\theta_i \cdot \mathrm{s}\,\psi_i & , & \mathrm{c}\,\phi_i \cdot \mathrm{c}\,\theta_i \end{pmatrix} \tag{3.16e}$$

with (2.81a)

$$\dot{\boldsymbol{p}}_i = \boldsymbol{T}_i(\boldsymbol{\eta}_i) \cdot \boldsymbol{v}_i, \tag{3.16f}$$

$$\boldsymbol{v} := \left(\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b} \;,\; \boldsymbol{\omega}_{LR}^{L} := \left(\boldsymbol{\omega}_{L_1R}^{L}, \ldots, \boldsymbol{\omega}_{L_{n_b}R}^{L}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b},$$

$$\boldsymbol{T}(\boldsymbol{p}) := \operatorname{diag}\left(\boldsymbol{T}_1(\boldsymbol{\eta}_1), \mathrm{T}_2(\boldsymbol{\eta}_2), \ldots, \boldsymbol{T}_{n_b}(\boldsymbol{\eta}_{n_b})\right),$$

and

$$\boldsymbol{T}_i(\boldsymbol{p}) = \boldsymbol{T}_i(\boldsymbol{\eta}_i) = \begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{H}_i(\boldsymbol{\eta}_i) \cdot \boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \end{pmatrix}. \tag{3.16g}$$

This yields the *constraint position equations*

$$\boldsymbol{g}(\boldsymbol{p}, t) = \mathbf{0}, \tag{3.17a}$$

the *constraint velocity equations*

$$\boldsymbol{g}_p(\boldsymbol{p}, t) \cdot \dot{\boldsymbol{p}} = \boldsymbol{g}_p(\boldsymbol{p}, t) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} = -\boldsymbol{g}_t(\boldsymbol{p}, t) =: \boldsymbol{\alpha}_{\mathrm{c}}(\boldsymbol{p}, t) \tag{3.17b}$$

and the *constraint acceleration equations*

$$\boldsymbol{g}_p(\boldsymbol{p}, t) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = -\left[\boldsymbol{g}_p(\boldsymbol{p}, t) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v}(\boldsymbol{p})\right]_p \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \tag{3.17c}$$
$$- 2 \cdot \boldsymbol{g}_{pt}(\boldsymbol{p}, t) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} - \boldsymbol{g}_{tt}(\boldsymbol{p}, t) =: \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}, t) \tag{3.17d}$$

with $\boldsymbol{T}_t(\boldsymbol{p}) \equiv \mathbf{0}$.

3.1.2.2 Kinematics in spatial dynamic analysis. In the dynamic analysis of mechanisms including n_c constraints, it is assumed that

$$\boldsymbol{g} : \mathbb{R}^{6n_b} \longrightarrow \mathbb{R}^{n_c} \quad , \quad n_c < 6n_b, \tag{3.18a}$$

and

$$\boldsymbol{g}_p(\boldsymbol{p}_0, t_0) \cdot \boldsymbol{T}(\boldsymbol{p}_0) \in \mathbb{R}^{n_c, 6n_b}. \tag{3.18b}$$

Then the mechanism is called *regular* if

$$\operatorname{rank}\Big(\boldsymbol{g}_p(\boldsymbol{p}_0\,,\,t_0)\cdot\boldsymbol{T}(\boldsymbol{p}_0)\Big)=n_c. \tag{3.18c}$$

Based on the previous discussion, kinematic constraint equations of planar and spatial joints will be derived in *Sections 5.1 and 5.2*, and in various examples in *Volume II.*

3.1.3 Singularity analysis of a planar slider–crank mechanism

The slider–crank mechanism of Figure 3.2 is important from both theoretical and practical point of view. It is included in a great variety of mechanisms from industrial practice and serves in various publications as a simple example for discussing the pathological or singular behavior of mechanisms ([40], [41], [55]). In this section the *singular behavior* of a planar two body *slider–crank* mechanism of Figure 3.2 will be systematically analyzed by means of matrix calculus, in order to study and illustrate pitfalls that may arise in kinematic and dynamic analysis even of simple but often-used mechanisms. This will be done for *two different driving functions* of the mechanism (Figure 3.2):

Case 1: $$g_1^a(\boldsymbol{p}\,,\,t)=\psi_i-a_1(t)=0. \tag{3.19a}$$

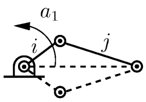
Case 2: $$g_2^a(\boldsymbol{p}\,,\,t)=x_{PO}^R+a_2(t)=x_{P_jO}^R+(l_j/2)\cdot\cos\psi_j+a_2(t)=0. \tag{3.19b}$$

In agreement with *Section 3.1*, the planar mechanism including the two bodies i (crank) and j (slider) is characterized by the generalized coordinate vector (Figure 3.2)

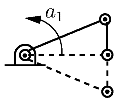
$$\boldsymbol{p}=\Big(x_{P_iO}^R\,,\,y_{P_iO}^R\,,\,\psi_i\,,\,x_{P_jO}^R\,,\,y_{P_jO}^R\,,\,\psi_j\Big)\quad,\quad\psi_\kappa:=\psi_{L_\kappa R}\quad;\quad\kappa=i\,,\,j. \tag{3.20}$$

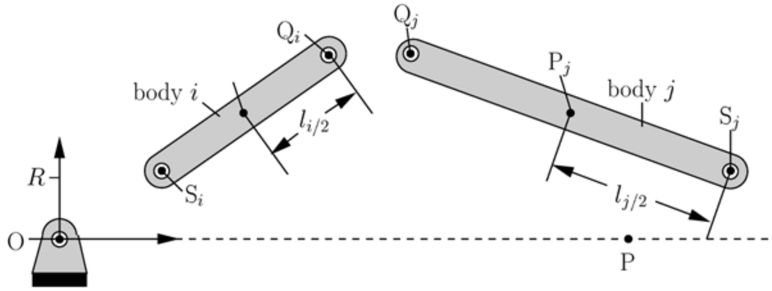
3.1.3.1 Identification of singularities by direct inspection. Most *singularities* of the mechanism are seen by *direct inspection* of Figure 3.2. The mechanism driven by $a_1(t)$ (Figure 3.2b) has

1. For $l_i<l_j$, *no lock-up* and *no bifurcation point.*

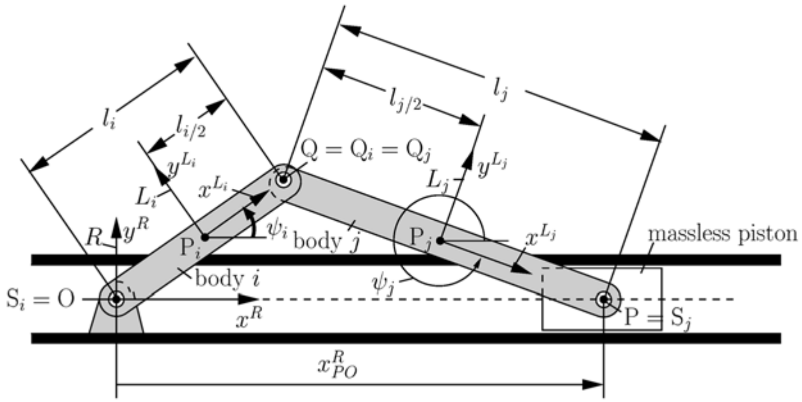


2. For $l_i>l_j$, *lock-ups* for $\psi_j\pm\frac{3}{2}\pi$.

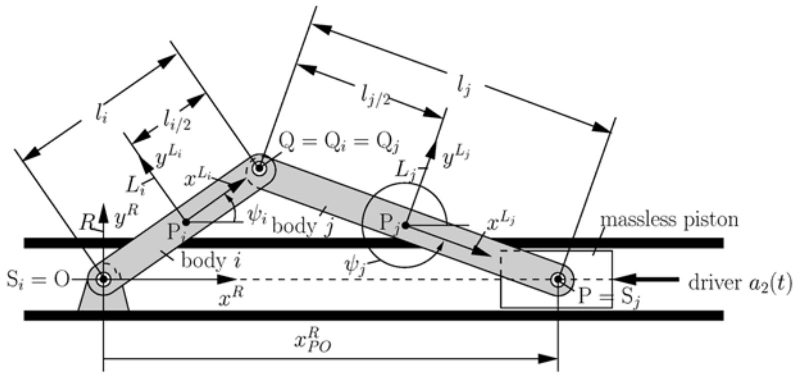




(a) Disassembled form



(b) Case 1: Mechanism with driving condition $\psi_i = a_1(t)$



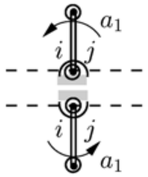
(c) Case 2: Mechanism with driving condition $x^R_{PO} = -a_2(t)$

Fig. 3.2: Slider–crank mechanism driven by two different active constraints

3. For $l_i = l_j$, *bifurcation points*

   for ($\psi_i = \pi/2$ and $\psi_j = \frac{3}{2}\pi$) and

   for ($\psi_i = \frac{3}{2}\pi$ and $\psi_j = \frac{5}{2}\pi$).

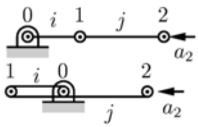


The mechanism driven by $a_2(t)$ (Figure 3.2c) has

1. For $l_i < l_j$, *lock-ups*

   for ($\psi_i = 0$ and $\psi_j = 2\pi$) and

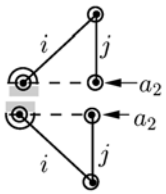
   for ($\psi_i = \pi$ and $\psi_j = 2\pi$).



2. For $l_i > l_j$, *lock-ups*

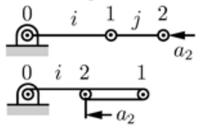
   for ($\psi_j = \frac{3}{2}\pi$),

   for ($\psi_j = \frac{5}{2}\pi$),



   for ($\psi_i = 0$ and $\psi_j = 2\pi$), and

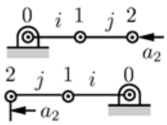
   for ($\psi_i = 0$ and $\psi_j = \pi$).



3. For $l_i = l_j$, *lock-ups*

   for ($\psi_i = 0$ and $\psi_j = 2\pi$) and

   for ($\psi_i = \pi$ and $\psi_j = \pi$).



A more refined singularity analysis of the mechanism can be easily obtained by a *geometrical analysis* using the minimal coordinates of the mechanism ([40]). These two approaches can hardly be applied to complex spatial mechanisms and cannot automatically be performed by computers. The above objective may be reached by a *formal singularity analysis* of the mechanism based on *matrix algebra*.

3.1.3.2 Local algebraic singularity analysis of the slider–crank mechanism. In a *first step* the kinematic constraint equations of the mechanism will be derived. In a *second step* the complete set of constraint equations of the mechanism will be analyzed for the two different driver functions selected (Cases 1 and 2).

Kinematic constraint equations. The geometry of the slider–crank mechanism of Figure 3.2 implies the following kinematic constraint equations:

1. The point S_i on body i coincides with the point O of frame R (Figure 3.2b). This implies the common-point constraint equation (vector loop equation)

$$\boldsymbol{r}_{P_iO}^{R} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}_{P_iO}^{L_i} = \boldsymbol{0},$$

and, together with the geometry relations

$$\boldsymbol{A}^{RL_i} = \begin{pmatrix} \cos\psi_i \ , & -\sin\psi_i \\ \sin\psi_i \ , & \cos\psi_i \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_{P_iO}^{L_i} = \begin{pmatrix} l_i/2 \\ 0 \end{pmatrix},$$

the constraint equations

$$x_{P_iO}^{R} - \frac{l_i}{2} \cdot \cos\psi_i = 0 \tag{3.21a}$$

and

$$y_{P_iO}^{R} - \frac{l_i}{2} \cdot \sin\psi_i = 0. \tag{3.21b}$$

2. The point S_j on body j lies on the x^R-axis (P $=$ S_j). This implies, together with the vector loop equation

$$-\boldsymbol{r}_{P_jO}^{R} + \boldsymbol{r}_{PO}^{R} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}_{P_jP}^{L_j} = \boldsymbol{0}$$

and the geometry relations

$$\boldsymbol{r}_{PO}^{R} = \begin{pmatrix} x_{PO}^{R} \\ 0 \end{pmatrix} , \ \boldsymbol{r}_{P_jP}^{R} = \begin{pmatrix} -l_j/2 \\ 0 \end{pmatrix} , \ \boldsymbol{A}^{RL_j} = \begin{pmatrix} \cos\psi_j \ , & -\sin\psi_j \\ \sin\psi_j \ , & \cos\psi_j \end{pmatrix},$$

the relations

$$\begin{pmatrix} x_{P_jO}^{R} \\ y_{P_jO}^{R} \end{pmatrix} = \begin{pmatrix} x_{PO}^{R} \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\psi_j \ , & -\sin\psi_j \\ \sin\psi_j \ , & \cos\psi_j \end{pmatrix} \cdot \begin{pmatrix} -l_j/2 \\ 0 \end{pmatrix}$$

or

$$x_{P_jO}^{R} - x_{PO}^{R} + (l_j/2) \cdot \cos\psi_j = 0 \tag{3.22a}$$

$$y_{P_jO}^{R} + 0 + (l_j/2) \cdot \sin\psi_j \qquad = 0. \tag{3.22b}$$

3. The point Q_i on the crank coincides with the point Q_j on the slider ($\mathrm{Q} = \mathrm{Q}_i = \mathrm{Q}_j$). Together with the vector loop equation

$$\boldsymbol{r}_{P_iO}^{R} + \boldsymbol{r}_{QP_i}^{R} + \boldsymbol{r}_{P_jQ}^{R} - \boldsymbol{r}_{P_jO}^{R} = \boldsymbol{0}$$

and the geometry relations

$$\boldsymbol{r}_{QP_i}^{R} = \boldsymbol{A}^{RL_i} \cdot \begin{pmatrix} l_i/2 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{r}_{P_jQ}^{R} = \boldsymbol{A}^{RL_j} \cdot \begin{pmatrix} l_j/2 \\ 0 \end{pmatrix},$$

this implies the relations

$$\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}x^R_{P_iO}\\y^R_{P_iO}\end{pmatrix}+\begin{pmatrix}\cos\psi_i\;,&-\sin\psi_i\\\sin\psi_i\;,&\cos\psi_i\end{pmatrix}\cdot\begin{pmatrix}l_i/2\\0\end{pmatrix}$$
$$+\begin{pmatrix}\cos\psi_j\;,&-\sin\psi_j\\\sin\psi_j\;,&\cos\psi_j\end{pmatrix}\cdot\begin{pmatrix}l_j/2\\0\end{pmatrix}-\begin{pmatrix}x^R_{P_jO}\\y^R_{P_jO}\end{pmatrix}$$

or

$$x^R_{P_iO}+(l_i/2)\cdot\cos\psi_i+(l_j/2)\cdot\cos\psi_j-x^R_{P_jO}=0 \tag{3.23a}$$
$$y^R_{P_iO}+(l_i/2)\cdot\sin\psi_i+(l_j/2)\cdot\sin\psi_j-y^R_{P_jO}=0. \tag{3.23b}$$

Kinematic constraint position equations. The assembled mechanism has a single DOF. Collecting (3.21), (3.22), and (3.23) provides the following five *kinematic constraint position equations*:

$$(0\,,0\,,0\,,0\,,0)^{\mathrm{T}}=\boldsymbol{g}^k(\boldsymbol{p})$$
$$=\begin{pmatrix}
x^R_{P_iO} & + & 0 & - & (l_i/2)\cdot\cos\psi_i & + & 0 & + & 0 & + & 0\\
0 & + & y^R_{P_iO} & - & (l_i/2)\cdot\sin\psi_i & + & 0 & + & 0 & + & 0\\
x^R_{P_iO} & + & 0 & + & (l_i/2)\cdot\cos\psi_i & - & x^R_{P_jO} & + & 0 & + & (l_j/2)\cdot\cos\psi_j\\
0 & + & y^R_{P_iO} & + & (l_i/2)\cdot\sin\psi_i & + & 0 & - & y^R_{P_jO} & + & (l_j/2)\cdot\sin\psi_j\\
0 & + & 0 & + & 0 & + & 0 & + & y^R_{P_jO} & + & (l_j/2)\cdot\sin\psi_j
\end{pmatrix}
\begin{matrix}(3.21a)\\(3.21b)\\(3.23a)\\(3.23b)\\(3.22b)\end{matrix}$$

and the following relation for the *relative coordinate* x^R_{PO}:

$$x^R_{PO}-x^R_{P_jO}-(l_j/2)\cdot\cos\psi_j=0. \tag{3.22a}$$

The complete constraint equations including the *kinematic* and *active constraints* of Cases 1 and 2 (3.19) are

$$\boldsymbol{g}_\kappa(\boldsymbol{p}\,,t)=\begin{pmatrix}\boldsymbol{g}^k(\boldsymbol{p})\\\boldsymbol{g}^a_\kappa(\boldsymbol{p}\,,t)\end{pmatrix}=\begin{pmatrix}\boldsymbol{0}\\\boldsymbol{0}\end{pmatrix}\quad,\quad\kappa=1\,,2\;. \tag{3.24}$$

Local algebraic analysis of the constraint equations.

Subsequently the constraint position, *velocity*, and *acceleration equations* of the slider–crank mechanism will be analyzed algebraically in two steps: for the *driving function* $a_1(t)$ *(Case 1)* and for the *driving function* $a_2(t)$ *(Case 2)*.

3.1.3.2.1 Local analysis of Case 1 ($\psi_i(t)=a_1(t)$). Written in components, the complete *constraint position equations* are

$$(0\,,0\,,0\,,0\,,0\,,0)^{\mathrm{T}} = \boldsymbol{g}_1(\boldsymbol{p}\,,t) \tag{3.25a}$$

$$= \begin{pmatrix} x^R_{P_iO} + 0 - (l_i/2)\cdot\cos\psi_i + 0 + 0 + 0 + 0 \\ 0 + y^R_{P_iO} - (l_i/2)\cdot\sin\psi_i + 0 + 0 + 0 + 0 \\ x^R_{P_iO} + 0 + (l_i/2)\cdot\cos\psi_i - x^R_{P_jO} + 0 + (l_j/2)\cdot\cos\psi_j + 0 \\ 0 + y^R_{P_iO} + (l_i/2)\cdot\sin\psi_i + 0 - y^R_{P_jO} + (l_j/2)\cdot\sin\psi_j + 0 \\ 0 + 0 + 0 + 0 + y^R_{P_jO} + (l_j/2)\cdot\sin\psi_j + 0 \\ 0 + 0 + \psi_i + 0 + 0 + 0 - a_1(t) \end{pmatrix}$$

with the *constraint Jacobian matrix*

$$\boldsymbol{g}_{1p}(\boldsymbol{p}) = \begin{pmatrix} 1\,, & 0\,, & (l_i/2)\cdot\sin\psi_i\,, & 0\,, & 0\,, & 0 \\ 0\,, & 1\,, & -(l_i/2)\cdot\cos\psi_i\,, & 0\,, & 0\,, & 0 \\ 1\,, & 0\,, & -(l_i/2)\cdot\sin\psi_i\,, & -1\,, & 0\,, & -(l_j/2)\cdot\sin\psi_j \\ 0\,, & 1\,, & (l_i/2)\cdot\cos\psi_i\,, & 0\,, & -1\,, & (l_j/2)\cdot\cos\psi_j \\ 0\,, & 0\,, & 0\,, & 0\,, & 1\,, & (l_j/2)\cdot\cos\psi_j \\ 0\,, & 0\,, & 1\,, & 0\,+ & 0\,+ & 0 \end{pmatrix}, \tag{3.25b}$$

the *constraint velocity equations*

$$\boldsymbol{g}_{1p}(\boldsymbol{p}\,,t)\cdot\dot{\boldsymbol{p}} = -\boldsymbol{g}_{1t}(\boldsymbol{p}\,,t) \quad \text{with} \quad \boldsymbol{g}_{1t}(\boldsymbol{p}\,,t) = -(0\,,0\,,0\,,0\,,0\,,\dot{a}_1(t))^{\mathrm{T}} \tag{3.25c}$$

and the *constraint acceleration equations*

$$\begin{aligned} \boldsymbol{g}_{1p}(\boldsymbol{p}\,,t)\cdot\ddot{\boldsymbol{p}} &= -\left(\boldsymbol{g}_{1p}(\boldsymbol{p}\,,t)\cdot\dot{\boldsymbol{p}}\right)_p\cdot\dot{\boldsymbol{p}} - 2\cdot\boldsymbol{g}_{1pt}(\boldsymbol{p}\,,t)\cdot\dot{\boldsymbol{p}} - \boldsymbol{g}_{1tt}(\boldsymbol{p}\,,t) \\ &=: \boldsymbol{\beta}_{\mathrm{c}_1}(\boldsymbol{p}\,,\dot{\boldsymbol{p}}\,,t) \end{aligned} \tag{3.25d}$$

with

$$-\boldsymbol{g}_{1tt} := +(0\,,0\,,0\,,0\,,0\,,\ddot{a}_1(t))^{\mathrm{T}} \quad , \quad \boldsymbol{g}_{1pt}(\boldsymbol{p}\,,t) = \boldsymbol{0}\in\mathbb{R}^6\,,$$

$$\boldsymbol{g}_{1p}(\boldsymbol{p}\,,t)\cdot\dot{\boldsymbol{p}} = \tag{3.25e}$$

$$\begin{pmatrix} \dot{x}^R_{P_iO} + \dot{\psi}_i\cdot(l_i/2)\cdot\sin\psi_i \\ \dot{y}^R_{P_iO} - \dot{\psi}_i\cdot(l_i/2)\cdot\cos\psi_i \\ \dot{x}^R_{P_iO} - \dot{\psi}_i\cdot(l_i/2)\cdot\sin\psi_i - \dot{x}^R_{P_jO} - \dot{\psi}_j\cdot(l_j/2)\cdot\sin\psi_j \\ \dot{y}^R_{P_iO} + \dot{\psi}_i\cdot(l_i/2)\cdot\cos\psi_i - \dot{y}^R_{P_jO} + \dot{\psi}_j\cdot(l_j/2)\cdot\cos\psi_j \\ \dot{y}^R_{P_jO} + \dot{\psi}_j\cdot(l_j/2)\cdot\cos\psi_j \\ \dot{\psi}_i \end{pmatrix},$$

$$\left(\boldsymbol{g}_{1p}(\boldsymbol{p}\,,\,t)\cdot\dot{\boldsymbol{p}}\right)_p\cdot\dot{\boldsymbol{p}} = \tag{3.25f}$$

$$\begin{pmatrix} 0\,,\,0\,, & (l_i/2)\cdot\dot\psi_i\cdot\cos\psi_i & ,\,0\,,\,0\,, & 0 \\ 0\,,\,0\,, & (l_i/2)\cdot\dot\psi_i\cdot\sin\psi_i & ,\,0\,,\,0\,, & 0 \\ 0\,,\,0\,, & -(l_i/2)\cdot\dot\psi_i\cdot\cos\psi_i & ,\,0\,,\,0\,, & -(l_j/2)\cdot\dot\psi_j\cdot\cos\psi_j \\ 0\,,\,0\,, & -(l_i/2)\cdot\dot\psi_i\cdot\sin\psi_i & ,\,0\,,\,0\,, & -(l_j/2)\cdot\dot\psi_j\cdot\sin\psi_j \\ 0\,,\,0\,, & 0 & ,\,0\,,\,0\,, & -(l_j/2)\cdot\dot\psi_j\cdot\sin\psi_j \\ 0\,,\,0\,, & 0 & ,\,0\,,\,0\,, & 0 \end{pmatrix} \cdot \begin{pmatrix} \dot x^R_{P_iO} \\ \dot y^R_{P_iO} \\ \dot\psi_i \\ \dot x^R_{P_jO} \\ \dot y^R_{P_jO} \\ \dot\psi_j \end{pmatrix},$$

and

$$\boldsymbol{\beta}_{c_1}(\boldsymbol{p}\,,\,\dot{\boldsymbol{p}}\,,\,t) = \begin{pmatrix} -\ (l_i/2)\cdot\dot\psi_i^2\cdot\cos\psi_i \\ -\ (l_i/2)\cdot\dot\psi_i^2\cdot\sin\psi_i \\ +\ (l_i/2)\cdot\dot\psi_i^2\cdot\cos\psi_i\ +\ (l_j/2)\cdot\dot\psi_j^2\cdot\cos\psi_j \\ +\ (l_i/2)\cdot\dot\psi_i^2\cdot\sin\psi_i\ +\ (l_j/2)\cdot\dot\psi_j^2\cdot\sin\psi_i \\ +\ (l_j/2)\cdot\dot\psi_j^2\cdot\sin\psi_i \\ +\ \ddot a_1(t) \end{pmatrix}. \tag{3.25g}$$

Local singularity analysis of the constraint Jacobian matrix. From (3.25b) it follows directly that

$$\det(\boldsymbol{g}_{1p}(\boldsymbol{p})) = -l_j\cdot\cos\psi_j. \tag{3.26a}$$

For $l_j > 0$ *singular situations* (Figure 3.3) occur for

$$\det(\boldsymbol{g}_{1p}(\boldsymbol{p})) = 0 \;\longleftrightarrow\; \cos\psi_j = 0 \;\longleftrightarrow\; \psi_j = (2n+1)\cdot\pi/2 =: \psi_j^*, \tag{3.26b}$$

$$n \in \boldsymbol{Z}\cup\{0\}.$$

Then

$$\operatorname{rank}(\boldsymbol{g}_{1p}(\boldsymbol{p}))\ \Big|_{\psi_j^* = (2n+1)\cdot\pi/2} = 5\ . \tag{3.26c}$$

Starting from the *initial condition*

$$\psi_j(0) = 2\pi \quad\text{and}\quad \psi_i(0) = a_1(0) = 0 \quad\text{together with } x^R_{PO}(0) = l_i + l_j,$$

the following situations occur (compare the different cases of Figure 3.3):

$$\begin{array}{llll}
\textit{Case 1.1:} & l_i < l_j & : & \frac{3}{2}\pi < \psi_j < \frac{5}{2}\pi\ . \\
\textit{Case 1.2:} & l_i > l_j & : & \psi_j = \frac{3}{2}\pi\ ,\quad \psi_i = \arcsin\frac{l_j}{l_i}, \qquad \textit{(Case 1.2a)} \\
 & & \text{and} & \\
 & & & \psi_j = \frac{5}{2}\pi\ ,\quad \psi_i = -\arcsin\frac{l_j}{l_i}\ . \qquad \textit{(Case 1.2b)} \\
\textit{Case 1.3:} & l_i = l_j & : & \psi_j = \frac{3}{2}\pi\ ,\quad \psi_i = \pi/2 \qquad \textit{(Case 1.3a)} \\
 & & \text{and} & \\
 & & & \psi_j = \frac{5}{2}\pi\ ,\quad \psi_i = \frac{3}{2}\pi\ . \qquad \textit{(Case 1.3b)}
\end{array}$$

The question that remains to be answered is which of the above identified singular situations are associated with *lock-ups*, and which with *bifurcation points* of motions of the slider–crank mechanism. Answers to this question will be obtained by the subsequent analysis of the *constraint velocity* and *constraint acceleration equations.*

Analysis of the constraint velocity equations. The *constraint velocity equation* (3.25c) is a linear equation in $\dot{\boldsymbol{p}}$:

$$\begin{pmatrix}
1\ , & 0\ , & (l_i/2)\cdot\sin\psi_i\ , & 0\ , & 0\ , & 0 \\
0\ , & 1\ , & -(l_i/2)\cdot\cos\psi_i\ , & 0\ , & 0\ , & 0 \\
1\ , & 0\ , & -(l_i/2)\cdot\sin\psi_i\ , & -1\ , & 0\ , & -(l_j/2)\cdot\sin\psi_j \\
0\ , & 1\ , & (l_i/2)\cdot\cos\psi_i\ , & 0\ , & -1\ , & (l_j/2)\cdot\cos\psi_j \\
0\ , & 0\ , & 0\ , & 0\ , & 1\ , & (l_j/2)\cdot\cos\psi_j \\
0\ , & 0\ , & 1\ , & 0\ + & 0\ + & 0
\end{pmatrix}\cdot\begin{pmatrix}\dot{x}^R_{P_iO}\\ \dot{y}^R_{P_iO}\\ \dot{\psi}_i\\ \dot{x}^R_{P_jO}\\ \dot{y}^R_{P_jO}\\ \dot{\psi}_j\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\ \dot{a}_1(t)\end{pmatrix}. \tag{3.26d}$$

In the *regular case* $(\det(\boldsymbol{g}_{1p}(\boldsymbol{p})) \neq 0$ or for $\cos\psi_j \neq 0)$, (3.26d) has the *unique finite solution*

$$\dot{\boldsymbol{p}} = \begin{pmatrix}\dot{x}^R_{P_iO}\\ \dot{y}^R_{P_iO}\\ \dot{\psi}_i\\ \dot{x}^R_{P_jO}\\ \dot{y}^R_{P_jO}\\ \dot{\psi}_j\end{pmatrix} = \begin{pmatrix}
\dfrac{\dot{a}_1(t)\cdot l_i\cdot\sin\psi_i}{2}\\
-\dfrac{\dot{a}_1(t)\cdot l_i\cdot\cos\psi_i}{2}\\
-\dot{a}_1(t)\\
-\dfrac{\dot{a}_1(t)\cdot l_i\cdot(\cos\psi_i\cdot\sin\psi_j - 2\cdot\sin\psi_i\cdot\cos\psi_j)}{2\cdot\cos\psi_j}\\
-\dfrac{\dot{a}_1(t)\cdot l_i\cdot\cos\psi_i}{2}\\
\dfrac{\dot{a}_1(t)\cdot l_i\cdot\cos\psi_i}{l_j\cdot\cos\psi_j}
\end{pmatrix}. \tag{3.26e}$$

This *solution tends to infinity* for $\cos\psi_j = 0$ *(singular case)* unless additional conditions are satisfied. These *singular situations* will be discussed below.

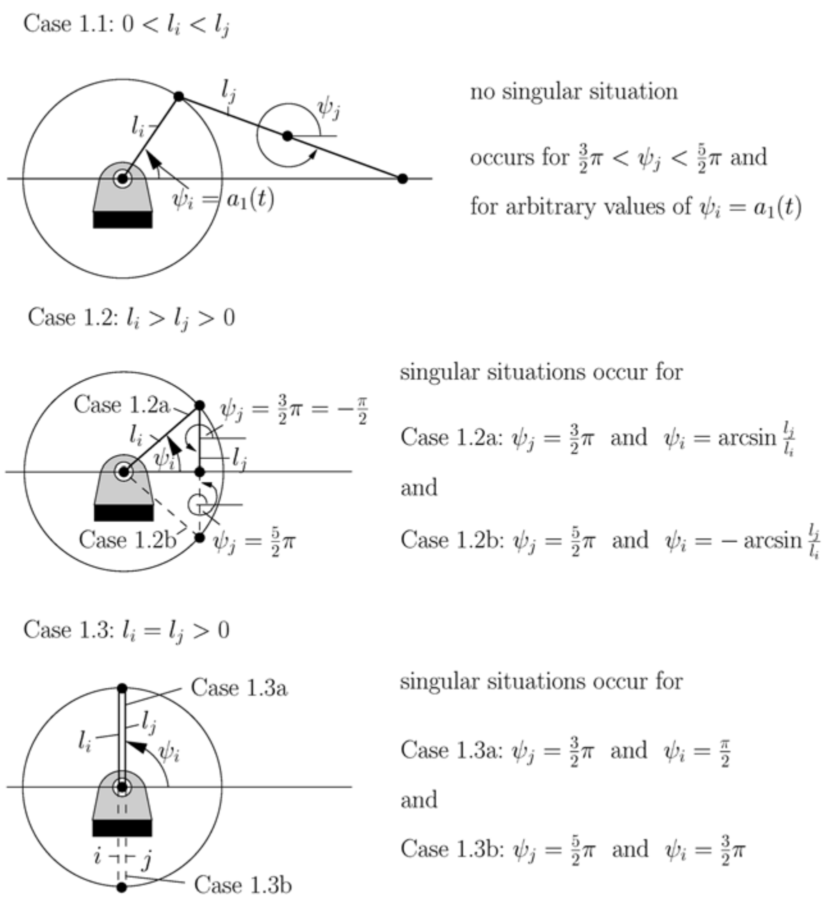


Fig. 3.3: Singular situations of Case 1 ($\psi_i(t) = a_1(t)$ for $\psi_j = \frac{3}{2}\pi$ or $\psi_j = \frac{5}{2}\pi$)

In the *singular case* ($\det(\boldsymbol{g}_{1p}(0)) = 0$ or $\cos\psi_j = 0$), (3.26d) does *not have* a finite solution $\dot{\boldsymbol{p}}(t)$ for

$$\operatorname{rank}\Big((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\Big|_{\psi_j^* = (2n+1)\cdot\pi/2}\Big) > \operatorname{rank}\Big((\boldsymbol{g}_{1p})\Big|_{\psi_j^* = (2n+1)\cdot\pi/2}\Big) = 5. \tag{3.26f}$$

On the other hand, the relation

$$\operatorname{rank}\Big((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\Big) = \operatorname{rank}\Big((\boldsymbol{g}_{1p})\Big)\Big|_{\psi_j^*} = 5 \tag{3.26g}$$

with the augmented matrix

$$\left(\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t}\right)\Big|_{\psi_j^*} = \tag{3.26h}$$

$$\left.\begin{pmatrix} 1, & 0, & (l_i/2)\cdot\sin\psi_i, & 0, & 0, & 0, & 0 \\ 0, & 1, & -(l_i/2)\cdot\cos\psi_i, & 0, & 0, & 0, & 0 \\ 1, & 0, & -(l_i/2)\cdot\sin\psi_i, & -1, & 0, & -(l_j/2)\cdot\sin\psi_j, & 0 \\ 0, & 1, & (l_i/2)\cdot\cos\psi_i, & 0, & -1, & (l_j/2)\cdot\cos\psi_j, & 0 \\ 0, & 0, & 0, & 0, & 1, & (l_j/2)\cdot\cos\psi_j, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & \dot{a}_1(t) \end{pmatrix}\right|_{\psi_j=\psi_j^*} =$$

$$\left.\begin{pmatrix} 1, & 0, & (l_i/2)\cdot\sin\psi_i, & 0, & 0, & 0, & 0 \\ 0, & 1, & -(l_i/2)\cdot\cos\psi_i, & 0, & 0, & 0, & 0 \\ 1, & 0, & -(l_i/2)\cdot\sin\psi_i, & -1, & 0, & -(l_j/2)\cdot\sin\psi_j, & 0 \\ 0, & 1, & (l_i/2)\cdot\cos\psi_i, & 0, & -1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0, & \dot{a}_1(t) \end{pmatrix}\right|_{\psi_j=\psi_j^*},$$

with

$$\operatorname{rank}\Big((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\Big) = \operatorname{rank}\Big((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\cdot(\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})^{\mathrm{T}}\Big), \tag{3.26i}$$

and with

$$\det\Big((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\cdot(\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})^{\mathrm{T}}\Big)\Big|_{\psi_j^*} = 0,$$

provides the relation

$$\det\left((\boldsymbol{g}_{1p}\,,\,-\boldsymbol{g}_{1t})\cdot\begin{pmatrix}\boldsymbol{g}_{1p}^{\mathrm{T}}\\ -\boldsymbol{g}_{1t}^{\mathrm{T}}\end{pmatrix}\right)\Bigg|_{\psi_j^*} = \tag{3.26j}$$

$$\begin{aligned}
&-\Big[\Big(4\cdot\dot{a}_1^2(t)\cdot l_i^2\cdot l_j^2\cdot\cos\psi_i\cdot\sin\psi_i\cdot\cos\psi_j\cdot\sin\psi_j \\
&+\Big(4\cdot\dot{a}_1^2(t)\cdot l_i^2\cdot l_j^2\cdot\cos\psi_i+\Big(-5\cdot\dot{a}_1^2(t)\cdot l_i^2-4\cdot\dot{a}_1^2(t)-4\Big)\cdot l_j^2\Big)\cdot\cos^2\psi_j \\
&+\Big(-4\cdot\dot{a}_1^2(t)\cdot l_i^2\cdot l_j^2-4\cdot\dot{a}_1^2(t)\cdot l_i^2\Big)\cdot\cos^2\psi_i\Big)\Big/4\Big]\Bigg|_{\psi_j^*} \\
&=\frac{1}{4}\cdot\dot{a}_1^2\cdot l_i^2\cdot(l_j^2+4)\cdot\cos^2\psi_i = 0.
\end{aligned}$$

Then (3.26d) has *(at least) two finite solutions* $\dot{\boldsymbol{p}}$ for

$$\frac{1}{4} \cdot l_i^2 \cdot (l_j^2 + 4) \cdot \cos^2 \psi_i \cdot \dot{a}_1^2 = 0 \quad , \quad l_i > 0 \quad , \quad l_j > 0 \ \text{ and for } \ \cos \psi_j^* = 0 \ . \tag{3.26k}$$

This equation is either satisfied for

$$\dot{a}_1 \neq 0 \qquad \text{and} \qquad \cos \psi_i = 0 \quad (\text{i.e., for } \psi_i = +\pi/2 \text{ or } \psi_i = -\pi/2) \tag{3.26l}$$

or for

$$\dot{a}_1(t) \equiv 0 \quad \text{and} \quad \cos \psi_i \neq 0 \tag{3.26m}$$

or for

$$\dot{a}_1(t) \equiv 0 \quad \text{together with} \quad \cos \psi_i = 0. \tag{3.26n}$$

Comment 3.1.1 (Constraint velocity analysis): As will be shown in the subsequent *constraint acceleration analysis*, the relation (3.26l) is associated with *Cases 1.3* and with the *bifurcation points*, whereas (3.26m) is associated with *Cases 1.2* and with *lock-up points*, and sometimes also with some *bifurcations* (Figure 3.4).

Analysis of the constraint acceleration equations. In the *regular case* (for $\cos \psi_j^* \neq 0$), the constraint acceleration equation

$$\boldsymbol{g}_{1p} \cdot \ddot{\boldsymbol{p}} = \boldsymbol{\beta}_{c_1} \quad \text{with} \quad \ddot{\boldsymbol{p}} = \Big(\ddot{x}^R_{P_iO} \,, \ddot{y}^R_{P_iO} \,, \ddot{\psi}_i \,, \ddot{x}^R_{P_jO} \,, \ddot{y}^R_{P_jO} \,, \ddot{\psi}_j \Big)^{\mathrm{T}} \tag{3.27a}$$

has the *unique finite solution*

$$\ddot{\boldsymbol{p}} = \tag{3.27b}$$

$$\begin{pmatrix} -\frac{l_i}{2} \cdot \Big(\ddot{a}_1(t) \cdot \sin \psi_i + \dot{\psi}_i^2 \cdot \cos \psi_i \Big) \\ -\frac{l_i}{2} \cdot \Big(\dot{\psi}_i^2 \cdot \sin \psi_i - \ddot{a}_1(t) \cdot \cos \psi_i \Big) \\ \ddot{a}_1(t) \\ -\frac{1}{2 \cdot \cos \psi_j} \Big\{ - (\dot{\psi}_i^2 \cdot l_j \cdot \sin \psi_i - \ddot{a}_1(t) \cdot l_i \cdot \cos \psi_i \Big) \cdot \sin \psi_j \\ \quad + \Big(2 \cdot \ddot{a}_1(t) \cdot l_i \cdot \sin \psi_i + 2 \cdot \dot{\psi}_i^2 \cdot l_j \cdot \cos \psi_i \Big) \cos \psi_j + \dot{\psi}_i^2 \cdot l_j \Big\} \\ -\frac{l_i}{2} \cdot \Big(\dot{\psi}_i^2 \cdot \sin \psi_i - \ddot{a}_1(t) \cdot \cos \psi_i \Big) \\ \dfrac{\dot{\psi}_i^2 \cdot l_j \cdot \sin \psi_j + l_i \cdot \Big(\dot{\psi}_i^2 \cdot \sin \psi_i - \ddot{a}_1(t) \cdot \cos \psi_i \Big)}{l_j \cdot \cos \psi_j} \end{pmatrix} .$$

In the *singular case* (for $\cos \psi_j^* = 0$), this solution tends to *infinity* unless additional conditions are satisfied. For

$$\operatorname{rank}\Big((\boldsymbol{g}_{1p} \,, \boldsymbol{\beta}_{c_1}) \Big) \Big|_{\psi_j^*} > \operatorname{rank}(\boldsymbol{g}_{1p}) \Big|_{\psi_j^*} = 5 \tag{3.28a}$$

or for

$$\det\Big((\boldsymbol{g}_{1p} \,, \boldsymbol{\beta}_{c_1}) \cdot (\boldsymbol{g}_{1p} \,, \boldsymbol{\beta}_{c_1})^{\mathrm{T}} \Big) \Big|_{\psi_j^*} \neq 0 \tag{3.28b}$$

with the augmented matrix

$$(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{\mathrm{c}_1}) =$$

$$\begin{pmatrix} 1\,,0\,, & (l_i/2)\cdot\sin\psi_i\,, & 0\,, & 0\,, & 0 & ,-(l_i/2)\cdot\dot\psi_i^2\cdot\cos\psi_i \\ 0\,,1\,,- & (l_i/2)\cdot\cos\psi_i\,, & 0\,, & 0\,, & 0 & ,-(l_i/2)\cdot\dot\psi_i^2\cdot\sin\psi_i \\ 1\,,0\,,- & (l_i/2)\cdot\sin\psi_i\,, & -1\,, & 0\,,- & (l_j/2)\cdot\sin\psi_j & ,+(l_i/2)\cdot\dot\psi_i^2\cdot\cos\psi_i+(l_j/2)\cdot\dot\psi_j^2\cdot\cos\psi_j \\ 0\,,1\,, & (l_i/2)\cdot\cos\psi_i\,, & 0\,, & -1\,, & (l_j/2)\cdot\cos\psi_j & ,+(l_i/2)\cdot\dot\psi_i^2\cdot\sin\psi_i+(l_j/2)\cdot\dot\psi_j^2\cdot\sin\psi_j \\ 0\,,0\,, & 0\,, & 0\,, & 1\,, & (l_j/2)\cdot\cos\psi_j & ,+(l_j/2)\cdot\dot\psi_j^2\cdot\sin\psi_j \\ 0\,,0\,, & 1\,, & 0\,, & 0\,, & 0 & ,\quad\ddot a_1(t) \end{pmatrix},$$

(3.27a) does not have a finite solution. For

$$\det\left((g_{1p}\,,\,\boldsymbol{\beta}_{\mathrm{c}_1})\cdot\begin{pmatrix} g_{1p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{\mathrm{c}_1}^{\mathrm{T}}\end{pmatrix}\right)\Bigg|_{\psi_j^*} =$$

$$\begin{aligned} &-\Big[\Big(\Big(\Big(\Big(4\cdot\dot\psi_i^4-4\cdot\ddot a_1^2(t)\Big)\cdot l_i^2\cdot l_j^2\cdot\cos\psi_i\cdot\sin\psi_i \\ &-8\cdot\ddot a_1(t)\cdot\dot\psi_i^2\cdot l_i^2\cdot l_j^2\cdot\cos^2\psi_i+4\cdot\ddot a_1(t)\cdot\dot\psi_i^2\cdot l_i^2\cdot l_j^2\Big)\cdot\cos\psi_j \\ &+\Big(2\cdot\dot\psi_i^2\cdot\dot\psi_j^2\cdot l_i\cdot l_j^3+8\cdot\dot\psi_i^2\cdot\dot\psi_j^2\cdot l_i\cdot l_j\Big)\cdot\sin\psi_i \\ &+\Big(-2\cdot\ddot a_1(t)\cdot\dot\psi_j^2\cdot l_i\cdot l_j^3-8\cdot\ddot a_1(t)\cdot\dot\psi_j^2\cdot l_i\cdot l_j^3\Big)\cdot\cos\psi_i\Big)\cdot\sin\psi_j \\ &+\Big(8\cdot\ddot a_1(t)\cdot\dot\psi_i^2\cdot l_i^2\cdot l_j^2\cdot\cos\psi_i\cdot\sin\psi_i+\Big(4\cdot\dot\psi_i^4-4\cdot\ddot a_1^2(t)\Big)\cdot l_i^2\cdot l_j^2\cdot\cos^2\psi_i \\ &+\Big(\Big(\dot\psi_i^4+5\cdot\ddot a_1^2(t)\Big)\cdot l_i^2-4\cdot\dot\psi_j^4+4\cdot\ddot a_1^2(t)+4\Big)\cdot l_j^2\Big)\cdot\cos^2\psi_j \\ &+\Big(4\cdot\ddot a_1(t)\cdot\dot\psi_j^2\cdot l_i\cdot l_j^3\cdot\sin\psi_i+4\cdot\dot\psi_i^2\cdot\dot\psi_j^2\cdot l_i\cdot l_j^3\cdot\cos\psi_i\Big)\cdot\cos\psi_j \\ &+\Big(-2\cdot\ddot a_1(t)\cdot\dot\psi_i^2\cdot l_i^2\cdot l_j^2-8\cdot\ddot a_1(t)\cdot\dot\psi_i^2\cdot l_i^2\Big)\cdot\cos\psi_i\cdot\sin\psi_i \\ &+\Big(\Big(\ddot a_1^2(t)-\dot\psi_i^4\Big)\cdot l_i^2\cdot l_j^2+\Big(4\cdot\ddot a_1^2(t)-4\cdot\dot\psi_i^4\Big)\cdot l_i^2\Big)\cdot\cos^2\psi_i+\dot\psi_j^4\cdot l_j^4 \\ &+\Big(\dot\psi_i^4\cdot l_i^2+4\cdot\dot\psi_i^4\dot\psi_j^4\Big)\cdot l_j^2+4\cdot\dot\psi_i^4\cdot l_i^2\Big)\Big/4\Big]\Bigg|_{\psi_j^*}=0, \end{aligned} \tag{3.29}$$

the *constraint acceleration equation* (3.27a) has at least *two finite solutions* $\ddot{\boldsymbol{p}}_i$. This will now be analyzed for the different singular situations of Figure 3.4.

In *Case 1.3a*, (3.29) implies, together with

$$l_i = l_j \quad , \quad \psi_j^* = \frac{3}{2}\pi$$

and for

$$\psi_i = \frac{\pi}{2} \quad \text{and} \quad \dot a_1 \neq 0, \tag{3.26l}$$

the relation

$$\det\left(\left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right)\cdot\left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right)^{\mathrm{T}}\right)\Bigg|_{\psi_j^*=\frac{3}{2}\pi}$$
$$=\left(\dot\psi_j+\dot\psi_i\right)^2\cdot\left(\dot\psi_j-\dot\psi_i\right)^2\cdot l_j^2\cdot\left(l_j^2+4\right)/4$$
$$=\left(\dot\psi_j+\dot a_1\right)^2\cdot\left(\dot\psi_j-\dot a_1\right)^2\cdot l_j^2\cdot\left(l_j^2+4\right)/4=0$$

iff

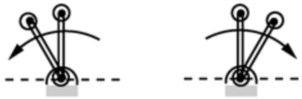
$$\dot\psi_j(t)=\quad\dot a_1(t)\neq0\qquad \textit{(Case 1.3a1)}\tag{3.30a}$$

or *(velocity analysis criteria)*

$$\dot\psi_j(t)=-\dot a_1(t)\neq0\qquad \textit{(Case 1.3a2)}.\tag{3.30b}$$

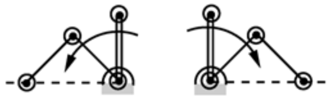
From this *singular configuration (Case 1.3a)*, four different motions may *bifurcate (bifurcation point)*:

$$\left.\begin{array}{l}\dot\psi_j\equiv\dot a_1(t)>0\ ,\\ \dot\psi_j\equiv\dot a_1(t)<0\ ,\end{array}\right.\quad \textit{(Case 1.3a1)}$$



and (3.31)

$$\left.\begin{array}{l}\dot\psi_j=-\dot a_1(t)>0,\\ \dot\psi_j=-\dot a_1(t)<0.\end{array}\right.\quad \textit{(Case 1.3a2)}$$



In *Case 1.3b*, (3.29) implies, together with

$$l_i=l_j\quad,\quad\psi_j^*=\frac{5}{2}\pi$$

and for

$$\psi_i=\frac{3}{2}\pi=-\frac{\pi}{2}\qquad\text{and}\qquad\dot a_1\neq0\tag{3.26l}$$

the relation

$$\det\left(\left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right)\cdot\left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right)^{\mathrm{T}}\right)\Bigg|_{\psi_j^*=\frac{5}{2}\pi}$$
$$=\left(\dot\psi_j+\dot a_1\right)^2\cdot\left(\dot\psi_j-\dot a_1\right)^2\cdot l_j^2\cdot\left(l_j^2+4\right)/4=0$$

iff

$$\dot\psi_j(t) = \quad \dot a_1(t) \neq 0 \quad \textit{(Case 1.3b1)} \tag{3.32a}$$

or *(velocity analysis criteria)*

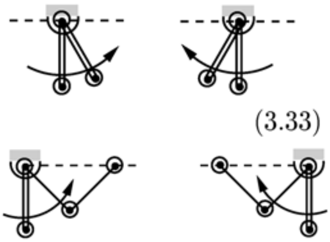
$$\dot\psi_j(t) = -\dot a_1(t) \neq 0 \quad \textit{(Case 1.3b2).} \tag{3.32b}$$

From this *singular configuration (Case 1.3b)*, again four different motions may *bifurcate (bifurcation point)*:

$$\left.\begin{array}{l}\dot\psi_j = \dot a_1(t) > 0, \\ \dot\psi_j = \dot a_1(t) < 0,\end{array}\right. \quad \textit{(Case 1.3b1)}$$

and

$$\left.\begin{array}{l}\dot\psi_j = -\dot a_1(t) > 0, \\ \dot\psi_j = -\dot a_1(t) < 0.\end{array}\right. \quad \textit{(Case 1.3b2)}$$



(3.33)

In *Case 1.2a*, (3.29) implies, together with

$$l_i > l_j \quad , \quad \psi_j^* = \frac{3}{2}\pi \quad , \quad \sin\psi_i = l_j/l_i \ \text{ and } \ \cos\psi_i = \sqrt{1-(l_j/l_i)^2},$$

the relation

$$\det\Big((\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{\mathrm{c}_1})\cdot(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{\mathrm{c}_1})^{\mathrm{T}}\Big)\ \Bigg|_{\psi_j^* = \frac{3}{2}\pi} \tag{3.34a}$$

$$= \underbrace{\left(l_j^2+4\right)}_{>0}\cdot\Big[2\cdot\ddot a_1(t)\cdot\underbrace{l_j\cdot\sqrt{l_i^2-l_j^2}}_{>0}\cdot\left(\dot\psi_j^2-\dot a_1^2\right)$$

$$+\left(\dot\psi_j^2-\dot a_1^2\right)^2\cdot l_j^2+\ddot a_1^2(t)\cdot\underbrace{(l_i^2-l_j^2)}_{>0}\Big]\Big/4 = 0.$$

The resulting equation,

$$\ddot a_1^2(t) + \frac{2l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)}\cdot\left(\dot\psi_j^2-\dot a_1^2\right) + \frac{l_j^2}{(l_i^2-l_j^2)}\cdot\left(\dot\psi_j^2-\dot a_1^2\right) = 0,$$

is quadratic in $\ddot a_1(t)$ and has two identical roots

$$\ddot a_{1\pm} = -\frac{l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)}\cdot\left(\dot\psi_j^2-\dot a_1^2\right)$$

$$\pm\underbrace{\sqrt{\frac{l_j^2\cdot(l_i^2-l_j^2)}{(l_i^2-l_j^2)^2}\cdot\left(\dot\psi_j^2-\dot a_1^2\right)^2-\frac{l_j^2\cdot(l_i^2-l_j^2)}{(l_i^2-l_j^2)^2}\cdot\left(\dot\psi_j^2-\dot a_1^2\right)^2}}_{=0}$$

or

$$\ddot{a}_{1\pm} = -\frac{l_j\sqrt{(l_i^2 - l_j^2)}}{(l_i^2 - l_j^2)} \cdot \left(\dot{\psi}_j^2 - \dot{a}_1^2\right) = \frac{l_j\sqrt{(l_i^2 - l_j^2)}}{(l_i^2 - l_j^2)} \cdot \left(\dot{a}_1^2 - \dot{\psi}_j^2\right). \tag{3.34b}$$

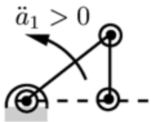
Together with

$$\dot{a}_1 = 0 \qquad \textit{(velocity analysis)} \tag{3.26m}$$

this implies

$$\ddot{a}_1 = -\underbrace{\frac{l_j\sqrt{(l_i^2 - l_j^2)}}{(l_i^2 - l_j^2)}}_{>0} \cdot \dot{\psi}_j^2 < 0\ . \tag{3.35}$$

This relation proves that the situation



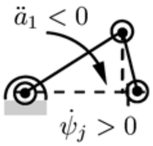
$$\ddot{a}_1 > 0 \quad \text{and} \quad \dot{a}_1 = 0 \tag{3.36}$$

cannot occur – *no such motion is possible.* This is a *lock-up* situation. On the other hand, the situation

$$\ddot{a}_1 < 0 \quad \text{and} \quad \dot{a}_1 = 0 \tag{3.37}$$

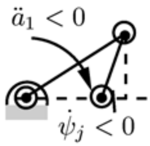
allows *two possible motions.* They may *bifurcate* from the singular configuration of *Case 1.2a*:

$$\begin{aligned} &\dot{\psi}_j > 0\ ,\quad \dot{a}_1(t) = 0\ ,\quad \ddot{a}_1 < 0, \\ &\psi_j = \tfrac{3}{2}\pi, \end{aligned} \qquad \textit{(Case 1.2a1)} \tag{3.38}$$



and

$$\begin{aligned} &\dot{\psi}_j < 0\ ,\quad \dot{a}_1(t) = 0\ ,\quad \ddot{a}_1 < 0, \\ &\psi_j = \tfrac{3}{2}\pi. \end{aligned} \qquad \textit{(Case 1.2a2)} \tag{3.39}$$



For *Case 1.2b*, (3.29) implies, together with

$$l_i > l_j\ ,\quad \psi_j^* = \frac{5}{2}\pi\ ,\quad \sin\psi_i = -\frac{l_j}{l_i},\ \text{ and }\ \cos\psi_i = \sqrt{1 - (l_j/l_i)^2},$$

the relation

$$\det\left(\left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right) \cdot \left(\boldsymbol{g}_{1p}\,,\,\boldsymbol{\beta}_{c_1}\right)^{\mathrm{T}}\right) \Bigg|_{\psi_j^* = \frac{5}{2}\pi}$$

$$= -\underbrace{\left(l_j^2+4\right)}_{>0}\cdot\Big[2\cdot\ddot{a}_1(t)\cdot\underbrace{l_j\cdot\sqrt{l_i^2-l_j^2}}_{>0}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)$$
$$-\underbrace{\left(\dot{\psi}_j^2-\dot{a}_1^2\right)^2\cdot l_j^2}_{>0}+\ddot{a}_1^2(t)\cdot\underbrace{(l_i^2-l_j^2)}_{>0}\Big]\Big/4=0.$$

The resulting quadratic equation,

$$\ddot{a}_1^2(t)+\frac{2l_j\sqrt{l_i^2-l_j^2}}{(l_i^2-l_j^2)}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)\cdot\ddot{a}_1+\left(\frac{l_j^2}{l_i^2-l_j^2}\right)\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)^2,$$

has two identical roots

$$\ddot{a}_{1\pm}=\frac{l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)$$
$$\pm\underbrace{\sqrt{\frac{l_j^2\cdot(l_i^2-l_j^2)}{(l_i^2-l_j^2)^2}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)^2-\frac{l_j^2\cdot(l_i^2-l_j^2)}{(l_i^2-l_j^2)^2}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right)^2}}_{=0}$$

or

$$\ddot{a}_{1\pm}=\frac{l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)}\cdot\left(\dot{\psi}_j^2-\dot{a}_1^2\right).$$

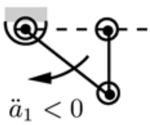
Together with

$$\dot{a}_1=0 \qquad \textit{(velocity analysis)} \tag{3.26m}$$

this implies

$$\ddot{a}_1=+\underbrace{\frac{l_j\sqrt{l_i^2-l_j^2}}{l_i^2-l_j^2}}_{>0}\cdot\dot{\psi}_j^2>0. \tag{3.40}$$

This relation proves that the situation



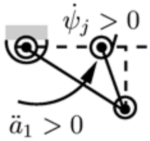
$$\ddot{a}_1<0 \quad , \quad \dot{a}_1=0 \tag{3.41}$$

cannot occur – *no such motion is possible.* This is a *lock-up* situation.
On the other hand, the situation

$$\ddot{a}_1>0 \quad , \quad \dot{a}_1=0 \tag{3.42}$$

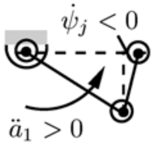
allows *two possible motions.* They may *bifurcate* from the singular configuration of *Case 1.2b*:

$$\dot{\psi}_j > 0 \quad , \quad \dot{a}_1(t) = 0, \ddot{a}_1 > 0, \quad \textit{(Case 1.2b1)} \tag{3.43}$$



and

$$\dot{\psi}_j < 0 \quad , \quad \dot{a}_1(t) = 0, \ddot{a}_1 > 0. \quad \textit{(Case 1.2b2)} \tag{3.44}$$



The above results of the singular situations of the slider–crank mechanism under the drive $a_1(t)$ are collected in Figure 3.4.

3.1.3.2.2 Local analysis of Case 2 ($x^R_{PO} = -a_2(t)$). Written in components, the *constraint position equations* are

$$(0\,,0\,,0\,,0\,,0\,,0)^{\mathrm{T}} = \boldsymbol{g}_2(\boldsymbol{p}\,,t) = \tag{3.45a}$$
$$\begin{pmatrix} x^R_{P_iO} + 0 - (l_i/2)\cdot\cos\psi_i + 0 + 0 + 0 + 0 \\ 0 + y^R_{P_iO} - (l_i/2)\cdot\sin\psi_i + 0 + 0 + 0 + 0 \\ x^R_{P_iO} + 0 + (l_i/2)\cdot\cos\psi_i - x^R_{P_jO} + 0 + (l_j/2)\cdot\cos\psi_j + 0 \\ 0 + y^R_{P_iO} + (l_i/2)\cdot\sin\psi_i + 0 - y^R_{P_jO} + (l_j/2)\cdot\sin\psi_j + 0 \\ 0 + 0 + 0 + 0 + y^R_{P_jO} + (l_j/2)\cdot\sin\psi_j + 0 \\ 0 + 0 + 0 + x^R_{P_jO} + 0 + (l_j/2)\cos\psi_j + a_2(t) \end{pmatrix}$$

with the *constraint Jacobian matrix*

$$\boldsymbol{g}_{2p}(\boldsymbol{p}) = \begin{pmatrix} 1\,, & 0\,, & (l_i/2)\cdot\sin\psi_i\,, & 0\,, & 0\,, & 0 \\ 0\,, & 1\,, & -(l_i/2)\cdot\cos\psi_i\,, & 0\,, & 0\,, & 0 \\ 1\,, & 0\,, & -(l_i/2)\cdot\sin\psi_i\,, & -1\,, & 0\,, & -(l_j/2)\cdot\sin\psi_j \\ 0\,, & 1\,, & (l_i/2)\cdot\cos\psi_i\,, & 0\,, & -1\,, & (l_j/2)\cdot\cos\psi_j \\ 0\,, & 0\,, & 0\,, & 0\,, & 1\,, & (l_j/2)\cdot\cos\psi_j \\ 0\,, & 0\,, & 0\,, & 1\,, & 0\,, & -(l_j/2)\cdot\sin\psi_j \end{pmatrix}. \tag{3.45b}$$

The *constraint velocity equations* are

$$\boldsymbol{g}_{2p}(\boldsymbol{p}\,,t)\cdot\dot{\boldsymbol{p}} = -\boldsymbol{g}_{2t}(\boldsymbol{p}\,,t) \tag{3.45c}$$

with

$$\boldsymbol{g}_{2t}(\boldsymbol{p}\,,t) = (0\,,0\,,0\,,0\,,0\,,\dot{a}_2(t))^{\mathrm{T}}. \tag{3.45d}$$

The constraint *acceleration equations* are

$$\boldsymbol{g}_{2p}(\boldsymbol{p}\,,t)\cdot\ddot{\boldsymbol{p}} = \boldsymbol{\beta}_{c_2}(\boldsymbol{p}\,,\dot{\boldsymbol{p}}\,,t) \tag{3.45e}$$

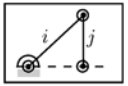
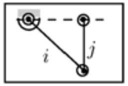
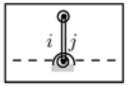
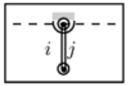
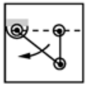
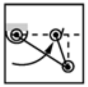
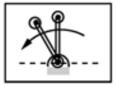
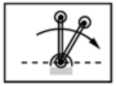
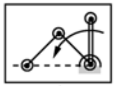
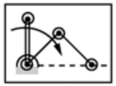
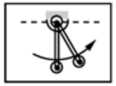
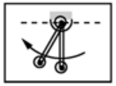
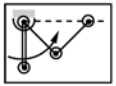
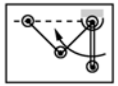
singularity analysis (Case 1: $\psi_i = a_1$)	$l_i > l_j$ (Case 1.2)		$l_i = l_j$ (Case 1.3)	
singular constraint Jacobian: $\psi_j = \psi_j^*, cos\psi_j^* = 0$ $\psi_j = (2n+1) \cdot \frac{\pi}{2}$	$\psi_j = \frac{3}{2}\pi$ (Case 1.2a) 	$\psi_j = \frac{5}{2}\pi$ (Case 1.2b) 	$\psi_j = \frac{3}{2}\pi$ (Case 1.3a) 	$\psi_j = \frac{5}{2}\pi$ (Case 1.3b) 
constraint velocity analysis: $\dot{a}_1 = 0$ or $\cos\psi_i = 0$	$\dot{a}_1 = 0$ $\cos\psi_i \neq 0$	$\dot{a}_1 = 0$ $\cos\psi_i \neq 0$	$\psi_i = \frac{\pi}{2}$ $\dot{a}_1 \neq 0$	$\psi_i = \frac{3}{2}\pi$ $\dot{a}_1 \neq 0$
constraint acceleration analysis: Cases 1.3 $\dot{\psi}_j = \dot{a}_1$ and $\dot{\psi}_j = -\dot{a}_1$ Cases 1.2a $\ddot{a}_1 = -\frac{l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)} \cdot \dot{\psi}_j^2 < 0$ Cases 1.2b $\ddot{a}_1 = +\frac{l_j\sqrt{(l_i^2-l_j^2)}}{(l_i^2-l_j^2)} \cdot \dot{\psi}_j^2 > 0$	$\ddot{a}_1 = -\frac{l_j \cdot (l_i^2 - l_j^2)^{\frac{1}{2}}}{l_i^2 - l_j^2} \cdot \dot{\psi}_j^2$ for $\ddot{a}_1 > 0$ lock-up  for $\ddot{a}_1 < 0, \dot{\psi}_j > 0$ (Case 1.2a1)  or for $\ddot{a}_1 < 0, \dot{\psi}_j < 0$ (Case 1.2a2)  bifurcations	$\ddot{a}_1 = \frac{l_j \cdot (l_i^2 - l_j^2)^{\frac{1}{2}}}{l_i^2 - l_j^2} \cdot \dot{\psi}_j^2$ for $\ddot{a}_1 < 0$ lock-up  for $\ddot{a}_1 > 0, \dot{\psi}_j > 0$ (Case 1.2b1)  or for $\ddot{a}_1 > 0, \dot{\psi}_j < 0$ (Case 1.2b2)  bifurcations	$\dot{\psi}_j = \dot{a}_1$ (Cases 1.3a1)   $\dot{\psi}_j = \dot{a}_1 > 0$ $\dot{\psi}_j = \dot{a}_1 < 0$ or $\dot{\psi}_j = -\dot{a}_1$ (Cases 1.3a2)   $\dot{a}_1 = -\dot{\psi}_j > 0$ $\dot{a}_1 = -\dot{\psi}_j < 0$ bifurcations	$\dot{\psi}_j = \dot{a}_1$ (Cases 1.3b1)   $\dot{\psi}_j = \dot{a}_1 > 0$ $\dot{\psi}_j = \dot{a}_1 < 0$ or $\dot{\psi}_j = -\dot{a}_1$ (Cases 1.3b2)   $\dot{a}_1 = -\dot{\psi}_j > 0$ $\dot{a}_1 = -\dot{\psi}_j < 0$ bifurcations

Fig. 3.4: Singular situations of the slider–crank mechanism of Case 1 (drive $a_1(t)$)

with the right hand side

$$\boldsymbol{\beta}_{c_2}(\boldsymbol{p}, \dot{\boldsymbol{p}}, t) = -\boldsymbol{g}_{2tt}(\boldsymbol{p}, t) - 2 \cdot \boldsymbol{g}_{2pt}(\boldsymbol{p}, t) \cdot \dot{\boldsymbol{p}} - \Big(\boldsymbol{g}_{2p}(\boldsymbol{p}, t) \cdot \dot{\boldsymbol{p}}\Big)_p \cdot \dot{\boldsymbol{p}}, \tag{3.45f}$$

with

$$-\boldsymbol{g}_{2tt}(\boldsymbol{p}, t) = (0, 0, 0, 0, 0, \dot{a}_2(t))^{\mathrm{T}} \quad , \quad -\boldsymbol{g}_{2pt}(\boldsymbol{p}, t) = \boldsymbol{0} \in \mathbb{R}^6, \tag{3.45g}$$

$$\boldsymbol{g}_{2p}(\boldsymbol{p}, t) \cdot \dot{\boldsymbol{p}} = \tag{3.45h}$$

$$\begin{pmatrix} \dot{x}^R_{P_iO} + \dot{\psi}_i \cdot (l_i/2) \cdot \sin\psi_i \\ \dot{y}^R_{P_iO} - \dot{\psi}_i \cdot (l_i/2) \cdot \cos\psi_i \\ \dot{x}^R_{P_iO} - \dot{\psi}_i \cdot (l_i/2) \cdot \sin\psi_i - \dot{x}^R_{P_jO} - \dot{\psi}_j \cdot (l_j/2) \cdot \sin\psi_j \\ \dot{y}^R_{P_iO} + \dot{\psi}_i \cdot (l_i/2) \cdot \cos\psi_i - \dot{y}^R_{P_jO} + \dot{\psi}_j \cdot (l_j/2) \cdot \cos\psi_j \\ \dot{y}^R_{P_jO} + \dot{\psi}_j \cdot (l_j/2) \cdot \cos\psi_j \\ \dot{x}^R_{P_jO} - \dot{\psi}_j \cdot (l_j/2) \cdot \sin\psi_j \end{pmatrix},$$

$$\Big(\boldsymbol{g}_{2p}(\boldsymbol{p}, t) \cdot \dot{\boldsymbol{p}}\Big)_p \cdot \dot{\boldsymbol{p}} = \tag{3.45i}$$

$$\begin{pmatrix} 0, 0, & (l_i/2) \cdot \dot{\psi}_i \cdot \cos\psi_i, & 0, 0, & 0 \\ 0, 0, & (l_i/2) \cdot \dot{\psi}_i \cdot \sin\psi_i, & 0, 0, & 0 \\ 0, 0, & -(l_i/2) \cdot \dot{\psi}_i \cdot \cos\psi_i, & 0, 0, & -(l_j/2) \cdot \dot{\psi}_j \cdot \cos\psi_j \\ 0, 0, & -(l_i/2) \cdot \dot{\psi}_i \cdot \sin\psi_i, & 0, 0, & -(l_j/2) \cdot \dot{\psi}_j \cdot \sin\psi_j \\ 0, 0, & 0, & 0, 0, & -(l_j/2) \cdot \dot{\psi}_j \cdot \sin\psi_j \\ 0, 0, & 0, & 0, 0, & -(l_j/2) \cdot \dot{\psi}_j \cdot \cos\psi_j \end{pmatrix} \cdot \begin{pmatrix} \dot{x}^R_{P_iO} \\ \dot{y}^R_{P_iO} \\ \dot{\psi}_i \\ \dot{x}^R_{P_jO} \\ \dot{y}^R_{P_jO} \\ \dot{\psi}_j \end{pmatrix},$$

and

$$\boldsymbol{\beta}_{c_2}(\boldsymbol{p}, \dot{\boldsymbol{p}}, t) = \begin{pmatrix} -(l_i/2) \cdot \dot{\psi}_i^2 \cdot \cos\psi_i \\ -(l_i/2) \cdot \dot{\psi}_i^2 \cdot \sin\psi_i \\ +(l_i/2) \cdot \dot{\psi}_i^2 \cdot \cos\psi_i + (l_j/2) \cdot \dot{\psi}_j^2 \cdot \cos\psi_j \\ +(l_i/2) \cdot \dot{\psi}_i^2 \cdot \sin\psi_i + (l_j/2) \cdot \dot{\psi}_j^2 \cdot \sin\psi_j \\ +(l_j/2) \cdot \dot{\psi}_j^2 \cdot \sin\psi_j \\ -\ddot{a}_2(t) + (l_j/2) \cdot \dot{\psi}_j^2 \cdot \cos\psi_j \end{pmatrix}. \tag{3.45j}$$

Local singularity analysis of the constraint Jacobian matrix. From (3.45b), it follows directly that

$$\det\Big(\boldsymbol{g}_{2p}(\boldsymbol{p})\Big) = l_i \cdot l_j \cdot \sin(\psi_i - \psi_j). \tag{3.45k}$$

For $l_i \cdot l_j > 0$ *singular situations* occur for

$$\det\Big(\boldsymbol{g}_{2p}(\boldsymbol{p})\Big) = 0 \;\longleftrightarrow\; \psi_i - \psi_j = \psi_i^* - \psi_j^* = n \cdot \pi \quad , \quad n \in \boldsymbol{Z} \cup \{0\}. \qquad (3.45l)$$

This implies

$$\operatorname{rank}\Big(\boldsymbol{g}_{2p}(\boldsymbol{p})\Big)\Big|_{\psi_i^* - \psi_j^*} = 5 \; .$$

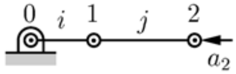
Starting from the *initial conditions*

$$\psi_j(0) = 2\pi \quad , \quad \psi_i(0) = 0 \quad , \quad \text{and} \quad x_{PO}^R = l_i + l_j = -a_2(0), \qquad (3.45m)$$

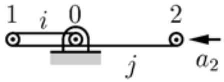
the following *singular situations* may occur for $(0 < \varepsilon < \frac{\pi}{2})$:

(3.45n)

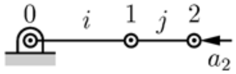
Case 2.1: $l_i < l_j$: $\psi_j = 2\pi \quad , \quad \psi_i = 0$ *(Case 2.1a)*



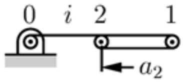
$\psi_j = 2\pi \quad , \quad \psi_i = \pi$ *(Case 2.1b)*



Case 2.2: $l_i > l_j$: $\psi_j = 2\pi \quad , \quad \psi_i = 0$ *(Case 2.2a)*

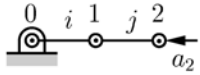


$\psi_j = \pi \quad , \quad \psi_i = 0$ *(Case 2.2b)*

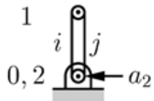


Case 2.3: $l_i = l_j$:

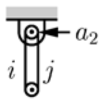
$\psi_j = 2\pi \quad , \quad \psi_i = 0$ *(Case 2.3a)*



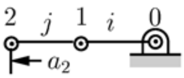
$\psi_j = \dfrac{3}{2}\pi \quad , \quad \psi_i = \dfrac{\pi}{2}$ *(Case 2.3b)*



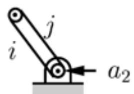
$\psi_j = \dfrac{5}{2}\pi \quad , \quad \psi_i = \dfrac{3}{2}\pi$ *(Case 2.3c)*



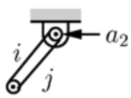
$\psi_j = \pi \quad , \quad \psi_i = \pi$ *(Case 2.3d)*



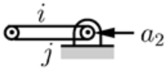
$\psi_j = \dfrac{3}{2}\pi + \varepsilon \quad , \quad \psi_i = \dfrac{\pi}{2} + \varepsilon$ *(Case 2.3e)*



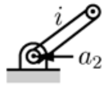
$\psi_j = \dfrac{5}{2}\pi - \varepsilon \quad , \quad \psi_i = \dfrac{3}{2}\pi - \varepsilon$ *(Case 2.3f)*



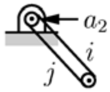
$$\psi_j = 2\pi \qquad , \quad \psi_i = \pi \qquad \textit{(Case 2.3g)}$$



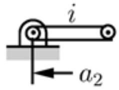
$$\psi_j = \frac{3}{2}\pi - \varepsilon, \quad \psi_i = \frac{\pi}{2} - \varepsilon \quad \textit{(Case 2.3h)}$$



$$\psi_j = \frac{5}{2}\pi + \varepsilon, \quad \psi_i = \frac{3}{2}\pi + \varepsilon \quad \textit{(Case 2.3i)}$$



$$\psi_j = \pi \qquad , \quad \psi_i = 0. \qquad \textit{(Case 2.3j)} \quad .$$



Analysis of the constraint velocity equations. Written in components, the *constraint velocity equations* (3.45c) are

$$\begin{pmatrix} 1 \;, 0 \;, & (l_i/2)\cdot\sin\psi_i \;, & 0 \;, & 0 \;, & 0 \\ 0 \;, 1 \;, & -(l_i/2)\cdot\cos\psi_i \;, & 0 \;, & 0 \;, & 0 \\ 1 \;, 0 \;, & -(l_i/2)\cdot\sin\psi_i \;, & -1 \;, & 0 \;, & -(l_j/2)\cdot\sin\psi_j \\ 0 \;, 1 \;, & (l_i/2)\cdot\cos\psi_i \;, & 0 \;, & -1 \;, & (l_j/2)\cdot\cos\psi_j \\ 0 \;, 0 \;, & 0 \;, & 0 \;, & 1 \;, & (l_j/2)\cdot\cos\psi_j \\ 0 \;, 0 \;, & 0 \;, & 1 \;, & 0 \;, & -(l_j/2)\cdot\sin\psi_j \end{pmatrix} \cdot \begin{pmatrix} \dot{x}^R_{P_iO} \\ \dot{y}^R_{P_iO} \\ \dot{\psi}_i \\ \dot{x}^R_{P_jO} \\ \dot{y}^R_{P_jO} \\ \dot{\psi}_j \end{pmatrix}$$
$$= (0\,,0\,,0\,,0\,,0\,,-\dot{a}_2(t))^{\mathrm{T}}. \tag{3.46}$$

In the *regular case*

$$\det\left(\boldsymbol{g}_{2p}(\boldsymbol{p})\right) = l_i \cdot l_j \cdot \sin(\psi_i - \psi_j) \neq 0 \tag{3.47}$$

(3.46) has the *unique* finite solution

$$\dot{\boldsymbol{p}} = \left(\dot{x}^R_{P_iO}\,,\,\dot{y}^R_{P_iO}\,,\,\dot{\psi}_i\,,\,\dot{x}^R_{P_jO}\,,\,\dot{y}^R_{P_jO}\,,\,\dot{\psi}_j\right)^{\mathrm{T}} \tag{3.48}$$
$$= \Bigg[\frac{\dot{a}_2(t)\cdot\sin(\psi_j+\psi_i) - \dot{a}_2(t)\cdot\sin(\psi_j-\psi_i)}{4\cdot\sin(\psi_j+\psi_i)}\,,$$
$$-\frac{\dot{a}_2(t)\cdot\cos(\psi_j+\psi_i) + \dot{a}_2(t)\cdot\cos(\psi_j-\psi_i)}{4\cdot\sin(\psi_j-\psi_i)}\,,$$
$$-\frac{\dot{a}_2(t)\cdot\cos\psi_j}{l_i\cdot\sin(\psi_j-\psi_i)}\,,\;\frac{\dot{a}_2(t)\cdot\sin(\psi_j+\psi_i) - 3\cdot\dot{a}_2(t)\cdot\sin(\psi_j-\psi_i)}{4\cdot\sin(\psi_j-\psi_i)}\,,$$
$$-\frac{\dot{a}_2(t)\cdot\cos(\psi_j+\psi_i) + \dot{a}_2(t)\cdot\cos(\psi_j-\psi_i)}{4\cdot\sin(\psi_j-\psi_i)}\,,\;\frac{\dot{a}_2(t)\cdot\cos\psi_i}{l_j\cdot\sin(\psi_j-\psi_i)}\Bigg]^{\mathrm{T}}.$$

For

$$\sin(\psi_i - \psi_j) = 0, \tag{3.49}$$

this *solution tends to infinity (singular case)* unless additional conditions are satisfied. These conditions will now be derived.

Elimination of the coordinates $\dot{x}^R_{P_jO}$ and $\dot{x}^R_{P_iO}$ from rows 1, 3, and 6 of the constraint velocity equations (3.46) provides the following expression for the driving velocity, $\dot{a}_2(t)$,

$$\dot{a}_2(t) = l_j \cdot \sin\psi_j \cdot \dot{\psi}_j + l_i \cdot \sin\psi_i \cdot \dot{\psi}_i \tag{3.50a}$$

and its time derivative

$$\begin{aligned}\ddot{a}_2(t) = \quad & \left(l_j \cdot \cos\psi_j \cdot \dot{\psi}_j^2 + l_i \cdot \cos\psi_i \cdot \dot{\psi}_i^2\right) \\ & + \left(l_j \cdot \sin\psi_j \cdot \ddot{\psi}_j + l_i \cdot \sin\psi_i \cdot \ddot{\psi}_i\right) .\end{aligned} \tag{3.50b}$$

These two equations will now be used in the discussion of the *singular situations of Case 2.*

In the *singular case* ($\det(\boldsymbol{g}_{2p}(\boldsymbol{p})) = 0$), (3.46) *does not have a finite solution* for

$$\operatorname{rank}\left(\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}\right)\Big|_{\psi_i^* - \psi_j^*} > \operatorname{rank}(\left(\boldsymbol{g}_{2p}\right)\Big|_{\psi_i^* - \psi_j^*} = 5\ . \tag{3.51a}$$

On the other hand, the relation

$$\operatorname{rank}\left(\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}\right)\Big|_{\psi_i^* - \psi_j^*} = \operatorname{rank}\left(\boldsymbol{g}_{2p}\right)\Big|_{\psi_i^* - \psi_j^*} = 5 \tag{3.51b}$$

implies

$$\begin{aligned}&\det\left(\left(\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}\right) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix}\right)\Bigg|_{\psi_i^* - \psi_j^*} = 0 = \\ &- \Big[\Big(\left(4 \cdot \dot{a}_2^2(t) + 8\right) \cdot l_i^2 \cdot l_j^2 \cdot \cos\psi_i \cdot \sin\psi_i \cdot \cos\psi_j \cdot \sin\psi_j \\ &+ \Big(\left(4 \cdot \dot{a}_2^2(t) + 8\right) \cdot l_i^2 \cdot l_j^2 \cdot \cos^2\psi_i + \Big(\left(-5 \cdot \dot{a}_2^2(t) - 4\right) \cdot l_i^2 - 4 \cdot \dot{a}_2^2(t) \cdot l_j^2\Big)\Big) \\ &\cdot \cos^2\psi_j + \Big(\left(-\dot{a}_2^2(t) - 4\right) \cdot l_i^2 \cdot l_j^2 - 4 \cdot \dot{a}_2^2(t) \cdot l_i^2\Big) \cdot \cos^2\psi_i\Big)\Big/4\Big]\Bigg|_{\psi_i^* - \psi_j^*}.\end{aligned} \tag{3.51c}$$

This implies for *Cases 2.1a* and *2.2a* ($\psi_j = 2\pi,\ \psi_i = 0,\ l_i < l_j$, and $l_i > l_j$) that

$$\begin{aligned}&\det\left(\left(\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}\right) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix}\right) = \dot{a}_2^2(t) \cdot \underbrace{\left((l_i^2 + 2) \cdot l_j^2 + 2 \cdot l_i^2\right)/2}_{>0} = 0 \\ &\qquad \text{iff} \quad \dot{a}_2(t) = 0,\end{aligned} \tag{3.52a}$$

which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up).* For *Case 2.1b* ($\psi_j = 2\pi,\ \psi_i = \pi$, and $l_i < l_j$), this implies again (3.52a) and enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, again *no motion is possible (lock-up).* For *Case 2.2b* ($\psi_j = \pi,\ \psi_i = 0$, and $l_i > l_j$), this implies again (3.52a), which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up).* For *Case 2.3a* ($\psi_j = 2\pi,\ \psi_i = 0$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \qquad (3.52\mathrm{b})$$

$$\text{iff} \quad \dot{a}_2(t) = 0,$$

which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up).* For *Case 2.3b* ($\psi_j = \frac{3}{2}\pi,\ \psi_i = \frac{\pi}{2}$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = 0 \qquad (3.52\mathrm{c})$$

for *arbitrary driving velocity* $\dot{a}_2(t)$, which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46) *(bifurcation).* For *Case 2.3c* ($\psi_j = \frac{5}{2}\pi,\ \psi_i = \frac{3}{2}\pi$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}] \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = 0 \qquad (3.52\mathrm{d})$$

for *arbitrary driving velocity* $\dot{a}_2(t)$, which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For *Case 2.3d* ($\psi_j = \pi,\ \psi_i = \pi$, and $l_i = l_j$) this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} \qquad (3.52\mathrm{e})$$

$$\text{iff} \quad \dot{a}_2(t) = 0,$$

which enables at least two finite solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up).*

The subsequent singular *Cases 2.3e to 2.3j* can be obtained by additional control inputs (in addition to the driving function $a_2(t)$):

For *Case 2.3e* ($\psi_j = \frac{3}{2}\pi + \varepsilon,\ \psi_i = \frac{\pi}{2} + \varepsilon,\ 0 < \varepsilon < \frac{\pi}{2}$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \sin^2\varepsilon \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \qquad (3.52\mathrm{f})$$

$$\text{iff} \quad \dot{a}_2(t) = 0.$$

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*. For *Case 2.3f* ($\psi_j = \frac{5}{2}\pi - \varepsilon,\ \psi_i = \frac{3}{2}\pi - \varepsilon,\ 0 < \varepsilon < \frac{\pi}{2}$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \sin^2\varepsilon \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \tag{3.52g}$$

iff $\dot{a}_2(t) = 0$.

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*. For *Case 2.3g* ($\psi_j = 2\pi,\ \psi_i = \pi$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \tag{3.52h}$$

iff $\dot{a}_2(t) = 0$.

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*. For *Case 2.3h* ($\psi_j = \frac{3}{2}\pi - \varepsilon,\ \psi_i = \frac{\pi}{2} - \varepsilon$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \sin^2\varepsilon \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \tag{3.52i}$$

iff $\dot{a}_2(t) = 0$.

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*. For *Case 2.3i* ($\psi_j = \frac{5}{2}\pi + \varepsilon,\ \psi_i = \frac{3}{2}\pi + \varepsilon$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \sin^2\varepsilon \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \tag{3.52j}$$

iff $\dot{a}_2(t) = 0$.

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*. For *Case 2.3j* ($\psi_j = \pi,\ \psi_i = 0$, and $l_i = l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\, -\boldsymbol{g}_{2t}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ -\boldsymbol{g}_{2t}^{\mathrm{T}} \end{pmatrix} \right) = \dot{a}_2^2(t) \cdot \underbrace{l_j^2 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0 \tag{3.52k}$$

iff $\dot{a}_2(t) = 0$.

This enables at least two solutions $\dot{\boldsymbol{p}}$ of (3.46). For $\dot{a}_2(t) \neq 0$, *no motion is possible (lock-up)*.

Analysis of the constraint acceleration equations. In the *regular case (for* $\sin(\psi_i - \psi_j) \neq 0$*)* the constraint acceleration equation

$$\boldsymbol{g}_{2p} \cdot \ddot{\boldsymbol{p}} = \boldsymbol{\beta}_{c_2} \tag{3.53a}$$

has a *unique finite solution*

$$\ddot{\boldsymbol{p}} = \left(\ddot{x}^R_{P_iO}\,,\,\ddot{y}^R_{P_iO}\,,\,\ddot{\psi}_i\,,\,\ddot{x}^R_{P_jO}\,,\,\ddot{y}^R_{P_jO}\,,\,\ddot{\psi}_j\right)^{\mathrm{T}} = \tag{3.53b}$$

$$\begin{pmatrix} \dfrac{\ddot{a}_2(t)\cdot\left(\sin(\psi_j+\psi_i)-\sin(\psi_j-\psi_i)\right)-2\cdot\left(\dot{\psi}_i^2\cdot l_i\cdot\,\sin\psi_j+\dot{\psi}_j^2\cdot l_j\cdot\,\sin\psi_i\right)}{4\cdot\,\sin(\psi_j-\psi_i)} \\ -\dfrac{\ddot{a}_2(t)\cdot\left(\cos(\psi_j+\psi_i)+\cos(\psi_j-\psi_i)\right)-2\cdot\left(\dot{\psi}_i^2\cdot l_i\cdot\,\cos\psi_j+\dot{\psi}_j^2\cdot l_j\cdot\,\cos\psi_i\right)}{4\cdot\,\sin(\psi_j-\psi_i)} \\ \dfrac{\dot{\psi}_i^2\cdot l_i\cdot\,\cos(\psi_j-\psi_i)-\ddot{a}_2(t)\cdot\,\cos\psi_j+\dot{\psi}_j^2\cdot l_j}{l_i\cdot\,\sin(\psi_j-\psi_i)} \\ \dfrac{\ddot{a}_2(t)\cdot\left(\sin(\psi_j+\psi_i)-3\cdot\,\sin(\psi_j-\psi_i)\right)-2\cdot\left(\dot{\psi}_i^2\cdot l_i\cdot\,\sin\psi_j+\dot{\psi}_j^2\cdot l_j\cdot\,\sin\psi_i\right)}{4\cdot\,\sin(\psi_j-\psi_i)} \\ -\dfrac{\ddot{a}_2(t)\cdot\left(\cos(\psi_j+\psi_i)+\cos(\psi_j-\psi_i)\right)-2\cdot\left(\dot{\psi}_i^2\cdot l_i\cdot\,\cos\psi_j+\dot{\psi}_j^2\cdot l_j\cdot\,\cos\psi_i\right)}{4\cdot\,\sin(\psi_j-\psi_i)} \\ -\dfrac{\dot{\psi}_j^2\cdot l_j\cdot\,\cos(\psi_j-\psi_i)-\ddot{a}_2(t)\cdot\,\cos\psi_i+\dot{\psi}_i^2\cdot l_i}{l_j\cdot\,\sin(\psi_j-\psi_i)} \end{pmatrix}.$$

In the *singular case* (for $\sin(\psi_i - \psi_j) = 0$) this solution tends to *infinity* unless additional conditions are satisfied. Then the *constraint acceleration equation* (3.53a) *does not have a finite solution* $\ddot{\boldsymbol{p}}(t)$ for

$$\det\left(\left(\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2}\right]\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)\Bigg|_{\psi_j^*} =$$

$$\begin{aligned}
&-\Big[\Big(\Big(\Big(\Big(8\cdot\dot{\psi}_j^4+8\cdot\dot{\psi}_i^4-4\cdot\ddot{a}_2^2(t)-8\Big)\cdot l_i^2\cdot l_j^2\cdot\,\cos\psi_i\\
&+\Big(6\cdot\ddot{a}_2(t)\cdot\dot{\psi}_i^2\cdot l_i^3+8\cdot\ddot{a}_2(t)\cdot\dot{\psi}_i^2\cdot l_i\Big)\cdot l_j^2\Big)\cdot\,\sin\psi_i\cdot\,\cos\psi_j\\
&+\Big(\Big(8\cdot\ddot{a}_2(t)\cdot\dot{\psi}_j^2\cdot l_i^2\cdot l_j-2\cdot\ddot{a}_2(t)\cdot\dot{\psi}_j^2\cdot l_i^2\cdot l_j^3\Big)\cdot\,\cos\psi_i\\
&+\Big(4\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i^3+8\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i\Big)\cdot l_j^3+8\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i^3\cdot l_j\Big)\cdot\,\sin\psi_i\Big)\cdot\,\sin\psi_j\\
&+\Big(\Big(8\cdot\dot{\psi}_j^4+8\cdot\dot{\psi}_i^4-4\cdot\ddot{a}_2^2(t)-8\Big)\cdot l_i^2\cdot l_j^2\cdot\,\cos\psi_i\\
&+\Big(6\cdot\ddot{a}_2(t)\cdot\dot{\psi}_i^2\cdot l_i^3+8\cdot\ddot{a}_2(t)\cdot\dot{\psi}_i^2\cdot l_i\Big)\cdot l_j^2\cdot\,\cos\psi_i\\
&+\Big(\Big(-4\cdot\dot{\psi}_i^4-4\cdot\dot{\psi}_i^4+5\cdot\ddot{a}_2^2(t)+4\Big)\cdot l_i^2+4\cdot\ddot{a}_2^2(t)\Big)\cdot l_j^2\Big)\cdot\,\cos\psi_i\\
&+\Big(\Big(8\cdot\ddot{a}_2(t)\cdot\dot{\psi}_j^2\cdot l_i^2\cdot l_j-2\cdot\ddot{a}_2(t)\cdot\dot{\psi}_j^2\cdot l_i^2\cdot l_j^3\Big)\cdot\,\cos^2\psi_i\\
&+\Big(\Big(4\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i^3+8\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i\Big)\cdot l_j^3+8\cdot\dot{\psi}_i^2\cdot\dot{\psi}_j^2\cdot l_i^3\cdot l_j\Big)\cdot\,\cos\psi_i
\end{aligned}$$

$$+\Big(6\cdot\ddot a_2(t)\cdot\dot\psi_j^2\cdot l_i^2+8\cdot\ddot a_2(t)\cdot\dot\psi_j^2\Big)\cdot l_j^3\Big)\cdot\cos\psi_j$$
$$+\Big(\Big(-4\cdot\dot\psi_j^4-4\cdot\dot\psi_i^4+\ddot a_2(t)+4\Big)\cdot l_i^2\cdot l_j^2+4\cdot\ddot a_2^2(t)\cdot l_i^2\Big)\cdot\cos^2\psi_i$$
$$+\Big(8\cdot\ddot a_2(t)\cdot\dot\psi_i^2\cdot l_i^3-2\cdot\ddot a_2(t)\cdot\dot\psi_i^2\cdot l_i^3\cdot l_j^2\Big)\cdot\cos\psi_i+\Big(2\cdot\dot\psi_j^4\cdot l_i^2+4\cdot\dot\psi_j^4\Big)\cdot l_j^4$$
$$+\Big(2\cdot\dot\psi_i^4\cdot l_i^4+\Big(4\cdot\dot\psi_j^4+4\cdot\dot\psi_i^4\Big)\cdot l_i^2\Big)\cdot l_j^2+4\cdot\dot\psi_i^4\cdot l_i^4\Big)\Big/4\Big]\Bigg|_{\psi_j^*}\neq 0.$$

For

$$\det\Big((\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2})\,,\,(\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2})^{\mathrm{T}}\Big)\Bigg|_{\psi_i^*-\psi_j^*}=0, \tag{3.54}$$

(3.53a) has at least two finite solutions. This situation will now be analyzed for the *different singular cases*. For *Cases 2.1a* and *2.2a* ($\psi_j=2\pi,\ \psi_i=0;\ l_i<l_j$, and $l_i>l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2})\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.55a}$$
$$\Big(\dot\psi_j^2\cdot l_j+\dot\psi_i^2\cdot l_i-\ddot a_2(t)\Big)^2\cdot\underbrace{\Big((l_i^2+2)\cdot l_j^2+2\cdot l_i^2\Big)\Big/2}_{>0}=0$$

for

$$\ddot a_2(t)=l_j\cdot\dot\psi_j^2+l_i\cdot\dot\psi_i^2>0 \tag{3.55b}$$

(*bifurcation* for $\dot\psi_j>0$, and $\dot\psi_i<0$, and for $\dot\psi_j<0$ and $\dot\psi_i>0$). For $\ddot a_2(t)<0$, no motion is possible *(lock-up)*. For *Case 2.1b* ($\psi_j=2\pi,\ \psi_i=\pi$, and $l_i<l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2})\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.55c}$$
$$\Big(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_i-\ddot a_2(t)\Big)^2\cdot\underbrace{\Big((l_i^2+2)\cdot l_j^2+2\cdot l_i^2\Big)\Big/2}_{>0}=0$$

for

$$\ddot a_2(t)=l_j\cdot\dot\psi_j^2-l_i\cdot\dot\psi_i^2. \tag{3.55d}$$

Due to $\dot a_2(t)=0$ and (3.50a),

$$\dot\psi_j=\frac{l_i}{l_j}\cdot\frac{\sin\psi_i}{\sin\psi_j}\cdot\dot\psi_i\ \Bigg|_{\begin{pmatrix}\sin\psi_i=0\\ \sin\psi_j=0\end{pmatrix}}=-\frac{l_i}{l_j}\cdot\frac{\cos\psi_i}{\cos\psi_j}\cdot\dot\psi_i\ \Bigg|_{\begin{pmatrix}\cos\psi_i=-1\\ \cos\psi_j=-1\end{pmatrix}}=+\frac{l_i}{l_j}\cdot\dot\psi_i. \tag{3.55e}$$

Inserting (3.55e) into (3.55d) yields

$$\ddot{a}_2(t) = l_j \cdot \left(\frac{l_i}{l_j}\right)^2 \cdot \dot{\psi}^2 - l_i \cdot \dot{\psi}^2 = l_i \cdot \underbrace{\left(\frac{l_i}{l_j} - 1\right)}_{<0} \cdot \dot{\psi}^2$$

or

$$\ddot{a}_2(t) < 0 \tag{3.55f}$$

(*bifurcation* for $\dot{\psi}_j < 0$ and $\dot{\psi}_i < 0$, and for $\dot{\psi}_j > 0$ and $\dot{\psi}_i > 0$). For $\ddot{a}_2(t) > 0$, no motion is possible *(lock-up)*. For *Case 2.2b* ($\psi_j = \pi,\ \psi_i = 0$, and $l_i > l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2}] \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}} \end{pmatrix}\right) = \tag{3.55g}$$

$$\left(\dot{\psi}_j^2 \cdot l_j - \dot{\psi}_i^2 \cdot l_i + \ddot{a}_2(t)\right)^2 \cdot \underbrace{\left((l_i^2 + 2) \cdot l_j^2 + 2 \cdot l_i^2\right) / 2}_{>0} = 0$$

for

$$\ddot{a}_2(t) = l_i \cdot \dot{\psi}_i^2 - l_j \cdot \dot{\psi}_j^2. \tag{3.55h}$$

Due to $\dot{a}_2(t) = 0$ and (3.48a),

$$\dot{\psi}_j = -\frac{l_i}{l_j} \cdot \frac{\sin\psi_i}{\sin\psi_j} \cdot \dot{\psi}_i \Bigg|_{\begin{pmatrix}\sin\psi_i = 0 \\ \sin\psi_j = 0\end{pmatrix}} = -\frac{l_i}{l_j} \cdot \frac{\cos\psi_i}{\cos\psi_j} \cdot \dot{\psi}_i \Bigg|_{\begin{pmatrix}\cos\psi_i = 1 \\ \cos\psi_j = -1\end{pmatrix}} = +\frac{l_i}{l_j} \cdot \dot{\psi}_i\ . \tag{3.55i}$$

Inserting (3.55i) into (3.56h) yields

$$\ddot{a}_2(t) = l_j \cdot \left(\frac{l_i}{l_j}\right)^2 \cdot \dot{\psi}^2 - l_i \cdot \dot{\psi}^2 = l_i \cdot \left(\frac{l_i}{l_j} - 1\right) \cdot \dot{\psi}^2 < 0$$

or

$$\ddot{a}_2(t) < 0 \tag{3.55j}$$

(*bifurcation solutions* for $\dot{\psi}_j > 0$ and $\dot{\psi}_i < 0$, and for $\dot{\psi}_j < 0$ and $\dot{\psi}_i > 0$). For $\ddot{a}_2(t) > 0$, no motion is possible *(lock-up)*. For *Case 2.3a* ($\psi_j = 2\pi,\ \psi_i = 0$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\,\boldsymbol{\beta}_{c_2}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}} \end{pmatrix}\right) = \tag{3.56a}$$

$$l_j^2 \cdot \left(\dot{\psi}_j^2 \cdot l_j + \dot{\psi}_i^2 \cdot l_j - \ddot{a}_2(t)\right)^2 \cdot \left(l_i^2 + 4\right) / 2 = 0$$

for

$$\ddot{a}_2(t) = l_j \cdot \left(\dot{\psi}_j^2 + \dot{\psi}_i^2\right) > 0 \tag{3.56b}$$

(*bifurcations* occur for $\dot{\psi}_j > 0$ and $\dot{\psi}_i < 0$, and for $\dot{\psi}_j < 0$ and $\dot{\psi}_i > 0$). For $\ddot{a}_2(t) > 0$, *no motion is possible (lock-up)*. For *Case 2.3b* ($\psi_j = \frac{3}{2}\pi$, $\psi_i = \frac{\pi}{2}$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,, \boldsymbol{\beta}_{c_2}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}} \end{pmatrix} \right) = \tag{3.56c}$$

$$\left(\dot{\psi}_j - \dot{\psi}_i\right)^2 \cdot \left(\dot{\psi}_j + \dot{\psi}_i\right)^2 \cdot \underbrace{l_j^4 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0$$

iff

$$\left.\begin{array}{lll} \dot{\psi}_j = \ \ \dot{\psi}_i & , & \dot{a}_2(t) \underset{(3.50a)}{=} l_i \cdot (-\dot{\psi}_j + \dot{\psi}_i) = 0 \\ & & \text{and} \\ & & \ddot{a}_2(t) = 0, \\ \text{or} & & \\ \dot{\psi}_j = -\dot{\psi}_i & , & \dot{a}_2(t) \underset{(3.50a)}{=} 2 \cdot l_i \cdot \dot{\psi}_j \\ & & \text{and} \\ & & \ddot{a}_2(t) = 2 \cdot l_i \cdot \ddot{\psi}_i \end{array}\right\} \textit{(bifurcation).} \quad \begin{array}{r} (3.56\text{d}) \\ \\ \\ (3.56\text{e}) \end{array}$$

For $\ddot{a}_2(t) > 0$, *no motion is possible* for (3.56d) *(lock-up)*. For *Case 2.3c* ($\psi_j = \frac{5}{2}\pi$, $\psi_i = \frac{3}{2}\pi$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,, \boldsymbol{\beta}_{c_2}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}} \end{pmatrix} \right) = \tag{3.56f}$$

$$\left(\dot{\psi}_j - \dot{\psi}_i\right)^2 \cdot \left(\dot{\psi}_j + \dot{\psi}_i\right)^2 \cdot \underbrace{l_j^4 \cdot \left(l_j^2 + 4\right)/2}_{>0} = 0$$

iff

$$\left.\begin{array}{lll} \dot{\psi}_i = \ \ \dot{\psi}_j & , & \dot{a}_2(t) = l_i \cdot (\dot{\psi}_j - \dot{\psi}_i) = 0 \\ & & \text{and} \\ & & \ddot{a}_2(t) = 0, \\ \text{or} & & \\ \dot{\psi}_i = -\dot{\psi}_j & , & \dot{a}_2(t) = 2 \cdot l_i \cdot \dot{\psi}_i \\ & & \text{and} \\ & & \ddot{a}_2(t) = 2 \cdot l_i \cdot \ddot{\psi}_i \end{array}\right\} \textit{(bifurcation).} \quad \begin{array}{r} (3.56\text{g}) \\ \\ \\ (3.56\text{h}) \end{array}$$

For $\ddot{a}_2(t) > 0$, *no motion is possible* for (3.56g) *(lock-up)*. For *Case 2.3d* ($\psi_j = \pi$, $\psi_i = \pi$, and $l_i = l_j$), this implies

$$\det\left((\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{c_2})\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.56i}$$

$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j+\dot\psi_i^2\cdot l_j+\ddot a_2(t)\right)^2\cdot\underbrace{\left(l_i^2+4\right)/2}_{>0}=0 \tag{3.56j}$$

iff

$$\ddot a_2(t)=-l_j\cdot\left(\dot\psi_j^2+\dot\psi_i^2\right)<0 \tag{3.56k}$$

(*bifurcation solutions* for $\dot\psi_i>0$ and $\dot\psi_j<0$, and for $\dot\psi_i<0$ and $\dot\psi_j>0$). For $\ddot a_2>0$, no motion is *possible.*

Additional singular cases are obtained by a *particular control strategy.* For *Case 2.3e* ($\psi_j=\frac{3}{2}\pi+\varepsilon,\ \psi_i=\frac{\pi}{2}+\varepsilon,\ 0<\varepsilon<\frac{\pi}{2}$, and $l_i=l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{c_2})\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.57a}$$
$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_j-\ddot a_2(t)\cdot\sin(\varepsilon)\right)^2\cdot\underbrace{\left(l_j^2+4\right)/2}_{>0}=0$$

iff

$$\ddot a_2(t)=l_j\cdot\left(\dot\psi_j^2-\dot\psi_i^2\right)\Big/\sin\varepsilon=0, \tag{3.57b}$$

where due to (3.50a), (3.52f), and $\psi_j=\psi_i+\pi$,

$$\dot\psi_j=\dot\psi_i. \tag{3.57c}$$

For $\ddot a_2>0$, *no motion is possible.* For *Case 2.3f* ($\psi_j=\frac{5}{2}\pi-\varepsilon,\ \psi_i=\frac{3}{2}\pi-\varepsilon,\ 0<\varepsilon<\frac{\pi}{2}$, and $l_i=l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{c_2})\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.57d}$$
$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_j-\ddot a_2(t)\cdot\sin(\varepsilon)\right)^2\cdot\underbrace{\left(l_j^2+4\right)/2}_{>0}=0$$

iff

$$\ddot a_2(t)=l_i\cdot\left(\dot\psi_j^2-\dot\psi_i^2\right)\Big/\sin\varepsilon=0, \tag{3.57e}$$

where due to (3.50a), (3.52g), and $\psi_j=\psi_i+\pi$,

$$\dot\psi_j=\dot\psi_i. \tag{3.57f}$$

For $\ddot a_2>0$ *no motion is possible.* For *Case 2.3g* ($\psi_j=2\pi,\ \psi_i=\pi$, and $l_i=l_j$), we obtain

$$\det\left(\left(\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{\mathrm{c}_2}\right)\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{\mathrm{c}_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.57g}$$

$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_j-\ddot a_2(t)\right)^2\cdot\left(l_j^2+4\right)/2=0$$

iff

$$\ddot a_2(t)=l_i\cdot\left(\dot\psi_j^2-\dot\psi_i^2\right)=0, \tag{3.57h}$$

where due to (3.50a), (3.52h), and $\psi_j=\psi_i+\pi$,

$$\dot\psi_i=\dot\psi_j. \tag{3.57i}$$

For $\ddot a_2>0$, *no motion is possible*. For *Case 2.3h* ($\psi_j=\frac{3}{2}\pi-\varepsilon,\ \psi_i=\frac{\pi}{2}-\varepsilon$, and $l_i=l_j$), we obtain

$$\det\left(\left(\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{\mathrm{c}_2}\right)\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{\mathrm{c}_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.57j}$$

$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_j+\ddot a_2(t)\cdot\sin(\varepsilon)\right)^2\cdot\underbrace{\left(l_j^2+4\right)/2}_{>0}=0$$

iff

$$\ddot a_2(t)=l_i\cdot\left(\dot\psi_i^2-\dot\psi_j^2\right)/\sin\varepsilon=0, \tag{3.57k}$$

where due to (3.50a), (3.52i), and $\psi_j=\pi+\psi_i$,

$$\dot\psi_i=\dot\psi_j. \tag{3.57l}$$

For $\ddot a_2>0$, *no motion is possible*. For *Case 2.3i* ($\psi_j=\frac{5}{2}\pi+\varepsilon,\ \psi_i=\frac{3}{2}\pi+\varepsilon$, and $l_i=l_j$), we obtain

$$\det\left(\left(\boldsymbol{g}_{2p}\,,\boldsymbol{\beta}_{\mathrm{c}_2}\right)\cdot\begin{pmatrix}\boldsymbol{g}_{2p}^{\mathrm{T}}\\ \boldsymbol{\beta}_{\mathrm{c}_2}^{\mathrm{T}}\end{pmatrix}\right)= \tag{3.57m}$$

$$l_j^2\cdot\left(\dot\psi_j^2\cdot l_j-\dot\psi_i^2\cdot l_j+\ddot a_2(t)\cdot\sin(\varepsilon)\right)^2\cdot\underbrace{\left(l_j^2+4\right)/2}_{>0}=0$$

iff

$$\ddot a_2(t)=l_i\cdot\left(\dot\psi_i^2-\dot\psi_j^2\right)/\sin\varepsilon=0, \tag{3.57n}$$

where due to (3.50a), (3.52j), and $\psi_j=\pi+\psi_i$,

$$\dot\psi_i=\dot\psi_j. \tag{3.57o}$$

For $\ddot a_2>0$, *no motion is possible (lock-up)*. For *Case 2.3j* ($\psi_j=\pi,\ \psi_i=0$, and $l_i=l_j$), we obtain

$$\det\left((\boldsymbol{g}_{2p}\,, \boldsymbol{\beta}_{c_2}) \cdot \begin{pmatrix} \boldsymbol{g}_{2p}^{\mathrm{T}} \\ \boldsymbol{\beta}_{c_2}^{\mathrm{T}} \end{pmatrix} \right) = \tag{3.57p}$$

$$l_j^2 \cdot \left(\dot{\psi}_j^2 \cdot l_j - \dot{\psi}_i^2 \cdot l_j + \ddot{a}_2(t) \right)^2 \cdot \underbrace{\left(l_j^2 + 4 \right) / 2}_{> 0} = 0$$

iff

$$\ddot{a}_2(t) = l_i \cdot \left(\dot{\psi}_i^2 - \dot{\psi}_j^2 \right) = 0, \tag{3.57q}$$

where due to (3.50a), (3.52k), and $\psi_j - \psi_i = \pi$,

$$\dot{\psi}_i = \dot{\psi}_j. \tag{3.57r}$$

For $\ddot{a}_2 > 0$, *no motion is possible (lock-up).*

The above results of *singular situations of the slider–crank mechanism under drive* $a_2(t)$ are collected in *Figures 3.5 and 3.6.*

Comment 3.1.2 (Singularity analysis of the constraint equations): The above singularity analysis of the slider–crank mechanism shows:

1. Pathological behavior is *not restricted to strange or peculiar mechanisms* invented by theoreticians. Even simple *often used mechanisms may have various singularities.*

2. Computational singularity checks are required in general purpose rigid body programs to identify both *singularities of mechanisms* and *inconsistent constraint equations.*

3. A complete singularity analysis, even of comparatively simple mechanisms, may need *comprehensive computational steps* and may be quite cumbersome.

4. The *constraint Jacobian* matrix $\boldsymbol{g}_p$ is of fundamental importance, both for *finding singular behavior* and for *solving the constraint equations.*

5. The occurrence of certain *types of singularities* can be numerically checked by computing determinants of certain matrices $[\boldsymbol{g}_p\,,\ (\boldsymbol{g}_p\,, -\boldsymbol{g}_t) \cdot (\boldsymbol{g}_p\,, -\boldsymbol{g}_t)^{\mathrm{T}},\ (\boldsymbol{g}_p\,, \boldsymbol{\beta}_{\mathrm{c}}) \cdot (\boldsymbol{g}_p\,, \boldsymbol{\beta}_{\mathrm{c}})^{\mathrm{T}}]$ or by checking the rank of some matrices. These computations may be done numerically and sometimes even symbolically as in the above simple slider–crank mechanism.

Besides checking algebraic criteria that include the constraint Jacobian matrix, *lock-up situations* or *bifurcation points* may also be empirically identified by observing a *rapid growth of some velocity* and/or *acceleration coordinates* of a mechanism.

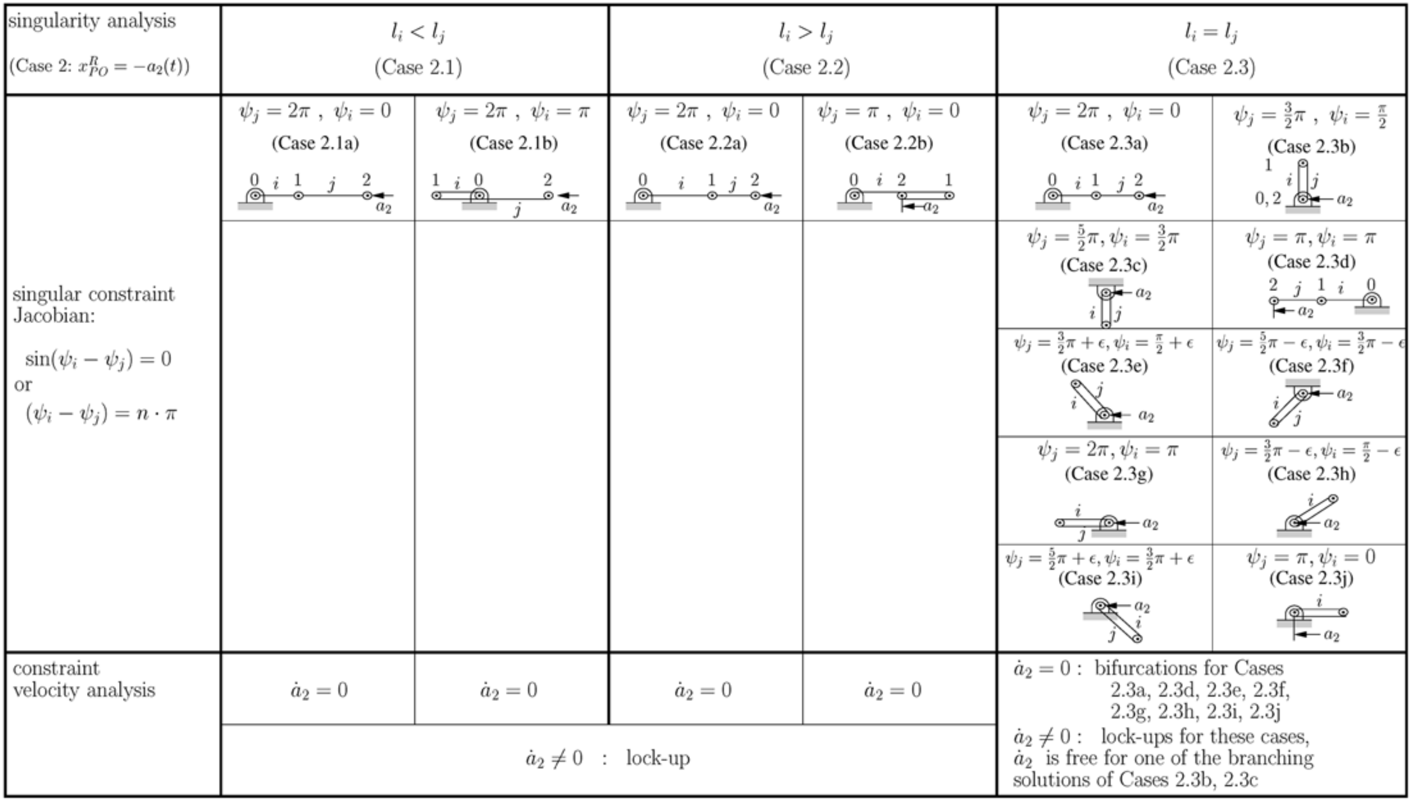


Fig. 3.5: Singular solutions of the slider–crank mechanism of Case 2 (drive $a_2(t)$)

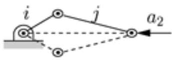
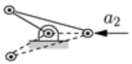
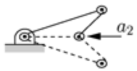
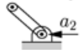
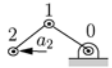
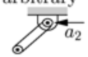
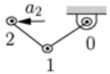
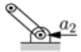
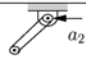
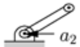
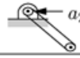
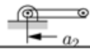
constraint acceleration analysis	$\ddot{a}_2 > 0$ together with and $\dot{\psi}_j > 0$ $\dot{\psi}_i < 0$; or with and $\dot{\psi}_j < 0$ $\dot{\psi}_i > 0$.  bifurcation	$\ddot{a}_2 < 0$ together with and $\dot{\psi}_j > 0$ $\dot{\psi}_i > 0$; or with and $\dot{\psi}_j < 0$ $\dot{\psi}_i < 0$.  bifurcation	$\ddot{a}_2 > 0$ together with and $\dot{\psi}_j > 0$ $\dot{\psi}_i < 0$; or with and $\dot{\psi}_j < 0$ $\dot{\psi}_i > 0$.  bifurcation	$\ddot{a}_2 < 0$ together with and $\dot{\psi}_j > 0$ $\dot{\psi}_i < 0$; or with and $\dot{\psi}_j < 0$ $\dot{\psi}_i > 0$.  bifurcation	Case 2.3a: $\ddot{a}_2 > 0$  for $\dot{\psi}_j > 0$ and $\dot{\psi}_i < 0$ or for $\dot{\psi}_j < 0$ and $\dot{\psi}_i > 0$ bifurcation $\ddot{a}_2 < 0$ lock-up Case 2.3b: $\ddot{a}_2$ arbitrary $\dot{a}_2 = 0$ or $\dot{a}_2 = 2 \cdot l_i \dot{\psi}_i$  or  bifurcation Case 2.3c: $\ddot{a}_2$ arbitrary $\dot{a}_2 = 0$ or $\dot{a}_2 = 2 \cdot l_i \dot{\psi}_i$  or  bifurcation Case 2.3d: (compare Case 2.3a) $\ddot{a}_2 < 0$ bifurcation  $\ddot{a}_2 > 0$ lock-up
	$\ddot{a}_2 < 0$ lock-up	$\ddot{a}_2 > 0$ lock-up	$\ddot{a}_2 < 0$ lock-up	$\ddot{a}_2 > 0$ lock-up	Case 2.3e: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$ Case 2.3f: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$ Case 2.3g: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$ Case 2.3h: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$ Case 2.3i: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$ Case 2.3j: $\ddot{a}_2 = 0$ lock-up  $\dot{\psi}_i = \dot{\psi}_j$

Fig. 3.6: Constraint acceleration analysis of the slider–crank mechanism of Case 2 (drive $a_2(t)$)

3.2 Constraint reaction forces and torques of mechanisms

In this section, basic relations of reaction forces and torques will be briefly discussed for *planar* and *spatial* mechanisms. By the definition of kinematic analysis, forces and torques are only considered in the *dynamic analysis* of mechanisms and not in pure kinematic analysis; this section is concerned with certain aspects of *dynamic analysis.*

As a consequence and in agreement with Sections 3.1.1.3 and 3.1.2.2, the inequality

$$n_c < 3n_b \qquad \textit{(planar case)} \tag{3.58a}$$

or

$$n_c < 6n_b \qquad \textit{(spatial case)} \tag{3.58b}$$

will be assumed.

3.2.1 Constraint reaction forces of *planar* mechanisms

Constraints connected to rigid bodies produce constraint reaction forces (and torques) that act on these bodies. Constraint reaction forces clearly depend on the associated constraint equations, and so it should be possible to express them in terms of these constraint equations. This can be achieved if the reactions of the constraint forces are expressed with respect to the same coordinates (frames) as the vectors of the generalized coordinates

$$\begin{aligned} \boldsymbol{p} &=: \left(\boldsymbol{r}^{\mathrm{T}}, \psi^{\mathrm{T}}\right)^{\mathrm{T}} && (3.59\mathrm{a}) \\ &= \left(x_{P_1O}, y_{P_1O}, \psi_1; x_{P_2O}, y_{P_2O}, \psi_2; \ldots; x_{P_{n_b}O}, y_{P_{n_b}O}, \psi_{n_b}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \end{aligned}$$

and the associated velocities

$$\boldsymbol{v} := \dot{\boldsymbol{p}} = \left(\dot{x}_{P_1O}, \dot{y}_{P_2O}, \dot{\psi}_1; \ldots; \dot{x}_{P_{n_b}O}, \dot{y}_{P_{n_b}O}, \dot{\psi}_{n_b}\right)^{\mathrm{T}} =: \left(\dot{\boldsymbol{r}}^{\mathrm{T}}, \dot{\psi}^{\mathrm{T}}\right)^{\mathrm{T}} \tag{3.59b}$$

with

$$\omega_{zi} := \dot{\psi}_i. \tag{3.59c}$$

It will be assumed that the n_c kinematic constraint equations are *consistent* and *independent.* Then, due to

$$\operatorname{rank} \boldsymbol{g}_p(\boldsymbol{p}) = n_c, \tag{3.60a}$$

the $3n_c$ constraint equations

$$\boldsymbol{g}(\boldsymbol{p}) \equiv \boldsymbol{0} \quad , \quad \boldsymbol{g}_p(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} = \boldsymbol{\alpha}_{\mathrm{c}}(\boldsymbol{p}) \quad \text{and} \quad \boldsymbol{g}_p(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}) \tag{3.60b}$$

can be uniquely solved with respect to the $3n_c$ (dependent) coordinates of p, $\boldsymbol{v} = \dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{v}} = \ddot{\boldsymbol{p}}$, for given independent coordinates of these vectors.

The *work* done by a force $\boldsymbol{F}$ providing a displacement $\boldsymbol{p}$ of a system is

$$W = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{p} \in \mathbb{R}^1 . \tag{3.61a}$$

A *virtual displacement* $\delta \boldsymbol{r}_P$ of a system is defined as a small *(infinitesimal)* variation in the location $\boldsymbol{r}_P$ of a system *at a fixed time instant* t*, consistent with the constraints and forces* acting on the system. Expressing $\boldsymbol{r}_P = \boldsymbol{r}_P(\boldsymbol{p}(t), t)$ in terms of the generalized coordinates $\boldsymbol{p}$, small *physical displacements* of $\boldsymbol{r}_{\mathrm{P}}$ are

$$\mathrm{d}\boldsymbol{r}_P = \frac{\partial \boldsymbol{r}_P}{\partial \boldsymbol{p}} \cdot \mathrm{d}\boldsymbol{p} + \frac{\partial \boldsymbol{r}_P}{\partial t} \cdot \mathrm{d}t, \tag{3.61b}$$

whereas a *virtual displacement* $\delta \boldsymbol{r}_P$ may be considered as a partial derivative of $\boldsymbol{r}_P$ with only the physical coordinates varied and the *time* t *kept constant*:

$$\delta \boldsymbol{r}_P = \frac{\partial \boldsymbol{r}_P}{\partial \boldsymbol{p}} \cdot \delta \boldsymbol{p}. \tag{3.61c}$$

The *virtual work* done by constraint reaction forces and torques ${}^{\mathrm{c}}\boldsymbol{f}$ is zero:

$$ {}^{\mathrm{c}}W = {}^{\mathrm{c}}\boldsymbol{f}^{\mathrm{T}} \cdot \delta \boldsymbol{p} = 0 \qquad \textit{(virtual work principle)}. \tag{3.62}$$

The *Taylor series expansion* of the constraint equations with respect to $\boldsymbol{p}$ provides, to a first-order approximation,

$$\boldsymbol{g}(\boldsymbol{p} + \delta \boldsymbol{p}) = \boldsymbol{g}(\boldsymbol{p}) + \boldsymbol{g}_p(\boldsymbol{p}) \cdot \delta \boldsymbol{p},$$

and together with the constraint equation

$$\boldsymbol{g}(\boldsymbol{p}) \equiv \boldsymbol{0},$$

and by the definition of the virtual displacement $\delta \boldsymbol{p}$, i.e., for

$$\boldsymbol{g}(\boldsymbol{p} + \delta \boldsymbol{p}) \equiv \boldsymbol{0}$$

the relation

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \delta \boldsymbol{p} \equiv \boldsymbol{0}. \tag{3.63}$$

The coordinate vector $\boldsymbol{p} \in \mathbb{R}^{3n_b}$ may be (conceptually) partitioned into n_c *dependent* coordinates $\boldsymbol{u} \in \mathbb{R}^{n_c}$ and $(3n_b - n_c)$ *independent* coordinates $\boldsymbol{w} \in \mathbb{R}^{3n_b - n_c}$. Then

$$\boldsymbol{p} = \left(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{w}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \delta \boldsymbol{p} = \left(\delta \boldsymbol{u}^{\mathrm{T}}, \delta \boldsymbol{w}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \text{and} \quad \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}} =: \boldsymbol{g}_p = (\boldsymbol{g}_u, \boldsymbol{g}_w), \tag{3.64}$$

with a regular matrix $\partial \boldsymbol{g}/\partial \boldsymbol{u} = \boldsymbol{g}_u \in \mathbb{R}^{n_c,n_c}$, and

$$^{\mathrm{c}}\boldsymbol{f} =: \left({}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}}, {}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad {}^{\mathrm{c}}_{u}\boldsymbol{f} \in \mathbb{R}^{n_c} \quad , \quad {}^{\mathrm{c}}_{w}\boldsymbol{f} \in \mathbb{R}^{3n_b - n_c}.$$

Using these notations, (3.62) and (3.63) can be written as

$${}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}} \cdot \delta\boldsymbol{u} = -{}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \cdot \delta\boldsymbol{w} \tag{3.65a}$$

and

$$\boldsymbol{g}_u \cdot \delta\boldsymbol{u} = -\boldsymbol{g}_w \cdot \delta\boldsymbol{w}, \tag{3.65b}$$

or as

$$\begin{pmatrix} {}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}_u \end{pmatrix} \cdot \delta\boldsymbol{u} = -\begin{pmatrix} {}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}_w \end{pmatrix} \cdot \delta\boldsymbol{w}, \tag{3.65c}$$

with

$$\begin{pmatrix} {}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}_u \end{pmatrix} \in \mathbb{R}^{n_c+1,n_c} \quad \text{and} \quad \begin{pmatrix} {}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}_w \end{pmatrix} \in \mathbb{R}^{n_c+1,3n_b-n_c}.$$

As the last n_c rows of the matrix $\begin{pmatrix} {}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}} \\ \boldsymbol{g}_u \end{pmatrix} \in \mathbb{R}^{n_c+1,n_c}$ are, by assumption, linearly independent, the first row ${}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}}$ of this matrix can be written as a linear combination of the last n_c rows with the coefficient vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{n_c})^{\mathrm{T}} \neq \boldsymbol{0}$. Then

$${}^{\mathrm{c}}_{u}\boldsymbol{f}^{\mathrm{T}} = \boldsymbol{\lambda}^{\mathrm{T}} \cdot \boldsymbol{g}_u \qquad \text{or} \qquad {}^{\mathrm{c}}_{u}\boldsymbol{f} = \boldsymbol{g}_u^{\mathrm{T}} \cdot \boldsymbol{\lambda}. \tag{3.66}$$

The coefficient vector $\boldsymbol{\lambda}$ is known as the vector of the *Lagrange multipliers.* Substituting (3.66) into (3.65a) yields

$$\boldsymbol{\lambda}^{\mathrm{T}} \cdot \boldsymbol{g}_u \cdot \delta\boldsymbol{u} = -{}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \cdot \delta\boldsymbol{w}, \tag{3.67a}$$

and together with (3.65b) yields

$$\boldsymbol{\lambda}^{\mathrm{T}} \cdot \boldsymbol{g}_w \cdot \delta\boldsymbol{w} = {}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \cdot \delta\boldsymbol{w}. \tag{3.67b}$$

The relation (3.67b) holds for *arbitrary* variations $\delta\boldsymbol{w}$ of the (*independent*) coordinates $\boldsymbol{w}$. Then

$$\boldsymbol{\lambda}^{\mathrm{T}} \cdot \boldsymbol{g}_w = {}^{\mathrm{c}}_{w}\boldsymbol{f}^{\mathrm{T}} \qquad \text{or} \qquad {}^{\mathrm{c}}_{w}\boldsymbol{f} = \boldsymbol{g}_w^{\mathrm{T}} \cdot \boldsymbol{\lambda}. \tag{3.68}$$

Appending (3.68) to (3.66) yields

$$\begin{pmatrix} {}^{\mathrm{c}}_{u}\boldsymbol{f} \\ {}^{\mathrm{c}}_{w}\boldsymbol{f} \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_u^{\mathrm{T}} \\ \boldsymbol{g}_w^{\mathrm{T}} \end{pmatrix} \cdot \boldsymbol{\lambda} \qquad \text{or} \qquad {}^{\mathrm{c}}\boldsymbol{f} = \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda} \tag{3.69}$$

as the *constraint reaction forces*, expressed in terms of the constraint Jacobian of the constraint equations and of a vector of Lagrange multipliers.

The *Lagrange multipliers* $\boldsymbol{\lambda}$ may be *computed* from the model equations in *DAE form*, or *eliminated* from the model equations to produce model equations in *DE form*.

In any case, the signs of the components λ_i of $\boldsymbol{\lambda}$ are not known before the model equations are solved. As a consequence, the directions of the constraint reaction forces and torques ${}^{c}\boldsymbol{f}$ are also not known in advance. This implies that the arrows of the constraint reaction forces drawn in a mechanical network of a system do not tell anything about their true directions. On the other hand, due to *Newton's law that "action is equal to reaction"*, the constraint reaction forces and torques, produced by a joint that connects two bodies i and j, are equal in magnitude and act in opposite directions on these bodies (Figure 3.7).

3.2.2 Constraint reaction forces of *spatial* mechanisms

Spatial mechanisms including n_b rigid bodies are specified by a *Cartesian coordinate vector*

$$\begin{aligned}\boldsymbol{p} = \Big(& x^R_{P_1O}\,,\, y^R_{P_1O}\,,\, z^R_{P_1O}\,,\, \varphi_1\,,\, \theta_1\,,\, \psi_1\,,\ldots,\\ & x^R_{P_{n_b}O}\,,\, y^R_{P_{n_b}O}\,,\, z^R_{P_{n_b}O}\,,\, \varphi_{n_b}\,,\, \theta_{n_b}\,,\, \psi_{n_b}\Big)^{\mathrm{T}} \in \mathbb{R}^{6n_b}\end{aligned} \tag{3.70a}$$

or

$$\boldsymbol{p} = \left(\boldsymbol{p}_1^{\mathrm{T}}, \boldsymbol{p}_2^{\mathrm{T}}, \ldots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b} \quad , \quad \boldsymbol{p}_i = \left({\boldsymbol{r}^R_{P_iO}}^{\mathrm{T}}\,,\, \boldsymbol{\eta}_i^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^6$$

with

$$\boldsymbol{r}^R_{P_iO} = \left(x^R_{P_iO}\,,\, y^R_{P_iO}\,,\, z_{P_iO}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{\eta}_i = (\varphi_i\,,\, \theta_i\,,\, \psi_i)^{\mathrm{T}}\,,$$

and with the *velocity vector*

$$\boldsymbol{v} := \left(\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b} \tag{3.70b}$$

with

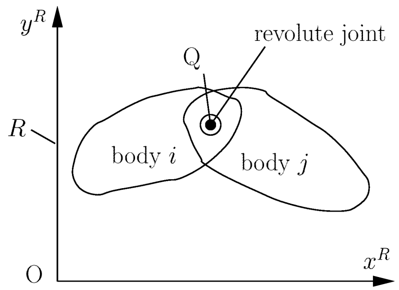
$$\dot{\boldsymbol{p}}_i = \left((\dot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\,,\, \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{v}_i = \left((\dot{\boldsymbol{r}}^R_{Q_iO})^{\mathrm{T}}\,,\, (\boldsymbol{\omega}^{L_i}_{L_iR})^{\mathrm{T}}\right)^{\mathrm{T}},$$

$$\boldsymbol{v} = \boldsymbol{T}(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{p}}, \tag{3.70c}$$

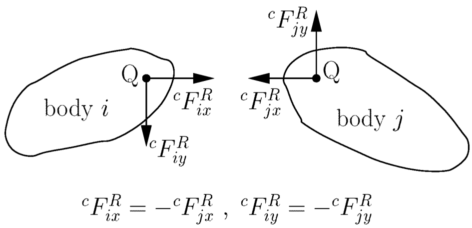
$$\boldsymbol{T}(\boldsymbol{p}) := \mathrm{diag}\,\left(\boldsymbol{T}_1(\boldsymbol{p}_1)\,,\, \cdots\,,\, \boldsymbol{T}_{n_b}(\boldsymbol{p}_{n_b})\right),$$

$$\boldsymbol{T}_i(\boldsymbol{p}_i) = \boldsymbol{T}_i(\boldsymbol{\eta}_i) = \mathrm{diag}\,\left(\mathbf{I}_3\,,\, \boldsymbol{H}_{i2}(\boldsymbol{\eta}_i) \cdot \boldsymbol{A}^{RL_i}\right),$$

$$\boldsymbol{\omega}^{L_i}_{L_iR} = \left(\omega^{L_i}_{xi}\,,\, \omega^{L_i}_{yi}\,,\, \omega^{L_i}_{zi}\right)^{\mathrm{T}} = \boldsymbol{A}^{L_iR} \cdot \boldsymbol{H}_i^{-1} \cdot (\boldsymbol{\eta}_i) \cdot \dot{\boldsymbol{\eta}}_i, \tag{3.70d}$$



(a) Two bodies connected by a revolute joint



(b) Free-body diagram for the constraint reaction forces

Fig. 3.7: Revolute joint and the associated constraint reaction forces

$$\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) = \begin{pmatrix} \mathrm{c}_{i2}\mathrm{c}_{i3} & , \ \mathrm{c}_{i1}\mathrm{s}_{i3} + s_{i1}\mathrm{s}_{i2}\mathrm{c}_{i3} & , \ \mathrm{s}_{i1}\mathrm{s}_{i3} - \mathrm{c}_{i1}\mathrm{s}_{i2}\mathrm{c}_{i3} \\ -\mathrm{c}_{i2}\mathrm{s}_{i3} & , \ \mathrm{c}_{i1}\mathrm{c}_{i3} - \mathrm{s}_{i1}\mathrm{s}_{i2}\mathrm{s}_{i3} & , \ \mathrm{s}_{i1}\mathrm{c}_{i3} + \mathrm{c}_{i1}\mathrm{s}_{i2}\mathrm{s}_{i3} \\ \mathrm{s}_{i2} & , \ -\mathrm{s}_{i1}\mathrm{c}_{i2} & , \ \mathrm{c}_{i1}c_{i2} \end{pmatrix}, \tag{3.70e}$$

$$\boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) := \begin{pmatrix} 1 \ , & 0 \ , & \mathrm{s}_{i2} \\ 0 \ , & \mathrm{c}_{i1} \ , & -\mathrm{s}_{i1}\cdot\mathrm{c}_{i2} \\ 0 \ , & \mathrm{s}_{i1} \ , & \mathrm{c}_{i1}\cdot\mathrm{c}_{i2} \end{pmatrix} \quad , \quad \dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot{\varphi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{pmatrix} \tag{3.70f}$$

and

$$\mathrm{s}_{i1} := \sin\varphi_{iLR} \ , \quad \mathrm{s}_{i2} := \sin\theta_{iLR} \ , \quad \mathrm{s}_{i3} := \sin\psi_{iLR} \ , \tag{3.70g}$$
$$\mathrm{c}_{i1} := \cos\varphi_{iLR} \ , \quad \mathrm{c}_{i2} := \cos\theta_{iLR} \ , \quad \text{and} \quad \mathrm{c}_{i3} := \cos\psi_{iLR} \ ,$$
for *Bryant angles* φ_{iLR} , θ_{iLR} , ψ_{iLR}.

Then the *principle of virtual work*

$$ {}^{\mathrm{c}}W = {}^{\mathrm{c}}\bar{\boldsymbol{f}}^{\mathrm{T}} \cdot \delta\boldsymbol{p} = 0 \tag{3.71a}$$

introduced in Section 3.2.1 with

$$^{\mathrm{c}}\bar{\boldsymbol{f}} = \left({}^{\mathrm{c}}\bar{\boldsymbol{f}}_1^{\mathrm{T}}\,, \ldots {}^{\mathrm{c}}\bar{\boldsymbol{f}}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}\,, \tag{3.71b}$$

$$^{\mathrm{c}}\bar{\boldsymbol{f}}_i = \left({}^{\mathrm{c}}\boldsymbol{f}_{xi}^R\,, {}^{\mathrm{c}}\boldsymbol{f}_{yi}^R\,, {}^{\mathrm{c}}\boldsymbol{f}_{zi}^R\,, {}^{\mathrm{c}}M_{xi}\,, {}^{\mathrm{c}}M_{yi}\,, {}^{\mathrm{c}}M_{zi}\right)^{\mathrm{T}} \in \mathbb{R}^6\,, \tag{3.71c}$$

and with the coordinate vector $\boldsymbol{p}$ (3.70a) can be written as

$$\begin{aligned} {}^{\mathrm{c}}W = \sum_{i=1}^{n_b} \Big(& {}^{\mathrm{c}}\boldsymbol{f}_{xi}^R \cdot \delta x_{P_i0}^R + {}^{\mathrm{c}}\boldsymbol{f}_{yi}^R \cdot \delta y_{P_i0}^R + {}^{\mathrm{c}}\boldsymbol{f}_{zi}^R \cdot \delta z_{P_i0}^R \\ & + {}^{\mathrm{c}}M_{xi} \cdot \delta\varphi_i + {}^{\mathrm{c}}M_{yi} \cdot \delta\theta_i + {}^{\mathrm{c}}M_{zi} \cdot \delta\psi_i \Big) \end{aligned} \tag{3.71d}$$

with

$$^{\mathrm{c}}M_{ji}\quad,\quad j = x\,, y\,, z\,,\quad \text{and}\quad \delta\varphi_i\,, \delta\theta_i\,, \delta\psi_i\quad,\quad i = 1\,, \ldots\,, n_b \tag{3.71e}$$

associated with the chosen rotation axes. Consider n_c *consistent and independent constraint position equations*

$$\begin{aligned} \boldsymbol{g} : \mathbb{R}^{6n_b} &\longrightarrow \mathbb{R}^{n_c} \\ \Uparrow \quad &\qquad \Uparrow \\ \boldsymbol{p} &\longmapsto \boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{0} \end{aligned} \tag{3.72a}$$

with

$$\operatorname{rank} \boldsymbol{g}_p(\boldsymbol{p}) = n_c, \quad \boldsymbol{g}_p \in \mathbb{R}^{n_c\,, 6n_b}. \tag{3.72b}$$

The *Taylor series expansion of the constraint equations* with respect to $\boldsymbol{p}$ provides, to a first order approximation (compare the derivation of Equation 3.63), the relation

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \delta\boldsymbol{p} \equiv \boldsymbol{0}. \tag{3.73a}$$

The coordinate vector $\boldsymbol{p} \in \mathbb{R}^{6n_b}$ may be (conceptually) partitioned into n_c *dependent* coordinates $\boldsymbol{u} \in \mathbb{R}^{n_c}$ and $(6n_b - n_c)$ *independent* coordinates $\boldsymbol{w} \in \mathbb{R}^{6n_b - n_c}$. Then

$$\boldsymbol{p} = \left(\boldsymbol{u}^{\mathrm{T}}\,, \boldsymbol{w}^{\mathrm{T}}\right)^{\mathrm{T}}, \quad \delta\boldsymbol{p} = \left(\delta\boldsymbol{u}^{\mathrm{T}}\,, \delta\boldsymbol{w}^{\mathrm{T}}\right)^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{g}_p = (\boldsymbol{g}_u\,, \boldsymbol{g}_w)\,, \tag{3.74a}$$

with the regular matrix $\boldsymbol{g}_u \in \mathbb{R}^{n_c, n_c}$, and with

$$^{\mathrm{c}}\bar{\boldsymbol{f}} = \left({}^{\mathrm{c}}_{u}\bar{\boldsymbol{f}}^{\mathrm{T}}\,, {}^{\mathrm{c}}_{w}\bar{\boldsymbol{f}}^{\mathrm{T}}\right)^{\mathrm{T}}\quad,\quad {}^{\mathrm{c}}_{u}\bar{\boldsymbol{f}} \in \mathbb{R}^{n_c}\quad,\quad {}^{\mathrm{c}}_{w}\bar{\boldsymbol{f}} \in \mathbb{R}^{6n_b - n_c}. \tag{3.74b}$$

Using these notations, provides (compare the derivation of Equation 3.69) the equation

$$\begin{pmatrix} {}^{\mathrm{c}}_{u}\bar{\boldsymbol{f}} \\ {}^{\mathrm{c}}_{w}\bar{\boldsymbol{f}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{g}_u^{\mathrm{T}} \\ \boldsymbol{g}_w^{\mathrm{T}} \end{pmatrix} \cdot \boldsymbol{\lambda} \qquad \text{or} \qquad {}^{\mathrm{c}}\bar{\boldsymbol{f}} = \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda} \tag{3.75}$$

as the *constraint reaction forces* expressed in terms of the constraint Jacobian matrix and of the vector of the Lagrange multipliers.

Divergent from the *planar* case, *spatial* equations of motion are written in the *velocity coordinate* $\boldsymbol{v}$ instead of $\dot{\boldsymbol{p}}$. This must be taken into account in the formulation of the constraint reaction forces. Starting from the *virtual power* represented in the coordinates $\dot{\boldsymbol{p}}$ and $\boldsymbol{v}$ yields, for a body i, the relations

$$({}^{\mathrm{c}}M_{xi}\,,\ {}^{\mathrm{c}}M_{yi}\,,\ {}^{\mathrm{c}}M_{zi}) \cdot \begin{pmatrix} \delta\dot{\varphi}_i \\ \delta\dot{\theta}_i \\ \delta\dot{\psi}_i \end{pmatrix} = \left({}^{\mathrm{c}}M_{xi}^{L_i}\,,\ {}^{\mathrm{c}}M_{yi}^{L_i}\,,\ {}^{\mathrm{c}}M_{zi}^{L_i}\right) \cdot \begin{pmatrix} \delta\omega_{xL_iR}^{L_i} \\ \delta\omega_{yL_iR}^{L_i} \\ \delta\omega_{zL_iR}^{L_i} \end{pmatrix} \tag{3.76a}$$

with

$$\delta\boldsymbol{\omega}_{L_iR}^{L_i} = \boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) \cdot \delta\dot{\boldsymbol{\eta}}_i. \tag{3.76b}$$

Inserting (3.76b) into (3.76a) yields

$$\begin{aligned} &({}^{\mathrm{c}}M_{xi}\,,\ {}^{\mathrm{c}}M_{yi}\,, {}^{\mathrm{c}}M_{zi}) \cdot \left(\delta\dot{\varphi}_i\,,\, \delta\dot{\theta}_i\,,\, \delta\dot{\psi}_i\right)^{\mathrm{T}} \\ &\quad = \left({}^{\mathrm{c}}M_{xi}^{L_i}\,,\ {}^{\mathrm{c}}M_{yi}^{L_i}\,,\ {}^{\mathrm{c}}M_{zi}^{L_i}\right) \cdot \boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) \cdot \left(\delta\dot{\varphi}_i\,,\, \delta\dot{\theta}_i\,,\, \delta\dot{\psi}_i\right)^{\mathrm{T}}. \end{aligned} \tag{3.76c}$$

As this equation holds for arbitrary variations $\delta\dot{\boldsymbol{\eta}}_i$, we have

$$({}^{\mathrm{c}}M_{xi}\,,\ {}^{\mathrm{c}}M_{yi}\,,\ {}^{\mathrm{c}}M_{zi})^{\mathrm{T}} = \left(\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i)\right)^{\mathrm{T}} \cdot \left({}^{\mathrm{c}}M_{xi}^{L_i}\,,\ {}^{\mathrm{c}}M_{yi}^{L_i}\,,\ {}^{\mathrm{c}}M_{zi}^{L_i}\right)^{\mathrm{T}}. \tag{3.76d}$$

Combining the constraint reaction forces and torques for the virtual translations and rotations yields, together with (3.71d) and (3.76d), the relation

$$\underbrace{\begin{pmatrix} {}^{\mathrm{c}}\boldsymbol{f}_{xi}^{R} \\ {}^{\mathrm{c}}\boldsymbol{f}_{yi}^{R} \\ {}^{\mathrm{c}}\boldsymbol{f}_{zi}^{R} \\ {}^{\mathrm{c}}M_{xi} \\ {}^{\mathrm{c}}M_{yi} \\ {}^{\mathrm{c}}M_{zi} \end{pmatrix}}_{=:\,{}^{\mathrm{c}}\bar{\boldsymbol{f}}_i} = \begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ & & \\ \mathbf{0}_{3,3} & , & \left(\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i)\right)^{\mathrm{T}} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} {}^{\mathrm{c}}\boldsymbol{f}_{xi}^{R} \\ {}^{\mathrm{c}}\boldsymbol{f}_{yi}^{R} \\ {}^{\mathrm{c}}\boldsymbol{f}_{zi}^{R} \\ {}^{\mathrm{c}}M_{xi}^{L_i} \\ {}^{\mathrm{c}}M_{yi}^{L_i} \\ {}^{\mathrm{c}}M_{zi}^{L_i} \end{pmatrix}}_{=:\,{}^{\mathrm{c}}\boldsymbol{f}_i}, \tag{3.76e}$$

or

$$
{}^{\mathrm{c}}\bar{\boldsymbol{f}}_i = \underbrace{\begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ & & \\ \mathbf{0}_{3,3} & , & \left(\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i)\right)^{\mathrm{T}} \end{pmatrix}}_{=: \left(\boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{\eta}_i)\right)^{-1}} \cdot {}^{\mathrm{c}}\boldsymbol{f}_i ,
$$

or

$$
{}^{\mathrm{c}}\bar{\boldsymbol{f}}_i = \left(\boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{\eta}_i)\right)^{-1} \cdot {}^{\mathrm{c}}\boldsymbol{f}_i \underset{(3.75)}{=} \boldsymbol{g}_{ip}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda}_i , \tag{3.76f}
$$

and finally the relation

$$
{}^{\mathrm{c}}\boldsymbol{f} = \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda} . \tag{3.77}
$$

4. Dynamics of planar and spatial rigid-body systems

In this chapter the equations of *spatial and planar motion of unconstrained rigid bodies* will be derived based on the *laws of Newton and Euler* (such as in standard textbooks like [44], [56], [57], [58], [59], [60], [61], [62], [63], [64], [65]). The equations of motion will be written with respect to a general body-fixed reference point P_i ($\mathrm{P}_i \neq \mathrm{C}_i$, C_i is the center of mass) of a body i. In *Section 4.1* the notions of *linear momentum* and *angular momentum* of a rigid body will be introduced and rewritten in a form suitable for representing the Newton–Euler equations in a desired form. Here the notions of the *center of mass* and the *inertia matrix* of a rigid body will also be introduced, and some properties of this matrix will be briefly discussed. In *Section 4.2* the *Newton–Euler equations* will be derived for planar and spatial motion of a rigid body represented with respect to the reference points P_i and C_i. (In *Appendix A.2* the *Lagrange formalism*, applied to a rigid body under spatial motion, will be briefly discussed). The equations of motion of unconstrained and constrained planar and spatial rigid-body mechanisms will be collected in *Section 4.3*, combining the *kinematic constraint equations* of *Section 3.1* and the *rigid-body equations* of *Section 4.2.4*. This provides model equations in *DE form* for *unconstrained* rigid bodies and in *DAE form* for *constrained rigid bodies* and *rigid-body mechanisms.* A few aspects concerning the *numerical solution of DAEs* will be briefly discussed in *Section 4.4*.

4.1 Linear momentum and angular momentum of a rigid body

This section considers a rigid body of mass m and volume V, a point P fixed on the body, and a vector $\boldsymbol{r}_{PO}$ from the origin O of the inertial frame R to P (cf. Figure 4.1).

4.1.1 Linear momentum

Consider a mass element of the rigid body of mass $\mathrm{d}m$ located at an arbitrary point Q of the rigid body, specified by the vector

$$\boldsymbol{r} := \boldsymbol{r}_{QO}. \tag{4.1}$$

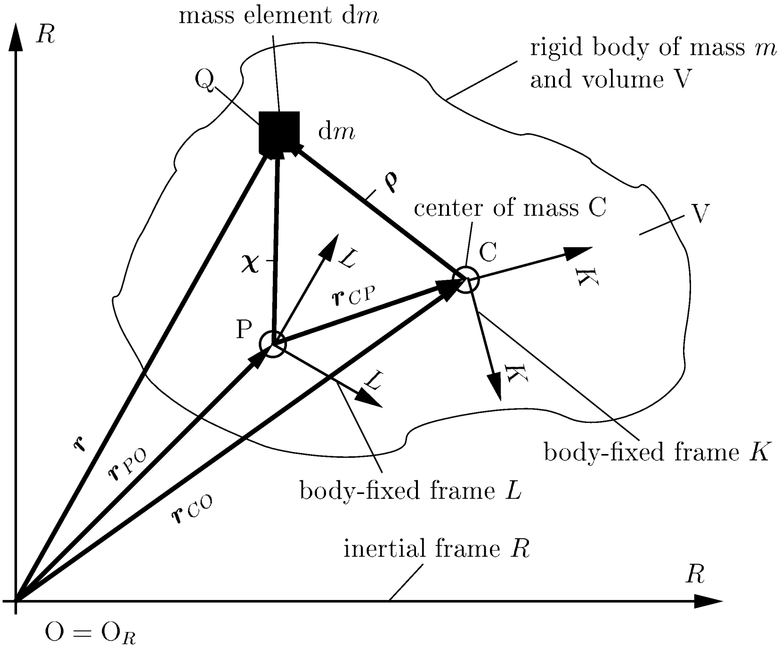


Fig. 4.1: Vector diagram used in the definitions of linear momentum and angular momentum

Then the expression

$$\mathrm{P}_O^R := \int {}^R\dot{\boldsymbol{r}}^R \, \mathrm{d}m \tag{4.2}$$

is called *linear momentum* of the rigid body with respect to the origin O of frame R, represented in R, and

$$\boldsymbol{L}_O^R := \int \tilde{\boldsymbol{r}}^R \cdot {}^R\dot{\boldsymbol{r}}^R \, \mathrm{d}m \tag{4.3}$$

is called *angular momentum* of the rigid body with respect to O of R and represented in R, where the velocity vector ${}^R\dot{\boldsymbol{r}}^R$ in (4.2) and (4.3) is measured in R and represented in R. Consider a second frame L with origin $\mathrm{O}_L = P$, and fixed on the body. Then the vector $\boldsymbol{r}$ can be written as (Figure 4.1)

$$\boldsymbol{r}^R := \boldsymbol{r}_{PO}^R + \boldsymbol{\chi}^R = \boldsymbol{r}_{PO}^R + \boldsymbol{A}^{RL} \cdot \boldsymbol{\chi}^L. \tag{4.4a}$$

The velocity vector ${}^R\dot{\boldsymbol{r}}^R$ is

$${}^R\dot{\boldsymbol{r}}^R = {}^R\dot{\boldsymbol{r}}_{PO}^R + {}^R\dot{\boldsymbol{\chi}}^R \tag{4.4b}$$

or

$$ {}^{R}\dot{\boldsymbol{r}}^{R} = {}^{R}\dot{\boldsymbol{r}}^{R}_{PO} + \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{\chi}^{L} + \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{\chi}}^{L}. \tag{4.4c} $$

Due to the *rigid-body property*

$$ {}^{L}\dot{\boldsymbol{\chi}}^{L} \equiv \mathbf{0} \tag{4.5} $$

and to the relation

$$ \dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR}, $$

the resulting velocity vector is

$$ {}^{R}\dot{\boldsymbol{r}}^{R} = {}^{R}\dot{\boldsymbol{r}}^{R}_{PO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \boldsymbol{\chi}^{L}. \tag{4.6} $$

Then the *linear momentum* $\boldsymbol{P}^{R}_{O}$ of the rigid body can be written as

$$ \boldsymbol{P}^{R}_{O} = \int {}^{R}\dot{\boldsymbol{r}}^{R}\,\mathrm{d}m = {}^{R}\dot{\boldsymbol{r}}^{R}_{PO} \cdot \int \mathrm{d}m + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \int \boldsymbol{\chi}^{L}\,\mathrm{d}m $$

or as

$$ \begin{aligned} \boldsymbol{P}^{R}_{O} &= m \cdot {}^{R}\dot{\boldsymbol{r}}^{R}_{PO} + m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \boldsymbol{r}^{L}_{CP} \\ &= m \cdot \left({}^{R}\dot{\boldsymbol{r}}^{R}_{PO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^{L}_{LR} \cdot \boldsymbol{r}^{L}_{CP} \right) \end{aligned} \tag{4.7a} $$

with C as the *center of mass* of the body, with

$$ \boldsymbol{r}^{L}_{CP} := \frac{1}{m} \cdot \int \boldsymbol{\chi}^{L}\,\mathrm{d}m \tag{4.8} $$

as the vector from the origin P of L to C, and with

$$ m := \int \mathrm{d}m \tag{4.9} $$

as the mass of the body. For the special case P = C, where the reference point P is chosen as the center of mass C ($\boldsymbol{r}_{CP} \equiv \mathbf{0}$), a simplified representation of $\boldsymbol{P}^{R}_{O}$ is obtained:

$$ \boldsymbol{P}^{R}_{O} = m^{R} \cdot {}^{R}\dot{\boldsymbol{r}}^{R}_{CO}. \tag{4.7b} $$

4.1.2 Angular momentum

The *angular momentum* of the rigid body

$$ \boldsymbol{L}^{R}_{O} = \int \tilde{\boldsymbol{r}}^{R} \cdot {}^{R}\dot{\boldsymbol{r}}^{R} \cdot \mathrm{d}m \tag{4.3} $$

can be written together with (4.4a) and (4.4b) as

$$\boldsymbol{L}_O^R = \int \left(\tilde{\boldsymbol{r}}_{PO}^R + \tilde{\boldsymbol{\chi}}^R\right) \cdot \left({}^R\dot{\boldsymbol{r}}_{PO}^R + {}^R\dot{\boldsymbol{\chi}}^R\right) \,\mathrm{d}m,$$

or as

$$\begin{aligned} \boldsymbol{L}_O^R = \; & \int \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m + \int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m \\ & + \int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m + \int \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m. \end{aligned} \tag{4.10}$$

The different terms of (4.10) can be rewritten as follows:

1. $$\int \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m = \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \cdot \int \mathrm{d}m = m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R. \tag{4.11}$$

2. $$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m = \left(\int \tilde{\boldsymbol{\chi}}^R \,\mathrm{d}m\right) \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R = \widetilde{\left(\int \boldsymbol{\chi}^R \,\mathrm{d}m\right)} \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R,$$

   which yields, together with the vector

   $$\boldsymbol{r}_{CP}^R := \frac{1}{m} \cdot \int \boldsymbol{\chi}^R \,\mathrm{d}m = \boldsymbol{A}^{RL} \cdot \boldsymbol{r}_{CP}^L, \tag{4.12}$$

   the relation

   $$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m = m \cdot \widetilde{\left(\frac{1}{m} \cdot \int \boldsymbol{\chi}^R \,\mathrm{d}m\right)} \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R = m \cdot \tilde{\boldsymbol{r}}_{CP}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R$$

   or

   $$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \,\mathrm{d}m = m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R. \tag{4.13}$$

3. $$\int \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m = \tilde{\boldsymbol{r}}_{PO}^R \cdot \left(\int {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m\right),$$

   which yields together with the rigid-body property (${}^L\dot{\boldsymbol{\chi}}^L \equiv \boldsymbol{0}$) and the relation

   $${}^R\dot{\boldsymbol{\chi}}^R = \dot{\boldsymbol{A}}^{RL} \cdot {}^L\boldsymbol{\chi}^L + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{\chi}}^L = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{\chi}^L, \tag{4.14}$$

   the relation

   $$\begin{aligned} & \int \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m \\ & \quad = \tilde{\boldsymbol{r}}_{PO}^R \cdot \int \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{\chi}^L \,\mathrm{d}m = \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \int \boldsymbol{\chi}^L \,\mathrm{d}m \\ & \quad = m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L = -m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot \boldsymbol{\omega}_{LR}^L. \end{aligned} \tag{4.15}$$

4. $$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m = \int \tilde{\boldsymbol{\chi}}^R \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\chi}^L \,\mathrm{d}m \tag{4.16}$$
$$= -\boldsymbol{A}^{RL} \cdot \left(\int \tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L \,\mathrm{d}m\right) \cdot \boldsymbol{\omega}^L_{LR}$$

or

$$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m = \underbrace{-\boldsymbol{A}^{RL} \cdot \overbrace{\left(\int \tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L \,\mathrm{d}m\right)}^{=:\,\boldsymbol{J}^L_P} \cdot \boldsymbol{A}^{LR}}_{=:\,\boldsymbol{J}^R_P} \cdot \underbrace{\boldsymbol{A}^{RL} \cdot \boldsymbol{\omega}^L_{LR}}_{=:\,\boldsymbol{\omega}^R_{LR}},$$

and finally

$$\int \tilde{\boldsymbol{\chi}}^R \cdot {}^R\dot{\boldsymbol{\chi}}^R \,\mathrm{d}m =: \boldsymbol{J}^R_P \cdot \boldsymbol{\omega}^R_{LR} = \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}^R_{LR}$$
$$= \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR}$$

with

$$\boldsymbol{J}^L_P := -\int \tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L \,\mathrm{d}m$$

and (4.17)

$$\boldsymbol{J}^R_P = \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{A}^{LR}$$

as the *inertia tensor (inertia matrix)* of the rigid body with respect to the point P, and represented in the local (body-fixed) frame L and in the global frame R, respectively.

Together with (4.11), (4.13), (4.15), and (4.16) the angular momentum $\boldsymbol{L}^R_O$ of the body can be written as

$$\begin{aligned}\boldsymbol{L}^R_O = \;& \boldsymbol{J}^R_P \cdot \boldsymbol{\omega}^R_{LR} + m \cdot \tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO} \\ &+ m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO} - m \cdot \tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{\omega}^L_{LR}\end{aligned} \tag{4.18a}$$

or as

$$\begin{aligned}\boldsymbol{L}^R_O = \;& \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR} + m \cdot \tilde{\boldsymbol{r}}^R_{PO} \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO} - \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{\omega}^L_{LR}\right) \\ &+ m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO}.\end{aligned} \tag{4.18b}$$

Together with

$${}^R\dot{\boldsymbol{r}}^R_{CO} = {}^R\dot{\boldsymbol{r}}^R_{PO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{CP} \tag{4.19}$$

this implies

$$\begin{aligned}\boldsymbol{L}^R_O = \;& \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR} + m \cdot \tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\dot{\boldsymbol{r}}^R_{CO} \\ &+ m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO}.\end{aligned} \tag{4.20}$$

For the special case P = C, $L \equiv K$, $\boldsymbol{r}_{CO} = \boldsymbol{r}_{PO}$, and $\boldsymbol{r}_{CP} \equiv \mathbf{0}$, the angular momentum $\boldsymbol{L}_O^R$ (4.20) is written as

$$L_O^R = \boldsymbol{A}^{RK} \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{\omega}_{KR}^K + m \cdot \tilde{\boldsymbol{r}}_{CO}^R \cdot \dot{\boldsymbol{r}}_{CO}^R \tag{4.21}$$

with the (constant) *inertia matrix* of the body with respect to point C

$$\boldsymbol{J}_C^K = \begin{pmatrix} J_{Cxx}^K \ , & -J_{Cyx}^K \ , & -J_{Cxz}^K \\ -J_{Cxy}^K \ , & J_{Cyy}^K \ , & -J_{Cyz}^K \\ -J_{Cxz}^K \ , & -J_{Cyz}^K \ , & J_{Czz}^K \end{pmatrix}, \tag{4.22a}$$

and with

$$J_{Cxx}^K := \int \left(\left(\rho_y^K\right)^2 + \left(\rho_z^K\right)^2 \right) \mathrm{d}m \quad \text{as a } \textit{moment of inertia} \tag{4.22b}$$

and

$$J_{Cxy}^K := \int \rho_x^K \cdot \rho_y^K \, \mathrm{d}m \quad \text{as a } \textit{product of inertia}$$

of the rigid body with respect of its center of mass C, and with

$$\boldsymbol{\rho}^K = \left(\rho_x^K \ , \rho_y^K \ , \rho_z^K\right) \tag{4.22c}$$

as the vector from point C to the location of the mass element $\mathrm{d}m$ (Figure 4.1).

4.1.3 Properties of the inertia matrix

4.1.3.1 Physical interpretation of $\boldsymbol{J}_P^L$. Using the relation

$$\tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L = -\left[\left(\boldsymbol{\chi}^L\right)^{\mathrm{T}} \cdot \boldsymbol{\chi}^L \cdot \mathbf{I}_3 - \boldsymbol{\chi}^L \cdot \left(\boldsymbol{\chi}^L\right)^{\mathrm{T}}\right] = \left(\tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L\right)^{\mathrm{T}} \tag{4.23}$$

implies, together with (4.17)

$$\boldsymbol{J}_P^L = \int \left[\left(\boldsymbol{\chi}^L\right)^{\mathrm{T}} \cdot \boldsymbol{\chi}^L \cdot \mathbf{I}_3 - \boldsymbol{\chi}^L \cdot \left(\boldsymbol{\chi}^L\right)^{\mathrm{T}}\right] \mathrm{d}m, \tag{4.24a}$$

or written in components with respect to frame L,

$$\boldsymbol{J}_P^L = \begin{pmatrix} J_{Pxx}^L \ , & -J_{Pxy}^L \ , & -J_{Pxz}^L \\ -J_{Pxy}^L \ , & J_{Pyy}^L \ , & -J_{Pyz}^L \\ -J_{Pxz}^L \ , & -J_{Pyz}^L \ , & J_{Pzz}^L \end{pmatrix} \in \mathbb{R}^3 \tag{4.24b}$$

with

$$J_{Pxx}^L := \int \left[\left(\chi_y^L\right)^2 + \left(\chi_z^L\right)^2\right] \mathrm{d}m, \text{ etc., as the } \textit{moments of inertia} \text{ and}$$
$$J_{Pxy}^L := \int \chi_x^L \cdot \chi_y^L \, \mathrm{d}m, \quad \text{etc., as the } \textit{products of inertia} \tag{4.24c}$$

of the rigid body with respect to a body-fixed point P, and with

$$\boldsymbol{\chi}^L = \left(\chi_x^L\,,\,\chi_y^L\,,\,\chi_z^L\right)^{\mathrm{T}}$$

as the vector from point P to the mass element dm located at the point Q (Figure 4.1).

4.1.3.2 Time dependence of $\boldsymbol{J_P^L}$ and $\boldsymbol{J_P^R}$. For rigid bodies of *constant mass* m, the inertia matrix $\boldsymbol{J}_P^L$ is a *constant matrix* when represented in frame L, whereas the matrix

$$\boldsymbol{J}_P^R = \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{A}^{LR}, \tag{4.25}$$

represented in frame R, is in general a *time-dependent matrix*, due to $\boldsymbol{A}^{RL} = \boldsymbol{A}^{RL}(\boldsymbol{\eta}(t))$.

4.1.3.3 Steiner–Huygens relation. Changing the reference point of $\boldsymbol{J}$ from point C (origin of frame K) to point P (origin of frame L) provides the following transformation of $\boldsymbol{J}$:

$$J_P^L = \boldsymbol{A}^{LK} \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{A}^{KL} + m \cdot \left((\boldsymbol{r}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^{\mathrm{T}}\right). \tag{4.26a}$$

Or, written in components,

$$\boldsymbol{J}_P^L = \underbrace{\begin{pmatrix} J_{Cxx}^L\,, & -J_{Cxy}^L\,, & -J_{Cxz}^L \\ -J_{Cxy}^L\,, & J_{Cyy}^L\,, & -J_{Cyz}^L \\ -J_{Cxz}^L\,, & -J_{Cyz}^L\,, & J_{Czz}^L \end{pmatrix}}_{=:\,\boldsymbol{J}_C^L} \tag{4.26b}$$

$$+\, m \cdot \begin{pmatrix} {y_{CP}^L}^2 + {z_{CP}^L}^2\,, & -x_{CP}^L y_{CP}^L\,, & -x_{CP}^L z_{CP}^L \\ -y_{CP}^L x_{CP}^L\,, & {x_{CP}^L}^2 + {z_{CP}^L}^2\,, & -y_{CP}^L z_{CP}^L \\ -z_{CP}^L x_{CP}^L\,, & -z_{CP}^L y_{CP}^L\,, & {x_{CP}^L}^2 + {y_{CP}^L}^2 \end{pmatrix}$$

with

$$\boldsymbol{r}_{CP}^L = \left(x_{CP}^L\,, y_{CP}^L\,, z_{CP}^L\right)^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{J}_C^L = \boldsymbol{A}^{LK} \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{A}^{KL}. \tag{4.26c}$$

***Planar case* (moment of inertia):** In the *planar case* only rotations about the z-axis perpendicular to the x–y plane occur. Then

$$J_P^R = J_P^L = J_{Pzz} =: J_P \in \mathbb{R}^1, \tag{4.27a}$$

and the *Steiner–Huygens* relation is

$$J_P^L = J_C^L + m \cdot \left({x_{CP}^L}^2 + {y_{CP}^L}^2\right) = J_C^L + m \cdot \left({x_{CP}^R}^2 + {y_{CP}^R}^2\right). \tag{4.27b}$$

Proof of the Steiner–Huygens relation (4.26):

Inserting the relation

$$\boldsymbol{\chi} = \boldsymbol{r}_{CP} + \boldsymbol{\rho} \tag{4.28}$$

(Figure 4.1) into (4.17) provides the relation

$$\boldsymbol{J}_P^L = -\int \tilde{\boldsymbol{\chi}}^L \cdot \tilde{\boldsymbol{\chi}}^L \, \mathrm{d}m = -\int \widetilde{\left(\boldsymbol{r}_{CP}^L + \boldsymbol{\rho}^L\right)} \cdot \widetilde{\left(\boldsymbol{r}_{CP}^L + \boldsymbol{\rho}^L\right)} \, \mathrm{d}m, \tag{4.29}$$

and, together with

$$\widetilde{(\boldsymbol{a} + \boldsymbol{b})} = \tilde{\boldsymbol{a}} + \tilde{\boldsymbol{b}},$$

the relation

$$\begin{aligned}\boldsymbol{J}_P^L = &-\int \tilde{\boldsymbol{\rho}}^L \cdot \tilde{\boldsymbol{\rho}}^L \, \mathrm{d}m - \tilde{\boldsymbol{r}}_{CP}^L \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot m \\ &- \tilde{\boldsymbol{r}}_{CP}^L \cdot \left(\int \tilde{\boldsymbol{\rho}}^L \, \mathrm{d}m\right) - \left(\int \tilde{\boldsymbol{\rho}}^L \, \mathrm{d}m\right) \cdot \tilde{\boldsymbol{r}}_{CP}^L.\end{aligned}$$

Together with

$$\begin{aligned}&\int \boldsymbol{\rho}^L \, \mathrm{d}m = \mathbf{0} \quad , \quad \boldsymbol{\rho}^L := \boldsymbol{r}_{QC}^L, \quad \text{and} \\ &\int \tilde{\boldsymbol{\rho}}^L \, \mathrm{d}m = \mathbf{0} \quad \text{(due to the \textit{definition of} C)},\end{aligned} \tag{4.30}$$

this yields the transformation relation

$$\boldsymbol{J}_P^L = \boldsymbol{J}_C^L - \tilde{\boldsymbol{r}}_{CP}^L \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot m \tag{4.31}$$

with

$$\boldsymbol{J}_C^L := -\int \tilde{\boldsymbol{\rho}}^L \cdot \tilde{\boldsymbol{\rho}}^L \, \mathrm{d}m = \int \left((\boldsymbol{\rho}^L)^{\mathrm{T}} \cdot \boldsymbol{\rho}^L \cdot \mathbf{I}_3 - \boldsymbol{\rho}^L \cdot (\boldsymbol{\rho}^L)^{\mathrm{T}}\right) \, \mathrm{d}m. \tag{4.32}$$

Due to

$$\boldsymbol{J}_C^L = \boldsymbol{A}^{LK} \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{A}^{KL} \tag{4.33}$$

and to

$$\tilde{\boldsymbol{r}}_{CP}^L \cdot \tilde{\boldsymbol{r}}_{CP}^L = -\left((\boldsymbol{r}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^{\mathrm{T}}\right),$$

the following relation holds

$$\boldsymbol{J}_P^L = \boldsymbol{A}^{LK} \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{A}^{KL} + \left((\boldsymbol{r}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^{\mathrm{T}}\right) \cdot m. \tag{4.34}$$

This proves (4.26a). □

4.2 Newton–Euler equations of an unconstrained rigid body

Based on the previously introduced *linear momentum* $\boldsymbol{P}_O^R$ and *angular momentum* $\boldsymbol{L}_O^R$, *Newton's* and *Euler's* laws can be compactly and precisely stated in symmetric form, when applied to an *unconstrained rigid body*, as

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_O^R = \sum_i \boldsymbol{F}_i^R \qquad \text{(Newton's law)} \tag{4.35}$$

and

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{L}_O^R = \sum_i \boldsymbol{M}_{iO}^R \qquad \text{(Euler's law)}, \tag{4.36}$$

with $\sum_i \boldsymbol{F}_i^R$ as the resultant *external force* acting on the body and $\sum_i \boldsymbol{M}_{iO}^R$ as the resultant *external force moment (torque)* with respect to the point $\mathrm{O} = \mathrm{O}_R$, acting on the body, with R as an *inertial frame*, and with ${}^R\dot{\boldsymbol{r}}_{PO}^R$ included in both, $\boldsymbol{P}_O^R$ and $\boldsymbol{L}_O^R$, where $\boldsymbol{r}_{PO}^R$ is *differentiated with respect to time in the inertial frame* R (${}^R\mathrm{d}/\mathrm{d}t$; compare Equation 2.28c).

4.2.1 Force moments and couples

A *torque* $\boldsymbol{M}$ may occur as a *force moment*, generated by a force, or as a *couple moment* (generated by two collinear forces of equal absolute values and of opposite signs).

Consider a force $\boldsymbol{F}_i$, acting on a body (as shown in Figure 4.2a), and a point P not located on the (dashed) line of action of $\boldsymbol{F}_i$. Consider two points P_i and Q_i on the line of action of $\boldsymbol{F}_i$. Then the *force moment* (moment, torque) $\boldsymbol{M}_{iF}$ of $\boldsymbol{F}_i$ acting on the body *with respect to the reference point* P is defined as

$$\boldsymbol{M}_{iPF}^R := \tilde{\boldsymbol{r}}_{P_iP}^R \cdot \boldsymbol{F}_i^R = \tilde{\boldsymbol{r}}_{Q_iP}^R \cdot \boldsymbol{F}_i^R. \tag{4.37}$$

Subsequently the point P_i will be used in $\boldsymbol{M}_{iPF}$ as the point on the line of action of the force $\boldsymbol{F}_i$. The torque $\boldsymbol{M}_j$ produced by two collinear forces $\boldsymbol{F}_j$ and $-\boldsymbol{F}_j$ of equal absolute values and opposite signs (a *"couple of forces"*) is known as *couple moment*. Due to the relations (Figure 4.2b)

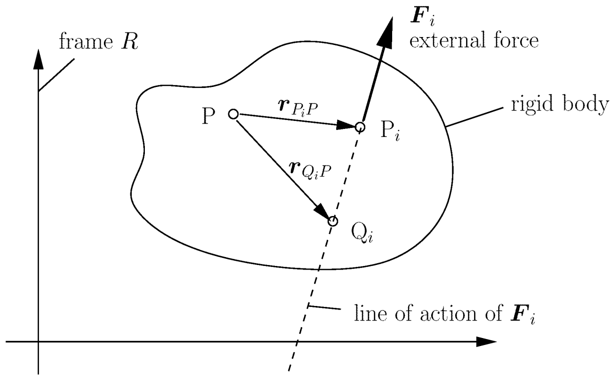
$$\boldsymbol{M}_{jP}^R = \tilde{\boldsymbol{r}}_{P_jP}^R \cdot \boldsymbol{F}_j^R - \tilde{\boldsymbol{r}}_{Q_jP}^R \cdot \boldsymbol{F}_j^R = \left(\tilde{\boldsymbol{r}}_{P_jP}^R - \tilde{\boldsymbol{r}}_{Q_jP}^R\right) \cdot \boldsymbol{F}_j^R$$

and

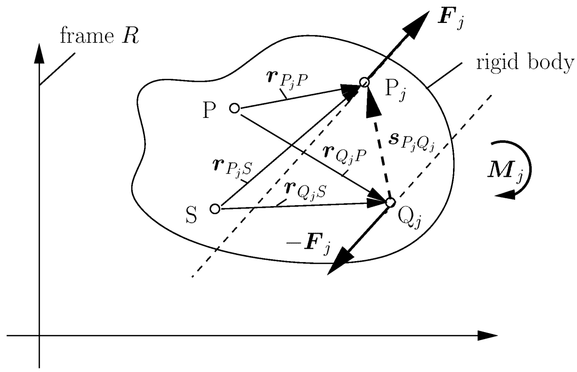
$$\boldsymbol{M}_{jS}^R = \left(\tilde{\boldsymbol{r}}_{P_jS}^R - \tilde{\boldsymbol{r}}_{Q_jS}^R\right) \cdot \boldsymbol{F}_j^R,$$

together with

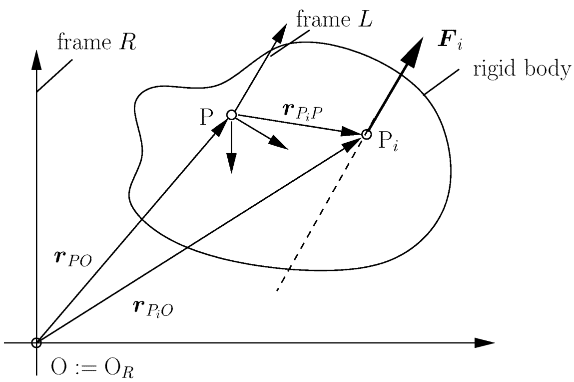
$$\tilde{\boldsymbol{s}}_{P_jQ_j}^R = \boldsymbol{r}_{P_jP}^R - \boldsymbol{r}_{Q_jP}^R = \boldsymbol{r}_{P_jS}^R - \boldsymbol{r}_{Q_jS}^R,$$



(a) Torque generated by a force



(b) Torque produced by a couple of forces



(c) Torque with respect to the origin O

Fig. 4.2: Forces and torques

the *couple moment*

$$\boldsymbol{M}_j^R := \tilde{\boldsymbol{s}}_{P_jQ_j}^R \cdot \boldsymbol{F}_j^R \tag{4.38}$$

is independent of the reference points P and Q. Thus the moment of a couple of forces is the same about arbitrary reference points. Therefore couple moments are called *free vectors*, whereas force moments are called *sliding vectors* in the sense that they are moments about an axis through a reference point.

Consider the vector diagram of Figure 4.2c, in which the *torque* $\boldsymbol{M}_{iF}$ *generated by a force* $\boldsymbol{F}_i$ *acting* on the rigid body can be taken with respect to the origin O of a reference frame R. This implies

$$\boldsymbol{M}_{iOF}^R = \tilde{\boldsymbol{r}}_{P_iO}^R \cdot \boldsymbol{F}_i^R = \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{F}_i^R + \tilde{\boldsymbol{r}}_{P_iP}^R \cdot \boldsymbol{F}_i^R = \tilde{\boldsymbol{r}}_{PO}^R \cdot \boldsymbol{F}_i^R + \boldsymbol{M}_{iPF}^R, \tag{4.39}$$

where

$$\boldsymbol{M}_{iPF}^R := \tilde{\boldsymbol{r}}_{P_iP}^R \cdot \boldsymbol{F}_i^R$$

is the moment of $\boldsymbol{F}_i$ with respect to point P on the rigid body, and $\boldsymbol{M}_{iOF}$ is the moment of $\boldsymbol{F}_i$ with respect to point O of the frame R. Then the total moment acting on a body and measured with respect to *point* O can be written in the form

$$\boldsymbol{M}_O^R = \sum_i \underbrace{\boldsymbol{M}_{iO}^R}_{\substack{\text{couple}\\ \text{moment}}} + \sum_i \underbrace{\tilde{\boldsymbol{r}}_{P_iO}^R \cdot \boldsymbol{F}_i^R}_{\substack{\text{moment of}\\ \text{the force } \boldsymbol{F}_i}} . \tag{4.40}$$

Together with

$$\boldsymbol{r}_{P_iO}^R = \boldsymbol{r}_{PO}^R + \boldsymbol{r}_{P_iP}^R, \tag{4.41}$$

this yields

$$\boldsymbol{M}_O^R = \underbrace{\sum_i \boldsymbol{M}_i^R + \sum_i \tilde{\boldsymbol{r}}_{P_iP}^R \cdot \boldsymbol{F}_i^R}_{=:\, \boldsymbol{M}_P^R} + \tilde{\boldsymbol{r}}_{PO}^R \cdot \sum_i \boldsymbol{F}_i^R$$

or

$$\boldsymbol{M}_O^R = \boldsymbol{M}_P^R + \tilde{\boldsymbol{r}}_{PO}^R \cdot \sum_i \boldsymbol{F}_i^R \tag{4.42a}$$

with

$$\boldsymbol{M}_P^R = \sum_i \boldsymbol{M}_i^R + \sum_i \tilde{\boldsymbol{r}}_{P_iP}^R \cdot \boldsymbol{F}_i^R \tag{4.42b}$$

as the total moment on the body measured with respect to the *point* P.

Planar case: The product $\tilde{\boldsymbol{r}}^R \cdot \boldsymbol{F}^R$ with the vectors $\boldsymbol{r}^R = (x^R\,,\, y^R\,,\, z^R)^{\mathrm{T}} = (x^R\,,\, y^R\,,\, 0)^{\mathrm{T}}$ and $\boldsymbol{F}^R = (F_x^R\,,\, F_y^R\,,\, F_z^R)^{\mathrm{T}} = (F_x^R\,,\, F_y^R\,,\, 0)^{\mathrm{T}}$ located inside the x^R–y^R plane of $\mathbb{R}^3$ provides the torque vector in $\mathbb{R}^3$:

$$
\begin{aligned}
\boldsymbol{M}^R = \tilde{\boldsymbol{r}}^R \cdot {}^R\boldsymbol{F}^R &= \begin{pmatrix} 0 & -z^R & y^R \\ z^R & 0 & -x^R \\ -y^R & x^R & 0 \end{pmatrix} \cdot \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & y^R \\ 0 & 0 & -x^R \\ -y^R & x^R & 0 \end{pmatrix} \cdot \begin{pmatrix} F_x^R \\ F_y^R \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\,y^R \cdot F_x^R + F_y^R \cdot x^R \end{pmatrix}
\end{aligned}
\tag{4.43}
$$

or

$$
\boldsymbol{M} = \left(-y^R \cdot F_x^R + F_y^R \cdot x^R\right) \cdot \boldsymbol{e}_{zR},
$$

that acts in the $\boldsymbol{e}_{zR}$-direction, perpendicular to the x^R–y^R plane.

4.2.2 Newton's law

Assuming that the mass m of the body is constant, and inserting (4.7a) into (4.35) yields, together with

$$
{}^R\dot{\boldsymbol{P}}_O^R = m \cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left({}^R\dot{\boldsymbol{r}}_{PO}^R + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L\right), \tag{4.44a}
$$

the relation

$$
{}^R\dot{\boldsymbol{P}}_O^R = m \cdot \left({}^R\ddot{\boldsymbol{r}}_{PO}^R + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L - \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot \dot{\boldsymbol{\omega}}_{LR}^L\right) = \sum_i \boldsymbol{F}_i^R \tag{4.44b}
$$

that can be written as

$$
\begin{aligned}
\left(m \cdot \mathbf{I}_3\,,\; -\,m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L\right) \cdot \begin{pmatrix} {}^R\ddot{\boldsymbol{r}}_{PO}^R \\ \dot{\boldsymbol{\omega}}_{LR}^L \end{pmatrix} \\
= \sum_i \boldsymbol{F}_i^R - m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L.
\end{aligned}
\tag{4.44c}
$$

The special case P = C, where the reference point P is chosen as the center of mass C of the body ($\boldsymbol{r}_{PC} \equiv \mathbf{0}$), yields the *simplified version of Newton's law as pure translational equation of motion*:

$$
m \cdot {}^R\ddot{\boldsymbol{r}}_{CO}^R = \sum_i \boldsymbol{F}_i^R, \tag{4.44d}
$$

that describes the acceleration of the *center of mass* C of the body with respect to the *inertial frame* R, measured in R. This does not explicitly depend on its angular velocity $\boldsymbol{\omega}_{LR}$.

4.2.3 Euler's law

Contrary to the linear momentum relation (4.44a), where the constant mass has been extracted from the bracket of $\boldsymbol{P}_O^R$ before the time differentiation, the time-dependent inertia matrix $\boldsymbol{J}_P^R = \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{A}^{LR}$ cannot be treated identically. This provides a more complex expression of Euler's law. Inserting (4.20) into (4.36) yields, together with

$$\dot{\boldsymbol{J}}_P^L \equiv \boldsymbol{0}, \tag{4.45}$$

the relation

$$\begin{aligned} {}^R\dot{\boldsymbol{L}}_O^R = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\Big(& \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{LR}^L + m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{CO}^R \\ & + m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \Big) \end{aligned}$$

or

$$\begin{aligned} {}^R\dot{\boldsymbol{L}}_O^R &= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{LR}^L + m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{CO}^R + m \cdot \tilde{\boldsymbol{r}}_{CP}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R \right) \\ &= \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{LR}^L + \boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{J}}_P^L \cdot \boldsymbol{\omega}_{LR}^L + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \dot{\boldsymbol{\omega}}_{LR}^L \\ &\quad + m \cdot {}^R\dot{\tilde{\boldsymbol{r}}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{CO}^R + m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\ddot{\boldsymbol{r}}_{CO}^R \\ &\quad + m \cdot {}^R\dot{\tilde{\boldsymbol{r}}}_{CP}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R + m \cdot \tilde{\boldsymbol{r}}_{CP}^R \cdot {}^R\ddot{\boldsymbol{r}}_{PO}^R, \end{aligned}$$

or

$$\begin{aligned} {}^R\dot{L}_0^R &= \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{RL}^L + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \dot{\boldsymbol{\omega}}_{LR}^L && (4.46) \\ &\quad + m \cdot \tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\ddot{\boldsymbol{r}}_{CO}^R + m \cdot {}^R\dot{\tilde{\boldsymbol{r}}}_{PO}^R \cdot {}^R\dot{\boldsymbol{r}}_{CO}^R && (4.47) \\ &\quad + m \cdot {}^R\dot{\tilde{\boldsymbol{r}}}_{CP}^R \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R + m \cdot \tilde{\boldsymbol{r}}_{CP}^R \cdot {}^R\ddot{\boldsymbol{r}}_{PO}^R, \end{aligned}$$

or

$$\begin{aligned} {}^R\dot{\boldsymbol{L}}_0^R &= \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{RL}^L + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_P^L \cdot \dot{\boldsymbol{\omega}}_{LR}^L \\ &\quad + m \cdot \left[\tilde{\boldsymbol{r}}_{PO}^R \cdot {}^R\ddot{\boldsymbol{r}}_{CO}^R + \underbrace{\underbrace{\left({}^R\dot{\tilde{\boldsymbol{r}}}_{CP}^R - {}^R\dot{\tilde{\boldsymbol{r}}}_{CO}^R \right)}_{=:\, -{}^R\dot{\tilde{\boldsymbol{r}}}_{PO}^R} \cdot {}^R\dot{\boldsymbol{r}}_{PO}^R + \tilde{\boldsymbol{r}}_{CP}^R \cdot {}^R\ddot{\boldsymbol{r}}_{PO}^R}_{=\boldsymbol{0}} \right]. \end{aligned}$$

Taking into account

$${}^R\dot{\boldsymbol{r}}_{CP}^R - {}^R\dot{\boldsymbol{r}}_{CO}^R = -{}^R\dot{\boldsymbol{r}}_{PO}^R \quad , \quad {}^R\dot{\tilde{\boldsymbol{r}}}_{CP}^R - {}^R\dot{\tilde{\boldsymbol{r}}}_{CO}^R = -{}^R\dot{\tilde{\boldsymbol{r}}}_{PO}^R, \tag{4.48}$$

and

$$ {}^R\dot{\tilde{\boldsymbol{r}}}^R_{PO} \cdot {}^R\dot{\boldsymbol{r}}^R_{PO} \equiv \mathbf{0} \tag{4.49}$$

provides

$$\begin{aligned} {}^R\dot{\boldsymbol{L}}^R_0 &= \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{RL} + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} \\ &\quad + m \cdot \left(\tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO} + \tilde{\boldsymbol{r}}^R_{CP} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO}\right), \end{aligned} \tag{4.50}$$

and finally yields *Euler's equation in the form*

$$\begin{aligned} {}^R\dot{\boldsymbol{L}}^R_0 &= m \cdot \left(\tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO} + \tilde{\boldsymbol{r}}^R_{CP} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO}\right) + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} \\ &\quad + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{RL} = \sum_i \boldsymbol{M}^R_{i0}. \end{aligned} \tag{4.51}$$

Inserting (4.42a) into (4.51) yields

$$\begin{aligned} &m \cdot \left(\tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO} + \tilde{\boldsymbol{r}}^R_{CP} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO}\right) + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} \\ &\quad + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR} = \sum_i \boldsymbol{M}^R_{iP} + \sum_i \tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{F}^R_i \end{aligned} \tag{4.52}$$

with

$$\boldsymbol{M}^R_{iP} = \boldsymbol{M}^R_i + \tilde{\boldsymbol{r}}^R_{P_iP} \cdot \boldsymbol{F}^R_i .$$

Taking into account (4.44d) yields

$$\begin{aligned} &m \cdot \left(\tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO} + \tilde{\boldsymbol{r}}^R_{CP} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO}\right) + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} \\ &\quad + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR} = \sum_i \boldsymbol{M}^R_{iP} + m \cdot \tilde{\boldsymbol{r}}^R_{PO} \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO}, \end{aligned} \tag{4.53}$$

and after multiplication of (4.53) by $\boldsymbol{A}^{LR}$, the relation

$$m \cdot \underbrace{\boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{r}}^R_{CP} \cdot \boldsymbol{A}^{RL}}_{=:\, \tilde{\boldsymbol{r}}^L_{CP}} \cdot \boldsymbol{A}^{LR} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO} + \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR} = \sum_i \boldsymbol{M}^L_{iP}$$

or

$$m \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{A}^{LR} \cdot {}^R\ddot{\boldsymbol{r}}^R_{PO} + \boldsymbol{J}^L_P \cdot \dot{\boldsymbol{\omega}}^L_{LR} = \sum_i \boldsymbol{M}^L_{iP} - \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR}, \tag{4.53a}$$

with

$$\sum_i \boldsymbol{M}^L_{iP} = \boldsymbol{A}^{LR} \cdot \sum_i \boldsymbol{M}^R_{iP} \tag{4.53b}$$

and

$$\sum_i \boldsymbol{M}^R_{iP} \equiv \sum_i \boldsymbol{M}^R_i + \sum_i \tilde{\boldsymbol{r}}^R_{P_iP} \cdot \boldsymbol{F}^R_i . \tag{4.42b}$$

Written in vector form, *Euler's equation of motion* is (for P $\neq$ C)

$$\left(m \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \boldsymbol{A}^{LR}, \boldsymbol{J}^L_P\right) \cdot \begin{pmatrix} {}^R\ddot{\boldsymbol{r}}^R_{PO} \\ \dot{\boldsymbol{\omega}}^L_{LR} \end{pmatrix} = \sum_i \boldsymbol{M}^L_{iP} - \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_P \cdot \boldsymbol{\omega}^L_{LR}. \tag{4.54a}$$

Special case (P = C): Choosing the reference point P as the center of mass C of the body yields, together with $\boldsymbol{r}_{CP} \equiv \mathbf{0}$ and $L = K$, the *simplified version of Euler's law* as the *equation of motion of pure rotations* of the body with respect to C:

$$\boldsymbol{J}_C^K \cdot \dot{\boldsymbol{\omega}}_{KR}^K = \sum_i M_{iC}^K - \tilde{\boldsymbol{\omega}}_{KR}^K \cdot \boldsymbol{J}_C^K \cdot \boldsymbol{\omega}_{KR}^K,$$

with (4.54b)

$$\boldsymbol{M}_{iC}^K = \boldsymbol{M}_i^K + \tilde{\boldsymbol{r}}_{P_iC}^K \cdot \boldsymbol{F}_i^K.$$

4.2.4 Newton-Euler equations of a rigid body under planar and spatial motion

In this section the *Newton–Euler equations of a rigid body* under planar and spatial motion will be collected.

4.2.4.1 *Spatial* motion. Combining the previous results (4.44c), (4.54a), (4.42b), and (4.26a) yields the *Newton–Euler equations of a rigid body* under *spatial* motion for P $\neq$ C:

$$\underbrace{\begin{bmatrix} m \cdot \mathbf{I}_3 & , & -m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^L \\ -m \cdot (\tilde{\boldsymbol{r}}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{A}^{LR} & , & \underbrace{\boldsymbol{J}_C^L + m \cdot \left((\boldsymbol{r}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^{\mathrm{T}}\right)}_{=: \boldsymbol{J}_P^L} \end{bmatrix}}_{=: \boldsymbol{M}}$$

$$\cdot \begin{pmatrix} {}^R\ddot{\boldsymbol{r}}_{PO}^R \\ \dot{\boldsymbol{\omega}}_{LR}^L \end{pmatrix} = \underbrace{\begin{pmatrix} \sum_i \boldsymbol{F}_i^R \\ \sum_i \boldsymbol{M}_i^L + \sum_i \tilde{\boldsymbol{r}}_{P_iP}^L \cdot \boldsymbol{F}_i^L \end{pmatrix}}_{=: \boldsymbol{f}} \tag{4.55a}$$

$$\underbrace{- \begin{bmatrix} m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L \\ \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \left[\boldsymbol{J}_C^L + m \cdot \left((\boldsymbol{r}_{CP}^L)^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^{\mathrm{T}}\right)\right] \cdot \boldsymbol{\omega}_{LR}^L \end{bmatrix}}_{=: \boldsymbol{q}_G}.$$

They are *coupled in* the coordinates of ${}^R\ddot{\boldsymbol{r}}_P^R$ and $\dot{\boldsymbol{\omega}}_{LR}^L$. The *matrices and vectors of (4.55a)* have (for a *body* i with P = P$_i$ $\neq$ C$_i$ = C) the form:

$$\left[\left(\ddot{\boldsymbol{r}}_{P_iO}^R\right)^{\mathrm{T}}, \left(\dot{\boldsymbol{\omega}}_{L_iR}^{L_i}\right)^{\mathrm{T}}\right]^{\mathrm{T}} = \left(\ddot{x}_{P_iO}^R, \ddot{y}_{P_iO}^R, \ddot{z}_{P_iO}^R, \dot{\omega}_{xL_iR}^{L_i}, \dot{\omega}_{yL_iR}^{L_i}, \dot{\omega}_{zL_iR}^{L_i}\right)^{\mathrm{T}}, \tag{4.55b}$$

$$
\boldsymbol{M}_i = \left[\begin{array}{c}
\begin{pmatrix} m_i \, , & 0 \, , & 0 \\ 0 \, , & m_i \, , & 0 \\ 0 \, , & 0 \, , & m_i \end{pmatrix} \qquad , \\[2ex]
m_i \cdot \begin{pmatrix} 0 & , -z^{L_i}_{\mathrm{c}\,i p_i} & , \; y^{L_i}_{\mathrm{c}\,i p_i} \\ z^{L_i}_{\mathrm{c}\,i p_i} & , \; 0 & , -x^{L_i}_{\mathrm{c}\,i p_i} \\ -y^{L_i}_{\mathrm{c}\,i p_i} & , \; x^{L_i}_{\mathrm{c}\,i p_i} & , \; 0 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{c}_{\,i2}\, \mathrm{c}_{\,i3} & , \; \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i3} + \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{c}_{\,i3} & , \; \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i3} - \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{c}_{\,i3} \\ -\mathrm{c}_{\,i2}\, \mathrm{s}_{\,i3} & , \; \mathrm{c}_{\,i1}\, \mathrm{c}_{\,i3} - \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{s}_{\,i3} & , \; \mathrm{s}_{\,i1}\, \mathrm{c}_{\,i3} + \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{s}_{\,i3} \\ \mathrm{s}_{\,i2} & , \; -\mathrm{s}_{\,i1}\, \mathrm{c}_{\,i2} & , \; \mathrm{c}_{\,i1}\, \mathrm{c}_{\,i2} \end{pmatrix} \; , \\[2ex]
m_i \cdot \begin{pmatrix} \mathrm{c}_{\,i2}\, \mathrm{c}_{\,i3} & , \; -\mathrm{c}_{\,i2}\, \mathrm{s}_{\,i3} & , \; \mathrm{s}_{\,i2} \\ \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i3} + \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{c}_{\,i3} & , \; \mathrm{c}_{\,i1}\, \mathrm{c}_{\,i3} - \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{s}_{\,i3} & , \; -\mathrm{s}_{\,i1}\, \mathrm{c}_{\,i2} \\ \mathrm{s}_{\,i1}\, \mathrm{s}_{\,i3} - \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{c}_{\,i3} & , \; \mathrm{s}_{\,i1}\, \mathrm{c}_{\,i3} + \mathrm{c}_{\,i1}\, \mathrm{s}_{\,i2}\, \mathrm{s}_{\,i3} & , \; \mathrm{c}_{\,i1}\, \mathrm{c}_{\,i2} \end{pmatrix} \cdot \begin{pmatrix} 0 & , \; z^{L_i}_{\mathrm{c}\,i p_i} & , -y^{L_i}_{\mathrm{c}\,i p_i} \\ -z^{L}_{\mathrm{c}\,i p_i} & , \; 0 & , \; x^{L_i}_{\mathrm{c}\,i p_i} \\ y^{L_i}_{\mathrm{c}\,i p_i} & , -x^{L_i}_{\mathrm{c}\,i p_i} & , \; 0 \end{pmatrix} \\[2ex]
\begin{pmatrix} J^{L_i}_{\mathrm{c}\,ix} & , -J^{L_i}_{\mathrm{c}\,ixy} & , -J^{L_i}_{\mathrm{c}\,ixz} \\ -J^{L_i}_{\mathrm{c}\,iyx} & , \; J^{L_i}_{\mathrm{c}\,iy} & , -J^{L_i}_{\mathrm{c}\,iyz} \\ -J^{L_i}_{\mathrm{c}\,izx} & , -J^{L_i}_{\mathrm{c}\,izy} & , \; J^{L_i}_{\mathrm{c}\,iz} \end{pmatrix} + m_i \cdot \begin{pmatrix} \left({y^{L_i}_{\mathrm{c}\,i p_i}}^2 + {z^{L_i}_{\mathrm{c}\,i p_i}}^2 \right) & , -x^{L_i}_{\mathrm{c}\,i p_i}\, y^{L_i}_{\mathrm{c}\,i p_i} & , -x^{L_i}_{\mathrm{c}\,i p_i}\, z^{L_i}_{\mathrm{c}\,i p_i} \\ -y^{L_i}_{\mathrm{c}\,i p_i}\, x^{L_i}_{\mathrm{c}\,i p_i} & , \left({x^{L_i}_{\mathrm{c}\,i p_i}}^2 + {z^{L_i}_{\mathrm{c}\,i p_i}}^2 \right) & , -y^{L_i}_{\mathrm{c}\,i p_i}\, z^{L_i}_{\mathrm{c}\,i p_i} \\ -z^{L_i}_{\mathrm{c}\,i p_i}\, x^{L_i}_{\mathrm{c}\,i p_i} & , -z^{L_i}_{\mathrm{c}\,i p_i}\, y^{L_i}_{\mathrm{c}\,i p_i} & , \left({x^{L_i}_{\mathrm{c}\,i p_i}}^2 + {y^{L_i}_{\mathrm{c}\,i p_i}}^2 \right) \end{pmatrix}
\end{array} \right] , \tag{4.55c}
$$

$$
\boldsymbol{q}_{G_i} = \left[\begin{array}{l}
m_i \cdot \begin{pmatrix}
\mathrm{c}_{i2}\,\mathrm{c}_{i3} & , & -\mathrm{c}_{i2}\,\mathrm{s}_{i3} & , & \mathrm{s}_{i2} \\
\mathrm{c}_{i1}\,\mathrm{s}_{i3} + \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , & \mathrm{c}_{i1}\,\mathrm{c}_{i3} - \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & , & -\mathrm{s}_{i1}\,\mathrm{c}_{i2} \\
\mathrm{s}_{i1}\,\mathrm{s}_{i3} - \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , & \mathrm{s}_{i1}\,\mathrm{c}_{i3} + \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & , & \mathrm{c}_{i1}\,\mathrm{c}_{i2}
\end{pmatrix} \\
\qquad \cdot \begin{pmatrix}
-\left({\omega^{L_i}_{zL_iR}}^2 + {\omega^{L_i}_{yL_iR}}^2\right) \cdot x^{L_i}_{\mathrm{c}\,ip_i} & + & \omega^{L_i}_{xL_iR} \cdot \omega^{L_i}_{yL_iR} \cdot y^{L_i}_{\mathrm{c}\,ip_i} & + & \omega^{L_i}_{xL_iR} \cdot \omega^{L_i}_{zL_iR} \cdot z^{L_i}_{\mathrm{c}\,ip_i} \\
\omega^{L_i}_{xL_iR} \cdot \omega^{L_i}_{yL_iR} \cdot x^{L_i}_{\mathrm{c}\,ip_i} & - & \left({\omega^{L_i}_{xL_iR}}^2 + {\omega^{L_i}_{zL_iR}}^2\right) \cdot y^{L_i}_{\mathrm{c}\,ip_i} & + & \omega^{L_i}_{yL_iR} \cdot \omega^{L_i}_{zL_iR} \cdot z^{L_i}_{\mathrm{c}\,ip_i} \\
\omega^{L_i}_{xL_iR} \cdot \omega^{L_i}_{zL_iR} \cdot x^{L_i}_{\mathrm{c}\,ip_i} & + & \omega^{L_i}_{yL_iR} \cdot \omega^{L_i}_{zL_iR} \cdot y^{L_i}_{\mathrm{c}\,ip_i} & - & \left({\omega^{L_i}_{xL_iR}}^2 + {\omega^{L_i}_{yL_iR}}^2\right) \cdot z^{L_i}_{\mathrm{c}\,ip_i}
\end{pmatrix} \\
\begin{pmatrix}
0 & , & -\omega^{L_i}_{zL_iR} & , & \omega^{L_i}_{yL_iR} \\
\omega^{L_i}_{zL_iR} & , & 0 & , & -\omega^{L_i}_{xL_iR} \\
-\omega^{L_i}_{yL_iR} & , & \omega^{L_i}_{xL_iR} & , & 0
\end{pmatrix} \cdot \left[\begin{pmatrix}
J^{L_i}_{\mathrm{c}\,ix} & , & -J^{L_i}_{\mathrm{c}\,ixy} & , & -J^{L_i}_{\mathrm{c}\,ixz} \\
-J^{L_i}_{\mathrm{c}\,iyx} & , & J^{L_i}_{\mathrm{c}\,iy} & , & -J^{L_i}_{\mathrm{c}\,iyz} \\
-J^{L_i}_{\mathrm{c}\,izx} & , & -J^{L_i}_{\mathrm{c}\,izy} & , & J^{L_i}_{\mathrm{c}\,iz}
\end{pmatrix} \right. \\
\qquad \left. + m_i \cdot \begin{pmatrix}
\left({y^{L_i}_{\mathrm{c}\,ip_i}}^2 + {z^{L_i}_{\mathrm{c}\,ip_i}}^2\right) & , & -x^{L_i}_{\mathrm{c}\,ip_i}\, y^{L_i}_{\mathrm{c}\,ip_i} & , & -x^{L_i}_{\mathrm{c}\,ip_i}\, z^{L_i}_{\mathrm{c}\,ip_i} \\
-y^{L_i}_{\mathrm{c}\,ip_i}\, x^{L_i}_{\mathrm{c}\,ip_i} & , & \left({x^{L_i}_{\mathrm{c}\,ip_i}}^2 + {z^{L_i}_{\mathrm{c}\,ip_i}}^2\right) & , & -y^{L_i}_{\mathrm{c}\,ip_i}\, z^{L_i}_{\mathrm{c}\,ip_i} \\
-z^{L_i}_{\mathrm{c}\,ip_i}\, x^{L_i}_{\mathrm{c}\,ip_i} & , & -z^{L_i}_{\mathrm{c}\,ip_i}\, y^{L_i}_{\mathrm{c}\,ip_i} & , & \left({x^{L_i}_{\mathrm{c}\,ip_i}}^2 + {y^{L_i}_{\mathrm{c}\,ip_i}}^2\right)
\end{pmatrix} \right] \cdot \begin{pmatrix}
\omega^{L_i}_{xL_iR} \\ \omega^{L_i}_{yL_iR} \\ \omega^{L_i}_{zL_iR}
\end{pmatrix}
\end{array}\right] \tag{4.55d}
$$

$$\text{with} \quad \begin{array}{llll} \mathrm{s}_{i1} := \sin\phi_{iLR} \;, & \mathrm{s}_{i2} := \sin\theta_{iLR} \;, & & \mathrm{s}_{i3} := \sin\psi_{iLR}, \\ \mathrm{c}_{i1} := \cos\phi_{iLR} \;, & \mathrm{c}_{i2} := \cos\theta_{iLR} & \text{and} & \mathrm{c}_{i3} := \cos\psi_{iLR}, \end{array} \tag{4.55e}$$

and

$$\boldsymbol{f}_i = \begin{bmatrix} \sum\limits_j \begin{pmatrix} F^R_{ijx} \\ F^R_{ijy} \\ F^R_{ijz} \end{pmatrix} \\ \sum\limits_j \begin{pmatrix} M^{L_i}_{ijx} \\ M^{L_i}_{ijy} \\ M^{L_i}_{ijz} \end{pmatrix} + \sum\limits_j \begin{pmatrix} 0 & , -z^{L_i}_{p_{ij}p_i} & , \; y^{L_i}_{p_{ij}p_i} \\ z^{L_i}_{p_{ij}p_i} & , \; 0 & , -x^{L_i}_{p_{ij}p_i} \\ -y^{L_i}_{p_{ij}p_i} & , \; x^{L_i}_{p_{ij}p_i} & , \; 0 \end{pmatrix} \cdot \begin{pmatrix} F^R_{ijx} \\ F^R_{ijy} \\ F^R_{ijz} \end{pmatrix} \end{bmatrix}. \tag{4.55f}$$

For P = C the Newton–Euler equations are *no longer coupled*:

$$\begin{aligned} &\begin{pmatrix} m \cdot \mathbf{I}_3 & , \; \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , \; \boldsymbol{J}^K_C \end{pmatrix} \cdot \begin{pmatrix} {}^R\ddot{\boldsymbol{r}}^R_{CO} \\ \dot{\boldsymbol{\omega}}^K_{KR} \end{pmatrix} \\ &= \begin{pmatrix} \sum\limits_i \boldsymbol{F}^R_i \\ \sum\limits_i \boldsymbol{M}^K_i + \sum\limits_i \tilde{\boldsymbol{r}}^K_{P_iC} \cdot \boldsymbol{F}^K_i - \tilde{\boldsymbol{\omega}}^K_{KR} \cdot \boldsymbol{J}^K_C \cdot \boldsymbol{\omega}^K_{KR} \end{pmatrix}. \end{aligned} \tag{4.56a}$$

The *matrices and vector functions of (4.56a)* are:

$$\boldsymbol{M}_i = \begin{bmatrix} \begin{pmatrix} m_i & , \; 0 & , \; 0 \\ 0 & , \; m_i & , \; 0 \\ 0 & , \; 0 & , \; m_i \end{pmatrix} & , \begin{pmatrix} 0 , 0 , 0 \\ 0 , 0 , 0 \\ 0 , 0 , 0 \end{pmatrix} \\ \begin{pmatrix} 0 , 0 , 0 \\ 0 , 0 , 0 \\ 0 , 0 , 0 \end{pmatrix} & , \begin{pmatrix} J^{L_i}_{c_ix} & , -J^{L_i}_{c_ixy} & , -J^{L_i}_{c_ixz} \\ -J^{L_i}_{c_iyx} & , \; J^{L_i}_{c_iy} & , -J^{L_i}_{c_iyz} \\ -J^{L_i}_{c_izx} & , -J^{L_i}_{c_izy} & , \; J^{L_i}_{c_iz} \end{pmatrix} \end{bmatrix}, \tag{4.56b}$$

$$\boldsymbol{f}_i = \begin{bmatrix} \sum\limits_j \begin{pmatrix} F^R_{ijx} \\ F^R_{ijy} \\ F^R_{ijz} \end{pmatrix} \\ \sum\limits_j \begin{pmatrix} M^{L_i}_{ijx} \\ M^{L_i}_{ijy} \\ M^{L_i}_{ijz} \end{pmatrix} + \sum\limits_i \begin{pmatrix} 0 & , -z^{L_i}_{p_{ij}p_i} & , \; y^{L_i}_{p_{ij}p_i} \\ z^{L_i}_{p_{ij}p_i} & , \; 0 & , -x^{L_i}_{p_{ij}p_i} \\ -y^{L_i}_{p_{ij}p_i} & , \; x^{L_i}_{p_{ij}p_i} & , \; 0 \end{pmatrix} \cdot \begin{pmatrix} F^{L_i}_{ijx} \\ F^{L_i}_{ijy} \\ F^{L_i}_{ijz} \end{pmatrix} \end{bmatrix}, \tag{4.56c}$$

and

$$\boldsymbol{q}_{G_i} = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & , -\omega^{L_i}_{zL_iR} & , \omega^{L_i}_{yL_iR} \\ \omega^{L_i}_{zL_iR} & , 0 & , -\omega^{L_i}_{xL_iR} \\ -\omega^{L_i}_{yL_iR} & , \omega^{L_i}_{xL_iR} & , 0 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{J}^{L_i}_{c_ix} & , -\boldsymbol{J}^{L_i}_{c_ixy} & , -\boldsymbol{J}^{L_i}_{c_ixz} \\ -\boldsymbol{J}^{L_i}_{c_iyx} & , \boldsymbol{J}^{L_i}_{c_iy} & , -\boldsymbol{J}^{L_i}_{c_iyz} \\ -\boldsymbol{J}^{L_i}_{c_izx} & , -\boldsymbol{J}^{L_i}_{c_izy} & , \boldsymbol{J}^{L_i}_{c_iz} \end{pmatrix} \cdot \begin{pmatrix} \omega^{L_i}_{xL_iR} \\ \omega^{L_i}_{yL_iR} \\ \omega^{L_i}_{zL_iR} \end{pmatrix} \right]. \tag{4.56d}$$

4.2.4.2 *Planar* motion. Using, in the *planar case for* (P $\neq$ C), the vectors

$$\begin{aligned} \boldsymbol{r}^L_{P_iP} &= \left(x^L_{P_iP}\,, y^L_{P_iP}\,, 0\right)^{\mathrm{T}} \quad , \quad \boldsymbol{F}^L_i = \left(F^L_{ix}\,, F^L_{iy}\,, 0\right)^{\mathrm{T}}, \\ \boldsymbol{r}^L_{CP} &= \left(x^L_{CP}\,, y^L_{CP}\,, 0\right)^{\mathrm{T}}, \\ \boldsymbol{\omega}^L_{LR} &= \left(0\,, 0\,, \omega^L_{zLR}\right)^{\mathrm{T}} = \left(0\,, 0\,, \dot{\psi}\right)^{\mathrm{T}}, \quad \dot{\boldsymbol{\omega}}^L_{LR} = \left(0\,, 0\,, \ddot{\psi}\right)^{\mathrm{T}}, \end{aligned} \tag{4.57a}$$

and the rotation angle $\psi := \psi_{LR}$ from R to L around the $\boldsymbol{e}_{zR}$-axis, provides the expressions

$$\begin{aligned} &\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{CP} \cdot \dot{\boldsymbol{\omega}}^L_{LR} \\ &= \begin{pmatrix} \cos\psi & , -\sin\psi & , 0 \\ \sin\psi & , \cos\psi & , 0 \\ 0 & , 0 & , 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & , 0 & , y^L_{CP} \\ 0 & , 0 & , -x^L_{CP} \\ -y^L_{CP} & , x^L_{CP} & , 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \ddot{\psi} \end{pmatrix} \\ &= \begin{pmatrix} 0 & , 0 & , \cos\psi \cdot y^L_{CP} + \sin\psi \cdot x^L_{CP} \\ 0 & , 0 & , \sin\psi \cdot y^L_{CP} - \cos\psi \cdot x^L_{CP} \\ -y^L_{CP} & , x^L_{CP} & , 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \ddot{\psi} \end{pmatrix} \\ &= \begin{pmatrix} \left(\cos\psi \cdot y^L_{CP} + \sin\psi \cdot x^L_{CP}\right) \cdot \ddot{\psi} \\ \left(\sin\psi \cdot y^L_{CP} - \cos\psi \cdot x^L_{CP}\right) \cdot \ddot{\psi} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi & , -\sin\psi & , 0 \\ \sin\psi & , \cos\psi & , 0 \\ 0 & , 0 & , 1 \end{pmatrix} \cdot \begin{pmatrix} y^L_{CP} \\ -x^L_{CP} \\ 0 \end{pmatrix} \cdot \ddot{\psi}, \end{aligned} \tag{4.57b}$$

$$\tilde{\boldsymbol{\omega}}_{LR}^{L} = \begin{pmatrix} 0 \ , & -\dot{\psi} \ , & 0 \\ \dot{\psi} \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & 0 \end{pmatrix} = \begin{pmatrix} 0 \ , & -1 \ , & 0 \\ 1 \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & 0 \end{pmatrix} \cdot \dot{\psi}, \tag{4.57c}$$

$$\begin{aligned}
&\boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \boldsymbol{r}_{CP}^{L} \\
&\quad = \begin{pmatrix} \cos\psi \ , & -\sin\psi \ , & 0 \\ \sin\psi \ , & \cos\psi \ , & 0 \\ 0 \ , & 0 \ , & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & -1 \ , & 0 \\ 1 \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} x_{CP}^{L} \\ y_{CP}^{L} \\ 0 \end{pmatrix} \cdot \dot{\psi}^2 \\
&\quad = \begin{pmatrix} \cos\psi \ , & -\sin\psi \ , & 0 \\ \sin\psi \ , & \cos\psi \ , & 0 \\ 0 \ , & 0 \ , & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \ , & 0 \ , & 0 \\ 0 \ , & -1 \ , & 0 \\ 0 \ , & 0 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{CP}^{L} \\ y_{CP}^{L} \\ 0 \end{pmatrix} \cdot \dot{\psi}^2 \\
&\quad = \begin{pmatrix} -\cos\psi \cdot x_{CP}^{L} + \sin\psi \cdot y_{CP}^{L} \\ -\sin\psi \cdot x_{CP}^{L} - \cos\psi \cdot y_{CP}^{L} \\ 0 \end{pmatrix} \\
&\quad = -\begin{pmatrix} \cos\psi \ , & -\sin\psi \ , & 0 \\ \sin\psi \ , & \cos\psi \ , & 0 \\ 0 \ , & 0 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{CP}^{L} \\ y_{CP}^{L} \\ 0 \end{pmatrix} \cdot \dot{\psi}^2,
\end{aligned} \tag{4.57d}$$

$$\begin{aligned}
\tilde{\boldsymbol{r}}_{P_iP}^{L} \cdot \boldsymbol{F}_i^{L} &= \begin{pmatrix} 0 \ , & 0 \ , & y_{P_iP}^{L} \\ 0 \ , & 0 \ , & -x_{P_iP}^{L} \\ -y_{P_iP}^{L} \ , & x_{P_iP}^{L} \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{ix}^{L} \\ F_{iy}^{L} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ -F_{ix}^{L} \cdot y_{P_iP}^{L} + F_{iy}^{L} \cdot x_{P_iP}^{L} \end{pmatrix}
\end{aligned}$$

or (4.57e)

$$\tilde{\boldsymbol{r}}_{P_iP} \cdot \boldsymbol{F}_i = \left(F_{ix}^{L} \cdot y_{P_iP}^{L} + F_{iy}^{L} \cdot x_{P_iP}^{L}\right) \cdot \boldsymbol{e}_{zR} = \left(-y_{P_iP}^{L} \ , \ x_{P_iP}^{L}\right) \cdot \begin{pmatrix} F_{ix}^{L} \\ F_{iy}^{L} \end{pmatrix} \cdot \boldsymbol{e}_{zL},$$

$$\left[\boldsymbol{J}_C^{L} + m \cdot \left((\boldsymbol{r}_{CP}^{L})^{\mathrm{T}} \cdot \boldsymbol{r}_{CP}^{L} \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^{L} \cdot (\boldsymbol{r}_{CP}^{L})^{\mathrm{T}}\right)\right] \cdot \dot{\boldsymbol{\omega}}_{LR}^{L} = \left[\begin{pmatrix} 0 \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & 0 \\ 0 \ , & 0 \ , & J_{Cz} \end{pmatrix}\right.$$

$$
+m\cdot\begin{pmatrix} \left({x_{CP}^{L}}^2+{y_{CP}^{L}}^2\right)-{x_{CP}^{L}}^2, & -x_{CP}^{L}\cdot y_{CP}^{L} & , & 0 \\ -x_{CP}^{L}\cdot y_{CP}^{L} & ,\left({x_{CP}^{L}}^2+{y_{CP}^{L}}^2\right)-{y_{CP}^{L}}^2, & & 0 \\ 0 & , & 0 & ,{x_{CP}^{L}}^2+{y_{CP}^{L}}^2 \end{pmatrix}\cdot
$$

$$
\left.\begin{pmatrix} 0 \\ 0 \\ \ddot{\psi} \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ J_{Cz}^{L}+m\left({x_{CP}^{L}}^2+{y_{CP}^{L}}^2\right)\cdot\ddot{\psi} \end{pmatrix}, \tag{4.57f}
$$

or

$$
\boldsymbol{J}_P\cdot\dot{\boldsymbol{\omega}}_{LR}=\left[J_{Cz}^{L}+m\cdot\left({x_{CP}^{L}}^2+{y_{CP}^{L}}^2\right)\right]\cdot\ddot{\psi}\cdot\boldsymbol{e}_{zR},
$$

and

$$
\tilde{\boldsymbol{\omega}}_{LR}^{L}\cdot\left[\boldsymbol{J}_C^{L}+m\cdot\left((\boldsymbol{r}_{CP}^{L})^{\mathrm{T}}\cdot\boldsymbol{r}_{CP}^{L}\cdot\mathbf{I}_3-\boldsymbol{r}_{CP}^{L}\cdot(\boldsymbol{r}_{CP}^{L})^{\mathrm{T}}\right)\right]\cdot\boldsymbol{\omega}_{LR}^{L} \tag{4.57g}
$$

$$
=\begin{pmatrix} 0 & , -\dot{\psi} & , 0 \\ \dot{\psi} & , \ 0 & , 0 \\ 0 & , \ 0 & , 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ J_{Cz}^{L}+m\cdot\left({x_{CP}^{L}}^2+{y_{CP}^{L}}^2\right)\cdot\ddot{\psi} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

This yields the *planar equations of motion* of an unconstrained rigid body for $\mathrm{P}\neq\mathrm{C}$:

$$
\begin{bmatrix} \begin{pmatrix} m & , 0 \\ 0 & , m \end{pmatrix} & , -m\cdot\begin{pmatrix} \cos\psi & , -\sin\psi \\ \sin\psi & , \cos\psi \end{pmatrix}\cdot\begin{pmatrix} y_{cp}^{L} \\ -x_{cp}^{L} \end{pmatrix} \\ -m\cdot\left(y_{cp}^{L}\ , \ -x_{cp}^{L}\right)\cdot\begin{pmatrix} \cos\psi & , \sin\psi \\ -\sin\psi & , \cos\psi \end{pmatrix} & , \quad J_{cz}^{L}+m\cdot\left({x_{cp}^{L}}^2+{y_{cp}^{L}}^2\right) \end{bmatrix}
$$

$$
\cdot\begin{bmatrix} \begin{pmatrix} \ddot{x}_{PO}^{R} \\ \ddot{y}_{PO}^{R} \end{pmatrix} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \sum\limits_i \begin{pmatrix} F_{ix}^{R} \\ F_{iy}^{R} \end{pmatrix} \\ \sum\limits_i M_{iz}^{L}+\left(-y_{p_ip}^{L}\ , \ x_{p_ip}^{L}\right)\cdot\left(F_{ix}^{L}\ , \ F_{iy}^{L}\right)^{\mathrm{T}} \end{bmatrix}
$$

$$
+\begin{bmatrix} m\cdot\begin{pmatrix} \cos\psi & , -\sin\psi \\ \sin\psi & , \cos\psi \end{pmatrix}\cdot\begin{pmatrix} x_{cp}^{L} \\ y_{cp}^{L} \end{pmatrix}\cdot\dot{\psi}^2 \\ 0 \end{bmatrix}, \tag{4.58a}
$$

or, written in expanded form for $\mathrm{P}\neq\mathrm{C}$,

$$
\begin{aligned} m\cdot\ddot{x}_{PO}^{R}+m\left(-\cos\psi\cdot y_{CP}^{L}-\sin\psi\cdot x_{CP}^{L}\right)\cdot\ddot{\psi} \\ =\sum_i F_{ix}^{R}+m\left(\cos\psi\cdot\dot{\psi}^2\cdot x_{CP}^{L}-\sin\psi\cdot\dot{\psi}^2\cdot y_{CP}^{L}\right), \end{aligned} \tag{4.58b}
$$

$$m \cdot \ddot{y}^R_{PO} + m\left(-\sin\psi \cdot y^L_{CP} + \cos\psi \cdot x^L_{CP}\right) \cdot \ddot{\psi} \qquad (4.58c)$$
$$= \sum_i F^R_{iy} + m\left(\sin\psi \cdot \dot{\psi}^2 \cdot x^L_{CP} + \cos\psi \cdot \dot{\psi}^2 \cdot y^L_{CP}\right),$$

and

$$m\Big[\Big(-y^L_{CP} \cdot \cos\psi - x^L_{CP} \cdot \sin\psi\Big) \cdot \ddot{x}^R_{PO}$$
$$+ \left(-y^L_{CP} \cdot \sin\psi + x^L_{CP} \cdot \cos\psi\right) \cdot \ddot{y}^R_{PO}\Big]$$
$$+ \left[J_{cz} + m\left(\left(x^L_{CP}\right)^2 + \left(y^L_{CP}\right)^2\right)\right] \cdot \ddot{\psi}$$
$$= \sum_i \left(M^L_{iz} - y^L_{P_iP} \cdot F^L_{xi} + x^L_{P_iP} \cdot F^L_{yi}\right). \qquad (4.58d)$$

For P = C, the *planar Newton–Euler equations of motion* have the decoupled *form*

$$\begin{bmatrix} \begin{pmatrix} m \ , & 0 \\ 0 \ , & m \end{pmatrix} , & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ (0 \ , \ 0) & , \ J^L_{cz} \end{bmatrix} \cdot \begin{bmatrix} \begin{pmatrix} \ddot{x}^R_{CO} \\ \ddot{y}^R_{CO} \end{pmatrix} \\ \ddot{\psi} \end{bmatrix} \qquad (4.59a)$$
$$= \begin{bmatrix} \sum_i \begin{pmatrix} F^R_{ix} \\ F^R_{iy} \end{pmatrix} \\ \sum_i \left(M^L_{iz} + \left(-y^L_{p_iC} \ , \ x^L_{p_iC}\right) \cdot \left(F^L_{ix} \ , \ F^L_{iy}\right)^{\mathrm{T}}\right) \end{bmatrix},$$

and the expanded form (P = C, $L = K$)

$$m \cdot \ddot{x}^R_{CO} = \sum_i F^R_{ix}, \qquad (4.59b)$$
$$m \cdot \ddot{y}^R_{CO} = \sum_i F^R_{iy}, \text{ and}$$
$$J^K_{Cz} \cdot \ddot{\psi}_{LR} = \sum_i \left[M^K_{iz} + \left(-y^K_{P_iC} \cdot F^K_{ix} + x^K_{P_iC} \cdot F^K_{iy}\right)\right].$$

4.3 Equations of motion of planar and spatial rigid-body mechanisms

In this section the model equations of unconstrained and constrained *planar* and *spatial* rigid bodies and rigid-body mechanisms will be collected. In the most general form they include: (1) *DEs* of the *kinematic and kinetic behavior* of the mechanism, (2) *algebraic equations* of the *kinematic and active constraints* of the mechanism, and (3) expressions of the associated *constraint reaction forces.*

The complete set of these equations and expressions is called the system of *DAEs.* These DAEs may be written in different forms; e.g., as

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \qquad \textit{(kinematic DEs)} \tag{4.60a}$$

$$\boldsymbol{M}(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = \boldsymbol{f} + {}^{c}\boldsymbol{f} + \boldsymbol{q}_G \qquad \textit{(kinetic DEs)} \tag{4.60b}$$

$$\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{0} \qquad \textit{(constraint equations)} \tag{4.60c}$$

$${}^{c}\boldsymbol{f} = \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda} \qquad \textit{(constraint reaction forces)}, \tag{4.60d}$$

or as

$$\dot{\boldsymbol{p}} \;=\; \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \tag{4.61a}$$

$$\begin{pmatrix} \boldsymbol{M}(\boldsymbol{p}) & , \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) & , \quad \boldsymbol{0}_{n_c,n_c} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f} + \boldsymbol{q}_G \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} \tag{4.61b, 4.61c}$$

with

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = \boldsymbol{\beta}_{\mathrm{c}}$$

as the *constraint acceleration equation*, with

$${}^{c}\boldsymbol{f} = \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda} \tag{4.61d}$$

as the *constraint reaction forces and torques*, with $\boldsymbol{f}$ as the applied forces and torques, and $\boldsymbol{q}_G$ as the centrifugal forces and gyroscopic terms. Theoretical analysis of the DAEs (Equations 4.60 and 4.61) shows that these two types of model equations have slightly different *analytical* and *numerical properties* ([66], [67], [68], [69]). The *analytical properties* of the DAEs will not be discussed here. A few aspects of the *numerical solution of DAEs* will be discussed in *Section 4.4*. Subsequently the DAEs will be written in the form (4.61).

4.3.1 Equations of *planar* motion of unconstrained rigid bodies in DE form and of constrained rigid-body systems in DAE form

Based on the results of Sections 3.1.1 and 3.2.1 (planar kinematics) and of Section 4.2.4 (dynamics), the equations of motion of rigid-body mechanisms under *planar motion* will be collected in the following order: (1) *single unconstrained planar rigid body*, (2) *system of unconstrained planar rigid bodies*, (3) *single constrained planar rigid body*, and (4) *system of constrained planar rigid bodies.*

4.3.1.1 A single unconstrained rigid body. Consider a rigid body i under planar motion, connected to other rigid bodies or to the ground only by force elements (e.g., springs, dampers, and actuators) and *not* by joints (Figure 4.3). Let $\boldsymbol{f}_i$ be the vector of all external forces and torques acting on the body. Then the translational and rotational equations of motion of this unconstrained planar rigid body can be compactly written as follows:*For* a reference point $\mathrm{P}_i \neq \mathrm{C}_i$ on body i (cf. Equation 4.58),

$$\begin{aligned} \dot{\boldsymbol{p}}_i &= \boldsymbol{v}_i && (4.62a)\\ \boldsymbol{M}_i \cdot \dot{\boldsymbol{v}}_i &= \boldsymbol{f}_i + \boldsymbol{q}_{G_i}, && (4.62b) \end{aligned}$$

with

$$\dot{\boldsymbol{p}}_i = \left(\dot{x}^R_{P_iO}\,,\, \dot{y}^R_{P_iO}\,,\, \dot{\psi}_i\right)^{\mathrm{T}} = \boldsymbol{v}_i \;\;,\;\; \dot{\boldsymbol{v}}_i = \ddot{\boldsymbol{p}}_i = \begin{bmatrix} \ddot{x}^R_{P_iO} \\ \ddot{y}^R_{P_iO} \\ \ddot{\psi}_i \end{bmatrix} \;,\; \psi_i := \psi_{L_iR}\,, \tag{4.62c}$$

$$\boldsymbol{M}_i = \tag{4.62d}$$

$$\begin{bmatrix} \begin{pmatrix} m_i \;,\; 0 \\ 0 \;,\; m_i \end{pmatrix} & ,\; -\,m_i\cdot \begin{pmatrix} \cos\psi_i \;,\; -\sin\psi_i \\ \sin\psi_i \;,\; \cos\psi_i \end{pmatrix}\cdot \begin{pmatrix} y^{L_i}_{C_iP_i} \\ -x^{L_i}_{C_iP_i} \end{pmatrix} \\ -\,m_i\cdot\left(y^{L_i}_{C_iP_i}\;,\; -x^{L_i}_{C_iP_i}\right)\cdot \begin{pmatrix} \cos\psi_i \;,\; \sin\psi_i \\ -\sin\psi_i \;,\; \cos\psi_i \end{pmatrix} \;, & J^{L_i}_{c_iz} + m_i\cdot\left({x^{L_i}_{C_iP_i}}^2 + {y^{L_i}_{C_iP_i}}^2\right) \end{bmatrix},$$

$$\boldsymbol{f}_i := \begin{bmatrix} \sum\limits_j \begin{pmatrix} F^R_{ijx} \\ F^R_{ijy} \end{pmatrix} \\ \sum\limits_j \left(M^{L_i}_{ijz} + \left(-y^{L_i}_{P_{ij}P_i}\;,\; x^{L_i}_{P_{ij}P_i}\right)\cdot\left(F^{L_i}_{ijx}\;,\; F^{L_i}_{ijy}\right)^{\mathrm{T}}\right) \end{bmatrix}, \tag{4.62e}$$

and

$$\boldsymbol{q}_{G_i} := \begin{bmatrix} m_i\cdot \begin{pmatrix} \cos\psi_i \;,\; -\sin\psi_i \\ \sin\psi_i \;,\; \cos\psi_i \end{pmatrix}\cdot \begin{pmatrix} x^{L_i}_{C_iP_i} \\ y^{L_i}_{C_iP_i} \end{pmatrix}\cdot \dot{\psi}_i^2 \\ 0 \end{bmatrix}. \tag{4.62f}$$

For a reference point $\mathrm{P}_i = \mathrm{C}_i$ on body i, the model equations are (cf. Equation 4.59):

$$\begin{aligned} \dot{\boldsymbol{p}}_i &= \boldsymbol{v}_i \\ \boldsymbol{M}_i \cdot \dot{\boldsymbol{v}}_i &= \boldsymbol{f}_i, \end{aligned} \tag{4.63a}$$

with

$$\dot{\boldsymbol{v}}_i := \ddot{\boldsymbol{p}}_i = \left(\ddot{x}^R_{C_iO}\,,\, \ddot{y}^R_{C_iO}\,,\, \ddot{\psi}_i\right)^{\mathrm{T}}, \tag{4.63b}$$

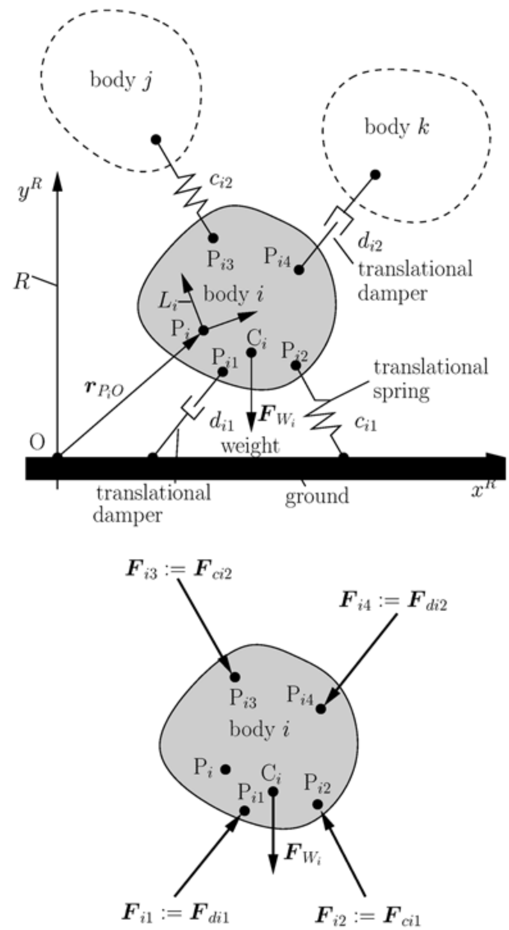


Fig. 4.3: Unconstrained planar rigid body i together with its free-body diagram

$$\boldsymbol{M}_i = \begin{bmatrix} \begin{pmatrix} m_i \ , & 0 \\ 0 \ , & m_i \end{pmatrix} & , \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ (0 \ , \ 0) & , \ J_{c_i z}^{L_i} \end{bmatrix} = \begin{pmatrix} m_i \ , & 0 \ , & 0 \\ 0 \ , & m_i \ , & 0 \\ 0 \ , & 0 \ , & J_{c_i z}^{L_i} \end{pmatrix},$$

$$\boldsymbol{f}_i = \begin{bmatrix} \sum\limits_j \begin{pmatrix} F_{ijx}^R \\ F_{ijy}^R \end{pmatrix} \\ \sum\limits_j \left(M_{ijz}^{L_i} + \left(-y_{P_{ij}C_i}^{L_i} \ , \ x_{P_{ij}C_i}^{L_i} \right) \cdot \left(F_{ijx}^{L_i} \ , \ F_{ijy}^{L_i} \right)^{\mathrm{T}} \right) \end{bmatrix}, \tag{4.63c}$$

and

$$\boldsymbol{q}_{G_i} = (0 \, , 0 \, , 0)^{\mathrm{T}} \, . \tag{4.63d}$$

4.3.1.2 System of unconstrained rigid bodies. Consider a system of n_b *unconstrained* planar rigid bodies connected by force elements (Figure 4.4). Taking into account (4.62a) and (4.62b), the equations of motion of this system are

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{4.64a}$$

$$\boldsymbol{M} \cdot \dot{\boldsymbol{v}} = \boldsymbol{f} + \boldsymbol{q}_G, \tag{4.64b}$$

with

$$\boldsymbol{p} := \left(\boldsymbol{p}_1^{\mathrm{T}}, \ldots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad \boldsymbol{p}_i = \left(x_{P_iO}^R, y_{P_iO}^R, \psi_i\right)^{\mathrm{T}} \in \mathbb{R}^3,$$

$$\boldsymbol{v} := \left(\boldsymbol{v}_1^{\mathrm{T}}, \ldots, \boldsymbol{v}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad \boldsymbol{v}_i^{\mathrm{T}} := \left(\dot{x}_{P_iO}^R, \dot{y}_{P_iO}^R, \dot{\psi}_i\right)^{\mathrm{T}} \in \mathbb{R}^3,$$

$$\boldsymbol{M} := \mathrm{diag}\left(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{n_b}\right) = \begin{pmatrix} \boldsymbol{M}_1 & & 0 \\ & \ddots & \\ \boldsymbol{0} & & \boldsymbol{M}_{n_b} \end{pmatrix} \in \mathbb{R}^{3n_b, 3n_b}, \tag{4.64c}$$

$$\boldsymbol{f} := \left(\boldsymbol{f}_1^{\mathrm{T}}, \ldots, \boldsymbol{f}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad \boldsymbol{q}_G := \left(\boldsymbol{q}_{G_1}^{\mathrm{T}}, \ldots, \boldsymbol{q}_{G_{n_b}}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b},$$

and with $\boldsymbol{v}_i$, $\boldsymbol{M}_i$, $\boldsymbol{f}_i$, and $\boldsymbol{q}_{G_i}$ as defined in (4.62c) to (4.62f), where $\boldsymbol{f}$ contains all external forces and torques acting on these bodies.

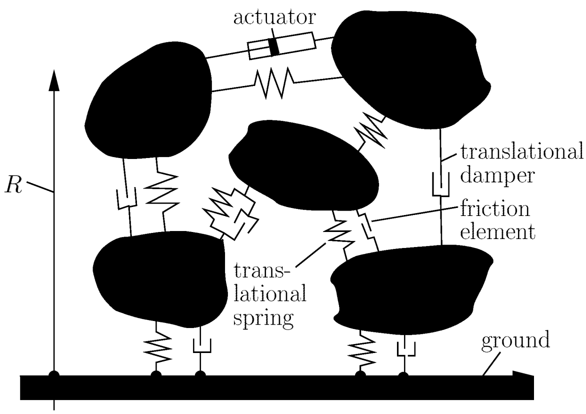


Fig. 4.4: System of unconstrained planar rigid bodies

4.3.1.3 A single rigid body constrained with respect to the base. A single unconstrained planar rigid body i has three DOFs ($3n_b = 3$). It still can move if it is constrained with respect to the base (Figure 4.5) by less than three consistent and independent constraint position equations ($n_c < 3n_b = 3$)

$$\boldsymbol{g}_i(\boldsymbol{p}_i) \equiv \boldsymbol{0}. \tag{4.65a}$$

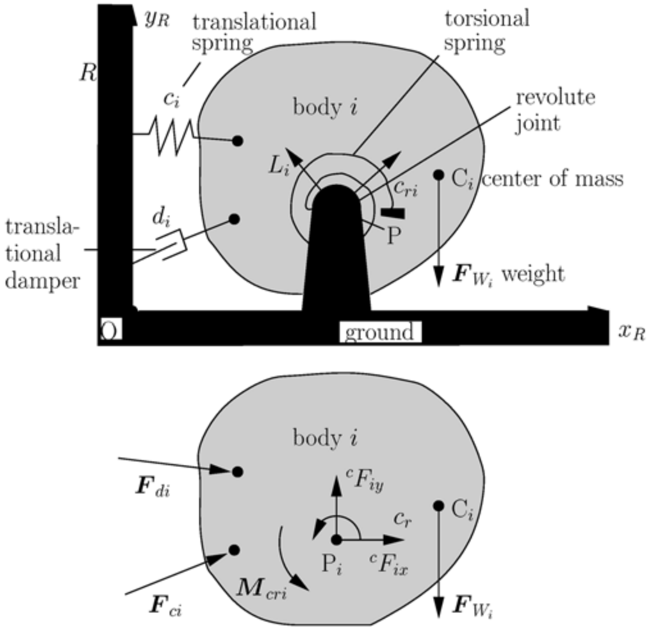


Fig. 4.5: A single constrained rigid body and its free-body diagram

The *kinetic equations of planar motion of a constrained rigid body i* are

$$\boldsymbol{M}_i \cdot \dot{\boldsymbol{v}}_i = \boldsymbol{f}_i + {}^{\mathrm{c}}\boldsymbol{f}_i + \boldsymbol{q}_{G_i} \tag{4.65b}$$

with

$$\dot{\boldsymbol{v}}_i = \ddot{\boldsymbol{p}}_i \quad \text{(planar motion)}, \tag{4.65c}$$

with *external forces* and *torques* $\boldsymbol{f}_i$, with (cf. Equation 3.69)

$${}^{\mathrm{c}}\boldsymbol{f}_i = \boldsymbol{g}_{ip_i}^{\mathrm{T}} \cdot \boldsymbol{\lambda}_i \quad , \quad (\boldsymbol{\lambda}_i \in \mathbb{R}^{n_c}) \, , \tag{4.65d}$$

as the vector of all *constraint reaction forces* and *torques* acting on the body, and with

$$\boldsymbol{g}_{ip_i}(\boldsymbol{p}_i) := \partial \boldsymbol{g}_i(p_i) \Big/ \partial \boldsymbol{p}_i \in \mathbb{R}^{n_c,3} \quad (\mathrm{rank}(\boldsymbol{g}_{ip_i}) = n_c) \tag{4.65e}$$

as the *constraint Jacobian matrix* of the constraint position equations. Then the *equations of motion* of the body are

$$\boldsymbol{M}_i \cdot \ddot{\boldsymbol{p}}_i - \boldsymbol{g}_{ip_i}^{\mathrm{T}} \cdot \boldsymbol{\lambda}_i = \boldsymbol{f}_i + \boldsymbol{q}_{G_i}$$

together with the *constraint position equation*

$$\boldsymbol{g}_i(\boldsymbol{p}_i) = \boldsymbol{0} \ . \tag{4.66a}$$

Taking into account the *constraint acceleration equation*

$$\boldsymbol{g}_{ip_i}(\boldsymbol{p}_i) \cdot \dot{\boldsymbol{v}}_i = \boldsymbol{\beta}_{\mathrm{c}_i} \tag{4.66b}$$

of (4.66a) yields, together with (4.61), the DAEs

$$\dot{\boldsymbol{p}}_i = \boldsymbol{v}_i \tag{4.67a}$$

$$\begin{pmatrix} \boldsymbol{M}_i \ , \ \boldsymbol{g}_{ip_i}^{\mathrm{T}} \\ \boldsymbol{g}_{ip_i} \ , \ \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{v}}_i \\ -\boldsymbol{\lambda}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{\beta}_{\mathrm{c}_i} \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_{G_i} \\ \boldsymbol{0} \end{pmatrix} , \tag{4.67b}$$

with $\boldsymbol{\beta}_{\mathrm{c}_i}$ as defined in (3.8c). These model equations are *DAEs* in $\boldsymbol{p}_i$, $\boldsymbol{v}_i$, and $\boldsymbol{\lambda}_i$.

4.3.1.4 System of constrained rigid bodies. Consider a system of n_b rigid bodies that are constrained by joints and modeled by n_c ($n_c < 3n_b$) algebraic constraint equations (Figure 4.6). For $\mathrm{P}_i \neq \mathrm{C}_i$, and in agreement with the above introduced notations, its model equations are *DAEs* in $\boldsymbol{p}$, $\dot{\boldsymbol{p}}$, and $\boldsymbol{\lambda}$. The kinetic equations of motion are

$$\boldsymbol{M} \cdot \ddot{\boldsymbol{p}} = \boldsymbol{f} + {}^{\mathrm{c}}\boldsymbol{f} + \boldsymbol{q}_G \tag{4.68a}$$

with

$$\boldsymbol{M} := \mathrm{diag}\,(\boldsymbol{M}_1 \, , \, \ldots \, , \, \boldsymbol{M}_{n_b}) \in \mathbb{R}^{3n_b, 3n_b} \ , \ \boldsymbol{M}_i \text{ defined in (4.62d)},$$

$$\boldsymbol{p} := \left(x^R_{P_iO} \, , \, y^R_{P_iO} \, , \, \psi_i \, , \, \ldots \, , \, x^R_{P_{n_b}O} \, , \, y^R_{P_{n_b}O} \, , \, \psi_{n_b}\right)^{\mathrm{T}} ,$$

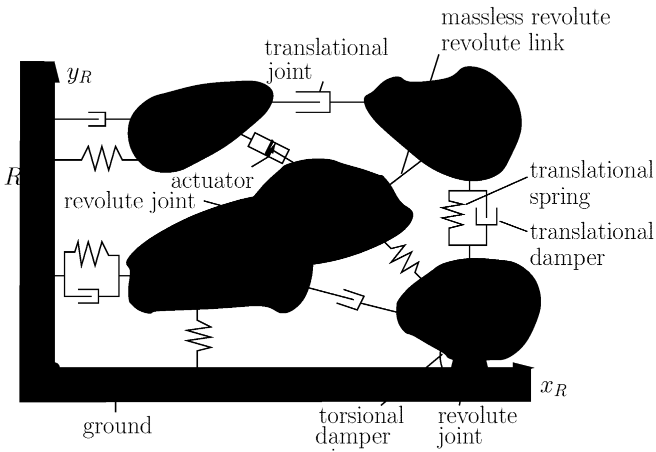


Fig. 4.6: System of constrained planar rigid bodies

$$\boldsymbol{q}_G := \left(\boldsymbol{q}_{G_1}^{\mathrm{T}}, \ldots, \boldsymbol{q}_{G_{n_b}}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{3n_b} \quad , \quad \boldsymbol{q}_{Gi} \text{ defined in (4.62f)},$$

$$\boldsymbol{f} := (\boldsymbol{f}_1^{\mathrm{T}}, \ldots, \boldsymbol{f}_{n_b}^{\mathrm{T}})^{\mathrm{T}} \quad , \quad \boldsymbol{f}_i \text{ defined in (4.62e)}, \tag{4.68b}$$

with

$${}^{\mathrm{c}}\boldsymbol{f} := \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda} \quad , \quad {}^{\mathrm{c}}\boldsymbol{f} := \left({}^{\mathrm{c}}\boldsymbol{f}_1, \ldots, {}^{\mathrm{c}}\boldsymbol{f}_{n_b}\right)^{\mathrm{T}} \in \mathbb{R}^{n_c} \quad \text{and} \quad \lambda \in \mathbb{R}^{n_c},$$

and with the constraint relations

$$\boldsymbol{g}(p) := (\boldsymbol{g}_1(\boldsymbol{p}), \ldots, \boldsymbol{g}_{n_c}(\boldsymbol{p}))^{\mathrm{T}} = \boldsymbol{0},$$

$$\boldsymbol{g}_p(p) \cdot \dot{\boldsymbol{p}} = \boldsymbol{0} \quad , \quad \boldsymbol{g}_p \in \mathbb{R}^{n_c, 3n_b},$$

and

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \ddot{\boldsymbol{p}} = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \dot{\boldsymbol{p}}) \tag{4.68c}$$

with

$$\boldsymbol{g}_p := \left[\left(\partial \boldsymbol{g}_1 / \partial \boldsymbol{p}\right)^{\mathrm{T}}, \ \ldots, \ \left(\partial \boldsymbol{g}_{n_c} / \partial \boldsymbol{p}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$$

and

$$\boldsymbol{\beta}_{\mathrm{c}} := \left(\boldsymbol{\beta}_{\mathrm{c}_1}^{\mathrm{T}}, \cdots, \boldsymbol{\beta}_{\mathrm{c}_{n_c}}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{n_c}.$$

The *DAEs* are compactly written in matrix form (cf. Equation 4.61)

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{4.69a}$$

$$\begin{pmatrix} \boldsymbol{M}(p) & , \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) & , \quad \boldsymbol{0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}, \dot{\boldsymbol{p}}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \dot{\boldsymbol{p}}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_G(\boldsymbol{p}, \dot{\boldsymbol{p}}) \\ \boldsymbol{0} \end{pmatrix}. \tag{4.69b}$$

For the special reference point $\mathrm{P}_i = \mathrm{C}_i$ (4.69) yields, together with

$$\boldsymbol{p} = \left(\boldsymbol{p}_1^{\mathrm{T}}, \ldots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}}, \ \boldsymbol{p}_i = \left(x_{C_iO}^{R}, y_{C_iO}^{R}, \psi_i\right)^{\mathrm{T}},$$

$$\boldsymbol{q}_{G_i} = \left(\boldsymbol{q}_{G_1}^{\mathrm{T}}, \ldots, \boldsymbol{q}_{G_{n_b}}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{0},$$

$$\boldsymbol{M} = \operatorname{diag}(\boldsymbol{M}_1, \ldots, \boldsymbol{M}_{n_b})^{\mathrm{T}}, \quad \text{and} \quad \boldsymbol{M}_i = \begin{pmatrix} m_i & , & 0 & , & 0 \\ 0 & , & m_i & , & 0 \\ 0 & , & 0 & , & J_{c_i z}^{L_i} \end{pmatrix},$$

and with

$$\boldsymbol{f} = \left(\boldsymbol{f}_1^{\mathrm{T}}, \ldots, \boldsymbol{f}_n^{\mathrm{T}}\right)^{\mathrm{T}} \tag{4.70}$$

and

$$\boldsymbol{f}_i = \begin{bmatrix} \sum_j \begin{pmatrix} F_{ijx}^{R} \\ F_{ijy}^{R} \end{pmatrix} \\ \sum_j \left(M_{ijz}^{L_i} + \left(-y_{P_jC_i}^{L_i}, \ x_{P_jC_i}^{L_i}\right) \cdot \left(F_{ijx}^{L_i}, \ F_{ijy}^{L_i}\right)^{\mathrm{T}}\right) \end{bmatrix},$$

the system of $(6n_b + n_c)$ *DAEs* in the $6n_b + n_c$ unknown variables $\boldsymbol{p}$, $\boldsymbol{v}$, and $\boldsymbol{\lambda}$, ($\boldsymbol{p} \in \mathbb{R}^{3n_b}$, $\boldsymbol{\lambda} \in \mathbb{R}^{n_c}$)

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{4.71a}$$

$$\begin{pmatrix} \boldsymbol{M}(p) & , \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(p) & , \quad \boldsymbol{0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}\,,\,\boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}\,,\,\boldsymbol{v}) \end{pmatrix}. \tag{4.71b}$$

4.3.2 Equations of *spatial* motion of unconstrained rigid bodies in DE form and of constrained rigid-body mechanisms in DAE form

The model equations of spatial mechanisms of this section will be written in complete analogy to the planar case of Section 4.3.1.

4.3.2.1 A single unconstrained rigid body. For a reference point $\mathrm{P}_i \neq \mathrm{C}_i$ on a body i, the following *model equations (in DE form)* hold (cf. Equation 4.55):

$$\dot{\boldsymbol{p}}_i = \boldsymbol{T}_i(\boldsymbol{p}_i) \cdot \boldsymbol{v}_i \qquad \textit{(kinematic DEs)} \tag{4.72a}$$

$$\boldsymbol{M}_i \cdot \dot{\boldsymbol{v}}_i = \boldsymbol{f}_i + \boldsymbol{q}_{G_i} \qquad \textit{(kinetic DEs)} \tag{4.72b}$$

with

$$\boldsymbol{p}_i = \left((\boldsymbol{r}_{P_iO}^R)^{\mathrm{T}}\,,\,\boldsymbol{\eta}_i^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^6 \quad , \quad \boldsymbol{r}_{P_iO}^R = \left(x_{P_iO}^R\,,\,y_{P_iO}^R\,,\,z_{P_iO}^R\right)^{\mathrm{T}}, \tag{4.73a}$$

$$\boldsymbol{\eta}_i = (\varphi_i\,,\,\theta_i\,,\,\psi_i)^{\mathrm{T}} \quad , \quad \dot{\boldsymbol{p}}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}\,,\,\dot{\boldsymbol{\eta}}_i^{\mathrm{T}}\right)^{\mathrm{T}}, \tag{4.73b}$$

$$\boldsymbol{v}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}\,,\,{\boldsymbol{\omega}_{L_iR}^{L_i}}^{\mathrm{T}}\right)^{\mathrm{T}}, \tag{4.73c}$$

$$\boldsymbol{\omega}_{L_iR}^{L_i} = \left(\omega_{xi}^{L_i}\,,\,\omega_{yi}^{L_i}\,,\,\omega_{zi}^{L_i}\right)^{\mathrm{T}} = \boldsymbol{A}^{L_iR} \cdot \boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) \cdot \dot{\boldsymbol{\eta}}_i,$$

$$\boldsymbol{A}^{L_iR} = \begin{pmatrix} \mathrm{c}_{i2}\,\mathrm{c}_{i3} & , c_{i1}\,\mathrm{s}_{i3} + s_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , \mathrm{s}_{i1}\,\mathrm{s}_{i3} - \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} \\ -\mathrm{c}_{i2}\,\mathrm{s}_{i3} & , \mathrm{c}_{i1}\,\mathrm{c}_{i3} - \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & , \mathrm{s}_{i1}\,\mathrm{c}_{i3} + \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} \\ \mathrm{s}_{i2} & , \quad -\mathrm{s}_{i1}\,\mathrm{c}_{i2} & , \quad \mathrm{c}_{i1}\,\mathrm{c}_{i2} \end{pmatrix}, \tag{4.73d}$$

$$\boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) := \begin{pmatrix} 1\,, & 0 \,, & \mathrm{s}_{i2} \\ 0\,, & \mathrm{c}_{i1}\,, & -\mathrm{s}_{i1}\cdot\mathrm{c}_{i2} \\ 0\,, & \mathrm{s}_{i1}\,, & \mathrm{c}_{i1}\cdot\mathrm{c}_{i2} \end{pmatrix} \quad , \quad \dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot{\varphi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{pmatrix} \tag{4.73e}$$

with

$$\boldsymbol{T}_i(\boldsymbol{p}_i) = \boldsymbol{T}_i(\boldsymbol{\eta}_i) = \mathrm{diag}(\mathbf{I}_3\;,\;\boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}), \tag{4.73f}$$

with $\mathrm{c}_{ij} := \cos\alpha_{ij}$ and $\mathrm{s}_{ij} := \sin\alpha_{ij}$ for $\alpha_{i1} = \varphi_i$, $\alpha_{i2} = \theta_i$, and $\alpha_{i3} = \psi_i$ for *Bryant angles* φ_i, θ_i, and ψ_i of the body i, and with $\boldsymbol{f}_i$, $\boldsymbol{M}_i$, and $\boldsymbol{q}_{G_i}$ as defined in (4.55c), (4.55d) and (4.55e). *For* $\mathrm{P}_i = \mathrm{C}_i$ (4.55b) yields

$$\dot{\boldsymbol{p}}_i = \boldsymbol{T}_i(\boldsymbol{p}_i) \cdot \boldsymbol{v}_i \tag{4.74a}$$

$$\boldsymbol{M}_i \cdot \dot{\boldsymbol{v}}_i = \boldsymbol{f}_i(\boldsymbol{p}_i\,,\, \boldsymbol{v}_i) + \boldsymbol{q}_{G_i}(\boldsymbol{p}_i\,,\, \boldsymbol{v}_i), \tag{4.74b}$$

with $\boldsymbol{H}_i$ and $\boldsymbol{T}_i$ as defined in (4.73e) and (4.73f), and with $\boldsymbol{M}_i$, $\boldsymbol{f}_i$ and $\boldsymbol{q}_{G_i}$ as defined in (4.55c), (4.55d) and (4.55f).

4.3.2.2 System of unconstrained rigid bodies. Consider a system of n_b unconstrained spatial rigid bodies connected by force elements. Taking into account (4.72a) and (4.72b), the equations of motion of this system are

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \tag{4.75a}$$

$$\boldsymbol{M} \cdot \dot{\boldsymbol{v}} = \boldsymbol{f}(\boldsymbol{p}\,,\, \boldsymbol{v}) + \boldsymbol{q}_G(\boldsymbol{p}\,,\, \boldsymbol{v}) \tag{4.75b}$$

with

$$\boldsymbol{v} := \left(\boldsymbol{v}_1^{\mathrm{T}}\,,\, \ldots\,,\, \boldsymbol{v}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \tag{4.76a}$$

and $\boldsymbol{v}_i$ defined in (4.73c), with

$$\boldsymbol{M} = \operatorname{diag}\left(\boldsymbol{M}_1\,,\, \ldots\,,\, \boldsymbol{M}_{n_b}\right) \in \mathbb{R}^{6n_b, 6n_b}, \tag{4.76b}$$

and $\boldsymbol{M}_i$ defined in (4.55c), with

$$\boldsymbol{T}(\boldsymbol{p}) = \operatorname{diag}\left(\boldsymbol{T}_1(\boldsymbol{p}_1)\,,\, \ldots\,,\, \boldsymbol{T}_{n_b}(\boldsymbol{p}_{n_b})\right), \tag{4.76c}$$

and $\boldsymbol{T}_i(\boldsymbol{p}_i)$ defined in (4.73f), with

$$\boldsymbol{f} = \left(\boldsymbol{f}_1^{\mathrm{T}}\,,\, \ldots\,,\, \boldsymbol{f}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \tag{4.76d}$$

and $\boldsymbol{f}_i$ defined in (4.55f), and with

$$\boldsymbol{q}_{G_i} = \left(\boldsymbol{q}_{G_1}^{\mathrm{T}}\,,\, \ldots\,,\, \boldsymbol{q}_{G_{n_b}}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \tag{4.76e}$$

and $\boldsymbol{q}_{G_i}$ defined in (4.55d).

4.3.2.3 A single rigid body constrained with respect to the base. An unconstrained spatial rigid body i has six DOFs ($n_b = 1,\ 6n_b = 6$). It can still move if it is constrained to the base by less than six independent constraints ($n_c < 6n_b$) with the constraint position equations

$$\boldsymbol{g}_i(\boldsymbol{p}_i) \equiv \boldsymbol{0},$$

with

$$\boldsymbol{p}_i = \left(x_{P_iO}^R\,,\, y_{P_iO}^R\,,\, z_{P_iO}^R\,,\, \varphi_i\,,\, \theta_i\,,\, \psi_i\right)^{\mathrm{T}} = \left((\boldsymbol{r}_{P_iO}^R)^{\mathrm{T}},\, \boldsymbol{\eta}_i^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^6, \tag{4.77a}$$

$$\boldsymbol{r}_{P_iO}^R = \left(x_{P_iO}^R\,,\, y_{P_iO}^R\,,\, z_{P_iO}^R\right)^{\mathrm{T}} \quad , \quad \boldsymbol{\eta}_i = (\varphi_i\,,\,\theta_i\,,\,\psi_i)^{\mathrm{T}}\,, \tag{4.77b}$$

$$\dot{\boldsymbol{p}}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}\,,\,\dot{\boldsymbol{\eta}}_i^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{v}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}\,,\,{\boldsymbol{\omega}_{L_iR}^{L_i}}^{\mathrm{T}}\right)^{\mathrm{T}}\,, \tag{4.77c}$$

$$\boldsymbol{\omega}_{L_iR}^{L_i} = \left(\omega_{xi}^{L_i}\,,\,\omega_{yi}^{L_i}\,,\,\omega_{zi}^{L_i}\right)^{\mathrm{T}} = \boldsymbol{A}^{L_iR}\cdot\boldsymbol{H}^{-1}(\boldsymbol{\eta}_i)\cdot\dot{\boldsymbol{\eta}}_i, \tag{4.77d}$$

$$\boldsymbol{A}^{L_iR} = \begin{pmatrix} \mathrm{c}_{i2}\,\mathrm{c}_{i3} & ,\mathrm{c}_{i1}\,\mathrm{s}_{i3} + \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & ,\mathrm{s}_{i1}\,\mathrm{s}_{i3} - \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} \\ -\mathrm{c}_{i2}\,\mathrm{s}_{i3} & ,\mathrm{c}_{i1}\,\mathrm{c}_{i3} - \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & ,\mathrm{s}_{i1}\,\mathrm{c}_{i3} + \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} \\ \mathrm{s}_{i2} & , \quad -\mathrm{s}_{i1}\,\mathrm{c}_{i2} & , \quad \mathrm{c}_{i1}\,\mathrm{c}_{i2} \end{pmatrix}, \tag{4.77e}$$

$$\boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) = \begin{pmatrix} 1\,,\; 0\;, & \mathrm{s}_{i2} \\ 0\,,\mathrm{c}_{i1}\,, & -\mathrm{s}_{i1}\cdot c_{i2} \\ 0\,,\mathrm{s}_{i1}\,, & \mathrm{c}_{i1}\cdot c_{i2} \end{pmatrix}, \quad \dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot{\varphi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{pmatrix}, \tag{4.77f}$$

and

$$\boldsymbol{T}_i(\boldsymbol{p}_i) = \boldsymbol{T}_i(\boldsymbol{\eta}_i) = \mathrm{diag}(\mathbf{I}_3\;,\;\boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}), \tag{4.77g}$$

where $c_{ij} := \cos\alpha_{ij}$, $s_{ij} := \sin\alpha_{ij}$ for $\alpha_{i1} = \varphi_i$, $\alpha_{i2} = \theta_i$, and $\alpha_{i3} = \psi_i$ for the *Bryant angles* φ_i, θ_i, ψ_i The *equations of spatial motion of a constrained rigid body* i for $\mathrm{P}_i \neq \mathrm{C}_i$ (written as *DAEs*) are (Equation 4.61):

$$\dot{\boldsymbol{p}}_i = \boldsymbol{T}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{v}_i \tag{4.77h}$$

$$\begin{pmatrix} \boldsymbol{M}_i(\boldsymbol{p}_i) & ,\; \boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{p}_i)\cdot\boldsymbol{g}_{ip_i}^{\mathrm{T}}(\boldsymbol{p}_i) \\ \boldsymbol{g}_{ip_i}(\boldsymbol{p}_i)\cdot\boldsymbol{T}_i(\boldsymbol{p}_i)\;, & \mathbf{0} \end{pmatrix}\cdot\begin{pmatrix} \dot{\boldsymbol{v}}_i \\ -\boldsymbol{\lambda}_i \end{pmatrix} \tag{4.77i}$$

$$= \begin{pmatrix} \boldsymbol{f}_i(\boldsymbol{p}_i\,,\,\boldsymbol{v}_i) \\ \boldsymbol{\beta}_{\mathrm{c}_i}(\boldsymbol{p}_i\,,\,\boldsymbol{v}_i) \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_{G_i}(\boldsymbol{p}_i\,,\,\boldsymbol{v}_i) \\ \mathbf{0} \end{pmatrix}$$

with

$\boldsymbol{g}_{\boldsymbol{ip}_i}(\boldsymbol{p}_i) \in \mathbb{R}^{n_c,6}$ (*constraint Jacobian matrix of* $\boldsymbol{g}_i$),

$\boldsymbol{\lambda}_i \in \mathbb{R}^{n_c}$ (*vector of the Lagrange multipliers*),

and (cf. Equation 4.55)

$$\boldsymbol{M}_i = \begin{pmatrix} m_i\cdot\mathbf{I}_3 & ,\; -m_1\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}_{C_iP_i}^{L_i} \\ m_1\cdot\tilde{\boldsymbol{r}}_{C_iP_i}^{L_i}\cdot\boldsymbol{A}^{L_iR}\;, & \boldsymbol{J}_{P_i}^{L_i} \end{pmatrix}, \tag{4.77j}$$

$$\boldsymbol{J}_{P_i}^{L_i} = \left[\boldsymbol{J}_{C_i}^{L_i} + m_i \cdot \left[(\boldsymbol{r}_{C_iP_i}^{L_i})^{\mathrm{T}} \cdot \boldsymbol{r}_{C_iP_i}^{L_i} \cdot \mathbf{I}_3 - \boldsymbol{r}_{C_iP_i}^{L_i} \cdot (\boldsymbol{r}_{C_iP_i}^{L_i})^{\mathrm{T}}\right]\right], \tag{4.77k}$$

$$\boldsymbol{q}_{G_i} = \begin{bmatrix} -m_i \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \cdot \tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \cdot \boldsymbol{r}_{C_iP_i}^{L_i} \\ -\tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \cdot \left[\boldsymbol{J}_{C_i}^{L_i} + m_i \cdot \left[(\boldsymbol{r}_{C_iP_i}^{L_i})^{\mathrm{T}} \cdot \boldsymbol{r}_{C_iP_i}^{L_i} \cdot \mathbf{I}_3 - \boldsymbol{r}_{C_iP_i}^{L_i} \cdot (\boldsymbol{r}_{C_iP_i}^{L_i})^{\mathrm{T}}\right]\right] \cdot \boldsymbol{\omega}_{L_iR}^{L_i} \end{bmatrix}, \tag{4.77l}$$

and with

$$^{\mathrm{c}}\boldsymbol{f}_i = \boldsymbol{T}_i^{\mathrm{T}}(\boldsymbol{p}_i) \cdot \boldsymbol{g}_{ip_i}^{\mathrm{T}}(\boldsymbol{p}_i) \cdot \boldsymbol{\lambda}_i \quad \textit{(constraint reaction forces and torques)}, \tag{4.77m}$$

introduced in (3.76f) and (3.77), $\boldsymbol{\beta}_{\mathrm{c}_i}(\boldsymbol{p}_i, \boldsymbol{v}_i)$ (right hand side of the constraint acceleration equation) defined in (3.8c).

4.3.2.4 System of constrained rigid bodies. Consider a mechanism that includes n_b rigid bodies that are constrained by joints modeled by n_c ($n_c < 6n_b$) algebraic constraint equations. In agreement with the above-introduced notation, the kinetic model equations and constraint position equations of this mechanism are:

$$\boldsymbol{M} \cdot \dot{\boldsymbol{v}} = \boldsymbol{f} + {}^{\mathrm{c}}\boldsymbol{f} + \boldsymbol{q}_G \quad \text{and} \quad \boldsymbol{g}(\boldsymbol{p}) \equiv \mathbf{0}, \tag{4.78a}$$

with

$$\boldsymbol{v} = \left(\boldsymbol{v}_1^{\mathrm{T}}, \dots, \boldsymbol{v}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b} \quad , \quad \boldsymbol{p} = \left(\boldsymbol{p}_1^{\mathrm{T}}, \dots, \boldsymbol{p}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \tag{4.78b}$$

$$\boldsymbol{p}_i = \left((\boldsymbol{r}_{P_iO}^R)^{\mathrm{T}}, \boldsymbol{\eta}_i^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^6 \quad , \quad \boldsymbol{r}_{P_iO}^R = \left(x_{P_iO}^R, y_{P_iO}^R, z_{P_iO}^R\right)^{\mathrm{T}}, \quad \text{and}$$

$$\boldsymbol{\eta}_i = (\varphi_i, \theta_i, \psi_i)^{\mathrm{T}}, \quad \dot{\boldsymbol{p}}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}, \dot{\boldsymbol{\eta}}_i^{\mathrm{T}}\right)^{\mathrm{T}}, \ \boldsymbol{v}_i = \left((\dot{\boldsymbol{r}}_{P_iO}^R)^{\mathrm{T}}, {\boldsymbol{\omega}_{L_iR}^{L_i}}^{\mathrm{T}}\right)^{\mathrm{T}}, \tag{4.78c}$$

with

$$\boldsymbol{\omega}_{L_iR}^{L_i} = \left(\omega_{xi}^{L_i}, \omega_{yi}^{L_i}, \omega_{zi}^{L_i}\right)^{\mathrm{T}} = \boldsymbol{A}^{L_iR} \cdot \boldsymbol{H}_i^{-1} \cdot (\boldsymbol{\eta}_i) \cdot \dot{\boldsymbol{\eta}}_i, \tag{4.78d}$$

$$\boldsymbol{H}_i^{-1}(\boldsymbol{\eta}_i) = \begin{pmatrix} 1, & 0, & \mathrm{s}_{i2} \\ 0, & \mathrm{c}_{i1}, & -\mathrm{s}_{i1} \cdot \mathrm{c}_{i2} \\ 0, & \mathrm{s}_{i1}, & \mathrm{c}_{i1} \cdot \mathrm{c}_{i2} \end{pmatrix} \quad , \quad \dot{\boldsymbol{\eta}} = \begin{pmatrix} \dot{\varphi}_i \\ \dot{\theta}_i \\ \dot{\psi}_i \end{pmatrix}, \tag{4.78e}$$

$$\boldsymbol{T}(\boldsymbol{p}) := \mathrm{diag}\left(\boldsymbol{T}_1(\boldsymbol{p}_1), \dots, \boldsymbol{T}_{n_b}(\boldsymbol{p}_{n_b})\right),$$

$$\boldsymbol{T}_i(\boldsymbol{p}_i) = \boldsymbol{T}_i(\boldsymbol{\eta}_i) = \mathrm{diag}(\mathbf{I}_3, \ \boldsymbol{H}_i(\boldsymbol{\eta}_i) \cdot \boldsymbol{A}^{RL_i}), \tag{4.78f}$$

$$\boldsymbol{M} := \mathrm{diag}\,(\boldsymbol{M}_1, \dots, \boldsymbol{M}_{n_b}) \in \mathbb{R}^{6n_b, 6n_b}, \quad \text{with } \boldsymbol{M}_i \text{ defined in (4.77j)},$$

$$\boldsymbol{H}(\boldsymbol{\eta}) := \mathrm{diag}\left(\boldsymbol{H}_1(\boldsymbol{\eta}_1), \dots, \boldsymbol{H}_{n_b}(\boldsymbol{\eta}_{n_b})\right) \in \mathbb{R}^{3n_b, 3n_b},$$

$$\boldsymbol{q}_G := \left(\boldsymbol{q}_{G_1}^{\mathrm{T}}, \ldots, \boldsymbol{q}_{G_{n_b}}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}\ , \text{ with } \ \boldsymbol{q}_{G_i} \text{ defined in (4.77l)}, \tag{4.78g}$$

$$\boldsymbol{f} := \left(\boldsymbol{f}_1^{\mathrm{T}}, \ldots, \boldsymbol{f}_{n_b}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \text{ with } \ \boldsymbol{f}_i \text{ defined in (4.55b)}\ , \tag{4.78h}$$

$${}^{\mathrm{c}}\boldsymbol{f} = \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda} \quad \textit{(constraint reaction forces and torques)},$$

$${}^{\mathrm{c}}\boldsymbol{f} := \left({}^{\mathrm{c}}\boldsymbol{f}_1^{\mathrm{T}}, \ldots, {}^{\mathrm{c}}\boldsymbol{f}_{n_c}^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{6n_b}, \tag{4.78i}$$

$$\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{n_c})^{\mathrm{T}} \in \mathbb{R}^{n_c} \quad \textit{(Lagrange multipliers)}, \tag{4.78j}$$

$$\boldsymbol{g}(\boldsymbol{p}) = \left(\boldsymbol{g}_i(\boldsymbol{p}), \ldots, \boldsymbol{g}_{n_c}(\boldsymbol{p})\right)^{\mathrm{T}} \in \mathbb{R}^{n_c},$$

$$\boldsymbol{g}_p^{\mathrm{T}} \in \mathbb{R}^{6n_b\,,\,n_c} \quad \textit{(constraint Jacobian matrix)}, \tag{4.78k}$$

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \quad \textit{(constraint acceleration equation)}, \tag{4.78l}$$

and

$$\boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) = \left(\boldsymbol{\beta}_{\mathrm{c}_1}, \ldots, \boldsymbol{\beta}_{\mathrm{c}_{n_c}}\right)^{\mathrm{T}} \in \mathbb{R}^{n_c} \ \textit{(right-hand side} \text{ of Equation 4.78l)}.$$

Compactly written in *DAE form*, these model equations are, for $\mathrm{P}_i \neq \mathrm{C}_i$,

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \tag{4.79a}$$

$$\begin{pmatrix} \boldsymbol{M}(p) & , \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \ , & \boldsymbol{0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} \tag{4.79b}$$
$$= \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{0} \end{pmatrix}.$$

4.4 Numerical solution of DAEs – a brief discussion

In this section a few aspects of common numerical DAE solvers will be discussed briefly. These may be divided into *algebraic* and *analytical* aspects. They will firstly be discussed for *ideal* situations (Section 4.4.1), and then for more *realistic* situations (Section 4.4.2). Before starting this discussion it should be mentioned that basic properties of DAEs may not be preserved when mapping time-continuous DAEs into time-discrete DAEs that are used in the numerical solution process. Among these, their *controllability* or even their *causality* may be lost (e.g., [70]).

4.4.1 Ideal situation

Here a few algebraic and analytical aspects underlying common DAE solvers will be briefly discussed, considering first an *ideal situation* which is characterized by the relations (4.81a), (4.81b), (4.81c), (4.81d), and (4.81e).

4.4.1.1 Algebraic aspects. Consider the DAEs written in the form (4.61)

$$\underbrace{\begin{pmatrix} \mathbf{I}_{6n_b} & , & \mathbf{0}_{6n_b,6n_b} & , & \mathbf{0}_{6n_b,n_c} \\ \mathbf{0}_{6n_b,6n_b} & , & \boldsymbol{M}(\boldsymbol{p}) & , & \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \mathbf{0}_{n_c,6n_b} & , & \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) & , & \mathbf{0}_{n_c,n_c} \end{pmatrix}}_{=:\,\boldsymbol{A}} \cdot \underbrace{\begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix}}_{=:\,\boldsymbol{x}} = \underbrace{\begin{pmatrix} \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \\ \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) + \boldsymbol{g}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix}}_{=:\,\boldsymbol{b}} . \tag{4.80a}$$

They may, as a *first step*, be treated as a *"linear" algebraic equation*

$$\boldsymbol{A} \cdot \boldsymbol{x} = \boldsymbol{b} \tag{4.80b}$$

in the unknown variables $\dot{\boldsymbol{p}}$, $\dot{\boldsymbol{v}}$, and $\boldsymbol{\lambda}$, or in $\boldsymbol{x}$, with

$$\boldsymbol{A} := \begin{pmatrix} \mathbf{I}_{6n_b} & , & \mathbf{0}_{6n_b,6n_b} & , & \mathbf{0}_{6n_b,n_c} \\ \mathbf{0}_{6n_b,6n_b} & , & \boldsymbol{M}(p) & , & \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \mathbf{0}_{n_c,6n_b} & , & \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) & , & \mathbf{0}_{n_c,n_c} \end{pmatrix} \in \mathbb{R}^{12n_b+n_c,12n_b+n_c}, \tag{4.80c}$$

$$\boldsymbol{b} := \begin{pmatrix} \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \\ \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) + \boldsymbol{g}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} \in \mathbb{R}^{12n_b+n_c},$$

and

$$\boldsymbol{x} := \begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} \in \mathbb{R}^{12n_b+n_c}.$$

Consider (4.80b) for $\boldsymbol{p}$ and $\boldsymbol{v}$ from a regime, where

$$\det(\boldsymbol{M}(\boldsymbol{p})) \neq 0 \qquad \textit{(regular mass matrix)}\ , \tag{4.81a}$$

$$\det(\boldsymbol{T}(\boldsymbol{p})) \neq 0 \qquad \textit{(regular kinematic matrix)}, \tag{4.81b}$$

$$\operatorname{rank}(\boldsymbol{g}_p(\boldsymbol{p})) = n_c \qquad \textit{(independent and consistent constraint equations)}, \tag{4.81c}$$

and for vectors $\boldsymbol{p}$ and $\boldsymbol{v}$ that are *consistent* with the *constraint position equations*

$$\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{0}, \tag{4.81d}$$

and with the *constraint velocity equations*

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} = \boldsymbol{0}. \tag{4.81e}$$

Then

$$\det(\boldsymbol{A}(\boldsymbol{p})) \neq 0, \tag{4.82a}$$

and (4.80a) and (4.80b) have a unique solution

$$\boldsymbol{x} = \boldsymbol{A}^{-1}(\boldsymbol{p}) \cdot \boldsymbol{b}. \tag{4.82b}$$

Proof of (4.82a) and (4.82b):

The relation (4.82a) holds iff

$$\det \begin{pmatrix} \boldsymbol{M} & , \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \ , & \boldsymbol{0}_{n_c,n_c} \end{pmatrix} \neq 0. \tag{4.83a}$$

This relation is satisfied iff the homogeneous linear equation

$$\begin{pmatrix} \boldsymbol{M} & , \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{g}_p^{\mathrm{T}} \\ \boldsymbol{g}_p \cdot \boldsymbol{T} & , \ \boldsymbol{0}_{n_c,n_c} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}}_h \\ -\boldsymbol{\lambda}_h \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \tag{4.83b}$$

has the unique solution

$$\left(\dot{\boldsymbol{v}}_h^{\mathrm{T}} \, , \, \boldsymbol{\lambda}_h^{\mathrm{T}}\right) = \left(\boldsymbol{0}_{n_b}^{\mathrm{T}} \, , \, \boldsymbol{0}_{n_c}^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{4.83c}$$

Multiplication of the first row of (4.83b) by $\dot{\boldsymbol{v}}_h^{\mathrm{T}}$ yields

$$\dot{\boldsymbol{v}}_h^{\mathrm{T}} \cdot \boldsymbol{M} \cdot \dot{\boldsymbol{v}}_h - \dot{\boldsymbol{v}}_h^{\mathrm{T}} \cdot \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda}_h \equiv 0. \tag{4.83d}$$

The second row of (4.83b) is

$$\boldsymbol{g}_p \cdot \boldsymbol{T} \cdot \dot{\boldsymbol{v}}_h = \boldsymbol{0} \quad \text{or} \quad \dot{\boldsymbol{v}}_h^{\mathrm{T}} \cdot \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{g}_p^{\mathrm{T}} = \boldsymbol{0}^{\mathrm{T}}. \tag{4.83e}$$

Inserting (4.83e) into (4.83d) yields the relation

$$\dot{\boldsymbol{v}}_h^{\mathrm{T}} \cdot \boldsymbol{M} \cdot \dot{\boldsymbol{v}}_h = 0$$

which, due to (4.81a), implies

$$\dot{\boldsymbol{v}}_h \equiv \boldsymbol{0}. \tag{4.83f}$$

Inserting (4.83f) into the first row of (4.83b) yields

$$\mathbf{0} + \boldsymbol{T}^{\mathrm{T}} \cdot \boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda}_h = \mathbf{0} \tag{4.83g}$$

and, together with (4.81b), the relation

$$\boldsymbol{g}_p^{\mathrm{T}} \cdot \boldsymbol{\lambda}_h = \mathbf{0}$$

or

$$\sum_{i=1}^{n_c} \boldsymbol{g}_{ip}^{\mathrm{T}} \cdot \lambda_{ih} = \mathbf{0} \quad , \quad \boldsymbol{g}_{ip}^{\mathrm{T}} \ i\text{th column of } \boldsymbol{g}_p^{\mathrm{T}}. \tag{4.83h}$$

Due to the assumed linear independence of the n_c columns $\boldsymbol{g}_{ip}^{\mathrm{T}}$ of $\boldsymbol{g}_p^{\mathrm{T}}$ (Equation 4.81c), (4.83h) only holds if the coefficient vector is zero; i.e., for

$$\boldsymbol{\lambda}_h = \mathbf{0}. \tag{4.83i}$$

This proves that the homogeneous linear equation (4.83b) only has the trivial solution (4.83c), and *that the relation (4.83a) holds.* □

The algebraic equation (4.80a) can then be uniquely solved with respect to $\boldsymbol{x}$. This provides the local solution

$$\begin{gathered} \dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \\ \begin{pmatrix} \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}(\boldsymbol{p}) & , \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \ , & \mathbf{0}_{n_c, n_c} \end{pmatrix}^{-1} \cdot \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) + \boldsymbol{g}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} \end{gathered} \tag{4.83j}$$

of (4.80a) for a-priori-given vectors $\boldsymbol{v}$ and $\boldsymbol{p}$, that are consistent with the constraint equations (4.81d) and (4.81e).

4.4.1.2 Numerical integration step. After having solved the linear algebraic equations (4.80a) as a *first step* with respect to $\dot{\boldsymbol{p}}$, $\dot{\boldsymbol{v}}$, and $\boldsymbol{\lambda}$ for consistent vectors $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{v}}$, all taken at a time instant t, a *numerical integration algorithm* is applied *in a second step* to compute $(\boldsymbol{p}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ at the time $(t+1)$ from $(\dot{\boldsymbol{p}}^{\mathrm{T}}, \dot{\boldsymbol{v}}^{\mathrm{T}})^{\mathrm{T}}$ at the time t. There is an important difference between the integration of initial value problems including *DEs* or *DAEs*: the *initial values* $(\boldsymbol{p}_0^{\mathrm{T}}, \boldsymbol{v}_0^{\mathrm{T}})^{\mathrm{T}}$ of a *DE* can be specified arbitrarily, whereas the initial values of a *DAE* must *satisfy* the *constraint position* and *velocity equations* (Equations 4.81d and 4.81e).

Clearly, the above approach is a *simple and crude method for solving DAEs* that does not take into account any modeling error or any error accumulation in the numerical solution process that may invalidade one or more of the above assumptions (4.81a), (4.81b), (4.81c), (4.81d), and (4.81e) caused by:

1. An *inconvenient engineering model* of the mechanism.

2. A *local singularity* (bifurcation, lock-up) *in the constraint equations.*
3. A *local kinematic singularity* in an operation point of the mechanism.
4. An *inappropriate choice of the initial conditions* of the mechanism that are *inconsistent* with (4.81d) or (4.81e).
5. Numerical errors in the *discretization process* of the time continuous equations.
6. Numerical solutions of the equations only providing *approximations to the theoretical solutions.*

4.4.2 More realistic situations

In this section, two critical situations will be briefly discussed: (1) a singular matrix $\boldsymbol{A}$ of (4.80a), and (2) the phenomenon of constraint violation.

4.4.2.1 Singular matrix $\boldsymbol{A}$. The matrix $\boldsymbol{A}$ of (4.80a) may be singular due to an inappropriate engineering model of a mechanism, due to a local singular kinematic matrix $\boldsymbol{T}(\boldsymbol{p})$, a local or global singularity in the constraint equations, or to numerical errors in the algebraic solution process of (4.80a). There exist well-approved and efficient numerical methods from linear algebra that provide reliable numerical solutions of the linear equations (4.80a) and check the regularity or singularity of $\boldsymbol{A}$. Among those, matrix factorization techniques like standard *Gaussian elimination* with full pivoting, *LU factorization*, *singular value decomposition*, *QR decomposition*, or *Gram–Schmidt procedure* offer well-approved numerical algorithms to perform this task. As the matrix $\boldsymbol{A}$ (4.80c) of a rigid-body mechanism usually includes many nullelements (i.e., it is sparse), *sparse matrix algorithms* have been developed for handling those systems.

Two such well-tested and documented computational algorithms are part of the United Kingdom Atomic Energy Authority Harwell science and mathematics library of FORTRAN subroutines ([71]). These programs provide the capability of finding a trade-off between optimal pivoting and preserving the sparsity of $\boldsymbol{A}$ in each solution step by selecting suitable values of control parameters. This algorithm has been implemented in the rigid-body program NUSTAR ([4]).

4.4.2.2 Constraint violation. On account of the numerical integration errors, solutions may be obtained that violate the constraint position and constraint velocity equations (4.81d) and (4.81e). Two main methods for circumventing this problem, called the *coordinate partitioning method* and the *constraint stabilization method*, will be briefly discussed. These methods use quite different ideas to control the accumulation of the numerical errors. In addition, algorithms have been developed that include both of the above approaches ([40]).

Coordinate partitioning

The coordinate partitioning method was first developed for planar motion in the computer program DADS-2D ([72]). Following the lines of [72], the idea behind this method may be briefly summarized as follows (for planar mechanisms with $n_p = 3n_b$):

The coordinate vector $\boldsymbol{p} \in \mathbb{R}^{n_p}$ of a planar mechanism may be partitioned as

$$\boldsymbol{p} = (\boldsymbol{u}^{\mathrm{T}}\,,\, \boldsymbol{w}^{\mathrm{T}})^{\mathrm{T}} \tag{4.84a}$$

with

$\boldsymbol{u} \in \mathbb{R}^{n_c}$ as the *vector of dependent coordinates* and

$\boldsymbol{w} \in \mathbb{R}^{n_p - n_c}$ as the *as vector of independent coordinates*, associated with $\boldsymbol{u}$.

Let

$$\boldsymbol{y} = (\boldsymbol{w}^{\mathrm{T}}\,,\, \dot{\boldsymbol{w}}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{2(n_p - n_c)} \quad \text{and} \quad \dot{\boldsymbol{y}} = (\dot{\boldsymbol{w}}^{\mathrm{T}}\,,\, \ddot{\boldsymbol{w}}^{\mathrm{T}})^{\mathrm{T}} \tag{4.84b}$$

be the *integration arrays*, defined in terms of the *independent coordinates* $\boldsymbol{w}$, the independent velocities $\dot{\boldsymbol{w}}$ and the independent accelerations $\ddot{\boldsymbol{w}}$.

Then the constraint position and velocity equations (Equations 4.81d and 4.81e) can be written as (Sections 3.2.1 and 3.2.2)

$$\boldsymbol{g}(\boldsymbol{u}\,,\, \boldsymbol{w}) = \boldsymbol{0} \tag{4.84c}$$

and

$$\boldsymbol{g}_u \cdot \dot{\boldsymbol{u}} = -\boldsymbol{g}_w \cdot \dot{\boldsymbol{w}}, \tag{4.84d}$$

where (4.84c) and (4.84d) are independent nonlinear algebraic equations in $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$, respectively. Then, at each integration step $\boldsymbol{y}_{t+1}$ is computed from $\dot{\boldsymbol{y}}_t$. Inserting $\boldsymbol{y}_{t+1}$ and $\dot{\boldsymbol{y}}_{t+1}$ in (4.84c) and (4.84d) provides the remainder coordinates $\boldsymbol{u}_{t+1}$ and $\dot{\boldsymbol{u}}_{t+1}$. In principle, this guarantees that the obtained *solution* of the DAEs is *consistent with the constraint equations.*

As is well known, reliable numerical solutions of the nonlinear equations (Equations 4.84c and 4.84d) – for instance by means of iterative methods like a Newton–Raphson algorithm – are critically dependent on a “good” estimate of the start value of $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ at each solution step.

In addition, the numerical error accumulation severely depends on a proper partitioning of $\boldsymbol{p}$ into $\boldsymbol{u}$ and $\boldsymbol{w}$. It may even be necessary to switch from a chosen set of independent coordinates $\boldsymbol{w}_{1t}$ to a different set $\boldsymbol{w}_{2t+1}$ during the integration process. For example, this may be necessary if the number of iterations in the integration steps of the Newton–Raphson process keep increasing from one time step to another.

Based on these criteria, *automatic techniques for partitioning* the coordinate vector $\boldsymbol{p}$ into $\boldsymbol{u}$ and $\boldsymbol{w}$ have been developed. They again use matrix factorization techniques, like those mentioned in Section 4.4.2.1. Then the selection of the dependent coordinates may depend critically on ([41]): the physical unit system chosen in the theoretical model equations, the type of pivoting, and the method of matrix factorization.

Further information on the numerical solution of DAEs can be found in [73], [74], [75], [76], [77], [78], [79], [80], and [81].

Constraint stabilization

The constraint stabilization method is another method to avoid numerical "solutions" of DAEs that do not satisfy the Equations (4.81d) and (4.81e). This method was introduced by Baumgarte in [82].

The idea behind this method may be described as follows: The DAE model (4.80a) of a rigid-body mechanism includes the constraint acceleration equation

$$\boldsymbol{g}_p \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} \equiv \mathbf{0} \quad \text{or} \quad \ddot{\boldsymbol{g}} = \mathbf{0}. \tag{4.85a}$$

This relation is a linear DE of second order in $\boldsymbol{g}$ with $2n_c$ eigenvalues that are located at the origin of the complex plane. Stability theory tells that such a system is extremely sensitive with respect to all types of disturbances and to the accumulation errors in the numerical solution process. In order to improve this situation, Baumgarte replaced the constraint relation (4.85a) by the extended relation

$$\ddot{\boldsymbol{g}} + \kappa_1 \cdot \dot{\boldsymbol{g}} + \kappa_2 \cdot \boldsymbol{g} = \mathbf{0} \tag{4.85b}$$

with free coefficients $\kappa_1\,,\ \kappa_2 \in \mathbb{R}^1$. The relation (4.85b) is a linear DE of second order in $\boldsymbol{g}$. Its $2n_c$ eigenvalues can be placed at arbitrary positions in the complex plane by apropriate choices of κ_1 and κ_2. This provides stable solutions $\boldsymbol{g}(t)$ of (4.85b) that converge asymptotically towards the desired equilibrium solution $\boldsymbol{g} \equiv \mathbf{0}$. As a consequence, (4.85b) is called the *stabilized constraint acceleration equation.* This equation may be written in the form

$$\boldsymbol{g}_p \cdot \boldsymbol{T} \cdot \dot{\boldsymbol{v}} = -\kappa_1 \cdot \dot{\boldsymbol{g}} - \kappa_2 \cdot \boldsymbol{g} + \boldsymbol{\beta}_{\mathrm{c}}$$

or (4.85c)

$$\boldsymbol{g}_p \cdot \boldsymbol{T} \cdot \dot{\boldsymbol{v}} = \underbrace{-\kappa_1 \cdot \boldsymbol{g}_p \cdot \boldsymbol{T} \cdot \boldsymbol{v} - \kappa_2 \cdot \boldsymbol{g} + \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}\,,\, \boldsymbol{v})}_{=:\, \bar{\boldsymbol{\beta}}_{\mathrm{c}}(\boldsymbol{p}\,,\, \boldsymbol{v}\,,\, \kappa_1\,,\, \kappa_2)}.$$

Combining the kinematic and kinetic DEs with (4.85c) provides the modified DAEs

$$\begin{pmatrix} \mathbf{I}_{6n_b} & , & \mathbf{0}_{6n_b,6n_b} & , & \mathbf{0}_{6n_b,n_c} \\ \mathbf{0}_{6n_b,6n_b} & , & \boldsymbol{M}(\boldsymbol{p}) & , & \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \mathbf{0}_{n_c,6n_b} & , & \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) & , & \mathbf{0}_{n_c,n_c} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{p}} \\ \dot{\boldsymbol{v}} \\ -\boldsymbol{\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{T}(\boldsymbol{p}) \cdot v \\ \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) + \boldsymbol{g}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \bar{\boldsymbol{\beta}}_c(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix}, \tag{4.85d}$$

with

$$\bar{\boldsymbol{\beta}}_c(\boldsymbol{p}, \boldsymbol{v}, \kappa_1, \kappa_2) := -\kappa_1 \cdot \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} - \kappa_2 \cdot \boldsymbol{g}(\boldsymbol{p}) + \boldsymbol{\beta}_c(\boldsymbol{p}, \boldsymbol{v}). \tag{4.85e}$$

These differ from the normal DAEs (4.80a) by a modified right-hand side of the constraint acceleration equations.

Comment 4.4.1 (Constraint stabilization method): The constraint stabilization method has been successfully implemented in the rigid-body program NUSTAR ([4]). Various applications of this program to rather complicated rigid-body mechanisms like off-road vehicles (trucks and tanks) under extreme driving maneuvers have provided stable and reliable numerical results, when the stabilization parameters κ_1 and κ_2 were chosen carefully.

The constraint stabilization method is an intuitive appealing ad hoc method. Though its functioning seems to be obvious, its theoretical justification is not trivial. A description of a basic problem behind this procedure was the follows.

The stabilized constraint relations are cross-coupled to the remainder kinematic and kinetic DEs in a closed loop. It must then be expected that the artificially introduced and modified eigenvalues of a subsystem (constraint equations) in the closed loop will also shift the eigenvalues of the remainder subsystems (kinematic and kinetic model equations). This would imply that the numerically motivated eigenvalues – that do not have any physical meaning – would shift the eigenvalues of the subsystems with a clear-cut physical meaning. If this would happen, the constrained stabilization method would not be reliable.

A theoretical analysis of the local dynamic behavior of the DAEs (4.85d) shows that here a *separation principle* holds that (to a first approximation) guarantees that the *numerical* and *physical eigenvalues* of the DAEs do not influence each other ([83]). This result provides a formal justification of the constraint stabilization method.

This situation is formally equivalent to a linear control loop that includes a *linear observer*. As a consequence, the stabilization parameters κ_1 and κ_2 and the resulting eigenvalues can be chosen by direct analogy to the choice of the eigenvalues of an observer in a control loop. The eigenvalues of the stabilized constraint relations shall provide well-damped transients of $\boldsymbol{g}(t)$. The absolute values of these eigenvalues should be chosen to be roughly three times larger than the absolute value of the largest significant eigenvalue of the kinetic and kinematic subsystems of the mechanism.

5. Model equations of planar and spatial joints

Joints prevent a body from either *moving along one* or *several axes*, or from *rotating around one* or *several axes*, or both. As a consequence, joints reduce the number of DOFs of a body or of a mechanism and thereby constrain their motion. Therefore *mathematical models* of joints are called *constraint equations* (*constraint position*, *velocity*, or *acceleration equations*). Theoretical models of joints (between a body and the ground) that constrain the motion of a body with respect to the *ground* are called *absolute constraint equations*, whereas models of joints (between two bodies) that constrain the *relative motion* of two bodies, are called *relative constraint equations.*

Joints are described by *geometric relations* between *absolute (global)* coordinates and/or *relative (local)* coordinates and body-fixed vectors and orientations. These geometric relations are included in the model equations of a mechanism by means of suitably chosen projections and representations of *vector loop equations* and/or *orientation loop equations.*

Vector and orientation loop equations may be written using quite different notations, some of which are suited to model specific joint types. It is of common practice to differ between *absolute* and *relative joint model equations.*

The constraint equations of *absolute joints* are always modeled by *absolute generalized coordinates.* The constraint equations of *relative joints* may include *relative coordinates*:

1. If these *relative coordinates* are *unconstrained* and appear in models of external forces (springs, dampers, actuators), they are calculated according to some compliant relations.
2. If the *relative coordinates* appearing in vector loop equations are *constrained* by geometric joints, they are either *eliminated* (by applying special projection operators to the constraint equations), or *isolated* (by applying alternative projection operators to the constraint equations) if they are needed for *monitoring* or *control* purposes.

To illustrate the above loop equations, consider the simple example of a *vector loop equation* and of an *orientation loop equation* between two bodies i and j in Figure 5.1. The *Vector loop equation* (represented in frame R) is:

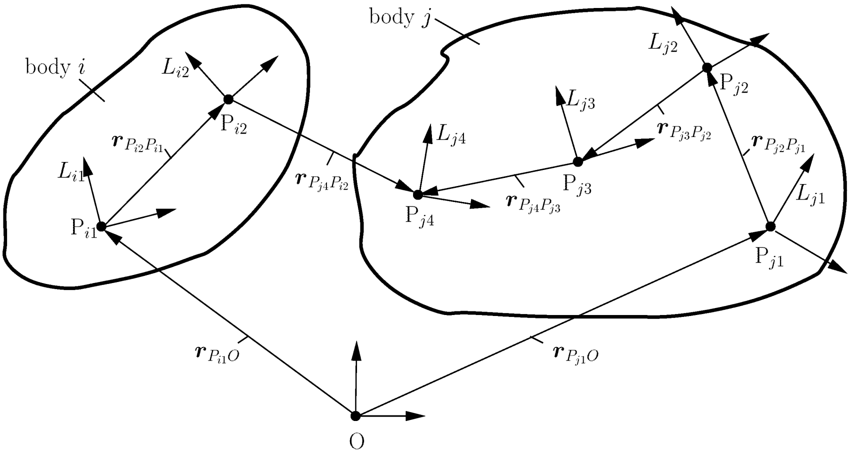


Fig. 5.1: Geometrical representation of a vector loop including two bodies

$$
\begin{aligned}
0 = {} & \underbrace{\boldsymbol{r}^R_{P_{i1}O}}_{\substack{\text{absolute}\\ \text{free vector}}} + \underbrace{\boldsymbol{A}^{RL_{i1}}}_{\substack{\text{free}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{r}^{L_{i1}}_{P_{i2}P_{i1}}}_{\substack{\text{constant}\\ \text{vector}}} + \boldsymbol{A}^{RL_{i1}} \cdot \underbrace{\boldsymbol{A}^{L_{i1}L_{i2}}}_{\substack{\text{fixed}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{r}^{L_{i2}}_{P_{j4}P_{i2}}}_{\substack{\text{relative}\\ \text{vector}}} \\
& - \underbrace{\boldsymbol{r}^R_{P_{j1}O}}_{\substack{\text{absolute}\\ \text{free vector}}} - \underbrace{\boldsymbol{A}^{RL_{j1}}}_{\substack{\text{free}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{r}^{L_{j1}}_{P_{j2}P_{j1}}}_{\substack{\text{constant}\\ \text{vector}}} - \boldsymbol{A}^{RL_{j1}} \cdot \underbrace{\boldsymbol{A}^{L_{j1}L_{j2}}}_{\substack{\text{fixed}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{r}^{L_{j2}}_{P_{j3}P_{j2}}}_{\substack{\text{constant}\\ \text{vector}}} \\
& - \boldsymbol{A}^{RL_{j1}} \cdot \boldsymbol{A}^{L_{j1}L_{j2}} \cdot \underbrace{\boldsymbol{A}^{L_{j2}L_{j3}}}_{\substack{\text{fixed}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{r}^{L_{j3}}_{P_{j4}P_{j3}}}_{\substack{\text{constant}\\ \text{vector}}} .
\end{aligned}
\tag{5.1a}
$$

The *orientation loop equation* (with respect to frame R) is:

$$
\begin{aligned}
& \underbrace{\boldsymbol{A}^{RL_{i1}}}_{\substack{\text{free}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{A}^{L_{i1}L_{i2}}}_{\substack{\text{fixed}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{A}^{L_{i2}L_{j4}}}_{\substack{\text{free relative}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{A}^{L_{j4}L_{j3}} \cdot \boldsymbol{A}^{L_{j3}L_{j2}} \cdot \boldsymbol{A}^{L_{j2}L_{j1}}}_{\substack{\text{fixed}\\ \text{orientation}}} \cdot \underbrace{\boldsymbol{A}^{L_{j1}R}}_{\substack{\text{free}\\ \text{orientation}}} \\
& = \mathbf{I}_3 .
\end{aligned}
\tag{5.1b}
$$

In the *spatial case*, each unconstrained rigid body i has six DOFs: three *rotational* and three *translational.* As a consequence:

1. A *vector loop equation* including three *independent* scalar equations may constrain one, two, or three *translational DOFs* of a body or of a mechanism (this may be achieved by suitable projections of the vector loop equation).
2. An *orientation loop equation* may constrain one, two, or three *rotational degrees of freedom* (this may again be achieved by suitable projections of the orientation loop equation).

If, for example, a single translational DOF is to be constrained, then either of the following my be used: a suitable projection of the vector loop equation, or a scalar product of a vector inside the vector loop with itself, or a function depending on the above equations of the vector loop, or a scalar function depending on one or both of the above scalar equations of the vector loop equations and of the orientation loop equation. The above loop equations provide a basis for deriving systematically algebraic constraint equations for a broad class of different joints. In particular, they suggest the use of scalar loop equations and scalar functions built from several of those as *building blocks*, from which a large number of common joint models can be set up. This has been done, to a certain extend, in [4], [40], and [41] (compare also the spatial case of *Section 5.2.1*).

Besides standard joint models, models of so called *massless links (composite joints)* play an important role in rigid-body dynamics. Various complex mechanisms from industrial practice include rigid bodies or groups of rigid bodies that are characterized by two properties:

1. Their only function within the mechanism is to suitably connect other bodies.
2. Their mass as well as their moments and products of inertia are much smaller than the inertia parameters of the adjacent rigid bodies.

These *couplers* of adjacent bodies are often considered as *composite joints without inertia properties* and are theoretically modeled by kinematic constraint equations. In *Section 5.1*, constraint equations of *planar joints* will be derived using the above-discussed approach. Among those, several absolute and relative planar joints will be discussed together with some technical realizations. In *Section 5.2*, constraint equations of several *building blocks* of *spatial* joints will be derived as a first step. In a second step, compositions of these building blocks will be set up that provide constraint equations and relative coordinates of several *spatial joints* that are commonly used in rigid-body systems.

5.1 Theoretical modeling of *planar* joints

In the *planar case* the number of different joint types is limited. To illustrate the previous discussion, *constraint position equations* (and for several joints, the associated *constraint velocity* and *acceleration equations*) will now be derived for a number of *common planar joints*. Among these two classes of kinematic constraints are of common interest: *absolute planar constraints* between a *body* and the *base (Section 5.1.1)* and *relative planar constraints* between *two bodies (Section 5.1.2)*.

5.1.1 Absolute constraints

A rigid body moving in a plane may have up to three DOFs. *Absolute* planar constraints of a body are those that occur between a body and the base (inertial frame R). They *constrain absolute coordinates* of the body with respect to frame R. Among various absolute constraint situations of a body in a plane, six commonly used configurations will be discussed (cf. Figure 5.2):

1. Absolute *orientation constraints* (*massless translational links* – two remaining translational DOFS).
2. Absolute *partial-position constraints*, (*massless revolute–translational joint*, or *pin* and *slot mechanism* – one remaining translational and one rotational DOF).
3. Absolute *complete-position constraints* (*revolute joint* – one remaining rotational DOF)),
4. Absolute *orientation* and *partial-position constraints* (*translational joint* – one remaining translational DOF),
5. An absolute *constant-distance constraint* (*massless revolute–revolute link* – two remaining DOFs).
6. A *combined absolute orientation/partial-position constraint* (*ball rolling on the ground* or *pulley* – a single remainder DOF).

5.1.1.1 Position constraints between a body and the base. Absolute x-position and/or y-position constraints require that the difference of the x-coordinates and/or of the y-coordinates between a point P_i on body i and a point Q_i on the base are kept constant. Then (Figure 5.3)

$$x^R_{P_iO} - x^R_{Q_iO} = c_x = \text{constant} \quad \textit{(x-position constraint)} \tag{5.2a}$$

and/or

$$y^R_{P_iO} - y^R_{Q_iO} = c_y = \text{constant} \quad \textit{(y-position constraint).} \tag{5.2b}$$

5.1.1.1.1 Partial-position constraint (massless revolute–translational link). An *absolute-position constraint* that only includes one of (5.2a) and (5.2b) is called (Figure 5.4) an *absolute partial-position constraint (massless revolute–translational link).* A mathematical model of this joint is obtained as follows. Consider the vector diagram of a body i connected to the ground by a massless revolute–translational link as in Figure 5.4a. The line of translation of the body is defined by the vector $\boldsymbol{t}_i$ between two noncoinciding points Q_i and S_i on the body i. The revolute joint has a constant distance c from the line of translation of the body i. The constraint position equation of this link is computed from the scalar product

$$(\boldsymbol{d}^R_{O_i})^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \left(\boldsymbol{t}^{L_i}_i\right)^{\perp} / ||\boldsymbol{t}_i|| = c \quad \text{with} \quad ||\boldsymbol{t}_i|| = ||\boldsymbol{t}^R_i|| = ||\boldsymbol{t}^{L_i}_i||, \tag{5.3a}$$

$$\boldsymbol{d}^R_{O_i} = \boldsymbol{r}^R_{Q_0Q_i} \quad , \quad \boldsymbol{t}^R_i = \boldsymbol{r}^R_{S_iQ_i} \quad , \quad \left(\boldsymbol{t}^{L_i}_i\right)^{\perp} = \mathbf{R} \cdot \boldsymbol{t}^{L_i}_i = \mathbf{R} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{r}^R_{S_iQ_i},$$

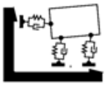
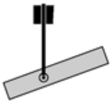
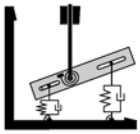
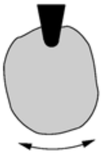
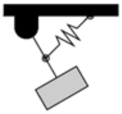
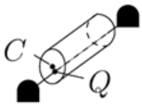
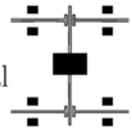
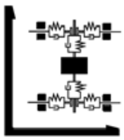
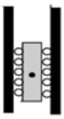
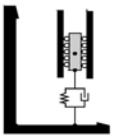
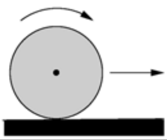
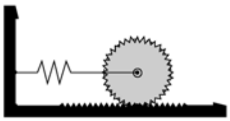
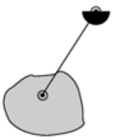
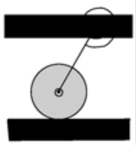
Type of (absolute) planar constraint	Joint type	Technical realization
no constraint	no joint pseudo-joint	airplane  test facility 
partial-position constraint	massless revolute–translational link; pin and slot 	vertical vehicle model 
complete-position constraint	revolute joint 	 pendulum  rotating unbalance
orientation constraint	orthogonal massless translational links 	mechanism for pure orthogonal translational motions 
orientation and partial-position constraint	translational joint 	translational mechanical oscillator 
combined orientation/ partial-position constraint	rack and pinion; rolling wheel 	 rack and pinion
constant-distance constraint	massless revolute–revolute link 	special wheel suspension 

Fig. 5.2: Common absolute joints

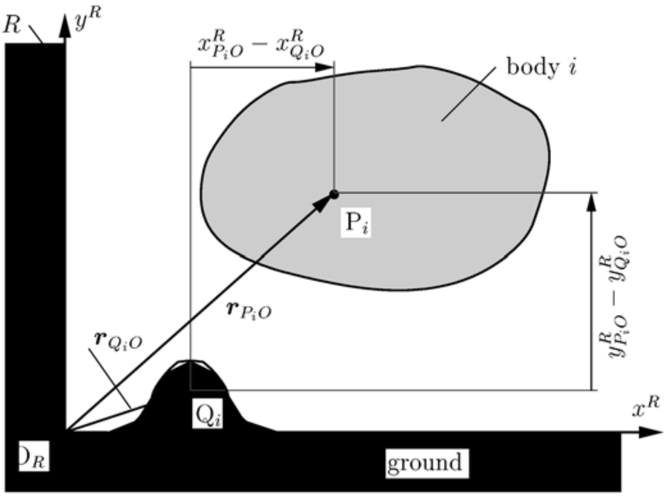


Fig. 5.3: Drawing associated with (5.2a) and (5.2b)

and

$$\mathbf{R} = \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix} \quad \textit{(orthogonal rotation matrix).}$$

Together with the vector loop equation

$$\boldsymbol{r}^R_{Q_0O} - \boldsymbol{r}^R_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{d}^R_{O_i} = \mathbf{0}, \tag{5.3b}$$

the *constraint position equation* is

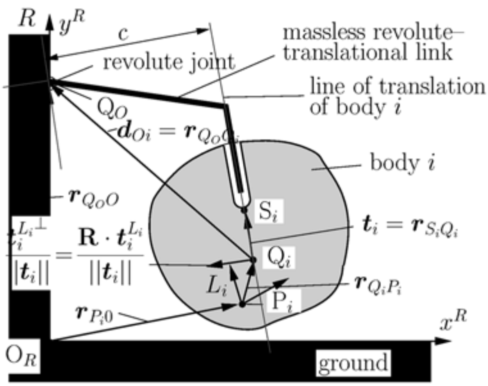
$$\frac{1}{||\boldsymbol{t}_i||} \cdot \left(\left(\boldsymbol{t}^{L_i}_i \right)^{\perp} \right)^{\mathrm{T}} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{d}^R_{O_i} - c = 0, \tag{5.3c}$$

or

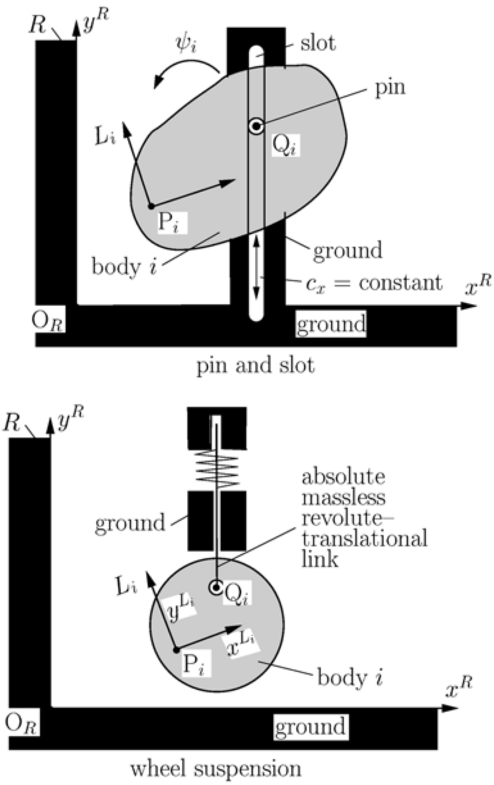
$$\begin{aligned} g = \frac{1}{||\boldsymbol{r}_{S_iQ_i}||} &\cdot (\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{L_iR} \\ &\cdot \left(\boldsymbol{r}^R_{Q_0O} - \boldsymbol{r}^R_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} \right) - c = 0, \end{aligned} \tag{5.3d}$$

or, together with

$$\begin{aligned} \boldsymbol{A}^{RL_i} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{L_iR} &= \begin{pmatrix} \cos\psi \ , & -\sin\psi \\ \sin\psi \ , & \cos\psi \end{pmatrix} \cdot \begin{pmatrix} 0 \ , & 1 \\ -1 \ , & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\psi \ , & \sin\psi \\ -\sin\psi \ , & \cos\psi \end{pmatrix} \\ &= \begin{pmatrix} 0 \ , & 1 \\ -1 \ , & 0 \end{pmatrix} = \mathbf{R}^{\mathrm{T}}, \end{aligned} \tag{5.3e}$$



(a) Vector diagram of an absolute massless revolute–translational link



(b) Technical realizations of mechanisms that include a massless revolute–translational link

Fig. 5.4: Vector diagram and technical realizations of mechanisms that include an absolute massless revolute–translational joint

it is

$$g = \frac{1}{||\boldsymbol{r}_{S_iQ_i}||} \cdot (\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{Q_0O} - \boldsymbol{r}^R_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right) - c = 0. \tag{5.3f}$$

The *constraint velocity equation* associated with (5.3f) is

$$\dot{g} = \frac{1}{||\boldsymbol{r}_{S_iQ_i}||} \cdot (\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \left(-\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right) = 0.$$

It may be written, together with (A.1.11c)

$$\dot{\boldsymbol{A}}^{RL_i} = \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} = \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \dot{\psi} \tag{5.3g}$$

as

$$0 = \underbrace{\left(-(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}\ ,\ -(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right)}_{=:\, \boldsymbol{g}_p(\boldsymbol{p})} \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^R_{P_iO} \\ \dot{\psi} \end{pmatrix} \tag{5.3h}$$

with the *constraint Jacobian matrix*

$$\boldsymbol{g}_p(\boldsymbol{p}) = \left(-(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}\ ,\ -(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right). \tag{5.3i}$$

The *constraint acceleration equation* is obtained from

$$\ddot{g} = 0 \tag{5.3j}$$

as

$$0 = (\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \left(-\ddot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{A}}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} \cdot \dot{\psi} - \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} \cdot \ddot{\psi}\right)$$

or as

$$\begin{aligned} &\underbrace{\left(-(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}\ ,\ -(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]}_{=:\, \boldsymbol{g}_p(\boldsymbol{p})} \cdot \begin{pmatrix} \ddot{\boldsymbol{r}}^R_{P_iO} \\ \ddot{\psi} \end{pmatrix} \\ &\qquad = \underbrace{(\boldsymbol{r}^R_{S_iQ_i})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \mathbf{R}^2 \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} \cdot \dot{\psi}^2}_{=:\, \boldsymbol{\beta}_{\mathrm{c}}} \end{aligned} \tag{5.3k}$$

(cf. Examples 3.3 and 3.4 of Volume II).

5.1.1.1.2 Complete-position constraint (revolute joint). An absolute position constraint that is modeled by both (5.2a) and (5.2b) is called an *absolute complete-position constraint (revolute joint)*. This joint is modeled in more detail by a *vector loop equation*

$$\boldsymbol{r}^{R}_{PO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{P_iP} - \boldsymbol{r}^{R}_{P_iO} = \mathbf{0} \tag{5.4a}$$

(Figure 5.5a), that together with the geometrical relations

$$\boldsymbol{A}^{RL_i} = \begin{pmatrix} \cos\psi_i\,, & -\sin\psi_i \\ \sin\psi_i\,, & \cos\psi_i \end{pmatrix}, \quad \boldsymbol{r}^{L_i}_{P_iP} = \begin{pmatrix} \eta_x \\ -\eta_y \end{pmatrix} = \text{constant}, \quad \text{and} \quad \boldsymbol{r}^{R}_{P_iO} = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} \tag{5.4b}$$

yields the *constraint position equations*

$$x^{R}_{PO} + \cos\psi_i \cdot \eta_x + \sin\psi_i \cdot \eta_y - \gamma_x = 0$$

and (5.4c)

$$y^{R}_{PO} + \sin\psi_i \cdot \eta_x - \cos\psi_i \cdot \eta_y - \gamma_y = 0.$$

The *constraint velocity equation*, associated with (5.4a) and (5.4c), is

$$\dot{\boldsymbol{r}}^{R}_{PO} + \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{P_iP} = \mathbf{0}.$$

Together with

$$\dot{\boldsymbol{A}}^{RL_i} = \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} = \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \dot{\psi}, \tag{5.4d}$$

this provides the equation

$$\underbrace{\left(\mathbf{I}_2\,, \;\; +\boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{P_iP}\right)}_{=:\, \boldsymbol{g}_p(\boldsymbol{p})} \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^{R}_{P_iO} \\ \dot{\psi} \end{pmatrix} = \mathbf{0}. \tag{5.4e}$$

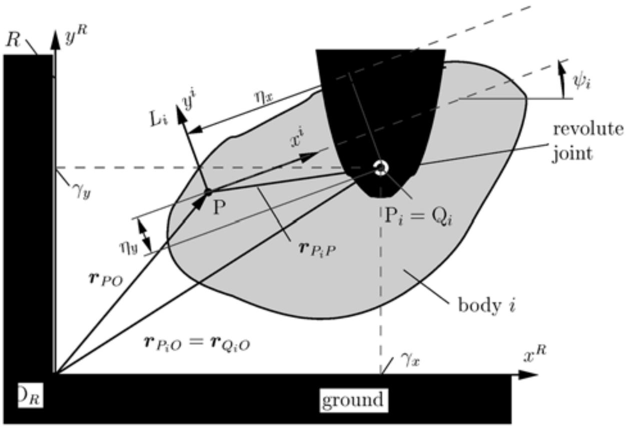
The associated *constraint acceleration equation* is

$$\ddot{\boldsymbol{r}}^{R}_{P_iO} + \dot{\boldsymbol{A}}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{P_iP} \cdot \dot{\psi} + \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{P_iP} \cdot \dot{\psi} = \mathbf{0}$$

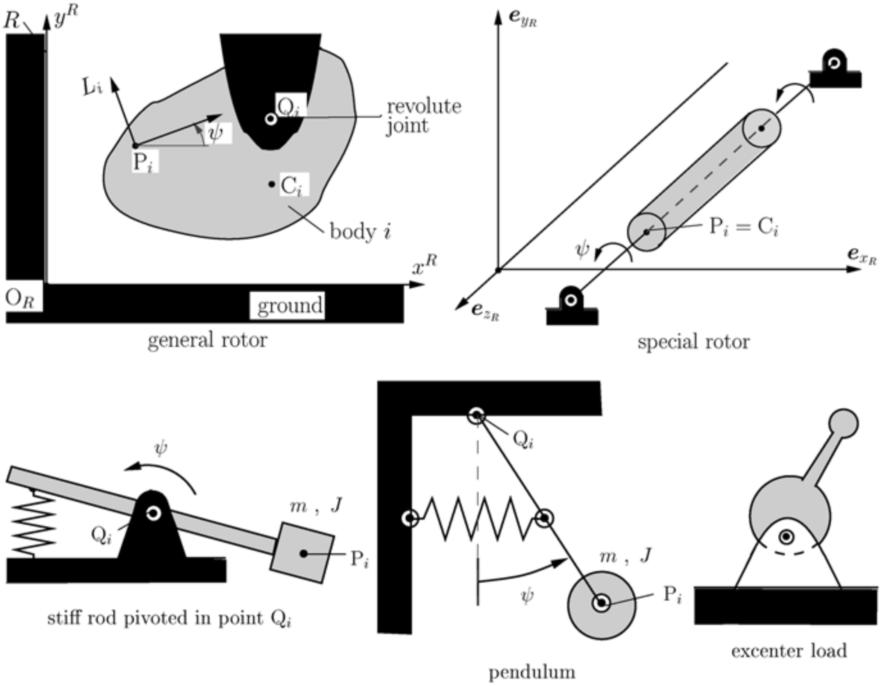
or

$$\underbrace{\left(\mathbf{I}_2\,, \;\; \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{P_iP}\right)}_{=:\, \boldsymbol{g}_p(\boldsymbol{p})} \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^{R}_{P_iO} \\ \dot{\psi} \end{pmatrix} = \underbrace{-\boldsymbol{A}^{RL_i} \cdot \mathbf{R}^2 \cdot \boldsymbol{r}^{L_i}_{P_iP} \cdot \dot{\psi}^2}_{=:\, \boldsymbol{\beta}_{\mathrm{c}}}. \tag{5.4f}$$

Absolute complete-position constraints are technically realized by *revolute joints* (Figures 5.5b and 5.2, and *Example 3.4 of Volume II*).



(a) Vector diagram of an absolute complete-position constraint (revolute joint)



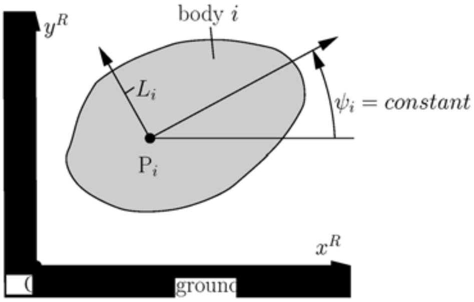
(b) Technical realizations of mechanism that include revolute joints

Fig. 5.5: Drawings of (absolute) revolute joints

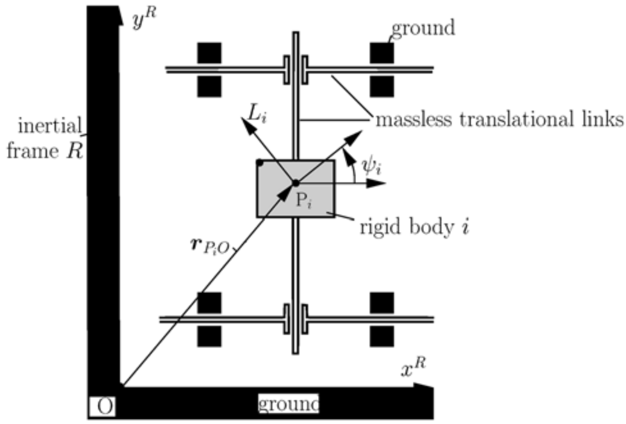
5.1.1.2 Orientation constraint (massless translational link). An absolute orientation constraint requires that the orientation $\psi_i = \psi_{L_iR}$ of a body i is kept constant with respect to the base (Figures 5.6a and 5.2). The associated constraint equation is

$$\psi_i = c_\psi = \text{constant}. \tag{5.5}$$

Technical realizations of this constraint are *combined massless translational links* (Figure 5.6b and *Example 3.1 of Volume II*).



(a) Absolute orientation constraint of a body i



(b) Technical realization of an absolute orientation constraint

Fig. 5.6: Drawings of an absolute orientation constraint

5.1.1.3 Orientation and partial-position constraint (translational joint). A simultaneous *absolute orientation* and *partial-position constraint* is technically realized by a *translational joint* between a body and the ground. This constraint requires that the *orientation* of body i with respect to the base is kept constant,

$$\psi_i = c_\psi = \text{constant}, \tag{5.6a}$$

and that, for example, the x-coordinate $x^R_{P_iO}$ of this body is constant,

$$x^R_{P_iO} = c_x = \text{constant}. \tag{5.6b}$$

Mechanisms that include *translational joints* are shown in Figures 5.7 and 5.2, and in *Example 3.2 of Volume II.*

A more general and more refined constraint equation than (5.6b) is obtained as follows. Consider Figure 5.7a with Q_0 , S_0 and Q_i , S_i as noncoincident points on the base and on the body i, respectively. These four points are located on a common straight line that is chosen as the direction of the translation of the body i with respect to the base. Then the vectors $\boldsymbol{r}_{S_0Q_0}$ and $\boldsymbol{r}_{S_iQ_i}$ must remain collinear, as well as the vectors

$$\boldsymbol{d}_{i0} := \boldsymbol{r}_{S_iQ_0} \quad \text{and} \quad \boldsymbol{t}_0 := \boldsymbol{r}_{S_0Q_0},$$

where $\boldsymbol{d}_{i0}$ is assumed to be non zero. The above requirement can be written as

$$(\boldsymbol{t}_0^\perp)^{\mathrm{T}} \cdot \boldsymbol{d}_{i0} = 0 \quad \textit{(orthogonality relation)} \tag{5.7a}$$

or as

$$\left(\mathbf{R} \cdot \boldsymbol{t}_0^R\right)^{\mathrm{T}} \cdot \boldsymbol{d}_{i0}^R = 0 \tag{5.7b}$$

with

$$\left(\boldsymbol{t}_0^R\right)^\perp := \mathbf{R} \cdot t_0^R \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} 0\ , & -1 \\ 1\ , & 0 \end{pmatrix}.$$

Taking into account the *vector loop equation* (Figure 5.7a)

$$\mathbf{0} = \boldsymbol{r}^R_{Q_0O} + \boldsymbol{d}^R_{i0} - \boldsymbol{A}^{RL_i} \cdot \left(\boldsymbol{r}^{L_i}_{S_iQ_i} + \boldsymbol{r}^{L_i}_{Q_iP_i}\right) - \boldsymbol{r}^R_{P_iO}$$

yields (5.7c)

$$\boldsymbol{d}^R_{i0} = \boldsymbol{A}^{RL_i} \cdot \left(\boldsymbol{r}^{L_i}_{S_iQ_i} + \boldsymbol{r}^{L_i}_{Q_iP_i}\right) + \boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{Q_0O}.$$

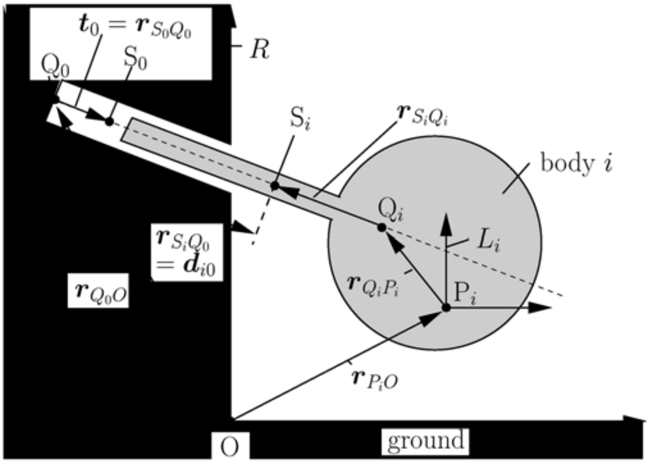
This provides the *constraint position equation*

$$0 = \left(\left(\boldsymbol{t}_0^R\right)^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}\right) \cdot \left[\boldsymbol{A}^{RL_i} \cdot \left(\boldsymbol{r}^{L_i}_{S_iQ_i} + \boldsymbol{r}^{L_i}_{Q_iP_i}\right) + \boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{QO}\right] \tag{5.8a}$$

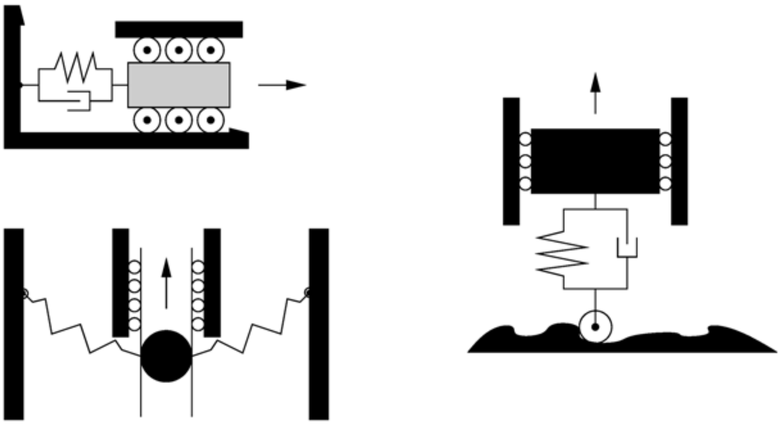
and

$$\psi_i = c_\psi = \text{constant} \tag{5.8b}$$

of an (absolute) *translational joint.* The *constraint velocity* and *acceleration equations* associated with (5.8) are obtained by analogy to the previous cases (see also the various planar mechanisms of Volume II).



(a) Vector diagram of a translational joint



(b) Technical realizations including an absolute translational joint

Fig. 5.7: Drawings of translational joints

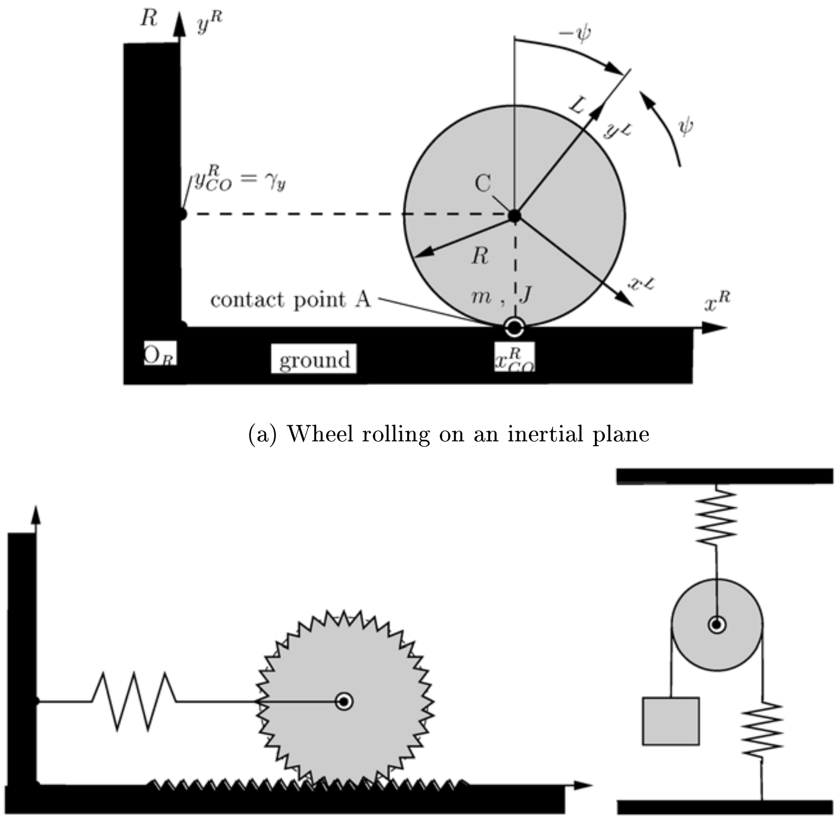
5.1.1.4 Combined orientation/partial-position constraint. A wheel rolling on an inertial plane without slippage is an example of an *absolute combined orientation/partial-position constraint* (Figure 5.8). This constraint is modeled by the *constraint velocity equation*

$$\dot{x}^R_{CO} + R \cdot \dot{\psi} = 0 \tag{5.9a}$$

that holds at the contact point A between the wheel and the base or by the associated *constraint position equation*

$$x^R_{CO} + R \cdot \psi - c_x = 0 \quad , \quad c_x = \text{constant} \tag{5.9b}$$

with R as radius of the wheel. Further technical situations that include this type of constraint are shown in Figures 5.8b and 5.8c, (see also *Example 3.3 of Volume II*).



(a) Wheel rolling on an inertial plane

(b) Rack and pinion mechanism

(c) Mass-spring pulley

Fig. 5.8: Mechanisms that include a combined absolute orientation/partial-position constraint

5.1.1.5 Constant-distance constraint (massless revolute–revolute link) . An absolute distance constraint between point P_i on a body i and point Q_i on the ground is modeled by the *constraint equation*

$$\boldsymbol{r}^{\mathrm{T}}_{P_iQ_i} \cdot \boldsymbol{r}_{P_iQ_i} = d^2 = \text{constant}, \tag{5.10a}$$

with d as the constant distance between points P_i and Q_i (Figure 5.9). Taking into account the *vector loop equation*

$$\boldsymbol{r}^R_{PO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{P_iP} - \boldsymbol{r}^R_{P_iQ_i} - \boldsymbol{r}^R_{OQ_i} = \boldsymbol{0} \tag{5.10b}$$

with

$$\boldsymbol{r}^R_{PO} = \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} \quad , \quad \boldsymbol{A}^{RL_i} = \begin{pmatrix} \cos\psi_i \ , & -\sin\psi_i \\ \sin\psi_i \ , & \cos\psi_i \end{pmatrix},$$

$$\boldsymbol{r}^{L_i}_{P_iP} = \begin{pmatrix} x^{L_i}_{P_iP} \\ 0 \end{pmatrix} = \text{constant, and} \quad \boldsymbol{r}^R_{Q_iO} = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} = \text{constant}$$

yields

$$\gamma_x + x^R_{PO} + \cos\psi_i \cdot x^{L_i}_{P_iP} = x^R_{P_iQ_i}$$

and

$$\gamma_y + y^R_{PO} + \sin\psi_i \cdot x^{L_i}_{P_iP} = y^R_{P_iQ_i},$$

and the relation

$$d = \left((\boldsymbol{r}^R_{P_iQ_i})^{\mathrm{T}} \cdot \boldsymbol{r}^R_{P_iQ_i}\right)^{1/2} = \left({x^R_{P_iQ_i}}^2 + {y^R_{P_iQ_i}}^2\right)^{1/2} = \text{constant},$$

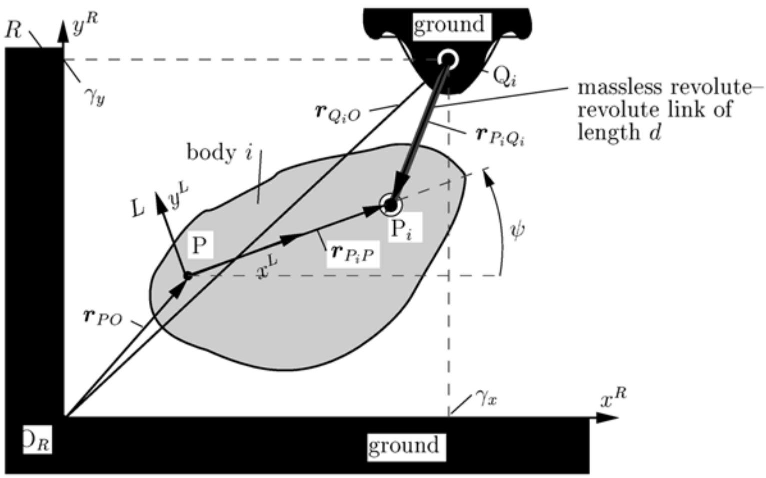


Fig. 5.9: Vector diagram of a mechanism with a constant-distance constraint (massless revolute–revolute link)

and finally the *constraint position equation*

$$g = \left[\left(\gamma_x + x_{PO}^R + x_{P_iP}^{L_i} \cdot \cos \psi_i \right)^2 + \left(\gamma_y + y_{PO}^R + x_{P_iP}^{L_i} \cdot \sin \psi_i \right)^2 \right]^{1/2} - d = 0 \tag{5.10c}$$

which is a function of the generalized coordinates x_{PO}^R, y_{PO}^R, and ψ_i of the body, and of the system constants γ_x, γ_y, $x_{P_iP}^{L_i}$, and $y_{P_iP}^{L_i}$. *Massless revolute–revolute links* are technical realizations of *constant-distance constraints* (see also *Example 3.4 of Volume II*).

5.1.2 Relative planar joints between two bodies

Relative constraints between two bodies i and j constrain relative motion of these bodies. Two rigid bodies (i and j) moving under relative constraints in a plane may together have between one and five degrees of freedom (DOFs). By analogy to the previous absolute constraints of a planar rigid body, the following *relative constraints* of two rigid bodies moving in a plane or in parallel planes are in common use:

1. *Partial relative position* constraint (*massless revolute–translational link* between two bodies).
2. *Complete* relative *position constraint* (*revolute joint* between two bodies).
3. Relative *orientation* and *partial-position constraint* (*translational joint* between two bodies).
4. *Relative orientation constraint* (combined *translational links* between two bodies).
5. *Relative constant-distance constraint* (*massless revolute–revolute link* between two bodies).
6. *Combined relative orientation/partial-position constraint* (*rack* and *pinion* between two bodies).

5.1.2.1 Position constraints. A relative x-position (or y-position) constraint requires that the difference between the x-coordinates (or y-coordinates) of a point Q_i on the body i and of a point Q_j on the body j is equal to a given constant c_x (or c_y); i.e.,

$$x_i^{L_i} - x_j^{L_i} = c_x \quad \textit{(relative x-position constraint)} \tag{5.11a}$$

or

$$y_i^{L_i} - y_i^{L_i} = c_y \quad \textit{(relative y-position constraint).} \tag{5.11b}$$

5.1.2.1.1 Partial-position constraint (massless revolute–translational link). A single of these two constraint equations provides a *relative partial-position constraint*. It is technically built by a *massless revolute–translational link*, as shown in Figures 5.10, 5.11 and 5.12. More refined mathematical model equations of a massless revolute–translational link are obtained as follows.

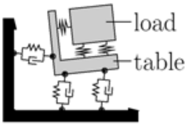
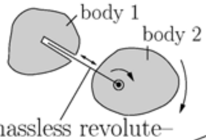
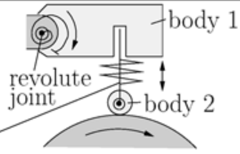
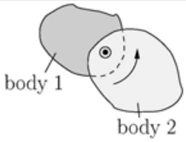
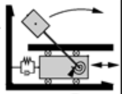
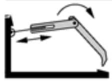
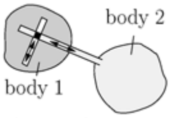
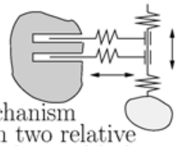
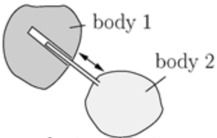
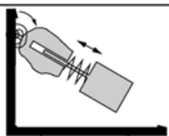
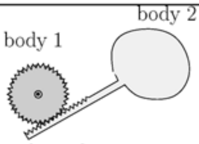
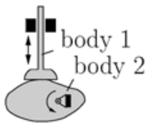
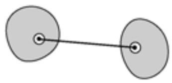
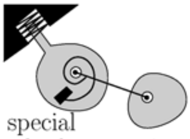
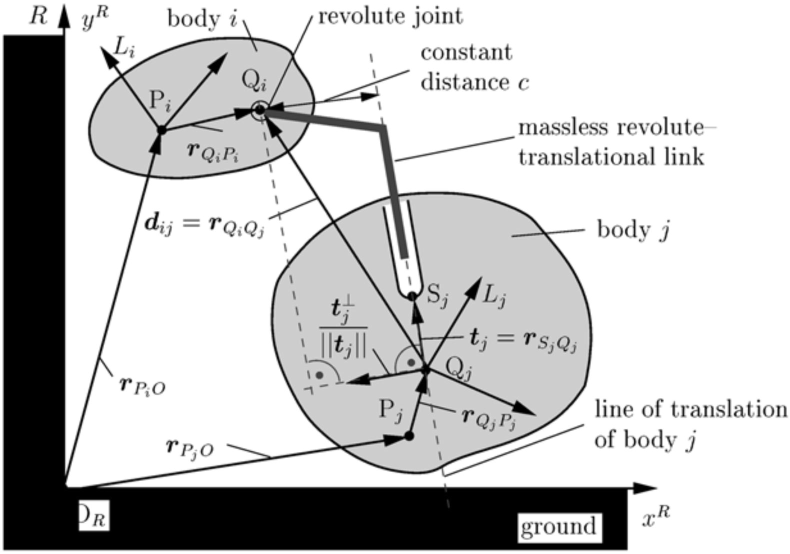
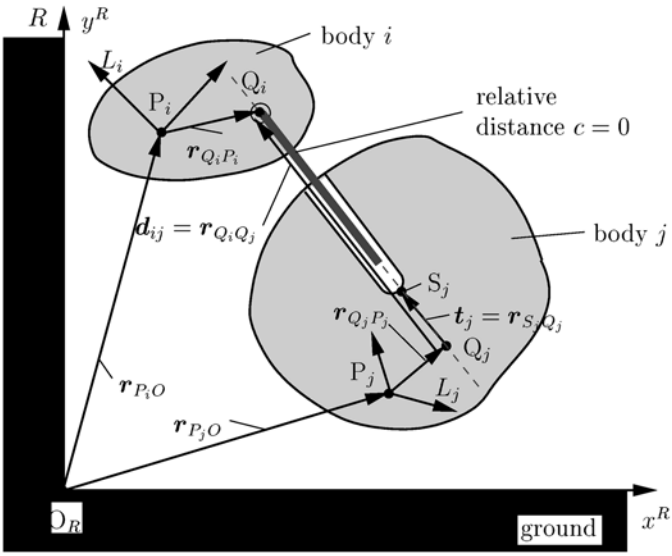
Type of (relative) planar constraint	Joint type	Technical realization
no constraint	no joint pseudo-joint	 load table loaded test facility
partial-position constraint	 body 1 body 2 massless revolute–translational link	 body 1 revolute joint body 2 roller rig
complete-position constraint	 body 1 body 2 revolute joint	cart loaded by a pendulum  excavator 
orientation constraint	 body 2 body 1 translational–translational links	 mechanism with two relative translational links
orientation and partial-position constraint	 body 1 body 2 translational joint	 two degrees of freedom mechanism
combined orientation/ partial-position constraint	 body 2 body 1 rack and pinion	 body 1 body 2 cam follower
constant-distance constraint	 massless revolute–revolute link	 special wheel suspension

Fig. 5.10: Common relative joints



(a) General massless revolute–translational link



(b) Special massless revolute–translational link

Fig. 5.11: Two types of massless revolute–translational links

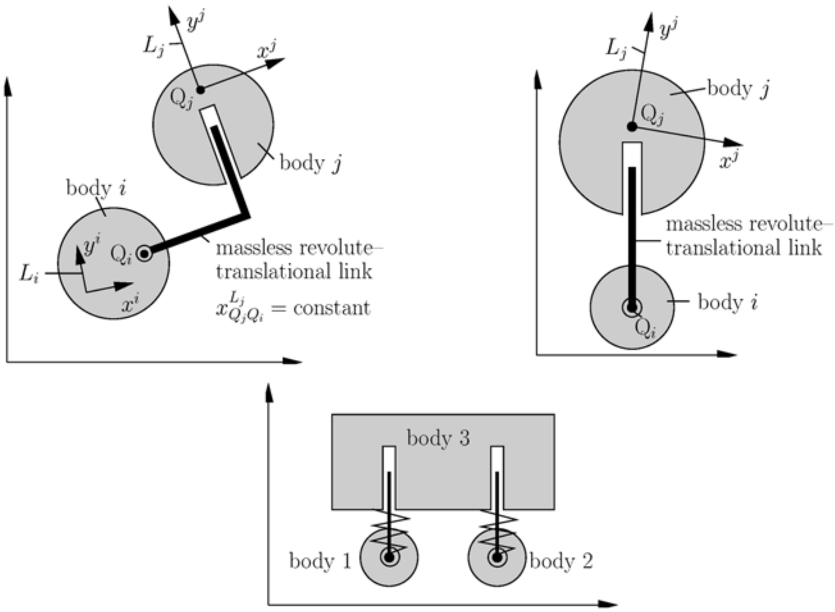


Fig. 5.12: Mechanisms (wheel suspensions) including relative partial-position constraints (massless revolute–translational links)

Consider the vector diagram of Figure 5.11a showing a massless link between two bodies with a revolute joint on body i and a translational joint on body j. The line of translation of body j is defined by two noncoinciding points Q_j and S_j on this body. The revolute axis is located in point Q_i on body i. The revolute joint is a constant distance c from the line of translation. Then the constraint position equation of this link can be written as the scalar product

$$(\boldsymbol{d}^R_{ij})^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{t}_j^{L_j \perp}/\|\boldsymbol{t}_j\| = c \quad (c = \text{constant relative distance}) \tag{5.12a}$$

with

$$\boldsymbol{d}^R_{ij} = \boldsymbol{r}^R_{Q_iQ_j} \quad , \quad \boldsymbol{t}^R_j = \boldsymbol{r}^R_{S_jQ_j} \quad , \quad \boldsymbol{t}_j^{L_j \perp} = \mathbf{R} \cdot \boldsymbol{t}_j^{L_j} = \mathbf{R} \cdot \boldsymbol{r}^{L_j}_{S_jQ_j},$$

$$\mathbf{R} = \begin{pmatrix} 0 \;, & -1 \\ 1 \;, & 0 \end{pmatrix} \quad \textit{(orthogonal rotation matrix)}, \quad \text{and} \quad \mathbf{R} \cdot \boldsymbol{A}^{RL_j} = \boldsymbol{A}^{RL_j} \cdot \mathbf{R}.$$

This provides, together with the *vector loop equation*

$$\mathbf{0} = \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{d}^{L_j}_{ij},$$

the *constraint position equation*

$$g = \frac{1}{\|\boldsymbol{r}_{S_jQ_j}\|} \cdot \left((\boldsymbol{r}^{L_j}_{S_jQ_j})^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{A}^{L_jR} \right) \tag{5.12b}$$
$$\cdot \left(\boldsymbol{r}^{R}_{P_iO} - \boldsymbol{r}^{R}_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} \right) - c = 0.$$

5.1.2.1.2 Complete-position constraint (revolute joint) . The above two constraint equations (5.11a) and (5.11b) together define a *complete relative-position constraint.* This *constraint* is technically built by a *revolute joint* shown in Figure 5.13a, with technical realizations shown in Figures 5.13b and 5.10.

More refined constraint position equations for a *revolute joint* at a common point $\mathrm{Q}_i = \mathrm{Q}_j$ of two bodies i and j are obtained from the *vector loop equation* (Figure 5.13a)

$$\boldsymbol{g} = \boldsymbol{r}^{R}_{P_iO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{r}^{R}_{P_jO} = \mathbf{0}. \tag{5.13a}$$

Together with the geometry relations

$$\boldsymbol{A}^{RL_\kappa} = \begin{pmatrix} \cos\psi_\kappa \;, & -\sin\psi_\kappa \\ \sin\psi_\kappa \;, & \cos\psi_\kappa \end{pmatrix} \quad , \qquad \kappa = i\,,\, j, \tag{5.13b}$$

and

$$\boldsymbol{r}^{L_i}_{Q_iP_i} = \begin{pmatrix} \lambda_x \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{r}^{L_j}_{Q_jP_j} = \begin{pmatrix} -\eta_x \\ \eta_y \end{pmatrix}, \tag{5.13c}$$

this provides the *constraint position equations of the revolute joint* between the bodies i and j represented in coordinate form:

$$x^{R}_{P_iO} - x^{R}_{P_jO} + \lambda_x \cdot \cos\psi_1 + \eta_x \cdot \cos\psi_2 + \eta_y \cdot \sin\psi_2 = 0$$

and (5.13d)

$$y^{R}_{P_iO} - y^{R}_{P_jO} + \lambda_x \cdot \sin\psi_1 + \eta_x \cdot \sin\psi_2 - \eta_y \cdot \cos\psi_2 = 0.$$

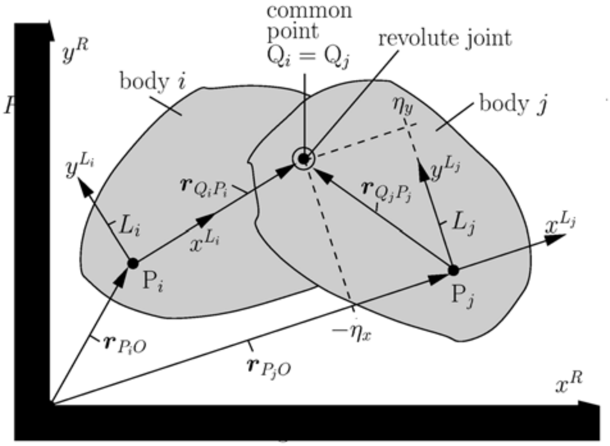
The *constraint velocity equations* associated with (5.13a) are

$$\dot{\boldsymbol{r}}^{R}_{P_iO} + \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} + \dot{\boldsymbol{A}}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \dot{\boldsymbol{r}}^{R}_{P_jO} = \mathbf{0}$$

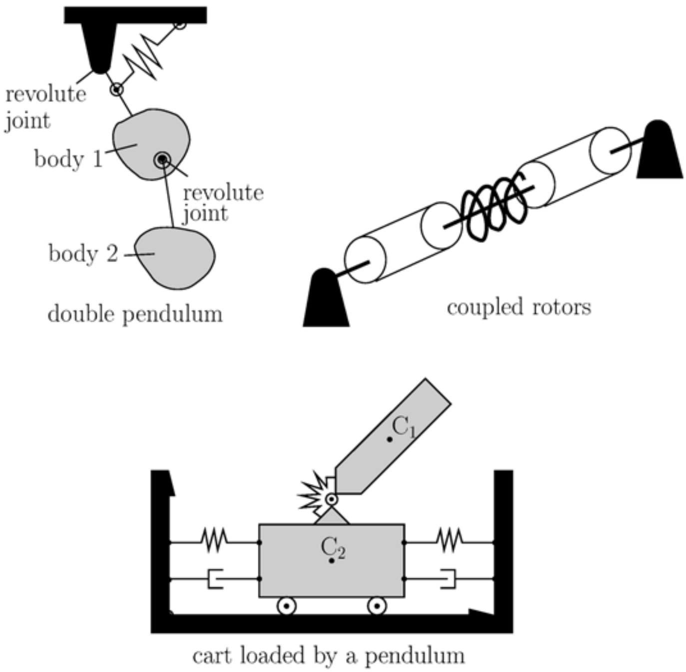
or (5.13e)

$$\underbrace{\left(\mathbf{I}_2 \;,\; \boldsymbol{A}^{RL_i} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} \;,\; -\mathbf{I}_2 \;,\; \boldsymbol{A}^{RL_j} \cdot \mathbf{R} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} \right)}_{=:\, \boldsymbol{g}_p(\boldsymbol{p})} \cdot \begin{pmatrix} \dot{\boldsymbol{r}}^{R}_{P_iO} \\ \dot{\psi}_i \\ \dot{\boldsymbol{r}}^{R}_{P_jO} \\ \dot{\psi}_j \end{pmatrix} = \mathbf{0}.$$

The associated *constraint acceleration equations* are



(a) Vector diagram of a (relative) revolute joint



(b) Technical realizations of planar mechanisms that include revolute joints

Fig. 5.13: Drawings of revolute joints between two bodies

$$\begin{aligned}
&\left(\mathbf{I}_2\ ,\ \boldsymbol{A}^{RL_i}\cdot\mathbf{R}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\ ,\ \ -\mathbf{I}_2\ ,\ \boldsymbol{A}^{RL_j}\cdot\mathbf{R}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\right)\\
&\qquad\cdot\left((\dot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\ ,\psi_i\ ,(\dot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}\ ,\psi_j\right)^{\mathrm{T}}\\
&=\underbrace{-\boldsymbol{A}^{RL_i}\cdot\mathbf{R}^2\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\cdot\dot{\psi}_i^2+\boldsymbol{A}^{RL_j}\cdot\mathbf{R}^2\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\cdot\dot{\psi}_j^2}_{=:\ \boldsymbol{\beta}_{\mathrm{c}}}.
\end{aligned}\tag{5.13f}$$

5.1.2.2 Orientation constraint (massless translational link). A relative orientation constraint between two bodies i and j implies the relation (Figure 5.14a)

$$\psi_i-\psi_j=c_\psi=\text{constant},\tag{5.14a}$$

with

$$\psi_i:=\psi_{L_iR}\qquad,\qquad\psi_j:=\psi_{L_jR}\tag{5.14b}$$

and with

$$\psi_{ij}:=\psi_i-\psi_j\tag{5.14c}$$

as the relative angle between the two bodies. It is related to the *orientation loop equation*

$$\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{L_iL_j}\cdot\boldsymbol{A}^{RL_i}=\mathbf{I}_2\tag{5.14d}$$

or

$$\boldsymbol{A}^{L_iL_j}=\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j},\tag{5.14e}$$

with

$$\begin{pmatrix}\cos\psi_{ij}\ ,&-\sin\psi_{ij}\\ \sin\psi_{ij}\ ,&\cos\psi_{ij}\end{pmatrix}=\begin{pmatrix}\cos\psi_i\ ,&\sin\psi_i\\ -\sin\psi_i\ ,&\cos\psi_i\end{pmatrix}\cdot\begin{pmatrix}\cos\psi_j\ ,&-\sin\psi_j\\ \sin\psi_j\ ,&\cos\psi_j\end{pmatrix},$$

or

$$\cos\psi_{ij}:=\cos\psi_i\cdot\cos\psi_j+\sin\psi_i\cdot\sin\psi_j=\cos\left(\psi_i-\psi_j\right),\tag{5.14f}$$

which implies the relation

$$\psi_{ij}=\psi_i-\psi_j.\tag{5.14g}$$

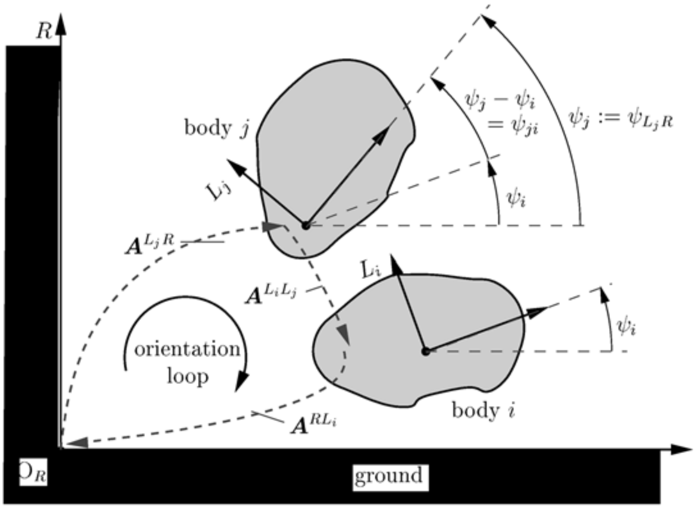
A more general relative orientation constraint is obtained by mechanisms that include *special gear sets* (Figure 5.15). The constraint equation of this mechanism is obtained by the velocity balance for rotations without slippage:

$$r_1\cdot\dot{\psi}_1=-r_2\cdot\dot{\psi}_2\quad\text{or}\quad r_1\cdot\dot{\psi}_1+r_2\cdot\dot{\psi}_2=0$$

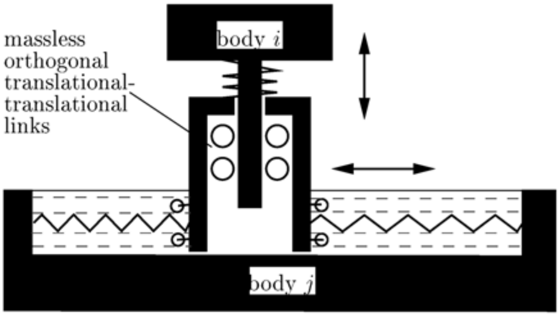
or

$$r_1\cdot\psi_1+r_2\cdot\psi_2=-\left(r_1\cdot\psi_{10}+r_2\cdot\psi_{20}\right)=\text{constant},$$

with ψ_{10} and ψ_{20} as initial orientations.



(a) Orientation loop associated with a relative orientation constraint equation (5.14d)



(b) Technical realization of a relative orientation constraint

Fig. 5.14: Drawings of a relative orientation constraint

5.1.2.3 Relative orientation and partial-position constraint (translational joint). A relative orientation and partial position constraint is technically realized by a *translational joint between two bodies i and j* (Figures 5.16 and 5.10). The constraint position equations of the *translational joint* are obtained from the requirements that this joint does not allow *relative rotations* of the two bodies i and j; i.e.,

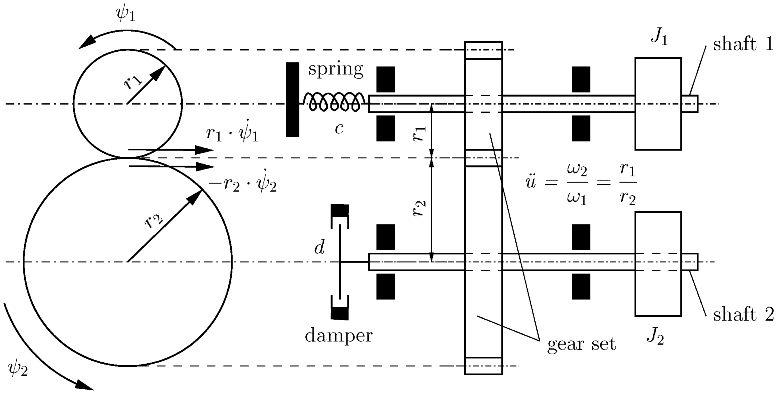


Fig. 5.15: Mechanism including a gear set

$$\psi_{ij} := \psi_i - \psi_j = c_\psi = \text{constant}$$

or (5.15a)

$$\boldsymbol{A}^{L_i L_j} = \begin{pmatrix} \cos\psi_{ij} \ , & -\sin\psi_{ij} \\ \sin\psi_{ij} \ , & \cos\psi_{ij} \end{pmatrix} = \text{constant},$$

but only *relative translations along a common axis*, where the latter requirement can be mathematically modeled as follows. Consider the geometrical situation of Figure 5.16a with noncoincident points S_i , Q_i, and S_j , Q_j on the bodies i and j, respectively, located on a common straight line that defines the direction of the relative translation of the bodies. Then the vectors $\boldsymbol{t}_i := \boldsymbol{r}_{S_i Q_i}$ and $\boldsymbol{r}_{S_j Q_j}$ must remain collinear as well as the vector $\boldsymbol{d}_{ji} := \boldsymbol{r}_{S_j Q_i}$ where the vector $\boldsymbol{r}_{S_j Q_i}$ is assumed to be nonzero. The above requirement can be written in terms of the orthogonality relation

$$\left(\boldsymbol{t}_i^{\perp}\right)^{\mathrm{T}} \cdot \boldsymbol{d}_{ji} = 0$$

or (5.15b)

$$\left(\mathbf{R} \cdot \boldsymbol{t}_i^{L_i}\right)^{\mathrm{T}} \cdot \boldsymbol{d}_{ji}^{L_i} = 0$$

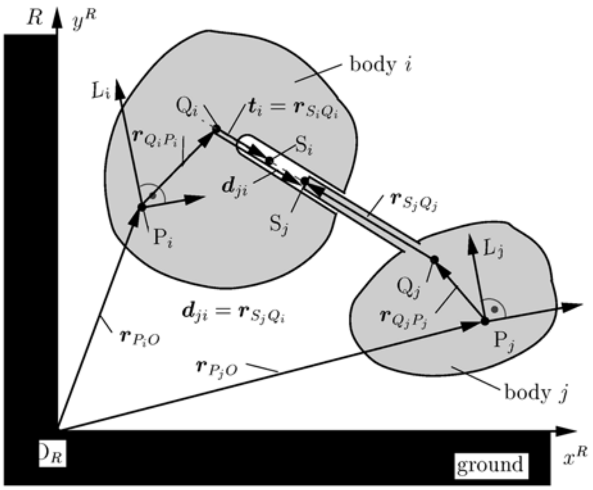
with

$$\boldsymbol{t}_i^{L_i \perp} := \mathbf{R} \cdot \boldsymbol{t}_i^{L_i} \quad \text{and} \quad \mathbf{R} := \begin{pmatrix} 0 \ , & -1 \\ 1 \ , & 0 \end{pmatrix}. \tag{5.15c}$$

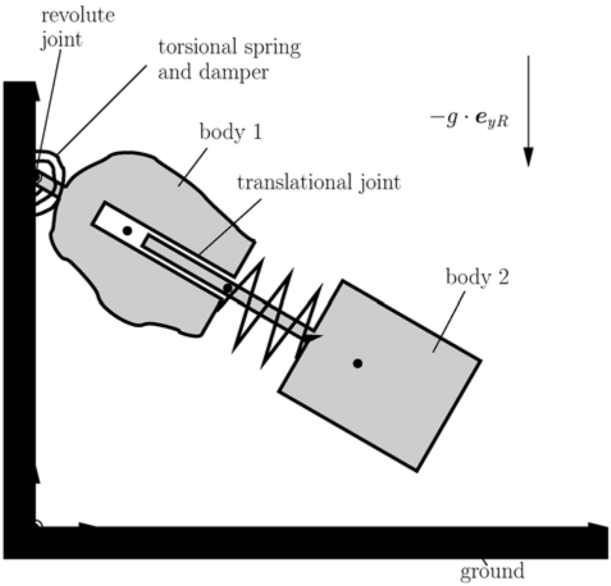
Taking into account the vector loop equation (Figure 5.16a)

$$\boldsymbol{r}_{P_i O}^R + \boldsymbol{r}_{Q_i P_i}^R + \boldsymbol{d}_{ji}^R - \boldsymbol{r}_{S_j Q_j}^R - \boldsymbol{r}_{Q_j P_j}^R - \boldsymbol{r}_{P_j O}^R = \boldsymbol{0}$$

yields



(a) Vector diagram associated with the constraint equations of a translational joint



(b) Mechanism including a translational joint between two bodies

Fig. 5.16: Drawings of (relative) translational joints

$$\boldsymbol{d}_{ji}^{Li} = \boldsymbol{A}^{L_iL_j} \cdot \boldsymbol{r}_{S_jQ_j}^{L_j} + \boldsymbol{A}^{L_iL_j} \cdot \boldsymbol{r}_{Q_jP_j}^{L_j} + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{r}_{P_jO}^{R} - \boldsymbol{A}^{L_iR} \cdot \boldsymbol{r}_{P_iO}^{R} - \boldsymbol{r}_{Q_iP_i}^{L_i} . \tag{5.15d}$$

This provides, together with (5.15c) and (5.15a), the *constraint position equations*

$$0 = \left(\left(\boldsymbol{t}_i^{L_i}\right)^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}}\right) \cdot \left(\boldsymbol{A}^{L_iL_j} \cdot \left(\boldsymbol{r}_{S_jQ_j}^{L_j} + \boldsymbol{r}_{Q_jP_j}^{L_j} + \boldsymbol{A}^{L_jR} \cdot \boldsymbol{r}_{P_jO}^{R}\right) - \boldsymbol{A}^{L_iR} \cdot \boldsymbol{r}_{P_iO}^{R} - \boldsymbol{r}_{Q_iP_i}^{L_i}\right) \tag{5.16a}$$

and $\psi_i - \psi_j = c_\psi = \text{constant}$. (5.16b)

5.1.2.4 Combined orientation/partial-position constraint. Combined relative orientation/partial-position constraints are theoretical models of *rack-and-pinion* mechanisms and *cam followers* (Figure 5.17). For the rack-and-pinion mechanism of Figure 5.17a, the noslippage condition guarantees identical velocities of bodies 1 and 2 at their contact point A. Then

$$\dot{y}_{P_1O}^{R} = -R \cdot \dot{\psi}_2 \tag{5.17a}$$

with R as the radius of the pinion. This provides the *constraint position equation*

$$y_{P_1O}^{R} - y_{P_1O}^{R}(0) = -R\,(\psi_2 - \psi_2(0)) \tag{5.17b}$$

with $y_{P_1O}^{R}(0)$ and $\psi_2(0)$ as starting conditions of $y_{P_1O}^{R}$ and ψ_2.

5.1.2.5 Constant-distance constraint (massless revolute–revolute link). A constant-distance constraint between two points P_i and P_j, located on bodies i and j, respectively, is modeled by the *constraint equation*

$$g = \boldsymbol{r}_{P_iP_j}^{\mathrm{T}} \cdot \boldsymbol{r}_{P_iP_j} - d^2 = 0, \tag{5.18a}$$

with d as the constant distance between the points P_i and P_j (Figure 5.18). Using the vector loop equation

$$\mathbf{0} = \boldsymbol{r}_{P_1O}^{R} - \boldsymbol{r}_{P_2O}^{R} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}_{P_iP_1}^{L_i} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}_{P_jP_i}^{L_i} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}_{P_2P_j}^{L_j}$$

and the geometry relations

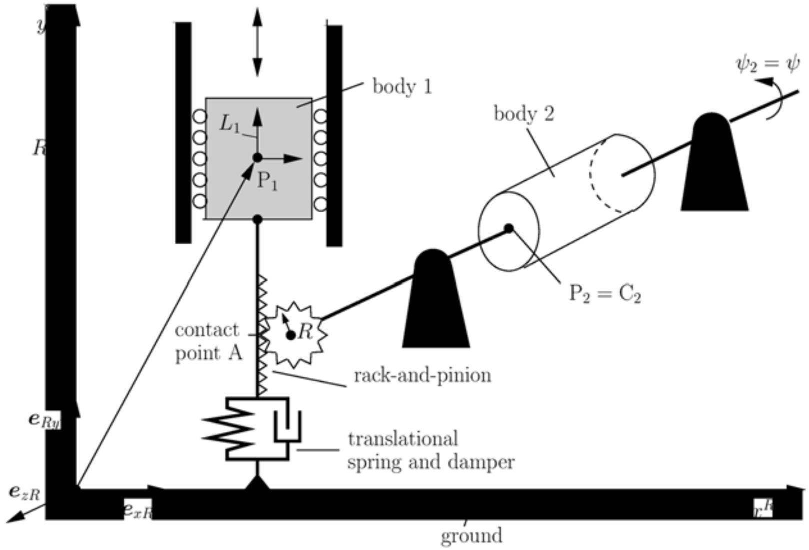
$$\boldsymbol{A}^{RL_\kappa} = \begin{pmatrix} \cos\psi_\kappa \;, & -\sin\psi_\kappa \\ \sin\psi_\kappa \;, & \cos\psi_\kappa \end{pmatrix} \quad , \quad \kappa = i\,,\, j, \tag{5.18b}$$

and

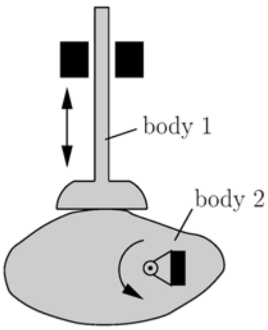
$$\boldsymbol{r}_{P_iP_1}^{L_i} = \begin{pmatrix} \kappa_x \\ -\kappa_y \end{pmatrix} \qquad , \qquad \boldsymbol{r}_{P_2P_j}^{L_j} = \begin{pmatrix} \eta_x \\ -\eta_y \end{pmatrix}$$

yields

$$y_{P_iP_j}^{L_i} \cdot \sin\psi_i - x_{P_iP_j}^{L_i} \cdot \cos\psi_i = x_{P_1O}^{R} - x_{P_2O}^{R} + \cos\psi_i \cdot \kappa_x + \sin\psi_i \cdot \kappa_y + \cos\psi_j \cdot \eta_x + \sin\psi_j \cdot \eta_y$$



(a) Rack-and-pinion mechanism



(b) Cam follower

Fig. 5.17: Technical realizations of joints, modeled by a combined relative orientation/partial-position constraint

and

$$-x^{L_i}_{P_iP_j} \cdot \sin\psi_i - y^{L_i}_{P_iP_j} \cdot \cos\psi_i = y^R_{P_1O} - y^R_{P_2O} + \sin\psi_i \cdot \kappa_x - \cos\psi_i \cdot \kappa_y + \sin\psi_j \cdot \eta_x - \cos\psi_j \cdot \eta_y. \tag{5.18c}$$

This provides the *constraint equation*

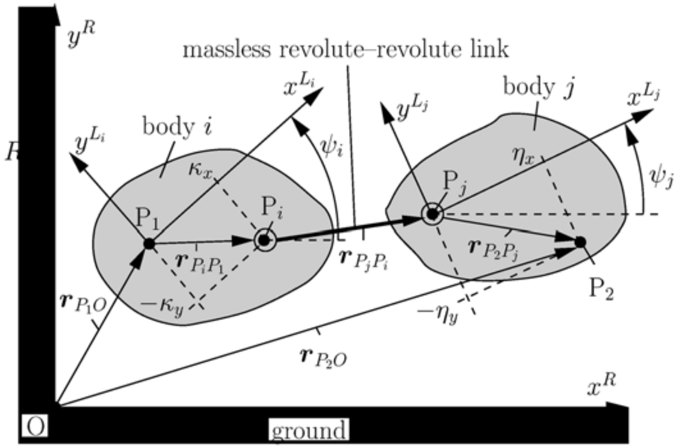


Fig. 5.18: Vector diagram of a mechanism with a constant-distance constraint

$$\left(x^{L_1\,2}_{P_iP_j} + y^{L_1\,2}_{P_iP_j}\right)^{1/2} - d = 0 \tag{5.18d}$$

with $x^{L_1}_{P_iP_j}$ and $y^{L_1}_{P_iP_j}$ obtained from (5.18c). *Massless revolute–revolute links* are technical realizations of *constant-distance constraints.*

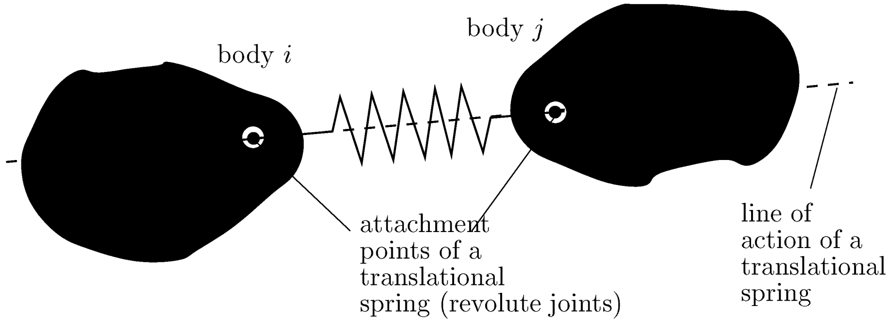
Comment 5.1.1 (Examples of planar joints): The above-discussed models of joints between two bodies under planar motion will be extensively used in the various examples of Section 4 in Volume II.

5.1.3 Pseudo-joint and force/torque elements

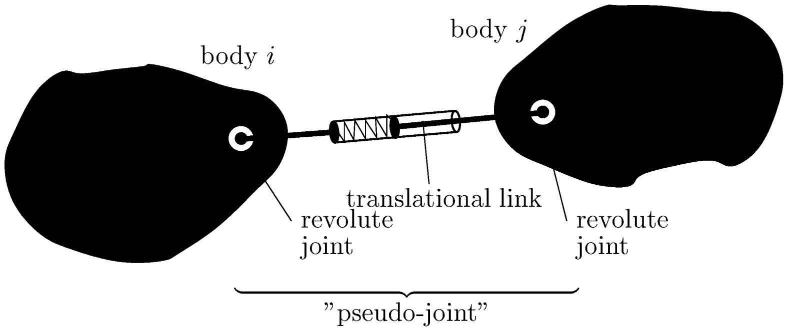
The various forces and torques applied to rigid bodies act in specific directions or around specific axes. Theoretical models of these forces and torques sometimes implicitly include joints or *"pseudo-joints"* (cf. *Section 6*).

5.1.3.1 Example of a translational spring element. Translational springs are assumed to act in the direction of a straight line, connecting their attachment points to the bodies (Figure 5.19a). Technically this can be achieved by connecting the attachment points of the spring by a *massless revolute–translational–revolute link* as a *"pseudo-joint"* that does not constrain any DOF of the bodies but forces the spring to act in the desired direction through the attachment points (Figures 5.19a and 5.19b).

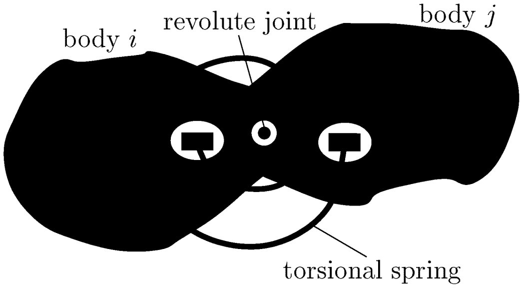
5.1.3.2 Example of a torsional spring. Torsional springs are always assumed to act around the axis of a revolute joint connecting two bodies. In contrast to the "pseudo-joint" of Section 5.1.3.1, this planar joint constrains two translational DOFs of adjacent bodies (Figure 5.19c).



(a) Line of action of a translational spring



(b) Model of a translational spring including a spring force element and a *"pseudo-joint"*, which does not constrain any DOF of the adjacent bodies but prescribes the line of action of the spring



(c) Model of a torsional spring acting around the axis of a revolute joint that constrains two relative translational DOFs of the adjacent bodies under planar motion

Fig. 5.19: Combinations of planar translational and torsional springs with joints

Comment 5.1.2 (Spatial torsional spring): Spatial as well as planar torsional springs are always assumed to act around the axis of a revolute joint connecting two bodies. In contrast to the "pseudo-joint" of Section 5.1.3.1, this joint constrains in the *spatial case* three translational DOFs and two rotational DOFs of the adjacent bodies.

5.2 Theoretical modeling of *spatial* joints

In this chapter mathematical models of *spatial joints* will be derived. As already shown in Section 5.1 for planar joints, models of spatial joints will be derived from *geometry relations* and from suitable *representations* and *projections* of *vector loop equations* and/or *orientation loop equations* that may serve as *constraint position/orientation relations.* Associated *relative coordinates* that may be used for *measurement* or *control* purposes are isolated by suitable (local or global) *representations* and *projections* of these vector or/and orientation loop equations.

As there exists a great variety of different *spatial joints* and *massless links*, only a few joints of common use in industrial applications will be discussed here. As a first step *(Section 5.2.1)* several *building blocks (BBs)* of joint models will be derived. Among those, constraint relations of a *common-point constraint (BB1)*, *parallel-axes constraint (BB2)*, *straight-line-point follower constraint (BB3)*, *rotation-blocker constraint (BB4)*, and *constant-distance constraint (BB5)* will be considered. From these building blocks, *constraint relations* and *relative coordinates* of the following standard *spatial joints* will be derived in *Section 5.2.2* (Table 5.1): *spherical joint (BB1)*, *massless spherical–spherical link (BB5)*, *translational joint (BB2, BB4)*, *universal joint (BB1, BB4)*, *revolute joint (BB1,BB2)*, *cylindrical joint (BB2, BB3)*, and *prismatic joint (BB2, BB3, BB4)*. A more general universal joint with *nonintersecting* and *nonorthogonal* rotation axes will be derived in *Appendix A.4*.

5.2.1 Building blocks of joint models

The discussion in Section 5.1 shows that combining different absolute and relative constraints of two rigid bodies under *planar* motion already enables the setting up of a large number of different planar mechanisms. Increasing the number of bodies allows the construction of a large variety of different planar kinematic and dynamic mechanisms and machines. It is easy to imagine that a tremendous variety of different *spatial* mechanisms can be built from only a small number of rigid bodies that move in *space* ($\mathbb{R}^3$). In this section certain *geometrical situations* associated with spatial joints will be theoretically modeled by constraint position, velocity, and acceleration equations. In Section 5.2.2 they serve as *building blocks* for modeling different *types of spatial joints* ([4], [40], [41]).

5.2.1.1 Common-point constraint (BB1; three constrained translational DOFs). A common-point constraint forces two points Q_i and Q_j of two rigid bodies i and j to a common position Q that may move in space (Figure 5.20). This constraint does not allow relative translational motions of the bodies i and j in the points Q_i and Q_j, but only relative rotations. Let R be an inertial frame and L_κ be a local frame with origin P_κ ($\kappa = i\,,\, j$) on a body κ. Let $\boldsymbol{A}^{RL_\kappa}$ be the orientation matrix of R with respect to L_κ ($\kappa = i\,,\, j$). This *geometrical situation* is described by the *vector loop relation* (see Figure 5.20)

$$\boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{QP_i} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{QP_j} = \boldsymbol{0}.$$

This provides the *common-point constraint position equation*

$$\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{QP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{QP_j} = \boldsymbol{0}, \tag{5.19a}$$

the associated *constraint velocity equation*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{g}(\boldsymbol{p})) &= \boldsymbol{g}_p(\boldsymbol{p}) \cdot \dot{\boldsymbol{p}} = \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} \\ &= {}^R\dot{\boldsymbol{r}}^R_{P_iO} - {}^R\dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{QP_i} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{QP_j} = \boldsymbol{0}\end{aligned}$$

or

$${}^R\dot{\boldsymbol{r}}^R_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{QP_i} \cdot \boldsymbol{\omega}^{L_i}_{L_iR} - {}^R\dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{QP_j} \cdot \boldsymbol{\omega}^{L_j}_{L_jR} = \boldsymbol{0}$$

or

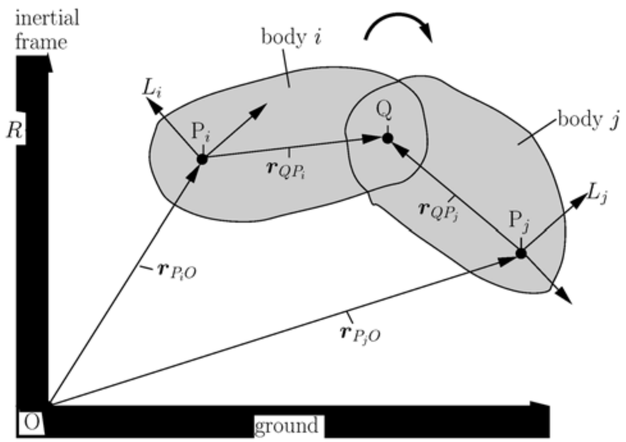


Fig. 5.20: Vector diagram of a common-point constraint

$$\underbrace{\left(\mathbf{I}_3\ ,\ \ -\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\ ,\ \ -\mathbf{I}_3\ ,\ \ +\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\right)}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}\cdot\underbrace{\begin{bmatrix}{}^R\dot{\boldsymbol{r}}^R_{P_iO}\\ \boldsymbol{\omega}^{L_i}_{L_iR}\\ {}^R\dot{\boldsymbol{r}}^R_{P_jO}\\ \boldsymbol{\omega}^{L_j}_{L_jR}\end{bmatrix}}_{=:\,(\boldsymbol{v}_1^{\mathrm{T}},\boldsymbol{v}_2^{\mathrm{T}})^{\mathrm{T}}} = \mathbf{0}, \tag{5.19b}$$

and the associated *constraint acceleration equation*

$$\begin{aligned}{}^R\ddot{\boldsymbol{r}}^R_{P_iO} - \boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\boldsymbol{\omega}^{L_i}_{L_iR} - \boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\dot{\boldsymbol{\omega}}^{L_i}_{L_iR} - {}^R\ddot{\boldsymbol{r}}^R_{P_jO}\\ + \boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\boldsymbol{\omega}^{L_j}_{L_jR} + \boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\dot{\boldsymbol{\omega}}^{L_j}_{L_jR} = \mathbf{0}\end{aligned}$$

or

$$\begin{aligned}&\underbrace{\left(\mathbf{I}_3\ ,\ \ -\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\ ,\ \ -\mathbf{I}_3\ ,\ \ +\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\right)}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}\cdot\begin{bmatrix}{}^R\ddot{\boldsymbol{r}}^R_{P_iO}\\ \dot{\boldsymbol{\omega}}^{L_i}_{L_iR}\\ {}^R\ddot{\boldsymbol{r}}^R_{P_jO}\\ \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}\end{bmatrix}\\ &= +\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\boldsymbol{\omega}^{L_i}_{L_iR} - \boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\boldsymbol{\omega}^{L_j}_{L_jR}\end{aligned} \tag{5.19c}$$

together with the *constraint Jacobian matrix*

$$\begin{aligned}\bar{\boldsymbol{g}}_p(\boldsymbol{p}) &:= \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\\ &= \left(\mathbf{I}_3\ ,\ \ -\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\ ,\ \ -\mathbf{I}_3\ ,\ \ \boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\right)\in\mathbb{R}^{3,12}.\end{aligned} \tag{5.19d}$$

The constraint acceleration equation (5.19c) will be used as a *building block (BB1)*. It provides a theoretical model of a *spherical joint* connecting the bodies i and j (see Section 5.2.2.1).

Comment 5.2.1 (BB1): In the case that relative rotation angles of the joint are needed, either for *sensing* or for *control* purposes, this building block must be reformulated.

Comment 5.2.2 (Alternative representation of the constraint Jacobian of BB1): The time derivative of the constraint position equation (5.19a)

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{g}(p) = \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{p}}\cdot\dot{\boldsymbol{p}} = \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\cdot\boldsymbol{v} \tag{5.19e}$$

may together with

$$\boldsymbol{p} = \left((\boldsymbol{r}^R_{P_iO})^{\mathrm{T}}\,,\,\boldsymbol{\eta}_i^{\mathrm{T}}\,,\,(\boldsymbol{r}^R_{P_jO})^{\mathrm{T}}\,,\,\boldsymbol{\eta}_j^{\mathrm{T}}\right)^{\mathrm{T}}\quad,\quad \boldsymbol{r}^R_{P_iO} = \left(x^R_{P_iO}\,,\,y^R_{P_iO}\,,\,z^R_{P_iO}\right)^{\mathrm{T}}$$

and

$$\boldsymbol{\eta}_i := (\varphi_i\,,\,\theta_i\,,\,\psi_i)^{\mathrm{T}}$$

be written in the form

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{g}(\boldsymbol{p}) = \left(\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}^R_{P_iO}}\,,\,\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\eta}_i}\,,\,\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}^R_{P_jO}}\,,\,\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{\eta}_j}\right)\cdot\dot{\boldsymbol{p}} \tag{5.19f}$$

$$= \left[\mathbf{I}_3\,,\,\frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\,,\,-\mathbf{I}_3\,,\,\frac{\partial\left(-\boldsymbol{A}^{RL_j}(\boldsymbol{\eta}_j)\cdot\boldsymbol{r}^{L_j}_{QP_j}\right)}{\partial\boldsymbol{\eta}_j}\right]\cdot\begin{bmatrix}\dot{\boldsymbol{r}}^R_{P_iO}\\ \dot{\boldsymbol{\eta}}_i\\ \dot{\boldsymbol{r}}^R_{P_jO}\\ \dot{\boldsymbol{\eta}}_j\end{bmatrix}.$$

Together with

$$\frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\cdot\dot{\boldsymbol{\eta}}_i = \frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\cdot H_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{\omega}^{L_i}_{L_iR},$$

this implies

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\cdot\boldsymbol{v} \tag{5.19g}$$

$$= \left[\mathbf{I}_3\,,\,\frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\,,\,-\mathbf{I}_3\,,\,\frac{\partial\left(-\boldsymbol{A}^{RL_j}(\boldsymbol{\eta}_j)\cdot\boldsymbol{r}^{L_j}_{QP_j}\right)}{\partial\boldsymbol{\eta}_j}\right]$$

$$\cdot\left[\begin{array}{cc|cc} \mathbf{I}_3 & \mathbf{0}_{3,3} & \multicolumn{2}{c}{\multirow{2}{*}{$\mathbf{0}_{6,6}$}} \\ \mathbf{0}_{3,3} & \boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$\mathbf{0}_{6,6}$}} & \mathbf{I}_3 & \mathbf{0}_{3,3} \\ & & \mathbf{0}_{3,3} & \boldsymbol{H}_j(\boldsymbol{\eta}_j)\cdot\boldsymbol{A}^{RL_j}(\boldsymbol{\eta}_j) \end{array}\right]\cdot\begin{bmatrix}\dot{\boldsymbol{r}}^R_{P_iO}\\ \boldsymbol{\omega}^{L_i}_{L_iR}\\ \hline \dot{\boldsymbol{r}}^R_{P_jO}\\ \boldsymbol{\omega}^{L_j}_{L_jR}\end{bmatrix}.$$

Comparing the block matrices of (5.19c) and (5.19g) yields the relation

$$-\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i} = \frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\cdot\boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \tag{5.19h}$$

or

$$\tilde{\boldsymbol{r}}^{L_i}_{QP_i} = -\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i)\cdot\frac{\partial\left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i)\cdot\boldsymbol{r}^{L_i}_{QP_i}\right)}{\partial\boldsymbol{\eta}_i}\cdot\boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i). \tag{5.19i}$$

Multiplying both sides of (5.19i) by $\boldsymbol{\omega}^{L_i}_{L_iR}$ yields

$$\tilde{\boldsymbol{r}}_{QP_i}^{L_i} \cdot \boldsymbol{\omega}_{L_iR}^{L_i} = -\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \frac{\partial \left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \boldsymbol{r}_{QP_i}^{L_i} \right)}{\partial \boldsymbol{\eta}_i} \cdot \underbrace{\boldsymbol{H}_i(\boldsymbol{\eta}_i) \cdot \boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \boldsymbol{\omega}_{L_iR}^{L_i}}_{= \dot{\boldsymbol{\eta}}_i} . \tag{5.19j}$$

Together with

$$\dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{r}_{QP_i}^{L_i} = \frac{\partial}{\partial \boldsymbol{\eta}_i} \left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \boldsymbol{r}_{QP_i}^{L_i} \right) \cdot \dot{\boldsymbol{\eta}}_i \tag{5.19k}$$

this yields

$$\begin{aligned} \tilde{\boldsymbol{r}}_{QP_i}^{L_i} \cdot \boldsymbol{\omega}_{L_iR}^{L_i} &= -\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \frac{\partial \left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \boldsymbol{r}_{QP_i}^{L_i} \right)}{\partial \boldsymbol{\eta}_i} \cdot \dot{\boldsymbol{\eta}}_i \\ &= -\boldsymbol{A}^{L_iR}(\boldsymbol{\eta}_i) \cdot \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{r}_{QP_i}^{L_i} . \end{aligned} \tag{5.19l}$$

or

$$\tilde{\boldsymbol{r}}_{QP_i}^{L_i} \cdot \boldsymbol{\omega}_{L_iR}^{L_i} = -\boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{A}^{RL_i}(\boldsymbol{\eta}_i) \cdot \tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \right) \cdot \boldsymbol{r}_{QP_i}^{L_i} , \tag{5.19m}$$

and finally the identity

$$\tilde{\boldsymbol{r}}_{QP_i}^{L_i} \cdot \boldsymbol{\omega}_{L_iR}^{L_i} = -\tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \cdot \boldsymbol{r}_{QP_i}^{L_i} . \tag{5.19n}$$

This proves the equivalence of the representations of the *constraint Jacobian matrix* in (5.19b) and (5.19g).

5.2.1.2 Parallel-axes constraint (BB2; two constrained rotational DOFs). This constraint allows two bodies i and j to perform rotations around two parallel axes only. Assume that these parallel axes are the x-axes of the frame L_{Q_i} fixed on the body i ($\boldsymbol{e}_{xQ_i}$), and of the frame L_{Q_j} fixed on the body j ($\boldsymbol{e}_{xQ_j}$). This *geometrical situation* will be compactly formulated by the *orientation loop equation* (Figure 5.21)

$$\boldsymbol{A}^{RL_j} \cdot \boldsymbol{A}^{L_jL_{Q_j}} \cdot \boldsymbol{A}^{L_{Q_j}L_{Q_i}} \cdot \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} = \mathbf{I}_3$$

or (5.20a)

$$\boldsymbol{A}^{L_{Q_j}L_{Q_i}} = \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}} = \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \boldsymbol{A}^{L_jL_i} \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}}$$

with

$$\boldsymbol{A}^{L_jL_i} = \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} .$$

Its *time derivative* is

$$\dot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = \boldsymbol{A}^{L_{Q_j}L_j} \cdot \dot{\boldsymbol{A}}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} + \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}}$$

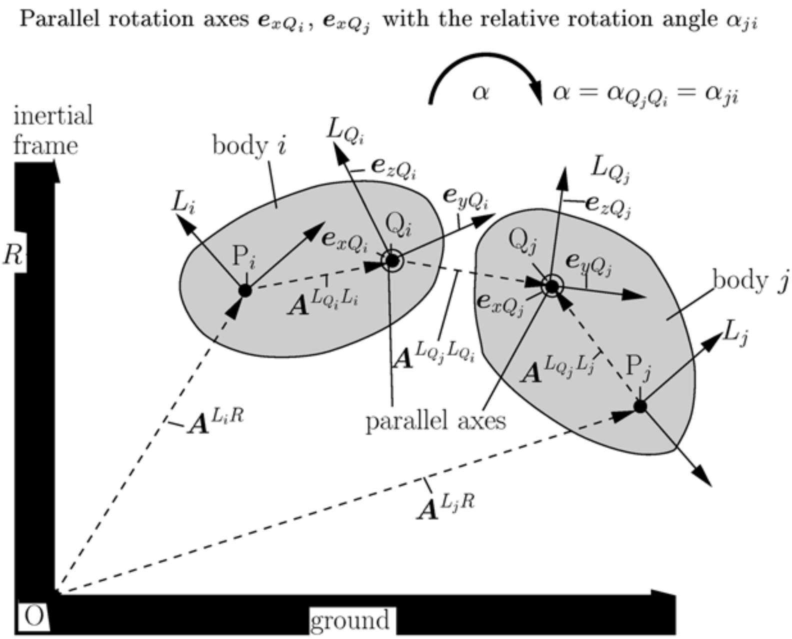


Fig. 5.21: Orientation diagram of a parallel-axes constraint

or

$$\begin{aligned}\dot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = & \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \boldsymbol{A}^{L_jR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL_j} \cdot \boldsymbol{A}^{RL_i} \cdot \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}} \\ & + \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}} .\end{aligned}$$

This yields, together with

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{and} \quad \dot{\boldsymbol{A}}^{LR} = \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL} = -\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR},$$

the relation

$$\begin{aligned}\dot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = & - \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}} \\ & + \underbrace{\boldsymbol{A}^{L_{Q_j}L_j}}_{\text{constant}} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \underbrace{\boldsymbol{A}^{L_iL_{Q_i}}}_{\text{constant}} .\end{aligned} \tag{5.20b}$$

The *second time derivative* of $\boldsymbol{A}^{L_{Q_j}L_{Q_i}}$ is

$$\ddot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = \boldsymbol{A}^{L_{Q_j}L_j} \cdot \Big(-\dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} + \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i}$$
$$- 2 \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}$$
$$+ \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}$$
$$+ \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_iR} \Big) \cdot \boldsymbol{A}^{L_iL_{Q_i}} . \tag{5.20c}$$

Due to the *geometrical assumptions* of Figure 5.21, the orientation matrix $\boldsymbol{A}^{L_{Q_j}L_{Q_i}}$ has the form

$$\boldsymbol{A}^{L_{Q_j}L_{Q_i}} = \begin{pmatrix} 1\,, & 0 & , \; 0 \\ 0\,, & \cos\alpha & , \; \sin\alpha \\ 0\,, & -\sin\alpha & , \; \cos\alpha \end{pmatrix} \quad , \quad \alpha := \varphi_{Q_jQ_i} = \varphi_{ji} = \alpha_{ji} . \tag{5.21a}$$

This implies

$$\dot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = \begin{pmatrix} 0\,, & 0 & , & 0 \\ 0\,, & -\sin\alpha & , & \cos\alpha \\ 0\,, & -\cos\alpha & , & -\sin\alpha \end{pmatrix} \cdot \dot{\alpha} \tag{5.21b}$$

and

$$\ddot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}} = \begin{pmatrix} 0\,, & 0 & , & 0 \\ 0\,, & -\cos\alpha & , & -\sin\alpha \\ 0\,, & \sin\alpha & , & -\cos\alpha \end{pmatrix} \cdot \dot{\alpha}^2 + \begin{pmatrix} 0\,, & 0 & , & 0 \\ 0\,, & -\sin\alpha & , & \cos\alpha \\ 0\,, & -\cos\alpha & , & -\sin\alpha \end{pmatrix} \cdot \ddot{\alpha} . \tag{5.21c}$$

As this building block only constrains two (relative rotational) DOFs of the bodies i and j: (1) only two independent equations of the acceleration equation (5.21c) are needed as *constraint acceleration equations*, and (2) a single equation is used for *computing the relative coordinate α.*

Due to common zeros in rows 2 and 3 and column 1 of the matrices $\boldsymbol{A}^{L_{Q_j}L_{Q_i}}$, $\dot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}}$, and $\ddot{\boldsymbol{A}}^{L_{Q_j}L_{Q_i}}$, and in columns 2 and 3 and row 1 of these matrices, the *relative coordinate α can be eliminated* from (5.20a), (5.20b), and (5.20c) by projecting these equations from the left by means of

$$\mathbf{P_r}^{\mathrm{T}}(y\,,\,z) := \begin{pmatrix} \boldsymbol{e}_y^{\mathrm{T}} \\ \boldsymbol{e}_z^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0\,,\,1\,,\,0 \\ 0\,,\,0\,,\,1 \end{pmatrix}$$

and from the right by

$$\mathbf{P}_r(x) = \boldsymbol{e}_x = \big(1\,,\,0\,,\,0\big)^{\mathrm{T}} .$$

This implies the following *constraint orientation*, *velocity*, and *acceleration* equations of this building block (BB2):

$$\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x)=\mathbf{0}, \tag{5.22a}$$

$$\begin{aligned}\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(&-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\\&+\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x)=\mathbf{0}\end{aligned} \tag{5.22b}$$

and

$$\begin{aligned}&\underbrace{\begin{bmatrix}\mathbf{0}_{2\,,3}\,, & \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\widetilde{[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)]}\Big\},\\ \mathbf{0}_{2\,,3}\,, & \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\cdot\widetilde{[\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)]}\Big\}\end{bmatrix}}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}\cdot\begin{bmatrix}\ddot{\boldsymbol{r}}^R_{P_iO}\\ \dot{\boldsymbol{\omega}}^{L_i}_{L_iR}\\ \ddot{\boldsymbol{r}}^R_{P_jO}\\ \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}\end{bmatrix}\\&=-\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x)\\&\quad+2\cdot\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x)\\&\quad-\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x)\end{aligned} \tag{5.22c}$$

with the *constraint Jacobian matrix*

$$\boldsymbol{g}_p\cdot\boldsymbol{T}(\boldsymbol{p})=\begin{bmatrix}\mathbf{0}_{2\,,3}\,, & -\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\widetilde{[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)]}\Big\},\\ \mathbf{0}_{2\,,3}\,, & +\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\widetilde{[\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)]}\Big\}\end{bmatrix}. \tag{5.22d}$$

The *relative coordinate* $\alpha := \alpha_{Q_jQ_i} = \alpha_{ji}$ of this building block and its first and second time derivatives are *isolated* from (5.20a), (5.20b), and (5.20c) by the projections

$$\sin\alpha = -\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\Delta\,)\,\mathbf{P}_r(y)\quad,\quad\cos\alpha = \mathbf{P_r}^{\mathrm{T}}(y)\,(\,\Delta\,)\,\mathbf{P}_r(y) \tag{5.23a}$$

as

$$\alpha=\alpha_{ji}=\alpha_{Q_jQ_i}=-\arctan\Big\{\Big[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\Delta\,)\,\mathbf{P}_r(y)\Big]\Big/\Big[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\Delta\,)\,\mathbf{P}_r(y)\Big]\Big\}$$

with

$$\Delta := \boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}},$$

$$\mathbf{P_r}^{\mathrm{T}}(z) := (0\,,0\,,1)^{\mathrm{T}}\quad,\quad\mathbf{P_r}^{\mathrm{T}}(y) := (0\,,1\,,0)^{\mathrm{T}},$$

$$\dot{\alpha} = \begin{cases} -\dfrac{1}{\sin\alpha} \cdot \left\{ \mathbf{P_r}^{\mathrm{T}}(y)\,(\,\square\,)\,\mathbf{P}_r(y) \right\} \\ \text{or} \\ -\dfrac{1}{\cos\alpha} \cdot \left\{ \mathbf{P_r}^{\mathrm{T}}(z)\,(\,\square\,)\,\mathbf{P}_r(y) \right\} \end{cases} \tag{5.23b}$$

with

$$\begin{aligned} \square := &- \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}, \end{aligned} \tag{5.23c}$$

and

$$\ddot{\alpha} = \begin{cases} -\dfrac{1}{\sin\alpha} \cdot \left\{ \left[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\bowtie\,)\,\mathbf{P}_r(y) \right] + \cos\alpha \cdot \dot{\alpha}^2 \right\} \\ \text{or} \\ -\dfrac{1}{\cos\alpha} \cdot \left\{ \left[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\bowtie\,)\,\mathbf{P}_r(y) \right] - \sin\alpha \cdot \dot{\alpha}^2 \right\} \end{cases} \tag{5.23d}$$

with

$$\begin{aligned} \bowtie := &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{L_iR} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &- \boldsymbol{A}^{L_{Q_j}L_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &- 2 \cdot \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}. \end{aligned}$$

5.2.1.3 Straight-line-point-follower constraint(BB3; two constrained translational DOFs). A straight-line-point-follower constraint allows – besides arbitrary rotations – translations of two bodies i and j only along a straight line. Assume that this straight line is oriented in the direction of the x-axis $\boldsymbol{e}_{xQ_i}$ of the frame L_{Q_i} fixed on body i with origin Q_i. Assume further that point Q_j on the body j can only move on this straight line. This *geometrical situation* is formulated by the vector loop equation (Figure 5.22)

$$\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \cdot \boldsymbol{r}^{L_{Q_i}}_{Q_jQ_i} = \boldsymbol{0},$$

or, *written in the local frame* L_{Q_i},

$$\begin{aligned} \hat{\boldsymbol{g}} = &\underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} \right) + \underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \underbrace{\boldsymbol{r}^{L_i}_{Q_iP_i}}_{\text{constant}} \\ &- \underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \underbrace{\boldsymbol{r}^{L_j}_{Q_jP_j}}_{\text{constant}} + \boldsymbol{r}^{L_{Q_i}}_{Q_jQ_i} \equiv \boldsymbol{0} \end{aligned} \tag{5.24a}$$

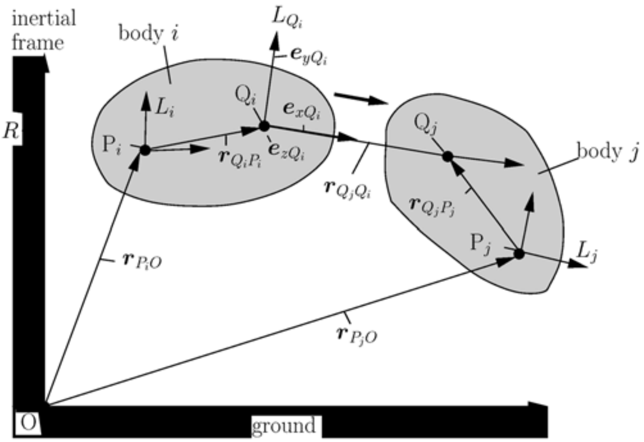


Fig. 5.22: Vector loop of a straight-line-point-follower constraint

with

$$\boldsymbol{r}^{L_{Q_i}}_{Q_jQ_i} := \left(x^{L_{Q_i}}_{Q_jQ_i}\,,\,0\,,\,0\right)^{\mathrm{T}} \quad , \quad (x^{L_{Q_i}}_{Q_jQ_i} \textit{ is the relative displacement}).$$

Differentiation of (5.24a) with respect to the time yields

$$\begin{aligned}\dot{\hat{\boldsymbol{g}}} = \;& \frac{^{L_{Q_i}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}) = \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL_i} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) \\ &+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO}\right) - \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL_i} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} \\ &- \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} + \dot{\boldsymbol{r}}^{L_{Q_i}}_{Q_jQ_i} \equiv \boldsymbol{0}\end{aligned}$$

or

$$\begin{aligned}\dot{\hat{\boldsymbol{g}}} = &- \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) \\ &+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO}\right) + \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} \\ &- \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} + \dot{\boldsymbol{r}}^{L_{Q_i}}_{Q_jQ_i} \equiv \boldsymbol{0} \qquad (5.24\mathrm{b})\end{aligned}$$

with

$$\dot{\boldsymbol{r}}^{L_{Q_i}}_{Q_jQ_i} := \left(\dot{x}^{L_{Q_i}}_{Q_jQ_i}\,,\,0\,,\,0\right)^{\mathrm{T}} \quad , \quad (\dot{x}^{L_{Q_i}}_{Q_jQ_i} \quad \textit{is the relative velocity}) \qquad (5.24\mathrm{c})$$

or

$$\frac{^{L_{Q_i}}\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{g}}) = - \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left[\left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]$$

$$+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left[\left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]$$
$$+ \left(\dot{x}^{L_{Q_i}}_{Q_jQ_i} , 0 , 0\right)^{\mathrm{T}} = (0 , 0 , 0)^{\mathrm{T}}. \tag{5.24d}$$

Differentiation of (5.24d) with respect to the time yields

$$\ddot{\hat{\boldsymbol{g}}} := \frac{{}^{L_{Q_i}}\mathrm{d}^2}{\mathrm{d}t^2}(\hat{\boldsymbol{g}}) = \boldsymbol{A}^{L_{Q_i}L_i} \cdot \widetilde{\left[\boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]} \cdot \dot{\boldsymbol{\omega}}^{L_i}_{L_iR}$$
$$+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left[\left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]$$
$$- 2 \cdot \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left[\left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]$$
$$+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left[\left(\ddot{\boldsymbol{r}}^R_{P_iO} - \ddot{\boldsymbol{r}}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]$$
$$+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j} \cdot \dot{\boldsymbol{\omega}}^{L_j}_{L_jR} + \left(\ddot{x}^{L_{Q_i}}_{Q_jQ_i} , 0 , 0\right)^{\mathrm{T}}. \tag{5.24e}$$

The purpose of the *straight-line-point-follower* is to *constrain two translational DOFs* of two bodies and to provide the relative coordinate $x^{L_{Q_i}}_{Q_jQ_i}$ of the remainder unconstrained translational DOF. Due to (5.24b) and (5.24c), the *constraint position, velocity and acceleration equations* are obtained as the projections

$$\boldsymbol{g} = \mathbf{P_r}^{\mathrm{T}}(y , z)\, \hat{\boldsymbol{g}} \quad \text{(elimination of } x^{L_{Q_i}}_{Q_jQ_i}\text{)},$$
$$\dot{\boldsymbol{g}} = \mathbf{P_r}^{\mathrm{T}}(y , z)\, \dot{\hat{\boldsymbol{g}}} \quad \text{(elimination of } \dot{x}^{L_{Q_i}}_{Q_jQ_i}\text{)}, \tag{5.25a}$$

and

$$\ddot{\boldsymbol{g}} = \mathbf{P_r}^{\mathrm{T}}(y , z)\, \ddot{\hat{\boldsymbol{g}}} \quad \text{(elimination of } \ddot{x}^{L_{Q_i}}_{Q_jQ_i}\text{), with } \mathbf{P_r}^{\mathrm{T}}(y , z) := \begin{pmatrix} 0 , 1 , 0 \\ 0 , 0 , 1 \end{pmatrix}.$$

The *relative coordinates* $x^{L_{Q_i}}_{Q_jQ_i}$, $\dot{x}^{L_{Q_i}}_{Q_jQ_i}$ and $\ddot{x}^{L_{Q_i}}_{Q_jQ_i}$ are obtained as the projections

$$\mathbf{P_r}^{\mathrm{T}}(x)\, \hat{\boldsymbol{g}} = -x^{L_{Q_i}}_{Q_jQ_i} \quad \text{(isolation of } x^{L_{Q_i}}_{Q_jQ_i}\text{)},$$
$$\mathbf{P_r}^{\mathrm{T}}(x)\, \dot{\hat{\boldsymbol{g}}} = -\dot{x}^{L_{Q_i}}_{Q_jQ_i} \quad \text{(isolation of } \dot{x}^{L_{Q_i}}_{Q_jQ_i}\text{)}, \tag{5.25b}$$

and

$$\mathbf{P_r}^{\mathrm{T}}(x)\, \ddot{\hat{\boldsymbol{g}}} = -\ddot{x}^{L_{Q_i}}_{Q_jQ_i} \quad \text{(isolation of } \ddot{x}^{L_{Q_i}}_{Q_jQ_i}\text{), with } \mathbf{P_r}^{\mathrm{T}}(x) := (1 , 0 , 0).$$

This provides the *constraint position equation*

$$\boldsymbol{g} = \mathbf{P_r}^{\mathrm{T}}(y , z) \Big[\underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) + \underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \underbrace{\boldsymbol{r}^{L_i}_{Q_iP_i}}_{\text{constant}}$$
$$- \underbrace{\boldsymbol{A}^{L_{Q_i}L_i}}_{\text{constant}} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j}\, \underbrace{\boldsymbol{r}^{L_j}_{Q_jP_j}}_{\text{constant}} \Big] = \left(0 , 0\right)^{\mathrm{T}}, \tag{5.25c}$$

the *constraint velocity equation*

$$\begin{aligned}\dot{\boldsymbol{g}} = \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{&-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big[\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]\\ &+\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big[\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]\Big\} = (0\,,\,0)^{\mathrm{T}},\end{aligned} \tag{5.25d}$$

and the *constraint acceleration equation*

$$\begin{aligned}&\underbrace{\begin{bmatrix}\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\,\Big(\boldsymbol{A}^{L_{Q_i}L_i}\boldsymbol{A}^{L_iR}\Big)\,,\;\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\,\Big[\boldsymbol{A}^{L_{Q_i}L_i}\,\widetilde{\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\boldsymbol{r}^{L_j}_{Q_jP_j}}\Big]\\ -\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\Big)\,,\;\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\,\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\Big)\end{bmatrix}}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\ \text{(constraint Jacobian matrix)}}\\ &\cdot\Big[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\,,\,(\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}} =\\ &\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big[\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]\\ &\qquad+2\cdot\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big[\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]\\ &\qquad+\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big\}.\end{aligned} \tag{5.25e}$$

The *relative coordinate* $x^{L_{Q_i}}_{Q_jQ_i}$ and its first and second time derivatives are computed by the relations

$$\begin{aligned}x^{L_{Q_i}}_{Q_jQ_i} = -\mathbf{P_r}^{\mathrm{T}}(x)\Big[&\,\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)+\boldsymbol{A}^{L_{Q_i}L_i}\cdot r^{L_i}_{Q_iP_i}\\ &-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}r^{L_j}_{Q_jP_j}\Big],\end{aligned} \tag{5.26a}$$

$$\begin{aligned}\dot{x}^{L_{Q_i}}_{Q_jQ_i} = \mathbf{P_r}^{\mathrm{T}}(x)\Big[&\,\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\\ &-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\Big],\end{aligned} \tag{5.26b}$$

and

$$\begin{aligned}\ddot{x}^{L_{Q_i}}_{Q_jQ_i} = -\,\mathbf{P_r}^{\mathrm{T}}(x)\Big\{&\boldsymbol{A}^{L_{Q_i}L_i}\cdot\widetilde{\Big[\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]}\cdot\dot{\boldsymbol{\omega}}^{L_i}_{L_iR}\\ &+\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\end{aligned}$$

$$
\begin{aligned}
& - 2 \cdot \boldsymbol{A}^{L_{Q_i} L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_i R} \cdot \boldsymbol{A}^{L_i R} \cdot \left(\dot{\boldsymbol{r}}^{R}_{P_i O} - \dot{\boldsymbol{r}}^{R}_{P_j O} - \boldsymbol{A}^{R L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \boldsymbol{r}^{L_j}_{Q_j P_j} \right) \\
& + \boldsymbol{A}^{L_{Q_i} L_i} \cdot \boldsymbol{A}^{L_i R} \cdot \left(\ddot{\boldsymbol{r}}^{R}_{P_i O} - \ddot{\boldsymbol{r}}^{R}_{P_j O} - \boldsymbol{A}^{R L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \boldsymbol{r}^{L_j}_{Q_j P_j} \right) \\
& + \boldsymbol{A}^{L_{Q_i} L_i} \cdot \boldsymbol{A}^{L_i R} \cdot \boldsymbol{A}^{R L_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_j P_j} \cdot \dot{\boldsymbol{\omega}}^{L_j}_{L_j R} \Big\} . \qquad (5.26c)
\end{aligned}
$$

5.2.1.4 Rotation-blocker constraint (BB4; one constrained rotational DOF). The constraint of this building block prevents two bodies i and j from rotating around a selected axis (axis ℓ in Figure 5.23).

Consider two orthogonal rotation axes (dashed lines in Figure 5.23) that keep their relative orientation while moving in the space. Consider the body i and the body-fixed frames L_i and L_{Q_i} with origins P_i and Q_i, respectively. Let the x-axis ($\boldsymbol{e}_{xQ_i}$) of frame L_{Q_i} be oriented in the direction of the first rotation axis, and assume that the body i may rotate around this axis by an angle $\varphi_x =: \varphi_1$. Consider the second body j and the body-fixed frames L_j and L_{Q_j} with origins P_j and Q_j, respectively.

Let the y-axis ($\boldsymbol{e}_{yQ_j}$) of frame L_{Q_j} be oriented in the direction of the second rotation axis, and assume that body j may rotate around this axis by an angle $\varphi_y =: \varphi_2$.

Consider the straight line h in Figure 5.23 that is parallel to rotation axis 2 and intersects rotation axis 1 at the point K. This point is the origin of a frame L_K with the x-axis $\boldsymbol{e}_{xK} = \boldsymbol{e}_{xL_K}$ oriented in the direction of $\boldsymbol{e}_{xQ_i}$, and with the y-axis $\boldsymbol{e}_{yK} = \boldsymbol{e}_{yL_K}$ oriented in the direction of $\boldsymbol{e}_{yQ_j}$ (Figure 5.23). Let

$$
\varphi_1 := \varphi_x := \varphi_{Q_i K} \qquad (5.27a)
$$

be the rotation angle from L_K to L_{Q_i} (rotation angle of L_{Q_i} with respect to L_K) around the common x-axes $\boldsymbol{e}_{xQ_i} = \boldsymbol{e}_{xK}$ of frames L_{Q_i} and L_K, and

$$
\varphi_2 := \varphi_y := \varphi_{Q_j K} \qquad (5.27b)
$$

be the rotation angle from L_K to L_{Q_j} (rotation angle of L_{Q_j} with respect to L_K) around the common y-axes $\boldsymbol{e}_{yQ_j} = \boldsymbol{e}_{yK}$ of frames L_{Q_j} and L_K. Then the *straight line* ℓ (dotted line in Figure 5.23) is oriented in the $\boldsymbol{e}_{zK}$-axis. Introducing the abbreviations

$$
\boldsymbol{A}^{Q_\kappa K} := \boldsymbol{A}^{L_{Q_\kappa} L_K} \quad (\kappa = i \, , \, j) \quad ,
$$

the geometrical situation of Figure 5.23 provides the orientation matrices

$$
\boldsymbol{A}^{Q_i K} = \begin{pmatrix} 1 \, , & 0 & , & 0 \\ 0 \, , & \cos\varphi_x & , & \sin\varphi_x \\ 0 \, , & -\sin\varphi_x & , & \cos\varphi_x \end{pmatrix} \, , \; \boldsymbol{A}^{Q_j K} = \begin{pmatrix} \cos\varphi_y & , \, 0 \, , & -\sin\varphi_y \\ 0 & , \, 1 \, , & 0 \\ \sin\varphi_y & , \, 0 \, , & \cos\varphi_y \end{pmatrix} ,
$$

and

$$
\boldsymbol{A}^{Q_i Q_j} = \boldsymbol{A}^{Q_i K} \cdot \boldsymbol{A}^{K Q_j} = \boldsymbol{A}^{Q_i K} \cdot \left(\boldsymbol{A}^{Q_j K} \right)^{\mathrm{T}}
$$

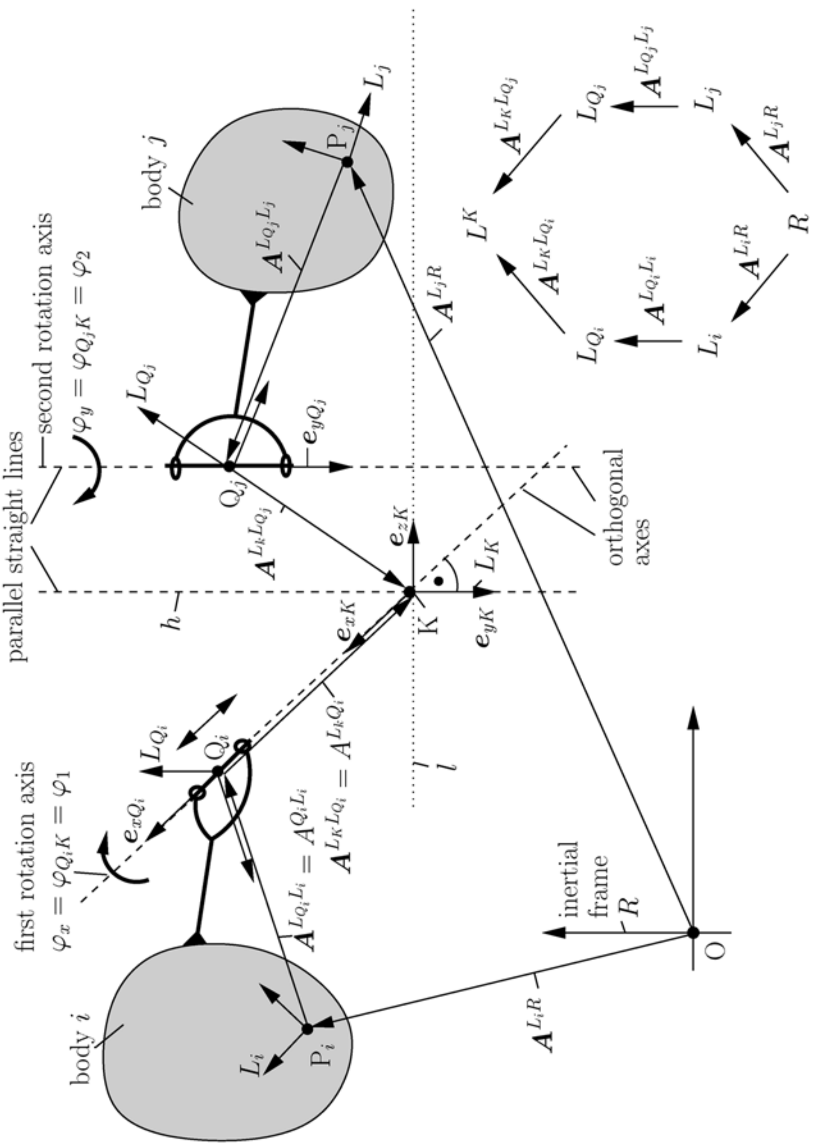


Fig. 5.23: Geometrical situation of a rotation-blocker constraint

or

$$\boldsymbol{A}^{Q_iQ_j} = \begin{pmatrix} 1 \ , & 0 & , \ 0 \\ 0 \ , & \cos\varphi_x & , \ \sin\varphi_x \\ 0 \ , & -\sin\varphi_x & , \ \cos\varphi_x \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi_y & , \ 0 \ , & \sin\varphi_y \\ 0 & , \ 1 \ , & 0 \\ -\sin\varphi_y & , \ 0 \ , & \cos\varphi_y \end{pmatrix},$$

and finally (5.28a)

$$\boldsymbol{A}^{Q_iQ_j} = \begin{pmatrix} \mathrm{c}\,\varphi_y & , \ 0 \ , & \mathrm{s}\,\varphi_y \\ -\mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ \mathrm{c}\,\varphi_x \ , & \mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \\ -\mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{s}\,\varphi_x \ , & \mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \end{pmatrix}, \quad \text{with} \quad \begin{array}{l} \mathrm{s}\,\varphi := \sin\varphi \\ \mathrm{c}\,\varphi := \cos\varphi, \end{array}$$

together with the associated time derivatives

$$\begin{aligned} \dot{\boldsymbol{A}}^{Q_iQ_j} = & \begin{pmatrix} -\mathrm{s}\,\varphi_y & , \ 0 \ , & \mathrm{c}\,\varphi_y \\ -\mathrm{s}\,\varphi_x \cdot c\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y \\ -\mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_y \cdot \mathrm{c}\,\varphi_x \end{pmatrix} \cdot \dot{\varphi}_y \\ & + \begin{pmatrix} 0 & , \ 0 \ , & 0 \\ -\mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{s}\,\varphi_x \ , & \mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \\ \mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{c}\,\varphi_x \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \end{pmatrix} \cdot \dot{\varphi}_x \end{aligned} \qquad (5.28b)$$

and

$$\begin{aligned} \ddot{\boldsymbol{A}}^{Q_iQ_j} = & \begin{pmatrix} -\mathrm{s}\,\varphi_y & , \ 0 \ , & \mathrm{c}\,\varphi_y \\ -\mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y \\ -\mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_y \cdot \mathrm{c}\,\varphi_x \end{pmatrix} \cdot \ddot{\varphi}_y \\ & + \begin{pmatrix} 0 & , \ 0 \ , & 0 \\ -\mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{s}\,\varphi_x \ , & \mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \\ \mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{c}\,\varphi_x \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \end{pmatrix} \cdot \ddot{\varphi}_x \\ & + \begin{pmatrix} -\mathrm{c}\,\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_y \\ \mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ 0 \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \\ \mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ 0 \ , & -\mathrm{c}\,\varphi_y \cdot \mathrm{c}\,\varphi_x \end{pmatrix} \cdot \dot{\varphi}_y^2 \\ & + \begin{pmatrix} 0 & , \ 0 \ , & 0 \\ \mathrm{s}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ -\mathrm{c}\,\varphi_x \ , & -\mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \\ \mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y & , \ \mathrm{s}\,\varphi_x \ , & -\mathrm{c}\,\varphi_x \cdot \mathrm{c}\,\varphi_y \end{pmatrix} \cdot \dot{\varphi}_x^2 \\ & + 2 \cdot \begin{pmatrix} 0 & , \ 0 \ , & 0 \\ -\mathrm{c}\,\varphi_x \cdot c\varphi_y & , \ 0 \ , & -\mathrm{c}\,\varphi_x \cdot \mathrm{s}\,\varphi_y \\ \mathrm{s}\,\varphi_x \cdot \mathrm{c}\,\varphi_y & , \ 0 \ , & \mathrm{s}\,\varphi_y \cdot \mathrm{s}\,\varphi_x \end{pmatrix} \cdot \dot{\varphi}_x \cdot \dot{\varphi}_y. \end{aligned} \qquad (5.28c)$$

Then the following *orientation loop* equation holds:

$$\boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iQ_i} \cdot \boldsymbol{A}^{Q_iQ_j} \cdot \boldsymbol{A}^{Q_jL_j} \cdot \boldsymbol{A}^{L_jR} = \mathbf{I}_3$$

or (5.29a)

$$\boldsymbol{A}^{Q_iQ_j} = \boldsymbol{A}^{Q_iL_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{A}^{L_jQ_j}.$$

This provides the *orientation relation*

$$\underbrace{\boldsymbol{A}^{Q_iL_i}}_{\text{constant}} \cdot \underbrace{\boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j}}_{\text{global}} \cdot \underbrace{\boldsymbol{A}^{L_jQ_j}}_{\text{constant}} - \underbrace{\boldsymbol{A}^{Q_iQ_j}}_{\text{local}} \equiv \mathbf{0} \tag{5.30a}$$

together with its *time derivative*

$$\underbrace{\boldsymbol{A}^{Q_iL_i}}_{\text{constant}} \cdot \dot{\boldsymbol{A}}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \underbrace{\boldsymbol{A}^{L_jQ_j}}_{\text{constant}} + \underbrace{\boldsymbol{A}^{Q_iL_i}}_{\text{constant}} \cdot \boldsymbol{A}^{L_iR} \cdot \dot{\boldsymbol{A}}^{RL_j} \cdot \underbrace{\boldsymbol{A}^{L_jQ_j}}_{\text{constant}} - \dot{\boldsymbol{A}}^{Q_iQ_j} \equiv \mathbf{0} \tag{5.30b}$$

and its *second time derivative*

$$\begin{aligned}
&\boldsymbol{A}^{Q_iL_i} \cdot \ddot{\boldsymbol{A}}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{A}^{L_jQ_j} + 2 \cdot \boldsymbol{A}^{Q_iL_i} \cdot \dot{\boldsymbol{A}}^{L_iR} \cdot \dot{\boldsymbol{A}}^{RL_j} \cdot \boldsymbol{A}^{L_jQ_j} \\
&+ \boldsymbol{A}^{Q_iL_i} \cdot \boldsymbol{A}^{L_iR} \cdot \ddot{\boldsymbol{A}}^{RL_j} \cdot \boldsymbol{A}^{L_jQ_j} - \ddot{\boldsymbol{A}}^{Q_iQ_j} \equiv \mathbf{0}.
\end{aligned} \tag{5.30c}$$

Inserting

$$\begin{aligned}
\dot{\boldsymbol{A}}^{L_iR} &= \boldsymbol{A}^{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{R}_{RL_i} = -\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR}, \\
\dot{\boldsymbol{A}}^{RL_j} &= \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}
\end{aligned}$$

and

$$\begin{aligned}
\ddot{\boldsymbol{A}}^{RL_j} &= \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} + \boldsymbol{A}^{RL_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR}, \\
\ddot{\boldsymbol{A}}^{L_iR} &= -\dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} + \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR}
\end{aligned}$$

into (5.30b) and (5.30c) yields the relations

$$\mathbf{0} = \boldsymbol{A}^{Q_iL_i} \left(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\right) \cdot \boldsymbol{A}^{L_jQ_j} - \dot{\boldsymbol{A}}^{Q_iQ_j} \tag{5.31a}$$

and

$$\begin{aligned}
\boldsymbol{A}^{Q_iL_i} \cdot \Big(&- \dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} + \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \\
&- 2 \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \\
&+ \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR}\Big) \cdot \boldsymbol{A}^{L_jQ_j} - \ddot{\boldsymbol{A}}^{Q_iQ_j} \equiv \mathbf{0}.
\end{aligned} \tag{5.31b}$$

In the next steps the *constraint equations* and *relative coordinates* of this building block will be derived:

1. As a *first step*, the *constraint equations* will be obtained by suitable projections of (5.30a), (5.31a), and (5.31b) that simultaneously *eliminate the relative coordinates* φ_x, $\dot{\varphi}_x$, $\ddot{\varphi}_x$, φ_y, $\dot{\varphi}_y$, *and* $\ddot{\varphi}_y$.
2. As a *second step*, the *relative coordinates* φ_x, φ_y, $\dot{\varphi}_x$, $\dot{\varphi}_y$, $\ddot{\varphi}_x$, *and* $\ddot{\varphi}_y$ *will be isolated* by alternative projections of (5.30a), (5.31a) and (5.31b).

Due to (5.28a), (5.28b), and (5.28c), the matrices $\boldsymbol{A}^{Q_iQ_j}$, $\dot{\boldsymbol{A}}^{Q_iQ_j}$, and $\ddot{\boldsymbol{A}}^{Q_iQ_j}$ have a common zero element in the first row and second column. Both *relative coordinates* φ_x *and* φ_y *can be eliminated* by projecting (5.30a), (5.31a), and (5.31b) from the left by means of $\mathbf{P_r}^{\mathrm{T}}(x) := (1\,,0\,,0)$ and from the right by means of $\mathbf{P}_r(y) := (0\,,1\,,0)^{\mathrm{T}}$. This provides the following *constraint orientation*, *velocity*, and *acceleration relations* of this building block:

$$\mathbf{P_r}^{\mathrm{T}}(x)\Big(\,\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\mathbf{P}_r(y)=0, \tag{5.32a}$$

$$\mathbf{P_r}^{\mathrm{T}}(x)\Big[\,\boldsymbol{A}^{Q_iL_i}\cdot\Big(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big) \tag{5.32b}$$

$$\cdot\,\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y)=0, \tag{5.32c}$$

and

$$\underbrace{\Big[\mathbf{0}_{1\,,3}\;,\;\mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\Big[\overbrace{\Big(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\mathbf{P}_r(y)}\Big]\Big\}\;,\;\mathbf{0}_{1\,,3}\;,\; -\mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\overbrace{\Big(\boldsymbol{A}^{L_jQ_j}\mathbf{P}_r(y)\Big)}\Big\}\Big]}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\,=\textit{constraint Jacobian matrix}} \tag{5.32d}$$

$$\cdot\;\Big[(\ddot{\boldsymbol{r}}^{R}_{P_iO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\,,\,(\ddot{\boldsymbol{r}}^{R}_{P_jO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}}$$
$$=-\mathbf{P_r}^{\mathrm{T}}(x)\Big[\boldsymbol{A}^{Q_iL_i}\cdot\Big(\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}$$
$$\cdot\,\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}-2\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big)\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y).$$

In agreement with matrices $\boldsymbol{A}^{Q_iQ_j}$, $\dot{\boldsymbol{A}}^{Q_iQ_j}$, and $\ddot{\boldsymbol{A}}^{Q_iQ_j}$ the following *isolation relations of the relative coordinates* hold: suitable projections of (5.30a) together with (5.28a) imply

$$\cos(\varphi_y)=\mathbf{P_r}^{\mathrm{T}}(x)\,\Big(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\,\mathbf{P}_r(x) \tag{5.33a}$$

and

$$\sin(\varphi_y)=\mathbf{P_r}^{\mathrm{T}}(x)\,\Big(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\,\mathbf{P}_r(z) \tag{5.33b}$$

as well as

$$\cos(\varphi_x)=\mathbf{P_r}^{\mathrm{T}}(y)\,\Big(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\,\mathbf{P}_r(y) \tag{5.33c}$$

and

$$\sin(\varphi_x) = -\mathbf{P_r}^{\mathrm{T}}(z)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(y). \tag{5.33d}$$

This implies

$$\varphi_x = -\arctan\left\{\frac{\mathbf{P_r}^{\mathrm{T}}(z)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(y)}{\mathbf{P_r}^{\mathrm{T}}(y)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(y)}\right\} \tag{5.34a}$$

and

$$\varphi_y = \arctan\left\{\frac{\mathbf{P_r}^{\mathrm{T}}(x)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(z)}{\mathbf{P_r}^{\mathrm{T}}(x)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(x)}\right\} \tag{5.34b}$$

for $-\pi/2 < \varphi_x < \pi/2$ and $-\pi/2 < \varphi_y < \pi/2$. By analogy with the above expressions, we obtain from (5.28b) together with (5.31a) the relations

$$\dot{\varphi}_y = \begin{cases} +\dfrac{1}{\cos\varphi_y}\cdot\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\Delta\,)\,\mathbf{P}_r(z) \\ \text{or} \\ -\dfrac{1}{\sin\varphi_y}\cdot\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\Delta\,)\,\mathbf{P}_r(x) \end{cases} \tag{5.35a}$$

and

$$\dot{\varphi}_x = \begin{cases} -\dfrac{1}{\cos\varphi_x}\cdot\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\Delta\,)\,\mathbf{P}_r(y) \\ \text{or} \\ -\dfrac{1}{\sin\varphi_x}\cdot\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\Delta\,)\,\mathbf{P}_r(y), \end{cases} \tag{5.35b}$$

with

$$\Delta := \boldsymbol{A}^{Q_iL_i}\cdot\left(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j} + \boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\right)\cdot\boldsymbol{A}^{L_jQ_j}, \tag{5.35c}$$

and finally from (5.28c) and (5.31b) the relations

$$\ddot{\varphi}_y = \begin{cases} \dfrac{1}{\cos\varphi_y}\cdot\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\bowtie\,)\,\mathbf{P}_r(z) + \dfrac{\sin\varphi_y}{\cos\varphi_y}\cdot\dot{\varphi}_y^2 \\ \text{or} \\ -\dfrac{1}{\sin\varphi_y}\cdot\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\bowtie\,)\,\mathbf{P}_r(x) - \dfrac{\cos\varphi_y}{\sin\varphi_y}\cdot\dot{\varphi}_y^2 \end{cases} \tag{5.36a}$$

and

$$\ddot{\varphi}_x = \begin{cases} -\dfrac{1}{\cos\varphi_x}\cdot\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\bowtie\,)\,\mathbf{P}_r(y) + \dfrac{\sin\varphi_x}{\cos\varphi_x}\cdot\dot{\varphi}_x^2 \\ \text{or} \\ -\dfrac{1}{\sin\varphi_x}\cdot\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\bowtie\,)\,\mathbf{P}_r(y) - \dfrac{\cos\varphi_x}{\sin\varphi_x}\cdot\dot{\varphi}_x^2, \end{cases} \tag{5.36b}$$

with

$$\bowtie = \boldsymbol{A}^{Q_iL_i} \cdot \Big(-\dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} + \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \\ - 2 \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \\ + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR} \Big) \cdot \boldsymbol{A}^{L_jQ_j}. \tag{5.36c}$$

5.2.1.5 Constant-distance constraint (BB5; one constrained translational DOF). This building block constrains the distance (ℓ) between two points Q_i and Q_j located on two bodies i and j. The associated constraint relation will be derived from the *vector loop equation* (Figure 5.24)

$$\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} + \boldsymbol{r}^R_{Q_jQ_i} = \boldsymbol{0}$$

or (5.37a)

$$\boldsymbol{r}_{Q_jQ_i} = \boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i},$$

with

$$\left[\left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{Q_jQ_i}\right)\right]^{1/2} = |\boldsymbol{r}^R_{Q_jQ_i}| = |\boldsymbol{r}^{L_i}_{Q_jQ_i}| = \ell = \text{constant} > 0. \tag{5.37b}$$

This provides the *constraint position equation of BB5*:

$$0 = -\ell + \Big\{ \left[\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}} \\ \cdot \Big[\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \underbrace{\boldsymbol{r}^{L_j}_{Q_jP_j}}_{\text{constant}} - \boldsymbol{A}^{RL_i} \cdot \underbrace{\boldsymbol{r}^{L_i}_{Q_iP_i}}_{\text{constant}} \Big]\Big\}^{1/2}. \tag{5.38a}$$

Differentiation of (5.38a) with respect to the time yields

$$0 = \frac{1}{2} \cdot 2 \cdot \left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right) \Big/ \left[\left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{Q_jQ_i}\right)\right]^{1/2}$$

or *approximately* (5.38b)

$$0 = \left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right) \Big/ \ell \quad , \quad \ell > 0.$$

This provides the following approximation of the *constraint velocity equation*:

$$\overbrace{\left[\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}}}^{=(\boldsymbol{r}^R_{Q_jQ_i})^{\mathrm{T}}} \\ \cdot \underbrace{\left[\dot{\boldsymbol{r}}^R_{P_jO} - \dot{\boldsymbol{r}}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]}_{=\dot{\boldsymbol{r}}^R_{Q_jQ_i}} \equiv 0. \tag{5.38c}$$

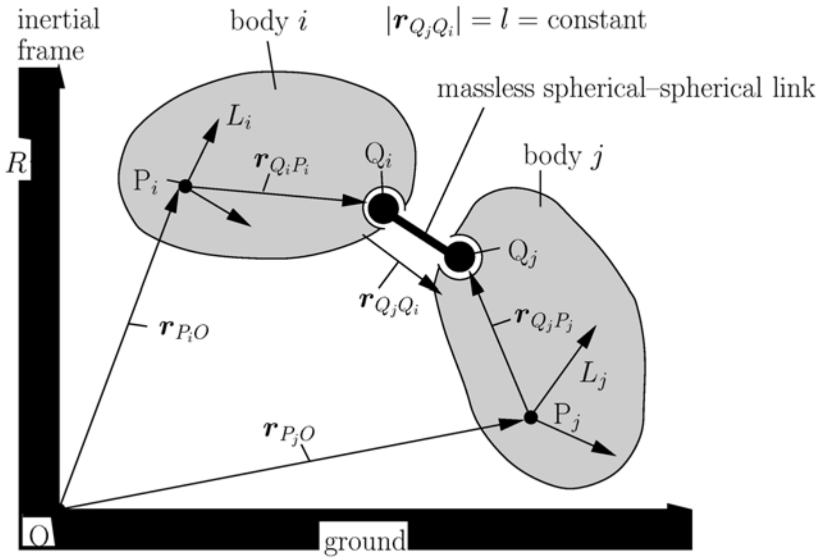


Fig. 5.24: Vector loop associated with a massless spherical–spherical link

The time derivative of (5.38b) is

$$\mathbf{0} = \left[\left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\ddot{\boldsymbol{r}}^R_{Q_jQ_i}\right) + \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right)\right] \Big/ \ell, \tag{5.38d}$$

with

$$\ddot{\boldsymbol{r}}^R_{Q_jQ_i} - \ddot{\boldsymbol{r}}^R_{P_jO} + \ddot{\boldsymbol{r}}^R_{P_iO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j} \cdot \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}$$
$$+ \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{Q_iP_i} \cdot \dot{\boldsymbol{\omega}}^{L_i}_{L_iR} = \mathbf{0}.$$

This provides the *constraint acceleration equation*

$$\underbrace{\left[-\left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}}, \left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{Q_iP_i}, \left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}}, -\left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\right]}_{=:\, \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \,=\, \textit{constraint Jacobian matrix}}$$

$$\cdot \left[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}, (\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}, (\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}, (\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\right]^{\mathrm{T}} = \left(\boldsymbol{r}^R_{Q_jQ_i}\right)^{\mathrm{T}} \tag{5.38e}$$

$$\cdot \left(\boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) - \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{r}}^R_{Q_jQ_i}\right)$$

or written explicitly:

$$\underbrace{\begin{aligned}&\Big[-\left(\boldsymbol{r}^R_{P_jO}-\boldsymbol{r}^R_{P_iO}+\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}}\ ,\\ &\quad\left(\boldsymbol{r}^R_{P_jO}-\boldsymbol{r}^R_{P_iO}+\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{Q_iP_i}\ ,\\ &\quad\left(\boldsymbol{r}^R_{P_jO}-\boldsymbol{r}^R_{P_iO}+\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}}\ ,\\ &\quad-\left(\boldsymbol{r}^R_{P_jO}-\boldsymbol{r}^R_{P_iO}+\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\Big]\end{aligned}}_{=:\ \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}$$

$$\cdot\left[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\ ,\ (\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\ ,\ (\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}\ ,\ (\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\right]^{\mathrm{T}} \tag{5.39}$$

$$\begin{aligned}=\ &\left[\boldsymbol{r}^R_{P_jO}-\boldsymbol{r}^R_{P_iO}+\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}}\\ &\cdot\left[\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\right]\\ &-\left[\dot{\boldsymbol{r}}^R_{P_jO}-\dot{\boldsymbol{r}}^R_{P_jO}+\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}}\\ &\cdot\left[\dot{\boldsymbol{r}}^R_{P_jO}-\dot{\boldsymbol{r}}^R_{P_jO}+\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}-\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}}.\end{aligned}$$

5.2.2 Theoretical models of common joints

In this section the constraint relations of several *spatial joints* in common use will be set up from the building blocks of Section 5.2.1. The spatial joints considered are collected in Table 5.1.

5.2.2.1 Spherical joint (BB1; constrains three translational DOFs). A spherical joint is completely modeled by the building block *BB1* (cf. Section 5.2.1.1). The *constraint position*, *velocity*, and *acceleration equations* of a *spherical joint* (Figure 5.25) are as follows.

Constraint position equations of a spherical joint

$$\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}+\boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{QP_i}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{QP_j}=\mathbf{0}. \tag{5.40a}$$

Constraint velocity equations of a spherical joint

$${}^R\dot{\boldsymbol{r}}^R_{P_iO}-\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\boldsymbol{\omega}^{L_i}_{L_iR}-{}^R\dot{\boldsymbol{r}}^R_{P_jO}+\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\boldsymbol{\omega}^{L_j}_{L_jR}=\mathbf{0}. \tag{5.40b}$$

Constraint acceleration equations of a spherical joint

$$\underbrace{\left[\mathbf{I}_3\ ,\ \ -\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\ ,\ \ -\mathbf{I}_3\ ,\ \ +\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\right]}_{=:\ \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}\cdot\begin{bmatrix}{}^R\ddot{\boldsymbol{r}}^R_{P_iO}\\ \dot{\boldsymbol{\omega}}^{L_i}_{L_iR}\\ {}^R\ddot{\boldsymbol{r}}^R_{P_jO}\\ \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}\end{bmatrix}$$

$$=+\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\boldsymbol{\omega}^{L_i}_{L_iR}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\boldsymbol{\omega}^{L_j}_{L_jR}. \tag{5.40c}$$

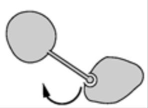
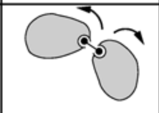
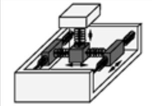
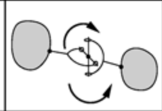
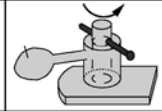
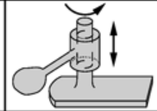
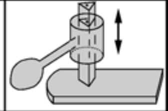
Constraint building blocks (BBs)	Spatial joints: spherical joint	massless spher.-spher. link	translational joint	universal joint	revolute joint	cylindrical joint	prismatic joint
							
BB 1 *common point*	X			X	X		
BB 2 *parallel axes*			X		X	X	X
BB 3 *straight line point follower*						X	X
BB 4 *rotation blocker*			X	X			X
BB 5 *constant distance*		X					

Table 5.1: Construction of spatial joint models from building blocks

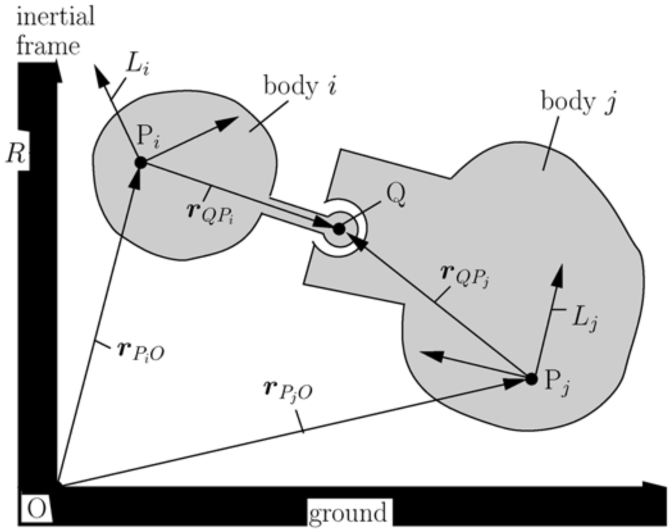


Fig. 5.25: Spherical joint

5.2.2.2 Massless spherical–spherical link (BB5; constrains one translational DOF). A massless spherical–spherical link is completely modeled by the building block *BB5* (cf. Section 5.2.1.5). Its constraint equations are as follows.

Constraint position equations of a massless spherical–spherical link

$$\begin{aligned}\Big\{\Big[&\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\Big]^{\mathrm{T}} \\ &\cdot\Big[\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j}\cdot\underbrace{\boldsymbol{r}^{L_j}_{Q_jP_j}}_{\text{constant}} - \boldsymbol{A}^{RL_i}\cdot\underbrace{\boldsymbol{r}^{L_i}_{Q_iP_i}}_{\text{constant}}\Big]\Big\}^{1/2} - \ell = 0.\end{aligned} \tag{5.41a}$$

Constraint velocity equations of a massless spherical–spherical link

$$\begin{aligned}\Big[&\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\Big]^{\mathrm{T}} \\ &\cdot\Big[\dot{\boldsymbol{r}}^R_{P_jO} - \dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\Big] = 0.\end{aligned} \tag{5.41b}$$

Constraint acceleration equations of a massless spherical–spherical link

$$\begin{aligned}\Big[&-\Big(\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\Big)^{\mathrm{T}}\ , \\ &\Big(\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i}\cdot\boldsymbol{r}^{L_i}_{Q_iP_i}\Big)^{\mathrm{T}}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{Q_iP_i}\ ,\end{aligned}$$

$$
\underbrace{\begin{aligned}
&\left(\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}} \; , \\
&- \left(\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\Big]
\end{aligned}}_{=:\, \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})}
$$

$$
\cdot \left[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}} \, , \, (\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}} \, , \, (\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}} \, , \, (\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\right]^{\mathrm{T}} \tag{5.41c}
$$

$$
\begin{aligned}
= \; & \left[\boldsymbol{r}^R_{P_jO} - \boldsymbol{r}^R_{P_iO} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}} \\
& \cdot \left[\boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right] \\
- & \left[\dot{\boldsymbol{r}}^R_{P_jO} - \dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}} \\
& \cdot \left[\dot{\boldsymbol{r}}^R_{P_jO} - \dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{r}^{L_i}_{Q_iP_i}\right]^{\mathrm{T}} .
\end{aligned}
$$

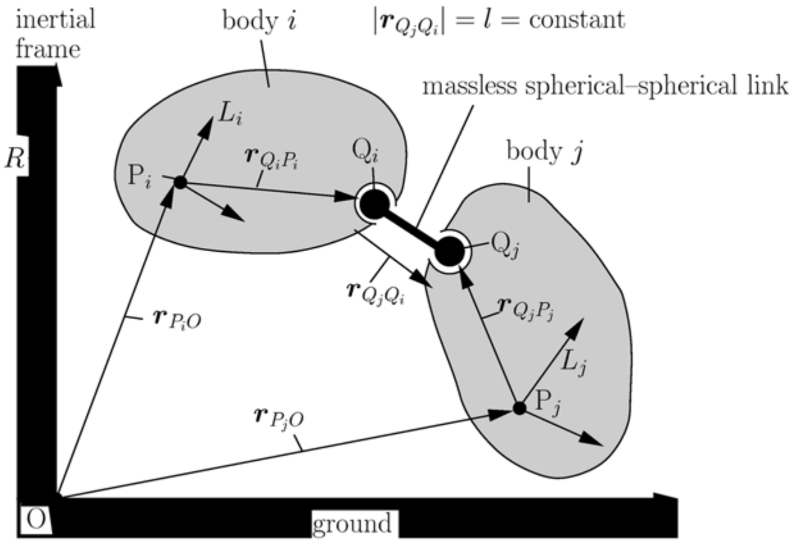


Fig. 5.26: Massless spherical–spherical link

5.2.2.3 Translational joint (BB2, BB4; constrains three rotational DOFs). A joint that allows two rigid bodies i and j to perform relative translational motions in three orthogonal directions and no relative rotation to each other (Figure 5.27) is called *spatial translational joint.* It is modeled by a combination of building block *BB4* and of a modified version of building block *BB2*, where the modified *BB2* constrains relative rotations around the common body-fixed x- and y-axes, and *BB4* constrains rotations around the remaining z-axis. As the modified BB2 allows relative rotations around the parallel z-axes $\boldsymbol{e}_{zQ_i}$ and $\boldsymbol{e}_{zQ_j}$ of bodies i and j, the matrix $\boldsymbol{A}^{L_{Q_j} L_{Q_i}}$ of (5.21a) is now modified to the form

$$\boldsymbol{A}^{L_{Q_j} L_{Q_i}} = \begin{pmatrix} \cos\alpha \ , & \sin\alpha \ , & 0 \\ -\sin\alpha \ , & \cos\alpha \ , & 0 \\ 0 \ , & 0 \ , & 1 \end{pmatrix} \ , \quad \alpha = \psi_{Q_j Q_i} = \psi_{ji} = \alpha_{ji} . \qquad (5.42a)$$

The constraint position equation of the *modified BB2* is obtained by the following projections of the orientation loop equation (5.20a):

$$\mathbf{P_r}^{\mathrm{T}}(x\,,\,y) \cdot \left(\boldsymbol{A}^{L_{Q_j} L_j} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \boldsymbol{A}^{L_i L_{Q_i}}\right) \cdot \mathbf{P}_r(z) = \mathbf{0} \qquad (5.42b)$$

with

$$\mathbf{P_r}^{\mathrm{T}}(x\,,\,y) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_r(z) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \qquad (5.42c)$$

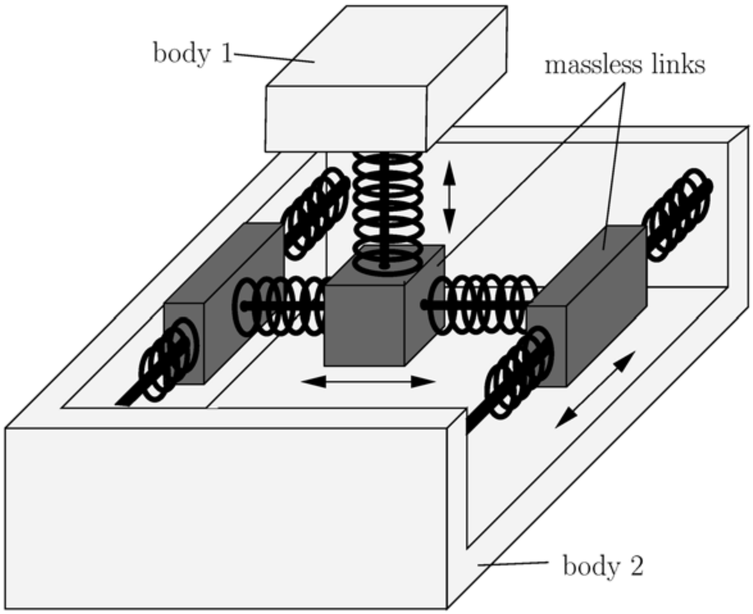


Fig. 5.27: Spatial translational joint

The *constraint position*, *velocity*, and *acceleration equations* of a *spatial translational joint* are as follows.

Constraint position equations of a spatial translational joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \mathbf{P_r}^{\mathrm{T}}(x)\Big(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\mathbf{P}_r(y)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_2\\ 0\end{bmatrix}. \tag{5.43a}$$

Constraint velocity equations of a spatial translational joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}} \\ \qquad +\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \mathbf{P_r}^{\mathrm{T}}(x)\Big[\boldsymbol{A}^{Q_iL_i}\cdot\Big(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j} \\ \qquad +\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\Big)\cdot\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y) \end{bmatrix} = \begin{bmatrix}\mathbf{0}_2\\ 0\end{bmatrix}. \tag{5.43b}$$

Constraint acceleration equations of a spatial translational joint

$$\underbrace{\begin{bmatrix} \begin{bmatrix} \mathbf{0}_{2\,,3} & , & \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big\{-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\overbrace{[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(z)]}\Big\} \\ \mathbf{0}_{1\,,3} & , & \mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\overbrace{\Big[\Big(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\mathbf{P}_r(y)\Big]}\Big\} \end{bmatrix} & , \\ \begin{bmatrix} \mathbf{0}_{2\,,3} & , & \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\cdot\overbrace{\Big[\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(z)\Big]}\Big\} \\ \mathbf{0}_{1\,,3} & , & -\mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\overbrace{\Big[\boldsymbol{A}^{L_jQ_j}\mathbf{P}_r(y)\Big]}\Big\} \end{bmatrix} & \end{bmatrix}}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\,\cdot\,\boldsymbol{T}(\boldsymbol{p})\,=\,\text{constraint Jacobian matrix}}$$

$$\cdot\;\Big[(\ddot{\boldsymbol{r}}^{R}_{P_iO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\,,\,(\ddot{\boldsymbol{r}}^{R}_{P_jO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}} \tag{5.43c}$$

$$= \begin{bmatrix} -\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \quad +2\cdot\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \quad -\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ -\mathbf{P_r}^{\mathrm{T}}(x)\Big[\boldsymbol{A}^{Q_iL_i}\cdot\Big(\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \\ \quad \cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}-2\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big)\cdot\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y) \end{bmatrix}.$$

5.2.2.4 Universal joint (BB1, BB4; constrains three translational and one rotational DOF). Consider the mechanism of Figure 5.28 comprising two rigid bodies i and j connected by a universal joint. Let Q be a point that is common to the bodies i and j. Assume that a frame L_{Q_i} is fixed on body i with dorigin Q, and with the x-axis $\boldsymbol{e}_{xQ_i}$ placed in the direction of the first rotation axis of the universal joint, and that a frame L_{Q_j} is fixed on body j with origin Q, and with the y-axis $\boldsymbol{e}_{yQ_j}$, placed in the direction of the second rotation axis of the universal joint, perpendicular to the first rotation axis. Consider a third frame L_k with origin Q fixed to the two (massless) rotation axes with unit vectors

$$\boldsymbol{e}_{xK} = \boldsymbol{e}_{xQ_i} \quad \text{and} \quad \boldsymbol{e}_{yK} = \boldsymbol{e}_{yQ_j}. \tag{5.44a}$$

Let

$$\varphi_1 = \varphi_x := \varphi_{Q_i K} \tag{5.44b}$$

be the angle of rotation of body i around the x-axis ($\boldsymbol{e}_{xK}$) of frame L_k and

$$\varphi_2 = \varphi_y := \varphi_{Q_j K} \tag{5.44c}$$

be the angle of rotation of body j around the y-axis ($\boldsymbol{e}_{yK}$) of frame L_k.

This so-called *universal joint* between bodies i and j constrains three translational DOFs and a single rotational DOF of the two bodies. The *three translational DOFs* are *eliminated* by a *vector loop equation* providing a *common-point* constraint relation of *BB1*. The *rotational DOF* is *eliminated* by a suitable projection of an *orientation loop equation* summarized in *BB4*. The *relative angles* φ_1 and φ_2 around the rotation axes of the universal joint are *isolated* for measurement or control purposes by suitable representations and projections of the orientation loop equation.

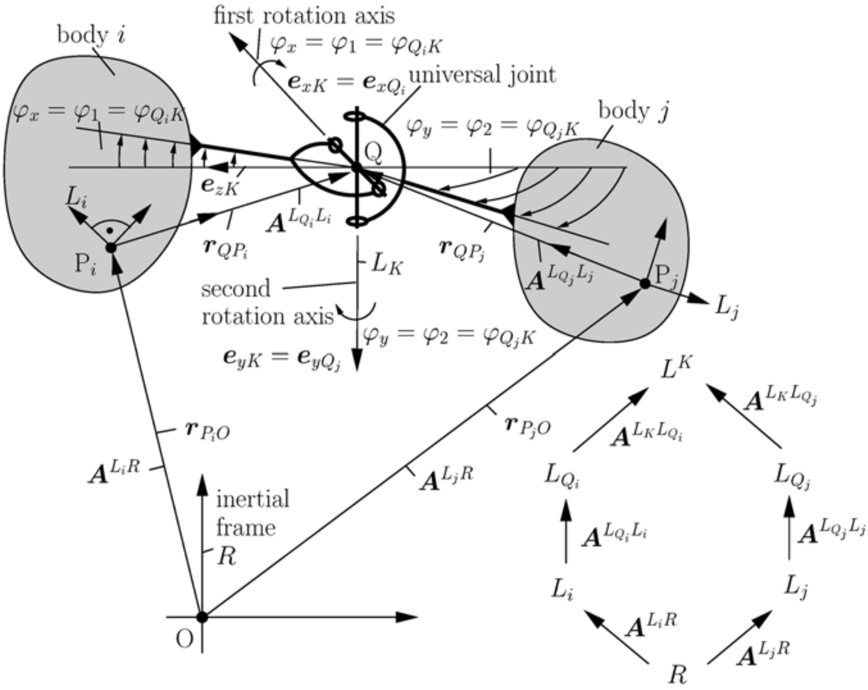
Collecting the relations included in the *BB1* and *BB4* provides the following *constraint position*, *velocity*, and *acceleration relations* and *relative rotation angles* of the two bodies i and j connected by a universal joint (Figure 5.28).

Constraint position equations of a universal joint

$$\begin{bmatrix} \boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{QP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{QP_j} \\ \mathbf{P_r}^{\mathrm{T}}(x) \left(\boldsymbol{A}^{Q_iL_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \boldsymbol{A}^{L_jQ_j} \right) \mathbf{P}_r(y) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_3 \\ 0 \end{bmatrix}. \tag{5.45a}$$

Constraint velocity equations of a universal joint

$$\begin{bmatrix} {}^R\dot{\boldsymbol{r}}^R_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{QP_i} \cdot \boldsymbol{\omega}^{L_i}_{L_iR} - {}^R\dot{\boldsymbol{r}}^R_{P_jO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{QP_j} \cdot \boldsymbol{\omega}^{L_j}_{L_jR} \\ \mathbf{P_r}^{\mathrm{T}}(x) \left[\boldsymbol{A}^{Q_iL_i} \cdot \left(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} + \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \right) \cdot \boldsymbol{A}^{L_jQ_j} \right] \mathbf{P}_r(y) \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0}_3 \\ 0 \end{bmatrix}. \tag{5.45b}$$



(a) Geometrical configuration of a universal joint

(b) Computer drawing of a universal joint

(c) Technical realization of a universal joint (built at RTS, University of Kassel)

Fig. 5.28: Drawings of a universal joint

Constraint acceleration equations of a universal joint

$$\underbrace{\begin{bmatrix} \mathbf{I}_3 & , & -\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i} & , & -\mathbf{I}_3 & , & +\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j} \\ \mathbf{0}_{1,3} & , & \mathbf{P_r}^{\mathrm{T}}(x)\left\{\boldsymbol{A}^{Q_iL_i}\cdot\overbrace{\left[\left(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(y)\right]}\right\} & , & \mathbf{0}_{1,3} & , & -\mathbf{P_r}^{\mathrm{T}}(x)\left\{\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\overbrace{\left[\boldsymbol{A}^{L_jQ_j}\mathbf{P}_r(y)\right]}\right\} \end{bmatrix}}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\,=\,\textit{constraint Jacobian matrix of the universal joint}\,\in\,\mathbb{R}^{4,12}}$$

$$\cdot\left[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}},\,(\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}},\,(\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}},\,(\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\right] \tag{5.45c}$$

$$=\begin{bmatrix} \boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{r}}^{L_i}_{QP_i}\cdot\boldsymbol{\omega}^{L_i}_{L_iR}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{r}}^{L_j}_{QP_j}\cdot\boldsymbol{\omega}^{L_j}_{L_jR} \\ -\mathbf{P_r}^{\mathrm{T}}(x)\Big[\,\boldsymbol{A}^{Q_iL_i}\cdot\Big(\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot \\ \qquad\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}-2\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big)\cdot\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y) \end{bmatrix}.$$

Relative coordinates φ_x and φ_y of a universal joint

$$\varphi_x = -\arctan\left[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\Delta\,)\,\mathbf{P}_r(y)\Big/\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\Delta\,)\,\mathbf{P}_r(y)\right] \tag{5.46a}$$

and

$$\varphi_y = \arctan\left[\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\Delta\,)\,\mathbf{P}_r(z)\Big/\mathbf{P_r}^{\mathrm{T}}(x)\,(\,\Delta\,)\,\mathbf{P}_r(x)\right] \tag{5.46b}$$

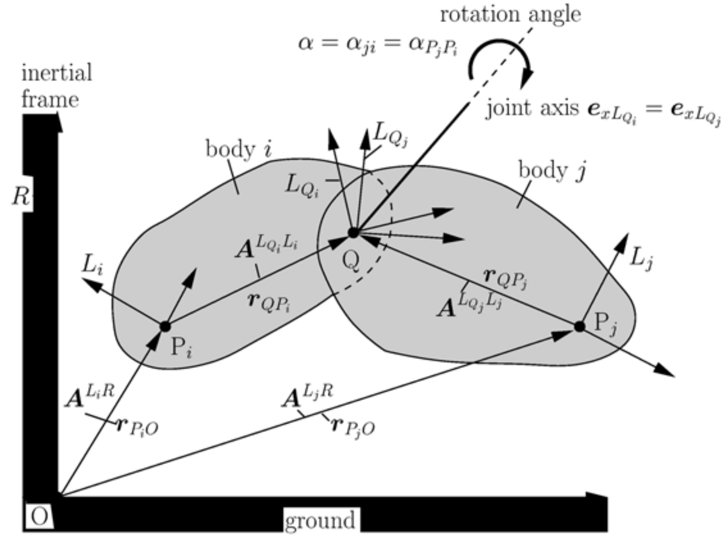
for $-\pi/2 < \varphi_x < \pi/2$ and $-\pi/2 < \varphi_y < \pi/2$, with

$$\Delta := \boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j} \tag{5.46c}$$

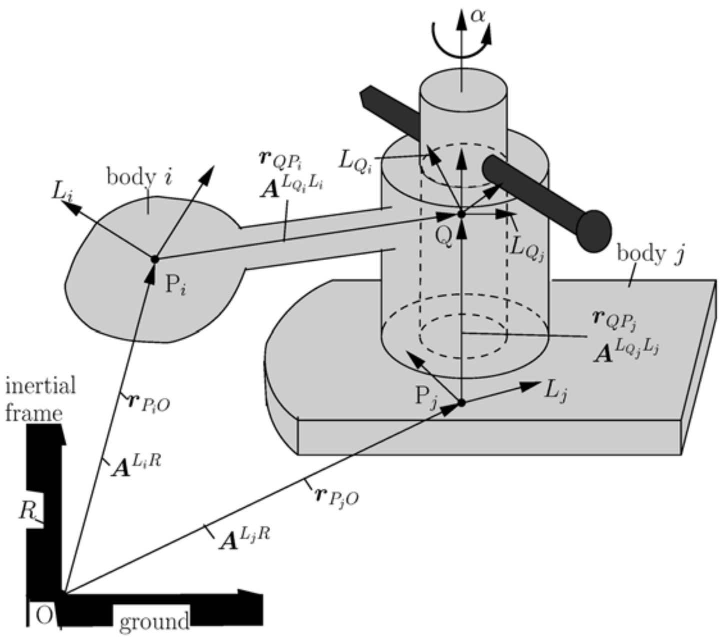
and

$$\mathbf{P}_r(x) := (1\,,0\,,0)^{\mathrm{T}}\quad,\quad\mathbf{P}_r(y) := (0\,,1\,,0)^{\mathrm{T}}\quad,\quad\mathbf{P}_r(z) := (0\,,0\,,1)^{\mathrm{T}}\,. \tag{5.46d}$$

5.2.2.5 Revolute joint (BB1, BB2; constrains three translational and two rotational DOFs). A revolute joint between the rigid bodies i and j constrains three translational DOFs *(BB1)* and two rotational DOFs *(BB2)* of the bodies (Figure 5.29). The three translational DOFs are *eliminated* by a *vector loop equation defining the common-point constraint relations of BB1*. The two rotational DOFs are *eliminated* by a suitable *projection of an orientation loop equation of BB2*. The *relative angle* around the rotation axis of the revolute joint is *isolated* by suitable *projections of the orientation loop equation.*



(a) Vector diagram



(b) Revolute joint

Fig. 5.29: Vector loop and orientation loop of a revolute joint

Collecting the relations included in *BB1* and *BB2* provides the following *joint relations.*

Constraint position equations of a revolute joint

$$\begin{bmatrix} \boldsymbol{r}^{R}_{P_iO} - \boldsymbol{r}^{R}_{P_jO} + \boldsymbol{A}^{RL_i} \cdot \boldsymbol{r}^{L_i}_{QP_i} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{QP_j} \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\left(\boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\right)\mathbf{P}_r(x) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{0}_2 \end{bmatrix}. \tag{5.47a}$$

Constraint velocity equations of a revolute joint

$$\begin{bmatrix} {}^R\dot{\boldsymbol{r}}^{R}_{P_iO} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{QP_i} \cdot \boldsymbol{\omega}^{L_i}_{L_iR} - {}^R\dot{\boldsymbol{r}}^{R}_{P_jO} + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{QP_j} \cdot \boldsymbol{\omega}^{L_j}_{L_jR} \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(- \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ \qquad + \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_3 \\ \mathbf{0}_2 \end{bmatrix}. \tag{5.47b}$$

Constraint acceleration equations of a revolute joint

$$\begin{gathered} \underbrace{\begin{bmatrix} \mathbf{I}_3 \,, & -\boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{r}}^{L_i}_{QP_i} \,, & -\mathbf{I}_3 \,, & +\boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{QP_j} \\ \mathbf{0}_{2\,,\,3} \,, & \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{ - \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \overbrace{\Big[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)\Big]}\Big\} \,, & \mathbf{0}_{2\,,\,3} \,, & \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big\{\boldsymbol{A}^{L_{Q_j}L_j} \cdot \overbrace{\Big[\boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)\Big]}\Big\} \end{bmatrix}}_{:= \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})} \\ \cdot \Big[(\ddot{\boldsymbol{r}}^{R}_{P_iO})^{\mathrm{T}} \,, (\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}} \,, (\ddot{\boldsymbol{r}}^{R}_{P_jO})^{\mathrm{T}} \,, (\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}} = \\ \begin{bmatrix} +\boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{r}}^{L_i}_{QP_i} \cdot \boldsymbol{\omega}^{L_i}_{L_iR} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{r}}^{L_j}_{QP_j} \cdot \boldsymbol{\omega}^{L_j}_{L_jR} \\ -\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\left(\boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\right)\mathbf{P}_r(x) \\ \quad +2 \cdot \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\left(\boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\right)\mathbf{P}_r(x) \\ \quad -\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\left(\boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\right)\mathbf{P}_r(x) \end{bmatrix}. \end{gathered} \tag{5.47c}$$

Relative coordinate of a revolute joint

$$\alpha = \alpha_{ji} = \alpha_{Q_jQ_i} = -\arctan\Big\{\Big[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\Delta\,)\,\mathbf{P}_r(y)\Big] \Big/ \Big[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\Delta\,)\,\mathbf{P}_r(y)\Big]\Big\},$$

with (5.48a)

$$\triangle := \boldsymbol{A}^{L_{Q_j} L_j} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \boldsymbol{A}^{L_i L_{Q_i}},$$

$$\dot{\alpha} = \begin{cases} -\dfrac{1}{\sin\alpha} \cdot \left[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\square\,)\,\mathbf{P}_r(y)\right] \\ \text{or} \\ -\dfrac{1}{\cos\alpha} \cdot \left[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\square\,)\,\mathbf{P}_r(y)\right] \end{cases} \tag{5.48b}$$

with

$$\square := \Big(-\boldsymbol{A}^{L_{Q_j} L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \boldsymbol{A}^{L_i L_{Q_i}} \tag{5.48c}$$
$$+\boldsymbol{A}^{L_{Q_j} L_j} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_i R} \cdot \boldsymbol{A}^{L_i L_{Q_i}}\Big),$$

and

$$\ddot{\alpha} = \begin{cases} -\dfrac{1}{\sin\alpha} \cdot \left\{\left[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\bowtie\,)\,\mathbf{P}_r(y)\right] + \cos\alpha \cdot \dot{\alpha}^2\right\} \\ \text{or} \\ -\dfrac{1}{\cos\alpha} \cdot \left\{\left[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\bowtie\,)\,\mathbf{P}_r(y)\right] - \sin\alpha \cdot \dot{\alpha}^2\right\} \end{cases} \tag{5.48d}$$

with

$$\begin{aligned} \bowtie := \Big(&+ \boldsymbol{A}^{L_{Q_j} L_j} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_i R} \cdot \boldsymbol{A}^{L_i L_{Q_i}} \\ &- \boldsymbol{A}^{L_{Q_j} L_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_j R} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \boldsymbol{A}^{L_i L_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j} L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \boldsymbol{A}^{L_i L_{Q_i}} \\ &- 2 \cdot \boldsymbol{A}^{L_{Q_j} L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_j R} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_i R} \cdot \boldsymbol{A}^{L_i L_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j} L_j} \cdot \boldsymbol{A}^{L_j R} \cdot \boldsymbol{A}^{R L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_i R} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_i R} \cdot \boldsymbol{A}^{L_i L_{Q_i}}\Big). \end{aligned}$$

5.2.2.6 Cylindrical joint (BB2, BB3; constrains two translational and two rotational DOFs). A cylindrical joint between bodies i and j constrains two rotational DOFs and two translational DOFs, where the common rotation axes of the two bodies point into the direction of the common translational motion. The common rotation axis of the two bodies are chosen in the $\boldsymbol{e}_{xL_{Q_i}}$-direction and in the $\boldsymbol{e}_{xL_{Q_j}}$-direction of frames L_{Q_i} and L_{Q_j} fixed on the bodies i and j, respectively (Figure 5.30).
Two translational DOFs are *eliminated* – and the relative coordinate $x^{L_i}_{Q_j Q_i}$ is *isolated* – by means of the model equations of *BB3* (straight-line-point-follower constraint in the x-direction).
Two rotational DOFs are *eliminated* – and the relative coordinate α_{ji} is *isolated* – by means of the model equations of *BB2* (parallel-axes constraint in the x-direction).
This provides the following model equations of a *cylindrical joint* with a relative motion in and around a common x-axis:.

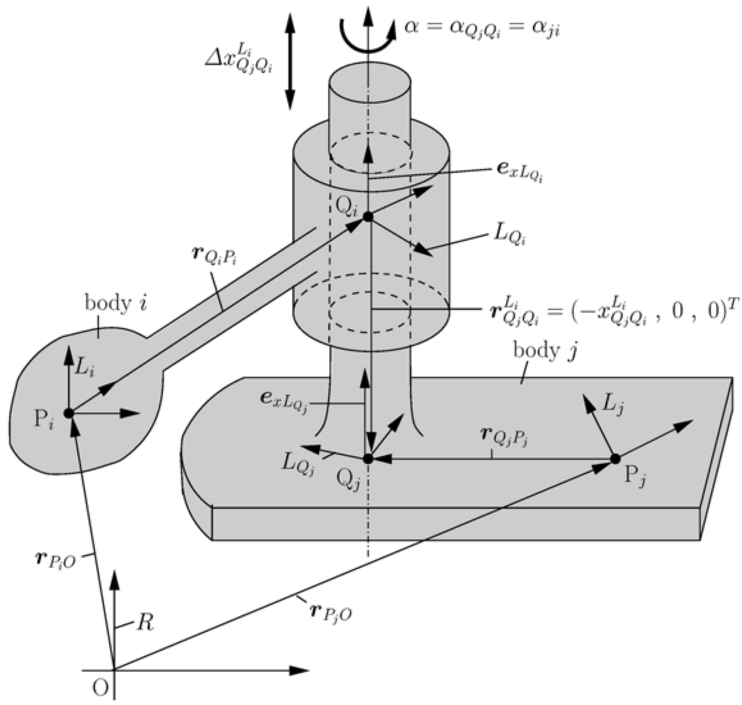


Fig. 5.30: Vector diagram of a cylindrical joint

Constraint position equations of a cylindrical joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big) \\ \qquad +\boldsymbol{A}^{L_{Q_i}L_i}\cdot r^{L_i}_{Q_iP_i}-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot r^{L_j}_{Q_jP_j}\Big] \end{bmatrix} = \begin{bmatrix}\mathbf{0}_2\\ \mathbf{0}_2\end{bmatrix} \tag{5.49a}$$

Constraint velocity equations of a cylindrical joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}} \\ \qquad +\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big) \\ \qquad +\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\Big] \end{bmatrix}$$

$$= \Big(\mathbf{0}_2^{\mathrm{T}}\,,\,\mathbf{0}_2^{\mathrm{T}}\Big)^{\mathrm{T}}. \tag{5.49b}$$

Constraint acceleration equations of a cylindrical joint

$$\underbrace{\left[\begin{array}{ll} \mathbf{0}_{2,3} & ,\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big\{-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\big[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)\big]\Big\}\ , \\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\Big) & ,\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big[\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big]\ , \\ \mathbf{0}_{2,3} & ,\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\cdot\big[\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(x)\big]\Big\} \\ -\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\Big) & ,\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\Big) \end{array}\right]}_{=:\ \boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}$$

$$\cdot\ \Big[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\,,\ (\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\,,\ (\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}\,,\ (\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}} = \tag{5.49c}$$

$$\left[\begin{array}{l} -\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \\ +2\cdot\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \\ -\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(x) \\ \mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big[-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\ \Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\Big] \\ +2\cdot\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big[\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\ \Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big] \\ +\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big) \end{array}\right].$$

Relative coordinates of a cylindrical joint

$$\alpha=\alpha_{ji}=\alpha_{Q_jQ_i}=-\arctan\Big\{\Big[\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(z)\,(\,\triangle\,)\,\mathbf{P}_r(y)\Big]\Big/\Big[\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y)\,(\,\triangle\,)\,\mathbf{P}_r(y)\Big]\Big\}\,,$$

with (5.50a)

$$\triangle:=\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\,,$$

and

$$x^{L_{Q_i}}_{Q_jQ_i}=-\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(x)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\ \Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big) + \boldsymbol{A}^{L_{Q_i}L_i}\cdot r^{L_i}_{Q_iP_i}-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot r^{L_j}_{Q_jP_j}\Big).$$

Relative velocities of a cylindrical joint

$$\dot{\alpha}=\begin{cases} -\dfrac{1}{\sin\alpha}\cdot\ \Big[\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(y)\,(\,\Box\,)\,\mathbf{P}_r(y)\Big] \\ \text{or} \\ -\dfrac{1}{\cos\alpha}\cdot\ \Big[\mathbf{P}_{\mathbf{r}}^{\mathrm{T}}(z)\,(\,\Box\,)\,\mathbf{P}_r(y)\Big]\,, \end{cases} \tag{5.50b}$$

with

$$\begin{aligned}\square := &- \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}\end{aligned}$$

and

$$\begin{aligned}\dot{x}^{L_{Q_i}}_{Q_jQ_i} = \mathbf{P_r}^{\mathrm{T}}(x)\Big[&\boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot r^{L_j}_{Q_jP_j}\right) \\ &- \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot r^{L_j}_{Q_jP_j}\right)\Big].\end{aligned}$$

Relative accelerations of a cylindrical joint

$$\ddot{\alpha} = \begin{cases} -\dfrac{1}{\sin\alpha} \cdot \left\{\left[\mathbf{P_r}^{\mathrm{T}}(y)\,(\,\bowtie\,)\,\mathbf{P}_r(y)\right] + \cos\alpha \cdot \dot{\alpha}^2\right\} \\ \text{or} \\ -\dfrac{1}{\cos\alpha} \cdot \left\{\left[\mathbf{P_r}^{\mathrm{T}}(z)\,(\,\bowtie\,)\,\mathbf{P}_r(y)\right] - \sin\alpha \cdot \dot{\alpha}^2\right\}, \end{cases} \tag{5.50c}$$

with

$$\begin{aligned}\bowtie := &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &- \boldsymbol{A}^{L_{Q_j}L_j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &- 2 \cdot \boldsymbol{A}^{L_{Q_j}L_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}} \\ &+ \boldsymbol{A}^{L_{Q_j}L_j} \cdot \boldsymbol{A}^{L_jR} \cdot \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iL_{Q_i}}.\end{aligned}$$

$$\begin{aligned}\ddot{x}^{L_{Q_i}}_{Q_jQ_i} = &- \mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{L_{Q_i}L_i} \cdot \widetilde{\left[\boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right]} \cdot \dot{\boldsymbol{\omega}}^{L_i}_{L_iR} \\ &+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\ &- 2 \cdot \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\ &+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\ddot{\boldsymbol{r}}^R_{P_iO} - \ddot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\ &+ \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j} \cdot \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big\}.\end{aligned} \tag{5.50d}$$

5.2.2.7 Prismatic joint (BB2, BB3, BB4; constrains three rotational and two translational DOFs). A prismatic joint between bodies i and j constrains three rotational DOFs and two translational DOFs. Two rotational DOFs around the x- and y-axes are constrained by the modified *BB2* (cf. Equation 5.42b). The remaining rotational DOF around the z-axis is constrained by *BB4*. The two translational DOFs in the y- and z-directions are constrained by *BB3* (Figure 5.31).

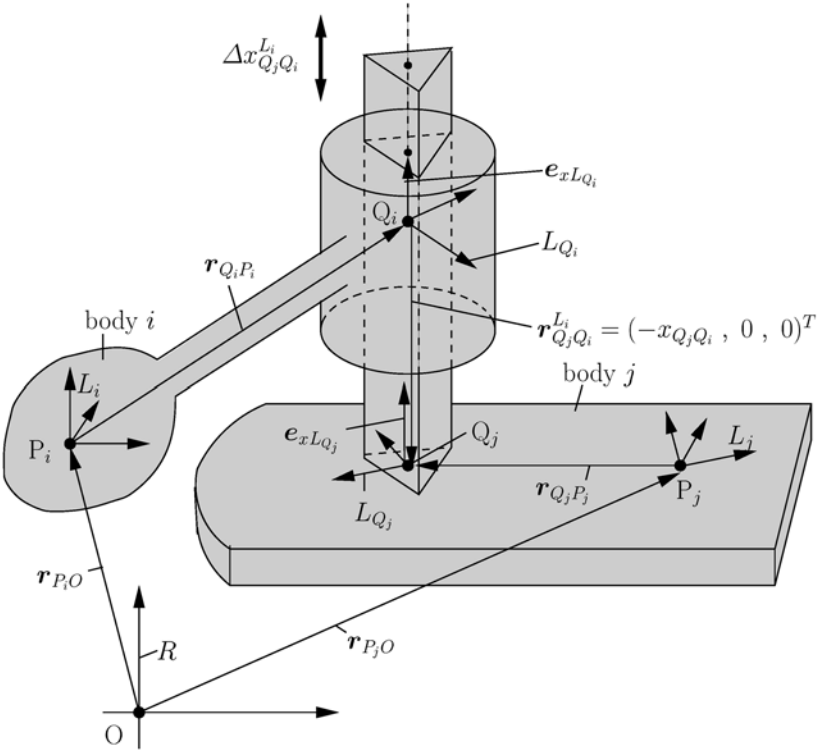


Fig. 5.31: Vector diagram of a translational (prismatic) joint

This provides the following model equations of a prismatic joint that may also be considered as a combination of the model equations of a translational joint (modified BB2, BB4) with the model equations of BB3.

Constraint position equations of a prismatic joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \mathbf{P_r}^{\mathrm{T}}(x)\left(\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\right)\mathbf{P}_r(y) \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\left(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\right)+\boldsymbol{A}^{L_{Q_i}L_i}\cdot r^{L_i}_{Q_iP_i} \\ \qquad -\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot r^{L_j}_{Q_jP_j}\Big] \end{bmatrix} = \begin{bmatrix}\mathbf{0}_2\\ \mathbf{0}_2\\ 0\end{bmatrix}. \tag{5.51a}$$

Constraint velocity equations of a prismatic joint

$$\begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}} \\ \qquad +\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \mathbf{P_r}^{\mathrm{T}}(x)\Big[\boldsymbol{A}^{Q_iL_i}\cdot\Big(-\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\Big)\cdot\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y) \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big) \\ \qquad +\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big)\Big] \end{bmatrix}$$

$$= \Big(\mathbf{0}_2^{\mathrm{T}}\,,\,\mathbf{0}_2^{\mathrm{T}}\,,\,0\Big)^{\mathrm{T}}\,. \tag{5.51b}$$

Constraint acceleration equations of a prismatic joint

$$\underbrace{\begin{bmatrix} \mathbf{0}_{2\,,\,3} & ,\ \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big\{-\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\widetilde{\Big[\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(z)\Big]}\Big\} & , & \mathbf{0}_{2\,,\,3} & ,\ \mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big\{\boldsymbol{A}^{L_{Q_j}L_j}\cdot\widetilde{\Big[\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\mathbf{P}_r(z)\Big]}\Big\} \\ \mathbf{0}_{1\,,\,3} & ,\ \mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\widetilde{\Big[\Big(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\boldsymbol{A}^{L_jQ_j}\Big)\mathbf{P}_r(y)\Big]}\Big\} & , & \mathbf{0}_{1\,,\,3} & ,\ -\mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{Q_iL_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\widetilde{\Big[\boldsymbol{A}^{L_jQ_j}\mathbf{P}_r(y)\Big]}\Big\} \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\Big) & ,\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[\boldsymbol{A}^{L_{Q_i}L_i}\cdot\widetilde{\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}\Big)-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}}\Big] & , & -\mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\Big) & ,\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big(\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j}\Big) \end{bmatrix}}_{=:\,\boldsymbol{g}_p(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})\,=\,\textit{constraint Jacobian matrix}}$$

$$\cdot\ \Big[(\ddot{\boldsymbol{r}}^R_{P_iO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_i}_{L_iR})^{\mathrm{T}}\,,\,(\ddot{\boldsymbol{r}}^R_{P_jO})^{\mathrm{T}}\,,\,(\dot{\boldsymbol{\omega}}^{L_j}_{L_jR})^{\mathrm{T}}\Big]^{\mathrm{T}} = \tag{5.51c}$$

$$\begin{bmatrix} -\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ +2\cdot\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ -\mathbf{P_r}^{\mathrm{T}}(x\,,\,y)\Big(\boldsymbol{A}^{L_{Q_j}L_j}\cdot\boldsymbol{A}^{L_jR}\cdot\boldsymbol{A}^{RL_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iL_{Q_i}}\Big)\mathbf{P}_r(z) \\ \\ -\mathbf{P_r}^{\mathrm{T}}(x)\Big[\boldsymbol{A}^{Q_iL_i}\cdot\Big(\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}+\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \\ -2\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big)\cdot\boldsymbol{A}^{L_jQ_j}\Big]\mathbf{P}_r(y) \\ \\ \mathbf{P_r}^{\mathrm{T}}(y\,,\,z)\Big[-\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\boldsymbol{r}^R_{P_iO}-\boldsymbol{r}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big) \\ +2\cdot\boldsymbol{A}^{L_{Q_i}L_i}\cdot\tilde{\boldsymbol{\omega}}^{L_i}_{L_iR}\cdot\boldsymbol{A}^{L_iR}\cdot\Big(\dot{\boldsymbol{r}}^R_{P_iO}-\dot{\boldsymbol{r}}^R_{P_jO}-\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big) \\ +\boldsymbol{A}^{L_{Q_i}L_i}\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{A}^{RL_j}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\tilde{\boldsymbol{\omega}}^{L_j}_{L_jR}\cdot\boldsymbol{r}^{L_j}_{Q_jP_j}\Big] \end{bmatrix}.$$

Relative coordinate of a prismatic joint

$$
\begin{aligned}
x^{L_{Q_i}}_{Q_jQ_i} = -\mathbf{P_r}^{\mathrm{T}}(x)\Big[\quad & \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}\right) \\
& + \boldsymbol{A}^{L_{Q_i}L_i} \cdot r^{L_i}_{Q_iP_i} - \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot r^{L_j}_{Q_jP_j}\Big].
\end{aligned}
\tag{5.52a}
$$

Relative velocity of a prismatic joint

$$
\begin{aligned}
\dot{x}^{L_{Q_i}}_{Q_jQ_i} = \mathbf{P_r}^{\mathrm{T}}(x)\Big[\quad & \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot r^{L_j}_{Q_jP_j}\right) \\
& - \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot r^{L_j}_{Q_jP_j}\right)\Big].
\end{aligned}
\tag{5.52b}
$$

Relative acceleration of a prismatic joint

$$
\begin{aligned}
\ddot{x}^{L_{Q_i}}_{Q_jQ_i} = & - \mathbf{P_r}^{\mathrm{T}}(x)\Big\{\boldsymbol{A}^{L_{Q_i}L_i} \cdot \left[\boldsymbol{A}^{L_iR} \cdot (\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO}) - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right] \cdot \dot{\boldsymbol{\omega}}^{L_i}_{L_iR} \\
& + \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\boldsymbol{r}^R_{P_iO} - \boldsymbol{r}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\
& - 2 \cdot \boldsymbol{A}^{L_{Q_i}L_i} \cdot \tilde{\boldsymbol{\omega}}^{L_i}_{L_iR} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\dot{\boldsymbol{r}}^R_{P_iO} - \dot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\
& + \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \left(\ddot{\boldsymbol{r}}^R_{P_iO} - \ddot{\boldsymbol{r}}^R_{P_jO} - \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \tilde{\boldsymbol{\omega}}^{L_j}_{L_jR} \cdot \boldsymbol{r}^{L_j}_{Q_jP_j}\right) \\
& + \boldsymbol{A}^{L_{Q_i}L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{r}}^{L_j}_{Q_jP_j} \cdot \dot{\boldsymbol{\omega}}^{L_j}_{L_jR}\Big\}.
\end{aligned}
\tag{5.52c}
$$

6. Constitutive relations of planar and spatial external forces and torques

In this chapter constitutive relations of some commonly used (passive and active) force and torque elements will be discussed. They usually act due to relative motion of system components. An important class of forces is associated with compliant elements such as *coil springs*, *leaf springs*, *dampers*, *tyres*, *shock absorbers.* and other *deformable media* that generate reaction forces and torques *between* – most times *two* – *bodies.* They may be a function of the relative position and velocity of the components. Sometimes they may also depend on their relative acceleration. Another class of forces and torques are those that act on a *single body* like, for example, the gravitational force.

In this chapter constitutive relations of the *gravitational force*, *translational springs* and *dampers*, *torsional springs* and *dampers*, and *actuators* and *motors* will be briefly discussed. In *Section 6.1* the model equations of forces and torques acting in a *plane* or in parallel planes will be considered. In *Section 6.2* the forces and torques acting in a *space* will be discussed. These constitutive relations will be extensively used in the applications of Volume II.

6.1 Constitutive relations of *planar* external forces and torques

An external force or torque applied to a rigid body is commonly represented (Figure 6.1 and Section 2.1.1.1) geometrically by an *arrow* ($\boldsymbol{F}$, $\boldsymbol{M}$) and formally by the relations

$$\boldsymbol{F} = F \cdot \boldsymbol{e}_F \quad \text{and} \quad \boldsymbol{M} = M \cdot \boldsymbol{e}_M \tag{6.1}$$

with the symbols of the *amplitudes* F of $\boldsymbol{F}$ and M of $\boldsymbol{M}$ possibly *changing their signs*, and with the unit vectors $\boldsymbol{e}_F$ (in the direction of $\boldsymbol{F}$) and $\boldsymbol{e}_M$ (in the direction of $\boldsymbol{M}$).

A sign convention states that F (or M) is counted positive, $F > 0$ (or $M > 0$), if it acts in the direction of the arrow, and negative otherwise. This implies that a *force* or *torque* is geometrically represented as an element of a *set of geometrical vectors*: (1) that are placed on a *common line of action* (in the

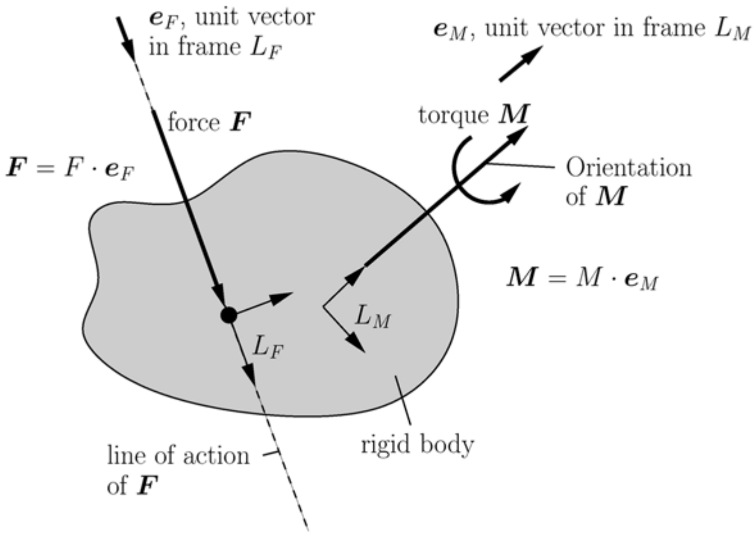


Fig. 6.1: Geometrical representation of a force $\boldsymbol{F}$ and a torque $\boldsymbol{M}$

case of F) and may have *equal* or *opposite directions*, or (2) that have *common orientation* (in the case of M) and may have *equal* or *opposite directions.*

The associated *algebraic vectors*, represented in frame L, are

$$\boldsymbol{F}^L = \boldsymbol{A}^{LL_F} \cdot \boldsymbol{F}^{L_F} \quad , \quad \boldsymbol{A}^{LL_F} = \begin{pmatrix} \cos\psi_F & , & -\sin\psi_F & , & 0 \\ \sin\psi_F & , & \cos\psi_F & , & 0 \\ 0 & , & 0 & , & 1 \end{pmatrix} ,$$

$$\boldsymbol{F}^{L_F} = (F\,,\,0\,,\,0)^{\mathrm{T}} \quad , \quad \psi_F := \psi_{L_F L}, \tag{6.2a}$$

and

$$\boldsymbol{M}^L = \boldsymbol{A}^{LL_M} \cdot \boldsymbol{M}^{L_M} \quad , \quad \boldsymbol{M}^{L_M} = (0\,,\,0\,,\,M)^{\mathrm{T}} , \tag{6.2b}$$

with F and M as the *amplitudes* of the *force* and *torque*, $\boldsymbol{e}_F$ and $\boldsymbol{e}_M$ as unit vectors in the direction of the force and torque arrows, $|F|$ and $|M|$ as the absolute values of F and M or as the length of $\boldsymbol{F}$ and $\boldsymbol{M}$, and $\boldsymbol{A}^{LL_F}$ or $\boldsymbol{A}^{LL_M}$ as mappings from frame L_F with the basis vectors $\{\boldsymbol{e}_{xF}\,,\,\boldsymbol{e}_{yF}\,,\,\boldsymbol{e}_{zF}\}$ with $\boldsymbol{e}_{xF} := \boldsymbol{e}_F$, or from frame L_M with the basis vectors $\{\boldsymbol{e}_{xM}\,,\,\boldsymbol{e}_{yM}\,,\,\boldsymbol{e}_{zM}\}$ with $\boldsymbol{e}_{zL} := \boldsymbol{e}_M$, respectively, into frame L with the basis vectors $(\boldsymbol{e}_{xL}\,,\,\boldsymbol{e}_{yL}\,,\,\boldsymbol{e}_{zL})$.

Then the force (or torque) vector points into the direction of the arrow $\boldsymbol{F}$ (or $\boldsymbol{M}$) for $F > 0$ (or $M > 0$), and in the opposite direction of the arrow for $F < 0$ (or $M < 0$).

Comment 6.1.1 (Characterization of forces and torques): The *action of a force on a body* is uniquely characterized by the *amplitude*, *direction*, and *line of action* of the force (force arrow). *Attachment points* of forces are not needed. The *action of a torque on a body* is uniquely characterized by the *amplitude* and *direction* of the torque (torque arrow). *Lines of action* and *attachment points* of torques are not needed.

6.1.1 Gravitational force (weight)

In this context the gravitational field will be assumed to act in the negative $\boldsymbol{e}_{yR}$ direction (Figure 6.2). Then the weight $\boldsymbol{F}_{W_i}$ of a body i is

$$\boldsymbol{F}_{W_i} = -m_i \cdot \mathrm{g} \cdot \boldsymbol{e}_{yR} \quad \text{or} \quad \boldsymbol{F}^R_{W_i} = \begin{pmatrix} 0 \\ -m_i \cdot \mathrm{g} \end{pmatrix} = \text{constant}, \tag{6.3a}$$

with g as the gravitational constant. Since the line of action of $\boldsymbol{F}_{W_i}$ meets the center of mass C_i of body i, the force $\boldsymbol{F}_{W_i}$ does not generate a moment with respect to C_i; i.e.,

$$\boldsymbol{M}^{L_i}_{C_i W_i} \equiv \boldsymbol{0}. \tag{6.3b}$$

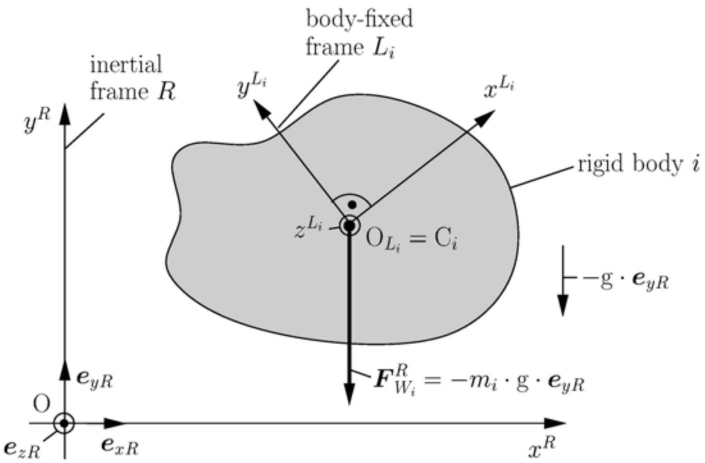


Fig. 6.2: Gravitational field acting on a body in the $(-\boldsymbol{e}_{yR})$ direction

6.1.2 Applied force and moment

Consider a force represented by an arrow $\boldsymbol{F}_i$ of length F_i through the point P_i on body i (Figure 6.3). Then

$$\boldsymbol{F}_i = F_i \cdot \boldsymbol{e}_{F_i} \tag{6.4a}$$

with the unit vector $\boldsymbol{e}_{F_i} = \boldsymbol{e}_{xF}$ of frame L_F and with

$$\boldsymbol{F}_i^{L_F} = \begin{pmatrix} F_i \\ 0 \end{pmatrix}$$

as the associated algebraic force vector, or

$$\boldsymbol{F}_i^R = \boldsymbol{A}^{RL_F} \cdot \boldsymbol{F}_i^{L_F} = \underbrace{\begin{pmatrix} \cos\psi_F & , -\sin\psi_F \\ \sin\psi_F & , \quad \cos\psi_F \end{pmatrix}}_{=:\, \boldsymbol{A}^{RL_F}} \cdot \begin{pmatrix} F_i \\ 0 \end{pmatrix}, \tag{6.4b}$$

and finally

$$\boldsymbol{F}_i = \boldsymbol{F}_{ix}^R \cdot \boldsymbol{e}_{xR} + \boldsymbol{F}_{iy}^R \cdot \boldsymbol{e}_{yR} \quad \text{and} \quad \boldsymbol{F}_i^R = F_i \cdot \begin{pmatrix} \cos\psi_F \\ \sin\psi_F \end{pmatrix} =: \begin{pmatrix} F_{ix}^R \\ F_{iy}^R \end{pmatrix},$$

with

$$F_{ix}^R := F_i \cdot \cos\psi_F \quad , \quad F_{iy}^R := F_i \cdot \sin\psi_F \quad \text{and} \quad \psi_F := \psi_{L_F R}.$$

The torque $\boldsymbol{M}_i$ of $\boldsymbol{F}_i$ with respect to the point O_{L_i}, and represented in frame L_i is

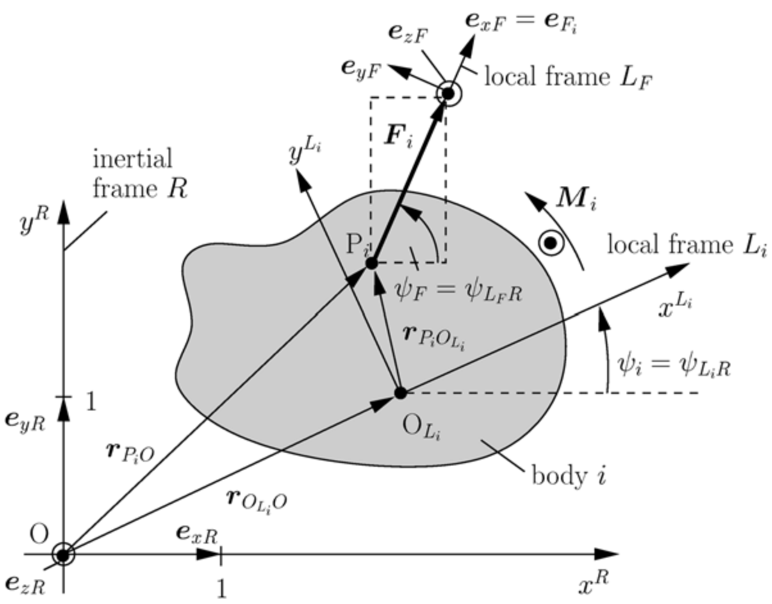


Fig. 6.3: A body acted upon by a constant force $\boldsymbol{F}_i$ and moment $\boldsymbol{M}_i$

$$\boldsymbol{M}_i^{L_i} := \boldsymbol{M}_{O_{L_i}}^{L_i} = \tilde{\boldsymbol{r}}_{P_iO_{L_i}}^{L_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{F}_i^R =$$

$$\left[\begin{pmatrix} 0 & , & 0 & , & y_{P_iO_{L_i}}^{L_i} \\ 0 & , & 0 & , & -x_{P_iO_{L_i}}^{L_i} \\ -y_{P_iO_{L_i}}^{L_i} & , & x_{P_iO_{L_i}}^{L_i} & , & 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\psi_i & , & \sin\psi_i & , & 0 \\ -\sin\psi_i & , & \cos\psi_i & , & 0 \\ 0 & , & 0 & , & 1 \end{pmatrix} \cdot \begin{pmatrix} F_{ix}^R \\ F_{iy}^R \\ 0 \end{pmatrix}\right],$$

or

$$\boldsymbol{M}_i^{L_i} =$$

$$\left[\begin{pmatrix} 0 & , & 0 & , & y_{P_iO_{L_i}}^{L_i} \\ 0 & , & 0 & , & -x_{P_iO_{L_i}}^{L_i} \\ -x_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i - y_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i & , & x_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i - y_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i & , & 0 \end{pmatrix} \cdot \begin{pmatrix} F_{ix}^R \\ F_{iy}^R \\ 0 \end{pmatrix}\right]$$

or

$$\begin{aligned} \boldsymbol{M}_i = &\Big[\Big(- x_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i - y_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i\Big) \cdot F_{ix}^R \\ &+ \Big(x_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i - y_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i\Big) \cdot F_{iy}^R\Big] \cdot \boldsymbol{e}_{zL_i} \end{aligned} \tag{6.4c}$$

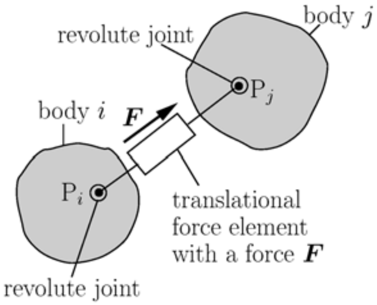
$$\begin{aligned} = F_i \cdot \Big[&\Big(- x_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i \cdot \cos\psi_F + x_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i \cdot \sin\psi_F\Big) \\ &+ \Big(- y_{P_iO_{L_i}}^{L_i} \cdot \cos\psi_i \cdot \cos\psi_F - y_{P_iO_{L_i}}^{L_i} \cdot \sin\psi_i \cdot \sin\psi_F\Big)\Big] \cdot e_{zL_i}, \end{aligned}$$

and finally

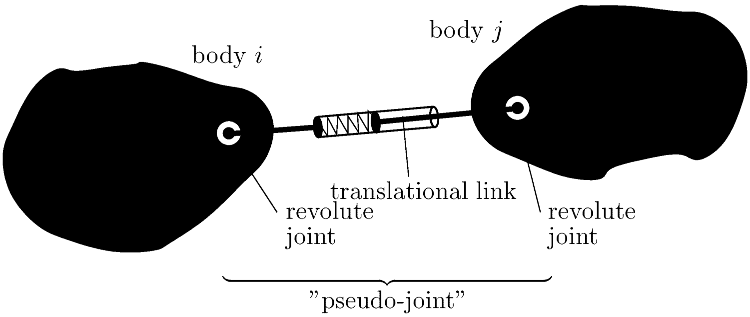
$$\boldsymbol{M}_i = F_i \cdot \Big[- x_{P_iO_{L_i}}^{L_i} \sin\Big(\psi_i - \psi_F\Big) - y_{P_iO_{L_i}}^{L_i} \cdot \cos\Big(\psi_F - \psi_i\Big)\Big] \cdot \boldsymbol{e}_{zL_i}. \tag{6.4d}$$

6.1.3 Translational force elements between two bodies

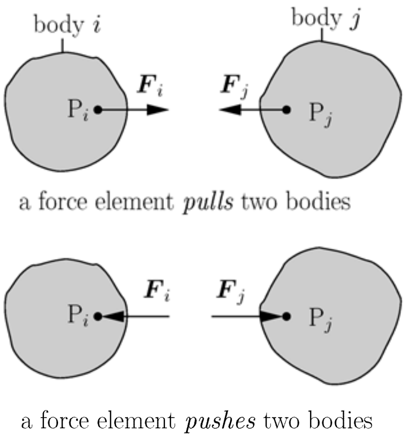
Consider a massless *translational force element* that exerts a force with an amplitude F along a line of action through the point P_i on a body i and point P_j on a body j (Figures 6.4a and 6.4b) *without imposing any kinematic constraint* on these bodies (cf. Section 5.1.3). The kinematics of this force element has been (symbolically) modeled by a *massless revolute–revolute–translational link* (*"pseudo-joint"*) that does not constrain any DOF of the bodies i and j (Figure 6.4b).



(a) Translational force element between two bodies i and j



(b) "Kinematics" of the massless revolute–revolute–translational link ("pseudo-joint") as part of a model of a translational force element between two bodies i and j



(c) Sign convention of translational forces between two bodies i and j

Fig. 6.4: Forces and vector diagram of a translational force element

Due to *Newton's third axiom (action ≡ reaction)* the forces $\boldsymbol{F}_i$ and $\boldsymbol{F}_j$ exerted from this element on the bodies i and j, respectively, have *equal absolute values* and *opposite signs*; i.e.,

$$\Big(\boldsymbol{F}_i = -\boldsymbol{F}_j\Big) \quad \text{or} \quad \Big(|\boldsymbol{F}_i| = |\boldsymbol{F}_j| \quad \text{and} \quad \operatorname{sign}(\boldsymbol{F}_i) = -\operatorname{sign}(\boldsymbol{F}_j)\Big). \tag{6.5}$$

The vector $\boldsymbol{d}_{ji}$ from point P_i to point P_j is computed from the vector loop equation (Figure 6.5)

$$\boldsymbol{d}_{ji} := \boldsymbol{r}_{P_j P_i} = \boldsymbol{r}_{O_j O} + \boldsymbol{s}_j - \boldsymbol{r}_{O_i O} - \boldsymbol{s}_i,$$

or represented in frame R as

$$\boldsymbol{d}_{ji}^{R} := \boldsymbol{r}_{O_j O}^{R} - \boldsymbol{r}_{O_i O}^{R} + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{s}_j^{L_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{s}_i^{L_i}. \tag{6.6}$$

The distance $\ell_{ij} = \ell_{ji}$ between points P_i and P_j (length of the force element) is

$$\ell_{ji} := \left(\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^{R}\right)\right)^{1/2} = \left(\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^{L_i}\right)\right)^{1/2} \in \mathbb{R}^1. \tag{6.7}$$

Then a unit vector

$$\boldsymbol{e}_{ji} := \frac{\boldsymbol{d}_{ji}}{|\boldsymbol{d}_{ji}|} \tag{6.8}$$

will be introduced (Figure 6.5). By a common sign convention, the pair of forces $(\boldsymbol{F}_i\,,\,\boldsymbol{F}_j)$ in Figure 6.4c is defined as:

1. *Positive* if the force element *pulls* the bodies (tends to draw the bodies together and to tension the force element).
2. *Negative* if the force element *pushes* the bodies (tends to increase the distance between the bodies and to compress the force element) (Figure 6.4c).

Then, in the case of *"pull"*, $\boldsymbol{F}_i$ (acting on the body i) has the same direction as $\boldsymbol{e}_{ji}$, and $\boldsymbol{F}_j$ (acting on the body j) has the opposite direction of $\boldsymbol{e}_{ji}$; i.e.,

$$\begin{aligned} \boldsymbol{F}_i &= F \cdot \boldsymbol{e}_{ji}, \\ &\text{and} \qquad \textit{(pull situation)} \\ \boldsymbol{F}_j &= -F \cdot \boldsymbol{e}_{ji}. \end{aligned} \tag{6.9a}$$

In the case of *"push"*, $\boldsymbol{F}_i$ has the opposite direction as $\boldsymbol{e}_{ji}$; i.e.,

$$\begin{aligned} \boldsymbol{F}_i &= -F \cdot \boldsymbol{e}_{ji}, \\ &\text{and} \qquad \textit{(push situation)} \\ \boldsymbol{F}_j &= F \cdot \boldsymbol{e}_{ji}. \end{aligned} \tag{6.9b}$$

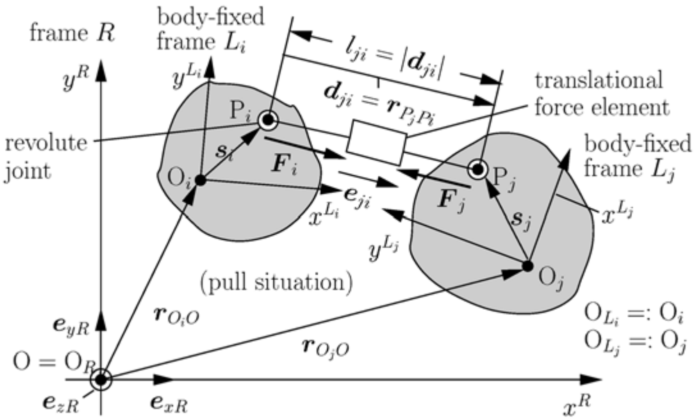


Fig. 6.5: Vector diagram of a translational force element

The above *translational force element* can be technically built by units such as *hydraulic* or *pneumatic actuators* or *bearings*, *electromagnetic actuators* or *bearings*, *spring elements*, and *damper elements*, that are connected to the bodies by revolute joints attached to each end of the force element, where a single or several of the above components may be included in a single translational force element.

6.1.3.1 Translational spring. *Translational* (point-to-point) *springs* are often used as (passive) force elements in rigid-body systems. The force of the spring of Figure 6.6 is defined as

$$\boldsymbol{F}_i := c_i \cdot (\ell_{ji} - \ell_{jiO}) \cdot \boldsymbol{e}_{ji} \;\;,\;\; c_i > 0 \;\;,\;\; \boldsymbol{e}_{ji} = \boldsymbol{d}_{ji}/|\boldsymbol{d}_{ji}| \;\;,\;\; \ell_{ji} = |\boldsymbol{d}_{ji}|, \quad (6.10a)$$

and may be written as

$$\boldsymbol{F}_i = c_i \cdot [1 - \ell_{jiO}\big/(\boldsymbol{d}_{ji}^{\mathrm{T}} \cdot \boldsymbol{d}_{ji})^{1/2}] \cdot \boldsymbol{d}_{ji} \quad \textit{(linear spring)}$$

or as

$$\boldsymbol{F}_i := c_i\,(\ell_{ji} - \ell_{jiO}) \cdot \boldsymbol{e}_{ji} \;\;,\;\; c_i : \mathbb{R}^1 \to \mathbb{R}^1 \quad \textit{(nonlinear spring)}, \qquad (6.10b)$$

with $c_i > 0$ as the *stiffness coefficient*, $c_i(\cdot)$ as the *stiffness characteristic*, ℓ_{ji} as the *deformed length* and ℓ_{jiO} as the *undeformed* length of the spring, and $\boldsymbol{d}_{ji} = \boldsymbol{r}_{P_jP_i}$ as the vector from the attachment point P_i of the spring on body i to the attachment point P_j on body j. These force relations are in agreement with the above sign convention:

1. For $(\ell_{ji} - \ell_{jiO}) > 0$ the two bodies are *pulled* and the spring is under *tension* (Figure 6.6). Then $\boldsymbol{F}_i$ acts in the direction of $\boldsymbol{e}_{ji}$ and $\boldsymbol{F}_j$ acts in the direction of $(-\boldsymbol{e}_{ji})$.

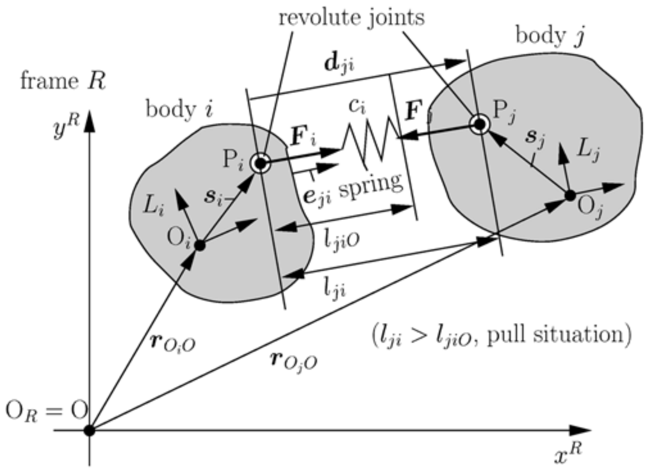


Fig. 6.6: Translational spring between points P_i and P_j of bodies i and j

2. For $(\ell_{ji} - \ell_{jiO}) < 0$ the two bodies are *pushed* and the spring is under *compression*. Then $\boldsymbol{F}_i$ acts in the $(-\boldsymbol{e}_{ji})$ direction, and $\boldsymbol{F}_j$ acts in the $\boldsymbol{e}_{ji}$ direction.

The deformed length ℓ_{ji} of the spring and the associated unit vector $\boldsymbol{e}_{ji}$ are computed by (6.6), (6.7), and (6.8).

6.1.3.2 Translational damper. Given a *translational* (point-to-point) *damper* between the bodies i and j, as shown in Figure 6.7. The damper force on body i is defined as

$$\boldsymbol{F}_i = d_i \cdot \dot{\ell}_{ji} \cdot \boldsymbol{e}_{ji} \ , \quad d_i > 0 \qquad \textit{(linear damper)} \tag{6.11a}$$

or as

$$\boldsymbol{F}_i = d_i \left(\dot{\ell}_{ji}\right) \cdot \boldsymbol{e}_{ji}, \quad d_i : \mathbb{R}^1 \to \mathbb{R}^1 \quad \textit{(nonlinear damper)}, \tag{6.11b}$$

with

$$\boldsymbol{e}_{ji} = \boldsymbol{r}_{P_jP_i}/|\boldsymbol{r}_{P_jP_i}| = \boldsymbol{d}_{ji}/|\boldsymbol{d}_{ji}| \quad \text{and} \quad \dot{\ell}_{ji} = |\dot{\boldsymbol{d}}_{ji}|,$$

with $d_i > 0$ as *damping coefficient*, $d_i(\cdot)$ as *damper characteristic*, and with

$$\begin{aligned} \ell_{ji}^2 &= \left(\boldsymbol{d}_{ji}^{\mathrm{T}}\right) \cdot (\boldsymbol{d}_{ji}) = \left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^{R}\right) \\ &= \left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{A}^{RL_i} \cdot \left(\boldsymbol{d}_{ji}^{L_i}\right) = (\boldsymbol{d}_{ji}^{L_i})^{\mathrm{T}} \cdot \boldsymbol{d}_{ji}^{L_i}, \end{aligned} \tag{6.12a}$$

and $\boldsymbol{d}_{ji}^{R}$ as defined in (6.6). Computation of the time derivative

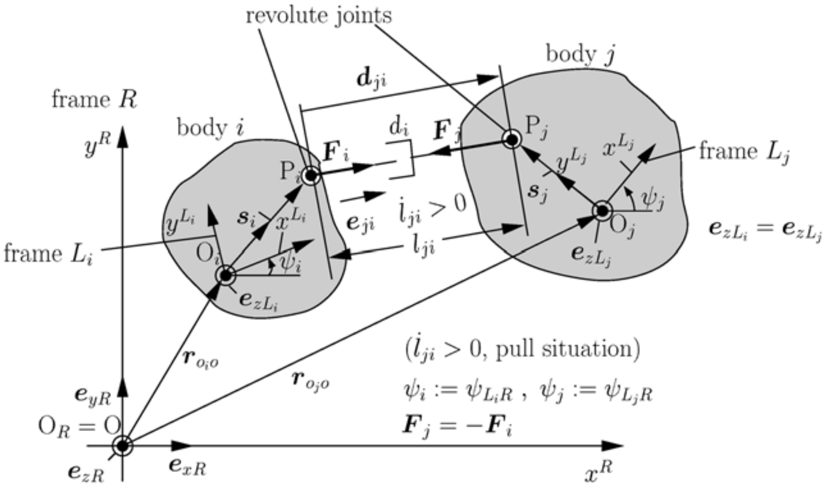


Fig. 6.7: Translational damper between points P_i and P_j of bodies i and j

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\ell_{ji}^2\right) &= 2\cdot\ell_{ji}\cdot\dot{\ell}_{ji} \underset{(6.12a)}{=} \frac{\mathrm{d}}{\mathrm{d}t}\left[(\boldsymbol{d}_{ji}^{L_i})^{\mathrm{T}}\cdot(\boldsymbol{d}_{ji})^{L_i}\right] \\
&= 2\cdot\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}}\cdot\left({}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i}\right) = 2\cdot\left(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{L_iR}\cdot\boldsymbol{d}_{ji}^{R}\right) \\
&= \left(2\cdot\boldsymbol{A}^{L_iR}\cdot\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left(\dot{A}^{L_iR}\cdot\boldsymbol{d}_{ji}^{R}+\boldsymbol{A}^{L_iR}\cdot{}^{R}\dot{\boldsymbol{d}}_{ji}^{R}\right) \\
&= 2\cdot\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\boldsymbol{A}^{RL_i}\cdot\left[\boldsymbol{A}^{L_iR}\cdot\left({}^{R}\dot{\boldsymbol{d}}_{ji}^{R}+\tilde{\boldsymbol{\omega}}_{RL_i}^{R}\cdot\boldsymbol{d}_{ji}^{R}\right)\right]^{\mathrm{T}} \\
&= 2\cdot\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left({}^{R}\dot{\boldsymbol{d}}_{ji}^{R}\right)-2\cdot\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left(\tilde{\boldsymbol{d}}_{ji}^{R}\right)\cdot\boldsymbol{\omega}_{RL_i}^{R}
\end{aligned}
\tag{6.12b}
$$

provides, together with

$$
(\boldsymbol{d}_{ji}^{R})^{\mathrm{T}}\cdot\tilde{\boldsymbol{d}}_{ji}^{R} = 0 \tag{6.12c}
$$

the relation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\ell_{ji}^2\right) = 2\cdot\ell_{ji}\cdot\dot{\ell}_{ji} = 2\cdot\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}}\cdot\left({}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i}\right) = 2\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left({}^{R}\dot{\boldsymbol{d}}_{ji}^{R}\right), \tag{6.12d}
$$

and finally the relation

$$
\dot{\ell}_{ji} = \frac{1}{\ell_{ji}}\cdot\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left({}^{R}\dot{\boldsymbol{d}}_{ji}^{R}\right) = \frac{1}{\ell_{ji}}\cdot\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}}\cdot{}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i} \tag{6.13}
$$

or

$$
\dot{\ell}_{ji} = \frac{\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left({}^{R}\dot{\boldsymbol{d}}_{ji}^{R}\right)}{\left(\left(\boldsymbol{d}_{ji}^{R}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{d}_{ji}^{R}\right)\right)^{1/2}} = \frac{\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}}\cdot\left({}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i}\right)}{\left(\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}}\cdot\left(\boldsymbol{d}_{ji}^{L_i}\right)\right)^{1/2}} = \boldsymbol{e}_{ji}^{\mathrm{T}}\cdot{}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i}.
$$

Equations (6.11) and (6.12d) yield the following expressions for the force of a *linear translational damper*:

$$\boldsymbol{F}_i = d_i \cdot \left(\frac{\left({}^R\dot{\boldsymbol{d}}_{ji}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^R\right)}{\left(\boldsymbol{d}_{ji}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^R\right)} \right) \cdot \boldsymbol{d}_{ji}^R = d_i \cdot \left({}^R\dot{\boldsymbol{d}}_{ji}^{R\,\mathrm{T}} \cdot \boldsymbol{e}_{ji} \right) \cdot \boldsymbol{e}_{ji} \tag{6.14a}$$

or

$$\boldsymbol{F}_i = d_i \cdot \left(\frac{\left({}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{d}_{ji}^{L_i}\right)}{\left(\boldsymbol{d}_{ji}^{L_i}\right)^{\mathrm{T}} \cdot \boldsymbol{d}_{ji}^{L_i}} \right) \cdot \boldsymbol{d}_{ji}^{L_i} = d_i \cdot \left({}^{L_i}\dot{\boldsymbol{d}}_{ji}^{L_i\,\mathrm{T}} \cdot \boldsymbol{e}_{ji} \right) \cdot \boldsymbol{e}_{ji}. \tag{6.14b}$$

These force expressions are usually much briefer for model equations written in *generalized coordinates* that include the *relative coordinate* $\boldsymbol{d}_{ji}$. For model equations written in absolute generalized coordinates, the following relations hold. Consider the vector

$$\boldsymbol{d}_{ji} = \boldsymbol{r}_{O_jO} - \boldsymbol{r}_{O_iO} + \boldsymbol{s}_j - \boldsymbol{s}_i, \tag{6.15a}$$

or represented in frame R,

$$\boldsymbol{d}_{ji}^R = \boldsymbol{r}_{O_jO}^R - \boldsymbol{r}_{O_iO}^R + \boldsymbol{A}^{RL_j} \cdot \boldsymbol{s}_j^{L_j} - \boldsymbol{A}^{RL_i} \cdot \boldsymbol{s}_i^{L_i}, \tag{6.15b}$$

with

$$\boldsymbol{A}^{RL_j} = \begin{pmatrix} \cos\psi_j \ , & -\sin\psi_j \\ \sin\psi_j \ , & \cos\psi_j \end{pmatrix}, \qquad \boldsymbol{A}^{RL_i} = \begin{pmatrix} \cos\psi_i \ , & -\sin\psi_i \\ \sin\psi_i \ , & \cos\psi_i \end{pmatrix}$$

and with

$$\psi_j := \psi_{L_jR}, \qquad \psi_i := \psi_{L_iR}.$$

The *time derivative of* $\boldsymbol{d}_{ji}^R$ with respect to the frame R is

$$\begin{aligned} \dot{\boldsymbol{d}}_{ji}^R := {}^R\dot{\boldsymbol{r}}_{O_jO}^R - {}^R\dot{\boldsymbol{r}}_{O_iO}^R + \dot{\boldsymbol{A}}^{RL_j} \cdot \boldsymbol{s}_j^{L_j} \\ + \boldsymbol{A}^{RL_j} \cdot {}^{L_j}\dot{\boldsymbol{s}}_j^{L_j} - \dot{\boldsymbol{A}}^{RL_i} \cdot \boldsymbol{s}_j^{L_i} - \boldsymbol{A}^{RL_i} \cdot {}^{L_i}\dot{\boldsymbol{s}}_i^{L_i}. \end{aligned}$$

Using

$$\dot{\boldsymbol{A}}^{RL_i} = \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \quad , \quad \dot{\boldsymbol{A}}^{RL_j} = \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}_{L_jR}^{L_j}, \tag{6.16a}$$

and the rigid-body property

$${}^{L_i}\dot{\boldsymbol{s}}_i^{L_i} \equiv {}^{L_j}\dot{\boldsymbol{s}}_j^{L_j} \equiv \boldsymbol{0} \tag{6.16b}$$

yields, together with

$$\dot{\boldsymbol{r}}_{O_iO}^R := {}^R\dot{\boldsymbol{r}}_{O_iO}^R \quad \text{and} \quad \dot{\boldsymbol{r}}_{O_jO}^R := {}^R\dot{\boldsymbol{r}}_{O_jO}^R, \tag{6.16c}$$

the time derivative of $\boldsymbol{d}_{ji}^R$:

$$\dot{\boldsymbol{r}}_{P_jP_i}^R = \dot{\boldsymbol{d}}_{ji}^R = \dot{\boldsymbol{r}}_{O_jO}^R - \dot{\boldsymbol{r}}_{O_iO}^R + \boldsymbol{A}^{RL_j} \cdot \tilde{\boldsymbol{\omega}}_{L_jR}^{L_j} \cdot \boldsymbol{s}_j^{L_j} - \boldsymbol{A}^{RL_i} \cdot \tilde{\boldsymbol{\omega}}_{L_iR}^{L_i} \cdot \boldsymbol{s}_i^{L_i}. \tag{6.17}$$

Then the above damper force relations (6.11a) and (6.11b) are in agreement with the previous sign definition. Since the damper opposes the relative motion of the two bodies when they move away from each other ($\dot{\ell}_{ji} > 0$), the damper forces exhibit a *pull* on the bodies. Then $\boldsymbol{F}_i$ acts in the direction of $\boldsymbol{e}_{ji}$ and $\boldsymbol{F}_j$ acts in the direction $(-\boldsymbol{e}_{ji})$. For ($\dot{\ell}_{ji} < 0$) the two bodies move towards each other and the damper forces exhibit a *push* on the bodies. Then $\boldsymbol{F}_i$ acts in $(-\boldsymbol{e}_{ji})$ direction and $\boldsymbol{F}_j$ acts in the direction of $\boldsymbol{e}_{ji}$.

6.1.3.3 Actuator. Forces between two bodies generated by *hydraulic*, *pneumatic*, or *magnetic* actuators are defined by complete analogy to the above sign conventions. They may be written in the form

$$\boldsymbol{F}_{ija} = \boldsymbol{F}_{ija}\left(x_i - x_j\,,\, |\dot{x}_i - \dot{x}_j|\,,\, t\right). \tag{6.18}$$

6.1.3.4 Torsional spring and damper. Torsional (rotational) springs and dampers between two bodies are always assumed to act around the axis of a revolute joint that connects these bodies. Consider two rigid bodies i and j connected at a point P by a revolute joint (Figure 6.8). Assume that a torsional spring and a torsional damper act around the rotation axis of the joint, and that they are attached to the arrow $\boldsymbol{r}_j$ fixed on body j and arrow $\boldsymbol{r}_i$ fixed on body i. This spring damper element exerts *torques of equal magnitude but opposite orientation on the bodies* i and j *(action* $\equiv$ *reaction)*. Let ψ_{ji} be the difference between the rotation angles of the bodies j and i (measured as the angle from $\boldsymbol{r}_i$ to $\boldsymbol{r}_j$), and let ψ_{jiO} be the angle of the undeformed spring. As a torque $\boldsymbol{M}_i$ is positive if it acts (for a compressed spring ($\psi_{ji} > \psi_{jiO}$)) counter-clockwise on the body i, and clockwise on the body j, it is formally described by the relation

$$\boldsymbol{M}_i := \left[c_r \cdot (\psi_{ji} - \psi_{jiO}) + d_r \cdot \dot{\psi}_{ji}\right] \cdot \boldsymbol{e}_{zR}, \tag{6.19}$$

with c_r *as the stiffness of the torsional spring* and d_r *as the damping coefficient of the torsional damper.*

6.1.3.5 Torque generated by a motor. Torques between two bodies that are generated by *electric*, *hydraulic*, or *pneumatic* motors are defined by analogy to the above sign convention of torsional spring-dampers. Those torques are usually written as

$$\boldsymbol{M}_{ija} = M_{ija}\left(\psi_{ji}\,,\, \dot{\psi}_{ji}\,,\, t\right) \cdot \boldsymbol{e}_{zR}. \tag{6.20}$$

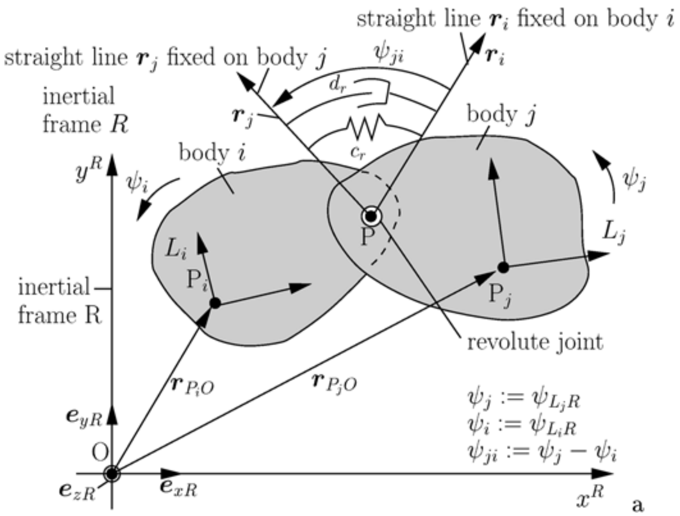


Fig. 6.8: Torsional spring and damper

6.2 Constitutive relations of *spatial* external forces and torques

The *planar* force/torque relations of Section 6.1 can be easily extended to *spatial* force/torque relations. The general force/torque relations (6.2a) and (6.2b) hold for *planar* and *spatial* vectors.

Gravitational force (weight)

Assuming that the gravitational field acts in the negative $\boldsymbol{e}_{yR}$-direction. Then the gravitational force on a body i of mass m_i is

$$\boldsymbol{F}_{W_i} = -m_i \cdot \mathrm{g} \cdot \boldsymbol{e}_{yR} \quad \text{or} \quad \boldsymbol{F}^R_{W_i} = \begin{pmatrix} 0 \\ -m_i \cdot \mathrm{g} \\ 0 \end{pmatrix} \quad = \quad \text{constant.} \qquad (6.21)$$

Applied force and torque

Consider an applied force represented by an arrow $\boldsymbol{F}_i$ of the amplitude F_i and acting on the body i along a line of action and through a point P_{F_i} (Figure 6.9). Let $\boldsymbol{e}_{F_i}$ be the unit vector in the direction of $\boldsymbol{F}_i$ on the line of action of $\boldsymbol{F}_i$. Then

$$\boldsymbol{F}^R_i = \left(F^R_{ix}\,, F^R_{iy}\,, F^R_{iz}\right)^{\mathrm{T}}$$

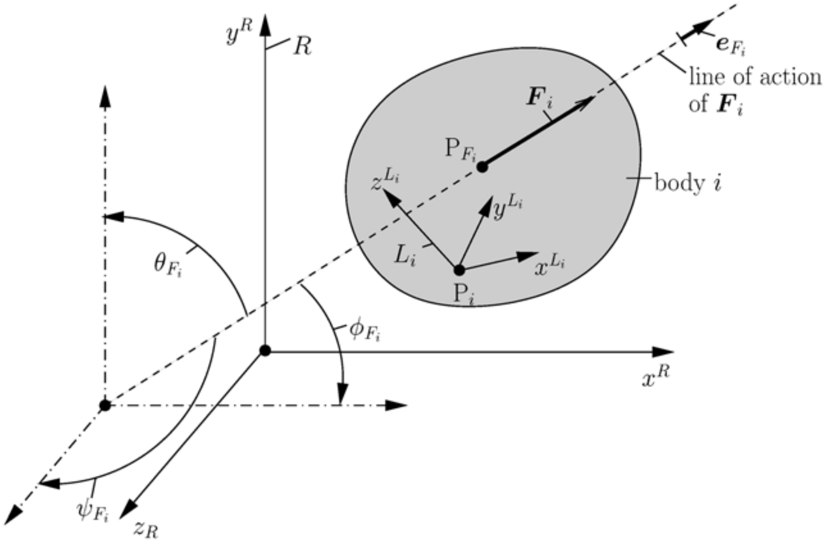


Fig. 6.9: Constant force $\boldsymbol{F}_i$ acting on a body i

or (6.22a)

$$\boldsymbol{F}_i = F_i \cdot \boldsymbol{e}_{F_i} = F^R_{ix} \cdot \boldsymbol{e}_{xR} + F^R_{iy} \cdot \boldsymbol{e}_{yR} + F^R_{iz} \cdot \boldsymbol{e}_{zR} \quad , \quad F_i \in \mathbb{R}^1 .$$

The torque of $\boldsymbol{F}_i$ with respect to the point P_i represented in frame L_i is

$$\boldsymbol{M}^{L_i}_{P_i} = \tilde{\boldsymbol{r}}^{L_i}_{P_{F_i}P_i} \cdot \boldsymbol{A}^{L_iR} \cdot \boldsymbol{F}^R_i = \left[\begin{pmatrix} 0 & , -z^{L_i}_{P_{F_i}P_i} & , \; y^{L_i}_{P_{F_i}P_i} \\ z^{L_i}_{P_{F_i}P_i} & , \; 0 & , -x^{L_i}_{P_{F_i}P_i} \\ -y^{L_i}_{P_{F_i}P_i} & , \; x^{L_i}_{P_{F_i}P_i} & , \; 0 \end{pmatrix} \right.$$

$$\left. \cdot \begin{pmatrix} \mathrm{c}_{i2}\,\mathrm{c}_{i3} & , \; \mathrm{c}_{i1}\,\mathrm{s}_{i3} + \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , \; \mathrm{s}_{i1}\,\mathrm{s}_{i3} - \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} \\ -\mathrm{c}_{i2}\,\mathrm{s}_{i3} & , \; \mathrm{c}_{i1}\,\mathrm{c}_{i3} - \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & , \; \mathrm{s}_{i1}\,\mathrm{c}_{i3} + \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} \\ \mathrm{s}_{i2} & , \; -\mathrm{s}_{i1}\,\mathrm{c}_{i2} & , \; \mathrm{c}_{i1}\,\mathrm{c}_{i2} \end{pmatrix} \right] \cdot \begin{pmatrix} F^R_{ix} \\ F^R_{iy} \\ F^R_{iz} \end{pmatrix}$$

(6.22b)

with the *Bryant angles* ϕ_{L_iR}, θ_{L_iR}, and ψ_{L_iR} and the abbreviations

$$\mathrm{s}_{i1} := \sin\phi_{L_iR} \quad , \quad \mathrm{s}_{i2} := \sin\theta_{L_iR} \quad , \quad \mathrm{s}_{i3} := \sin\psi_{L_iR},$$

$$\mathrm{c}_{i1} := \cos\phi_{L_iR} \quad , \quad \mathrm{c}_{i2} := \cos\theta_{L_iR} \quad , \quad \mathrm{c}_{i3} := \cos\psi_{L_iR}.$$

Translational springs, dampers, and actuators between two bodies

The relations (6.5) to (6.9b) of planar forces also hold for *spatial forces*, taking into account *spatial vectors* $\boldsymbol{d}_{ji}$, $\boldsymbol{r}_{P_jP_i}$, $\boldsymbol{s}_j$, $\boldsymbol{s}_i$, etc., and the *spatial* *transformation matrices*:

$$\boldsymbol{A}^{RL_i} = \begin{pmatrix} \mathrm{c}_{i2}\,\mathrm{c}_{i3} & , & -\mathrm{c}_{i2}\,\mathrm{s}_{i3} & , & \mathrm{s}_{i2} \\ \mathrm{c}_{i1}\,\mathrm{s}_{i3} + \mathrm{s}_{i3}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , & \mathrm{c}_{i1}\,\mathrm{c}_{i3} - \mathrm{s}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , & -\mathrm{s}_{i1}\,\mathrm{c}_{i2} \\ \mathrm{s}_{i1}\,\mathrm{s}_{i3} - \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{c}_{i3} & , & \mathrm{s}_{i1}\,\mathrm{c}_{i3} + \mathrm{c}_{i1}\,\mathrm{s}_{i2}\,\mathrm{s}_{i3} & , & \mathrm{c}_{i1}\,\mathrm{c}_{i2} \end{pmatrix} \tag{6.23a}$$

The same holds for the relations (6.10a) to (6.18) of *translational springs*, *dampers*, and *actuators*.

Torsional springs, dampers, and motors

Torsional springs, dampers, and motors are in the planar case, as well as in the *spatial case*, assumed to act around the axis of a revolute joint. Then the torque relations (6.19) and (6.20) of *planar* torsional elements also hold for the *spatial* case.

A. Appendix

This appendix includes four sections. In *Section A.1* special *vector and matrix operations* used in rigid-body dynamics will be discussed. The *Lagrange equations* of a rigid body under spatial motion are briefly discussed in *Section A.2* as an alternative to the *Newton–Euler equations.* In *Section A.3* vector and matrix notations together with the model equations of *planar and spatial mechanisms* are compared with each other, followed by the *constraint equations* of a *general universal joint*, derived in *Section A.4*.

A.1 Special vector and matrix operations used in mechanics

In this section some special vector and matrix operations that are often used in rigid-body dynamics will be briefly discussed. Starting with some basic definitions and properties of *Euclidean vector spaces* in Section *A.1.1, algebraic and geometric properties* of the *scalar product* and of the *vector product* (or *cross product*) of *planar* vectors will be discussed in Section *A.1.2.* In *Section A.1.3, cross product* operations of *spatial vectors* will be proved. *Time derivatives* of *matrices* and *vectors* represented in *different frames* will be discussed in *Sections A.1.4* and *A.1.5* for *planar* and *spatial* situations, respectively, followed by a brief review of the derivatives of the vector functions (*gradient*, *Jacobian matrix*) in *Section A.1.6.*

A.1.1 Euclidean vector space

In the sense of linear algebra, a real *Euclidean vector space* is defined as a quintuple

$$(V\,,\,K\,,\,+\,,\,\cdot\,,\,\bullet)\,, \tag{A.1.1}$$

which includes the following entities: a *field of real numbers* (scalars) K, a *commutative group* $(V\,,+)$ of vectors V, and the mappings

$$\begin{array}{lll} V\times V \xrightarrow{+} V & & \textit{(addition of vectors)} \\ \cup\!\!\!\!\cup \quad \cup\!\!\!\!\cup \qquad \cup\!\!\!\!\cup & & \\ (\boldsymbol{x},\ \boldsymbol{y}) \longmapsto \boldsymbol{z} := \boldsymbol{x}+\boldsymbol{y} & = \ \boldsymbol{y}+\boldsymbol{x} & (\textit{commutative law}) \end{array}$$

$$
\begin{array}{ccccl}
V & \times & K & \xrightarrow{\cdot} & V \qquad \textit{(multiplication of a vector by a scalar)} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\boldsymbol{x} & , & \lambda) & \longmapsto & \boldsymbol{z} := \lambda \cdot \boldsymbol{x},
\end{array}
$$

that satisfy the *properties*

$$(\lambda \cdot \mu) \cdot \boldsymbol{x} = \lambda \cdot (\mu \cdot \boldsymbol{x}) \quad ; \quad \lambda, \mu \in K \ ; \ \boldsymbol{x} \in V \qquad \textit{(associative law)}$$

$$
\left.
\begin{array}{ll}
\lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y} \ ; \ \lambda \in K & ; \ \boldsymbol{x}, \boldsymbol{y} \in V \\
(\lambda + \mu) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \mu \cdot \boldsymbol{x} \ ; \ \lambda, \mu \in K & ; \ \boldsymbol{x} \in V
\end{array}
\right. \quad \textit{(distributive laws)}
$$

and

$$1 \cdot \boldsymbol{x} = \boldsymbol{x} \qquad ; \ 1 \in K \quad ; \quad \boldsymbol{x} \in V \quad \textit{(existence of a unity).}$$

Then

$$(V, K, +, \cdot)$$

is called *linear vector space.* A mapping

$$\varphi : V_1 \longrightarrow V_2 \tag{A.1.2}$$

between two linear vector spaces V_1 and V_2 is called *linear* iff

$$
\begin{array}{ll}
\varphi(\boldsymbol{x}_1 + \boldsymbol{x}_2) = \varphi(\boldsymbol{x}_1) + \varphi(\boldsymbol{x}_2) & , \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in V_1 \\
\varphi(\lambda \cdot \boldsymbol{x}_1) = \lambda \cdot \varphi(\boldsymbol{x}_1) & , \quad \boldsymbol{x} \in V_1 \ , \ \lambda \in K.
\end{array}
$$

A *linear mapping* φ (A.1.2) can be represented with respect to two bases L of V_1 and R of V_2 by means of a matrix $\boldsymbol{A}^{RL}$:

$$
\begin{array}{cccc}
\boldsymbol{A}^{RL} : & V_1 & \longrightarrow & V_2 \\
& \Downarrow & & \Downarrow \\
& \boldsymbol{r}^L & \longmapsto & \boldsymbol{r}^R := \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L
\end{array}
$$

with $\boldsymbol{r}^L$ and $\boldsymbol{r}^R$ as algebraic vectors, represented in L and R, and with $\boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L$ as the product of the matrix $\boldsymbol{A}^{RL}$ with the algebraic vector $\boldsymbol{r}^L$, where the dot in the product $\boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L$ is sometimes omitted. Introducing a *bilinear mapping*

$$
\begin{array}{ccccc}
\beta : & V \times V & \xrightarrow{\bullet} & K \\
& \Downarrow \quad \Downarrow & & \Downarrow \\
& (\boldsymbol{x}, \boldsymbol{y}) & \longmapsto & \lambda = \beta(\boldsymbol{x}, \boldsymbol{y}) =: \boldsymbol{x} \bullet \boldsymbol{y}
\end{array}
$$

which is assumed to be *linear in both arguments* ($\boldsymbol{x}$ and $\boldsymbol{y}$); i.e.,

$$\beta(\boldsymbol{x}_1 + \boldsymbol{x}_2\,,\,\boldsymbol{y}) = \beta(\boldsymbol{x}_1 + \boldsymbol{y}) + \beta(\boldsymbol{x}_2 + \boldsymbol{y}),$$

$$\beta(\boldsymbol{x}\,,\,\boldsymbol{y}_1 + \boldsymbol{y}_2) = \beta(\boldsymbol{x} + \boldsymbol{y}_1) + \beta(\boldsymbol{x} + \boldsymbol{y}_2)$$

and

$$\beta(\lambda \cdot \boldsymbol{x}\,,\,\boldsymbol{y}) = \lambda \cdot \beta(\boldsymbol{x}\,,\,\boldsymbol{y}) = \beta(\boldsymbol{x}\,,\,\lambda \cdot \boldsymbol{y}),$$

and postulating the additional properties

$$\beta(\boldsymbol{x}\,,\,\boldsymbol{y}) = \beta(\boldsymbol{y}\,,\,\boldsymbol{x}) \quad ; \quad \boldsymbol{x} \in V \quad ; \quad \boldsymbol{y} \in V \qquad \textit{(symmetry)}$$

and

$$\beta(\boldsymbol{x}\,,\,\boldsymbol{x}) > 0 \quad \text{for} \quad \boldsymbol{x} \neq \boldsymbol{0} \qquad \textit{(definiteness)}$$

provides a *scalar product* "$\bullet$" on V and associates with the *linear vector space* $(V\,,\,K\,,\,+\,,\,\cdot)$ a *Euclidean space* $(V\,,\,K\,,\,+\,,\,\cdot\,,\,\bullet)$.

The *scalar product* enables to introduce a *norm* into a vector space. This allows defininition of the *length* of a vector and relative *angles* between two vectors as well as the concept of orthogonality. This introduces topological properties into a vector space V and enables an analysis in V.

Comment A.1.1 (Vector product): As is easily seen from the above definitions (A.1.1), the vector-space concept of linear algebra does not explicitly include the notion of a *vector product* or *cross product*; i.e., no mapping

$$\begin{array}{llll} \times : & V \times V & \xrightarrow{\times} & V \\ & \Psi \quad \Psi & & \Psi \\ & (\boldsymbol{y}\,,\,\boldsymbol{w}) & \longmapsto & \boldsymbol{z} =: (\boldsymbol{y} \times \boldsymbol{w}). \end{array}$$

On the other hand, it is common in rigid-body dynamics, to define a vector product $\boldsymbol{y}^R \times \boldsymbol{w}^R$ of algebraic vectors $\boldsymbol{y}^R$ and $\boldsymbol{w}^R$ of V with respect to a common basis R as the result of a *matrix multiplication* (linear mapping)

$$\begin{array}{llll} \tilde{\boldsymbol{y}}^R : & V & \longrightarrow & V \\ & \Psi & & \Psi \\ & \boldsymbol{w}^R & \longmapsto & \boldsymbol{z}^R = \tilde{\boldsymbol{y}}^R(\boldsymbol{w}) := \tilde{\boldsymbol{y}}^R \cdot \boldsymbol{w}^R := (\boldsymbol{y}^R \times \boldsymbol{w}^R) \quad , \quad \boldsymbol{y}^R \in V \end{array} \tag{A.1.3}$$

with the skew-symmetric matrix $\tilde{\boldsymbol{y}}^R$ constructed from a given algebraic vector $\boldsymbol{y}^R \in V$ that provides for each $\boldsymbol{w}^R \in V$ the same result as the standard vector product (compare Equations A.1.5e and A.1.7b of Sections A.1.2 and A.1.3). This notation provides both *conceptual clearness* of the vector product defined as a linear mapping between vector spaces, and the possibility to compute a *vector product in the framework* of *standard matrix multiplication.*

A.1.2 Scalar product and cross product of planar vectors

Scalar product

Given two planar geometrical vectors $\boldsymbol{r}_{QO}$ and $\boldsymbol{r}_{PO}$. The *scalar product* of these *geometrical vectors* is defined as the scalar (Figure A.1.1)

$$\boldsymbol{r}_{QO} \bullet \boldsymbol{r}_{PO} := |\boldsymbol{r}_{QO}| \cdot |\boldsymbol{r}_{PO}| \cdot \cos\varphi_{QP} \in \mathbb{R}^1. \tag{A.1.4a}$$

Algebraic computation of the scalar product

The *scalar product* (A.1.4a) of two algebraic vectors is computed as

$$\begin{aligned}\left(\boldsymbol{r}^R_{QO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^R_{PO}\right) &= \left(\boldsymbol{r}^L_{QO}\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}^L_{PO}\right) \\ &= x^R_{QO} \cdot x^R_{PO} + y^R_{QO} \cdot y^R_{PO} = x^L_{QO} \cdot x^L_{PO} + y^L_{QO} \cdot y^L_{PO}\end{aligned} \tag{A.1.4b}$$

from algebraic vectors represented in frames R or L.

Proof:

Representing $\boldsymbol{r}_{QO}$ and $\boldsymbol{r}_{PO}$ in frame R yields

$$\begin{aligned}|\boldsymbol{r}^R_{QO}| \cdot |\boldsymbol{r}^R_{PO}| \cdot \cos\varphi_{QP} &= \left|\begin{matrix} x^R_{QO} \\ y^R_{QO}\end{matrix}\right| \cdot \left|\begin{matrix} x^R_{PO} \\ y^R_{PO}\end{matrix}\right| \cdot \cos\varphi_{QP} \\ &= |x^R_{QO} \cdot \boldsymbol{e}_{xR} + y^R_{QO} \cdot \boldsymbol{e}_{yR}| \cdot |x^R_{PO} \cdot \boldsymbol{e}_{xR} + y^R_{PO} \cdot \boldsymbol{e}_{yR}| \cdot \cos\varphi_{QP}.\end{aligned} \tag{A.1.4c}$$

Then (cf. Figure A.1.1)

$$\begin{aligned}\sin\varphi_{QR} &= \frac{y^R_{QO}}{|\boldsymbol{r}^R_{QO}|} = \frac{y^R_{QO}}{|\boldsymbol{r}^R_{QO}|}, \qquad \cos\varphi_{QR} = \frac{x^R_{QO}}{|\boldsymbol{r}^R_{QO}|} = \frac{x^R_{QO}}{|\boldsymbol{r}^R_{QO}|}, \\ \sin\varphi_{PR} &= \frac{y^R_{PO}}{|\boldsymbol{r}^R_{PO}|} = \frac{y^R_{PO}}{|\boldsymbol{r}^R_{PO}|}, \qquad \cos\varphi_{PR} = \frac{x^R_{PO}}{|\boldsymbol{r}^R_{PO}|} = \frac{x^R_{PO}}{|\boldsymbol{r}^R_{PO}|},\end{aligned} \tag{A.1.4d}$$

and

$$\begin{aligned}\cos(\varphi_{QR} - \varphi_{PR}) &= \cos(\varphi_{QP}) \\ &= \cos\varphi_{QR} \cdot \cos\varphi_{PR} + \sin\varphi_{QR} \cdot \sin\varphi_{PR}.\end{aligned} \tag{A.1.4e}$$

Furthermore,

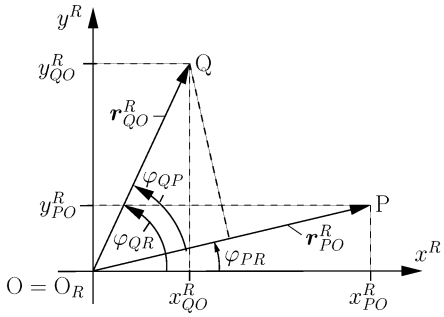


Fig. A.1.1: Vector diagram of a scalar product

$$\cos(\varphi_{QP}) = \frac{x^R_{QO}}{|\boldsymbol{r}^R_{QO}|} \cdot \frac{x^R_{PO}}{|\boldsymbol{r}^R_{PO}|} + \frac{y^R_{QO}}{|\boldsymbol{r}^R_{QO}|} \cdot \frac{y^R_{PO}}{|\boldsymbol{r}^R_{PO}|} \tag{A.1.4f}$$

or

$$\cos(\varphi_{QP}) = \frac{x^R_{QO} \cdot x^R_{PO} + y^R_{QO} \cdot y^R_{PO}}{|\boldsymbol{r}^R_{QO}| \cdot |\boldsymbol{r}^R_{PO}|} \ .$$

Inserting (A.1.4f) into (A.1.4c) yields

$$|\boldsymbol{r}^R_{QO}| \cdot |\boldsymbol{r}^R_{PO}| \cdot \cos\varphi_{QP} = |\boldsymbol{r}^R_{QO}| \cdot |\boldsymbol{r}^R_{PO}| \cdot \frac{x^R_{QO} \cdot x^R_{PO} + y^R_{QO} \cdot y^R_{PO}}{|\boldsymbol{r}^R_{QO}| \cdot |\boldsymbol{r}^R_{PO}|} \tag{A.1.4g}$$

$$= x^R_{QO} \cdot x^R_{PO} + y^R_{QO} \cdot y^R_{PO} = \left(x^R_{QO} \cdot y^R_{QO}\right) \cdot \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix} =: \left(\boldsymbol{r}^R_{QO}\right)^{\mathrm{T}} \cdot \boldsymbol{r}^R_{PO}.$$

□

Comment A.1.2 (Scalar product and orthogonality of vectors): Due to (A.1.4a), two vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are *orthogonal* to each other iff their scalar product is zero.

Vector product

The *vector product* of two geometrical vectors $\boldsymbol{r}_{QO}$ and $\boldsymbol{r}_{PO}$ from the x–y plane (Figure A.1.2), is defined as the vector

$$\boldsymbol{r}_c := \boldsymbol{r}_{QO} \times \boldsymbol{r}_{PO} = |\boldsymbol{r}_{QO}| \cdot |\boldsymbol{r}_{PO}| \cdot \sin\varphi_{QP} \cdot \boldsymbol{e}_{zR} = |\boldsymbol{r}_c| \cdot \boldsymbol{e}_{zR}, \tag{A.1.5a}$$

with $\boldsymbol{e}_{zR}$ as the unit vector perpendicular to the $\boldsymbol{e}_{xR}$–$\boldsymbol{e}_{yR}$ plane.

Algebraic computation of the vector product

The vector product $\boldsymbol{r}_c$ is computed from the *vectors*

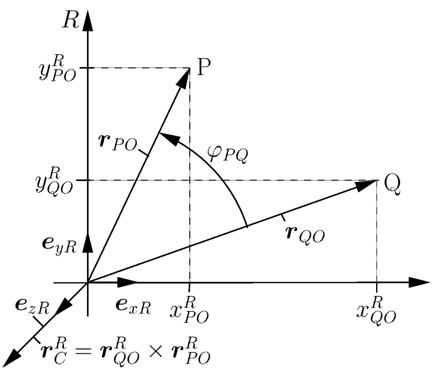


Fig. A.1.2: Vector diagram of a vector product

$$\boldsymbol{r}_{QO} = x^R_{QO} \cdot \boldsymbol{e}_{xR} + y^R_{QO} \cdot \boldsymbol{e}_{zR} \qquad \text{or} \qquad \boldsymbol{r}^R_{QO} = \begin{pmatrix} x^R_{QO} \\ y^R_{QO} \end{pmatrix} \tag{A.1.5b}$$

and

$$\boldsymbol{r}_{PO} = x^R_{PO} \cdot \boldsymbol{e}_{xR} + y^R_{PO} \cdot \boldsymbol{e}_{yR} \qquad \text{or} \qquad \boldsymbol{r}^R_{PO} = \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \end{pmatrix}$$

as

$$\boldsymbol{r}^R_c = \tilde{\boldsymbol{r}}^R_{QO} \cdot \boldsymbol{r}^R_{PO}, \tag{A.1.5c}$$

with the algebraic vectors $\boldsymbol{r}^R_{QO}$ and $\boldsymbol{r}^R_{PO}$ *extended* to the $\mathbb{R}^3$; i.e.,

$$\boldsymbol{r}_{PO} = x^R_{PO} \cdot \boldsymbol{e}_{xR} + y^R_{PO} \cdot \boldsymbol{e}_{yR} + 0 \cdot \boldsymbol{e}_{zR} = \left(x^R_{PO}\,,\, y^R_{PO}\,,\, 0\right)^{\mathrm{T}}$$

and

$$\boldsymbol{r}^R_{QO} = x^R_{QO} \cdot \boldsymbol{e}_{xR} + y^R_{QO} \cdot \boldsymbol{e}_{yR} + 0 \cdot \boldsymbol{e}_{zR} = \left(x^R_{QO}\,,\, y^R_{QO}\,,\, 0\right)^{\mathrm{T}},$$

and with the *skew-symmetric* spatial *matrix*

$$\tilde{\boldsymbol{r}}^R_{QO} = \begin{pmatrix} 0 & , & 0 & , & y^R_{QO} \\ 0 & , & 0 & , & -x^R_{QO} \\ -y^R_{QO} & , & x^R_{QO} & , & 0 \end{pmatrix} \tag{A.1.5d}$$

built from the extended spatial vector $\boldsymbol{r}^R_{QO}$. Then

$$\boldsymbol{r}^R_c = \tilde{\boldsymbol{r}}^R_{QO} \cdot \boldsymbol{r}^R_{PO} = \begin{pmatrix} 0 & , & 0 & , & y^R_{QO} \\ 0 & , & 0 & , & -x^R_{QO} \\ -y^R_{QO} & , & x^R_{QO} & , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^R_{PO} \\ y^R_{PO} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ x_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R \end{pmatrix} \tag{A.1.5e}$$

or

$$\boldsymbol{r}_c = \left(x_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R \right) \cdot \boldsymbol{e}_{zR} =: z_c \cdot \boldsymbol{e}_{zR}.$$

Proof:

By the definition of the vector product, the relation

$$\boldsymbol{r}_c^R = |\boldsymbol{r}_{QO}^R| \cdot |\boldsymbol{r}_{PO}^R| \cdot \sin \varphi_{QP} \tag{A.1.5a}$$

holds. Then

$$|\boldsymbol{r}_c^R|^2 = |\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 \cdot \sin^2 \varphi_{QP} = |\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 \cdot (1 - \cos^2 \varphi_{QP}).$$

Together with the scalar product

$$\left(\boldsymbol{r}_{QO}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}_{PO}^R\right) = |\boldsymbol{r}_{QO}^R| \cdot |\boldsymbol{r}_{PO}^R| \cdot \cos \varphi_{QP}$$

and

$$\cos^2 \varphi_{QP} = \left[\left(\boldsymbol{r}_{QO}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}_{PO}^R\right) \right]^2 \Big/ \left(|\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 \right),$$

this yields

$$\begin{aligned} |\boldsymbol{r}_c^R|^2 = \Big\{ & |\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 \\ & \cdot \left[|\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 - \left(\left(\boldsymbol{r}_{QO}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}_{PO}^R\right) \right) \right]^2 \Big\} \Big/ \left(|\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 \right), \end{aligned}$$

or

$$\begin{aligned} |\boldsymbol{r}_c^R|^2 &= |\boldsymbol{r}_{QO}^R|^2 \cdot |\boldsymbol{r}_{PO}^R|^2 - \left[\left(\boldsymbol{r}_{QO}^R\right)^{\mathrm{T}} \cdot \left(\boldsymbol{r}_{PO}^R\right) \right]^2 \\ &= \left[\left(x_{QO}^R\right)^2 + \left(y_{QO}^R\right)^2 \right] \cdot \left[\left(x_{PO}^R\right)^2 + \left(y_{PO}^R\right)^2 \right] \\ &\quad - \left(x_{QO}^R \cdot x_{PO}^R + y_{QO}^R \cdot y_{PO}^R \right)^2 \\ &= \left(x_{QO}^R\right)^2 \cdot \left(x_{PO}^R\right)^2 + \left(y_{QO}^R\right)^2 \cdot \left(y_{PO}^R\right)^2 \\ &\quad + \left(x_{QO}^R\right)^2 \cdot \left(y_{PO}^R\right)^2 + \left(x_{PO}^R\right)^2 \cdot \left(y_{QO}^R\right)^2 - \left(x_{QO}^R\right)^2 \cdot \left(x_{PO}^R\right)^2 \\ &\quad - \left(y_{QO}^R\right)^2 \cdot \left(y_{PO}^R\right)^2 - 2 \cdot x_{QO}^R \cdot x_{PO}^R \cdot y_{QO}^R \cdot y_{PO}^R, \end{aligned}$$

and finally

$$\left|\boldsymbol{r}_c^R\right|^2 = \left(z_c^R\right)^2 = \left(x_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R\right)^2$$

or

$$z_c = \lambda \cdot \left(x_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R\right) \cdot \boldsymbol{e}_{zR} \quad , \quad \lambda = +1 \text{ or } -1. \tag{A.1.6a}$$

On the other hand

$$\boldsymbol{r}_{QO}^R \times \boldsymbol{r}_{PO}^R = \tilde{\boldsymbol{r}}_{QO}^R \cdot \boldsymbol{r}_{PO}^R = \begin{pmatrix} 0 & , & 0 & , & y_{QO}^R \\ 0 & , & 0 & , & -x_{QO}^R \\ -y_{QO}^R & , & x_{QO}^R & , & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{PO}^R \\ y_{PO}^R \\ 0 \end{pmatrix} \tag{A.1.6b}$$

$$= \begin{pmatrix} 0 \\ 0 \\ x_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R \end{pmatrix} = (y_{QO}^R \cdot y_{PO}^R - y_{QO}^R \cdot x_{PO}^R) \cdot \boldsymbol{e}_{zR}.$$

Comparing (A.1.6a) and (A.1.6b) proves (A.1.5e) for $\lambda = +1$. □

A.1.3 Cross product of spatial vectors

The vector product of two spatial vectors $\boldsymbol{r}_{PO}$ and $\boldsymbol{r}_{QO}$, represented in R, is in agreement with *Appendix A.1.2*:

$$\begin{aligned} \boldsymbol{r}_{PO} \times \boldsymbol{r}_{QO} = \quad & \left(y_{PO}^R \cdot z_{QO}^R - z_{PO}^R \cdot y_{QO}^R\right) \cdot \boldsymbol{e}_{xR} \\ + & \left(z_{PO}^R \cdot x_{QO}^R - x_{PO}^R \cdot z_{QO}^R\right) \cdot \boldsymbol{e}_{yR} \\ + & \left(x_{PO}^R \cdot y_{QO}^R - y_{PO}^R \cdot x_{QO}^R\right) \cdot \boldsymbol{e}_{zR} \end{aligned} \tag{A.1.7a}$$

or

$$\boldsymbol{r}_{PO}^R \times \boldsymbol{r}_{QO}^R = \begin{pmatrix} y_{PO}^R \cdot z_{QO}^R - z_{PO}^R \cdot y_{QO}^R \\ z_{PO}^R \cdot x_{QO}^R - x_{PO}^R \cdot z_{QO}^R \\ x_{PO}^R \cdot y_{QO}^R - y_{PO}^R \cdot x_{QO}^R \end{pmatrix}. \tag{A.1.7b}$$

Introducing the skew-symmetric matrix

$$\tilde{\boldsymbol{r}}_{PO}^R := \begin{pmatrix} 0 & , & -z_{PO}^R & , & y_{PO}^R \\ z_{PO}^R & , & 0 & , & -x_{PO}^R \\ -y_{PO}^R & , & x_{PO}^R & , & 0 \end{pmatrix} \tag{A.1.7c}$$

constructed from the vector $\boldsymbol{r}_{PO}^R$, the product

$$\tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{r}^R_{QO} = \begin{pmatrix} 0 & , -z^R_{PO} \, , & y^R_{PO} \\ z^R_{PO} \, , & 0 & , -x^R_{PO} \\ -y^R_{PO} \, , & x^R_{PO} \, , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^R_{QO} \\ y^R_{QO} \\ z^R_{QO} \end{pmatrix} \tag{A.1.7d}$$

$$= \begin{pmatrix} y^R_{PO} \cdot z^R_{QO} - z^R_{PO} \cdot y^R_{QO} \\ z^R_{PO} \cdot x^R_{QO} - x^R_{PO} \cdot z^R_{QO} \\ x^R_{PO} \cdot y^R_{QO} - y^R_{PO} \cdot x^R_{QO} \end{pmatrix}$$

yields the same vector as the cross product (A.1.7a). As a consequence, the vector product (A.1.7b) can be replaced by the matrix relation (A.1.7d)

$$\boldsymbol{r}^R_{PO} \times \boldsymbol{r}^R_{QO} = \tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{r}^R_{QO} \in \mathbb{R}^3. \tag{A.1.8}$$

The *vector product* of two vectors (in $\mathbb{R}^3$), represented in frames R or L, satisfies the relation

$$\tilde{\boldsymbol{r}}^R_{PO} \cdot \boldsymbol{r}^R_{QO} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{PO} \underbrace{\boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL}}_{= \mathbf{I}_3} \cdot \boldsymbol{r}^L_{QO} = \boldsymbol{A}^{RL} \cdot \left(\tilde{\boldsymbol{r}}^L_{PO} \cdot \boldsymbol{r}^L_{QO} \right) \tag{A.1.9a}$$

with

$$\tilde{\boldsymbol{r}}^R_{PO} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}^L_{PO} \cdot \boldsymbol{A}^{LR}. \tag{A.1.9b}$$

Furthermore, the following relations hold:

1. $$\left(\tilde{\boldsymbol{r}}^R_{PO} \right)^{\mathrm{T}} = -\tilde{\boldsymbol{r}}^R_{PO} = \tilde{\boldsymbol{r}}^R_{OP}. \tag{A.1.9c}$$
2. $$\widetilde{\left(\boldsymbol{r}^R_{PO_L} + \boldsymbol{r}^R_{O_L O} \right)} = \tilde{\boldsymbol{r}}^R_{PO_L} + \tilde{\boldsymbol{r}}^R_{O_L O}. \tag{A.1.9d}$$
3. $$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \boldsymbol{r}^R_{PO_L} = \mathbf{0}. \tag{A.1.9e}$$
4. $$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L} = \left(\tilde{\boldsymbol{r}}^R_{PO_L} \right)^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{r}}^R_{PO_L} \right)^{\mathrm{T}} = \left(\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L} \right)^{\mathrm{T}}. \tag{A.1.9f}$$
5. $$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L} = - \left[\underbrace{\left(\boldsymbol{r}^R_{PO_L} \right)^{\mathrm{T}} \cdot \boldsymbol{r}^R_{PO_L}}_{\textit{scalar product}} \cdot \mathbf{I}_3 - \underbrace{\boldsymbol{r}^R_{PO_L} \cdot \left(\boldsymbol{r}^R_{PO_L} \right)^{\mathrm{T}}}_{\textit{dyadic product}} \right]. \tag{A.1.9g}$$

This latter relation will be used in representations of the *inertia matrix* of a rigid body in *Chapter 4*.

Proof of A.1.9c:

By definition,

$$\left(\tilde{\boldsymbol{r}}^R_{PO}\right)^{\mathrm{T}} := \begin{pmatrix} 0 & , -z^R_{PO} & , & y^R_{PO} \\ z^R_{PO} & , & 0 & , -x^R_{PO} \\ -y^R_{PO} & , & x^R_{PO} & , & 0 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} 0 & , & z^R_{PO} & , -y^R_{PO} \\ -z^R_{PO} & , & 0 & , & x^R_{PO} \\ y^R_{PO} & , -x^R_{PO} & , & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & , -z^R_{PO} & , & y^R_{PO} \\ z^R_{PO} & , & 0 & , -x^R_{PO} \\ -y^R_{PO} & , & x^R_{PO} & , & 0 \end{pmatrix} = -\tilde{\boldsymbol{r}}^R_{PO} = \tilde{\boldsymbol{r}}^R_{OP}.$$

This proves *(A.1.9c)*. □

Proof of A.1.9d:

The skew-symmetric matrix associated to the vector sum

$$\boldsymbol{r}^R_{PO_L} + \boldsymbol{r}^R_{O_LO} = \begin{pmatrix} x^R_{PO_L} + x^R_{O_LO} \\ y^R_{PO_L} + y^R_{O_LO} \\ z^R_{PO_L} + z^R_{O_LO} \end{pmatrix}$$

is

$$\widetilde{\left(\boldsymbol{r}^R_{PO_L} + \boldsymbol{r}^R_{O_LO}\right)} = \begin{pmatrix} 0 & , -z^R_{PO_L} - z^R_{O_LO} & , & y^R_{PO_L} + y^R_{O_LO} \\ z^R_{PO_L} + z^R_{O_LO} & , & 0 & , -x^R_{PO_L} - x^R_{O_LO} \\ -y^R_{PO_L} - y^R_{O_LO} & , & x^R_{PO} + x^R_{O_LO} & , & 0 \end{pmatrix}. \tag{A.1.10a}$$

On the other hand

$$\tilde{\boldsymbol{r}}^R_{PO_L} = \begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , & 0 & , -x^R_{PO_L} \\ -y^R_{PO_L} & , & x^R_{PO_L} & , & 0 \end{pmatrix},$$

$$\tilde{\boldsymbol{r}}^R_{O_LO} = \begin{pmatrix} 0 & , -z^R_{O_LO} & , & y^R_{O_LO} \\ z^R_{O_LO} & , & 0 & , -x^R_{O_LO} \\ -y^R_{O_LO} & , & x^R_{O_LO} & , & 0 \end{pmatrix},$$

and

$$\tilde{\boldsymbol{r}}^R_{PO_L} + \tilde{\boldsymbol{r}}^R_{O_LO} = \begin{pmatrix} 0 & , -z^R_{PO_L} - z^R_{O_LO} & , & y^R_{PO_L} + y^R_{O_LO} \\ z^R_{PO_L} + z^R_{O_LO} & , & 0 & , -x^R_{PO_L} - x^R_{O_LO} \\ -y^R_{PO_L} - y^R_{O_LO} & , & x^R_{PO} + x^R_{O_LO} & , & 0 \end{pmatrix}. \tag{A.1.10b}$$

Relation (A.1.10a) together with (A.1.10b) implies

$$\tilde{\boldsymbol{r}}^R_{PO_L} + \tilde{\boldsymbol{r}}^R_{O_LO} = \widetilde{\boldsymbol{r}^R_{PO_L} + \boldsymbol{r}^R_{O_LO}}.$$ □

Proof of (A.1.9e):

The relation

$$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \boldsymbol{r}^R_{PO_L} = \begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , \quad 0 & , & -x^R_{PO_L} \\ -y^R_{PO_L} & , \quad x^R_{PO_L} & , & 0 \end{pmatrix} \cdot \begin{pmatrix} x^R_{PO_L} \\ y^R_{PO_L} \\ z^R_{PO_L} \end{pmatrix}$$

$$= \begin{pmatrix} -z^R_{PO_L} \cdot y^R_{PO_L} + y^R_{PO_L} \cdot z^R_{PO_L} \\ z^R_{PO_L} \cdot x^R_{PO_L} - x^R_{PO_L} \cdot z^R_{PO_L} \\ -y^R_{PO_L} \cdot x^R_{PO_L} + x^R_{PO_L} \cdot y^R_{PO_L} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

proves *(A.1.9e).* □

Proof of (A.1.9f):

The expressions

$$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L} = \begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , \quad 0 & , & -x^R_{PO_L} \\ -y^R_{PO_L} & , \quad x^R_{PO_L} & , & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , \quad 0 & , & -x^R_{PO_L} \\ -y^R_{PO_L} & , \quad x^R_{PO_L} & , & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\left(z^R_{PO_L}\right)^2 - \left(y^R_{PO_L}\right)^2 & , \quad y^R_{PO_L} \cdot x^R_{PO_L} & , \quad z^R_{PO_L} \cdot y^R_{PO_L} \\ x^R_{PO_L} \cdot y^R_{PO_L} & , -\left(z^R_{PO_L}\right)^2 - \left(x^R_{PO_L}\right)^2 & , \quad z^R_{PO_L} \cdot y^R_{PO_L} \\ x^R_{PO_L} \cdot z^R_{PO_L} & , \quad y^R_{PO_L} \cdot z^R_{PO_L} & , -\left(y^R_{PO_L}\right)^2 - \left(x^R_{PO_L}\right)^2 \end{pmatrix},$$

and

$$\left(\tilde{\boldsymbol{r}}^R_{PO_L}\right)^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{r}}^R_{PO_L}\right)^{\mathrm{T}} =$$
$$\begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , \quad 0 & , & -x^R_{PO_L} \\ -y^R_{PO_L} & , \quad x^R_{PO_L} & , & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & , -z^R_{PO_L} & , & y^R_{PO_L} \\ z^R_{PO_L} & , \quad 0 & , & -x^R_{PO_L} \\ -y^R_{PO_L} & , \quad x^R_{PO_L} & , & 0 \end{pmatrix}$$

imply that the relation

$$\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L} = \left(\tilde{\boldsymbol{r}}^R_{PO_L}\right)^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{r}}^R_{PO_L}\right)^{\mathrm{T}}$$

holds. This proves *(A.1.9f).* □

Proof of (A.1.9g):

$$\left[\left(\boldsymbol{r}^R_{PO_L}\right)^{\mathrm{T}} \cdot \boldsymbol{r}^R_{PO_L} \cdot \mathbf{I}_3 - \boldsymbol{r}^R_{PO_L} \cdot \left(\boldsymbol{r}^R_{PO_L}\right)^{\mathrm{T}}\right] =$$

$$\left(x^R_{PO_L}\, , \, y^R_{PO_L}\, , \, z^R_{PO_L}\right) \cdot \begin{pmatrix} x^R_{PO_L} \\ y^R_{PO_L} \\ z^R_{PO_L} \end{pmatrix} \cdot \begin{pmatrix} 1\, , \, 0\, , \, 0 \\ 0\, , \, 1\, , \, 0 \\ 0\, , \, 0\, , \, 1 \end{pmatrix}$$

$$- \begin{pmatrix} x^R_{PO_L} \\ y^R_{PO_L} \\ z^R_{PO_L} \end{pmatrix} \cdot \left(x^R_{PO_L}\, , \, y^R_{PO_L}\, , \, z^R_{PO_L}\right) =$$

$$\begin{pmatrix} (x^R_{PO_L})^2 + (y^R_{PO_L})^2 + (z^R_{PO_L})^2, & 0 & , & 0 \\ 0 & ,(x^R_{PO_L})^2 + (y^R_{PO_L})^2 + (z^R_{PO_L})^2, & & 0 \\ 0 & , & 0 & ,(x^R_{PO_L})^2 + (y^R_{PO_L})^2 + (z^R_{PO_L})^2 \end{pmatrix}$$

$$- \begin{pmatrix} (x^R_{PO_L})^2 & ,y^R_{PO_L} \cdot x^R_{PO_L} & ,z^R_{PO_L} \cdot x^R_{PO_L} \\ x^R_{PO_L} \cdot y^R_{PO_L}, & (y^R_{PO_L})^2 & ,z^R_{PO_L} \cdot y^R_{PO_L} \\ x^R_{PO_L} \cdot z^R_{PO_L} & ,y^R_{PO_L} \cdot z^R_{PO_L}, & (z^R_{PO_L})^2 \end{pmatrix} =$$

$$\begin{pmatrix} (y^R_{PO_L})^2 + (z^R_{PO_L})^2, & -y^R_{PO_L} \cdot x^R_{PO_L} & , & -z^R_{PO_L} \cdot x^R_{PO_L} \\ -x^R_{PO_L} \cdot y^R_{PO_L} & ,(x^R_{PO_L})^2 + (z^R_{PO_L})^2, & & -z^R_{PO_L} \cdot y^R_{PO_L} \\ -x^R_{PO_L} \cdot z^R_{PO_L} & , & -y^R_{PO_L} \cdot z^R_{PO_L} & ,(x^R_{PO_L})^2 + (y^R_{PO_L})^2 \end{pmatrix} = -\tilde{\boldsymbol{r}}^R_{PO_L} \cdot \tilde{\boldsymbol{r}}^R_{PO_L},$$

due to the proof of (A.1.9f). This proves *(A.1.9g)*. □

A.1.4 Time derivatives of planar orientation matrices and of planar vectors in different frames

In this section the following relations will be proved:

$$^{R}\dot{\boldsymbol{r}}^L_{PO_L} := \frac{^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) = {}^{L}\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L}, \tag{A.1.11a}$$

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} = \tilde{\boldsymbol{\omega}}^L_{LR} = \mathbf{R} \cdot \dot{\psi}_{LR}, \tag{A.1.11b}$$

or

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} = \boldsymbol{A}^{RL} \cdot \mathbf{R} \cdot \dot{\psi}_{LR}, \quad \mathbf{R} = \begin{pmatrix} 0\, , & -1 \\ 1\, , & 0 \end{pmatrix} \tag{A.1.11c}$$

with

$$\boldsymbol{\omega}_{LR} = \dot{\psi}_{LR} \cdot \boldsymbol{e}_{zL}, \quad \tilde{\boldsymbol{\omega}}^L_{LR} = \mathbf{R} \cdot \dot{\psi}_{LR}, \tag{A.1.11d}$$

$$^{R}\dot{\boldsymbol{r}}^L_{PO_L} = {}^{L}\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}, \tag{A.1.11e}$$

and

$$
\begin{aligned}
{}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} &:= \boldsymbol{A}^{\boldsymbol{RL}} \cdot {}^{\boldsymbol{R}}\dot{\boldsymbol{r}}^{\boldsymbol{L}}_{\boldsymbol{PO_L}} = \boldsymbol{A}^{\boldsymbol{RL}} \cdot \left({}^{\boldsymbol{L}}\dot{\boldsymbol{r}}^{\boldsymbol{L}}_{\boldsymbol{PO_L}} + \tilde{\boldsymbol{\omega}}^{\boldsymbol{L}}_{\boldsymbol{LR}} \cdot \boldsymbol{r}^{\boldsymbol{L}}_{\boldsymbol{PO_L}}\right) \qquad \text{(A.1.11f)} \\
&= \boldsymbol{A}^{RL} \cdot \left({}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} + \mathbf{R} \cdot \dot{\psi}_{LR} \cdot \boldsymbol{r}^{L}_{PO_L}\right) = \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} + \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^{L}_{PO_L}.
\end{aligned}
$$

Proof of (A.1.11):

Assume that the vector $\boldsymbol{r}_{PO_L}(t)$ with P fixed on a moving frame L (see Figure 2.6) is represented in L, but differentiated with respect to the time in frame R. Then

$$
{}^{R}\dot{\boldsymbol{r}}_{PO_L} := \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t) \cdot \boldsymbol{e}_{xL}(t) + y^{L}_{PO_L}(t) \cdot \boldsymbol{e}_{yL}(t)\right) \qquad \text{(A.1.12a)}
$$

with *basis vectors* $\boldsymbol{e}_{iL}(t)$ *that are not constant with respect to* R. Applying the product rule of differentiation to (A.1.12a) provides the relation

$$
\begin{aligned}
{}^{R}\dot{\boldsymbol{r}}_{PO_L}(t) = &\left(\frac{{}^{R}\mathrm{d}}{\mathrm{d}t} x^{L}_{PO_L}(t)\right) \cdot \underline{\boldsymbol{e}}_{xL} + x^{L}_{PO_L}(t) \cdot \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{xL}(t)\right) \qquad \text{(A.1.12b)} \\
&+ \left(\frac{{}^{R}\mathrm{d}}{\mathrm{d}t} y^{L}_{PO_L}(t)\right) \cdot \underline{\boldsymbol{e}}_{yL} + y^{L}_{PO_L}(t) \cdot \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{yL}(t)\right).
\end{aligned}
$$

By definition of the product rule of differentiation the *basis vectors* $\boldsymbol{e}_{iL}$, marked by underlined letters in (A.1.12b) *are considered to be constant.* The time derivatives of the *scalar functions* $x^{L}_{PO_L}$ and $y^{L}_{PO_L}$ are

$$
\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}\right) =: \dot{x}^{L}_{PO_L}(t)
$$

and (A.1.13)

$$
\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(y^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(y^{L}_{PO_L}\right) =: \dot{y}^{L}_{PO_L}(t).
$$

This implies

$$
\begin{aligned}
{}^{R}\dot{\boldsymbol{r}}_{PO_L}(t) = \;&\dot{x}^{L}_{PO_L}(t) \cdot \boldsymbol{e}_{xL}(t) + x^{L}_{PO_L}(t) \cdot \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\overline{\boldsymbol{e}}_{xL}(t)\right) \qquad \text{(A.1.14)} \\
&+ \dot{y}^{L}_{PO_L}(t) \cdot \boldsymbol{e}_{yL}(t) + y^{L}_{PO_L}(t) \cdot \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\overline{\boldsymbol{e}}_{yL}(t)\right),
\end{aligned}
$$

where the *overlined basis vectors* $\overline{\boldsymbol{e}}_{iL}$ $(i = x\,,\, y)$ in (A.1.14) may change with respect to frame R. They *are no longer constant in time* – they rotate with respect to frame R. Representation of the basis vectors $\boldsymbol{e}_{iL}$ of L $(i = x\,,\, y)$ in frame R according to (2.12b) and (2.12c) yields, together with

$$
\boldsymbol{e}_{xL}(t) = \cos\psi \cdot \boldsymbol{e}_{x_R} + \sin\psi \cdot \boldsymbol{e}_{y_R}, \quad \boldsymbol{e}_{yL}(t) = -\sin\psi \cdot \boldsymbol{e}_{x_R} + \cos\psi \cdot \boldsymbol{e}_{y_R}, \qquad \text{(A.1.15)}
$$

the relations

$$\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{xL}(t)\right) = \frac{^R\mathrm{d}}{\mathrm{d}t}\left(\cos\psi \cdot \boldsymbol{e}_{xR} + \sin\psi \cdot \boldsymbol{e}_{yR}\right) \tag{A.1.16a}$$

$$= \dot{\overline{\cos\psi}} \cdot \boldsymbol{e}_{xR} + \dot{\overline{\sin\psi}} \cdot \boldsymbol{e}_{yR} + \cos\psi \cdot {^R\dot{\boldsymbol{e}}_{xR}} + \sin\psi \cdot {^R\dot{\boldsymbol{e}}_{yR}}$$

and

$$\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{yL}(t)\right) = \frac{^R\mathrm{d}}{\mathrm{d}t}\left(-\sin\psi \cdot \boldsymbol{e}_{xR}\right) + \frac{^R\mathrm{d}}{\mathrm{d}t}\left(\cos\psi \cdot \boldsymbol{e}_{yR}\right) \tag{A.1.16b}$$

$$= -\dot{\overline{\sin\psi}} \cdot \boldsymbol{e}_{xR} - (\sin\psi) \cdot {^R\dot{\boldsymbol{e}}_{xR}} + \dot{\overline{\cos\psi}} \cdot \boldsymbol{e}_{yR} + \cos\psi \cdot {^R\dot{\boldsymbol{e}}_{yR}}.$$

Since the *basis vectors* $\boldsymbol{e}_{iR}$ *are constant in frame* R, the relations

$${^R\dot{\boldsymbol{e}}_{jR}} = \frac{^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{jR} \equiv \mathbf{0} \quad \text{for} \quad j = x\,,\, y \tag{A.1.17}$$

imply

$$\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{xL}(t)\right) = \left[(-\sin\psi) \cdot \boldsymbol{e}_{xR} + \cos\psi \cdot \boldsymbol{e}_{yR}\right] \cdot \dot{\psi}$$

and

$$\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{yL}(t)\right) = \left[(-\cos\psi) \cdot \boldsymbol{e}_{xR} - \sin\psi \cdot \boldsymbol{e}_{yR}\right] \cdot \dot{\psi}.$$

Inserting these relations into (A.1.14) yields the relation

$$\begin{aligned}\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) = {} & \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL}\right) \\ & + x^L_{PO_L} \cdot \left[(-\sin\psi)\,\boldsymbol{e}_{xR} + (\cos\psi)\,\boldsymbol{e}_{yR}\right] \cdot \dot{\psi} \\ & + y^L_{PO_L} \cdot \left[(-\cos\psi)\,\boldsymbol{e}_{xR} - (\sin\psi)\,\boldsymbol{e}_{yR}\right] \cdot \dot{\psi}\end{aligned} \tag{A.1.18}$$

or

$$\begin{aligned}\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) = {} & \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL}\right) \\ & + \Big[\left(x^L_{PO_L} \cdot (-\sin\psi) + y^L_{PO_L} \cdot (-\cos\psi)\right) \cdot \boldsymbol{e}_{xR} \\ & + \left(x^L_{PO_L} \cdot \cos\psi + y^L_{PO_L} \cdot (-\sin\psi)\right) \cdot \boldsymbol{e}_{yR}\Big] \cdot \dot{\psi}.\end{aligned} \tag{A.1.19a}$$

Representing the basis vectors $\boldsymbol{e}_{xR}$, $\boldsymbol{e}_{yR}$ according to (2.13) in frame L yields

$$\begin{aligned}\frac{^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) = {} & \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL}\right) \\ + & \Big[\left(x^L_{PO_L} \cdot (-\sin\psi) + y^L_{PO_L} \cdot (-\cos\psi)\right) \cdot \left(\cos\psi \cdot \boldsymbol{e}_{xL} - \sin\psi \cdot \boldsymbol{e}_{yL}\right)\Big] \cdot \dot{\psi} \\ + & \Big[\left(x^L_{PO_L} \cdot \cos\psi + y^L_{PO_L} \cdot (-\sin\psi)\right) \cdot \left(\sin\psi \cdot \boldsymbol{e}_{xL} + \cos\psi \cdot \boldsymbol{e}_{yL}\right)\Big] \cdot \dot{\psi}\end{aligned} \tag{A.1.19b}$$

or

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) = \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL}\right) \tag{A.1.19c}$$
$$+ \Big[(-\sin\psi) \cdot (\cos\psi) \cdot x^L_{PO_L} + (-\cos\psi) \cdot (\cos\psi) \cdot y^L_{PO_L}$$
$$+ (\cos\psi) \cdot (\sin\psi) \cdot x^L_{PO_L} + (-\sin\psi) \cdot (\sin\psi) \cdot y^L_{PO_L}\Big] \cdot \boldsymbol{e}_{xL} \cdot \dot{\psi}$$
$$+ \Big[(-\sin\psi) \cdot (-\sin\psi) \cdot x^L_{PO_L} + (-\cos\psi) \cdot (-\sin\psi) \cdot y^L_{PO_L}$$
$$+ (\cos\psi) \cdot (\cos\psi) \cdot x^L_{PO_L} + (-\sin\psi) \cdot (\cos\psi) \cdot y^L_{PO_L}\Big] \cdot \boldsymbol{e}_{yL} \cdot \dot{\psi}.$$

In order to obtain the formal expression (A.1.11a), several terms of (A.1.19c) that cancel will not be dropped in the next steps. Then (A.1.19c) may be written as

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) = \begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \end{pmatrix} \tag{A.1.20}$$
$$+ \dot{\psi} \cdot \left[\begin{array}{l} [(\cos\psi) \cdot (-\sin\psi) + (\cos\psi \cdot \sin\psi)] \cdot x^L_{PO_L} \\ [(-\sin\psi) \cdot (-\sin\psi) + (\cos\psi) \cdot (\cos\psi)] \cdot x^L_{PO_L} \\ + \quad [\cos\psi \cdot (-\cos\psi) + (\sin\psi) \cdot (-\sin\psi)] \cdot y^L_{PO_L} \\ + \ [(-\sin\psi) \cdot (-\cos\psi) + (\cos\psi) \cdot (-\sin\psi)] \cdot y^L_{PO_L} \end{array}\right]$$

or

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) = \begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \end{pmatrix} +$$
$$\begin{pmatrix} \cos\psi \cdot (-\sin\psi) + \cos\psi \cdot \sin\psi \ , & \cos\psi \cdot (-\cos\psi) + \sin\psi \cdot (-\sin\psi) \\ (-\sin\psi) \cdot (-\sin\psi) + \cos\psi \cdot \cos\psi \ , & (-\sin\psi) \cdot (-\sin\psi) + \cos\psi \cdot (-\sin\psi) \end{pmatrix}$$
$$\cdot \begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \end{pmatrix} \cdot \dot{\psi}$$

or as

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) \tag{A.1.21a}$$
$$= \underbrace{\begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \end{pmatrix}}_{=:\ {}^L\dot{\boldsymbol{r}}^L_{PO_L}} + \underbrace{\begin{pmatrix} \cos\psi \ , & \sin\psi \\ -\sin\psi \ , & \cos\psi \end{pmatrix}}_{=:\ \boldsymbol{A}^{LR}} \cdot \underbrace{\begin{pmatrix} -\sin\psi \ , & -\cos\psi \\ \cos\psi \ , & -\sin\psi \end{pmatrix} \cdot \dot{\psi}}_{=:\ \dot{\boldsymbol{A}}^{RL}} \cdot \underbrace{\begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \end{pmatrix}}_{=:\ \boldsymbol{r}^L_{PO_L}},$$

and together with (cf. Equation 2.32a)

$$\dot{\boldsymbol{r}}^L_{PO_L} := {}^L\dot{\boldsymbol{r}}^L_{PO_L} = \left(\dot{x}^L_{PO_L} \ , \ \dot{y}^L_{PO_L}\right)^{\mathrm{T}}, \tag{A.1.21b}$$

as

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) + \boldsymbol{A}^{LR}\cdot\dot{\boldsymbol{A}}^{RL}\cdot\boldsymbol{r}^L_{PO_L}, \tag{A.1.22}$$

and finally as

$${}^R\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{LR}\cdot\dot{\boldsymbol{A}}^{RL}\cdot\boldsymbol{r}^L_{PO_L} \tag{A.1.23}$$

together with

$$\boldsymbol{A}^{RL}\cdot{}^R\dot{\boldsymbol{r}}^L_{PO_L} = \boldsymbol{A}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L} + \dot{\boldsymbol{A}}^{RL}\cdot\boldsymbol{r}^L_{PO_L}. \tag{A.1.24}$$

This *proves (A.1.11a).* □

Using

$$\dot{\boldsymbol{A}}^{RL} = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\cos\psi\ , & -\sin\psi\\ \sin\psi\ , & \cos\psi\end{pmatrix} = \begin{pmatrix}-\sin\psi\ , & -\cos\psi\\ \cos\psi\ , & -\sin\psi\end{pmatrix}\cdot\dot{\psi}, \quad \psi = \psi_{LR},$$

provides

$$\begin{aligned}\boldsymbol{A}^{LR}\cdot\dot{\boldsymbol{A}}^{RL} &= \begin{pmatrix}\cos\psi\ , & \sin\psi\\ -\sin\psi\ , & \cos\psi\end{pmatrix}\cdot\dot{\overline{\begin{pmatrix}\cos\psi\ , & -\sin\psi\\ \sin\psi\ , & \cos\psi\end{pmatrix}}} \\ &= \begin{pmatrix}\cos\psi\ , & \sin\psi\\ -\sin\psi\ , & \cos\psi\end{pmatrix}\cdot\begin{pmatrix}-\sin\psi\ , & -\cos\psi\\ \cos\psi\ , & -\sin\psi\end{pmatrix}\cdot\dot{\psi} \\ &= \underbrace{\begin{pmatrix}0\ , & -1\\ 1\ , & 0\end{pmatrix}}_{=\mathbf{R}}\cdot\dot{\psi} = \begin{pmatrix}0\ , & -\dot{\psi}\\ \dot{\psi}\ , & 0\end{pmatrix} = \mathbf{R}\cdot\dot{\psi}_{LR} =: \tilde{\boldsymbol{\omega}}^L_{LR},\end{aligned} \tag{A.1.25a}$$

proves *(A.1.11b)*, and yields (A.1.11c) directly,with the *planar orthogonal rotation matrix* $\mathbf{R}$ defined in (2.22a), and (cf. Comment A.1.3) with

$$\boldsymbol{\omega}^L_{LR} := \begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\dot{\psi}_{LR} = \begin{pmatrix}0\\0\\\dot{\psi}_{LR}\end{pmatrix} \quad \text{or} \quad \boldsymbol{\omega}_{LR} = \dot{\psi}_{LR}\cdot\boldsymbol{e}_{zL}$$

as the *angular velocity* around the $\boldsymbol{e}_{zL}$–axis. This *proves (A.1.11d).* □

Together with (A.1.11b), (A.1.11a) yields

$${}^R\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \mathbf{R}\cdot\boldsymbol{r}^L_{PO_L}\cdot\dot{\psi}_{LR}$$

or, written in components,

$$\begin{pmatrix} {}^{R}\dot{x}^{L}_{PO_L} \\ {}^{R}\dot{y}^{L}_{PO_L} \end{pmatrix} = \begin{pmatrix} {}^{L}\dot{x}^{L}_{PO_L} \\ {}^{L}\dot{y}^{L}_{PO_L} \end{pmatrix} + \begin{pmatrix} 0 \;, & -1 \\ 1 \;, & 0 \end{pmatrix} \cdot \begin{pmatrix} x^{L}_{PO_L} \\ y^{L}_{PO_L} \end{pmatrix} \cdot \dot{\psi}_{LR}, \tag{A.1.25b}$$

or

$${}^{R}\dot{x}^{L}_{PO_L} = {}^{L}\dot{x}^{L}_{PO_L} - y^{L}_{PO_L} \cdot \dot{\psi}_{LR} \tag{A.1.25c}$$

and

$${}^{R}\dot{y}^{L}_{PO_L} = {}^{L}\dot{y}^{L}_{PO_L} + x^{L}_{PO_L} \cdot \dot{\psi}_{LR}.$$

This *proves (A.1.11e).* □

Comment A.1.3 (Formal notation $\tilde{\boldsymbol{\omega}}^{L}_{LR}$): The above notation using $\tilde{\boldsymbol{\omega}}_{LR}$ is motivated by the intention to write the equations of motions of *planar* rigid-body systems in a form that is identical to the equations of motion of *spatial* mechanisms, discussed in Section A.1.5 (compare Section A.1.5 and Equation 2.27).

In the *spatial case*,

$$\boldsymbol{\omega}^{L}_{LR} := \begin{pmatrix} \boldsymbol{\omega}^{L}_{xLR} \\ \boldsymbol{\omega}^{L}_{yLR} \\ \boldsymbol{\omega}^{L}_{zLR} \end{pmatrix}^{L} \tag{A.1.26a}$$

is the angular velocity vector of frame L with respect to frame R, represented in L. Then $\tilde{\boldsymbol{\omega}}^{L}_{LR}$ is defined as the skew-symmetric matrix

$$\tilde{\boldsymbol{\omega}}^{L}_{LR} := \begin{pmatrix} 0 & , -\omega^{L}_{zLR} \;, & \omega^{L}_{yLR} \\ \omega^{L}_{zLR} \;, & 0 & , -\omega^{L}_{xLR} \\ -\omega^{L}_{yLR} \;, & \omega^{L}_{xLR} \;, & 0 \end{pmatrix}^{L}. \tag{A.1.26b}$$

In the *planar case*, only vectors $\boldsymbol{r} = x^{L} \cdot \boldsymbol{e}_{xL} + y^{L} \cdot \boldsymbol{e}_{yL}$ or $\boldsymbol{r}^{L} = (x^{L} \,,\, y^{L})^{\mathrm{T}}$ with basis vectors $\{\boldsymbol{e}_{xL} \,,\, \boldsymbol{e}_{yL}\}$ of the reference frame L inside the x–y plane occur, where frame L can only rotate around the $\boldsymbol{e}_{zL}$–axis perpendicular to this plane by an angle ψ_{LR}. Then the angular velocity vector $\boldsymbol{\omega}^{L}_{LR}$ associated with this rotation around $\boldsymbol{e}_{zL}$ may be written in an extended form in $\mathbb{R}^3$ as

$$\boldsymbol{\omega}^{L}_{LR}\Big|_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 0 \\ \omega^{L}_{zLR} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi}_{LR} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \dot{\psi}_{LR} = \dot{\psi}_{LR} \cdot \boldsymbol{e}_{zL}$$

with (A.1.27)

$$\tilde{\boldsymbol{\omega}}^{L}_{LR}\Big|_{\mathbb{R}^3} = \left(\begin{array}{cc|c} 0 \;, & -\dot{\psi}_{LR} & 0 \\ \dot{\psi}_{LR} \;, & 0 & 0 \\ \hline 0 \;, & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} 0 \;, & -1 & 0 \\ 1 \;, & 0 & 0 \\ \hline 0 \;, & 0 & 0 \end{array}\right) \cdot \dot{\psi}_{LR}.$$

Projecting $\tilde{\boldsymbol{\omega}}_{LR}^{L}\Big|_{I\!R^3}$ into the x–y plane yields, in agreement with (A.1.11b),

$$\tilde{\boldsymbol{\omega}}_{LR}^{L} := \tilde{\boldsymbol{\omega}}_{LR}^{L}\Big|_{I\!R^2=<e_{xL}\,,\,e_{yL}>} = \begin{pmatrix} 0\,, & -1 \\ 1\,, & 0 \end{pmatrix} \cdot \dot{\psi}_{LR} \underset{(A.1.25a)}{=} \mathbf{R} \cdot \dot{\psi}_{LR}, \quad \text{(A.1.28)}$$

with $\mathbf{R}$ as the planar orthogonal rotation matrix (2.22e).

Mapping the vector ${}^{R}\dot{\boldsymbol{r}}_{PO_L}^{L}$ by means of $\boldsymbol{A}^{RL}$ into frame R yields, together with (A.1.11a) and (A.1.11b), the relation

$$\boldsymbol{A}^{RL} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO_L}^{L} = \boldsymbol{A}^{RL}\left({}^{L}\dot{\boldsymbol{r}}_{PO_L}^{L} + \tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \boldsymbol{r}_{PO_L}^{L}\right). \quad \text{(A.1.29a)}$$

On the other hand

$${}^{R}\dot{\boldsymbol{r}}_{PO_L}^{R} = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL} \cdot \boldsymbol{r}_{PO_L}^{L}\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix} \cos\psi_{LR}\,, & -\sin\psi_{LR} \\ \sin\psi_{LR}\,, & \cos\psi_{LR} \end{pmatrix} \cdot \begin{pmatrix} x_{PO_L}^{L} \\ y_{PO_L}^{L} \end{pmatrix}\right]$$

or

$$\begin{aligned} \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_{PO_L} = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\Big[&\left(\cos\psi_{LR} \cdot x_{PO_L}^{L} - \sin\psi_{LR} \cdot y_{PO_L}^{L}\right) \cdot \boldsymbol{e}_{xR} \\ &+ \left(\sin\psi_{LR} \cdot x_{PO_L}^{L} + \cos\psi_{LR} \cdot y_{PO_L}^{L}\right) \cdot \boldsymbol{e}_{yR}\Big]. \end{aligned}$$

Together with

$$\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{xR}\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{yR}\right) = \mathbf{0} \quad \text{and} \quad \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x_{PO_L}^{L}\right) = \dot{x}_{PO_L}^{L}, \quad \text{(A.1.29b)}$$

this yields

$${}^{R}\dot{\boldsymbol{r}}_{PO_L}^{R} = \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}(\cos\psi_{LR} \cdot x_{PO_L}^{L} - \sin\psi_{LR} \cdot y_{PO_L}^{L}) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\sin\psi_{LR} \cdot x_{PO_L}^{L} + \cos\psi_{LR} \cdot y_{PO_L}^{L}) \end{pmatrix},$$

$$\begin{aligned} \begin{pmatrix} \dot{x}_{PO_L}^{R} \\ \dot{y}_{PO_L}^{R} \end{pmatrix} = &\begin{pmatrix} -\sin\psi_{LR} \cdot x_{PO_L}^{L} - \cos\psi_{LR} \cdot y_{PO_L}^{L} \\ +\cos\psi_{LR} \cdot x_{PO_L}^{L} - \sin\psi_{LR} \cdot y_{PO_L}^{L} \end{pmatrix} \cdot \dot{\psi}_{LR} \\ &+ \begin{pmatrix} \cos\psi_{LR} \cdot \dot{x}_{PO_L}^{L} - \sin\psi_{LR} \cdot \dot{y}_{PO_L}^{L} \\ \sin\psi_{LR} \cdot \dot{x}_{PO_L}^{L} + \cos\psi_{LR} \cdot \dot{y}_{PO_L}^{L} \end{pmatrix}, \end{aligned}$$

or

$$\begin{pmatrix} \dot{x}^R_{PO} \\ \dot{y}^R_{PO} \end{pmatrix} = \underbrace{\begin{pmatrix} -\sin\psi_{LR} \ , & -\cos\psi_{LR} \\ \cos\psi_{LR} \ , & -\sin\psi_{LR} \end{pmatrix} \cdot \dot{\psi}_{LR}}_{=:\ \dot{\boldsymbol{A}}^{RL}} \cdot \underbrace{\begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \end{pmatrix}}_{=:\ \boldsymbol{r}^L_{PO_L}}$$

$$+ \underbrace{\begin{pmatrix} \cos\psi_{LR} \ , & -\sin\psi_{LR} \\ \sin\psi_{LR} \ , & \cos\psi_{LR} \end{pmatrix}}_{=:\ \boldsymbol{A}^{RL}} \cdot \underbrace{\begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \end{pmatrix}}_{=:\ {}^L\dot{\boldsymbol{r}}^L_{PO_L}},$$

and finally

$${}^R\dot{\boldsymbol{r}}^R_{PO_L} = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} = \boldsymbol{A}^{RL} \underbrace{\left(\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + {}^L\dot{\boldsymbol{r}}^L_{PO_L}\right)}_{= {}^R\dot{\boldsymbol{r}}^L_{PO_L}}. \tag{A.1.29c}$$

The relations (A.1.29a), (A.1.11b), and (A.1.29c) imply

$$\begin{aligned} {}^R\dot{\boldsymbol{r}}^R_{PO_L} &= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^R_{PO_L}\right) = \boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L} = \boldsymbol{A}^{RL} \cdot (\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} + {}^L\dot{\boldsymbol{r}}^L_{PO_L}) \\ &= \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L}. \end{aligned} \tag{A.1.29d}$$

This *proves (A.1.11f).* □

On the other hand (A.1.29d) also implies

$$\boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L} \neq \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{r}}^L_{PO_L} \quad \text{for} \quad \boldsymbol{A}^{RL} \neq \text{constant}. \tag{A.1.30}$$

□

A.1.5 Time derivatives of spatial orientation matrices and of spatial vectors in different frames

In this section the following *spatial* relations will be proved:

$${}^R\dot{\boldsymbol{r}}^L_{PO_L} := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L}, \tag{A.1.31a}$$

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} = \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR}, \tag{A.1.31b}$$

$${}^R\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}, \tag{A.1.31c}$$

$${}^R\dot{\boldsymbol{r}}^R_{PO_L} = \boldsymbol{A}^{RL} \cdot \underbrace{\left(\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L} + {}^L\dot{\boldsymbol{r}}^L_{PO_L}\right)}_{{}^R\dot{\boldsymbol{r}}^L_{PO_L}} = \boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L}, \tag{A.1.31d}$$

$$^{R}\dot{\boldsymbol{r}}^{L}_{PO_L} = \boldsymbol{A}^{LR} \cdot {}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L}, \tag{A.1.31e}$$

and

$$^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^{L}_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} \tag{A.1.31f}$$

except

$$^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} \neq \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} \quad \text{for} \quad \boldsymbol{\omega}^{L}_{LR} \neq \boldsymbol{0}. \tag{A.1.31g}$$

The *formal vector* $\boldsymbol{\omega}^{L}_{LR}$ introduced here does *not yet have any physical meaning.* A *physical interpretation of* $\boldsymbol{\omega}^{L}_{LR}$ is introduced in Section 2.2.2.3.

Proof of (A.1.31):

Consider the time derivative of a vector $\boldsymbol{r}^{L}_{PO_L}$ represented in frame L and differentiated with respect to time t in frame R. Then

$$\begin{aligned} {}^{R}\dot{\boldsymbol{r}}_{PO_L} &:= \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) \qquad\qquad \text{(A.1.32a)} \\ &= \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t)\cdot\boldsymbol{e}_{xL}(t) + y^{L}_{PO_L}(t)\cdot\boldsymbol{e}_{yL}(t) + z^{L}_{PO_L}(t)\cdot\boldsymbol{e}_{zL}(t)\right), \end{aligned}$$

with the *basis vectors* $\boldsymbol{e}_{iL}$ of L $(i = x, y, z)$ *that are not constant in time* when measured in frame R. They may rotate with respect to frame R. Applying the product rule of differentiation to (A.1.32a) yields

$$\begin{aligned} {}^{R}\dot{\boldsymbol{r}}_{PO_L}(t) = & \left(\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}x^{L}_{PO_L}(t)\right)\cdot\underline{\boldsymbol{e}}_{xL} + x^{L}_{PO_L}(t)\cdot\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{xL}(t) \\ & + \left(\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}y^{L}_{PO_L}(t)\right)\cdot\underline{\boldsymbol{e}}_{yL} + y^{L}_{PO_L}(t)\cdot\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{yL}(t) \\ & + \left(\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}z^{L}_{PO_L}(t)\right)\cdot\underline{\boldsymbol{e}}_{zL} + z^{L}_{PO_L}(t)\cdot\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{zL}(t). \end{aligned} \tag{A.1.32b}$$

The basis vectors $\boldsymbol{e}_{iL}$ of L, marked by underlined letters in (A.1.32b), are *kept constant* with respect to frame R by definition of the product rule of differentiation. The time derivatives of the *scalar functions* x_{PO_L}, y_{PO_L}, and z_{PO_L} are

$$\begin{aligned} \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t)\right) &= \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(x^{L}_{PO_L}(t)\right) =: \dot{x}^{L}_{PO_L}(t), \\ \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(y^{L}_{PO_L}(t)\right) &= \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(y^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(y^{L}_{PO_L}(t)\right) =: \dot{y}^{L}_{PO_L}(t) \end{aligned}$$

and (A.1.32c)

$$\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(z^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(z^{L}_{PO_L}(t)\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(z^{L}_{PO_L}(t)\right) =: \dot{z}^{L}_{PO_L}(t).$$

This implies

$$
\begin{aligned}
{}^R\dot{\boldsymbol{r}}_{PO_L}(t) &= \dot{x}^L_{PO_L}(t)\cdot e_{xL} + \dot{y}^L_{PO_L}(t)\cdot e_{yL} + \dot{z}^L_{PO_L}(t)\cdot e_{zL} \qquad \text{(A.1.32d)}\\
&+ x^L_{PO_L}(t)\cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\overline{\boldsymbol{e}}_{xL}(t)\right) + y^L_{PO_L}(t)\cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\overline{\boldsymbol{e}}_{yL}(t)\right) + z^L_{PO_L}(t)\cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\overline{\boldsymbol{e}}_{zL}(t)\right),
\end{aligned}
$$

where the *overlined basis vectors* in (A.1.32d) may change with respect to frame R and are therefore *no longer* treated as *constants*. Representation of a basis vector $\boldsymbol{e}_{iL}$ of L in frame R according to (2.46) yields, together with

$$
\boldsymbol{e}_{iL}(t) = \ell_i\cdot\boldsymbol{e}_{x_R} + m_i\cdot\boldsymbol{e}_{y_R} + n_i\cdot\boldsymbol{e}_{z_R} \quad \text{or} \quad \boldsymbol{e}^R_{iL}(t) = (\ell_i\,,\, m_i\,,\, n_i)^{\mathrm{T}} \qquad \text{(A.1.32e)}
$$

and

$$
\ell_i = |\boldsymbol{e}_{iL}|\cdot|\boldsymbol{e}_{xR}|\cdot\cos\alpha = \cos\alpha \ , \ \ \alpha = \text{angle between } \boldsymbol{e}_{iL} \text{ and } \boldsymbol{e}_{iR}, \qquad \text{(A.1.32f)}
$$

the relations

$$
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{iL}(t)\right) = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\ell_i\cdot\boldsymbol{e}_{xR} + m_i\cdot\boldsymbol{e}_{yR} + n_i\cdot\boldsymbol{e}_{zR}\right)
$$

and

$$
\begin{aligned}
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{iL}(t)\right) = \quad & \dot{\ell}_i\cdot\boldsymbol{e}_{xR} + \dot{m}_i\cdot\boldsymbol{e}_{yR} + \dot{n}_i\cdot\boldsymbol{e}_{zR} \qquad \text{(A.1.32g)}\\
&+ \ell_i\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{xR}\right) + m_i\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{yR}\right) + n_i\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{zR}\right).
\end{aligned}
$$

As the basic vectors $\boldsymbol{e}_{iR}$ of (A.1.32g) *are constant in frame* R, the relations

$$
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\boldsymbol{e}_{jR} \equiv \boldsymbol{0}, \quad \text{for} \quad j = x\,,\, y\,,\, z \qquad \text{(A.1.32h)}
$$

imply

$$
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}_{iL}(t)\right) = \dot{\ell}_i\cdot\boldsymbol{e}_{xR} + \dot{m}_i\cdot\boldsymbol{e}_{yR} + \dot{n}_i\cdot\boldsymbol{e}_{zR}, \quad i = x\,,\, y\,,\, z, \qquad \text{(A.1.32i)}
$$

with the time derivatives of the scalar functions $\ell_i\,,\, m_i\,,\, n_i$ defined as

$$
\dot{\ell}_i := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\ell_i\right) = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\ell_i\right) = \frac{{}^R\mathrm{d}}{\mathrm{d}t}\ell_i, \quad \text{etc.} \qquad \text{(A.1.32j)}
$$

Then (A.1.32b) can be written as

$$
\begin{aligned}
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{PO_L}(t)\right) &= \dot{x}^L_{PO_L}\cdot\boldsymbol{e}_{xL} + \dot{y}^L_{PO_L}\cdot\boldsymbol{e}_{yL} + \dot{z}^L_{PO_L}\cdot\boldsymbol{e}_{zL} \qquad \text{(A.1.33a)}\\
&+ x^L_{PO_L}\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}^R_{xL}(t)\right) + y^L_{PO_L}\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}^R_{yL}(t)\right) + z^L_{PO_L}\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{e}^R_{zL}(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL} + \dot{z}^L_{PO_L} \cdot \boldsymbol{e}_{zL}\right) \\
&\quad + x^L_{PO_L} \cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\ell_x \cdot \boldsymbol{e}_{xR} + m_x \cdot \boldsymbol{e}_{yR} + n_x \cdot \boldsymbol{e}_{zR}\right) \\
&\quad + y^L_{PO_L} \cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\ell_y \cdot \boldsymbol{e}_{xR} + m_y \cdot \boldsymbol{e}_{yR} + n_y \cdot \boldsymbol{e}_{zR}\right) \\
&\quad + z^L_{PO_L} \cdot \frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\ell_z \cdot \boldsymbol{e}_{xR} + m_z \cdot \boldsymbol{e}_{yR} + n_z \cdot \boldsymbol{e}_{zR}\right).
\end{aligned}
$$

This implies, together with (A.1.32i), the relation

$$
\begin{aligned}
\frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}_{PO_L}(t)\right) = \; & \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL} + \dot{z}^L_{PO_L} \cdot \boldsymbol{e}_{zL}\right) \\
& + \left(x^L_{PO_L} \cdot \dot{\ell}_x + y^L_{PO_L} \cdot \dot{\ell}_y + z^L_{PO_L} \cdot \dot{\ell}_z\right) \cdot \boldsymbol{e}_{xR} \\
& + \left(x^L_{PO_L} \cdot \dot{m}_x + y^L_{PO_L} \cdot \dot{m}_y + z^L_{PO_L} \cdot \dot{m}_z\right) \cdot \boldsymbol{e}_{yR} \\
& + \left(x^L_{PO_L} \cdot \dot{n}_x + y^L_{PO_L} \cdot \dot{n}_y + z^L_{PO_L} \cdot \dot{n}_z\right) \cdot \boldsymbol{e}_{zR}.
\end{aligned}
\tag{A.1.33b}
$$

Representing the basis vectors $\boldsymbol{e}_{xR}$, $\boldsymbol{e}_{yR}$, and $\boldsymbol{e}_{zR}$ by means of (2.47a) in frame L,

$$
\boldsymbol{e}_{xR} = \sum_{i=x,y,z} \ell_i \cdot \boldsymbol{e}_{iL} \quad , \quad \boldsymbol{e}_{yR} = \sum_{i=x,y,z} m_i \cdot \boldsymbol{e}_{iL} \quad , \quad \boldsymbol{e}_{zR} = \sum_{i=x,y,z} n_i \cdot \boldsymbol{e}_{iL}
$$

yields the relations

$$
\begin{aligned}
&\frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}_{PO_L}(t)\right) = \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL} + \dot{z}^L_{PO_L} \cdot \boldsymbol{e}_{zL}\right) \\
&\quad + \left(x^L_{PO_L} \cdot \dot{\ell}_x + y^L_{PO_L} \cdot \dot{\ell}_y + z^L_{PO_L} \cdot \dot{\ell}_z\right) \cdot \left(\ell_x \cdot \boldsymbol{e}_{xL} + \ell_y \cdot \boldsymbol{e}_{yL} + \ell_z \cdot \boldsymbol{e}_{zL}\right) \\
&\quad + \left(x^L_{PO_L} \cdot \dot{m}_x + y^L_{PO_L} \cdot \dot{m}_y + z^L_{PO_L} \cdot \dot{m}_z\right) \cdot \left(m_x \cdot \boldsymbol{e}_{xL} + m_y \cdot \boldsymbol{e}_{yL} + m_z \cdot \boldsymbol{e}_{zL}\right) \\
&\quad + \left(x^L_{PO_L} \cdot \dot{n}_x + y^L_{PO_L} \cdot \dot{n}_y + z^L_{PO_L} \cdot \dot{n}_z\right) \cdot \left(n_x \cdot \boldsymbol{e}_{xL} + n_y \cdot \boldsymbol{e}_{yL} + n_z \cdot \boldsymbol{e}_{zL}\right)
\end{aligned}
\tag{A.1.34a}
$$

or

$$
\begin{aligned}
\frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}_{PO_L}(t)\right) = \; & \left(\dot{x}^L_{PO_L} \cdot \boldsymbol{e}_{xL} + \dot{y}^L_{PO_L} \cdot \boldsymbol{e}_{yL} + \dot{z}^L_{PO_L} \cdot \boldsymbol{e}_{zL}\right) \\
& + \Big(\dot{\ell}_x \cdot \ell_x \cdot x^L_{PO_L} + \dot{\ell}_y \cdot \ell_x \cdot y^L_{PO_L} + \dot{\ell}_z \cdot \ell_x \cdot z^L_{PO_L} \\
& \quad + \dot{m}_x \cdot m_x \cdot x^L_{PO_L} + \dot{m}_y \cdot m_x \cdot y^L_{PO_L} + \dot{m}_z \cdot m_x \cdot z^L_{PO_L} \\
& \quad + \dot{n}_x \cdot n_x \cdot x^L_{PO_L} + \dot{n}_y \cdot n_x \cdot y^L_{PO_L} + \dot{n}_z \cdot n_x \cdot z^L_{PO_L}\Big) \cdot \boldsymbol{e}_{xL}
\end{aligned}
\tag{A.1.34b}
$$

$$
\begin{aligned}
&+ \Big(\dot{\ell}_x \cdot \ell_y \cdot x^L_{PO_L} + \dot{\ell}_y \cdot \ell_y \cdot y^L_{PO_L} + \dot{\ell}_z \cdot \ell_y \cdot z^L_{PO_L} \\
&\quad + \dot{m}_x \cdot m_y \cdot x^L_{PO_L} + \dot{m}_y \cdot m_y \cdot y^L_{PO_L} + \dot{m}_z \cdot m_y \cdot z^L_{PO_L} \\
&\quad + \dot{n}_x \cdot n_y \cdot x^L_{PO_L} + \dot{n}_y \cdot n_y \cdot y^L_{PO_L} + \dot{n}_z \cdot n_y \cdot z^L_{PO_L} \Big) \cdot \boldsymbol{e}_{yL} \\
&+ \Big(\dot{\ell}_x \cdot \ell_z \cdot x^L_{PO_L} + \dot{\ell}_y \cdot \ell_z \cdot y^L_{PO_L} + \dot{\ell}_z \cdot \ell_z \cdot z^L_{PO_L} \\
&\quad + \dot{m}_x \cdot m_z \cdot x^L_{PO_L} + \dot{m}_y \cdot m_z \cdot y^L_{PO_L} + \dot{m}_z \cdot m_z \cdot z^L_{PO_L} \\
&\quad + \dot{n}_x \cdot n_z \cdot x^L_{PO_L} + \dot{n}_y \cdot n_z \cdot y^L_{PO_L} + \dot{n}_z \cdot n_z \cdot z^L_{PO_L} \Big) \cdot \boldsymbol{e}_{zL}
\end{aligned}
$$

or

$$
\frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}^L_{PO_L}(t) \right) \tag{A.1.34c}
$$

$$
= \begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \\ \dot{z}^L_{PO_L} \end{pmatrix} + \begin{pmatrix}
(\dot{\ell}_x \cdot \ell_x + \dot{m}_x \cdot m_x + \dot{n}_x \cdot n_x) \cdot x^L_{PO_L} \\
(\dot{\ell}_x \cdot \ell_y + \dot{m}_x \cdot m_y + \dot{n}_x \cdot n_y) \cdot x^L_{PO_L} \\
(\dot{\ell}_x \cdot \ell_z + \dot{m}_x \cdot m_z + \dot{n}_x \cdot n_z) \cdot x^L_{PO_L} \\
+ (\dot{\ell}_y \cdot \ell_x + \dot{m}_y \cdot m_x + \dot{n}_y \cdot n_x) \cdot y^L_{PO_L} \\
+ (\dot{\ell}_y \cdot \ell_y + \dot{m}_y \cdot m_y + \dot{n}_y \cdot n_y) \cdot y^L_{PO_L} \\
+ (\dot{\ell}_y \cdot \ell_z + \dot{m}_y \cdot m_z + \dot{n}_y \cdot n_z) \cdot y^L_{PO_L} \\
+ (\dot{\ell}_z \cdot \ell_x + \dot{m}_z \cdot m_x + \dot{n}_z \cdot n_x) \cdot z^L_{PO_L} \\
+ (\dot{\ell}_z \cdot \ell_y + \dot{m}_z \cdot m_y + \dot{n}_z \cdot n_y) \cdot z^L_{PO_L} \\
+ (\dot{\ell}_z \cdot \ell_z + \dot{m}_z \cdot m_z + \dot{n}_z \cdot n_z) \cdot z^L_{PO_L}
\end{pmatrix}
$$

$$
= \begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \\ \dot{z}^L_{PO_L} \end{pmatrix} + \begin{pmatrix}
\dot{\ell}_x \cdot \ell_x + \dot{m}_x \cdot m_x + \dot{n}_x \cdot n_x , \\
\dot{\ell}_x \cdot \ell_y + \dot{m}_x \cdot m_y + \dot{n}_x \cdot n_y , \\
\dot{\ell}_x \cdot \ell_z + \dot{m}_x \cdot m_z + \dot{n}_x \cdot n_z , \\
\dot{\ell}_y \cdot \ell_x + \dot{m}_y \cdot m_x + \dot{n}_y \cdot n_x , \\
\dot{\ell}_y \cdot \ell_y + \dot{m}_y \cdot m_y + \dot{n}_y \cdot n_y , \\
\dot{\ell}_y \cdot \ell_z + \dot{m}_y \cdot m_z + \dot{n}_y \cdot n_z , \\
\dot{\ell}_z \cdot \ell_x + \dot{m}_z \cdot m_x + \dot{n}_z \cdot n_x \\
\dot{\ell}_z \cdot \ell_y + \dot{m}_z \cdot m_y + \dot{n}_z \cdot n_y \\
\dot{\ell}_z \cdot \ell_z + \dot{m}_z \cdot m_z + \dot{n}_z \cdot n_z
\end{pmatrix} \cdot \begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \\ z^L_{PO_L} \end{pmatrix},
$$

or, together with (2.53a),

$$
\frac{{}^R\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{r}^L_{PO_L}(t) \right) \tag{A.1.34d}
$$

$$= \underbrace{\begin{pmatrix} \dot{x}^L_{PO_L} \\ \dot{y}^L_{PO_L} \\ \dot{z}^L_{PO_L} \end{pmatrix}}_{=: {}^L\dot{\boldsymbol{r}}^L_{PO_L}} + \underbrace{\begin{pmatrix} \ell_x \ , \ m_x \ , \ n_x \\ \ell_y \ , \ m_y \ , \ n_y \\ \ell_z \ , \ m_z \ , \ n_z \end{pmatrix}}_{=: \boldsymbol{A}^{LR}} \cdot \underbrace{\begin{pmatrix} \dot{\ell}_x \ , \ \dot{\ell}_y \ , \ \dot{\ell}_z \\ \dot{m}_x \ , \ \dot{m}_y \ , \ \dot{m}_z \\ \dot{n}_x \ , \ \dot{n}_y \ , \ \dot{n}_z \end{pmatrix}}_{=: \dot{\boldsymbol{A}}^{RL}} \cdot \underbrace{\begin{pmatrix} x^L_{PO_L} \\ y^L_{PO_L} \\ z^L_{PO_L} \end{pmatrix}}_{=: \boldsymbol{r}^L_{PO_L}}$$

and finally with

$$\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} = \left(\dot{x}^L_{PO_L}, \dot{y}^L_{PO_L}, \dot{z}^L_{PO_L}\right)^{\mathrm{T}}, \tag{A.1.34e}$$

the relation

$$\begin{gathered} {}^R\dot{\boldsymbol{r}}^L_{PO_L} := \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) = \frac{{}^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) + \boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} \\ \text{or} \\ {}^R\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^L_{PO_L} . \end{gathered} \tag{A.1.35}$$

This proves (A.1.31a). □

Differentiation of the *orthogonality relation*

$$\boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL} = \mathbf{I}_3 \tag{A.1.36}$$

with respect to time yields the relations

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} + \dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL} = \mathbf{0} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} = -\boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL} \tag{A.1.37}$$

or

$$\begin{aligned} \boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} &= -\dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL} \\ &= -\left(\left(\boldsymbol{A}^{RL}\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{A}}^{LR}\right)^{\mathrm{T}}\right)^{\mathrm{T}} = -\left(\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL}\right)^{\mathrm{T}} . \end{aligned}$$

This proves that the matrices

$$\left(\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL}\right) \quad \text{and} \quad \left(\dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL}\right)$$

are *skew-symmetric*; i.e., that

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} = -\left(\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL}\right)^{\mathrm{T}} \quad \text{and} \quad \dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL} = -\left(\dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{A}^{RL}\right)^{\mathrm{T}} .$$

Each *skew-symmetric* matrix $\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL}$ can be generated by a *formal vector*

$$\boldsymbol{\omega}^L_{LR} = \left(\omega^L_{xLR}, \omega^L_{yLR}, \omega^L_{zLR}\right)^{\mathrm{T}}, \tag{A.1.38}$$

represented in frame L according to the relation

$$\boldsymbol{A}^{LR} \cdot \dot{\boldsymbol{A}}^{RL} = \tilde{\boldsymbol{\omega}}^L_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \tag{A.1.39a}$$

with the *skew-symmetric matrix*

$$\tilde{\boldsymbol{\omega}}^L_{LR} := \begin{pmatrix} 0 & , -\omega^L_{zLR} & , \quad \omega^L_{yLR} \\ \omega^L_{zLR} & , \quad 0 & , -\omega^L_{xLR} \\ -\omega^L_{yLR} & , \quad \omega^L_{xLR} & , \quad 0 \end{pmatrix} =: -\tilde{\boldsymbol{\omega}}^L_{RL}. \tag{A.1.39b}$$

This proves (A.1.31b). □

Inserting (A.1.39a) into (A.1.35) yields the relation

$$^R\dot{\boldsymbol{r}}^L_{PO_L} = {}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}. \tag{A.1.40a}$$

This proves (A.1.31c). □

Interchanging the indices R and L in (A.1.40a) yields

$$^L\dot{\boldsymbol{r}}^R_{PO_L} = {}^R\dot{\boldsymbol{r}}^R_{PO_L} + \tilde{\boldsymbol{\omega}}^R_{RL} \cdot \boldsymbol{r}^R_{PO_L} = {}^R\dot{\boldsymbol{r}}^R_{PO_L} - \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}. \tag{A.1.40b}$$

Using the similarity transformation of matrices

$$\boldsymbol{A}^{LR} \cdot \boldsymbol{B}^R \left(\boldsymbol{A}^{LR}\right)^{-1} = \boldsymbol{B}^L \quad ; \quad \boldsymbol{B}\,,\ \boldsymbol{A}^{LR} \in \mathbb{R}^{3,3} \tag{A.1.41a}$$

and the *orthogonality property*

$$\left(\boldsymbol{A}^{LR}\right)^{-1} = \left(\boldsymbol{A}^{LR}\right)^{\mathrm{T}} = \boldsymbol{A}^{RL}$$

yields (compare also Equation 2.41c of Section 2.2.2)

$$\boldsymbol{B}^L = \boldsymbol{A}^{LR} \cdot \boldsymbol{B}^R \cdot \boldsymbol{A}^{RL} \quad \text{and} \quad \tilde{\boldsymbol{\omega}}^R_{LR} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}. \tag{A.1.41b}$$

Interchanging the indices R and L in (A.1.39a) yields

$$\boldsymbol{A}^{RL} \cdot \dot{\boldsymbol{A}}^{LR} = \tilde{\boldsymbol{\omega}}^R_{RL} = -\tilde{\boldsymbol{\omega}}^R_{LR} \quad \text{or} \quad \dot{\boldsymbol{A}}^{LR} = \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL} = -\boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{LR}$$

together with (A.1.41b),

$$\dot{\boldsymbol{A}}^{LR} = \boldsymbol{A}^{LR} \cdot \tilde{\boldsymbol{\omega}}^R_{RL} \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{A}^{LR} = \tilde{\boldsymbol{\omega}}^L_{RL} \cdot \boldsymbol{A}^{LR} = -\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{A}^{LR}. \tag{A.1.41c}$$

Multiplication of (A.1.40a) from the the left-hand side by $\boldsymbol{A}^{RL}$ yields

$$\boldsymbol{A}^{RL} \cdot {}^R\dot{\boldsymbol{r}}^L_{PO_L} = \boldsymbol{A}^{RL} \left({}^L\dot{\boldsymbol{r}}^L_{PO_L} + \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{PO_L}\right). \tag{A.1.42}$$

On the other hand,

$$\frac{^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right) = {}^L\dot{\boldsymbol{r}}^L_{PO_L} = \frac{^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{LR} \cdot \boldsymbol{r}^R_{PO_L}\right).$$

Taking into account *Bryant angles* yields, together with (cf. Equation 2.56)

$$\boldsymbol{A}^{LR} = \begin{pmatrix} \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{c}\,\varphi \cdot \mathrm{s}\,\psi + \mathrm{s}\,\varphi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{s}\,\varphi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\varphi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \\ -\mathrm{c}\,\theta \cdot s\psi & , \mathrm{c}\,\varphi \cdot \mathrm{c}\,\psi - s\varphi \cdot s\theta \cdot s\psi & , s\varphi \cdot \mathrm{c}\,\psi + \mathrm{c}\,\varphi \cdot s\theta \cdot s\psi \\ s\theta & , -s\varphi \cdot \mathrm{c}\,\theta & , \mathrm{c}\,\varphi \cdot \mathrm{c}\,\theta \end{pmatrix}, \tag{A.1.43a}$$

and (cf. Equation 2.70)

$$\dot{\boldsymbol{A}}^{LR} = \tag{A.1.43b}$$

$$\left(\begin{matrix} -\mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\theta} - \mathrm{c}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\psi} & , \\ \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\theta} - \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\psi} & , \\ \mathrm{c}\,\theta \cdot \dot{\theta} & , \end{matrix} \right.$$

$$\begin{matrix} \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\phi} + \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\psi} + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\phi} + \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\theta} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\psi} & , \\ \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\phi} - \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\psi} - \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\phi} - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\theta} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\psi} & , \\ -\mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot{\phi} + \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \dot{\theta} & , \end{matrix}$$

$$\left. \begin{matrix} +\mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\phi} + \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\psi} + \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\phi} - \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\theta} + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\psi} \\ +\mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\phi} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\psi} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\phi} + \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\theta} + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\psi} \\ -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot{\phi} - \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \dot{\theta} \end{matrix} \right),$$

the relation

$$\frac{{}^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) =: \frac{{}^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{LR} \cdot \boldsymbol{r}^R_{PO_L}\right) = \tag{A.1.43c}$$

$$\frac{{}^L\mathrm{d}}{\mathrm{d}t}\left[\begin{pmatrix} \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi + \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi & , \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi - \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \\ -\mathrm{c}\,\theta \cdot \mathrm{s}\,\psi & , \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi & , \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \\ \mathrm{s}\,\theta & , -\mathrm{s}\,\phi \cdot \mathrm{c}\,\theta & , \mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \end{pmatrix} \cdot \begin{pmatrix} x^R_{PO_L} \\ y^R_{PO_L} \\ z^R_{PO_L} \end{pmatrix}\right]$$

$$= \left(\begin{matrix} -\mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\theta} - \mathrm{c}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\psi} & , \\ \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\theta} - \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\psi} & , \\ \mathrm{c}\,\theta \cdot \dot{\theta} & , \end{matrix} \right. \tag{A.1.43d}$$

$$\begin{matrix} \mathrm{s}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\phi} + \mathrm{c}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\psi} + \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\phi} + \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\theta} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\psi} & , \\ \mathrm{s}\,\phi \cdot \mathrm{c}\,\psi \cdot \dot{\phi} - \mathrm{c}\,\phi \cdot \mathrm{s}\,\psi \cdot \dot{\psi} - \mathrm{c}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\phi} - \mathrm{s}\,\phi \cdot \mathrm{c}\,\theta \cdot \mathrm{s}\,\psi \cdot \dot{\theta} - \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \mathrm{c}\,\psi \cdot \dot{\psi} & , \\ -\mathrm{c}\,\phi \cdot \mathrm{c}\,\theta \cdot \dot{\phi} + \mathrm{s}\,\phi \cdot \mathrm{s}\,\theta \cdot \dot{\theta} & , \end{matrix}$$

$$\underbrace{\begin{matrix} +\mathrm{c}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot{\phi}+\mathrm{s}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot{\psi}+\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot{\phi}-\mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot{\theta}+\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot{\psi} \\ +\mathrm{c}\,\phi\cdot\mathrm{c}\,\psi\cdot\dot{\phi}-\mathrm{s}\,\phi\cdot\mathrm{s}\,\psi\cdot\dot{\psi}-\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot{\phi}+\mathrm{c}\,\phi\cdot\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi\cdot\dot{\theta}+\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi\cdot\dot{\psi} \\ -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta\cdot\dot{\phi}-\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\dot{\theta} \end{matrix}\Bigg)}_{=\dot{\boldsymbol{A}}^{LR}}$$

$$\cdot\underbrace{\begin{pmatrix} x^R_{PO_L} \\ y^R_{PO_L} \\ z^R_{PO_L} \end{pmatrix}}_{=\boldsymbol{r}^R_{PO_L}}$$

$$+\underbrace{\begin{pmatrix} \mathrm{c}\,\theta\cdot\mathrm{c}\,\psi & , \mathrm{c}\,\phi\cdot\mathrm{s}\,\psi+\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi & , \mathrm{s}\,\phi\cdot\mathrm{s}\,\psi-\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \\ -\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi & , \mathrm{c}\,\phi\cdot\mathrm{c}\,\psi-\mathrm{s}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi & , \mathrm{s}\,\phi\cdot\mathrm{c}\,\psi+\mathrm{c}\,\phi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \\ \mathrm{s}\,\theta & , -\mathrm{s}\,\phi\cdot\mathrm{c}\,\theta & , \mathrm{c}\,\phi\cdot\mathrm{c}\,\theta \end{pmatrix}}_{=\boldsymbol{A}^{LR}}\cdot\underbrace{\begin{pmatrix} \dot{x}^R_{PO_L} \\ \dot{y}^R_{PO_L} \\ \dot{z}^R_{PO_L} \end{pmatrix}}_{={}^R\dot{\boldsymbol{r}}^R_{PO_L}}$$

and the relation

$$\begin{aligned} {}^L\dot{\boldsymbol{r}}^L_{PO_L} &= \frac{{}^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}\right) \\ &= \dot{\boldsymbol{A}}^{LR}\cdot\boldsymbol{r}^R_{PO_L}+\boldsymbol{A}^{LR}\cdot\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^R_{PO_L}\right)=\dot{\boldsymbol{A}}^{LR}\cdot\boldsymbol{r}^R_{PO_L}+\boldsymbol{A}^{LR}\cdot{}^R\dot{\boldsymbol{r}}^R_{PO_L} \end{aligned}$$

or

$$\boldsymbol{A}^{RL}\cdot\frac{{}^L\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^L_{PO_L}(t)\right)=\boldsymbol{A}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L}=\underbrace{\boldsymbol{A}^{RL}\cdot\dot{\boldsymbol{A}}^{LR}\cdot\boldsymbol{r}^R_{PO_L}+{}^R\dot{\boldsymbol{r}}^R_{PO_L}}_{=:\frac{{}^L\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r}^R_{PO_L})={}^L\dot{\boldsymbol{r}}^R_{PO_L}}, \tag{A.1.44}$$

and finally, together with (A.1.40b), the relations

$${}^L\dot{\boldsymbol{r}}^R_{PO_L}=\boldsymbol{A}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L} \quad \text{or} \quad {}^L\dot{\boldsymbol{r}}^L_{PO_L}=\boldsymbol{A}^{LR}\cdot{}^L\dot{\boldsymbol{r}}^R_{PO_L}, \tag{A.1.45}$$

and

$${}^R\dot{\boldsymbol{r}}^L_{PO_L}=\boldsymbol{A}^{LR}\cdot{}^R\dot{\boldsymbol{r}}^R_{PO_L} \quad \text{or} \quad {}^R\dot{\boldsymbol{r}}^R_{PO_L}=\boldsymbol{A}^{RL}\cdot{}^R\dot{\boldsymbol{r}}^L_{PO_L}. \tag{A.1.46}$$

This proves (A.1.31d) and (A.1.31e). □

Inserting (A.1.31c) into (A.1.31d) yields

$${}^R\dot{\boldsymbol{r}}^R_{PO_L}=\boldsymbol{A}^{RL}\cdot{}^R\dot{\boldsymbol{r}}^L_{PO_L}\underset{(\text{A.1.31c})}{=}\boldsymbol{A}^{RL}\cdot{}^L\dot{\boldsymbol{r}}^L_{PO_L}+\boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L} \tag{A.1.47}$$

or

$${}^R\dot{\boldsymbol{r}}^R_{PO_L}=\boldsymbol{A}^{RL}\cdot\left(\tilde{\boldsymbol{\omega}}^L_{LR}\cdot\boldsymbol{r}^L_{PO_L}+{}^L\dot{\boldsymbol{r}}^L_{PO_L}\right), \tag{A.1.48}$$

and, together with (A.1.39a), the relation

$$ {}^{R}\dot{\boldsymbol{r}}^{R}_{PO_L} = \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{r}^{L}_{PO_L} + \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L} . \tag{A.1.49} $$

This proves (A.1.31f). □

Then

$$ \frac{{}^{R}\mathrm{d}}{\mathrm{d}t} \boldsymbol{r}^{R}_{PO_L}(t) \neq \boldsymbol{A}^{RL} \cdot \frac{{}^{L}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}^{L}_{PO_L}(t)\right) = \boldsymbol{A}^{RL} \cdot {}^{L}\dot{\boldsymbol{r}}^{L}_{PO_L}(t) \quad \text{for} \quad \boldsymbol{\omega}^{L}_{LR} \neq \mathbf{0}. \tag{A.1.31g} $$

This proves (A.1.31g). □

A.1.6 Derivatives of vector functions

In the kinematics and dynamics of mechanical systems, vectors representing the positions of points on bodies, or equations describing the geometry of the dynamics of the motion are often functions of several variables. When analyzing these equations, partial derivatives of vectors and functions with respect to these variables occur. These situations can be efficiently handled by *vector* and *matrix calculus* notation.

Gradient of a mapping

Let

$$ \boldsymbol{p} \in \mathbb{R}^n $$

and

$$ \begin{array}{cccl} g: \mathbb{R}^n & \rightarrow & \mathbb{R}^1 & , \quad g \in \mathcal{C}^1(\mathbb{R}^n) \quad (g \text{ is a differentiable function}) \\ \Cup & & \Cup & \\ \boldsymbol{p} & \mapsto & g(\boldsymbol{p}) = g(p_1, \ldots, p_n). & \end{array} $$

Then $\boldsymbol{g}_p := \partial g / \partial \boldsymbol{p}$ is a row vector (called the *gradient* of g):

$$ \boldsymbol{g}_p = (\partial g/\partial p_1, \ldots, \partial g/\partial p_n) . $$

Example A.1.1 (Gradient of a mapping): Let

$$ \boldsymbol{p} = (x_1, x_2)^{\mathrm{T}} \quad , \quad g := \mathbb{R}^2 \rightarrow \mathbb{R}^1 \quad \text{and} \quad g = x_1^2 + 3x_2x_1 . \tag{A.1.50a} $$

Then

$$ \boldsymbol{g}_p = \left(\frac{\partial g}{\partial p_1}, \frac{\partial g}{\partial p_2}\right) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}\right) = (2 \cdot x_1 + 3 \cdot x_2 \ , \ 3 \cdot x_1) . \tag{A.1.50b} $$

The *gradient* of the *special functions*

$$\begin{array}{cccccl} g_1 : \mathbb{R}^n & \times & \mathbb{R}^n & \longrightarrow & \mathbb{R}^1 & \\ \cup\!\!\!| & & \cup\!\!\!| & & \cup\!\!\!| & \\ (\boldsymbol{y} & , & \boldsymbol{x}) & \longmapsto & g_1(\boldsymbol{y}\,,\,\boldsymbol{x}) := \boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x} \quad , \quad \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}, \end{array} \tag{A.1.51a}$$

and

$$\begin{array}{cccccl} g_2 : \mathbb{R}^n & \times & \mathbb{R}^n & \longrightarrow & \mathbb{R}^1 & \\ \cup\!\!\!| & & \cup\!\!\!| & & \cup\!\!\!| & \\ (\boldsymbol{x} & , & \boldsymbol{x}) & \longmapsto & g_2(\boldsymbol{x}\,,\,\boldsymbol{x}) := \boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x} \quad , \quad \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} \end{array} \tag{A.1.51b}$$

is

$$\boldsymbol{g}_{1x} = \left[\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right] = (\boldsymbol{A} \cdot \boldsymbol{y})^{\mathrm{T}} = \boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \tag{A.1.52}$$

and

$$\boldsymbol{g}_{2x} = \left[\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right] = 2 \cdot (\boldsymbol{A} \cdot \boldsymbol{x}) = 2 \cdot \boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A}. \tag{A.1.53}$$

Proof of (A.1.52):

$$\begin{aligned} \boldsymbol{g}_{1x} &= \left[\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right] = \left[\frac{\partial}{\partial x_1}\left(\boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\ ,\ \ldots\ ,\ \frac{\partial}{\partial x_n}\left(\boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right] \\ &= \left[\frac{\partial}{\partial x_1}\left(\sum_{i\,,\,j=1}^{n} a_{ij} \cdot y_i \cdot x_j\right)\ ,\ \ldots\ ,\ \frac{\partial}{\partial x_n}\left(\sum_{i\,,\,j=1}^{n} a_{ij} \cdot y_i \cdot x_j\right)\right] \\ &= \left[\sum_{i=1}^{n} a_{i1} y_i\ ,\ \ldots\ ,\ \sum_{i=1}^{n} a_{in} y_i\right] \\ &= \left[(a_{11}\,,\,\ldots\,,\,a_{n1}) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\ ,\ \ldots\ ,\ (a_{1n}\,,\,\ldots\,,\,a_{nn}) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}\right] \\ &= (y_1\,,\,\ldots\,,\,y_n) \cdot \begin{pmatrix} a_{11} & ,\ \ldots\ , & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & ,\ \ldots\ , & a_{nn} \end{pmatrix} = \boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A}. \end{aligned}$$

Then

$$\boldsymbol{g}_{1x}^{\mathrm{T}} = \left[\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{y}\right)\right]^{\mathrm{T}} = \left[\frac{\partial}{\partial \boldsymbol{x}}\left(\boldsymbol{y}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right]^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \cdot \boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{y}. \tag{A.1.54a}$$

$\square$

Proof of (A.1.53):

$$
\begin{aligned}
\boldsymbol{g}_{2x} &= \frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right) \\
&= \left[\frac{\partial}{\partial x_1}\left(\sum_{i\,,\,j=1}^{n} a_{ij} \cdot x_i \cdot x_j\right) \;,\; \ldots \;,\; \frac{\partial}{\partial x_n}\left(\sum_{i\,,\,j=1}^{n} a_{ij} \cdot x_i \cdot x_j\right)\right] \\
&= \left[\sum_{i=1}^{n} a_{i1} \cdot x_i + \sum_{j=1}^{n} a_{1j} \cdot x_j \;,\; \ldots \;,\; \sum_{i=1}^{n} a_{in} \cdot x_i + \sum_{j=1}^{n} a_{nj} \cdot x_j\right] \\
&= \Big[(a_{11}\,, \ldots\,, a_{n1}) \cdot \boldsymbol{x} + (a_{11}\,, \ldots\,, a_{1n}) \cdot \boldsymbol{x} \;,\; \ldots \;, \\
&\qquad\qquad (a_{1n}\,, \ldots\,, a_{nn}) \cdot \boldsymbol{x} + (a_{n1}\,, \ldots\,, a_{nn}) \cdot \boldsymbol{x}\Big] \\
&= \boldsymbol{x}^{\mathrm{T}} \cdot \begin{pmatrix} a_{11} & , \ldots , & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & , \ldots , & a_{nn} \end{pmatrix} + \boldsymbol{x}^{\mathrm{T}} \cdot \begin{pmatrix} a_{11} & , \ldots , & a_{n1} \\ \vdots & \vdots & \vdots \\ a_{1n} & , \ldots , & a_{nn} \end{pmatrix}
\end{aligned}
$$

or

$$
\begin{aligned}
\boldsymbol{g}_{2x} &= \frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right) = \boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} + \boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A}^{\mathrm{T}} \\
&= \boldsymbol{x}^{\mathrm{T}} \cdot \left(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\right) = 2 \cdot \boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A}
\end{aligned}
$$

and

$$
\boldsymbol{g}_{2x}^{\mathrm{T}} = \left[\frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right]^{\mathrm{T}} = 2 \cdot \boldsymbol{A} \cdot \boldsymbol{x}. \tag{A.1.54b}
$$

□

Jacobian matrix of a mapping

Let

$$
\boldsymbol{p} \in \mathbb{R}^n
$$

and

$$
\begin{aligned}
\boldsymbol{g} : \mathbb{R}^n &\to \mathbb{R}^m \quad , \quad \boldsymbol{g} \in \mathcal{C}^1(\mathbb{R}^n) \\
\cup\!\!\!\!\!\! \quad & \quad \cup\!\!\!\!\!\! \\
\boldsymbol{p} &\mapsto \boldsymbol{g}(\boldsymbol{p}) = (g_1(\boldsymbol{p})\,, \ldots, g_m(\boldsymbol{p}))^{\mathrm{T}} \\
&\qquad\quad = (g_1(p_1\,, \ldots\,, p_n)\,, \ldots\,, g_m(p_1\,, \ldots\,, p_n))^{\mathrm{T}}\,.
\end{aligned} \tag{A.1.55a}
$$

Then

$$\boldsymbol{g}_p := \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}} = \begin{pmatrix} \partial g_1/\partial p_1 & , \ \dots \ , & \partial g_1/\partial p_n \\ \vdots & & \vdots \\ \partial g_m/\partial p_1 & , \ \dots \ , & \partial g_m/\partial p_n \end{pmatrix} \in \mathbb{R}^{m,n} \tag{A.1.55b}$$

is called the *Jacobian (matrix)* of $\boldsymbol{g}(p)$.

Example A.1.2 (Jacobian matrix of a mapping): Let

$$\boldsymbol{p} = (x_1, x_2, x_3)^{\mathrm{T}}, \quad \boldsymbol{g} : \mathbb{R}^3 \to \mathbb{R}^2, \quad \text{and} \quad \boldsymbol{g} = (g_1, g_2)^{\mathrm{T}},$$

with

$$\begin{aligned} g_1(\boldsymbol{p}) &= x_1^3 + 2x_1x_3 + x_2x_3^2 + x_3^4 \quad \text{and} \\ g_2(\boldsymbol{p}) &= 4 + x_1^2x_2^3 + x_1^4x_3 + x_2x_3^3. \end{aligned} \tag{A.1.56a}$$

Then

$$\boldsymbol{g}_{\boldsymbol{p}} := \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}} = \begin{pmatrix} 3x_1^2 + 2x_3 & , & x_3^2 & , & 2x_1 + 2x_2x_3 + 4x_3^3 \\ 2x_1x_2^3 + 4x_1^3x_3 & , & 3x_1^2x_2^2 + x_3^3 & , & x_1^4 + 3x_2x_3^2 \end{pmatrix} \in \mathbb{R}^{2,3}. \tag{A.1.56b}$$

The following relations are easily proved by direct calculations. Let

$$\boldsymbol{p} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} : \mathbb{R}^n \to \mathbb{R}^n \quad ; \quad \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{C}^1(\mathbb{R}^n). \tag{A.1.57}$$

Then:

1. $(\boldsymbol{a} \bullet \boldsymbol{b})_p := \dfrac{\partial}{\partial \boldsymbol{p}}(\boldsymbol{a} \bullet \boldsymbol{b}) = \boldsymbol{b} \bullet \boldsymbol{a}_p + \boldsymbol{a} \bullet \boldsymbol{b}_p \in \mathbb{R}^{1,m}.$ (A.1.58a)
2. $(\boldsymbol{a} \bullet \boldsymbol{b})_p \neq \boldsymbol{b}_p \bullet \boldsymbol{a} + \boldsymbol{a} \bullet \boldsymbol{b}_p$ (a not well-defined right-hand side). (A.1.58b)
3. $\left(\tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}^R\right)_p = \dfrac{\partial}{\partial \boldsymbol{p}}\left(\tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}^R\right) = \left(\tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}_p^R - \tilde{\boldsymbol{b}} \cdot \boldsymbol{a}_p^R\right).$ (A.1.58c)

Proof of (A.1.58a):

Select an arbitrary basis of R in $\mathbb{R}^m$. Then for $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n)^{\mathrm{T}}$

$$\begin{aligned} \left[(\boldsymbol{a}^R)^{\mathrm{T}} \cdot \boldsymbol{b}^R\right]_p &= \frac{\partial}{\partial \boldsymbol{p}} \left[(\boldsymbol{a}^R)^{\mathrm{T}} \cdot \boldsymbol{b}^R\right] = \frac{\partial}{\partial \boldsymbol{p}} \left(\sum_{i=1}^m a_i^R b_i^R\right) \\ &= \left[\frac{\partial}{\partial p_1}\left(\sum_{i=1}^m a_i^R b_i^R\right) \ , \ \dots \ , \ \frac{\partial}{\partial p_n}\left(\sum_{i=1}^m a_i^R b_i^R\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \left[\sum_{i=1}^{m}\left(\frac{\partial a_i^R}{\partial p_1}\cdot b_i^R\right)\ ,\ \ldots\ ,\ \sum_{i=1}^{m}\left(\frac{\partial a_i^R}{\partial p_n}\cdot b_i^R\right)\right]\\
&\quad + \left[\sum_{i=1}^{m}\left(a_i^R\cdot\frac{\partial b_i^R}{\partial p_1}\right)\ ,\ \ldots\ ,\ \sum_{i=1}^{m}\left(a_i^R\cdot\frac{\partial b_i^R}{\partial p_n}\right)\right]\\
&= \left(b_1^R,\ldots,b_m^R\right)\cdot\begin{pmatrix}\partial a_1^R\big/\partial p_1\ ,\ \ldots\ ,\ \partial a_1^R\big/\partial p_n\\ \partial a_m^R\big/\partial p_1\ ,\ \ldots\ ,\ \partial a_m^R\big/\partial p_n\end{pmatrix}\\
&\quad + \left(a_1^R,\ldots,a_m^R\right)\cdot\begin{pmatrix}\partial b_1^R\big/\partial p_1\ ,\ \ldots\ ,\ \partial b_1^R\big/\partial p_n\\ \partial b_m^R\big/\partial p_1\ ,\ \ldots\ ,\ \partial b_m^R\big/\partial p_n\end{pmatrix}\\
&= \left(\boldsymbol{b}^R\right)^{\mathrm{T}}\cdot\boldsymbol{a}_p^R+\left(\boldsymbol{a}^R\right)^{\mathrm{T}}\cdot\boldsymbol{b}_p^R.
\end{aligned}
$$

This proves (A.1.58a). □

Example A.1.3 (Example related to Equation A.1.58a): Let

$$
\boldsymbol{a}^R := \begin{pmatrix} x_1\cdot x_2^2\\ x_1+2\cdot x_1\cdot x_2\\ x_1+x_2+2\cdot x_1^2\cdot x_2^2\end{pmatrix},\ \boldsymbol{b}^R := \begin{pmatrix} x_1\cdot x_3\\ x_2^3\cdot x_1\\ x_1^2\cdot x_3\end{pmatrix},\ \text{and } \boldsymbol{p} := \begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}. \tag{A.1.59a}
$$

Computation of $\dfrac{\partial[(\boldsymbol{a}^R)^{\mathrm{T}}\cdot\boldsymbol{b}^R]}{\partial\boldsymbol{p}}$:

$$
(\boldsymbol{a}^R)^{\mathrm{T}}\cdot\boldsymbol{b}^R = \left(x_1\cdot x_2^2\,,\ x_1+2\cdot x_1\cdot x_2\,,\ x_1+x_2+2\cdot x_1^2\cdot x_2^2\right)\cdot\begin{pmatrix} x_1\cdot x_3\\ x_2^3\cdot x_1\\ x_1^2\cdot x_3\end{pmatrix}
$$

$$
= \left(x_1^2\cdot x_2^2\cdot x_3+x_2^3\cdot x_1^2+2\cdot x_1^2\cdot x_2^4+x_1^3\cdot x_3+x_1^2\cdot x_2\cdot x_3+2\cdot x_1^4\cdot x_2^2\cdot x_3\right)
$$

and

$$
\frac{\partial[(\boldsymbol{a}^R)^{\mathrm{T}}\cdot\boldsymbol{b}^R]}{\partial\boldsymbol{p}} \tag{A.1.59b}
$$

$$
= \begin{pmatrix}
2\cdot x_1\cdot x_2^2\cdot x_3+2\cdot x_1\cdot x_2^3+4\cdot x_1\cdot x_2^4+3\cdot x_1^2\cdot x_3\\
\qquad +2\cdot x_1\cdot x_2\cdot x_3+8\cdot x_1^3\cdot x_2^2\cdot x_3\\
2\cdot x_2\cdot x_1^2\cdot x_3+3\cdot x_2^2\cdot x_1^2+8\cdot x_2^3\cdot x_1^2+x_1^2\cdot x_3\\
\qquad +4\cdot x_2\cdot x_1^4\cdot x_3\\
x_1^2\cdot x_2^2+x_1^3+x_1^2\cdot x_2+2\cdot x_1^4\cdot x_2^2
\end{pmatrix}^{\mathrm{T}}
$$

$$
= \begin{bmatrix} x_1 \cdot \left(2 \cdot x_2^2 \cdot x_3 + 2 \cdot x_2^3 + 4 \cdot x_2^4 + 3 \cdot x_1 \cdot x_3 + 2 \cdot x_2 \cdot x_3 + 8 \cdot x_1^2 \cdot x_2^2 \cdot x_3\right) \\ x_1^2 \cdot \left(2 \cdot x_2 \cdot x_3 + 3 \cdot x_2^2 + 8 \cdot x_2^3 + x_3 + 4 \cdot x_2 \cdot x_1^2 \cdot x_3\right) \\ x_1^2 \cdot \left(x_2^2 + x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2\right) \end{bmatrix}^{\mathrm{T}} .
$$

Computation of $(\boldsymbol{b}^R)^{\mathrm{T}} \cdot \dfrac{\partial \boldsymbol{a}^R}{\partial \boldsymbol{p}} + (\boldsymbol{a}^R)^{\mathrm{T}} \cdot \dfrac{\partial \boldsymbol{b}^R}{\partial \boldsymbol{p}}$:

$$
\frac{\partial \boldsymbol{a}^R}{\partial \boldsymbol{p}} = \begin{bmatrix} x_2^2 & , & 2 \cdot x_1 \cdot x_2 & , & 0 \\ (1 + 2 \cdot x_2) & , & 2 \cdot x_1 & , & 0 \\ (1 + 4 \cdot x_1 \cdot x_2^2) & , & (1 + 4 \cdot x_1^2 \cdot x_2) & , & 0 \end{bmatrix} ,
$$

$$
\frac{\partial \boldsymbol{b}^R}{\partial \boldsymbol{p}} = \begin{bmatrix} x_3 & , & 0 & , & x_1 \\ x_2^3 & , & 3 \cdot x_2^2 \cdot x_1 & , & 0 \\ 2 \cdot x_1 \cdot x_3 & , & 0 & , & x_1^2 \end{bmatrix} , \tag{A.1.59c}
$$

$$
(\boldsymbol{b}^R)^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{a}^R}{\partial \boldsymbol{p}} \tag{A.1.59d}
$$

$$
= \begin{bmatrix} x_1 \cdot x_3 \cdot x_2^2 + x_2^3 \cdot x_1 \cdot (1 + 2 \cdot x_2) + x_1^2 \cdot x_3 \cdot (1 + 4 \cdot x_1 \cdot x_2^2) \\ x_1 \cdot x_3 \cdot (2 \cdot x_1 \cdot x_2) + x_2^3 \cdot x_1 \cdot (2 \cdot x_1) + x_1^2 \cdot x_3 \cdot (1 + 4 \cdot x_1^2 \cdot x_2) \\ 0 \end{bmatrix}^{\mathrm{T}}
$$

$$
= \begin{bmatrix} x_1 \cdot \left(x_2^2 \cdot x_3 + x_2^3 \cdot (1 + 2 \cdot x_2) + x_1 \cdot x_3 \cdot (1 + 4 \cdot x_1 \cdot x_2^2)\right) \\ x_1^2 \cdot \left(2 \cdot x_2 \cdot x_3 + 2 \cdot x_2^3 + x_3 \cdot (1 + 4 \cdot x_1^2 \cdot x_2)\right) \\ 0 \end{bmatrix}^{\mathrm{T}} ,
$$

$$
(\boldsymbol{a}^R)^{\mathrm{T}} \cdot \frac{\partial \boldsymbol{b}^R}{\partial \boldsymbol{p}} \tag{A.1.59e}
$$

$$
= \begin{bmatrix} x_1 \cdot x_2^2 \cdot x_3 + x_1 \cdot (1 + 2 \cdot x_2) \cdot x_2^3 + (x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2) \cdot 2 \cdot x_1 \cdot x_3 \\ x_1 \cdot (1 + 2 \cdot x_2) \cdot 3 \cdot x_2^2 \cdot x_1 \\ x_1^2 \cdot x_2^2 + (x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2) \cdot x_1^2 \end{bmatrix}^{\mathrm{T}}
$$

$$
= \begin{bmatrix} x_1 \cdot \left(x_2^2 \cdot x_3 + x_2^3 \cdot (1 + 2 \cdot x_2) + 2 \cdot x_3 \cdot (x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2)\right) \\ x_1^2 \cdot (1 + 2 \cdot x_2) \cdot 3 \cdot x_2^2 \\ x_1^2 \cdot \left(x_2^2 + x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2\right) \end{bmatrix}^{\mathrm{T}} ,
$$

and finally

$$(\boldsymbol{b}^R)^{\mathrm{T}} \cdot \frac{\partial(\boldsymbol{a}^R)}{\partial \boldsymbol{p}} + (\boldsymbol{a}^R)^{\mathrm{T}} \cdot \frac{\partial(\boldsymbol{b}^R)}{\partial \boldsymbol{p}} = \tag{A.1.59f}$$

$$= \begin{bmatrix} x_1 \cdot \big(x_2^2 \cdot x_3 + x_2^3 \cdot (1 + 2 \cdot x_2) + x_1 \cdot x_3 \cdot (1 + 4 \cdot x_1 \cdot x_2^2) \\ \qquad + x_2^2 \cdot x_3 + x_2^3 \cdot (1 + 2 \cdot x_2) + 2 \cdot x_3 \cdot (x_1 + x_2^2 \cdot 2 \cdot x_1 \cdot x_3)\big) \\ x_1^2 \cdot \big(2 \cdot x_2 \cdot x_3 + 2 \cdot x_2^3 + x_3 \cdot (1 + 4 \cdot x_1^2 \cdot x_2) + 3 \cdot x_2^2 \cdot (1 + 2 \cdot x_1)\big) \\ x_1^2 \cdot \big(x_2^2 + x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2\big) \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} x_1 \cdot \big(2 \cdot x_2^2 \cdot x_3 + 2 \cdot x_2^3 \cdot (1 + 2 \cdot x_2) + 3 \cdot x_1 \cdot x_3 + 2 \cdot x_2 \cdot x_3 + 8 \cdot x_1^2 \cdot x_2^2 \cdot x_3\big) \\ x_1^2 \cdot \big(2 \cdot x_2 \cdot x_3 + 3 \cdot x_2^2 + 8 \cdot x_2^3 + x_3 + 4 \cdot x_1^2 \cdot x_2 \cdot x_3\big) \\ x_1^2 \cdot \big(x_2^2 + x_1 + x_2 + 2 \cdot x_1^2 \cdot x_2^2\big) \end{bmatrix}^{\mathrm{T}} .$$

Proof of A.1.58c: Let $\tilde{\boldsymbol{a}}^R := \begin{pmatrix} \tilde{\boldsymbol{a}}_1^R \\ \vdots \\ \tilde{\boldsymbol{a}}_m^R \end{pmatrix}$, $\tilde{\boldsymbol{a}}_i^R = i$th row vector of $\tilde{\boldsymbol{a}}^R$. Then

$$\frac{\partial}{\partial \boldsymbol{p}} \left(\tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}^R\right) = \begin{bmatrix} \partial/\partial \boldsymbol{p} \left(\tilde{\boldsymbol{a}}_1^R \cdot \boldsymbol{b}^R\right) \\ \vdots \\ \partial/\partial \boldsymbol{p} \left(\tilde{\boldsymbol{a}}_m^R \cdot \boldsymbol{b}^R\right) \end{bmatrix} \underset{(A.1.58a)}{=} \begin{bmatrix} (\boldsymbol{b}^R)^{\mathrm{T}} \cdot (\tilde{\boldsymbol{a}}_1^R)_p^{\mathrm{T}} + \tilde{\boldsymbol{a}}_1^R \cdot \boldsymbol{b}_p^R \\ \vdots \\ (\boldsymbol{b}^R)^{\mathrm{T}} \cdot (\tilde{\boldsymbol{a}}_m^R)_p^{\mathrm{T}} + \tilde{\boldsymbol{a}}_m^R \cdot \boldsymbol{b}_p^R \end{bmatrix}$$

$$= \begin{pmatrix} \tilde{\boldsymbol{a}}_{1p}^R \cdot \boldsymbol{b}^R + \tilde{\boldsymbol{a}}_1^R \cdot \boldsymbol{b}_p^R \\ \vdots \\ \tilde{\boldsymbol{a}}_{mp}^R \cdot \boldsymbol{b}^R + \tilde{\boldsymbol{a}}_m^R \cdot \boldsymbol{b}_p^R \end{pmatrix} = \tilde{\boldsymbol{a}}_p^R \cdot \boldsymbol{b}^R + \tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}_p^R = \tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}_p^R - \tilde{\boldsymbol{b}}^R \cdot \boldsymbol{a}_p^R .$$

This proves (A.1.58c). □

Relations used in constraint equations

The subsequent relations will be extensively used in the *kinematic constraint relations* and in the relations of the *constraint reaction forces* (see *Section 3*).

Given a vector $\boldsymbol{p}(t) = (p_1(t), \ldots, p_n(t)) \in \mathcal{C}^2(\mathbb{R}^n)$ and a vector function

$$\begin{array}{ccccccc} \boldsymbol{g}: & \mathbb{R}^n & \times & \mathbb{R}^1 & \longrightarrow & \mathbb{R}^m & , \quad \boldsymbol{g} \in \mathcal{C}^2(\mathbb{R}^{n+1}) \\ & \Psi & & \Psi & & \Psi & \\ & (\boldsymbol{p}(t) & , & t) & \longmapsto & \boldsymbol{g}(\boldsymbol{p}(t)\,,\, t) = \begin{pmatrix} \boldsymbol{g}_1(p\,,\, t) \\ \vdots \\ \boldsymbol{g}_m(p\,,\, t) \end{pmatrix}, & \end{array} \tag{A.1.60a}$$

then

$$\dot{\boldsymbol{g}}(\boldsymbol{p}(t)\,,\, t) := \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{g}(\boldsymbol{p}(t)\,,\, t)\right) = \boldsymbol{g}_p(\boldsymbol{p}(t)\,,\, t) \cdot \dot{\boldsymbol{p}}(t) + \boldsymbol{g}_t(\boldsymbol{p}(t)\,,\, t),$$

with (A.1.60b)

$$\boldsymbol{g}_p = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}} = \begin{pmatrix} \partial g_1 \big/ \partial p_1 , \ldots , \partial g_1 \big/ \partial p_n \\ \vdots \\ \partial g_m \big/ \partial p_1 , \ldots , \partial g_m \big/ \partial p_n \end{pmatrix} \in \mathbb{R}^{m,n}$$

and

$$\boldsymbol{g}_t = \begin{pmatrix} \partial g_1 \big/ \partial t \\ \vdots \\ \partial g_m \big/ \partial t \end{pmatrix} \in \mathbb{R}^m .$$

Furthermore,

$$\begin{aligned} \ddot{\boldsymbol{g}}(\boldsymbol{p}(t)\,,\,t) &:= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,(\boldsymbol{g}(\boldsymbol{p}(t)\,,\,t)) = \frac{\mathrm{d}}{\mathrm{d}t}\,\Big(\boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_t(\boldsymbol{p}(t)\,,\,t)\Big) \\ &= \Big(\boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t)\Big)_p\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\ddot{\boldsymbol{p}}(t) \\ &\quad + \boldsymbol{g}_{pt}(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_{tp}(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_{tt}(\boldsymbol{p}(t)\,,\,t) \end{aligned}$$

or (A.1.60c)

$$\begin{aligned} \ddot{\boldsymbol{g}}(\boldsymbol{p}(t)\,,\,t) &:= \boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\ddot{\boldsymbol{p}}(t) + \Big(\boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t)\Big)_p\cdot\dot{\boldsymbol{p}}(t) \\ &\quad + 2\cdot\boldsymbol{g}_{tp}(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_{tt}(\boldsymbol{p}(t)\,,\,t). \end{aligned}$$

Then the equation

$$\ddot{\boldsymbol{g}}(\boldsymbol{p}(t)\,,\,t) = \boldsymbol{0} \tag{A.1.61a}$$

may be written as

$$\boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\ddot{\boldsymbol{g}}(t) = \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}(t)\,,\,\dot{\boldsymbol{p}}(t)\,,\,t), \tag{A.1.61b}$$

with

$$\begin{aligned} \boldsymbol{\beta}_{\mathrm{c}} := -\,&\Big(\boldsymbol{g}_p(\boldsymbol{p}(t)\,,\,t)\cdot\ddot{\boldsymbol{p}}(t) + \Big(\boldsymbol{g}_p(p(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t)\Big)_p\cdot\dot{\boldsymbol{p}}(t) \\ &+ 2\cdot\boldsymbol{g}_{tp}(\boldsymbol{p}(t)\,,\,t)\cdot\dot{\boldsymbol{p}}(t) + \boldsymbol{g}_{tt}(\boldsymbol{p}(t)\,,\,t)\Big) \end{aligned} \tag{A.1.61c}$$

and with $\boldsymbol{g}_p$ as the Jacobian matrix of $\boldsymbol{g}$.

A.2 Lagrange formalism of a rigid body under *spatial* motion

The *Lagrange formalism* is an alternative approach for deriving the equations of motion of rigid-body systems. It is equivalent to the *Newton–Euler* approach. Due to its *"integral character"*, the application of *Lagrange formalism* to examples sometimes appears to be easier compared to the *Newton–Euler approach*, which is characterized by its *"differential character"*. It may happen if a mechanism includes passive components of different energy species (mechanical, electrical, magnetic, or thermodynamic; ([56], [65])).

The *Lagrange equations* of a mechanism may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right)^{\mathrm{T}} - \left(\frac{\partial L}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} = \boldsymbol{F}_g \quad , \quad \overset{= \mathbf{0}}{\cancel{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\lambda}}}\right)^{\mathrm{T}}}} - \left(\frac{\partial L}{\partial \boldsymbol{\lambda}}\right)^{\mathrm{T}} = \mathbf{0}, \tag{A.2.1a}$$

with

$$\boldsymbol{p} = (\boldsymbol{p}_1^{\mathrm{T}}\,, \dots\,, \boldsymbol{p}_{n_b}^{\mathrm{T}})^{\mathrm{T}} \;\;,\;\; \boldsymbol{p}_i = (\boldsymbol{r}_i^{\mathrm{T}}\,, \boldsymbol{\eta}_i^{\mathrm{T}})^{\mathrm{T}},$$

as the vector of the *generalized coordinates of the mechanism*,

$$\boldsymbol{v} = (\boldsymbol{v}_1^{\mathrm{T}}\,, \dots\,, \boldsymbol{v}_{n_b}^{\mathrm{T}})^{\mathrm{T}} \;\;,\;\; \boldsymbol{v}_i = \boldsymbol{T}_i^{-1}(\boldsymbol{\eta}_i) \cdot \dot{\boldsymbol{p}}_i,$$

as the vector of the *velocities*, with

$$\boldsymbol{T}(\boldsymbol{p}) = \mathrm{diag}(\boldsymbol{T}_1\,, \dots\,, \boldsymbol{T}_{n_b}) \quad , \quad \boldsymbol{T}_i = \begin{pmatrix} \mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{H}_i \cdot \boldsymbol{A}^{RL} \end{pmatrix},$$

with $\boldsymbol{F}_g$ as the vector of the *generalized forces* and *torques* associated with $\boldsymbol{p}$ and $\boldsymbol{v}$, with ${}^{\mathrm{c}}\boldsymbol{f} = \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{\lambda}$ as the vector of the *constraint reaction forces*, and with L as the *Lagrange function*, where usually

$$L = T_k(\boldsymbol{v}\,, \boldsymbol{p}) - U(\boldsymbol{p}) + \boldsymbol{g}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda}, \tag{A.2.1b}$$

with $T_k(\boldsymbol{v}\,, \boldsymbol{p})$ as the *kinetic energy* of the mechanism, $U(p)$ as the *potential energy* of the mechanism, $\mathbf{0} = \boldsymbol{g}(\boldsymbol{p}) \in \mathbb{R}^{n_c}$ as the *constraint equations of holonomic constraints*, and $\boldsymbol{\lambda} \in \mathbb{R}^{n_c}$ as the vector of the *Lagrange multipliers*.

Comment A.2.1 (Lagrange equations): In case that the generalized coordinates $\boldsymbol{p}$ are *minimal coordinates*, the term $\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda}$ does not appear in L. The basic idea behind this approach is defining the generalized forces $\boldsymbol{F}_g$ associated to the generalized coordinates and velocities.

A.2.1 Kinetic energy of an unconstrained rigid body

Consider a "point mass" with the mass m located at the point Q of an inertial frame R (Figure A.2.1a). The *kinetic energy* of the *point mass* with respect to R is

$$\begin{aligned} T_k &= \frac{1}{2}\left(\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{QO}^R\right)\right)^{\mathrm{T}} \cdot \left(\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{r}_{QO}^R\right)\right) \cdot m \\ &= \frac{1}{2}\left({}^R\dot{\boldsymbol{r}}_{QO}^R\right)^{\mathrm{T}} \cdot \left({}^R\dot{\boldsymbol{r}}_{QO}^R\right) \cdot m \in \mathbb{R}^1 . \end{aligned} \tag{A.2.2}$$

Consider an unconstrained rigid body of mass m, with center of mass C and volume V, with a reference point P fixed on the body, and a vector $\boldsymbol{r}_{PO}$ from the origin O of R to the point P (Figure A.2.1b).

Consider a mass element of mass dm of the body at an arbitrary point Q on the body, specified by the vector

$$\boldsymbol{r} := \boldsymbol{r}_{QO}. \tag{A.2.3}$$

The *kinetic energy* of the body is

$$T_k = \frac{1}{2} \cdot \int \left({}^R\dot{\boldsymbol{r}}_{QO}^R\right)^{\mathrm{T}} \cdot \left({}^R\dot{\boldsymbol{r}}_{QO}^R\right) \,\mathrm{d}m\left(\boldsymbol{r}_{QO}\right), \tag{A.2.4}$$

where the velocity vector ${}^R\dot{\boldsymbol{r}}^R := {}^R\dot{\boldsymbol{r}}_{QO}^R$ is measured and represented in R. Consider a second frame L with origin $\mathrm{O}_L = \mathrm{P}$ fixed on the body. Then the vector $\boldsymbol{r} = \boldsymbol{r}_{QO}$ can be written as (Figure A.2.1b)

$$\boldsymbol{r}^R := \boldsymbol{r}_{QO}^R = \boldsymbol{r}_{PO}^R + \boldsymbol{\chi}^R = \boldsymbol{r}_{PO}^R + \boldsymbol{A}^{RL} \cdot \boldsymbol{\chi}^L. \tag{A.2.5a}$$

The velocity vector ${}^R\dot{\boldsymbol{r}}^R$ is

$${}^R\dot{\boldsymbol{r}}^R = {}^R\dot{\boldsymbol{r}}_{PO}^R + {}^R\dot{\boldsymbol{\chi}}^R \tag{A.2.5b}$$

or

$${}^R\dot{\boldsymbol{r}}^R = {}^R\dot{\boldsymbol{r}}_{PO}^R + \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{\chi}^L + \boldsymbol{A}^{RL} \cdot {}^L\dot{\boldsymbol{\chi}}^L. \tag{A.2.5c}$$

Due to the *rigid-body property*

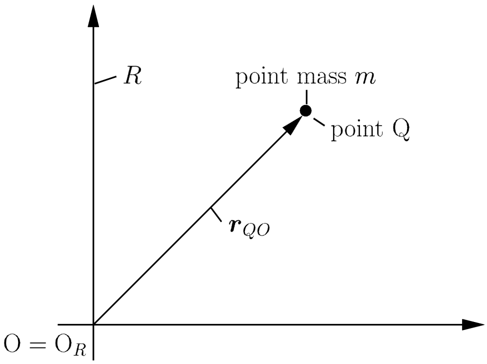
$${}^L\dot{\boldsymbol{\chi}}^L \equiv \mathbf{0} \tag{A.2.6}$$

and to the relation

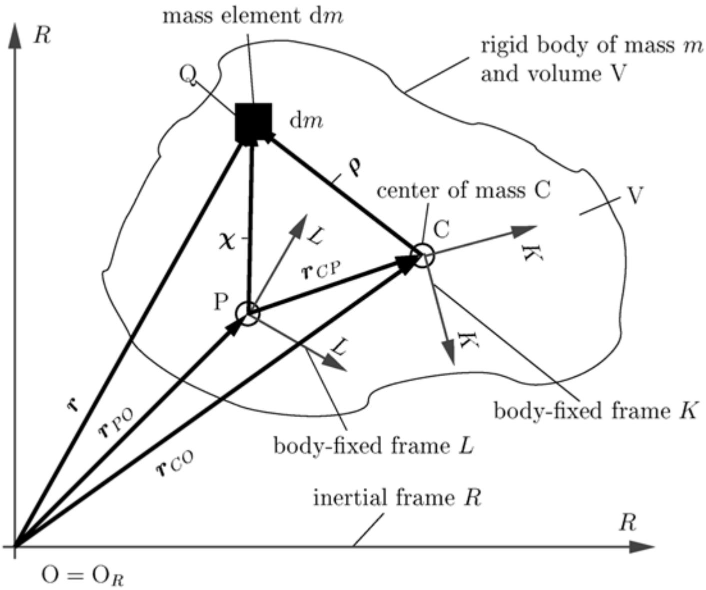
$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \tag{A.2.7a}$$

the resulting velocity vector is

$${}^R\dot{\boldsymbol{r}}^R = {}^R\dot{\boldsymbol{r}}_{PO}^R + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{\chi}^L. \tag{A.2.7b}$$



(a) Point mass in space



(b) Unconstrained rigid body in space

Fig. A.2.1: Vector diagram used in the definition of the kinetic energy of a rigid body

Inserting (A.2.7b) into (A.2.4) yields the following expressions for the *kinetic energy* of the rigid body, written with respect to the reference point P:

$$T_{kP} = \frac{1}{2} \int \left({}^R\dot{\boldsymbol{r}}^R\right)^{\mathrm{T}} \cdot \left({}^R\dot{\boldsymbol{r}}^R\right) \, \mathrm{d}m \tag{A.2.8}$$
$$= \frac{1}{2} \int \left({}^R\dot{\boldsymbol{r}}^R_{PO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\chi}^L\right)^{\mathrm{T}} \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO} + \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\chi}^L\right) \, \mathrm{d}m$$
$$= \frac{1}{2} \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO}\right)^{\mathrm{T}} \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO}\right) \cdot \int \mathrm{d}m$$
$$+ \frac{1}{2} \int \left((\boldsymbol{\chi}^L)^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{\omega}}^L_{LR}\right)^{\mathrm{T}} \cdot \underbrace{\boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL}}_{= \mathbf{I}_3} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\chi}^L\right) \mathrm{d}m$$
$$+ \left({}^R\dot{\boldsymbol{r}}^R_{PO}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \int \boldsymbol{\chi}^L \, \mathrm{d}m.$$

Together with

$$\boldsymbol{r}^L_{CP} := \frac{1}{m} \int \boldsymbol{\chi}^L \, \mathrm{d}m \tag{A.2.9a}$$

as the vector from the origin P of L to the *center of mass* C of the body, with

$$\boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL} = \mathbf{I}_3 \tag{A.2.9b}$$

and

$$\int \mathrm{d}m = m \tag{A.2.9c}$$

this yields

$$T_{kP} = \frac{1}{2} \cdot m \cdot \left\|{}^R\dot{\boldsymbol{r}}^R_{PO}\right\|^2 + m \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{r}^L_{CP}$$
$$+ \frac{1}{2} \int \boldsymbol{\chi}^{LT} \cdot \left(\tilde{\boldsymbol{\omega}}^L_{LR}\right)^{\mathrm{T}} \cdot \left(\tilde{\boldsymbol{\omega}}^L_{LR}\right) \cdot \boldsymbol{\chi}^L \, \mathrm{d}m$$

or

$$T_{kP} = \frac{1}{2} \cdot m \cdot \left\|{}^R\dot{\boldsymbol{r}}^R_{PO}\right\|^2 + m \cdot \left({}^R\dot{\boldsymbol{r}}^R_{PO}\right)^{\mathrm{T}} \cdot \tilde{\boldsymbol{\omega}}^R_{LR} \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{r}^L_{CP} \tag{A.2.10}$$
$$+ \frac{1}{2} \int \left\|\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\chi}^L\right\|^2 \, \mathrm{d}m.$$

Using the vector relation

$$\|\tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\chi}\|^2 = (\boldsymbol{\omega} \times \boldsymbol{\chi})^{\mathrm{T}} \cdot \underbrace{(\boldsymbol{\omega} \times \boldsymbol{\chi})}_{=: \boldsymbol{d}} = (\boldsymbol{\omega} \times \boldsymbol{\chi})^{\mathrm{T}} \cdot \boldsymbol{d} = \boldsymbol{\omega}^{\mathrm{T}} \cdot (\boldsymbol{\chi} \times \boldsymbol{d})$$
$$= \boldsymbol{\omega}^{\mathrm{T}} \cdot \left[\boldsymbol{\chi} \times \left(\boldsymbol{\omega} \times \boldsymbol{\chi}\right)\right] = \boldsymbol{\omega}^{\mathrm{T}} \cdot \left[\left(\boldsymbol{\chi}^{\mathrm{T}} \cdot \boldsymbol{\chi}\right) \cdot \boldsymbol{\omega} - \left(\boldsymbol{\chi} \cdot \boldsymbol{\chi}^{\mathrm{T}}\right) \cdot \boldsymbol{\omega}\right]$$
$$= \boldsymbol{\omega}^{\mathrm{T}} \cdot \left[\left(\boldsymbol{\chi}^{\mathrm{T}} \cdot \boldsymbol{\chi}\right) \cdot \mathbf{I}_3 - \boldsymbol{\chi} \cdot \boldsymbol{\chi}^{\mathrm{T}}\right] \cdot \boldsymbol{\omega}$$

yields

$$\left\|\tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \boldsymbol{\chi}^{L}\right\|^{2} \qquad = \left(\boldsymbol{\omega}_{LR}^{L}\right)^{\mathrm{T}} \cdot \left[(\boldsymbol{\chi}^{L})^{\mathrm{T}} \cdot \boldsymbol{\chi}^{L} \cdot \mathbf{I}_{3} - \boldsymbol{\chi}^{L} \cdot (\boldsymbol{\chi}^{L})^{\mathrm{T}}\right] \cdot \boldsymbol{\omega}_{LR}^{L}. \tag{A.2.11a}$$

Together with

$$\begin{aligned} &\frac{1}{2}\int \left\|\tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \boldsymbol{\chi}^{L}\right\|^{2} \,\mathrm{d}m \\ =&\frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{L}\right)^{\mathrm{T}} \cdot \left\{\int \left[(\boldsymbol{\chi}^{L})^{\mathrm{T}} \cdot \boldsymbol{\chi}^{L} \cdot \mathbf{I}_{3} - \boldsymbol{\chi}^{L} \cdot (\boldsymbol{\chi}^{L})^{\mathrm{T}}\right] \,\mathrm{d}m\right\} \cdot \boldsymbol{\omega}_{LR}^{L} \end{aligned} \tag{A.2.11b}$$

and with the abbreviations (4.17), (4.23) and (4.24a)

$$\boldsymbol{J}_{P}^{L} := -\int \tilde{\boldsymbol{\chi}}^{L} \cdot \tilde{\boldsymbol{\chi}}^{L} \,\mathrm{d}m = \int \left[(\boldsymbol{\chi}^{L})^{\mathrm{T}} \cdot \boldsymbol{\chi}^{L} \cdot \mathbf{I}_{3} - \boldsymbol{\chi}^{L} \cdot (\boldsymbol{\chi}^{L})^{\mathrm{T}}\right] \,\mathrm{d}m, \tag{A.2.11c}$$

this yields, the following equivalent expressions for the *kinetic energy of a rigid body* under *spatial motion*, written with respect to an arbitrary *reference point* P:

$$\begin{aligned} T_{kP} =& \frac{1}{2} \cdot m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R} \qquad\qquad \text{(A.2.12a)} \\ &+ \frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{L}\right)^{\mathrm{T}} \cdot \boldsymbol{J}_{P}^{L} \cdot \boldsymbol{\omega}_{LR}^{L} + m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^{L} \cdot \boldsymbol{r}_{CP}^{L} \\ =& \frac{1}{2} \cdot m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R} \\ &+ \frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{J}_{P}^{L} \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{A}^{RL} \cdot \left(\boldsymbol{\omega}_{LR}^{L}\right) \\ &+ m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot \tilde{\boldsymbol{\omega}}_{LR}^{R} \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{r}_{CP}^{L} \\ =& \frac{1}{2} \cdot m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R} \\ &+ \frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{J}_{P}^{R} \cdot \left(\boldsymbol{\omega}_{LR}^{R}\right) - m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^{L} \cdot \boldsymbol{\omega}_{LR}^{L} \\ =& \frac{1}{2} \cdot m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R} \\ &+ \frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{J}_{P}^{R} \cdot \left(\boldsymbol{\omega}_{LR}^{R}\right) - m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^{L} \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}_{LR}^{R} \\ =& \frac{1}{2} \cdot m \cdot \left({}^{R}\dot{\boldsymbol{r}}_{PO}^{R}\right)^{\mathrm{T}} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R} \\ &+ \frac{1}{2}\left(\boldsymbol{\omega}_{LR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{J}_{P}^{R} \cdot \left(\boldsymbol{\omega}_{LR}^{R}\right) + m \cdot \left(\boldsymbol{\omega}_{LR}^{R}\right)^{\mathrm{T}} \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{r}}_{CP}^{L} \cdot \boldsymbol{A}^{LR} \cdot {}^{R}\dot{\boldsymbol{r}}_{PO}^{R}. \end{aligned}$$

A.2.2 Spatial equations of motion of an unconstrained rigid body for $\boldsymbol{P} = \boldsymbol{C}$

The *kinetic energy* of a rigid body with respect to the *center of mass* C is obtained from (A.2.12a) for P = C and for $\boldsymbol{r}_{PC} = 0$ as

$$T_{kC} = \frac{1}{2} \cdot m \cdot \|{}^R\dot{\boldsymbol{r}}^R_{CO}\|^2 + \frac{1}{2}\left(\boldsymbol{\omega}^R_{LR}\right)^{\mathrm{T}} \cdot \boldsymbol{J}^R_C \cdot \left(\boldsymbol{\omega}^R_{LR}\right)$$

or (A.2.12b)

$$T_{kC} = \frac{1}{2} \cdot m \cdot \|{}^R\dot{\boldsymbol{r}}^R_{CO}\|^2 + \frac{1}{2}\left(\boldsymbol{\omega}^L_{LR}\right)^{\mathrm{T}} \cdot \boldsymbol{J}^L_C \cdot \left(\boldsymbol{\omega}^L_{LR}\right).$$

Choosing

$$\boldsymbol{p} := \left(x^R_{CO}\,,\, y^R_{CO}\,,\, z^R_{CO}\,,\, \phi\,,\, \theta\,,\, \psi\right)^{\mathrm{T}} = \left(\left(\boldsymbol{r}^R_{CO}\right)^{\mathrm{T}}\ ,\ \boldsymbol{\eta}^{\mathrm{T}}\right)^{\mathrm{T}} \tag{A.2.12c}$$

as the vector of the generalized coordinates together with the associated velocity vector

$$\boldsymbol{v}_{CO} = \left(\left(\dot{\boldsymbol{r}}^R_{CO}\right)^{\mathrm{T}}\ ,\ \left(\boldsymbol{\omega}^L_{LR}\right)^{\mathrm{T}}\right)^{\mathrm{T}} \quad , \quad \boldsymbol{\omega}^L_{LR} = \boldsymbol{A}^{LR}(\boldsymbol{\eta}) \cdot \boldsymbol{H}^{-1}(\boldsymbol{\eta}) \cdot \dot{\boldsymbol{\eta}}$$

provides the following expressions of the Lagrange equations for P = C.

Equation (A.1.53) of *Appendix A.1.6*

$$\frac{1}{2}\left(\frac{\partial}{\partial x}\left(\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{A} \cdot \boldsymbol{x}\right)\right)^{\mathrm{T}} = \boldsymbol{A} \cdot \boldsymbol{x} \quad , \quad \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}, \tag{A.2.12d}$$

applied to (A.2.12b) yields

$$\left(\frac{\partial T_{kC}}{\partial\left(\dot{\boldsymbol{r}}^R_{CO}\right)}\right)^T = m \cdot {}^R\dot{\boldsymbol{r}}^R_{CO} \quad , \quad \left(\frac{\partial T_{kC}}{\partial \boldsymbol{\omega}^R_{LR}}\right)^{\mathrm{T}} = \boldsymbol{J}^R_C \cdot \boldsymbol{\omega}^R_{LR},$$

$$\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_{kC}}{\partial\left(\dot{\boldsymbol{r}}^R_{CO}\right)}\right)^T = m \cdot {}^R\ddot{\boldsymbol{r}}^R_{CO}, \tag{A.2.12e}$$

and

$$\begin{aligned}
\frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_{kC}}{\partial \boldsymbol{\omega}^R_{LR}}\right)^{\mathrm{T}} &= \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{J}^R_C \cdot \boldsymbol{\omega}^R_{LR}\right) =: \frac{{}^R\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_C \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}^R_{LR}\right) \\
&= \dot{\boldsymbol{A}}^{RL} \cdot \boldsymbol{J}^L_C \cdot \boldsymbol{\omega}^L_{LR} + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_C \cdot \dot{\boldsymbol{A}}^{LR} \cdot \boldsymbol{\omega}^R_{LR} + \boldsymbol{A}^{RL} \cdot \boldsymbol{J}^L_C \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{\omega}}^R_{LR} \\
&= \boldsymbol{A}^{RL} \cdot \Big(\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{J}^L_C \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \underbrace{- \boldsymbol{J}^L_C \cdot \tilde{\boldsymbol{\omega}}^L_{LR} \cdot \underbrace{\boldsymbol{A}^{LR} \cdot \boldsymbol{\omega}^R_{LR}}_{= \boldsymbol{\omega}^L_{LR}}}_{= \mathbf{0}} + \boldsymbol{J}^L_C \cdot \boldsymbol{A}^{LR} \cdot {}^R\dot{\boldsymbol{\omega}}^R_{LR}\Big).
\end{aligned}$$

Together with

$$\tilde{\boldsymbol{\omega}}^L_{LR} \cdot \boldsymbol{\omega}^L_{LR} = \mathbf{0}$$

and

$$
\begin{aligned}
{}^{R}\dot{\boldsymbol{\omega}}_{LR}^{R} &= \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{\omega}_{LR}^{R}\right) = \frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{A}^{RL}\cdot\boldsymbol{\omega}_{LR}^{L}\right) = \dot{\boldsymbol{A}}^{LR}\cdot\boldsymbol{\omega}_{LR}^{L} + \boldsymbol{A}^{RL}\cdot{}^{L}\dot{\boldsymbol{\omega}}_{LR}^{L} \\
&= \boldsymbol{A}^{RL}\cdot\underbrace{\tilde{\boldsymbol{\omega}}_{LR}^{L}\cdot\boldsymbol{\omega}_{LR}^{L}}_{=\,\mathbf{0}} + \boldsymbol{A}^{RL}\cdot{}^{L}\dot{\boldsymbol{\omega}}_{LR}^{L} = \boldsymbol{A}^{RL}\cdot{}^{L}\dot{\boldsymbol{\omega}}_{LR}^{L},
\end{aligned}
$$

this yields the relation

$$
\frac{{}^{R}\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T_{kC}}{\partial \boldsymbol{\omega}_{LR}^{R}}\right)^{\mathrm{T}} = \boldsymbol{A}^{RL}\cdot\left(\boldsymbol{J}_{C}^{L}\cdot\dot{\boldsymbol{\omega}}_{LR}^{L} + \tilde{\boldsymbol{\omega}}_{LR}^{L}\cdot\boldsymbol{J}_{C}^{L}\cdot\boldsymbol{\omega}_{LR}^{R}\right). \tag{A.2.12f}
$$

Together with

$$
\frac{\partial L_C}{\partial \boldsymbol{r}_{CO}^{R}} = -\frac{\partial U}{\partial \boldsymbol{r}_{CO}^{R}} \quad \text{and} \quad \frac{\partial L_C}{\partial \boldsymbol{\eta}} = -\frac{\partial U}{\partial \boldsymbol{\eta}}, \tag{A.2.12g}
$$

this provides the *equations of motion of an unconstrained rigid body for* P = C (cf. Equation 4.56a of *Section 4.2.4.1*):

$$
\begin{pmatrix} m\cdot\mathbf{I}_3 & , & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3} & , & \boldsymbol{J}_{C}^{L} \end{pmatrix} \cdot \begin{pmatrix} \ddot{\boldsymbol{r}}_{CO}^{R} \\ \dot{\boldsymbol{\omega}}_{LR}^{L} \end{pmatrix} = \tag{A.2.13a}
$$

$$
\begin{pmatrix} \mathbf{0} \\ -\tilde{\boldsymbol{\omega}}_{LR}^{L}\cdot\boldsymbol{J}_{C}^{L}\cdot\boldsymbol{\omega}_{LR}^{L} \end{pmatrix} - \begin{pmatrix} \left(\partial U/\partial \boldsymbol{r}_{CO}^{R}\right)^{\mathrm{T}} \\ \boldsymbol{A}^{LR}\cdot\left(\partial U/\partial \boldsymbol{\eta}\right)^{\mathrm{T}} \end{pmatrix}.
$$

A.2.3 Spatial equations of motion of a constrained rigid body

The *equations of motion of a rigid body under constrained spatial motion* for P = C in *DAE form* are (cf. Equation 4.77i of *Section 4.3.2.3*)

$$
\left[\begin{array}{c|c|c}
m\cdot\mathbf{I}_3 & \mathbf{0}_{3,3} & (\partial \boldsymbol{g}/\partial \boldsymbol{r}_{CO}^{R})^{\mathrm{T}} \\
\mathbf{0}_{3,3} & \boldsymbol{J}_{C}^{L} & \boldsymbol{A}^{LR}(\boldsymbol{\eta})\cdot\boldsymbol{H}^{\mathrm{T}}(\boldsymbol{\eta})\cdot(\partial \boldsymbol{g}/\partial \boldsymbol{\eta})^{\mathrm{T}} \\
\hdashline
\partial \boldsymbol{g}/\partial \boldsymbol{r}_{CO}^{R} & (\partial \boldsymbol{g}/\partial \boldsymbol{\eta})\cdot\boldsymbol{H}(\boldsymbol{\eta})\cdot\boldsymbol{A}^{RL}(\boldsymbol{\eta}) & \mathbf{0}
\end{array}\right]
$$

$$
\cdot\begin{bmatrix} \ddot{\boldsymbol{r}}_{CO}^{R} \\ \dot{\boldsymbol{\omega}}_{LR}^{L} \\ \hdashline -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\tilde{\boldsymbol{\omega}}_{LR}^{L}\cdot\boldsymbol{J}_{C}^{L}\cdot\boldsymbol{\omega}_{LR}^{L} \\ \hdashline \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\left(\partial U/\partial \boldsymbol{r}_{CO}^{R}\right)^{\mathrm{T}} \\ -\left(\partial U/\partial \boldsymbol{\eta}\right)^{\mathrm{T}} \\ \hdashline \boldsymbol{\beta}_{\mathrm{c}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{F}_{g} \\ \hdashline \mathbf{0} \end{bmatrix}. \tag{A.2.13b}
$$

They include the *constraint acceleration equations* of the holonomic constraints

$$\left[\partial \boldsymbol{g} \big/ \partial \boldsymbol{r}_{CO}^{R}\ ,\ \left(\partial \boldsymbol{g} \big/ \partial \boldsymbol{\eta}\right) \cdot \boldsymbol{H}(\boldsymbol{\eta}) \cdot \boldsymbol{A}^{RL}(\boldsymbol{\eta})\right] \cdot \begin{pmatrix} \ddot{\boldsymbol{r}}_{CO}^{R} \\ \dot{\boldsymbol{\omega}}_{LR}^{L} \end{pmatrix} = \boldsymbol{\beta}_{\mathrm{c}} \tag{A.2.13c}$$

and the *constraint reaction forces and torques*

$$^{\mathrm{c}}\boldsymbol{f}_i = \begin{pmatrix} \left(\partial \boldsymbol{g} \big/ \partial \boldsymbol{r}_{CO}^{R}\right)^{\mathrm{T}} \\ \boldsymbol{A}^{LR}(\boldsymbol{\eta}) \cdot \boldsymbol{H}^{\mathrm{T}}(\boldsymbol{\eta}) \left(\partial \boldsymbol{g} \big/ \partial \boldsymbol{\eta}\right)^{\mathrm{T}} \end{pmatrix} \cdot \boldsymbol{\lambda}\ . \tag{A.2.13d}$$

A.3 Model equations of *planar* and *spatial* mechanisms

In this section relevant *differences* between vector notations and equations of the motion of *planar* and *spatial* mechanisms will be briefly summarized:

1. The dimension of *geometrical displacement*, *velocity*, *acceleration*, and *force vectors* is increased from two (in the *planar case*) to three (in the *spatial case*).

2. *Angular velocity* and *torque vectors* of *planar* mechanisms have a *fixed direction perpendicular to the plane* ($\mathbb{R}^2$). *Angular velocity* and *torque vectors in spatial mechanisms* may have *arbitrary directions* in $\mathbb{R}^3$.

3. Contrary to $\mathbb{R}^2$ (considered as a subspace of $\mathbb{R}^3$), $\mathbb{R}^3$ is *closed under "vector product operations"*; i.e.,

$$\begin{aligned} \tilde{\boldsymbol{a}}^R \cdot \boldsymbol{b}^R \quad & \in \mathbb{R}^3 \quad \text{for} \quad \boldsymbol{a}\,,\boldsymbol{b} \in \mathbb{R}^3 \\ \boldsymbol{M}^L = \tilde{\boldsymbol{r}}^L \cdot \boldsymbol{A}^{LR} \cdot \boldsymbol{F}^R & \in \mathbb{R}^3 \quad \text{for} \quad \boldsymbol{r}\,,\boldsymbol{F} \in \mathbb{R}^3 . \end{aligned} \tag{A.3.1}$$

4. All *matrices* (including orientation matrices) are extended from elements of $\mathbb{R}^{2,2}$ in the *planar case* to elements of $\mathbb{R}^{3,3}$ in the *spatial case.*

5. The transformation matrix

$$\boldsymbol{A}^{RL}(\psi) = \begin{pmatrix} \cos\psi \;, & -\sin\psi \\ \sin\psi \;, & \cos\psi \end{pmatrix} \in \mathbb{R}^{2,2} \quad , \quad \psi := \psi_{LR}, \tag{A.3.2}$$

of the *planar case* is extended to a product of three elementary transformation matrices,

$$\boldsymbol{A}^{RL}(\varphi\,,\theta\,,\psi) = \boldsymbol{A}^{RL_1}(\psi)\cdot\boldsymbol{A}^{L_1L_2}(\theta)\cdot\boldsymbol{A}^{L_2L}(\varphi) \in \mathbb{R}^{3,3}, \tag{A.3.3a}$$

with the rotation angles φ, θ, ψ chosen to be around suitable axes in the *spatial case.* In the special case of *Bryant angles*, this yields

$$\boldsymbol{A}^{RL} = \begin{pmatrix} \mathrm{c}\,\theta\cdot\mathrm{c}\,\psi & ,\mathrm{c}\,\varphi\cdot\mathrm{s}\,\psi+\mathrm{s}\,\varphi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi & ,\mathrm{s}\,\varphi\cdot\mathrm{s}\,\psi-\mathrm{c}\,\varphi\cdot\mathrm{s}\,\theta\cdot\mathrm{c}\,\psi \\ -\mathrm{c}\,\theta\cdot\mathrm{s}\,\psi & ,\mathrm{c}\,\varphi\cdot\mathrm{c}\,\psi-\mathrm{s}\,\varphi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi & ,\mathrm{s}\,\varphi\cdot\mathrm{c}\,\psi+\mathrm{c}\,\varphi\cdot\mathrm{s}\,\theta\cdot\mathrm{s}\,\psi \\ \mathrm{s}\,\theta & , \quad -\mathrm{s}\,\varphi\cdot\mathrm{c}\,\theta & , \quad \mathrm{c}\,\varphi\cdot\mathrm{c}\,\theta \end{pmatrix}^{\mathrm{T}} . \tag{A.3.3b}$$

6. A basic *difference* between *planar* rotations and *spatial* rotations is that: in the *planar* case, $\boldsymbol{\omega}_{zLR}$ is the time derivative of a rotation angle ψ_{LR} around the z-axis perpendicular to the x–y plane

$$\boldsymbol{\omega}_{zLR} = \dot{\psi}_{LR}\cdot\boldsymbol{e}_{zL} = \dot{\psi}_{LR}\cdot\boldsymbol{e}_{zR} \;\text{ or }\; \boldsymbol{\omega}_{zLR}^{L} = (0,0,1)^{\mathrm{T}}\cdot\dot{\psi}_{LR}, \tag{A.3.4}$$

whereas in the *spatial case*, $\boldsymbol{\omega}_{LR}^{L}$ is related to the time derivatives of the angles φ, θ, ψ by the (nonlinear) *kinematic differential equation*

$$\boldsymbol{\omega}_{LR}^{L} = \boldsymbol{A}^{LR}(\varphi\,,\,\theta\,,\,\psi)\cdot\boldsymbol{H}^{-1}(\varphi\,,\,\theta\,,\,\psi)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\,(\varphi,\theta,\psi)^{\mathrm{T}}\,, \tag{A.3.5}$$

with $\boldsymbol{H}(\varphi\,,\,\theta\,,\,\psi)$ depending on the special choice of $\boldsymbol{A}^{RL}$. For *Bryant angles* this yields

$$\boldsymbol{H}^{-1}(\varphi\,,\,\theta\,,\,\psi) = \begin{pmatrix} 1\;, & 0\;\;, & \mathrm{s}\,\theta \\ 0\;, & \mathrm{c}\,\varphi\;, & -\mathrm{s}\,\varphi\cdot\mathrm{c}\,\theta \\ 0\;, & \mathrm{s}\,\varphi\;, & \mathrm{c}\,\varphi\cdot\mathrm{c}\,\theta \end{pmatrix} \tag{A.3.6}$$

with $\mathrm{s}\,\alpha := \sin\alpha$ and $\mathrm{c}\,\alpha := \cos\alpha$.

7. The *kinematic differential equation* (Equation A.3.5) of the *spatial case* must be solved simultaneously with the kinetic differential equations of the mechanism.

8. The vector of *Cartesian coordinates* of a body under *planar motion*

$$\boldsymbol{p} = \left(x_{PO}^{R}\,,\,y_{PO}^{R}\,,\,\psi_{LR}\right)^{\mathrm{T}} \in \mathbb{R}^{3} \tag{A.3.7a}$$

is extended to the vector

$$\boldsymbol{p} = \left(x_{PO}^{R}\,,\,y_{PO}^{R}\,,\,z_{PO}^{R}\,,\,\varphi\,,\,\theta\,,\,\psi\right)^{\mathrm{T}} \in \mathbb{R}^{6} \tag{A.3.7b}$$

in the case of *spatial motions.*

9. The *velocity equation*

$$\dot{\boldsymbol{p}} = \left(\dot{x}_{PO}^{R}\,,\,\dot{y}_{PO}^{R}\,,\,\dot{\psi}\right)^{\mathrm{T}} =: \boldsymbol{v} \tag{A.3.8a}$$

of the *planar case* is extended to the velocity equation

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(\boldsymbol{p})\cdot\boldsymbol{v} \tag{A.3.8b}$$

of the *spatial case*, with

$$\dot{\boldsymbol{p}} = \left(\dot{x}_{PO}^{R}\,,\,\dot{y}_{PO}^{R}\,,\,\dot{z}_{PO}^{R}\,,\,\dot{\varphi}\,,\,\dot{\theta}\,,\,\dot{\psi}\right)^{\mathrm{T}}, \tag{A.3.8c}$$

$$\boldsymbol{v} = \left(\dot{x}_{PO}^{R}\,,\,\dot{y}_{PO}^{R}\,,\,\dot{z}_{PO}^{R}\,,\,\omega_{x}^{L}\,,\,\omega_{y}^{L}\,,\,\omega_{z}^{L}\right)^{\mathrm{T}}, \tag{A.3.8d}$$

and

$$\boldsymbol{T}(\boldsymbol{p}) = \begin{pmatrix} \mathbf{I}_3 \;, & \mathbf{0}_{3,3} \\ \mathbf{0}_{3,3}\;, & \boldsymbol{H}\,(\varphi\,,\,\theta\,,\,\psi)\cdot\boldsymbol{A}^{RL}\,(\varphi\,,\,\theta\,,\,\psi) \end{pmatrix}. \tag{A.3.8e}$$

10. The time derivative of an orientation matrix

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL}\cdot\tilde{\boldsymbol{\omega}}_{LR}^{L} \in \mathbb{R}^{3} \tag{A.3.9a}$$

reduces, in the *planar case*, to

$$\dot{\boldsymbol{A}}^{RL} = \boldsymbol{A}^{RL} \cdot \mathbf{R} \cdot \dot{\psi}_{LR} \in \mathbb{R}^2 \quad , \quad \mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{A.3.9b}$$

11. The *Steiner–Huygens formula* is extended from

$$J_P^L = \left[J_C^L + m \cdot \left(x_{CP}^2 + y_{CP}^2 \right) \right] \in \mathbb{R}^1 \tag{A.3.10a}$$

in the *planar case (moment of inertia)* to

$$\boldsymbol{J}_P^L = \left[\boldsymbol{J}_C^L + m \cdot \left[(\boldsymbol{r}_{CP}^L)^T \cdot \boldsymbol{r}_{CP}^L \cdot \mathbf{I}_3 - \boldsymbol{r}_{CP}^L \cdot (\boldsymbol{r}_{CP}^L)^T \right] \right] \in \mathbb{R}^{3,3} \tag{A.3.10b}$$

in the *spatial case (inertia matrix)*, where $\boldsymbol{J}_C^L$ includes three moments of inertia, $J_{C_{ii}}^L$, and three products of inertia, $J_{C_{ij}}$, $i \neq j$.

12. The vector of *centrifugal forces* and *gyroscopic terms* of the *spatial case*

$$\boldsymbol{q}_G(\boldsymbol{p}\,,\,\boldsymbol{v}) = \begin{pmatrix} -m \cdot \boldsymbol{A}^{RL} \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{r}_{CP}^L \\ -\tilde{\boldsymbol{\omega}}_{LR}^L \cdot \boldsymbol{J}_P^L \cdot \boldsymbol{\omega}_{LR}^L \end{pmatrix} \in \mathbb{R}^6 \tag{A.3.11a}$$

reduces, in the *planar case*, to

$$\begin{aligned} \boldsymbol{q}_G(\boldsymbol{p}\,,\,\boldsymbol{v}) &= \begin{pmatrix} -m \cdot \boldsymbol{A}^{RL} \cdot \mathbf{R}^2 \cdot \boldsymbol{r}_{CP}^L \cdot \dot{\psi}_{LR}^2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} +m \cdot \boldsymbol{A}^{RL} \cdot \boldsymbol{r}_{CP}^L \cdot \dot{\psi}_{LR}^2 \\ 0 \end{pmatrix} \in \mathbb{R}^3 \quad , \quad \mathbf{R}^2 = -\mathbf{I}_2, \end{aligned} \tag{A.3.11b}$$

(no gyroscopic terms).

13. Holonomic *kinematic constraint position, velocity* and *acceleration equations* of a *planar mechanism* have the form

$$\boldsymbol{g}(\boldsymbol{p}) = \boldsymbol{0} \qquad \textit{(constraint position equation)}, \tag{A.3.12a}$$

$$\underbrace{\boldsymbol{g}_p(\boldsymbol{p})}_{\text{constraint Jacobian}} \cdot \boldsymbol{v} = \boldsymbol{0} \qquad \textit{(constraint velocity equation)}, \tag{A.3.12b}$$

and

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = -\left(\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{v} \right)_p \cdot \boldsymbol{v} =: \boldsymbol{\beta}_c(\dot{\boldsymbol{p}}\,,\,\boldsymbol{p}) \tag{A.3.12c}$$

(constraint acceleration equation).

For *spatial mechanisms* these equations have the form

$$\boldsymbol{g}(\boldsymbol{p}) = \mathbf{0} \qquad \textit{(constraint position equation)}, \tag{A.3.13a}$$

$$\underbrace{\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})}_{\substack{\text{constraint}\\ \text{Jacobian}}} \cdot \boldsymbol{v} = \mathbf{0} \qquad \textit{(constraint velocity equation)}, \tag{A.3.13b}$$

and

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \dot{\boldsymbol{v}} = -\left(\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v}\right)_p \cdot \boldsymbol{T}(\boldsymbol{p}) \cdot \boldsymbol{v} =: \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \tag{A.3.13c}$$

$$\textit{(constraint acceleration equation)}.$$

14. In agreement with the above constraint equations, the *constraint reaction forces* and *torques* are computed in the *planar case* by the relation

$$^{\mathrm{c}}\boldsymbol{f} = \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{\lambda} \in \mathbb{R}^3 \tag{A.3.14a}$$

from the *Lagrange multipliers* $\boldsymbol{\lambda}$, and in the *spatial case* by the relation

$$^{\mathrm{c}}\boldsymbol{f} = \left(\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})\right)^{\mathrm{T}} \cdot \boldsymbol{\lambda} = \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{\lambda} \in \mathbb{R}^6 . \tag{A.3.14b}$$

15. The *DAEs* of a *planar mechanism* are

$$\dot{\boldsymbol{p}} = \boldsymbol{v} \tag{A.3.15a}$$

$$\begin{pmatrix} \boldsymbol{M}(\boldsymbol{p}) \,, \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \,, \quad \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \mathbf{0} \end{pmatrix} .$$

For *spatial mechanisms*, they have the form

$$\dot{\boldsymbol{p}} = \boldsymbol{T}(p) \cdot \boldsymbol{v} \tag{A.3.15b}$$

$$\begin{pmatrix} \boldsymbol{M}(\boldsymbol{p}) & , \boldsymbol{T}^{\mathrm{T}}(\boldsymbol{p}) \cdot \boldsymbol{g}_p^{\mathrm{T}}(\boldsymbol{p}) \\ \boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) \,, & \mathbf{0} \end{pmatrix} \cdot \begin{pmatrix} \dot{\boldsymbol{v}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) \\ \boldsymbol{\beta}_{\mathrm{c}}(\boldsymbol{p}, \boldsymbol{v}) \end{pmatrix} + \begin{pmatrix} \boldsymbol{q}_G(\boldsymbol{p}, \boldsymbol{v}) \\ \mathbf{0} \end{pmatrix} .$$

16. The number of different *joint types* is much higher in the *spatial case* than in the *planar case.* In the *spatial* case there exist many more combinations of projections and representations of vector loop equations and orientation loop equations. The reader should be aware that formal constraint relations with identical names may theoretically model quite *different planar* and *spatial joints.* For instance, the *common-point relation* models a *revolute joint* in the *planar* case and a *spherical joint* in the *spatial* case.

17. *Planar mechanisms* have

$$n_p = 3n_b - n_c \tag{A.3.16a}$$

DOFs whereas *spatial mechanisms* have

$$n_p = 6n_b - n_c \tag{A.3.16b}$$

DOFs, where n_b is the *number of rigid bodies*, and n_c is the *number* of (independent) *constraint equations* of the mechanism.

A.4 Constraint equations of a general universal joint

In this appendix model equations of a *general universal joint* with nonintersecting and nonorthogonal rotation axes will be derived. This joint model provides the approximate constraint equations of mechanisms that include a chain of three rigid bodies that are coupled by two revolute joints according to the drawing of Figure A.4.1. A precise theoretical model of this 8 DOF mechanism is a system of *18 kinematic DEs*, *18 kinetic DEs*, and *10 constraint equations.*

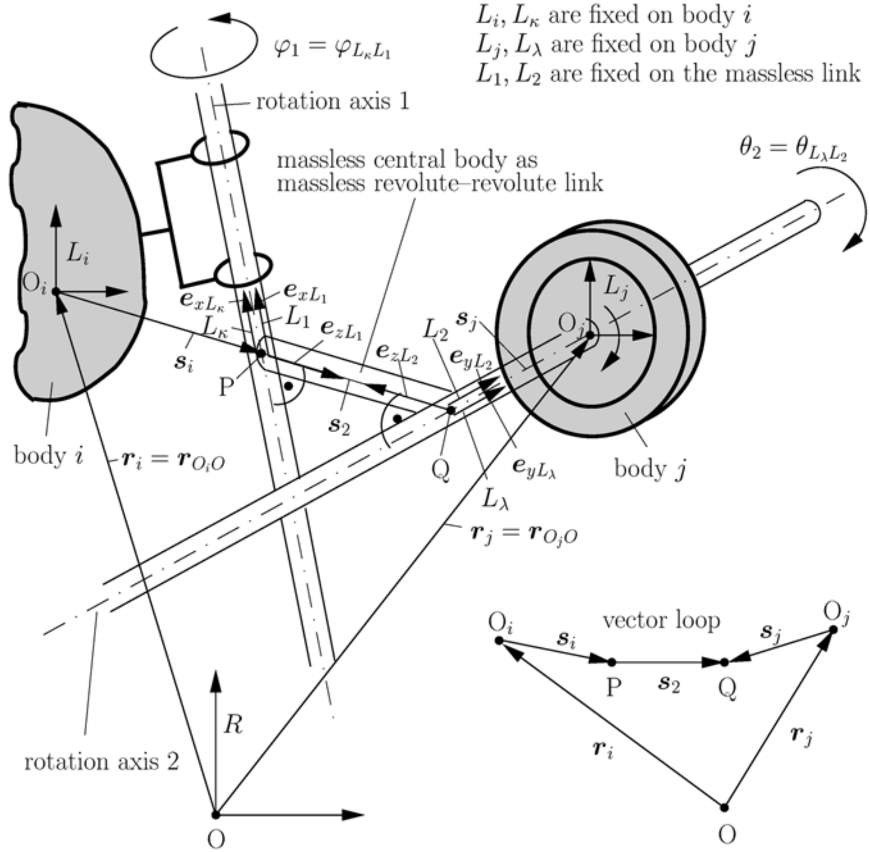


Fig. A.4.1: Drawing of a general universal joint (massless revolute–revolute link) connecting two rigid bodies

In such mechanisms of machines and vehicles the central body connecting the two other bodies has negligible inertia compared to the adjacent two bodies. This rigid body can therefore be treated as a *massless revolute–revolute* link

(Figure A.4.1). This 8 DOF mechanism will be *approximately modeled* by *12 kinematic DEs*, *12 kinetic DEs*, and *4 constraint equations.*

The *four constraint equations* together with the *two relative rotation angles* of this mechanism will now be derived for the general case of rotation axes of the joint (massless revolute–revolute link) that neither intersect each other nor are orthogonal to each other.

A.4.1 Notation and abbreviations

Two rigid bodies i and j have reference points O_i (origin of the local frame L_i on body i) and O_j (origin of the local frame L_j on body j), (Figures A.4.1 and A.4.2). Let R be an inertial frame with origin O. Let

$$\boldsymbol{s}_2 := \boldsymbol{r}_{QP} \tag{A.4.1a}$$

be the shortest vector between the two rotation axes, with P located on rotation axis 1 and Q located on rotation axis 2 (this distance will be computed in Section A.4.3). Consider two frames L_κ and L_1 with origins in P, and two frames L_λ and L_2 with origins in Q. (To obtain a clearer picture of this situation, the two frames L_κ and L_1 are drawn with different origins P_i and P, where the distance $\boldsymbol{s}_1 := \boldsymbol{r}_{PP_i}$ is set to zero (P_i = P). The same representation is chosen for (L_λ , L_2) with origins (Q and Q_j), and $\boldsymbol{s}_3 := \boldsymbol{r}_{Q_jQ} = \mathbf{0}$ (Q = Q_j).) Assume that the $\boldsymbol{e}_{xL_1}$-axis of L_1 is oriented in the direction of rotation axis 1, and that its $\boldsymbol{e}_{zL_1}$-axis is oriented in the direction of $\boldsymbol{r}_{QP}$ (then L_1 is fixed to the massless link). Let L_κ be fixed to body i with its $\boldsymbol{e}_{xL_\kappa}$-axis oriented parallel to $\boldsymbol{e}_{xL_1}$. The *rotation angle* of L_κ relative to L_1 around their common x-axes is called $\varphi_1 := \varphi_{L_\kappa L_1}$. Assume that the $\boldsymbol{e}_{zL_2}$-axis of L_2 is oriented in the direction of $\boldsymbol{r}_{PQ}$ (i.e., $\boldsymbol{e}_{zL_2} = -\boldsymbol{e}_{zL_1}$), and that its $\boldsymbol{e}_{yL_2}$-axis is oriented in the direction of rotation axis 2 (then L_2 is fixed on the massless link). Let L_λ be fixed on body j with its $\boldsymbol{e}_{yL_\lambda}$-axis oriented parallel to $\boldsymbol{e}_{yL_2}$. The *rotation angle* of L_λ relative to L_2 around their common y-axes is called $\theta_2 := \theta_{L_\lambda L_2}$. Let

$$\boldsymbol{r}_i := \boldsymbol{r}_{O_iO} \quad \text{and} \quad \boldsymbol{r}_j := \boldsymbol{r}_{O_jO} \tag{A.4.1b}$$

be the displacement vector of O_ℓ from O ($\ell = i\,,\, j$), and let

$$\boldsymbol{s}_i^i := \boldsymbol{s}_i^{L_i} := \boldsymbol{r}_{P_iO_i}^{L_i} := \boldsymbol{r}_{PO_i}^{L_i} \quad \text{and} \quad \boldsymbol{s}_j^j := \boldsymbol{s}_j^{L_j} := \boldsymbol{r}_{Q_jO_j}^{L_j} := \boldsymbol{r}_{QO_j}^{L_j} \tag{A.4.1c}$$

be vectors fixed on bodies i and j, respectively. The vector $\boldsymbol{s}_2 := \boldsymbol{r}_{QP}$ is represented as $\boldsymbol{s}_2^1 := \boldsymbol{s}_2^{L_1} = \boldsymbol{s}_{PQ}^{L_1} = (0\,,\,0\,,\,s_{z2})^{\mathrm{T}}$. The *rotation angles* φ_1 and θ_2 are *time dependent*, whereas the relative rotation angle $\psi_{21} := \psi_{L_2L_1}$ of L_2 with respect to L_1 around the common z-axes is a *constant angle*. The different orientation matrices introduced in Figure A.4.2 are

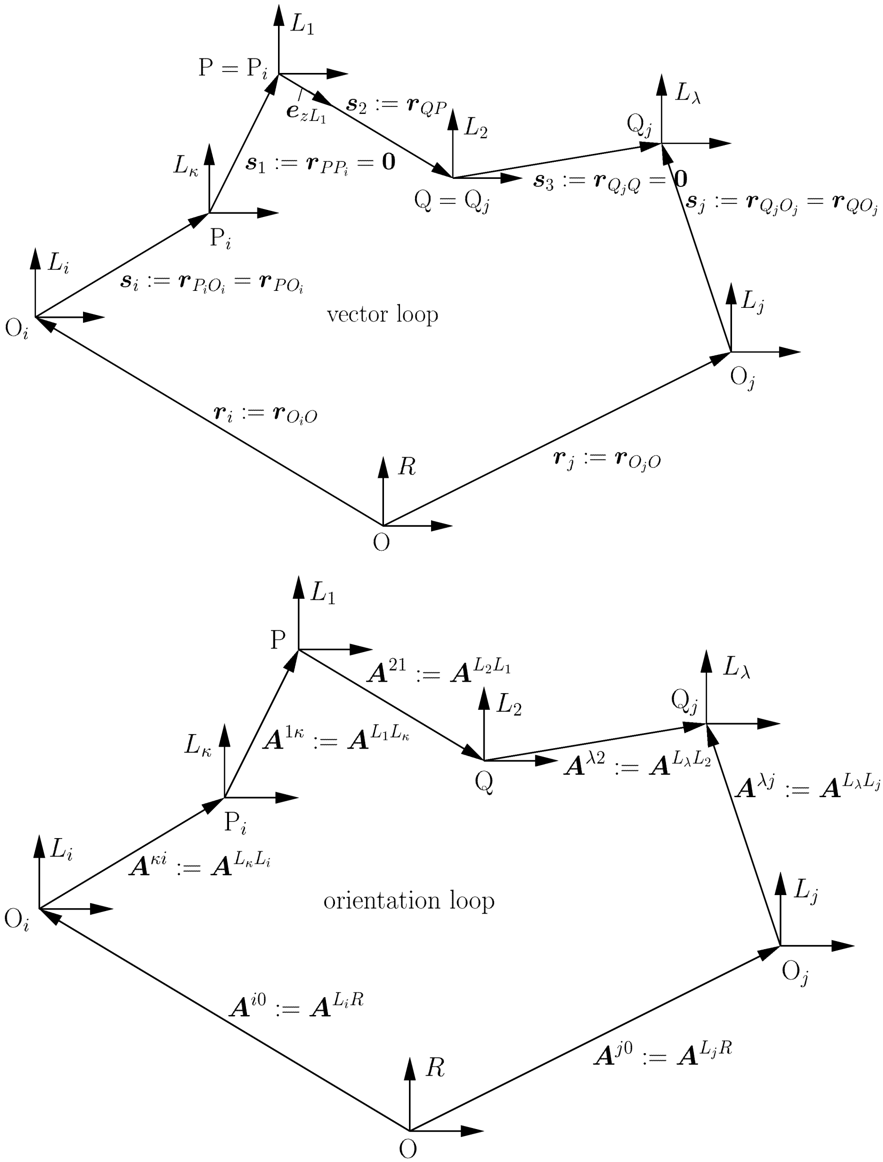


Fig. A.4.2: Diagrams for deriving the *vector loop* and *orientation loop* equations of the mechanism

$$\boldsymbol{A}^{i0} := \boldsymbol{A}^{L_iR} \qquad \text{(variable)},$$
$$\boldsymbol{A}^{j0} := \boldsymbol{A}^{L_jR} \qquad \text{(variable)},$$
$$\boldsymbol{A}^{ij} := \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \qquad \text{(variable)},$$

$$
\begin{aligned}
\boldsymbol{A}^{\kappa i} &:= \boldsymbol{A}^{L_\kappa L_i} && \text{(constant)},\\
\boldsymbol{A}^{\lambda j} &:= \boldsymbol{A}^{L_\lambda L_j} && \text{(constant)},\\
\boldsymbol{A}^{21} &:= \boldsymbol{A}^{L_2 L_1}(\psi_{21}) && \text{(constant)},\\
\boldsymbol{A}^{1\kappa}(\varphi_1) &:= \boldsymbol{A}^{L_1 L_\kappa}(\varphi_1) && \text{(variable), and}\\
\boldsymbol{A}^{2\lambda}(\theta_2) &:= \boldsymbol{A}^{L_2 L_\lambda}(\theta_2) && \text{(variable).}
\end{aligned}
\tag{A.4.1d}
$$

The following abbreviations will be used for *angular velocities*,

$$
\boldsymbol{\omega}_{j0}^{j} = \boldsymbol{\omega}_{L_j R}^{L_j}, \quad \boldsymbol{\omega}_{i0}^{i} = \boldsymbol{\omega}_{L_i R}^{L_i}, \quad \text{and} \quad \boldsymbol{\omega}_{ij}^{i} = \boldsymbol{\omega}_{L_i L_j}^{L_i}, \tag{A.4.1e}
$$

and *projection operators*,

$$
\mathbf{P_r}^{\mathrm{T}}(x) = (1\,,0\,,0), \quad \mathbf{P_r}^{\mathrm{T}}(y) = (0\,,1\,,0), \quad \text{and} \quad \mathbf{P_r}^{\mathrm{T}}(z) = (0\,,0\,,1). \tag{A.4.1f}
$$

A.4.2 Computation of constraint equations

The constraint equations

$$
\boldsymbol{g} = (g_1\,,g_2\,,g_3\,,g_4)^{\mathrm{T}} = \mathbf{0}_4 \tag{A.4.2a}
$$

of this mechanism include *four scalar constraint equations*

$$
g_i = 0 \quad , \quad (i = 1\,,2\,,3\,,4), \tag{A.4.2b}
$$

that will now be derived together with the two *relative coordinates* φ_1 and θ_2 from suitable representations and projections of *vector loop* and *orientation loop equations*, obtained from the geometrical situations discussed above and shown in Figures A.4.1 and A.4.2.

A.4.2.1 First constraint equation. The *first constraint equation* that *eliminates a rotational DOF* between the two bodies i and j is obtained by the *orientation loop equation* (Figure A.4.2)

$$
\boldsymbol{A}^{\lambda\kappa} \cdot \boldsymbol{A}^{\kappa 1} \cdot \boldsymbol{A}^{12} \cdot \boldsymbol{A}^{2\lambda} = \mathbf{I}_3 \tag{A.4.3a}
$$

or

$$
\bar{\boldsymbol{g}}_1 := \boldsymbol{A}^{\lambda\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1) - \boldsymbol{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21}(\psi_{21}) \equiv \mathbf{0}. \tag{A.4.3b}
$$

Together with

$$
\boldsymbol{A}^{\lambda\kappa} = \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{A}^{i\kappa} = \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa}, \tag{A.4.3c}
$$

this yields

$$
\bar{\boldsymbol{g}}_1 = \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1) - \boldsymbol{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21}(\psi_{21}) = \mathbf{0}. \tag{A.4.4a}
$$

Its first and second time derivatives are, respectively,

$$\begin{aligned}\dot{\boldsymbol{g}}_1 = \; & \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \tilde{\boldsymbol{\omega}}^i_{ij} \cdot \boldsymbol{A}^{i\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1) \\ & + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa} \cdot \dot{\boldsymbol{A}}^{\kappa 1}(\varphi_1) - \dot{\boldsymbol{A}}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21} = \boldsymbol{0},\end{aligned} \tag{A.4.4b}$$

and

$$\begin{aligned}\ddot{\boldsymbol{g}}_1 = \; & \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \left(\tilde{\boldsymbol{\omega}}^i_{ij} \cdot \tilde{\boldsymbol{\omega}}^i_{ij} \cdot \dot{\tilde{\boldsymbol{\omega}}}^i_{ij}\right) \cdot \boldsymbol{A}^{i\kappa} \cdot \boldsymbol{A}^{\kappa 1} \\ & + 2 \cdot \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \tilde{\boldsymbol{\omega}}^i_{ij} \cdot \boldsymbol{A}^{i\kappa} \cdot \dot{\boldsymbol{A}}^{\kappa 1}(\varphi_1) \\ & + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa} \cdot \ddot{A}^{\kappa 1}(\varphi_1) - \ddot{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21} = \boldsymbol{0}.\end{aligned} \tag{A.4.4c}$$

Using the abbreviations $\mathrm{c}\,\varphi := \cos\varphi$ and $\mathrm{s}\,\varphi := \sin\varphi$ yields, together with $\varphi_1 := \varphi_{L_\kappa L_1}$ as the rotation angle around $\boldsymbol{e}_{xL_1} = \boldsymbol{e}_{xL_\kappa}$, the relation (cf. Equation 2.55)

$$\begin{aligned}\boldsymbol{A}^{\kappa 1} &= \begin{pmatrix} 1\,, & 0\,, & 0 \\ 0\,, & \mathrm{c}\,\varphi_1\,, & \mathrm{s}\,\varphi_1 \\ 0\,, & -\,\mathrm{s}\,\varphi_1\,, & \mathrm{c}\,\varphi_1 \end{pmatrix}, \\ \dot{\boldsymbol{A}}^{\kappa 1} &= \begin{pmatrix} 0\,, & 0\,, & 0 \\ 0\,, & -\,\mathrm{s}\,\varphi_1\,, & \mathrm{c}\,\varphi_1 \\ 0\,, & -\,\mathrm{c}\,\varphi_1\,, & -\,\mathrm{s}\,\varphi_1 \end{pmatrix} \cdot \dot{\varphi}_1, \quad \text{and} \\ \ddot{A}^{\kappa 1} &= \begin{pmatrix} 0\,, & 0\,, & 0 \\ 0\,, & -\,\mathrm{s}\,\varphi_1\,, & \mathrm{c}\,\varphi_1 \\ 0\,, & -\,\mathrm{c}\,\varphi_1\,, & -\,\mathrm{s}\,\varphi_1 \end{pmatrix} \cdot \ddot{\varphi}_1 + \begin{pmatrix} 0\,, & 0\,, & 0 \\ 0\,, & -\,\mathrm{c}\,\varphi_1\,, & -\,\mathrm{s}\,\varphi_1 \\ 0\,, & \mathrm{s}\,\varphi_1\,, & -\,\mathrm{c}\,\varphi_1 \end{pmatrix} \cdot \dot{\varphi}_1^2,\end{aligned} \tag{A.4.5a}$$

and, with $\theta_2 = \theta_{L_\lambda L_2}$ as the rotation angle around $\boldsymbol{e}_{yL_2} = \boldsymbol{e}_{yL_\lambda}$, the relation

$$\begin{aligned}\boldsymbol{A}^{\lambda 2} &= \begin{pmatrix} \mathrm{c}\,\theta_2\,, & 0\,, & -\,\mathrm{s}\,\theta_2 \\ 0\,, & 1\,, & 0 \\ \mathrm{s}\,\theta_2\,, & 0\,, & \mathrm{c}\,\theta_2 \end{pmatrix}, \\ \dot{\boldsymbol{A}}^{\lambda 2} &= \begin{pmatrix} -\,\mathrm{s}\,\theta_2\,, & 0\,, & -\,\mathrm{c}\,\theta_2 \\ 0\,, & 0\,, & 0 \\ \mathrm{c}\,\theta_2\,, & 0\,, & -\,\mathrm{s}\,\theta_2 \end{pmatrix} \cdot \dot{\theta}_2, \quad \text{and} \\ \ddot{A}^{\lambda 2} &= \begin{pmatrix} -\,\mathrm{c}\,\theta_2\,, & 0\,, & \mathrm{s}\,\theta_2 \\ 0\,, & 0\,, & 0 \\ -\,\mathrm{s}\,\theta_2\,, & 0\,, & -\,\mathrm{c}\,\theta_2 \end{pmatrix} \cdot \dot{\theta}_2^2 + \begin{pmatrix} -\,\mathrm{s}\,\theta_2\,, & 0\,, & -\,\mathrm{c}\,\theta_2 \\ 0\,, & 0\,, & 0 \\ \mathrm{c}\,\theta_2\,, & 0\,, & -\,\mathrm{s}\,\theta_2 \end{pmatrix} \cdot \ddot{\theta}_2.\end{aligned} \tag{A.4.5b}$$

Using $\psi_{21} := \psi_{L_2 L_1}$ as the constant rotation angle of L_2 with respect to L_1 about the common $\boldsymbol{e}_{zL_1} = \boldsymbol{e}_{zL_2}$ axis yields the relations

$$\boldsymbol{A}^{21} = \begin{pmatrix} \mathrm{c}\,\psi_{21} & , \mathrm{s}\,\psi_{21} & , 0 \\ -\,\mathrm{s}\,\psi_{21} & , \mathrm{c}\,\psi_{21} & , 0 \\ 0 & , \quad 0 & , 1 \end{pmatrix} \quad \text{and} \quad \dot{\boldsymbol{A}}^{21} \equiv \mathbf{0}. \tag{A.4.5c}$$

Then terms $\boldsymbol{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21}(\psi_{21})$ and $\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1)$ of (A.4.4a) are

$$\boldsymbol{A}^{\lambda 2} \cdot \boldsymbol{A}^{21} = \begin{pmatrix} \mathrm{c}\,\theta_2 & , 0 & , -\,\mathrm{s}\,\theta_2 \\ 0 & , 1 & , \quad 0 \\ \mathrm{s}\,\theta_2 & , 0 & , \quad \mathrm{c}\,\theta_2 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{c}\,\psi_{21} & , \mathrm{s}\,\psi_{21} & , 0 \\ -\,\mathrm{s}\,\psi_{21} & , \mathrm{c}\,\psi_{21} & , 0 \\ 0 & , \quad 0 & , 1 \end{pmatrix} \tag{A.4.6a}$$

and

$$\boldsymbol{A}^{\lambda\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1) = \begin{pmatrix} a_{11} & , a_{12} & , a_{13} \\ a_{21} & , a_{22} & , a_{23} \\ a_{31} & , a_{32} & , a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 , & \quad 0 & , \quad 0 \\ 0 , & \quad \mathrm{c}\,\varphi_1 & , \mathrm{s}\,\varphi_1 \\ 0 , & -\,\mathrm{s}\,\varphi_1 & , \mathrm{c}\,\varphi_1 \end{pmatrix}, \tag{A.4.6b}$$

respectively. Multiplication of the left-hand side of (A.4.6a) and (A.4.6b) by $\mathbf{P_r}^{\mathrm{T}}(y)$ and right-hand side by $\mathbf{P}_r(x)$ yields the projections

$$\mathbf{P_r}^{\mathrm{T}}(y)\boldsymbol{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21}\,\mathbf{P}_r(x) = -\sin\psi_{21} \tag{A.4.7a}$$

and

$$\mathbf{P_r}^{\mathrm{T}}(y)\boldsymbol{A}^{\lambda\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1)\,\mathbf{P}_r(x) = a_{21} = \mathbf{P_r}^{\mathrm{T}}(y)\boldsymbol{A}^{\lambda\kappa}\,\mathbf{P}_r(x) \tag{A.4.7b}$$

that are *independent of* φ_1. This *eliminates* the yet unknown relative coordinates φ_1 and θ_2 from (A.4.4a) and provides the *first constraint orientation equation*

$$0 \equiv g_1 := \mathbf{P_r}^{\mathrm{T}}(y)\,\boldsymbol{A}^{\lambda\kappa} \cdot \boldsymbol{A}^{\kappa 1}(\varphi_1)\,\mathbf{P}_r(x) - \mathbf{P_r}^{\mathrm{T}}\,\boldsymbol{A}^{\lambda 2}(\theta_2) \cdot \boldsymbol{A}^{21}(\psi_{21})\,\mathbf{P}_r(x)$$

or (A.4.8a)

$$g_1 := \mathbf{P_r}^{\mathrm{T}}(y)\,\boldsymbol{A}^{\lambda\kappa}\,\mathbf{P}_r(x) + \sin\psi_{21} \equiv 0$$

or (A.4.8b)

$$g_1 = \mathbf{P_r}^{\mathrm{T}}(y)\,\left(\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \boldsymbol{A}^{i\kappa}\right)\,\mathbf{P}_r(x) + \sin\psi_{21} = 0 \quad , \quad \psi_{21} = \text{constant},$$

and the associated *constraint velocity* and *acceleration equations*

$$\dot{g}_1 = \mathbf{P_r}^{\mathrm{T}}(y)\,\left(\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \tilde{\boldsymbol{\omega}}^{i}_{ji} \cdot \boldsymbol{A}^{i\kappa}\right)\,\mathbf{P}_r(x) = 0 \tag{A.4.9a}$$

and

$$\ddot{g}_1 = \mathbf{P_r}^{\mathrm{T}}(y)\,\left[\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \left(\tilde{\boldsymbol{\omega}}^{i}_{ij} \cdot \tilde{\boldsymbol{\omega}}^{i}_{ij} + \dot{\tilde{\boldsymbol{\omega}}}^{i}_{ji}\right) \cdot \boldsymbol{A}^{i\kappa}\right]\,\mathbf{P}_r(x) = 0. \tag{A.4.9b}$$

Together with

$$\begin{aligned}
\boldsymbol{A}^{ij} &= \boldsymbol{A}^{i0}\cdot\boldsymbol{A}^{0j} \quad , \quad \boldsymbol{\omega}^{i}_{ij} = \boldsymbol{\omega}^{i}_{i0} - \boldsymbol{\omega}^{i}_{j0} = \boldsymbol{\omega}^{i}_{i0} - \boldsymbol{A}^{ij}\cdot\boldsymbol{\omega}^{j}_{j0}, && \text{(A.4.10a)}\\
\dot{\boldsymbol{\omega}}^{i}_{ji} &= \dot{\boldsymbol{\omega}}^{i}_{i0} - \boldsymbol{A}^{ij}\cdot\tilde{\boldsymbol{\omega}}^{j}_{ji}\cdot\boldsymbol{\omega}^{j}_{j0} - \boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}\\
&= \dot{\boldsymbol{\omega}}^{i}_{i0} - \boldsymbol{A}^{ij}\cdot\left(\tilde{\boldsymbol{\omega}}^{j}_{j0} - \tilde{\boldsymbol{\omega}}^{j}_{i0}\right)\cdot\boldsymbol{\omega}^{j}_{j0} - \boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}\\
&= \dot{\boldsymbol{\omega}}^{i}_{i0} + \boldsymbol{A}^{ij}\cdot\tilde{\boldsymbol{\omega}}^{j}_{i0}\cdot\boldsymbol{\omega}^{j}_{j0} - \boldsymbol{A}^{ij}\cdot\tilde{\boldsymbol{\omega}}^{j}_{j0}\cdot\boldsymbol{\omega}^{j}_{j0} - \boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}\\
&= \dot{\boldsymbol{\omega}}^{i}_{i0} + \tilde{\boldsymbol{\omega}}^{i}_{i0}\cdot\boldsymbol{A}^{ij}\cdot\boldsymbol{\omega}^{j}_{j0} - \boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}, && \text{(A.4.10b)}\\
\tilde{\dot{\boldsymbol{\omega}}}^{i}_{ji} &= \tilde{\dot{\boldsymbol{\omega}}}^{i}_{i0} + \widetilde{\tilde{\boldsymbol{\omega}}^{i}_{i0}\cdot\boldsymbol{A}^{ij}\cdot\boldsymbol{\omega}^{j}_{j0}} - \widetilde{\boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}}, && \text{(A.4.10c)}
\end{aligned}$$

and

$$\dot{\tilde{\boldsymbol{\omega}}} = \tilde{\dot{\boldsymbol{\omega}}}, \tag{A.4.10d}$$

the *constraint acceleration* equation is

$$\begin{aligned}
0 = \ddot{g}_1 &= \mathbf{P_r}^{\mathrm{T}}(y)\left(\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\tilde{\boldsymbol{\omega}}^{i}_{ij}\cdot\tilde{\boldsymbol{\omega}}^{i}_{ij}\cdot\boldsymbol{A}^{i\kappa}\right)\mathbf{P}_r(x)\\
&+ \mathbf{P_r}^{\mathrm{T}}(y)\left[\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\left(\tilde{\dot{\boldsymbol{\omega}}}^{i}_{i0} - \widetilde{\boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}} + \widetilde{\tilde{\boldsymbol{\omega}}^{i}_{i0}\cdot\boldsymbol{A}^{ij}\cdot\boldsymbol{\omega}^{j}_{j0}}\right)\cdot\boldsymbol{A}^{i\kappa}\right]\mathbf{P}_r(x)
\end{aligned}$$

or

$$\begin{aligned}
\ddot{g}_1 = \quad & \mathbf{P_r}^{\mathrm{T}}(y)\left(\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\tilde{\dot{\boldsymbol{\omega}}}^{i}_{i0}\cdot\boldsymbol{A}^{i\kappa}\right)\mathbf{P}_r(x)\\
&- \mathbf{P_r}^{\mathrm{T}}(y)\left(\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\widetilde{\boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}}\cdot\boldsymbol{A}^{i\kappa}\right)\mathbf{P}_r(x)\\
&+ \mathbf{P_r}^{\mathrm{T}}(y)\left(\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\tilde{\boldsymbol{\omega}}^{i}_{ij}\cdot\tilde{\boldsymbol{\omega}}^{i}_{ij}\cdot\boldsymbol{A}^{i\kappa}\right)\mathbf{P}_r(x)\\
&+ \mathbf{P_r}^{\mathrm{T}}(y)\left[\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\widetilde{\left(\tilde{\boldsymbol{\omega}}^{i}_{i0}\cdot\boldsymbol{A}^{ij}\cdot\boldsymbol{\omega}^{j}_{j0}\right)}\cdot\boldsymbol{A}^{i\kappa}\right]\mathbf{P}_r(x) = 0.
\end{aligned}$$

Taking into account the relations

$$\tilde{\dot{\boldsymbol{\omega}}}^{i}_{i0}\cdot\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x) = -\widetilde{\left(\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x)\right)}\cdot\dot{\boldsymbol{\omega}}^{i}_{i0}$$

and (A.4.10e)

$$\widetilde{\left(\boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}\right)}\cdot\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x) = -\widetilde{\left(\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x)\right)}\cdot\boldsymbol{A}^{ij}\cdot\dot{\boldsymbol{\omega}}^{j}_{j0}$$

yields the final form of the *constraint acceleration equation*

$$\underbrace{\left[0\,,\,-\mathbf{P_r}^{\mathrm{T}}(y)\left[\boldsymbol{A}^{\lambda j}\cdot\boldsymbol{A}^{ji}\cdot\widetilde{\left(\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x)\right)}\right]\,,\,0\,,\,\mathbf{P_r}^{\mathrm{T}}(y)\left[\boldsymbol{A}^{\lambda j}\cdot\widetilde{\left(\boldsymbol{A}^{i\kappa}\,\mathbf{P}_r(x)\right)}\cdot\boldsymbol{A}^{ij}\right]\right]}_{=:\,\boldsymbol{g}_{1p}(\boldsymbol{p})\cdot\boldsymbol{T}(\boldsymbol{p})}$$

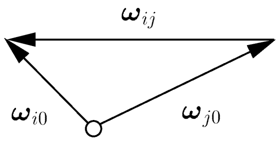
$$
\cdot \begin{bmatrix} \ddot{\boldsymbol{r}}_i^0 \\ \dot{\boldsymbol{\omega}}_{i0}^i \\ \ddot{\boldsymbol{r}}_j^0 \\ \dot{\boldsymbol{\omega}}_{j0}^j \end{bmatrix} = -\mathbf{P_r}^{\mathrm{T}}(y) \left(\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \tilde{\boldsymbol{\omega}}_{ij}^i \cdot \tilde{\boldsymbol{\omega}}_{ij}^i \cdot \boldsymbol{A}^{i\kappa} \cdot \mathbf{P}_r(x) \right) \tag{A.4.11}
$$

$$
- \mathbf{P_r}^{\mathrm{T}}(y) \left[\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \left(\tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{A}^{ij} \cdot \boldsymbol{\omega}_{j0}^j \right) \cdot \boldsymbol{A}^{i\kappa}\, \mathbf{P}_r(x) \right]
$$

$$
= \underbrace{-\mathbf{P_r}^{\mathrm{T}}(y) \left\{ \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{ji} \cdot \left[\tilde{\boldsymbol{\omega}}_{ij}^i \cdot \tilde{\boldsymbol{\omega}}_{ij}^i + \left(\tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{A}^{ij} \cdot \boldsymbol{\omega}_{j0}^j \right) \right] \right\} \mathbf{P}_r(x),}_{=:\, \beta_{c1}}
$$

where

$$
\begin{aligned}
\boldsymbol{A}^{ij} &= \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0i}, \\
\boldsymbol{\omega}_{ij}^i &= \boldsymbol{\omega}_{i0}^i - \boldsymbol{A}^{ij} \cdot \boldsymbol{\omega}_{j0}^j, \\
\dot{\boldsymbol{\omega}}_{ij}^i &= \dot{\boldsymbol{\omega}}_{i0}^i - \boldsymbol{A}^{ij} \cdot \dot{\boldsymbol{\omega}}_{j0}^j - \boldsymbol{A}^{ij} \cdot \tilde{\boldsymbol{\omega}}_{ji}^j \cdot \boldsymbol{\omega}_{j0}^j, \\
\boldsymbol{A}^{\lambda j} &= \boldsymbol{A}^{\lambda 2} \cdot \boldsymbol{A}^{2j}, \quad \text{and} \\
\boldsymbol{A}^{\kappa i} &= \boldsymbol{A}^{\kappa 1} \cdot \boldsymbol{A}^{1i}.
\end{aligned} \tag{A.4.12}
$$



Comment A.4.1 (Computation of φ_1 and θ_2): The *relative coordinates $\varphi_1(t)$ and $\theta_2(t)$* may be isolated by suitable projections of the preceding orientation loop equations. Here these relative rotation angles will be computed later by means of proper projections of suitable *vector loop equations* of the mechanism.

A.4.2.2 Second constraint equation. The *second constraint equation* that eliminates a translational DOF of the mechanism is computed from a suitable projection of the vector loop equation (Figure A.4.2)

$$
\mathbf{0} = \boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 + \boldsymbol{s}_i^0 - \boldsymbol{s}_j^0 + \boldsymbol{s}_2^0. \tag{A.4.13}
$$

Projecting this equation into frame L_1 yields, together with

$$
\boldsymbol{s}_2^1 = (0\,,\,0\,,\,s_{z2})\,, \quad s_{z2} > 0, \tag{A.4.14a}
$$

the relation

$$
\mathbf{0} = \boldsymbol{s}_2^1 + \boldsymbol{A}^{1\kappa} \cdot \boldsymbol{A}^{\kappa 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j \right] \tag{A.4.14b}
$$

or

$$
\mathbf{0} = \boldsymbol{A}^{\kappa 1} \cdot \boldsymbol{s}_2^1 + \boldsymbol{A}^{\kappa 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j \right]. \tag{A.4.14c}
$$

Together with

$$\boldsymbol{A}^{\kappa 1} \cdot \boldsymbol{s}_2^1 = \begin{pmatrix} 1\,, & 0 & , \; 0 \\ 0\,, & \mathrm{c}\,\varphi_1 & , \; \mathrm{s}\,\varphi_1 \\ 0\,, & -\,\mathrm{s}\,\varphi_1 & , \; \mathrm{c}\,\varphi_1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ s_{z2} \end{pmatrix} = \begin{pmatrix} 0 \\ s_{z2} \cdot \sin\varphi_1 \\ s_{z2} \cdot \cos\varphi_1 \end{pmatrix}, \tag{A.4.14d}$$

this yields the relation

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ s_{z2} \cdot \sin\varphi_1 \\ s_{z2} \cdot \cos\varphi_1 \end{pmatrix} + \bar{\boldsymbol{g}}_2, \tag{A.4.15a}$$

with

$$\begin{aligned} \bar{\boldsymbol{g}}_2 :&= \boldsymbol{A}^{\kappa 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j \right] \\ &= \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{s}_j^j. \end{aligned} \tag{A.4.15b}$$

These three scalar equations *only include the relative coordinate* φ_1. They will be used to *derive the second constraint equation* and to *isolate the relative coordinate* φ_1. The projection

$$g_2 + \mathbf{P_r}^{\mathrm{T}}(x) \left[\boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{s}_j^j \right] := 0 \tag{A.4.16a}$$

of (A.4.16a) is chosen as the *second constraint position equation.* It no longer includes the relative coordinate $\varphi_1(t)$. The associated *constraint velocity equation* is

$$\begin{aligned} \dot{g}_2 = \mathbf{P_r}^{\mathrm{T}}(x) \Big\{ & \boldsymbol{A}^{\kappa i} \cdot \Big[\dot{\boldsymbol{A}}^{i0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\ & - \dot{\boldsymbol{A}}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{i0} \cdot \dot{\boldsymbol{A}}^{0j} \boldsymbol{s}_j^j \Big] \Big\} = 0 \end{aligned}$$

or

$$\begin{aligned} \dot{g}_2(t) = \mathbf{P_r}^{\mathrm{T}}(x) \Big\{ & \boldsymbol{A}^{\kappa i} \cdot \Big[\boldsymbol{A}^{i0} \cdot \tilde{\boldsymbol{\omega}}_{0i}^0 \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\ & - \boldsymbol{A}^{i0} \cdot \left(\tilde{\boldsymbol{\omega}}_{0i}^0 \cdot \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j + \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{s}_j^j \right) \Big] \Big\} = 0, \end{aligned}$$

and finally

$$\begin{aligned} \dot{g}_2(t) = \mathbf{P_r}^{\mathrm{T}}(x) \Big\{ & \boldsymbol{A}^{\kappa i} \cdot \Big[-\tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\ & - \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{s}_j^j \Big] \Big\} = 0. \end{aligned} \tag{A.4.16b}$$

The associated constraint acceleration equation is

$$
\begin{aligned}
\ddot{g}_2 = \mathbf{P_r}^{\mathrm{T}}(x)\, \boldsymbol{A}^{\kappa i} \cdot \Big\{ & -\dot{\tilde{\boldsymbol{\omega}}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j\right) - \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \tilde{\boldsymbol{\omega}}^{0}_{0i} \cdot \left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j\right) \\
& - \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}^0_i - \dot{\boldsymbol{r}}^0_j\right) + \boldsymbol{A}^{i0} \cdot \tilde{\boldsymbol{\omega}}^{0}_{0i} \cdot \left(\dot{\boldsymbol{r}}^0_i - \dot{\boldsymbol{r}}^0_j\right) \\
& + \boldsymbol{A}^{i0} \cdot \left(\ddot{r}^0_i - \ddot{r}^0_j\right) + \dot{\tilde{\boldsymbol{\omega}}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j \\
& + \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \tilde{\boldsymbol{\omega}}^{0}_{0i} \cdot \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j + \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \\
& - \boldsymbol{A}^{i0} \cdot \tilde{\boldsymbol{\omega}}^{0}_{0i} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j - \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \\
& - \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \dot{\tilde{\boldsymbol{\omega}}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \Big\} = 0
\end{aligned}
$$

or

$$
\begin{aligned}
\ddot{g}_2 = \; \mathbf{P_r}^{\mathrm{T}}(x)\, \boldsymbol{A}^{\kappa i} \cdot \Big\{ \; & \boldsymbol{A}^{i0} \cdot \ddot{r}^0_i + \widetilde{\boldsymbol{A}^{i0} \cdot \left[\left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j\right) - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j\right]} \cdot \dot{\boldsymbol{\omega}}^{i}_{i0} \\
& - \boldsymbol{A}^{i0} \cdot \ddot{r}^0_j + \left(\boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{s}}^j_j\right) \cdot \dot{\boldsymbol{\omega}}^{j}_{j0} \Big\} \\
+ \mathbf{P_r}^{\mathrm{T}}(x)\, \boldsymbol{A}^{\kappa i} \cdot \Big\{ \; & \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j\right) \\
& - 2 \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}^0_i - \dot{\boldsymbol{r}}^0_j\right) - \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \\
& + 2 \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \Big\} = 0,
\end{aligned}
$$

and finally

$$
\begin{aligned}
& \underbrace{\begin{aligned} \Big[\mathbf{P_r}^{\mathrm{T}}(x) \left(\boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0}\right), \mathbf{P_r}^{\mathrm{T}}(x) \Big\{\boldsymbol{A}^{\kappa i} \cdot \widetilde{\left[\boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j\right) - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j\right]}\Big\}, \\ -\mathbf{P_r}^{\mathrm{T}}(x) \left(\boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0}\right), \mathbf{P_r}^{\mathrm{T}}(x) \left(\boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{s}}^j_j\right)\Big] \end{aligned}}_{=: \, \boldsymbol{g}_{2p}(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})} \\
& \cdot \begin{bmatrix} \ddot{\boldsymbol{r}}^0_i \\ \dot{\boldsymbol{\omega}}^{i}_{i0} \\ \ddot{\boldsymbol{r}}^0_j \\ \dot{\boldsymbol{\omega}}^{j}_{j0} \end{bmatrix} = \underbrace{\begin{aligned} & -\mathbf{P_r}^{\mathrm{T}}(x) \Big\{\boldsymbol{A}^{\kappa i} \cdot \Big[\tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\boldsymbol{r}^0_i - \boldsymbol{r}^0_j - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}^j_j\right) \\ & -\boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j + 2 \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}^{j}_{j0} \cdot \boldsymbol{s}^j_j \\ & -2 \cdot \tilde{\boldsymbol{\omega}}^{i}_{i0} \cdot \boldsymbol{A}^{i0} \cdot \left(\dot{\boldsymbol{r}}^0_i - \dot{\boldsymbol{r}}^0_j\right)\Big]\Big\} \, . \end{aligned}}_{=: \, \beta_{c2}}
\end{aligned}
\tag{A.4.17}
$$

Isolation of the relative coordinate $\varphi_1(t)$

The relative coordinate $\varphi_1(t)$ is isolated by projecting the vector loop equation (A.4.15a) to its second and third component. This yields the relations

$$\sin\varphi_1 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(y)\ (\bar{\boldsymbol{g}}_2) \quad \text{for} \quad s_{z2} \neq 0 \tag{A.4.18a}$$

and

$$\cos\varphi_1 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ (\bar{\boldsymbol{g}}_2) \quad \text{for} \quad s_{z2} \neq 0, \tag{A.4.18b}$$

with

$$\bar{\boldsymbol{g}}_2 = \boldsymbol{A}^{\kappa i} \cdot \boldsymbol{A}^{i0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0\right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j\right]. \tag{A.4.18c}$$

This yields the *relative rotation angle*

$$\varphi_1 = \varphi_{\kappa 1} = \arctan\left[\frac{\mathbf{P_r}^{\mathrm{T}}(y)\,(\bar{\boldsymbol{g}}_2)}{\mathbf{P_r}^{\mathrm{T}}(z)\,(\bar{\boldsymbol{g}}_2)}\right] \quad \text{for} \quad -\pi \leq \varphi_1 \leq \pi. \tag{A.4.19}$$

The time derivatives of (A.4.18a) and (A.4.18b),

$$\dot{\varphi}_1 \cdot \cos\varphi_1 = -\frac{1}{s_{z2}}\, \mathbf{P_r}^{\mathrm{T}}(y)\ \left(\dot{\bar{\boldsymbol{g}}}_2\right) \tag{A.4.20a}$$

and

$$\dot{\varphi}_1 \cdot \sin\varphi_1 = \frac{1}{s_{z2}}\, \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\dot{\bar{\boldsymbol{g}}}_2\right), \tag{A.4.20b}$$

finally yield the relations

$$\dot{\varphi}_1 = \begin{cases} -\dfrac{1}{s_{z2} \cdot \cos\varphi_1} \cdot \mathbf{P_r}^{\mathrm{T}}(y)\ \left(\dot{\bar{\boldsymbol{g}}}_1\right) & \text{for} \quad 0 < |\sin\varphi_1| < \varepsilon \\[2ex] \dfrac{1}{s_{z2} \cdot \sin\varphi_1} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\dot{\bar{\boldsymbol{g}}}_1\right) & \text{for} \quad 0 < |\cos\varphi_1| < \varepsilon. \end{cases} \tag{A.4.21}$$

The time derivatives of (A.4.20a) and (A.4.20b),

$$\ddot{\varphi}_1 \cdot \cos\varphi_1 - \dot{\varphi}_1^2 \cdot \sin\varphi_1 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(y)\,\left(\ddot{\bar{\boldsymbol{g}}}_2\right)$$

and

$$\ddot{\varphi}_1 \cdot \sin\varphi_1 + \dot{\varphi}_1^2 \cdot \cos\varphi_1 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\,\left(\ddot{\bar{\boldsymbol{g}}}_2\right),$$

yield the relation

$$\begin{pmatrix} \cos\varphi_1 \ , & -\sin\varphi_1 \\ \sin\varphi_1 \ , & \cos\varphi_1 \end{pmatrix} \cdot \begin{pmatrix} \ddot{\varphi}_1 \\ \dot{\varphi}_1^2 \end{pmatrix} = \frac{1}{s_{z2}} \cdot \begin{pmatrix} \mathbf{P_r}^{\mathrm{T}}(y)\ \left(\ddot{\bar{\boldsymbol{g}}}_2\right) \\ \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\ddot{\bar{\boldsymbol{g}}}_2\right) \end{pmatrix}, \tag{A.4.22a}$$

and with

$$\cos^2\varphi_1 + \sin^2\varphi_1 = 1 \neq 0, \tag{A.4.22b}$$

its solution

$$\begin{pmatrix} \ddot{\varphi}_1 \\ \dot{\varphi}_1^2 \end{pmatrix} = \frac{1}{s_{z2}} \cdot \begin{pmatrix} \cos\varphi_1 \ , \ \sin\varphi_1 \\ -\sin\varphi_1 \ , \ \cos\varphi_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{P_r}^{\mathrm{T}}(y) \ \left(\ddot{\boldsymbol{g}}_2\right) \\ \mathbf{P_r}^{\mathrm{T}}(z) \ \left(\ddot{\boldsymbol{g}}_2\right) \end{pmatrix},$$

and the final result

$$\ddot{\varphi}_1(t) = \frac{1}{s_{z2}} \cdot \left[\cos\varphi_1 \cdot \mathbf{P_r}^{\mathrm{T}}(y) \ \left(\ddot{\boldsymbol{g}}_2\right) + \sin\varphi_1 \cdot \mathbf{P_r}^{\mathrm{T}}(z) \ \left(\ddot{\boldsymbol{g}}_2\right)\right], \quad s_{z2} \neq 0. \tag{A.4.23}$$

A.4.2.3 Third constraint equation. The third constraint equation that eliminates another translational DOF of the mechanism is computed from another representation and projection of the vector loop equation (A.4.13)

$$\mathbf{0} = \boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 + \boldsymbol{s}_i^0 - \boldsymbol{s}_j^0 + \boldsymbol{s}_2^0. \tag{A.4.13}$$

Representation of this equation in L_2 yields

$$\mathbf{0} = \boldsymbol{s}_2^2 + \boldsymbol{A}^{2\lambda}(\theta_2) \cdot \boldsymbol{A}^{\lambda 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0\right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j\right] \tag{A.4.24a}$$

with the constant vector

$$\boldsymbol{s}_2^2 = (0\,,\,0\,,\,-s_{z2})^{\mathrm{T}}\,, \tag{A.4.24b}$$

the rotation matrix

$$\boldsymbol{A}^{\lambda 2}(\theta_2) = \begin{pmatrix} \mathrm{c}\,\theta_2 \ , \ 0 \ , \ -\ \mathrm{s}\,\theta_2 \\ 0 \ \ , \ 1 \ , \ \ 0 \\ \mathrm{s}\,\theta_2 \ , \ 0 \ , \ \ \mathrm{c}\,\theta_2 \end{pmatrix} \tag{A.4.24c}$$

and the relation

$$\boldsymbol{A}^{\lambda 2} \cdot \boldsymbol{s}_2^2 = \begin{pmatrix} \mathrm{c}\,\theta_2 \ , \ 0 \ , \ -\ \mathrm{s}\,\theta_2 \\ 0 \ \ , \ 1 \ , \ \ 0 \\ \mathrm{s}\,\theta_2 \ , \ 0 \ , \ \ \mathrm{c}\,\theta_2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -s_{z2} \end{pmatrix} = \begin{pmatrix} +\ s_{z2} \cdot \sin\theta_2 \\ 0 \\ -\ s_{z2} \cdot \cos\theta_2 \end{pmatrix}. \tag{A.4.24d}$$

This provides together with

$$\mathbf{0} = \boldsymbol{A}^{\lambda 2} \cdot \boldsymbol{s}_2^2 + \boldsymbol{A}^{\lambda 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0\right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j\right] \tag{A.4.25}$$

the relation

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} + \; s_{z2} \cdot \sin\theta_2 \\ 0 \\ - \; s_{z2} \cdot \cos\theta_2 \end{pmatrix} + \bar{\boldsymbol{g}}_3, \tag{A.4.26a}$$

with

$$\begin{aligned} \bar{\boldsymbol{g}}_3 :&= \boldsymbol{A}^{\lambda 0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j \right] \\ &= \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) - \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{s}_j^j + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i. \end{aligned} \tag{A.4.26b}$$

These three scalar equations only include the relative coordinate $\theta_2(t)$. They will be used to derive the *third constraint equation* and to isolate the *relative coordinate* $\theta_2(t)$. The projection

$$\boldsymbol{g}_3 := \mathbf{P_r}^{\mathrm{T}}(y) \left[\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) - \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{s}_j^j + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i \right] = 0 \tag{A.4.27a}$$

of (A.4.26a) is chosen as the *third constraint equation* of the mechanism. It no longer includes the relative coordinate $\theta_2(t)$. The associated *constraint velocity equation* is

$$\begin{aligned} \dot{\boldsymbol{g}}_3(t) = \mathbf{P_r}^{\mathrm{T}}(y) \Big\{ \; \boldsymbol{A}^{\lambda j} \cdot \Big[& \boldsymbol{A}^{j0} \cdot \tilde{\boldsymbol{\omega}}_{0j}^0 \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{j0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\ & + \boldsymbol{A}^{j0} \cdot \tilde{\boldsymbol{\omega}}_{0j}^0 \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i - \boldsymbol{0} \Big] \Big\} = 0 \end{aligned}$$

or

$$\begin{aligned} \dot{g}_3 = \mathbf{P_r}^{\mathrm{T}}(y) \Big\{ \; \boldsymbol{A}^{\lambda j} \cdot \Big[& - \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{j0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\ & - \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \Big] \Big\} = 0. \end{aligned} \tag{A.4.27b}$$

The associated *constraint acceleration equation* is

$$\begin{aligned} \ddot{g}_3 = \mathbf{P_r}^{\mathrm{T}}(y) \, \boldsymbol{A}^{\lambda j} \cdot \Big[& - \dot{\tilde{\boldsymbol{\omega}}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) \\ & - 2 \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) + \boldsymbol{A}^{j0} \cdot \left(\ddot{\boldsymbol{r}}_i^0 - \ddot{\boldsymbol{r}}_j^0 \right) \\ & - \dot{\tilde{\boldsymbol{\omega}}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i + \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i \\ & - \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \\ & - \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \dot{\tilde{\boldsymbol{\omega}}}_{i0}^i \cdot \boldsymbol{s}_i^i \Big] = 0 \end{aligned}$$

or

$$
\begin{aligned}
\ddot{g}_3 = \; & \mathbf{P_r}^{\mathrm{T}}(y) \; \Big\{ \; \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \, , \, -\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{s}}_i^i \, , \, -\boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \, , \\
& \qquad \boldsymbol{A}^{\lambda j} \cdot \left[\boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^R - \boldsymbol{r}_j^R \right) \right] \Big\} \cdot \\
& \qquad \left[\left(\ddot{r}_i^0 \right)^{\mathrm{T}} , \left(\dot{\boldsymbol{\omega}}_{i0}^i \right)^{\mathrm{T}} , \left(\ddot{r}_j^0 \right)^{\mathrm{T}} , \left(\dot{\boldsymbol{\omega}}_{j0}^j \right)^{\mathrm{T}} \right]^{\mathrm{T}} \\
& + \mathbf{P_r}^{\mathrm{T}}(y) \; \Big\{ \; \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) \\
& \qquad -2 \cdot \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) \\
& \qquad + \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i \\
& \qquad -2 \cdot \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \\
& \qquad + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \qquad \Big\} = 0,
\end{aligned}
$$

and finally

$$
\begin{aligned}
& \underbrace{\begin{array}{r} \Big[\; \mathbf{P_r}^{\mathrm{T}}(y) \, \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \, , \, -\mathbf{P_r}^{\mathrm{T}}(y) \, \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{s}}_i^i \, , \, -\mathbf{P_r}^{\mathrm{T}}(y) \, \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \, , \\ \mathbf{P_r}^{\mathrm{T}}(y) \, \boldsymbol{A}^{\lambda j} \cdot \left\{ \boldsymbol{A}^{j0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i \right] \right\} \; \Big] \end{array}}_{=: \, \boldsymbol{g}_{3p} \cdot \boldsymbol{T}(\boldsymbol{p})} \\
& \cdot \left[\left(\ddot{\boldsymbol{r}}_i^0 \right)^{\mathrm{T}} , \left(\dot{\boldsymbol{\omega}}_{i0}^i \right)^{\mathrm{T}} , \left(\ddot{\boldsymbol{r}}_j^0 \right)^{\mathrm{T}} , \left(\dot{\boldsymbol{\omega}}_{j0}^j \right)^{\mathrm{T}} \right]^{\mathrm{T}} \\
& \underbrace{\begin{array}{l} = -\mathbf{P_r}^{\mathrm{T}}(y) \; \Big\{ \; \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left[\left(\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i \right] \\ \qquad + \boldsymbol{A}^{\lambda j} \cdot \boldsymbol{A}^{j0} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \\ \qquad -2 \cdot \boldsymbol{A}^{\lambda j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{A}^{j0} \cdot \left[\left(\dot{\boldsymbol{r}}_i^0 - \dot{\boldsymbol{r}}_j^0 \right) + \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i \right] \Big\}. \end{array}}_{=: \, \beta_{c3}}
\end{aligned}
\tag{A.4.27c}
$$

Isolation of the relative coordinate $\theta_2(t)$

The relative coordinate $\theta_2(t)$ is isolated by projecting the vector loop equation (A.4.26a) to its first and third component. This yields the relations

$$
\sin\theta_2 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(x) \; (\bar{\boldsymbol{g}}_3) \tag{A.4.28a}
$$

and

$$
\cos\theta_2 = \frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(z) \; (\bar{\boldsymbol{g}}_3) \, . \tag{A.4.28b}
$$

This yields the relative rotation angle

$$\theta_2 = \theta_{\lambda 2} = -\arctan\left\{\frac{\mathbf{P_r}^{\mathrm{T}}(y)\,(\bar{\boldsymbol{g}}_3)}{\mathbf{P_r}^{\mathrm{T}}(z)\,(\bar{\boldsymbol{g}}_3)}\right\} \quad \text{for} \quad -\pi < \theta_2 < \pi. \tag{A.4.29}$$

The time derivatives of (A.4.28a) and (A.4.28b),

$$\dot{\theta}_2 \cdot \cos\theta_2 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\dot{\bar{\boldsymbol{g}}}_3\right) \tag{A.4.30a}$$

and

$$\dot{\theta}_2 \cdot \sin\theta_2 = -\frac{1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\dot{\bar{\boldsymbol{g}}}_3\right), \tag{A.4.30b}$$

finally yield the relations

$$\dot{\theta}_2 = \begin{cases} \dfrac{-1}{s_{z2} \cdot \cos\theta_2} \cdot \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\dot{\bar{\boldsymbol{g}}}_3\right) & \text{for} \quad 0 < \sin\theta_2 < \varepsilon \\[2ex] \dfrac{-1}{s_{z2} \cdot \sin\theta_2} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\dot{\bar{\boldsymbol{g}}}_3\right) & \text{for} \quad 0 < \cos\theta_2 < \varepsilon. \end{cases} \tag{A.4.31}$$

The time derivatives of (A.4.30a) and (A.4.30b) yield

$$\ddot{\theta}_2 \cdot \cos\theta_2 - \dot{\theta}_2^2 \cdot \sin\theta_2 = \frac{-1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right),$$
$$\ddot{\theta}_2 \cdot \sin\theta_2 + \dot{\theta}_2^2 \cdot \cos\theta_2 = \frac{-1}{s_{z2}} \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right),$$

and, together, the relation

$$\begin{pmatrix} \cos\theta_2 & , & -\sin\theta_2 \\ \sin\theta_2 & , & \cos\theta_2 \end{pmatrix} \cdot \begin{pmatrix} \ddot{\theta}_2 \\ \dot{\theta}_2^2 \end{pmatrix} = \frac{-1}{s_{z2}} \cdot \begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right) \\ \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right) \end{bmatrix},$$

with the solution

$$\begin{pmatrix} \ddot{\theta}_2 \\ \dot{\theta}_2^2 \end{pmatrix} = \frac{-1}{s_{z2}} \cdot \begin{pmatrix} \cos\theta_2 & , & \sin\theta_2 \\ -\sin\theta_2 & , & \cos\theta_2 \end{pmatrix} \cdot \begin{bmatrix} \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right) \\ \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right) \end{bmatrix}$$

and finally

$$\ddot{\theta}_2 = \frac{-1}{s_{z2}} \cdot \left[\cos\theta_2 \cdot \mathbf{P_r}^{\mathrm{T}}(x)\ \left(\ddot{\bar{\boldsymbol{g}}}_3\right) + \sin\theta_2 \cdot \mathbf{P_r}^{\mathrm{T}}(z)\ \left(\ddot{\bar{\boldsymbol{g}}}_2\right)\right]. \tag{A.4.32}$$

A.4.2.4 Fourth constraint equation. The *fourth constraint equation* is again computed from the vector loop equation (A.4.13)

$$\boldsymbol{r}_i + \boldsymbol{s}_i + \boldsymbol{r}_{QP} - \boldsymbol{r}_j - \boldsymbol{s}_j = \boldsymbol{0} \quad , \quad \boldsymbol{r}_{QP} = \boldsymbol{s}_2,$$

or, represented in frame R,

$$\boldsymbol{r}_i^0 - \boldsymbol{r}_j^0 + \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j + \boldsymbol{r}_{QP}^0 = \boldsymbol{0}. \tag{A.4.33}$$

Taking into account the constant distance of the mechanism between the points P and Q, and introducing this distance as

$$\ell_0 := |\boldsymbol{r}_{QP}| = |\boldsymbol{s}_2| = |s_{z2}| = \text{constant}, \tag{A.4.34}$$

the following *constant-distance constraint relation* is introduced as the *fourth constraint position equation* of the mechanism (compare with the vector loop of Figure A.4.1 and *BB5* of *Chapter 5.2*):

$$g_4 := \left((\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{r}_{QP}^0\right)^{1/2} - \ell_0 \equiv 0 \quad , \quad \boldsymbol{r}_{QP}^0 = \left(r_{QP_1}^0 \, , \, r_{QP_2}^0 \, , \, r_{QP_3}^0\right). \tag{A.4.35}$$

Its time derivative is

$$\dot{g}_4 = \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[\sum_{i=1}^{3}(r_{QP_i}^0)^2\right]^{1/2} - \ell_0\right\} = \frac{\frac{1}{2}\cdot 2 \cdot \sum\limits_{i=1}^{3} r_{QP_i}^0 \cdot \dot{r}_{QP_i}^0}{\left[\sum\limits_{i=1}^{3}(r_{QP_i}^0)^2\right]} \tag{A.4.36a}$$

or

$$\dot{g}_4 = \frac{(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \dot{\boldsymbol{r}}_{QP}^0}{\left[(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{r}_{QP_i}^0\right]^{1/2}}. \tag{A.4.36b}$$

Assuming only small deviations of $|\boldsymbol{r}_{QP}|$ from ℓ_0, the substitution

$$\ell_0 = \left[(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{r}_{QP}^0\right]^{1/2} \tag{A.4.36c}$$

is introduced into (A.4.36b), which provides the following approximations of the *constraint velocity* and *acceleration equations*:

$$\dot{\bar{g}}_4 \approx \frac{(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \dot{\boldsymbol{r}}_{QP}^0}{\ell_0} \tag{A.4.37}$$

and

$$\ddot{\bar{g}}_4 \approx \frac{(\dot{\boldsymbol{r}}_{QP}^0)^{\mathrm{T}} \cdot \dot{\boldsymbol{r}}_{QP}^0}{\ell_0} + \frac{(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \ddot{\boldsymbol{r}}_{QP}^0}{\ell_0}. \tag{A.4.38}$$

This implies the *constraint position equation*

$$\begin{aligned} g_4 := \ell_0 + \Big\{ & \left[\boldsymbol{r}_j^0 - \boldsymbol{r}_i^0 + \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i\right]^{\mathrm{T}} \\ & \cdot \left[\boldsymbol{r}_j^0 - \boldsymbol{r}_i^0 + \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i\right] \Big\}^{1/2} = 0, \end{aligned} \tag{A.4.39}$$

the associated *constraint velocity equation*

$$\dot{g}_4 := (\dot{\bar{g}}_4) \cdot \ell_0 = 0 + \overbrace{\left[\boldsymbol{r}_j^0 - \boldsymbol{r}_i^0 + \boldsymbol{A}^{0j} \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{0i} \cdot \boldsymbol{s}_i^i\right]^{\mathrm{T}}}^{=(\boldsymbol{r}_{QP}^0)^{\mathrm{T}}} \cdot \left[\dot{\boldsymbol{r}}_j^0 - \dot{\boldsymbol{r}}_i^0 + \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i\right] = 0$$

or

$$\dot{g}_4 = 0 + \underbrace{\left[-(\boldsymbol{r}_{QP}^0)^{\mathrm{T}}\,,\, (\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{s}}_i^i\,,\, (\boldsymbol{r}_{QP}^0)^{\mathrm{T}}\,,\, -(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{s}}_j^j\right]}_{=:\,\boldsymbol{g}_{4p}(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})} \cdot \left[(\dot{\boldsymbol{r}}_i^0)^{\mathrm{T}}\,,\, (\boldsymbol{\omega}_{i0}^i)^{\mathrm{T}}\,,\, (\dot{\boldsymbol{r}}_j^0)^{\mathrm{T}}\,,\, (\boldsymbol{\omega}_{j0}^j)^{\mathrm{T}}\right]^{\mathrm{T}} = 0 \tag{A.4.40}$$

and, together with

$$\begin{aligned}\ddot{\boldsymbol{r}}_{QP}^0 = \; & \ddot{\boldsymbol{r}}_j^0 - \ddot{\boldsymbol{r}}_i^0 + \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{s}_j^j - \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{s}}_j^j \cdot \dot{\boldsymbol{\omega}}_{j0}^j \\ & - \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i + \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{s}}_i^i \cdot \dot{\boldsymbol{\omega}}_{i0}^i,\end{aligned} \tag{A.4.41}$$

the associated *constraint acceleration equation*

$$\begin{aligned}\ddot{g}_4 = & \underbrace{\left[-(\boldsymbol{r}_{QP}^0)^{\mathrm{T}}\,,\, (\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{s}}_i^i\,,\, (\boldsymbol{r}_{QP}^0)^{\mathrm{T}}\,,\, -(\boldsymbol{r}_{QP}^0)^{\mathrm{T}} \cdot \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{s}}_j^j\right]}_{=:\,\boldsymbol{g}_{4p}(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p})} \\ & \cdot \left[(\ddot{\boldsymbol{r}}_i^0)^{\mathrm{T}}\,,\, (\dot{\boldsymbol{\omega}}_{i0}^i)^{\mathrm{T}}\,,\, (\ddot{\boldsymbol{r}}_j^0)^{\mathrm{T}}\,,\, (\dot{\boldsymbol{\omega}}_{j0}^j)^{\mathrm{T}}\right]^{\mathrm{T}} \\ = & \underbrace{\left(\boldsymbol{r}_{QP}^0\right)^{\mathrm{T}} \cdot \left(\boldsymbol{A}^{0i} \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \tilde{\boldsymbol{\omega}}_{i0}^i \cdot \boldsymbol{s}_i^i - \boldsymbol{A}^{0j} \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \tilde{\boldsymbol{\omega}}_{j0}^j \cdot \boldsymbol{s}_j^j\right) - \left(\dot{\boldsymbol{r}}_{QP}^0\right)^{\mathrm{T}} \cdot \left(\dot{\boldsymbol{r}}_{QP}^0\right)}_{=:\,\beta_{c4}}.\end{aligned} \tag{A.4.42}$$

Comment A.4.2 (Complete set of model equations of the above joint): The complete set of model equations of the above general universal joint is built from (A.4.8b), (A.4.9a), (A.4.11), (A.4.16a), (A.4.16b), (A.4.17), (A.4.27a), (A.4.27b), (A.4.27c), (A.4.39), (A.4.40), and (A.4.42). The complete *constraint Jacobian matrix* is (cf. Equations A.4.11, A.4.17, A.4.27c, and A.4.42)

$$\boldsymbol{g}_p(\boldsymbol{p}) \cdot \boldsymbol{T}(\boldsymbol{p}) = \left(\boldsymbol{g}_{1p}^{\mathrm{T}}(\boldsymbol{p})\,,\, \boldsymbol{g}_{2p}^{\mathrm{T}}(\boldsymbol{p})\,,\, \boldsymbol{g}_{3p}^{\mathrm{T}}(\boldsymbol{p})\,,\, \boldsymbol{g}_{4p}^{\mathrm{T}}(\boldsymbol{p})\right)^{\mathrm{T}} \cdot \boldsymbol{T}(\boldsymbol{p}), \tag{A.4.43a}$$

with

$$\boldsymbol{T}(\boldsymbol{p}) = \mathrm{diag}\left(\boldsymbol{T}_1(\boldsymbol{p}_1)\,,\, \boldsymbol{T}_2(\boldsymbol{p}_2)\right) \tag{A.4.43b}$$

and

$$\boldsymbol{T}_i(\boldsymbol{p}_i) = \mathrm{diag}\left(\mathbf{I}_3\,,\, \boldsymbol{H}_i(\boldsymbol{\eta}_i)\cdot \boldsymbol{A}^{RL_i}\right). \tag{A.4.43c}$$

The complete *right-hand side* of the total constraint acceleration equation is (according to Equations A.4.11, A.4.17, A.4.27c, and A.4.42)

$$\boldsymbol{\beta}_{\mathrm{c}} = \left(\beta_{c1}^{\mathrm{T}}\,,\, \beta_{c2}^{\mathrm{T}}\,,\, \beta_{c3}^{\mathrm{T}}\,,\, \beta_{c4}^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{A.4.43d}$$

A.4.3 Computation of the shortest distance between two rotation axes

Consider the following description of the two rotation axes in Figure A.4.1:

$$\boldsymbol{y}_1 = \tau\cdot \boldsymbol{a}_1 + \boldsymbol{b}_1 \quad \text{(rotation axis 1)} \tag{A.4.44a}$$

and

$$\boldsymbol{y}_2 = \sigma\cdot \boldsymbol{a}_2 + \boldsymbol{b}_2 \quad \text{(rotation axis 2),} \tag{A.4.44b}$$

with $\boldsymbol{a}_1$, $\boldsymbol{b}_1$ and $\boldsymbol{a}_2$, $\boldsymbol{b}_2$ as *known constant vectors* and with τ and σ as free scalar parameters (cf. Figure A.4.3). Let $|\boldsymbol{r}_{QP}| = |\boldsymbol{s}_2| = s_{z2}$ (cf. Equation A.4.24b) be the *shortest distance* between the rotation axes 1 and 2 with $\boldsymbol{r}_{QP}$ as the vector from point P on axis 1 to point Q on axis 2. Then

$$\boldsymbol{r}_{PO} = \tau_0\cdot \boldsymbol{a}_1 + \boldsymbol{b}_1 \tag{A.4.45a}$$

and

$$\boldsymbol{r}_{QO} = \sigma_0\cdot \boldsymbol{a}_2 + \boldsymbol{b}_2, \tag{A.4.45b}$$

with the parameter values

$$\tau_0 = \frac{\left[(\boldsymbol{b}_1-\boldsymbol{b}_2)^{\mathrm{T}}\cdot \boldsymbol{a}_2\right]\cdot\left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_1\right] - \left[(\boldsymbol{b}_1-\boldsymbol{b}_2)^{\mathrm{T}}\cdot \boldsymbol{a}_1\right]\cdot\left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_2\right]}{\left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_2\right]\cdot\left[\boldsymbol{a}_1^{\mathrm{T}}\cdot \boldsymbol{a}_1\right] - \left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_1\right]^2} \tag{A.4.46a}$$

and

$$\sigma_0 = \frac{\left[(\boldsymbol{b}_1-\boldsymbol{b}_2)^{\mathrm{T}}\cdot \boldsymbol{a}_2\right]\cdot\left[\boldsymbol{a}_1^{\mathrm{T}}\cdot \boldsymbol{a}_1\right] - \left[(\boldsymbol{b}_1-\boldsymbol{b}_2)^{\mathrm{T}}\cdot \boldsymbol{a}_1\right]\cdot\left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_1\right]}{\left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_2\right]\cdot\left[\boldsymbol{a}_1^{\mathrm{T}}\cdot \boldsymbol{a}_1\right] - \left[\boldsymbol{a}_2^{\mathrm{T}}\cdot \boldsymbol{a}_1\right]^2}\,. \tag{A.4.46b}$$

This provides the shortest distance between axes 1 and 2 as

$$|\boldsymbol{r}_{QP}| = |\boldsymbol{s}_2| = s_{z2} = |\boldsymbol{r}_{QO} - \boldsymbol{r}_{PO}|$$

or

$$s_{z2} = |\boldsymbol{s}_2| = \Big|\,(\sigma_0\cdot \boldsymbol{a}_2 + \boldsymbol{b}_2) - (\tau_0\cdot \boldsymbol{a}_1 + \boldsymbol{b}_1)\,\Big|. \tag{A.4.47}$$

The parameter values τ_0 and σ_0 are computed as the solution of the linear equation

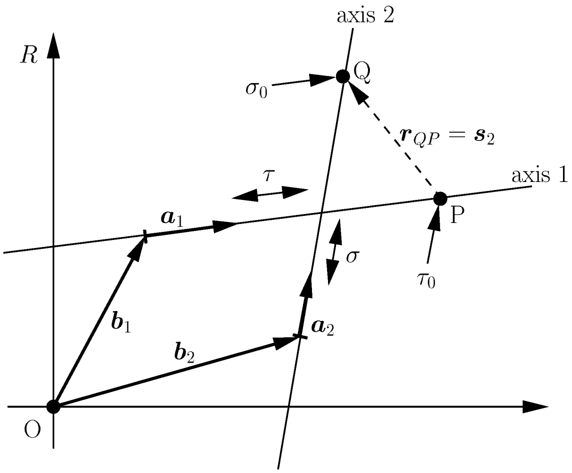


Fig. A.4.3: Shortest distance between two rotation axes

$$\begin{pmatrix} \boldsymbol{a}_1^{\mathrm{T}} \cdot \boldsymbol{a}_1 \ , & -\boldsymbol{a}_1^{\mathrm{T}} \cdot \boldsymbol{a}_2 \\ \boldsymbol{a}_2^{\mathrm{T}} \cdot \boldsymbol{a}_1 \ , & -\boldsymbol{a}_2^{\mathrm{T}} \cdot \boldsymbol{a}_2 \end{pmatrix} \cdot \begin{pmatrix} \tau_0 \\ \sigma_0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1^{\mathrm{T}} \cdot \boldsymbol{b}_2 \ , & -\boldsymbol{a}_1^{\mathrm{T}} \cdot \boldsymbol{b}_1 \\ \boldsymbol{a}_2^{\mathrm{T}} \cdot \boldsymbol{b}_2 \ , & -\boldsymbol{a}_2^{\mathrm{T}} \cdot \boldsymbol{b}_1 \end{pmatrix}, \tag{A.4.48}$$

which follows from the geometrical condition that vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are both orthogonal to vector $\boldsymbol{r}_{QP} = (\boldsymbol{r}_{QO} - \boldsymbol{r}_{PO})$.

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