

Introduction

First consideration is the Universal Approximation Theorem, which establish in general, that a standard multilayer feed-forward neural network with just only a single hidden layer, with a finite number of neurons, can approximate continuous function in a compact subset of the space of real vectors.

Initially, if we think about the logistic discrete equation:

$$x(n+1) = r * x(n) * (1 - x(n))$$

And we try for r between 3.6 and 4 to predict the value of $x(n)$ given $x(0)$ and n .

We might do it just iterating the equation, or we can try to create and train a ML with thousands of cases and try to apply to any $x(n)$.

Let's focus on the case $r=4$. For that value, Schröder found an analytic solution:

$$x(n) = \sin^2 (\sin^2(x(0)) * 2^n)$$

We want to use the logistic function in, for instance, the interval of $n=0,1,2,\dots, 10^{20}$ and $x(0)$ between 0 and 1 including both. From a mathematical point of view, the subset $\{0,1,2,3,\dots,10^{20}\} \times [0,1]$ into $\mathbb{R} \times \mathbb{R}$ is closed and bounded, so it's compact.

And obviously, Schröder solution is continuous (even differentiable) in $\mathbb{R} \times \mathbb{R}$.

According to Universal Approximation Theorem, we can approximate this function with a RNN (the core of ML). Nice!

We can think if a model able to approximate sin/cos functions might approximate Schröders solution.

To do that, I've implemented a LSTM model that works really well for $\sin(x)$, $\cos(x)$, $\sin(x^2)$

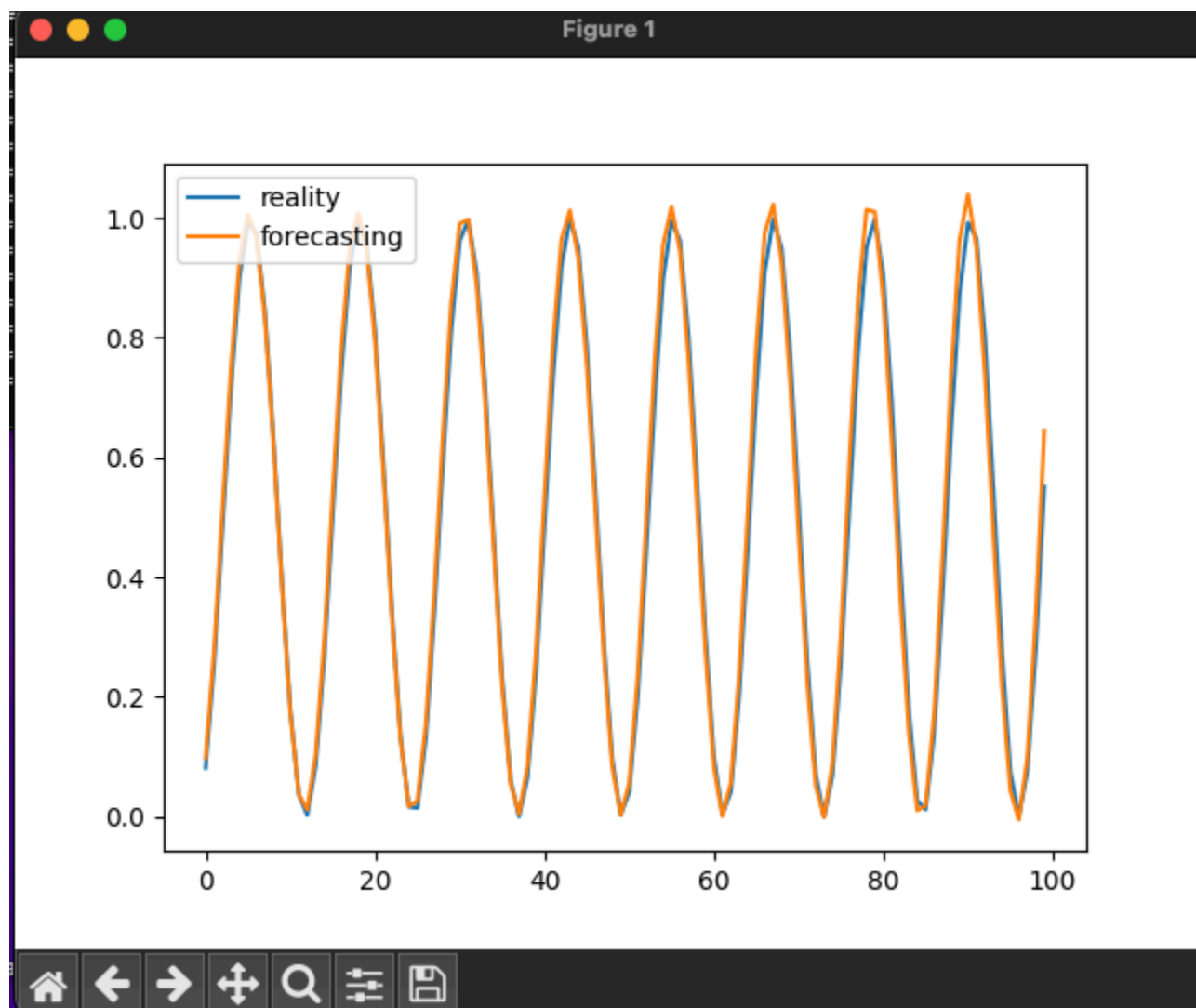
What is expected from the chaotic functions, as they have sensibility to initial conditions. The calculated sequence might vary a lot due to numerical instabilities.

For the starting point I'm going to consider two choices:

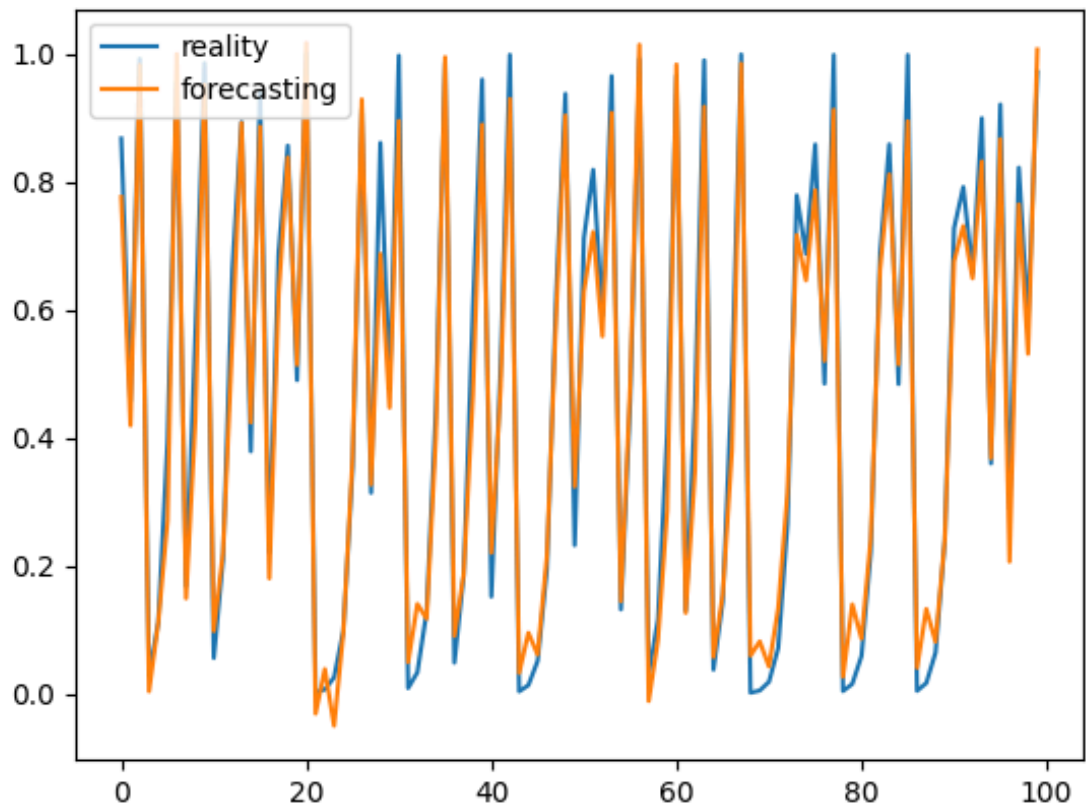
- With 32 neurons and 2 layers the fit is almost perfect for sin (and the training time is 115s)
- With 4 neurons and 2 layers the fit is almost perfect also (training time 109s)

Some experimental results

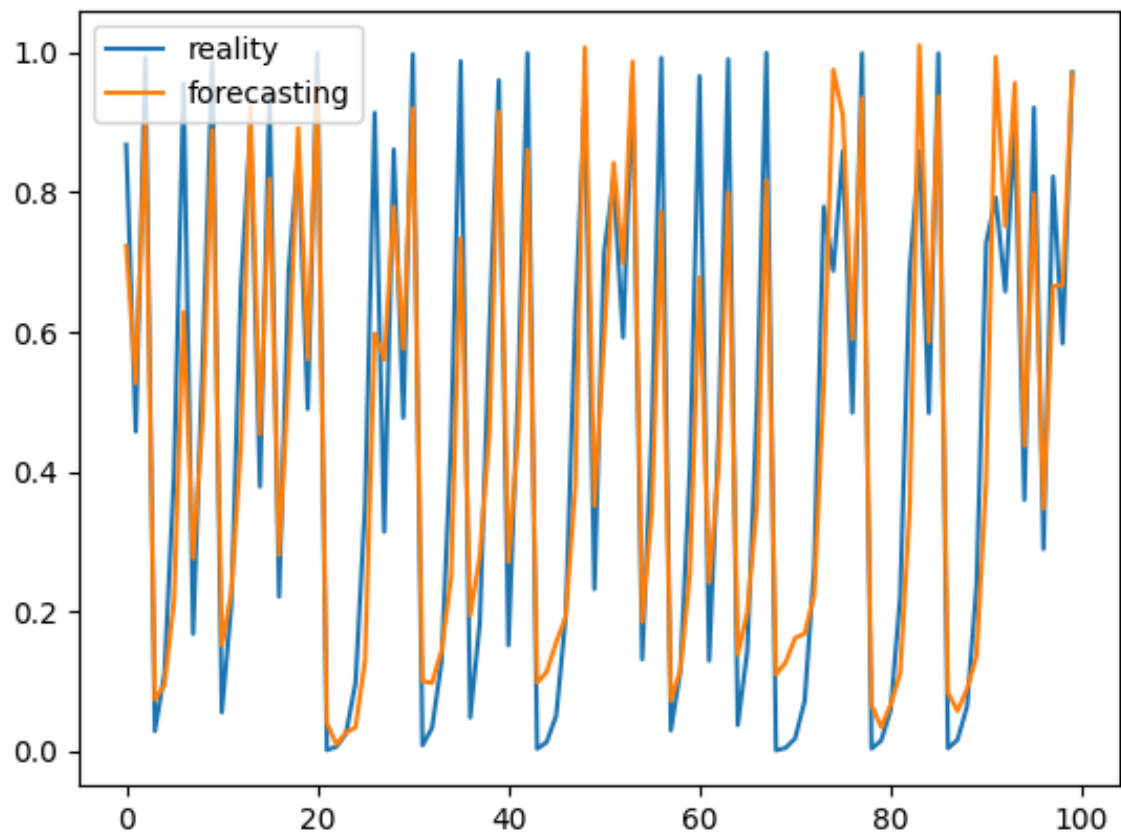
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or $\sin(x^2)$ functions I used 2 layers of 32 neurons. The adjust is almost perfect:



Trying to forecast Schrödingers analytic solution to logistic discrete equation ($R=4$) and $x_0=0.1234$ using the same schema as for the previous cases, the results are not really bad:

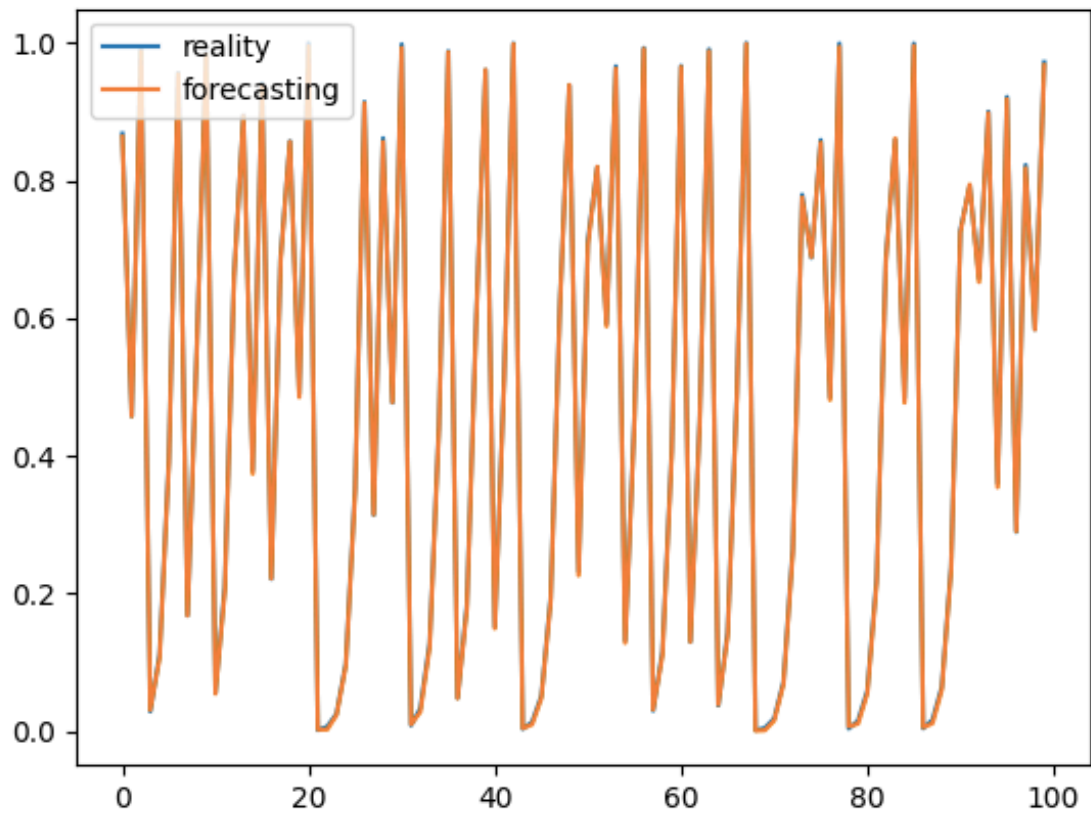


We can try again with more layers and neurons (4 layer and 128 neurons):



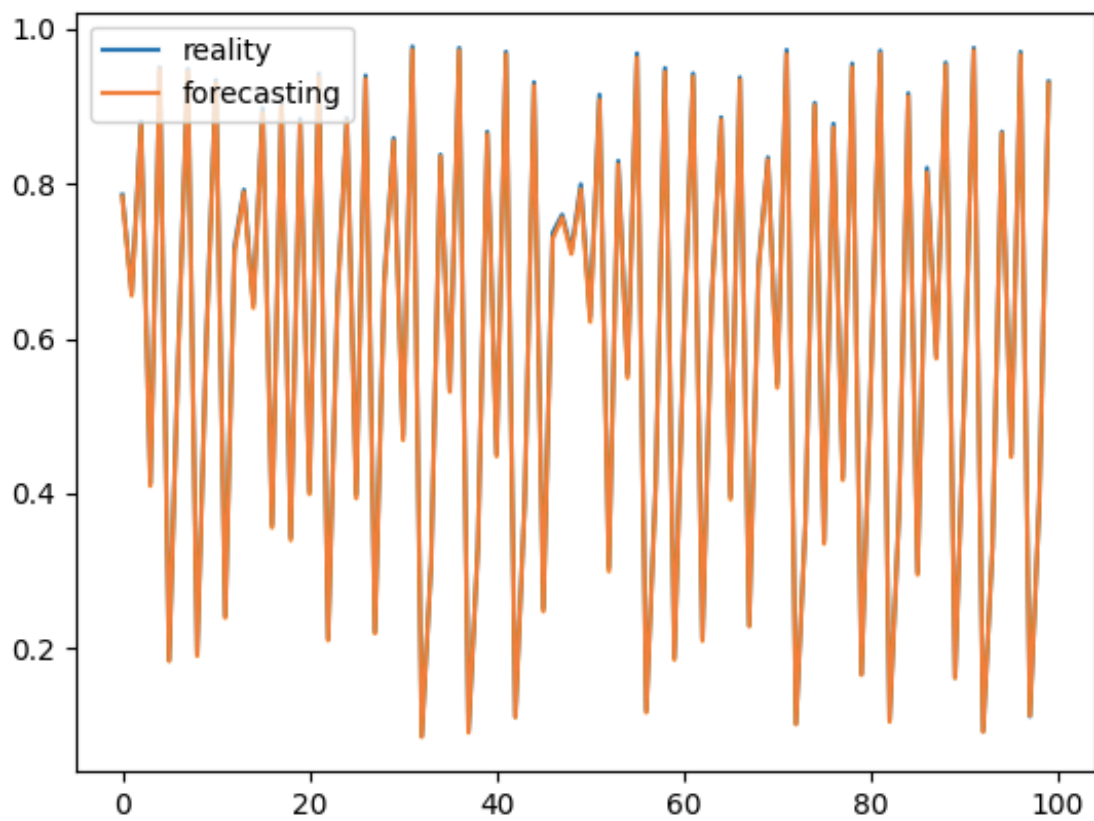
What I've observed it's not a question of having more neurons and layers but more train attempts. With 1000 epochs, the error loss has the opportunity to decrease to 0.001, that guarantees a good approach. With an error during training of around 0.1, there will be no valid forecasting...

The result with 1000 epochs and only 32 neurons in 2 layers:



With only 32 neurons, 2 layers, but 1.000 epochs of training, we are able to fit perfectly the chaotic logistic equation

$$x(n+1) = 3.92 * x(n) * (1-x(n))$$
$$x(0) = 0.1234$$



References

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