NP-Complete Problems, Puzzles, and Games

Samantha Shultz
Advisor Dr. Li Feng

NP-Complete Problems, Puzzles, and Games

Samantha Shultz
Advisor Dr. Li Feng

Abstract

Puzzles and Games that are fun are also challenging, many times this is due to their complexity. There are games whose complexity has been proven to be complex enough to be classified as NP-complete. NP-complete problems are those that are both NP and NP-Hard. These puzzles and games were classified by being reduced into to more commonly studied and understood NP-complete problems, proving them to also be NP-complete.

Intro

P, NP, NP-complete, and NP hard are all ways of categorizing problems and how they can be solved. P stands for polynomial time, meaning that if a problem is size n then it should be solved in $n^{O(1)}$.[3] NP problems are those that can be solved in non-deterministic polynomial time. NP hard problems are those that may be NP-complete but may not be NP, and NP-complete is a problem that is both NP and NP-hard. It's important to note that all NP-complete problems are NP-hard but not all NP-hard problems are NP-complete.[2] NP-complete are some of the hardest problems to solve and make up many many challenging puzzles and games. These puzzles include Cryptarithms, Dots and Boxes, Instant Insanity, Kplumber, Mastermind, Mindsweeper, Nonograms, and Same Game. Each of these game are quite different some started out as being played on a computer as mini games like Minesweeper Same Game, and Kplumber, and other have been around much longer like Minesweeper and Dots and Boxes.

No matter how different the puzzles may be or how they were created they all have been proved to be NP-complete by being reduced into other problems thats have be further studied and are more widely known including Traveling Salesman, Knapsack, Satisfiably, and Partitions.

1. Example Puzzles

1.1 Cryptarithms

Cryptarithms are NP-Complete puzzles that use letters as numbers like variables.[4] Typically in cryptarithms there are two words that use some sort calculation, for example, addition to equal another word like in figure 1, this causes every letter to represent some number value. Part of what make this problem difficult is that the value of a letter can not change and the same number can not be set to multiple letters. For example in SEND + MORE = MONEY, every E must have the same value and the value is only used for E, creating a one to one correspondence between letter and values.[4] Another common rule is that numbers do not start with leading zeros.

Figure 1 shows two classic examples of cryptarithms and their solutions. Notice that some problems may only have one solution like SEND + MORE = MONEY, a few like TWO + TWO = FOUR, or even thousand solutions.

SEND + MORE MONEY	➡	9567 +1085 10652	TWO +TWO FOUR	➡	734+734 = 1468 765+765 = 1530 836+836 = 1672 846+846 = 1692 867+867 = 1734 928+928 = 1856 938+938 = 1876

Figure 1: Example of two cryptarithms and their solutions

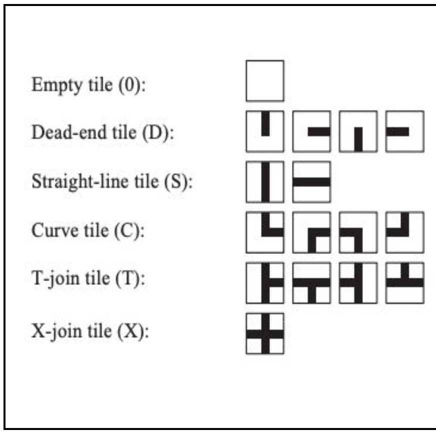
Cryptarithm are NP-Complete because in most cases the only way to solve them is to use brute force and try all of the possible solutions. While there are many algorithms to solve cryptarithms, they all use brute force methods and are not solved in polynomial time.

1.2 Dots and boxes

Dots and boxes is a game that can have any number of rows and columns of dots where players take turns connecting any horizontal or vertical set of dots with the goal of completing a square or "box" and gaining a point. Once a player completes a box they are then able to have another turn and connect two new dots. The player with the most boxes in the end wins. Dots and Boxes is an impartial game, meaning that both players have the same options when making moves.

1.3 KPlumber

KPlumber features a box of tiles that can be rotated with each tile containing a pipe. The goal is for the user to rotate the tiles so that the pipes all successfully line up. In "It Is Tough to Be a Plumber" the authors went through the problem of giving each type of tile a name, as shown in the first box below (figure 2). They then tested the complexity based off of the tiles present. Surprisingly, they found that removing straight line tiles allowed the puzzle to be solved in polynomial time. Additionally, removing dead-ends and T-joints, or removing dead-ends and curves also resulted in a polynomial complexity, as long as the straight line tile was still present. They also found that the complexity of the game does not depend on whether there are empty tiles and X-join tiles present. Although, there are still several cases where the problem has yet to be solved in polynomial time and therefore remains classified as NP-complete. [7]



The summary of our complexity results

O	C	D	S	T	X	Complexity
★	★	★	×	★	★	Polynomial (Theorem 2)
★	★	×	√	×	★	Polynomial (Theorem 3)
★	√	√	√	★	★	NP-complete (Theorem 8)
★	√	★	√	√	★	NP-complete (Theorem 8)
★	×	×	√	★	★	Polynomial (Theorem 4)
★	×	√	√	★	★	The same (unknown) complexity (Theorem 9)

The sign √ means that the corresponding type of tiles is present, the sign × means that it is not present and ★ means that it does not matter.

Figure 2: Right: The six different types are tiles used and each possible rotation, **Left**: summary of “It Is Tough to Be a Plumber.” complexity results

1.4 Mastermind

Mastermind is a two player game with one player being a code maker and the other the code breaker. In this game one player (the code maker) selects four colored pegs out of the six colors available and places them in any order, hiding them from the second player. The goal of the second player (the code breaker) is find the correct sequence that player one is hiding. Player two is given ten turns to show the first player their guess. The first player then responds with the number of correct colors and the number of colors in the correct spot. This allows the code breaker to test possibilities and learn from their guesses to narrow down the options.

When using a variant of Mastermind called Static Mastermind where all the guesses are given at once instead of one at a time. From there all of the hints or responses can be evaluated at once. With putting a bound on the number of guesses the problem can be turned into satisfiability problem(SAT). [9]

1.5 Minesweeper

Minesweeper is a single player puzzle game that starts with gridded rectangular board with the goal of locating all of the mines in the shortest time. Each square in the grid can be clicked to reveal a mine, a safe spot, or a number 0-8 representing how many mines are in the eight squares around the square that was clicked. The numbers can be used as a hint to narrow down the location of each mine. The game ends if the player clicks on a mine and is “blown up.” Mindsweeper can be reduced to a SAT problem using boolean circuits making it NP-complete. [6]

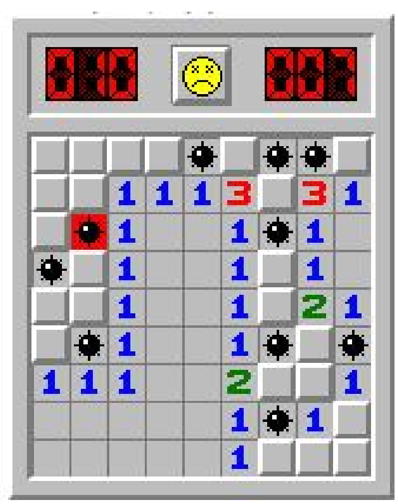


Figure 3: 9x9 Minesweeper game

1.6 Nonogram (Griddler, Paint-by-numbers)

Nonograms are logic puzzles where the player is given hints to try and solve the puzzle by coloring in some hidden picture. They are often shown as an *n x m* grid where each column and row are given a number, or several numbers, representing how many boxes should be colored in for that column or row. For example. Even when playing these games as a human it is a lot of trial and error trying to solve the puzzle.

In figure 4, the right shows an unsolved nonogram and the left shows the completed nonogram with a constraints satisfied. This example with pulled from a proof of Nonogram being classified as NP-complete by using 3-Dimensional mapping. [10]

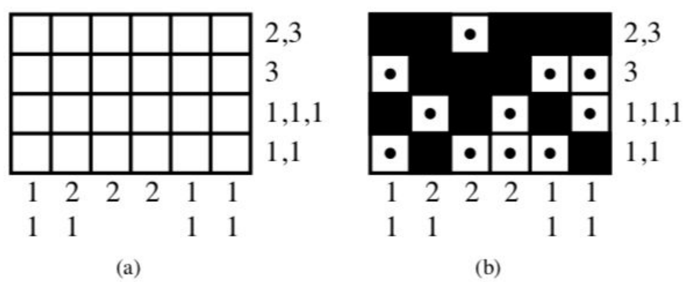


Figure 4: Nonogram examples from (Ueda, 1996)

1.7 Same Game (Chain-Shot!, Clickomania)

Same Game was originally created by Kuniaki "Morisuke" Moribe in 1985 with his version being call called *Chain-Shot*! [1] The game can now be found all over as with various versions but with the same basic concept. The games consist of a gridded rectangle filled with colored tiles with the goal of clearing those tiles by selecting ones surrounded by tiles of the same color. Once selected, or clicked, the like-colored tiles will disappear and any tiles above will then fall below replacing the space of the selected tiles that are removed.

There are typically two variations of winning the game, one being "elimination variant" winning by clearing the entire board of all tiles and the other is "score variant" where points are given base of the number of tiles removed in a single click, the more tiles removed the more points received. [1] Worst case, finishing a game with *n* tiles takes *n* clicks. Same Game can be reduced to a 3-Partition problem and has been proved to be NP-Complete as long as there are at least two colors and two columns in a game, but if there are less it can be solved in polynomial time. [1]

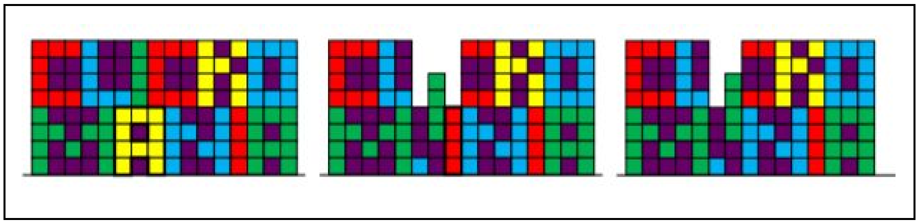


Figure 5: Same Game variation "ClickoMania" example (from Adler, Aviv, et al., 2017). [1]

2.) Reducibility

The puzzles and games explained earlier can reduced into other more well known NP-complete problems, including the traveling salesman, knapsack, 3-Partition, SAT, and 3SAT. These are all problems that have been studied extensively and have pre-existing algorithms that are believed to be efficient.

2.1 Traveling Salesman

The Traveling Salesman is a classic problem consisting of a salesman and a list of cities. The goal is for the salesman to visit each city once and to start and end at the same city while traveling the shortest possible distance. Finding the shortest path is an optimization problem.

In figure 6, it is easy to find the shortest path just by looking at the graph because this example is so small; there are only six vertices and seven edges making it easy to count the length of each path. However, what if there are hundreds of possible paths or more? It then becomes very time consuming to add up the length of each path in your head and an algorithm is needed.

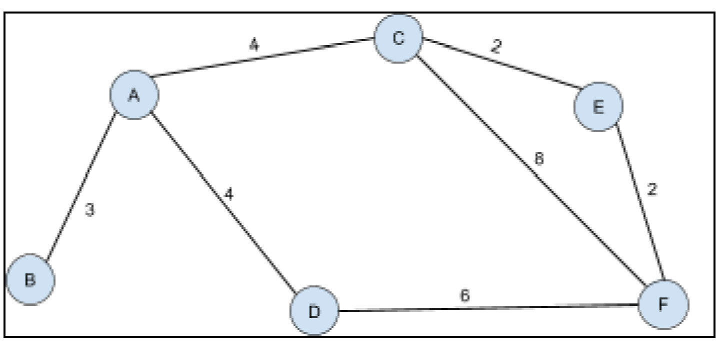


Figure 6: Traveling Salesman graph example

2.2 Knapsack

Think of the knapsack problem like packing a bag for a flight where every item has a set weight and size. The bag can only be so heavy and can only fit so much, so the person going on the trip must decide what items are most important when packing the bag. In every knapsack problem there are n number of items each with a given set of two attributes. Other examples include grocery shopping where items each cost some amount and the customer can only carry so much. The problem lies within trying figure out how to load the knapsack with a combination of units of the specified types of items that yields the greatest total value.[8] Packing knapsack for a flight can be shown in figure 7.

wt = the weight of each type-t item, for t = 1, 2, . . ., n,
vt = the value associated with each type-t item, for t = 1, 2, . . ., n,
c = the weight capacity of the knapsack.

Figure 7: Representations of knapsack problem

2.3 3-Partition

The goal of the 3-Partition is to find three disjoint subsets of one larger set, all with equal sums. In figure 8, S is the set with subsets a,b, and c whose sums all equal 42. Notice that each number only occurs once and does not appear in more than one subset.

$$S = \{1, 2, 8, 9, 10, 12, 14, 21, 29\},\ a = \{21, 12, 9\}\ b = \{1, 2, 39\}\ c = \{10, 14, 8\}\ \sum_{a,b,c} = 42$$

Figure 8: Representation of 3-Partition

2.4 Satisfiability (SAT)

Satisfiability, commonly known as SAT tries to check if some boolean expression is satisfiable, meaning is there a truth assignment to the variables in some expression that makes the expression true.

Conclusion

P, NP, NP-complete, and NP hard are all ways of categorizing problems and how they can be solved. Many games and puzzles and fit it to the category or NP-complete making them both NP and NP-Hard. These puzzles and games have been reduced to other known NP-complete problems. For example, Minesweeper and MasterMind can be be reduced into a satisfiability problem. The puzzles and problems discussed in this paper are only a few of the many already studied and proved, there are so many other games that have yet to be classified and are yet to come. Studying these problems allow for a better understanding of other problems and their complexity.

Bibliography

1. Adler, Aviv, et al. “Clickomania Is Hard, Even with Two Colors and Columns.” *The Mathematics of Various Entertaining Subjects*, 2017, doi:10.23943/princeton/9780691171920.003.0018.

2. Bhatia, Sanjiv K. “NP Completeness.” *NP Completeness*, 2013, www.cs.umsl.edu/~sanjiv/classes/cs5130/lectures/np.pdf. Biedl, Therese C., et al. "The complexity of Clickomania." *More games of no chance* 42 (2002): 389-404.

3. Demaine, Erik. *Lecture 16: NP-Completeness*, MIT OpenCourseWare, 2015, ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-046j-design-and-analysis-of-algorithms-spring-2015/lecture-notes/MIT6_046JS15_lec16.pdf.

4. Eppstein, David. “On the NP-Completeness of Cryptarithms.” *On the NP-Completeness of Cryptarithms*, Columbia University, 8 June 2000, www.ics.uci.edu/~eppstein/pubs/Epp-SN-87.pdf.

5. Hanratty, Patrick. *Computational Complexity of Games and Puzzles*, The Bren School of Information and Computer Sciences, www.ics.uci.edu/~eppstein/cgt/hard.html.

6. Kaye, Richard. “Minesweeper Is NP-Complete.” *Minesweeper Is NP-Complete*, 2000, simon.bailey.at/random/kaye.minesweeper.pdf.

7. Král’, Daniel, et al. “It Is Tough to Be a Plumber.” *Theoretical Computer Science*, vol. 313, no. 3, 2004, pp. 473–484., doi:10.1016/j.tcs.2002.12.002.

8. Niu, Shun-Chen. *The Knapsack Problem*. The University of Texas at Dallas, 2004, www.utdallas.edu/~scniu/OPRE-6201/documents/DP3-Knapsack.pdf.

9. Stuckman , Jeff, and Guo-Qiang Zhang. *Mastermind Is NP-Complete*. Case Western Reserve University, 10 Apr. 2016, notendur.hi.is/hh/kennsla/rrr/h06/mastermind.pdf.

10. Ueda, Nobuhisa, and Tadaaki Nagao. “NP-Completeness Results for NONOGRAM .” *Parsimonious Reductions* , 1996.