Time Dependency of Laser Entropy

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1 Introduction

In this note, we study the evolution of an atoms-field system, in which the atoms are continuously pumping from the ground state to the excited state. Particularly, the evolutions of the photon statistics and the von Neumann entropy for the field are studied. It has to be pointed out that we initialize the field from a vacuum state, therefore only the main diagonal elements of the density operator can be non-zero during the evolution. We compare the evolution curves for the entropy of lasers operated at different levels above the threshold, meanwhile their steady average photon numbers is kept fixed to be 200.

2 Theory

2.1 Equations of Motion

Consider that we have a laser of which the cavity field evolves due to the addition of many excited atoms. The parameters which define the system are

- r_a: the effective pumping rate;
- g: the interaction strength between atoms and the field;
- γ: the decay rate of atoms; and
- C: the decay rate of the field.

Here we focus our study on the cavity field. The coarse-grained equation of motion for the density matrix of the field is given by [1]

$$\begin{aligned}\dot{\rho}_{nm} =& -\frac{M_{nm}A}{1+N_{nm}B/A}\rho_{nm} + \frac{\sqrt{nm}A}{1+N_{n-1,m-1}B/A}\rho_{n-1,m-1} \\ & -\frac{C}{2}(n+m)\rho_{nm} + C\sqrt{(n+1)(m+1)}\rho_{n+1,m+1},\end{aligned} \tag{1}$$

where the linear gain coefficient A, the self-saturation coefficient B, and the dimensionless factors M_{nm} and N_{nm} are defined by [1]

$$\begin{aligned} A &= \frac{2r_a g^2}{\gamma^2}, \quad & M_{nm} &= \frac{1}{2}(n+m+2) + (n-m)^2\frac{B}{8A}, \\ B &= \frac{4g^2}{\gamma^2}A, \quad & N_{nm} &= \frac{1}{2}(n+m+2) + (n-m)^2\frac{B}{16A}. \end{aligned} \tag{2}$$

For the matrix elements ρ_{ij}, it's noticed that only those with the same $i-j$ values are coupled with each other.

Starting from a Vacuum State $\rho = |0\rangle\langle 0|$

In this note, we consider the case where the cavity field is initially a vacuum state $|0\rangle$, i.e. only ρ_{00} is non-zero at the beginning. Therefore all off-diagonal elements will keep zero during evolution, we just need to solve the equation of motion for the main diagonal elements $\rho_{nn} \equiv p(n)$, which is given by [1]

$$\begin{aligned} \dot{p_n} = &- \frac{(n+1)A}{1+(n+1)B/A}p_n + \frac{nA}{1+nB/A}p_{n-1} \\ &- Cnp_n + C(n+1)p_{n+1}. \end{aligned} \tag{3}$$

By solving this equation, we can find how photon statistics evolve with respect to time.

2.2 Von Neumann Entropy

The von Neumann entropy can quantify the uncertainty of a quantum state, which is defined by

$$S = -\mathrm{Tr}\left(\rho \ln \rho\right). \tag{4}$$

The logarithm of a matrix can be calculated after diagonalizing. Suppose ρ can be diagonalized by the unitary operator U, i.e. $\rho = UDU^\dagger$ and $D = \mathrm{Diag}(\lambda_0, \ldots, \lambda_{n-1})$. Then we have

$$S = -\mathrm{Tr}\left(U\rho U^\dagger U \ln \rho U^\dagger\right) = -\mathrm{Tr}\left(D \ln D\right) = -\sum_n \lambda_n \ln \lambda_n. \tag{5}$$

If we start from a vacuum state, the density matrices will always be diagonalized, i.e. $\lambda_n = \rho_{nn} \equiv p_n$. Then it's simple to calculate the von Neumann entropy by

$$S = -\sum_n p_n \ln p_n. \tag{6}$$

Actually now it reduced to the classical Shannon entropy.

3 Setup for Numeric Simulations

Here we solve the equation of motion Eq. 3 numerically. There are a few parameters in the equation. The methods how we setup these parameters are explained as follows.

Lasers are operated above the threshold ($A/C = 1$) by different levels, meanwhile their steady average photon numbers $\bar{n}$ are fixed to be 200. And $\bar{n}$ is given by

$$\bar{n} = \frac{A}{C}\frac{A-C}{B} = \frac{A}{C}\frac{A-C}{4g^2 A/\gamma^2} = (\alpha - 1)\frac{\gamma^2}{4g^2}, \tag{7}$$

where we define $\alpha \equiv A/C$. So given a pair of $\bar{n}$ and α, we can get $\gamma^2/(4g^2)$. If C is also known, A and B can be obtained as $A = \alpha C$ and $B = A \times 4g^2/\gamma^2 = \alpha(\alpha-1)C/\bar{n}$.

Recall that the effective pumping rate should be smaller than 1, i.e. $r_a < 1$. From Eq. 7 and the definition of A in Eq. 2, we get $r_a = 2\bar{n}C\frac{\alpha}{\alpha-1} < 1$. Then we get a inequality for α,

$$\alpha > \frac{1}{1 - 2\bar{n}C}. \tag{8}$$

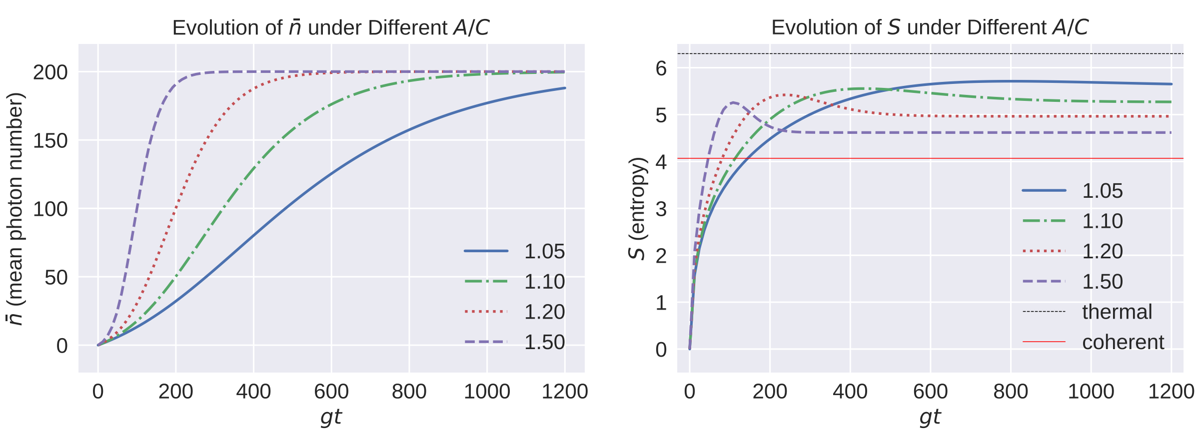
If we would like to study a laser operated above the threshold by a little amount, i.e. $\alpha \approx 1_+$, we have to make sure that $2\bar{n}C$ is small enough. Then the lower bound for α would be approximated by $1 + 2\bar{n}C$. We have chosen $\bar{n} = 200$, then we set $C = 1 \times 10^{-4}$ which seems to be an appropriate value. Now the lower bound for α is 1.04.

In summary, here is how we setup parameters: $\bar{n} = 200$; $C = 1 \times 10^{-4}$; the values of α are chosen in [1.05, 1.1, 1.2, 1.5, 2, 3, 5, 16, 64, 256, 1042]; A and B are obtained by $A = \alpha C$ and $B = \alpha(\alpha-1)C/\bar{n}$.

4 Results

4.1 Starting From a Vacuum State

Here I shown several results of numeric simulations. The evolution of average photon numbers and entropy are illustrated in Fig. 1, Fig. 2, and Fig. 3 for cases with different $\alpha(A/C)$ values.

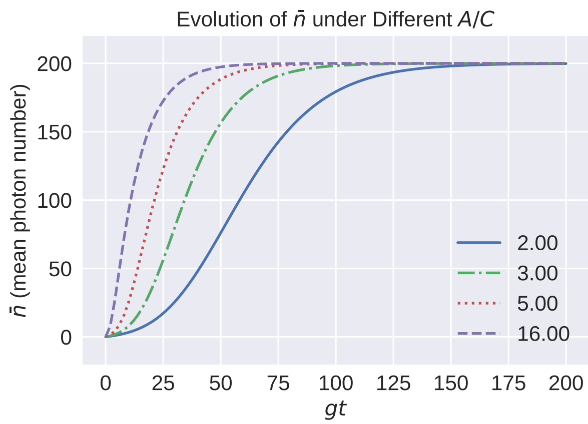


(a) Average Photon Numbers

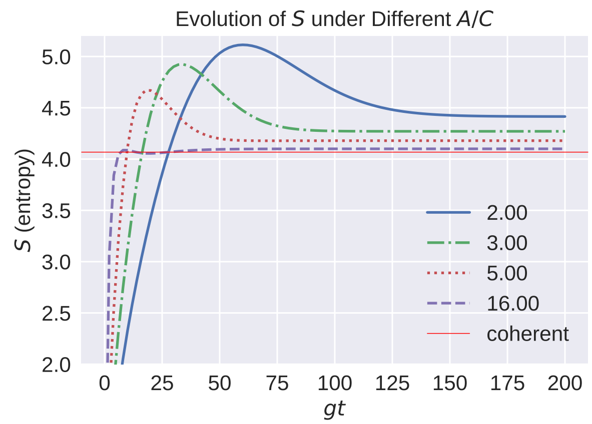
(b) Von Neumann Entropy

Figure 1: Result under Small Values of A/C

As for the mean photon numbers, all of them approach 200 asymptotically. The larger A/C is, the faster the mean photon numbers reach 200. In Fig. 3a, it's shown that $A/C = 64$ is large enough to be considered to be far above threshold, as the evolution curves of photon statistics are closely to each other for $A/C = 64, 256, 1024$.

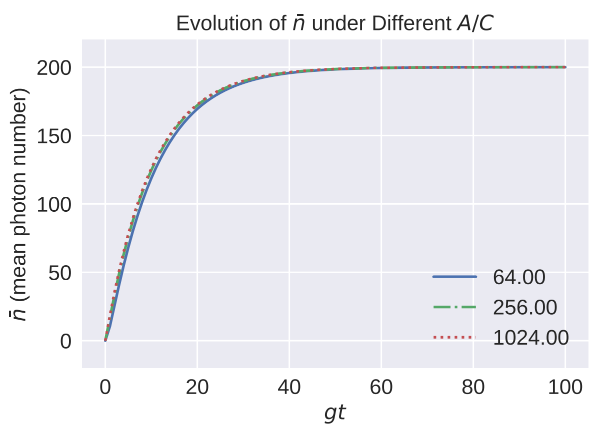


(a) Average Photon Numbers

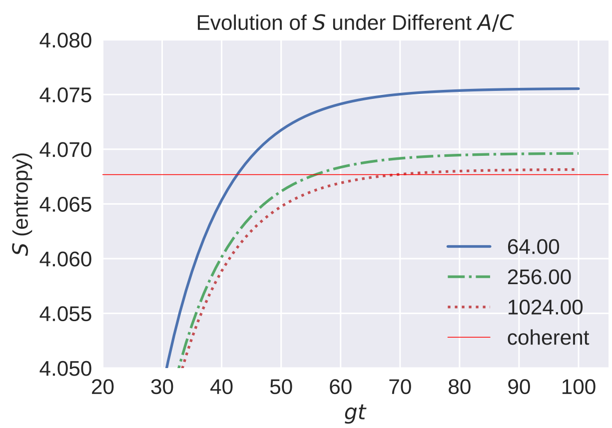


(b) Von Neumann Entropy

Figure 2: Result under Medium Values of A/C



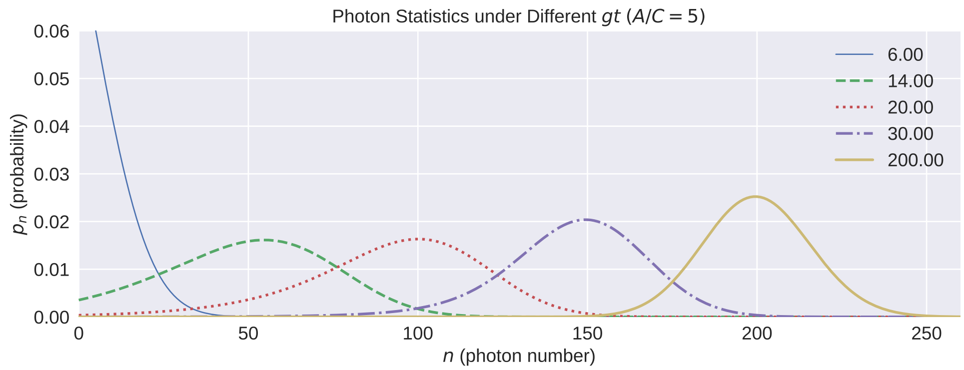
(a) Average Photon Numbers



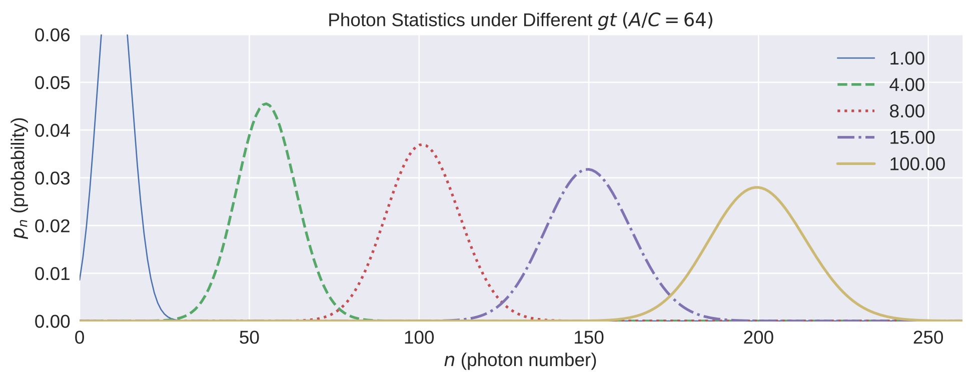
(b) Von Neumann Entropy

Figure 3: Result under Large Values of A/C

As for the von Neumann entropy, the evolutions curves show different behavior for lasers operated under different values of A/C. We also show the entropy of a thermal state ($\bar{n} = 200$) and a coherent state ($\bar{n} = 200$) with completely random phase. $A/C \sim 16$ seems to be a critical point. When $A/C < 16$, the entropy curves have a peak before them falls to steady values asymptotically. When $A/C > 16$, the entropy will increase monotonically to approach the steady values. And similar to the mean photon numbers, the larger A/C is, the faster the entropy will increase with respect to time. In Fig. 3b, it's shown that $A/C = 1024$ is large enough to regarded to be far above threshold, as the steady entropy of it is almost the same with that for a Poisson distribution with the average number of 200, i.e. a mixed state with coherent states with completely random phase.



(a) r'Photon Statistics vs. Time ($A/C = 5$)'

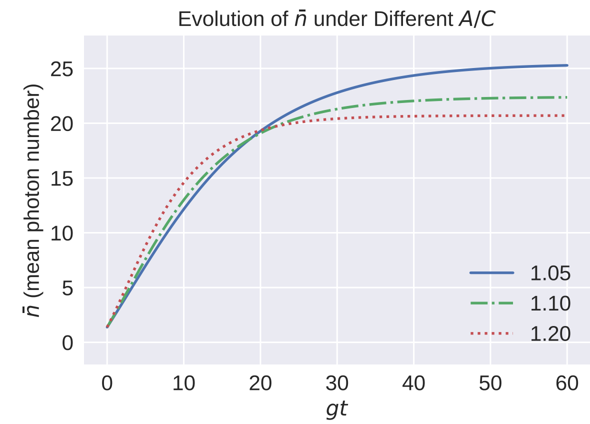


(b) r'Photon Statistics vs. Time ($A/C = 64$)'

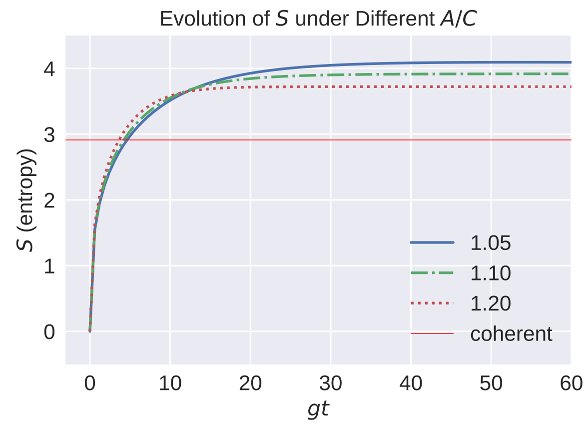
Figure 4: Evolution of Photon Statistics

To show why there are peaks for entropy curves under small and medium values of A/C. I show the photon statistics with respect to time in Fig. 4 for $A/C = 5$ and $A/C = 64$. It's seen that for $A/C = 5$, the photon number distribution spread widely before it becomes steady, therefore the entropy will increase first and then decrease. Meanwhile, for $A/C = 64$, the photon statistics will soon have a similar shape like Poisson distribution with narrow width. As time goes by, the mean photon numbers change, however the shapes of the curve do not change much. The height will decrease a little, and the width increase a little, therefore the entropy increases.

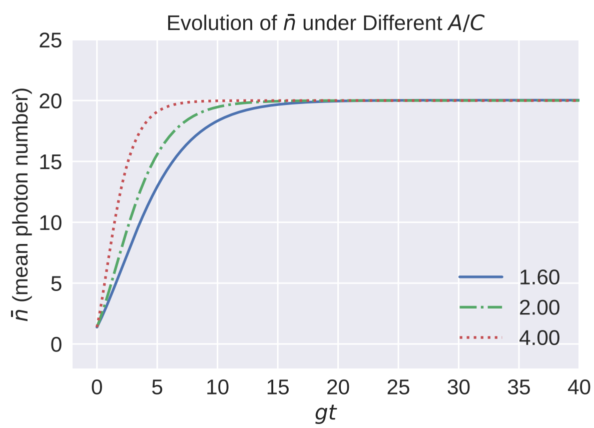
4.2 Starting From a Squeezed Vacuum State



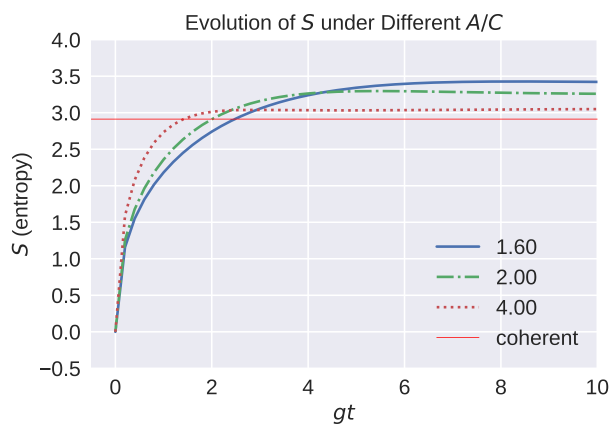
(a) Average Photon Numbers



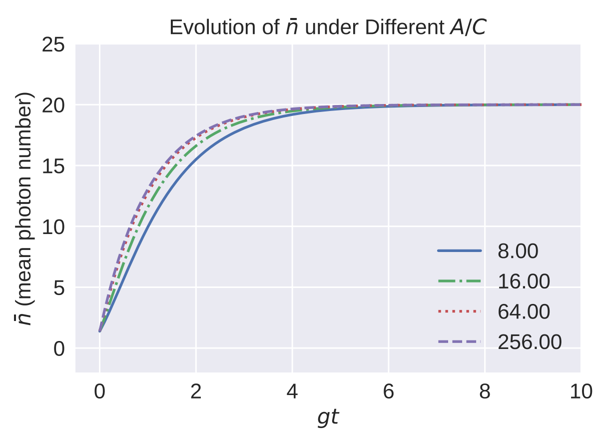
(b) Von Neumann Entropy



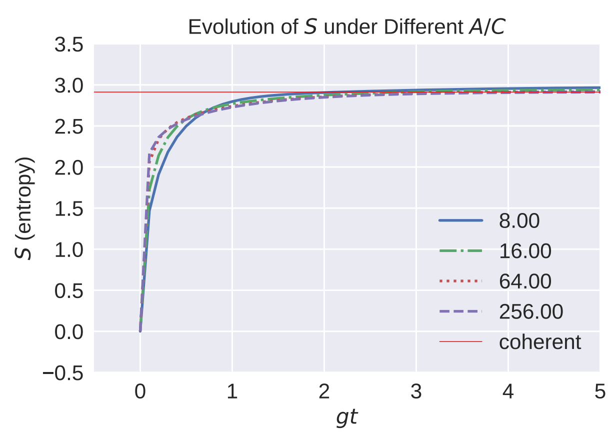
(c) Average Photon Numbers



(d) Von Neumann Entropy

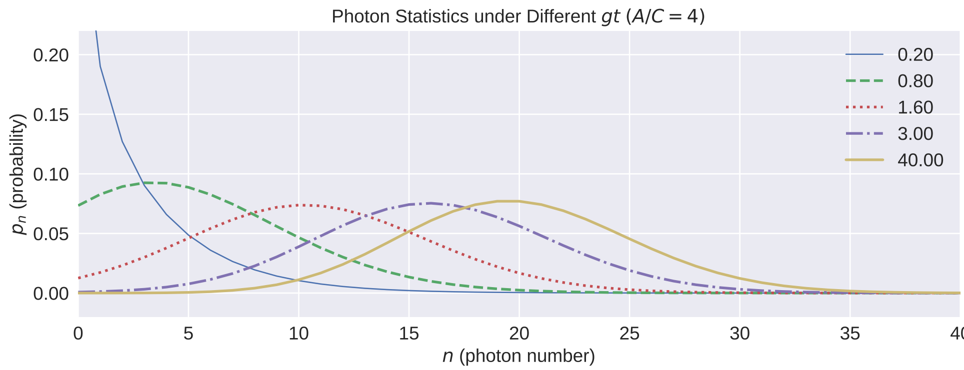


(e) Average Photon Numbers

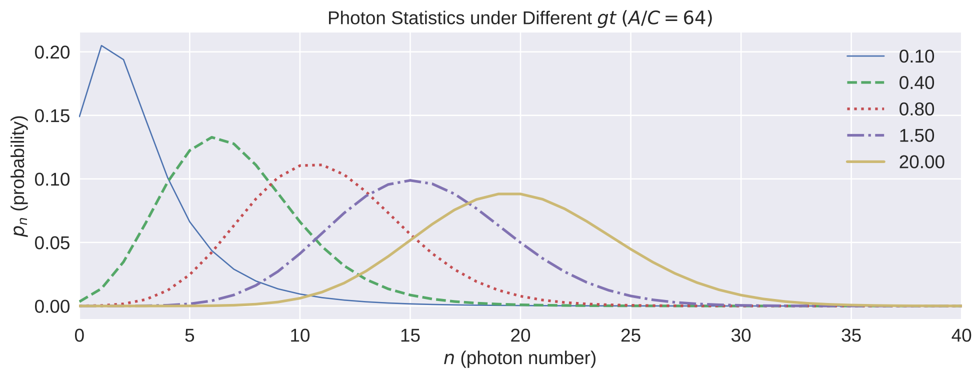


(f) Von Neumann Entropy

Figure 5: Result under Different Values of A/C



(a) Photon Statistics vs. Time ($A/C = 4$)



(b) Photon Statistics vs. Time ($A/C = 64$)

Figure 6: Evolution of Photon Statistics

References

[1] M. O. Scully and M. S. Zubairy. Quantum optics, 1999.