05-Bases-Introduction

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```
In [19]: from latools import *
         from sympy import *
         init_printing(use_latex=True)
```

1 Example 1

This is an example of a basis in $\mathbb{R}^2$

Let:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Is it possible to represent any vector in $\mathbb{R}^2$ in terms of $\{\mathbf{v}_1, \mathbf{v}_2\}$? Equivalently:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} -1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Matrix formulation:

$$\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Keeping the variables straight: In the system above, we must interpret the variables as follows:

- x and y are given numbers.
- a and b are the unknowns in the system, and must be found in terms of x and y.

We solve the system in the standard way, but use symbols for the right-hand side:

```
In [20]: x, y = symbols('x,y')
         A = matrix_to_rational([[ 1,-1, x],
                                 [ 2, 2, y]])
         A
```

Out[20]:

$$\begin{bmatrix} 1 & -1 & x \\ 2 & 2 & y \end{bmatrix}$$

```
In [21]: A1 = rop(A, 'R1*(-2)+R2=>R2')
         A1
```

Out[21]:

$$\begin{bmatrix}1 & -1 & x\\ 0 & 4 & -2x+y\end{bmatrix}$$

```
In [22]: A2 = rop(A1, 'R2*(1/4)=>R2')
         A2
```

Out[22]:

$$\begin{bmatrix}1 & -1 & x\\ 0 & 1 & -\frac{x}{2}+\frac{y}{4}\end{bmatrix}$$

```
In [23]: A3 = rop(A2, 'R2*(1)+R1=>R1')
         A3
```

Out[23]:

$$\begin{bmatrix}1 & 0 & \frac{x}{2}+\frac{y}{4}\\ 0 & 1 & -\frac{x}{2}+\frac{y}{4}\end{bmatrix}$$

```
In [24]: A4 = rop(A3, 'R2*(2)+R1=>R1')
         A4
```

Out[24]:

$$\begin{bmatrix}1 & 2 & -\frac{x}{2}+\frac{3y}{4}\\ 0 & 1 & -\frac{x}{2}+\frac{y}{4}\end{bmatrix}$$

Interpreting the result of the computations, we get:

$$a=-\frac{1}{2}x+\frac{1}{2}yb=\frac{1}{4}x+\frac{1}{4}y$$

Notice that we can write this in terms of matrix multiplication:

$$\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

Unsurprisingly, the matrix above is the inverse of the original matrix.

1.1 Conclusion

Given the vectors:

$$\mathbf{v}_1=\begin{bmatrix}-1\\ 2\end{bmatrix}\quad \mathbf{v}_1=\begin{bmatrix}1\\ 2\end{bmatrix}$$

any vector in $\mathbb{R}^2$ can be expressed in terms of $\{\mathbf{v}_1,\mathbf{v}_2\}$:

$$\begin{bmatrix}x\\ y\end{bmatrix}=a\begin{bmatrix}-1\\ 2\end{bmatrix}+b\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}-1 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}x\\ y\end{bmatrix}$$

where:

$$\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{4} & \frac{1}{4}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

When this happens, we say that $\{\mathbf{v}_1,\mathbf{v}_2\}$ is a *basis* of $\mathbb{R}^2$.

2 Example 2

Let's now to an example in $\mathbb{R}^4$. The given vectors are:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We want to express an arbitrary vector in $\mathbb{R}^4$ as:

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This is equivalent to the system:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$$

Let's solve this system using Gaussian Elimination:

```
In [25]: x, y, z, t = symbols('x,y,z,t')
         A = matrix_to_rational([[ 1,  0, 2, 1, x],
                                 [ 2,  1, 0, 1, y],
                                 [-1,  3, 1, 1, z],
                                 [ 1, -1, 1, 1, t]])
         A
```

Out[25]:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & x \\ 2 & 1 & 0 & 1 & y \\ -1 & 3 & 1 & 1 & z \\ 1 & -1 & 1 & 1 & t \end{bmatrix}$$

```
In [26]: A1 = rop(A, 'R1*(-2)+R2=>R2', 'R1*(1)+R3=>R3', 'R1*(-1)+R4=>R4')
         A1
```

Out[26]:

$$\begin{bmatrix} 1 & 0 & 2 & 1 & x \\ 0 & 1 & -4 & -1 & -2x+y \\ 0 & 3 & 3 & 2 & x+z \\ 0 & -1 & -1 & 0 & t-x \end{bmatrix}$$

```
In [27]: A2 = rop(A1, 'R2*(-3)+R3=>R3', 'R2*(1)+R4=>R4')
         A2
```

```
Out[27]:
```

$$\left[\begin{matrix}1 & 0 & 2 & 1 & x\\0 & 1 & -4 & -1 & -2x + y\\0 & 0 & 15 & 5 & 7x - 3y + z\\0 & 0 & -5 & -1 & t - 3x + y\end{matrix}\right]$$

```
In [28]: A3 = rop(A2, 'R3*(1/15)=>R3')
         A3
```

```
Out[28]:
```

$$\left[\begin{matrix}1 & 0 & 2 & 1 & x\\0 & 1 & -4 & -1 & -2x + y\\0 & 0 & 1 & \frac{1}{3} & \frac{7x}{15} - \frac{y}{5} + \frac{z}{15}\\0 & 0 & -5 & -1 & t - 3x + y\end{matrix}\right]$$

```
In [29]: A4 = rop(A3, 'R3*(-2)+R1=>R1', 'R3*(4)+R2=>R2', 'R3*(5)+R4=>R4')
         A4
```

```
Out[29]:
```

$$\left[\begin{matrix}1 & 0 & 0 & \frac{1}{3} & \frac{x}{15} + \frac{2y}{5} - \frac{2z}{15}\\0 & 1 & 0 & \frac{1}{3} & -\frac{2x}{15} + \frac{y}{5} + \frac{4z}{15}\\0 & 0 & 1 & \frac{1}{3} & \frac{7x}{15} - \frac{y}{5} + \frac{z}{15}\\0 & 0 & 0 & \frac{2}{3} & t - \frac{2x}{3} + \frac{z}{3}\end{matrix}\right]$$

```
In [30]: A5 = rop(A4, 'R4*(3/2)=>R4')
         A5
```

```
Out[30]:
```

$$\left[\begin{matrix}1 & 0 & 0 & \frac{1}{3} & \frac{x}{15} + \frac{2y}{5} - \frac{2z}{15}\\0 & 1 & 0 & \frac{1}{3} & -\frac{2x}{15} + \frac{y}{5} + \frac{4z}{15}\\0 & 0 & 1 & \frac{1}{3} & \frac{7x}{15} - \frac{y}{5} + \frac{z}{15}\\0 & 0 & 0 & 1 & \frac{3t}{2} - x + \frac{z}{2}\end{matrix}\right]$$

```
In [31]: A6 = rop(A5, 'R4*(-1/3)+R1=>R1', 'R4*(-1/3)+R2=>R2', 'R4*(-1/3)+R3=>R3')
         A6
```

```
Out[31]:
```

$$\left[\begin{matrix}1 & 0 & 0 & 0 & -\frac{t}{2} + \frac{2x}{5} + \frac{2y}{5} - \frac{3z}{10}\\0 & 1 & 0 & 0 & -\frac{t}{2} + \frac{x}{5} + \frac{y}{5} + \frac{z}{10}\\0 & 0 & 1 & 0 & -\frac{t}{2} + \frac{4x}{5} - \frac{y}{5} - \frac{z}{10}\\0 & 0 & 0 & 1 & \frac{3t}{2} - x + \frac{z}{2}\end{matrix}\right]$$

We conclude that it is always possible to find the representation, and:

$$\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}\frac{2}{5} & \frac{2}{5} & -\frac{3}{10} & -\frac{1}{2}\\\frac{1}{5} & \frac{1}{5} & \frac{1}{10} & -\frac{1}{2}\\\frac{4}{5} & -\frac{1}{5} & -\frac{1}{10} & -\frac{1}{2}\\-1 & 0 & \frac{1}{2} & \frac{3}{2}\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}$$

It follows that the given vectors are a basis of $\mathbb{R}^4$

3 Example 3

Determine if the following vectors form a basis of $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

Solution: We need to check if an arbitrary vector can be expressed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$$

We set this up as a linear system:

$$\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & -2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Next, solve the system by Gaussian Elimination:

```
In [32]: x, y, z = symbols('x,y,z')
         A = matrix_to_rational([[ 1, 2,  0, x],
                                 [-1, 0, -2, y],
                                 [ 3, 1,  5, z]])
         A
```

Out[32]:

$$\begin{bmatrix} 1 & 2 & 0 & x \\ -1 & 0 & -2 & y \\ 3 & 1 & 5 & z \end{bmatrix}$$

```
In [33]: A1 = rop(A, 'R1*(1)+R2=>R2', 'R1*(-3)+R3=>R3')
         A1
```

Out[33]:

$$\begin{bmatrix} 1 & 2 & 0 & x \\ 0 & 2 & -2 & x+y \\ 0 & -5 & 5 & -3x+z \end{bmatrix}$$

```
In [34]: A2 = rop(A1, 'R2*(1/2)=>R2')
         A2
```

Out[34]:

$$\begin{bmatrix} 1 & 2 & 0 & x \\ 0 & 1 & -1 & \frac{x}{2}+\frac{y}{2} \\ 0 & -5 & 5 & -3x+z \end{bmatrix}$$

```
In [35]: A3 = rop(A2, 'R2*(-2)+R1=>R1', 'R2*(5)+R3=>R3')
         A3
```

Out[35]:

$$\begin{bmatrix}1 & 0 & 2 & -y\\0 & 1 & -1 & \frac{x}{2}+\frac{y}{2}\\0 & 0 & 0 & -\frac{x}{2}+\frac{5y}{2}+z\end{bmatrix}$$

When we translate this augmented matrix back to system form we get:

$$\begin{aligned}a+2c &= -y\\ b-c &= \frac{1}{2}x+\frac{1}{2}y\\ 0 &= -\frac{1}{2}x+\frac{5}{2}y+z\end{aligned}$$

The important equation to look here is the third one:

$$0 = -\frac{1}{2}x+\frac{5}{2}y+z$$

There are two possibilites:

- If $-\frac{1}{2}x+\frac{5}{2}y+z=0$, the system has solutions (infinitely many, actually)
- If $-\frac{1}{2}x+\frac{5}{2}y+z\neq 0$, the system is inconsistent.

We conclude that *there are* values of x, y and z for which the system will not have solutions. For example, if:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\6\\0\end{bmatrix}$$

the system has no solutions, since $-\frac{1}{2}x+\frac{5}{2}y+z=-\frac{1}{2}2+\frac{5}{2}6+0=14\neq 0$. It follows that it is *not* possible to represent this vector in terms of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. We conclude that these vectors do not constitute a basis of $\mathbb{R}^{\nvDash}$

4 Provisional Definition of Basis

We say that a set of n vectors in form a *basis* of $\mathbb{R}^n$ if it is possible to represent any vector in $\mathbb{R}^n$ in terms of the vectors in the given set.

In practice: to check if $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ a basis of $\mathbb{R}^n$, write the system:

$$\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}=a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_n\mathbf{v}_n$$

- If it is always possible to solve the system, finding values of a_1, a_2,…,a_n that represent the vector $[x_1,x_2,\ldots,x_n]$, then the set is a basis.
- Otherwise, the set is not a basis.

5 Exercises

5.1 1

For each set of vectors given below, determine if it is a basis or not.

5.1.1 (a)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad \text{in } \mathbb{R}^3$$

```
In [ ]:
```

5.1.2 (b)

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} \quad \text{in } \mathbb{R}^4$$

```
In [ ]:
```

5.2 (c)

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -2 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 4 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 5 \\ -8 \\ -10 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ 8 \\ 4 \\ -2 \end{bmatrix} \quad \text{in } \mathbb{R}^4$$

```
In [ ]:
```

6 2

Suppose that two vectors in $\mathbb{R}^2$ are given by:

$$\mathbf{v}_1 = \begin{bmatrix} r \\ s \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} t \\ u \end{bmatrix}$$

When are these vectors a basis of $\mathbb{R}^2$? Your answer will be in the form of an algebraic relationship for r, s, t, u.

Note: It is possible to do this symbolically using the computer, but you might find it easier to just use pencil and paper.