15-Gram-Schmidt-Orthogonalization-Algorithm

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```
In [1]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

1 Description of the Algorithm

Input: $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ a linearly independent set of vectors in $\mathbb{R}^n$
Output: $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, and orthogonal basis of span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.
Algorithm:

- Let $\mathbf{v}_1 = \mathbf{u}_1$
- Let $\mathbf{v}_j = \mathbf{u}_j - \dfrac{\mathbf{u}_j \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \dfrac{\mathbf{u}_j \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \dfrac{\mathbf{u}_j \cdot \mathbf{v}_{j-1}}{\mathbf{v}_{j-1} \cdot \mathbf{v}_{j-1}}\mathbf{v}_{j-1}$ for $j = 2, \ldots, k$

2 Example

Apply the Gram-Schmidt diagonalization procedure to the following vectors in $\mathbb{R}^3$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Start by defining the vectors:

```
In [2]: u1 = Matrix([1,1,0])
        u2 = Matrix([-2,0,1])
        u3 = Matrix([1,1,1])
```

2.0.1 Step 1:

```
In [3]: v1=u1
        v1
```

Out[3]:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2.0.2 Step 2:

Project $\mathbf{u}_2$ onto the direction of $\mathbf{v}_1$

```
In [4]: v2 = u2 - (u2.dot(v1))/(v1.dot(v1))*v1
        v2
```

Out[4]:

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

2.0.3 Step 3:

Project $\mathbf{u}_3$ onto span$\{\mathbf{v}_1, \mathbf{v}_2\}$

```
In [5]: v3 = u3 - (u3.dot(v1))/(v1.dot(v1))*v1 - (u3.dot(v2))/(v2.dot(v2))*v2
        v3
```

Out[5]:

$$\begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

3 Application - Diagonalization of Symmetric Matrices

If A is a $n \times n$ symmetric matrix, it is always possible to find an *orthonormal* basis of $\mathbb{R}^n$ consisting of eigenvectors of A. This example shows how to proceed to find such basis.

As an example, let's find an orthonormal basis of eigenvectors of the matrix:

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{3}{2} & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

We start by finding the eigenvalues and eigenvectors of A.

```
In [6]: A = matrix_to_rational([[3/2, 1/2, -1],
                                [1/2, 3/2,  1],
                                [ -1,   1,  0]])
        A
```

Out[6]:

$$\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & \frac{3}{2} & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

```
In [7]: lbd = symbols('lambda')
        p = det(A - lbd*eye(3))
        p
```

Out[7]:

$$-\lambda^3 + 3\lambda^2 - 4$$

```
In [8]: factor(p)
```

Out[8]:

$$-(\lambda - 2)^2 (\lambda + 1)$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$. We next find a basis for each of the eigenspaces:

3.0.1 Eigenspace of $\lambda_1 = 2$

```
In [9]: R = reduced_row_echelon_form(A - 2*eye(3))
        R
```

Out[9]:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to the RREF has a single equation:

$$x_1 - x_2 + 2x_3 = 0$$

There are two free variables, x_2 and x_3. So, the eigenspace $E(2)$ has dimension 2. To find a basis for $E(2)$ we let:

$$x_2 = 1, x_3 = 0, \text{ so that } x_1 = 1 - 2 \times 0 = 1, \text{ and we let: } \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = 0, x_3 = 1, \text{ so that } x_1 = 0 - 2 \times 1 = -2:, \text{ and we let: } \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

3.1 Eigenspace of $\lambda_2 = -1$

```
In [10]: R = reduced_row_echelon_form(A - (-1)*eye(3))
         R
```

Out[10]:

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We now get the system:

$$x_1 - \frac{1}{2}x_3 = 0$$
$$x_2 + \frac{1}{2}x_3 = 0$$

There is only one free variable, x_3, so the eigenspace $E(-1)$ has dimension 1. Letting $x_3 = 2$ we get $x_1 = 1$ and $x_2 = -1$, which gives us the eigenvector:

$$\mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Summarizing our findings:

$$\text{Eigenvalue } \lambda_1 = 2; \quad \text{Eigenvectors: } \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Eigenvalue } \lambda_2 = -1; \quad \text{Eigenvector: } \mathbf{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Notice that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is *not* orthogonal, since $\mathbf{u}_1 \cdot \mathbf{u}_2 \neq 0$. However:

$\mathbf{u}_3$ is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$

This is because they are eigenvectors that correspond to different eigenvalues.
To get an orthogonal basis, we apply the Gram-Schimidt procedure to the set $\{\mathbf{u}_1, \mathbf{u}_2\}$

```
In [11]: u1 = Matrix([1,1,0])
         u2 = Matrix([-2,0,1])
         v1 = u1
         v2 = u2 - u2.dot(v1)/v1.dot(v1)*v1
         v2
```

```
Out[11]:
```

$$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Checking:

```
In [12]: v1.dot(v2)
```

```
Out[12]:
```

$$0$$

To complete the orthonormal basis, we define:

```
In [13]: v3=Matrix([1,-1,2])
         v3
```

Out[13]:

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

To check that the three vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are indeed orthogonal we can compute:

```
In [14]: print(v1.dot(v2), v1.dot(v3), v2.dot(v3))
```

```
0 0 0
```

To get an orthononormal basis, we simply normalize the basis:

```
In [15]: n1 = v1*1/v1.norm()
         n1
```

Out[15]:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

```
In [16]: n2 = v2*1/v2.norm()
         n2
```

Out[16]:

$$\begin{bmatrix} -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$

```
In [17]: n3 = v3*1/v3.norm()
         n3
```

Out[17]:

$$\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$

Let's now to check our work, build the change of basis matrix:

```
In [18]: P = Matrix.hstack(n1,n2,n3)
         P
```

Out[18]:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \end{bmatrix}$$

P must be an orthogonal matrix, so we compute:

```
In [19]: P.T * P
```

Out[19]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we can check the diagonalization:

```
In [20]: P.T * A * P
```

Out[20]:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

```
In [ ]:
```