06-Practice-with-Bases-and-Matrix-Inverses

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```
In [2]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

0.1 Determine if a Set of Vectors is a Basis

0.1.1 Example 1

Determine if the vectors below form a basis of $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix}1\\3\\0\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}2\\2\\1\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}-1\\4\\1\end{bmatrix}$$

Solution We want to find c_1, c_2, c_3 such that:

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = c_1\begin{bmatrix}1\\3\\0\end{bmatrix} + c_2\begin{bmatrix}2\\2\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\4\\1\end{bmatrix}$$

Equivalently, we need to solve the system:

$$\begin{bmatrix}1 & 2 & -1\\3 & 2 & 4\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$$

We solve the system using Gaussian Elimination:

```
In [3]: x, y, z = symbols('x, y, z')
        A = matrix_to_rational([[1,  2, -1, x],
                                [3,  2,  4, y],
                                [0, -1,  1, z]])
        A
```

Out[3]:

$$\begin{bmatrix}1 & 2 & -1 & x\\3 & 2 & 4 & y\\0 & -1 & 1 & z\end{bmatrix}$$

```
In [4]: A1 = rop(A, 'R1*(-3)+R2=>R2')
        A1
```

Out[4]:

$$\begin{bmatrix} 1 & 2 & -1 & x \\ 0 & -4 & 7 & -3x+y \\ 0 & -1 & 1 & z \end{bmatrix}$$

```
In [5]: A2 = rop(A1, 'R2<=>R3')
        A2
```

Out[5]:

$$\begin{bmatrix} 1 & 2 & -1 & x \\ 0 & -1 & 1 & z \\ 0 & -4 & 7 & -3x+y \end{bmatrix}$$

```
In [6]: A3 = rop(A2, 'R2*(-1)=>R2')
        A3
```

Out[6]:

$$\begin{bmatrix} 1 & 2 & -1 & x \\ 0 & 1 & -1 & -z \\ 0 & -4 & 7 & -3x+y \end{bmatrix}$$

```
In [7]: A4 = rop(A3, 'R2*(-2)+R1=>R1', 'R2*(4)+R3=>R3')
        A4
```

Out[7]:

$$\begin{bmatrix} 1 & 0 & 1 & x+2z \\ 0 & 1 & -1 & -z \\ 0 & 0 & 3 & -3x+y-4z \end{bmatrix}$$

```
In [8]: A5 = rop(A4, 'R3*(1/3)=>R3')
        A5
```

Out[8]:

$$\begin{bmatrix} 1 & 0 & 1 & x+2z \\ 0 & 1 & -1 & -z \\ 0 & 0 & 1 & -x+\frac{y}{3}-\frac{4z}{3} \end{bmatrix}$$

```
In [9]: A6 = rop(A5, 'R3*(-1)+R1=>R1', 'R3*(1)+R2=>R2')
        A6
```

Out[9]:

$$\begin{bmatrix}1 & 0 & 0 & 2x - \frac{y}{3} + \frac{10z}{3}\\0 & 1 & 0 & -x + \frac{y}{3} - \frac{7z}{3}\\0 & 0 & 1 & -x + \frac{y}{3} - \frac{4z}{3}\end{bmatrix}$$

This is in RREF, so we get the solution:

$$c_1 = 2x - \frac{1}{3}y + \frac{10}{3}z$$

$$c_2 = -x + \frac{1}{3}y - \frac{7}{3}z$$

$$c_3 = -x + \frac{1}{3}y - \frac{4}{3}z$$

This can also be written in matrix form:

$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}2 & -\frac{1}{3} & \frac{10}{3}\\-1 & \frac{1}{3} & -\frac{7}{3}\\-1 & \frac{1}{3} & -\frac{4}{3}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Since we can find a solution for any given x, y and z, we conclude that the given set of vectors is a basis.

0.1.2 Example 2

Determine if the vectors below form a basis of $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix}0\\2\\1\\1\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}0\\3\\0\\3\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}-2\\1\\-2\\1\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

Solution We want to find c_1, c_2, c_3 such that:

$$\begin{bmatrix}r\\s\\t\\u\end{bmatrix} = c_1\begin{bmatrix}0\\2\\1\\1\end{bmatrix} + c_2\begin{bmatrix}0\\3\\0\\3\end{bmatrix} + c_3\begin{bmatrix}-2\\1\\-2\\1\end{bmatrix} + c_4\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

Equivalently, we need to solve the system:

$$\begin{bmatrix}0 & 0 & -2 & 0\\2 & 3 & 1 & 0\\1 & 0 & -2 & 1\\1 & 3 & 1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = \begin{bmatrix}r\\s\\t\\u\end{bmatrix}$$

We solve the system using Gaussian Elimination:

```
In [10]: r, s, t, u = symbols('r,s,t,u')
         A = matrix_to_rational([[0,  0, -2,  0, r],
                                 [2,  3,  1,  0, s],
                                 [1,  0, -2,  1, t],
                                 [1,  3,  1, -1, u]])
         A
```

Out[10]:

$$\begin{bmatrix}0 & 0 & -2 & 0 & r\\2 & 3 & 1 & 0 & s\\1 & 0 & -2 & 1 & t\\1 & 3 & 1 & -1 & u\end{bmatrix}$$

```
In [11]: A1 = rop(A, 'R1<=>R3')
         A1
```

Out[11]:

$$\begin{bmatrix}1 & 0 & -2 & 1 & t\\2 & 3 & 1 & 0 & s\\0 & 0 & -2 & 0 & r\\1 & 3 & 1 & -1 & u\end{bmatrix}$$

```
In [12]: A2 = rop(A1, 'R1*(-2)+R2=>R2', 'R1*(-1)+R4=>R4')
         A2
```

Out[12]:

$$\begin{bmatrix}1 & 0 & -2 & 1 & t\\0 & 3 & 5 & -2 & s-2t\\0 & 0 & -2 & 0 & r\\0 & 3 & 3 & -2 & -t+u\end{bmatrix}$$

```
In [13]: A3 = rop(A2, 'R2*(1/3)=>R2')
         A3
```

Out[13]:

$$\begin{bmatrix}1 & 0 & -2 & 1 & t\\0 & 1 & \frac{5}{3} & -\frac{2}{3} & \frac{s}{3}-\frac{2t}{3}\\0 & 0 & -2 & 0 & r\\0 & 3 & 3 & -2 & -t+u\end{bmatrix}$$

```
In [14]: A4 = rop(A3, 'R2*(-3)+R4=>R4')
         A4
```

Out[14]:

$$\begin{bmatrix}1 & 0 & -2 & 1 & t\\0 & 1 & \frac{5}{3} & -\frac{2}{3} & \frac{s}{3}-\frac{2t}{3}\\0 & 0 & -2 & 0 & r\\0 & 0 & -2 & 0 & -s+t+u\end{bmatrix}$$

```
In [15]: A5 = rop(A4, 'R3*(-1/2)=>R3')
         A5
```

Out[15]:

$$\begin{bmatrix} 1 & 0 & -2 & 1 & t \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & \frac{s}{3} - \frac{2t}{3} \\ 0 & 0 & 1 & 0 & -\frac{r}{2} \\ 0 & 0 & -2 & 0 & -s + t + u \end{bmatrix}$$

```
In [16]: A6 = rop(A5, 'R3*(2)+R1=>R1', 'R3*(-5/3)+R2=>R2', 'R3*(2)+R4=>R4')
         A6
```

Out[16]:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -r + t \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{5r}{6} + \frac{s}{3} - \frac{2t}{3} \\ 0 & 0 & 1 & 0 & -\frac{r}{2} \\ 0 & 0 & 0 & 0 & -r - s + t + u \end{bmatrix}$$

Thus, the system is equivalent to:

$$c_1 + c_4 = -r + t$$

$$c_2 - \frac{2}{3}c_4 = \frac{5}{6}r + \frac{1}{3}s - \frac{2}{3}t$$

$$c_3 = -\frac{1}{2}r$$

$$0 = -r - s + t + u$$

The last equation is impossible if $-r - w + t + u \neq 0$, so the given set is not a basis.

0.2 Matrix Inversion

0.2.1 Example 1

Determine if the matrix below is invertible and, if so, find its inverse.

$$\begin{bmatrix} 2 & 3 & 1 & -2 \\ 3 & -1 & 2 & 2 \\ 1 & 5 & -2 & -3 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

Solution:

```
In [17]: A = matrix_to_rational([[2,  3,  1, -2, 1, 0, 0, 0],
                                 [3, -1,  2,  2, 0, 1, 0, 0],
                                 [1,  5, -2, -3, 0, 0, 1, 0],
                                 [1,  2,  1,  1, 0, 0, 0, 1]])
         A
```

Out[17]:

$$\begin{bmatrix}2 & 3 & 1 & -2 & 1 & 0 & 0 & 0\\3 & -1 & 2 & 2 & 0 & 1 & 0 & 0\\1 & 5 & -2 & -3 & 0 & 0 & 1 & 0\\1 & 2 & 1 & 1 & 0 & 0 & 0 & 1\end{bmatrix}$$

```
In [18]: A1 = rop(A, 'R1<=>R3')
         A1
```

Out[18]:

$$\begin{bmatrix}1 & 5 & -2 & -3 & 0 & 0 & 1 & 0\\3 & -1 & 2 & 2 & 0 & 1 & 0 & 0\\2 & 3 & 1 & -2 & 1 & 0 & 0 & 0\\1 & 2 & 1 & 1 & 0 & 0 & 0 & 1\end{bmatrix}$$

```
In [19]: A2 = rop(A1, 'R1*(-3)+R2=>R2', 'R1*(-2)+R3=>R3', 'R1*(-1)+R4=>R4')
         A2
```

Out[19]:

$$\begin{bmatrix}1 & 5 & -2 & -3 & 0 & 0 & 1 & 0\\0 & -16 & 8 & 11 & 0 & 1 & -3 & 0\\0 & -7 & 5 & 4 & 1 & 0 & -2 & 0\\0 & -3 & 3 & 4 & 0 & 0 & -1 & 1\end{bmatrix}$$

```
In [20]: A3 = rop(A2, 'R2*(-1/16)=>R2')
         A3
```

Out[20]:

$$\begin{bmatrix}1 & 5 & -2 & -3 & 0 & 0 & 1 & 0\\0 & 1 & -\frac{1}{2} & -\frac{11}{16} & 0 & -\frac{1}{16} & \frac{3}{16} & 0\\0 & -7 & 5 & 4 & 1 & 0 & -2 & 0\\0 & -3 & 3 & 4 & 0 & 0 & -1 & 1\end{bmatrix}$$

```
In [21]: A4 = rop(A3, 'R2*(-5)+R1=>R1', 'R2*(7)+R3=>R3', 'R2*(3)+R4=>R4')
         A4
```

Out[21]:

$$\begin{bmatrix}1 & 0 & \frac{1}{2} & \frac{7}{16} & 0 & \frac{5}{16} & \frac{1}{16} & 0\\0 & 1 & -\frac{1}{2} & -\frac{11}{16} & 0 & -\frac{1}{16} & \frac{3}{16} & 0\\0 & 0 & \frac{3}{2} & -\frac{13}{16} & 1 & -\frac{7}{16} & -\frac{11}{16} & 0\\0 & 0 & \frac{3}{2} & \frac{31}{16} & 0 & -\frac{3}{16} & -\frac{7}{16} & 1\end{bmatrix}$$

```
In [22]: A5 = rop(A4, 'R3*(2/3)=>R3')
         A5
```

Out[22]:

$$\begin{bmatrix}1 & 0 & \frac{1}{2} & \frac{7}{16} & 0 & \frac{5}{16} & \frac{1}{16} & 0\\0 & 1 & -\frac{1}{2} & -\frac{11}{16} & 0 & -\frac{1}{16} & \frac{3}{16} & 0\\0 & 0 & 1 & -\frac{13}{24} & \frac{2}{3} & -\frac{7}{24} & -\frac{11}{24} & 0\\0 & 0 & \frac{3}{2} & \frac{31}{16} & 0 & -\frac{3}{16} & -\frac{7}{16} & 1\end{bmatrix}$$

```
In [23]: A6 = rop(A5, 'R3*(-1/2)+R1=>R1', 'R3*(1/2)+R2=>R2', 'R3*(-3/2)+R4=>R4')
         A6
```

Out[23]:

$$\begin{bmatrix}1 & 0 & 0 & \frac{17}{24} & -\frac{1}{3} & \frac{11}{24} & \frac{7}{24} & 0\\0 & 1 & 0 & -\frac{23}{24} & \frac{1}{3} & -\frac{5}{24} & -\frac{1}{24} & 0\\0 & 0 & 1 & -\frac{13}{24} & \frac{2}{3} & -\frac{7}{24} & -\frac{11}{24} & 0\\0 & 0 & 0 & \frac{11}{4} & -1 & \frac{1}{4} & \frac{1}{4} & 1\end{bmatrix}$$

```
In [24]: A7 = rop(A6, 'R4*(4/11)=>R4')
         A7
```

Out[24]:

$$\begin{bmatrix}1 & 0 & 0 & \frac{17}{24} & -\frac{1}{3} & \frac{11}{24} & \frac{7}{24} & 0\\0 & 1 & 0 & -\frac{23}{24} & \frac{1}{3} & -\frac{5}{24} & -\frac{1}{24} & 0\\0 & 0 & 1 & -\frac{13}{24} & \frac{2}{3} & -\frac{7}{24} & -\frac{11}{24} & 0\\0 & 0 & 0 & 1 & -\frac{4}{11} & \frac{1}{11} & \frac{1}{11} & \frac{4}{11}\end{bmatrix}$$

```
In [25]: A8 = rop(A7, 'R4*(-17/24)+R1=>R1', 'R4*(23/24)+R2=>R2', 'R4*(13/24)+R3=>R3
         A8
```

Out[25]:

$$\begin{bmatrix}1 & 0 & 0 & 0 & -\frac{5}{66} & \frac{13}{33} & \frac{5}{22} & -\frac{17}{66}\\0 & 1 & 0 & 0 & -\frac{1}{66} & -\frac{4}{33} & \frac{1}{22} & \frac{23}{66}\\0 & 0 & 1 & 0 & \frac{31}{66} & -\frac{8}{33} & -\frac{9}{22} & \frac{13}{66}\\0 & 0 & 0 & 1 & -\frac{4}{11} & \frac{1}{11} & \frac{1}{11} & \frac{4}{11}\end{bmatrix}$$

Since the RREF of the augmented matrix has the identity matrix on its left half, the given matrix is invertible. Its inverse can be found on the right half of the augmented matrix:

$$\begin{bmatrix}-\frac{5}{66} & \frac{13}{33} & \frac{5}{22} & -\frac{17}{66}\\-\frac{1}{66} & -\frac{4}{33} & \frac{1}{22} & \frac{23}{66}\\\frac{31}{66} & -\frac{8}{33} & -\frac{9}{22} & \frac{13}{66}\\-\frac{4}{11} & \frac{1}{11} & \frac{1}{11} & \frac{4}{11}\end{bmatrix}$$

0.2.2 Example 2

Determine if the matrix below is invertible and, if so, find its inverse.

$$\begin{bmatrix}1 & 2 & 0 & -2\\-1 & 2 & -6 & -8\\-1 & 0 & 3 & 3\\0 & 2 & -3 & -5\end{bmatrix}$$

Solution:

```
In [26]: A = matrix_to_rational([[ 1, 2,  0, -2, 1, 0, 0, 0],
                                [-1, 2, -6, -8, 0, 1, 0, 0],
                                [-1, 0,  3,  3, 0, 0, 1, 0],
                                [ 0, 2, -3, -5, 0, 0, 0, 1]])
         A
```

Out[26]:

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 \\ -1 & 2 & -6 & -8 & 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -3 & -5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
In [27]: A1 = rop(A, 'R1*(1)+R2=>R2', 'R1*(1)+R3=>R3')
         A1
```

Out[27]:

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 4 & -6 & -10 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & -5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
In [28]: A2 = rop(A1, 'R2*(1/4)=>R2')
         A2
```

Out[28]:

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{5}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -3 & -5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
In [29]: A3 = rop(A2, 'R2*(-2)+R1=>R1', 'R2*(-2)+R3=>R3', 'R2*(-2)+R4=>R4')
         A3
```

Out[29]:

$$\begin{bmatrix} 1 & 0 & 3 & 3 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{5}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 6 & 6 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

Since we got a row of zeros, we can stop here, and the matrix will not be invertible.

```
In [ ]:
```