14-Diagonalization

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```
In [23]: from latools import *
         from sympy import *
         init_printing(use_latex=True)
```

1 Diagonalization

1.1 Example 1

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 9x - 18y + 6z \\ 6x - 11y + 2z \\ 2x - 6y + 5z \end{bmatrix}$$

Find, if possible, a basis of $\mathbf{R^3}$ that diagonalizes L, and the diagonalization of L.

Solution: We start by writing the matrix of L on the standard basis:

$$M = \begin{bmatrix} 9 & -18 & 6 \\ 6 & -11 & 2 \\ 2 & -6 & 5 \end{bmatrix}$$

We have to determine if it is possible to find a basis of eigenvectors of M. We start by finding the eigenvalues:

```
In [24]: M = matrix_to_rational([[9, -18, 6],
                                 [6, -11, 2],
                                 [2,  -6, 5]])
         M
```

Out[24]:

$$\begin{bmatrix} 9 & -18 & 6 \\ 6 & -11 & 2 \\ 2 & -6 & 5 \end{bmatrix}$$

```
In [25]: lbd = symbols('lambda')
         p = det(M - lbd*eye(3))
         p
```

Out[25]:

$$-\lambda^3 + 3\lambda^2 + \lambda - 3$$

```
In [26]: factor(p)
```

Out[26]:

$$-(\lambda - 3)(\lambda - 1)(\lambda + 1)$$

The eigenvalues of M are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 3$. Next, we find a basis for each eigenspace:

Eigenspace of $\lambda_1 = -1$:

```
In [27]: R = reduced_row_echelon_form(M - (-1)*eye(3))
         R
```

Out[27]:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The system for the RREF matrix is:

$$\begin{aligned} x_1 - 3x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

Letting the free variable $x_3 = 1$ we get $x_1 = 3$ and $x_2 = 2$, so we get following basis for the eigenspace $E(-1)$:

$$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Eigenspace of $\lambda_2 = 1$:

```
In [28]: R = reduced_row_echelon_form(M - (1)*eye(3))
         R
```

Out[28]:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The system for the RREF matrix is:

$$x_1 - 3x_3 = 0$$
$$x_2 - \frac{5}{3}x_3 = 0$$

Letting $x_3 = 3$ we get $x_1 = 9$ and $x_2 = 5$, and the following basis for the eigenspace $E(1)$:

$$\left\{ \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix} \right\}$$

Eigenspace of $\lambda_3 = 3$:

In [29]:
```
R = reduced_row_echelon_form(M - (3)*eye(3))
R
```

Out[29]:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The system for the RREF matrix is:

$$x_1 - 2x_3 = 0$$
$$x_2 - x_3 = 0$$

Letting $x_3 = 1$ we get $x_1 = 2$ and $x_2 = 1$, and the following basis for the eigenspace $E(1)$:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Summary:

Eigenvalue	Basis of Eigenspace
$\lambda_1 = -1$	$\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$
$\lambda_2 = 1$	$\left\{ \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix} \right\}$
$\lambda_3 = 3$	$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

The sum of the dimensions of the eigenspaces is $1 + 1 + 1 = 3$, so we get a basis of $\mathbb{R}^3$:

$$B = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

It is guaranteed that these three vectors form a basis, because:

- They are linearly independent, since they are eigenvectors corresponding to different eigenvalues.
- Since the dimension of $\mathbb{R}^3$ is 3, any set of three linearly independent vectors is a basis.

We conclude that the matrix is diagonalizable. To find the matrix of the linear transformation on basis B, we define the change of basis matrix:

```
In [30]: P = matrix_to_rational([[3, 9, 2],
                                 [2, 5, 1],
                                 [1, 3, 1]])
         P
```

```
Out[30]:
```

$$\begin{bmatrix} 3 & 9 & 2 \\ 2 & 5 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

The matrix of the linear transformation on the new basis is:

```
In [31]: D = P**(-1) * M * P
         D
```

```
Out[31]:
```

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We get a diagonal matrix with the eigenvalues at the diagonal, as expected.

```
In [32]: D = matrix_to_rational([[1,0,0],[0,-1,0],[0,0,3]])
         D
```

```
Out[32]:
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

1.2 Example 2

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by:

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -8x + 25y \\ -4x + 12y \end{bmatrix}$$

Find, if possible, a basis of $\mathbf{R^3}$ that diagonalizes L, and the diagonalization of L.

Solution: We start by writing the matrix of L on the standard basis:

$$M = \begin{bmatrix} -8 & 25 \\ -4 & 12 \end{bmatrix}$$

We have to determine if it is possible to find a basis of eigenvectors of M. We start by finding the eigenvalues:

```
In [33]: M = matrix_to_rational([[-8, 25],
                                  [-4, 12]])
         M
```

Out[33]:

$$\begin{bmatrix} -8 & 25 \\ -4 & 12 \end{bmatrix}$$

```
In [34]: lbd = symbols('lambda')
         p = det(M - lbd*eye(2))
         factor(p)
```

Out[34]:

$$(\lambda - 2)^2$$

The matrix has only one eigenvalue, $\lambda_1 = 2$. We next find a basis for the eigenspace of the eigenvalue:

```
In [35]: R = reduced_row_echelon_form(M - 2*eye(2))
         R
```

Out[35]:

$$\begin{bmatrix} 1 & -\frac{5}{2} \\ 0 & 0 \end{bmatrix}$$

The system corresponding to the RREF has a single equation:

$$x_1 - \frac{5}{2}x_2 = 0$$

Letting $x_2 = 2$ we get $x_1 = 5$, which yields the following basis for the eigenspace $E(2)$:

$$\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$$

This means that we have *at most one* linearly independent eigenvector, so it is not possible to have a basis of eigenvectors (because the dimension of $\mathbb{R}^2$ is 2). We conclude that *this matrix is not diagonalizable*.

```
In [ ]:
```