Problem 1.(20 points.) Find a basis for the subspace of $\mathbb{R}^3$ spanned by the vectors:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} -4 \\ 4 \\ -8 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} -4 \\ -2 \\ 13 \end{bmatrix}$$

Solution: Form the matrix with the given vectors in it's columns and find its reduced row echelon form:

$$\begin{bmatrix} 1 & -4 & 2 & -4 \\ -1 & 4 & 0 & -2 \\ 2 & -8 & -3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The reduced row echelon matrix has two pivot columns, column 1 and column 3. So, the corresponding vectors, $\mathbf{u}_1$ and $\mathbf{u}_2$ are a basis of the subspace:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right\}$$

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Problem 2.(20 points.) Answer the following items for the matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 11 & 7 & 1 \\ 1 & 1 & 4 & 0 \end{bmatrix}$$

(a) (10 points.) Find a basis for the range of A, and determine the dimension of the range.

Solution: The reduced row echelon form of A is:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{5}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

A basis for the range is given by the columns of A corresponding to pivot columns in the reduced row echelon form, that is, columns 1, 2 and 3. So, a basis for the range is:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\}$$

Since a basis for the range has two vectors, the range has dimension 3. ☺

(b) (10 points.) Find a basis for the kernel of A, and determine the dimension of the kernel.

Solution: From the reduced row echelon form, we conclude that vectors $\mathbf{x}$ in the kernel satisfy the equations:

$$x_1 - \frac{5}{3}x_4 = 0$$
$$x_2 + \frac{1}{3}x_4 = 0$$
$$x_3 + \frac{1}{3}x_4 = 0$$

The only free variable is x_4, so, letting $s = x_4$ we get the following representation for the kernel:

$$\left\{ \begin{bmatrix} \frac{5}{3}s \\ -\frac{1}{3}s \\ -\frac{1}{3}s \\ s \end{bmatrix} : x \in \mathbb{R} \right\}$$

By choosing an arbitrary $s \neq 0$ we get a basis for the kernel. With $s = 1$ we get the basis:

$$\left\{ \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right\}$$

Since the basis has one vector, the kernel has dimension 1. ☺

Problem 3. (16 points.) Find scalars a and b that make the following matrix identity true.

$$\begin{bmatrix} a & 1 & 2 \\ 2 & 2 & b \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & -1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 11 & -\frac{4}{3} \\ 5 & -8 \end{bmatrix}$$

Solution: The product of the matrices is:

$$\begin{bmatrix} 2a+7 & 2a-5 \\ 3b+6 & -2b+2 \\ 5 & -8 \end{bmatrix}$$

So, we must have:

$$2a + 7 = 8$$
$$3b + 6 = 11$$
$$2a - 5 = -4$$
$$-2b + 2 = -\frac{4}{3}$$

The first equation gives:

$$a = \frac{8-7}{2} = \frac{1}{2}.$$

The second equation gives:

$$b = \frac{11-6}{3} = \frac{5}{3}.$$

Plugging in these values in the third and fourth equation, we see that all equations are satisfied, so we get the solution:

$$a = \frac{1}{2}, \quad b = \frac{5}{3}.$$

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Problem 4. (16 points.) Find the determinants for the following matrices. Explain the method you used to find the determinant, showing all computations. Do not use Python for this problem.

(a) (8 points.) $\det \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \\ 3 & 4 & 1 \end{bmatrix}$

Solution: Using the "shortcut" for 3×3 determinants we get:

$$1 \times (-2) \times 1 + 0 \times 1 \times 3 + 3 \times 2 \times 4 - 3 \times (-2) \times 3 - 1 \times 1 \times 4 - 0 \times 2 \times 1 = -2 + 0 + 24 + 12 - 4 + 0 = 36$$

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(b) (8 points.) $\det \begin{bmatrix} 2 & 3 & -5 & 9 & 0 \\ 0 & -1 & 3 & -4 & 2 \\ 0 & 0 & 3 & 7 & -11 \\ 0 & 0 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}$

Solution: The matrix is triangular, so the determinant is the product of the diagonal entries:

$$2 \times (-1) \times 3 \times \frac{1}{2} \times 10 = -30$$

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Problem 5. (18 points.) Determine if each of the following statements is true or false, and provide a brief justification for your answer.

(a) (3 points.) A linear transformation $T : \mathbb{R}^8 \to \mathbb{R}^4$ must be onto.

Solution: False. The range of T is a subspace of $\mathbb{R}^4$ with dimension at most 4. If the dimension of the range is less than 4, then the linear transformation is not onto.

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(b) (3 points.) A linear transformation $T : \mathbb{R}^8 \to \mathbb{R}^4$ can be one-to-one.

Solution: False. Since the dimension of the domain is 8 and the dimension of the codomain is 4, and $8 > 4$, the transformation can't be one-to-one.

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(c) (3 points.) There is exactly one value of a for which the matrix below is singular:

$$\begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix}$$

Solution: False. The matrix is singular if and only if:

$$\det \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} = a^2 - 4 = 0.$$

This equation has two solutions, $a = 2$ and $a = -2$. ☺

(d) (3 points.) The two matrices below have the same determinant.

$$\begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix} \quad \begin{bmatrix} a & b + 2a & c \\ d & f + 2d & g \\ h & i + 2h & j \end{bmatrix}$$

emphSolution: True. The second matrix is obtained from the first by adding to the second column the first columns multiplied by 2. ☺

(e) (3 points.) If $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are distinct vectors in $\mathbb{R}^4$, then the subspace spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ has dimension 3.

Solution: False. If the vectors are linearly dependent, the spanned subspace will have dimension *less* than 3. ☺

(f) (3 points.) If all entries of a matrix are positive, then the matrix is invertible.

Solution: False. The matrix:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

has all entries positive but its determinant is $1 \times 4 - 2 \times 2 = 0$, so it is not invertible. ☺

Problem 6. (10 points.) Find the area of the parallelogram in $\mathbb{R}^2$ determined by the vectors

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Solution: The area is:

$$\left| \det \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix} \right| = |2 \times 4 - 3 \times (-2)| = 14$$