02-EROs-and-Matrix-Multiplication

January 6, 2017

In [1]:
```
from latools import *
from sympy import *
init_printing(use_latex=True)
```

1 Solving Systems Using Matrix Multiplication

We want to solve the linear system

$$A\mathbf{x} = \mathbf{v}$$

where A is the matrix below:

In [2]:
```
A = matrix_to_rational([[2,  1, -2],
                        [2, -1,  2],
                        [1,  1,  1]])
A
```

Out[2]:

$$\begin{bmatrix} 2 & 1 & -2 \\ 2 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

We want the right-hand side of the matrix, $\mathbf{v}$ to be a generic vector. We can do this using the symbolic features of `sympy`:

In [3]:
```
a, b, c = symbols('a, b, c')
v = Matrix([a,b,c])
v
```

Out[3]:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Let's now perform a sequence of row operations to get the RREF of A:

In [4]:
```
A1 = rop(A, 'R1*(1/2)=>R1')
A1
```

Out [4]:

$$\begin{bmatrix}1 & \frac{1}{2} & -1\\ 2 & -1 & 2\\ 1 & 1 & 1\end{bmatrix}$$

```
In [5]: A2 = rop(A1, 'R1*(-2)+R2=>R2', 'R1*(-1)+R3=>R3')
        A2
```

Out [5]:

$$\begin{bmatrix}1 & \frac{1}{2} & -1\\ 0 & -2 & 4\\ 0 & \frac{1}{2} & 2\end{bmatrix}$$

```
In [6]: A3 = rop(A2, 'R2*(-1/2)=>R2')
        A3
```

Out [6]:

$$\begin{bmatrix}1 & \frac{1}{2} & -1\\ 0 & 1 & -2\\ 0 & \frac{1}{2} & 2\end{bmatrix}$$

```
In [7]: A4 = rop(A3, 'R2*(-1/2)+R1=>R1', 'R2*(-1/2)+R3=>R3')
        A4
```

Out [7]:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 3\end{bmatrix}$$

```
In [8]: A5 = rop(A4, 'R3*(1/3)=>R3')
        A5
```

Out [8]:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -2\\ 0 & 0 & 1\end{bmatrix}$$

```
In [9]: A6 = rop(A5, 'R3*(2)+R2=>R2')
        A6
```

Out [9]:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Notice that the RREF of matrix A is the identity matrix. Let's also notice that we can define a sequence of row operations and obtain the RREF of A in a single function call:

```
In [10]: rop_seq = ['R1*(1/2)=>R1', 'R1*(-2)+R2=>R2', 'R1*(-1)+R3=>R3',
                    'R2*(-1/2)=>R2', 'R2*(-1/2)+R1=>R1', 'R2*(-1/2)+R3=>R3',
                    'R3*(1/3)=>R3', 'R3*(2)+R2=>R2'
                   ]
         R = rop(A, *rop_seq)
         R
```

Out[10]:

$$\left[\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}\right]$$

Now a key insight: to get the solution of the system, we apply the \emph{the same sequence of row operations to the vector **v**:

```
In [11]: x = rop(v, *rop_seq)
         x
```

Out[11]:

$$\left[\begin{matrix}\frac{a}{4} + \frac{b}{4}\\- \frac{b}{3} + \frac{2 c}{3}\\- \frac{a}{4} + \frac{b}{12} + \frac{c}{3}\end{matrix}\right]$$

```
In [12]: print(latex(x))
```

```
\left[\begin{matrix}\frac{a}{4} + \frac{b}{4}\\- \frac{b}{3} + \frac{2 c}{3}\\- \fr
```

Lets check that indeed we have a solution:

```
In [13]: A * x
```

Out[13]:

$$\left[\begin{matrix}a\\b\\c\end{matrix}\right]$$

Now, let's notice the following: the vector x can alternatively be obtained from multiplying a matrix by **v**:

```
In [14]: E = matrix_to_rational([[ 1/4,  1/4,   0],
                                 [   0, -1/3, 2/3],
                                 [-1/4, 1/12, 1/3]])
         E
```

Out[14]:

$$\left[\begin{matrix}\frac{1}{4} & \frac{1}{4} & 0\\0 & - \frac{1}{3} & \frac{2}{3}\\- \frac{1}{4} & \frac{1}{12} & \frac{1}{3}\end{matrix}\right]$$

```
In [15]: E * v
```

Out[15]:

$$\begin{bmatrix} \frac{a}{4} + \frac{b}{4} \\ -\frac{b}{3} + \frac{2c}{3} \\ -\frac{a}{4} + \frac{b}{12} + \frac{c}{3} \end{bmatrix}$$

Now for the punchline: what happens when we multiply the matrix E by A?

```
In [16]: E * A
```

Out[16]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, we notice that a more practical way to find what is the matrix E is to perform the sequence of row operations to the 3×3 identity matrix:

```
In [17]: E = rop(eye(3), *rop_seq)
         E
```

Out[17]:

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{4} & \frac{1}{12} & \frac{1}{3} \end{bmatrix}$$

We say that the matrix E is the *inverse* of the matrix A, and denote it by A^{-1}.

2 The Inverse of a Matrix

Let's start with the following 4×4 matrix:

```
In [18]: A = matrix_to_rational([[ 0,  4, -1, 0],
                                 [ 0, -2, -2, 1/3],
                                 [-1,  1,  0, 1/12],
                                 [ 2,- 1,  3, 0]])
         A
```

Out[18]:

$$\begin{bmatrix} 0 & 4 & -1 & 0 \\ 0 & -2 & -2 & \frac{1}{3} \\ -1 & 1 & 0 & \frac{1}{12} \\ 2 & -1 & 3 & 0 \end{bmatrix}$$

We augment the matrix by appending the 4×4 identity:

```
In [19]: M = Matrix.hstack(A,eye(4))
         M
```

Out[19]:

$$\begin{bmatrix} 0 & 4 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & \frac{1}{3} & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & \frac{1}{12} & 0 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We now perform row operations to (if possible) reduce A to the identity matrix.

```
In [20]: M1 = rop(M, 'R1<=>R3')
         M1
```

Out[20]:

$$\begin{bmatrix} -1 & 1 & 0 & \frac{1}{12} & 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & \frac{1}{3} & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
In [21]: M2 = rop(M1, 'R1*(-1)=>R1')
         M2
```

Out[21]:

$$\begin{bmatrix} 1 & -1 & 0 & -\frac{1}{12} & 0 & 0 & -1 & 0 \\ 0 & -2 & -2 & \frac{1}{3} & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
In [22]: M3 = rop(M2, 'R1*(-2)+R4=>R4')
         M3
```

Out[22]:

$$\begin{bmatrix} 1 & -1 & 0 & -\frac{1}{12} & 0 & 0 & -1 & 0 \\ 0 & -2 & -2 & \frac{1}{3} & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & \frac{1}{6} & 0 & 0 & 2 & 1 \end{bmatrix}$$

```
In [23]: M4 = rop(M3, 'R2*(-1/2)=>R2')
         M4
```

Out[23]:

$$\begin{bmatrix} 1 & -1 & 0 & -\frac{1}{12} & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{6} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 4 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & \frac{1}{6} & 0 & 0 & 2 & 1 \end{bmatrix}$$

```
In [24]: M5 = rop(M4, 'R2*(1)+R1=>R1', 'R2*(-4)+R3=>R3', 'R2*(-1)+R4=>R4')
         M5
```

Out[24]:

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{6} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -5 & \frac{2}{3} & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{3} & 0 & \frac{1}{2} & 2 & 1 \end{bmatrix}$$

```
In [25]: M6 = rop(M5, 'R3*(-1/5)=>R3')
         M6
```

Out[25]:

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{6} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 2 & \frac{1}{3} & 0 & \frac{1}{2} & 2 & 1 \end{bmatrix}$$

```
In [26]: M7 = rop(M6, 'R3*(-1)+R1=>R1', 'R3*(-1)+R2=>R2', 'R3*(-2)+R4=>R4')
         M7
```

Out[26]:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{60} & \frac{1}{5} & -\frac{1}{10} & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{30} & \frac{1}{5} & -\frac{1}{10} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{5} & \frac{2}{5} & \frac{13}{10} & 2 & 1 \end{bmatrix}$$

```
In [27]: M8 = rop(M7, 'R4*(5/3)=>R4')
         M8
```

Out[27]:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{7}{60} & \frac{1}{5} & -\frac{1}{10} & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{30} & \frac{1}{5} & -\frac{1}{10} & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{13}{6} & \frac{10}{3} & \frac{5}{3} \end{bmatrix}$$

```
In [28]: M9 = rop(M8, 'R4*(7/60)+R1=>R1', 'R4*(1/30)+R2=>R2', 'R4*(2/15)+R3=>R3')
         M9
```

Out[28]:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{5}{18} & \frac{11}{72} & -\frac{11}{18} & \frac{7}{36} \\ 0 & 1 & 0 & 0 & \frac{2}{9} & -\frac{1}{36} & \frac{1}{9} & \frac{1}{18} \\ 0 & 0 & 1 & 0 & -\frac{1}{9} & -\frac{1}{9} & \frac{4}{9} & \frac{2}{9} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{13}{6} & \frac{10}{3} & \frac{5}{3} \end{bmatrix}$$

The matrix that appears in the position originally holding the identity matrix is:

In [29]: E = M9[:,4:8]
         E

Out[29]:

$$\left[\begin{matrix}\frac{5}{18} & \frac{11}{72} & -\frac{11}{18} & \frac{7}{36}\\\frac{2}{9} & -\frac{1}{36} & \frac{1}{9} & \frac{1}{18}\\-\frac{1}{9} & -\frac{1}{9} & \frac{4}{9} & \frac{2}{9}\\\frac{2}{3} & \frac{13}{6} & \frac{10}{3} & \frac{5}{3}\end{matrix}\right]$$

In [30]: E*A

Out[30]:

$$\left[\begin{matrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{matrix}\right]$$

In [31]: A*E

Out[31]:

$$\left[\begin{matrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{matrix}\right]$$

In [32]: v = Matrix([2,-3,5,-1])
         v

Out[32]:

$$\left[\begin{matrix}2\\-3\\5\\-1\end{matrix}\right]$$

In [33]: x = E * v
         x

Out[33]:

$$\left[\begin{matrix}-\frac{227}{72}\\\frac{37}{36}\\\frac{19}{9}\\\frac{59}{6}\end{matrix}\right]$$

In [34]: A * x

Out[34]:

$$\begin{bmatrix} 2 \\ -3 \\ 5 \\ -1 \end{bmatrix}$$

In [ ]: