16-Range-Kernel-Rank-Nullity

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```
In [1]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

1 How to find the range and kernel of a matrix

1.1 Example

Find the range, kernel, rank and nullity of the matrix:

$$A = \begin{bmatrix} 2 & 4 & 1 & 1 & -5 \\ 2 & 4 & 2 & 0 & -4 \\ -3 & -6 & 0 & -3 & 9 \\ 0 & 0 & -2 & 2 & -2 \end{bmatrix}$$

Solution: First notice that A represents a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^4$
Start by finding the RREF of A.

```
In [2]: A = matrix_to_rational([[ 2,  4,  1,  1, -5],
                                [ 2,  4,  2,  0, -4],
                                [-3, -6,  0, -3,  9],
                                [ 0,  0, -2,  2, -2]])
        A
```

Out[2]:

$$\begin{bmatrix} 2 & 4 & 1 & 1 & -5 \\ 2 & 4 & 2 & 0 & -4 \\ -3 & -6 & 0 & -3 & 9 \\ 0 & 0 & -2 & 2 & -2 \end{bmatrix}$$

```
In [3]: R = reduced_row_echelon_form(A)
        R
```

Out[3]:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the RREF we have:

- The free variables are x_2, x_4 and x_5.
- The pivot variables are x_1 and x_3.

Thus, a basis of the range is given by the pivot columns *in the original matrix*:

$$\operatorname{range}(A) = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

It also follows that $\operatorname{rank}(A) = \dim(\operatorname{range}(A)) = 2$.

To find a basis of the kernel of A we need to find a basis of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$. To do this, let's write the system associated with the RREF:

$$\begin{aligned} x_1 + 2x_2 + x_4 - 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0 \end{aligned}$$

Writing the pivot variables as functions of the free variables we get:

$$\begin{aligned} x_1 &= -2x_2 - x_4 + 3x_5 \\ x_3 &= x_4 - x_5 \end{aligned}$$

Since there are 3 free variables, the dimension of the solution set is 3. To find a basis, we construct the following table:

Variable			
x_2	1	0	0
x_4	0	1	0
x_5	0	0	1
$x_1 = -2x_2 - x_4 + 3x_5$	-2	-1	-3
$x_3 = x_4 - x_5$	0	1	-1

We conclude that:

$$\operatorname{kernel}(A) = \operatorname{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Finally, $\operatorname{nullity}(A) = 3$ (Dimension of the kernel.)

It is recommended that we check the results, as shown in the following cells:

```
In [4]: A*Matrix([-2,1,0,0,0])
```

Out[4]:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

```
In [5]: A*Matrix([-1,0,1,1,0])
```

Out[5]:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

```
In [6]: A*Matrix([3,0,-1,0,1])
```

Out[6]:

$$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

```
In [ ]:
```