07-Determinants-Definition

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In [3]:
```
from latools import *
from sympy import *
init_printing(use_latex=True)
from itertools import permutations
```

In [4]:
```
def row_scale(M, i, c):
    M[i,:] *= c
    for k in range(M.cols):
        M[i,k] = cancel(M[i,k])

def row_scale_add(M, i, j, c):
    M[j,:] += c*M[i,:]
    for k in range(M.cols):
        M[j,k] = cancel(M[j,k])

def row_swap(M, i, j):
    M[i,:], M[j,:] = M[j,:], M[i,:]
```

In [5]:
```
a, b, c, d = symbols('a, b, c, d')
A = Matrix([[a, b],[c, d]])
M = A.row_join(eye(2))
M
```

Out[5]:

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

In [6]:
```
row_scale(M, 0, 1/a)
M
```

Out[6]:

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

In [7]:
```
row_scale_add(M, 0, 1, -c)
M
```

Out[7]:

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{1}{a}(ad-bc) & -\frac{c}{a} & 1 \end{bmatrix}$$

In [8]:
```
row_scale(M, 1, a/(a*d-b*c))
M
```

Out[8]:

$$\begin{bmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

In [9]:
```
row_scale_add(M, 1, 0, -b/a)
M
```

Out[9]:

$$\begin{bmatrix} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

In [10]:
```
Ainv = M[:, 2:]
Ainv
```

Out[10]:

$$\begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

In [11]:
```
Adet = denom(Ainv[0,0])
Adet
```

Out[11]:

$$ad - bc$$

In [12]:
```
C = Adet*Ainv
C
```

Out[12]:

$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In [13]:
```
n = 2
A = Matrix([symbols(','.join(['a_{}{}'.format(i+1,j+1) for j in range(n)])
```

In [14]:
```
print(latex(A))
```

```
\left[\begin{matrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{matrix}\right]
```

In [15]:
```
n = 2
A = Matrix([symbols(','.join(['a_{}{}'.format(i+1,j+1) for j in range(n)])
M = A.row_join(eye(n))
for i in range(M.rows):
    row_scale(M, i, 1/M[i,i])
    for j in range(M.rows):
        if i != j:
            row_scale_add(M, i, j, -M[j,i])
Ainv = M[:, n:]
Adet = denom(Ainv[0,0])
C = Adet * Ainv
```

In [16]:
```
Ainv
```

Out[16]:

$$\begin{bmatrix} \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}} & -\frac{a_{12}}{a_{11}a_{22}-a_{12}a_{21}} \\ -\frac{a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & \frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}} \end{bmatrix}$$

In [17]:
```
Adet
```

Out[17]:

$$a_{11}a_{22} - a_{12}a_{21}$$

In [18]:
```
print(latex(Adet))
```

```
a_{11} a_{22} - a_{12} a_{21}
```

In [19]:
```
C
```

Out[19]:

$$\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

In [20]:
```
n = 3
A = Matrix([symbols(','.join(['a_{}{}'.format(i+1,j+1) for j in range(n)])
                for i in range(n)])
M = A.row_join(eye(n))
for i in range(M.rows):
    row_scale(M, i, 1/M[i,i])
    for j in range(M.rows):
        if i != j:
            row_scale_add(M, i, j, -M[j,i])
Ainv = M[:,n:]
Adet = denom(Ainv[0,0])
C = Adet * Ainv
```

In [21]: A

Out[21]:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

In [22]: print(latex(A))

```
\left[\begin{matrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_
```

In [23]: Adet

Out[23]:

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

In [24]: print(latex(Adet))

```
a_{11} a_{22} a_{33} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{12} a_{23}
```

In [25]: C

Out[25]:

$$\begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & -a_{12}a_{33} + a_{13}a_{32} & a_{12}a_{23} - a_{13}a_{22} \\ -a_{21}a_{33} + a_{23}a_{31} & a_{11}a_{33} - a_{13}a_{31} & -a_{11}a_{23} + a_{13}a_{21} \\ a_{21}a_{32} - a_{22}a_{31} & -a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

In [26]: print(latex(C))

```
\left[\begin{matrix}a_{22} a_{33} - a_{23} a_{32} & - a_{12} a_{33} + a_{13} a_{32}
```

In [27]: simplify(A*Ainv)

Out[27]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In [28]:
```
n = 4
A = Matrix([symbols(','.join(['a_{}{}'.format(i+1,j+1) for j in range(n)])
           for i in range(n)])
M = A.row_join(eye(n))
for i in range(M.rows):
    row_scale(M, i, 1/M[i,i])
```

```
            for j in range(M.rows):
                if i != j:
                    row_scale_add(M, i, j, -M[j,i])
        Ainv = M[:,n:]
        Adet = denom(Ainv[0,0])
        C = Adet * Ainv
```

In [29]:
```
Adet
```

Out[29]:

$$a_{11}a_{22}a_{33}a_{44}-a_{11}a_{22}a_{34}a_{43}-a_{11}a_{23}a_{32}a_{44}+a_{11}a_{23}a_{34}a_{42}+a_{11}a_{24}a_{32}a_{43}-a_{11}a_{24}a_{33}a_{42}-a_{12}a_{21}a_{33}a_{44}+a_{12}a_{21}a_{34}a_{43}+$$

In [30]:
```
simplify(A*Ainv)
```

Out[30]:

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

In [31]:
```
C
```

Out[31]:

$$\begin{bmatrix} a_{22}a_{33}a_{44} - a_{22}a_{34}a_{43} - a_{23}a_{32}a_{44} + a_{23}a_{34}a_{42} + a_{24}a_{32}a_{43} - a_{24}a_{33}a_{42} & -a_{12}a_{33}a_{44} + a_{12}a_{34}a_{43} + a_{13}a_{32}a_{44} - \\ -a_{21}a_{33}a_{44} + a_{21}a_{34}a_{43} + a_{23}a_{31}a_{44} - a_{23}a_{34}a_{41} - a_{24}a_{31}a_{43} + a_{24}a_{33}a_{41} & a_{11}a_{33}a_{44} - a_{11}a_{34}a_{43} - a_{13}a_{31}a_{44} + a \\ a_{21}a_{32}a_{44} - a_{21}a_{34}a_{42} - a_{22}a_{31}a_{44} + a_{22}a_{34}a_{41} + a_{24}a_{31}a_{42} - a_{24}a_{32}a_{41} & -a_{11}a_{32}a_{44} + a_{11}a_{34}a_{42} + a_{12}a_{31}a_{44} - \\ -a_{21}a_{32}a_{43} + a_{21}a_{33}a_{42} + a_{22}a_{31}a_{43} - a_{22}a_{33}a_{41} - a_{23}a_{31}a_{42} + a_{23}a_{32}a_{41} & a_{11}a_{32}a_{43} - a_{11}a_{33}a_{42} - a_{12}a_{31}a_{43} + a \end{bmatrix}$$

In [32]:
```
print(latex(Adet))
```

```
a_{11} a_{22} a_{33} a_{44} - a_{11} a_{22} a_{34} a_{43} - a_{11} a_{23} a_{32} a_
```

In [33]:
```
A = Matrix([[1,0,-3,-1],
            [2,-1,-4,1],
            [3,1,1,2],
            [1,2,1,1]])
A
```

Out[33]:

$$\begin{bmatrix}1 & 0 & -3 & -1\\2 & -1 & -4 & 1\\3 & 1 & 1 & 2\\1 & 2 & 1 & 1\end{bmatrix}$$

In [34]:
```
print(latex(A))
```

```
\left[\begin{matrix}1 & 0 & -3 & -1\\2 & -1 & -4 & 1\\3 & 1 & 1 & 2\\1 & 2 & 1 & 1\
```

In [35]:
```
A = Matrix([[1,2],[3,-1]])
A
```

Out[35]:

$$\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$$

In [36]:
```
A = Matrix([[2,-1,0],[3,2,4],[2,1,4]])
```

In [37]:
```
A = Matrix([[Rational(1,2),  0, 0, 0],
            [3, -1, 0, 0],
            [1,  4, 5, 0],
            [2, -3, 4, 3]])
A
```

Out[37]:

$$\begin{bmatrix}\frac{1}{2} & 0 & 0 & 0\\3 & -1 & 0 & 0\\1 & 4 & 5 & 0\\2 & -3 & 4 & 3\end{bmatrix}$$

In [38]:
```
print(latex(A))
```

```
\left[\begin{matrix}\frac{1}{2} & 0 & 0 & 0\\3 & -1 & 0 & 0\\1 & 4 & 5 & 0\\2 & -3
```

In [39]:
```
B = Matrix([[6, -3, -2, 0],
            [0,  4,  2, 1],
            [0,  0, -8, 0],
            [0,  0,  0, 4]])
B
```

Out[39]:

$$\begin{bmatrix}6 & -3 & -2 & 0\\0 & 4 & 2 & 1\\0 & 0 & -8 & 0\\0 & 0 & 0 & 4\end{bmatrix}$$

In [40]:
```
print(latex(B))
```

```
\left[\begin{matrix}6 & -3 & -2 & 0\\0 & 4 & 2 & 1\\0 & 0 & -8 & 0\\0 & 0 & 0 & 4\e
```

In [41]:
```
6*4*-8*4
```

Out[41]:

$$-768$$

In [42]:
```
n = A.rows
sout = """\\begin{tabular}{|c|l|c|c|}\\hline
Permutation & Inversions & Term in Determinant\\\\\\hline
"""
detA = 0
for p in permutations(range(n)):
    sout += '${}$ & '.format(str(tuple([k+1 for k in p])))
    sout += ' $'
    ninv = 0
    for i in range(n):
        for j in range(i, n):
            if p[i]>p[j]:
                sout += '({},{})\;'.format(p[i]+1,p[j]+1)
                ninv += 1
    sout += '$ & '
    sout += ' $'
    sout += '+' if ninv % 2 == 0 else '-'
    for i in range(n):
        sout += 'a_{{{}{}}}'.format(i+1, p[i]+1)
    sout += '='
    sout += '+' if ninv % 2 == 0 else '-'
    v = 1
    for i in range(n):
        t = A[i, p[i]]
        v *= t
        sout += '({})'.format(t)
    detA += v if ninv % 2 == 0 else -v
    sout += '={} $ '.format(v)
    sout += '\\\\\\hline\n'
sout += '&& Sum: ${}$\\\\\\hline\n'.format(detA)
sout +="""\\end{tabular}
"""
```

In [43]:
```
print(sout)
```

```
\begin{tabular}{|c|l|c|c|}\hline
Permutation & Inversions & Term in Determinant\\\hline
$(1, 2, 3, 4)$ &  $$ &  $+a_{11}a_{22}a_{33}a_{44}=+(1/2)(-1)(5)(3)=-15/2 $ \\\hlin
$(1, 2, 4, 3)$ &  $(4,3)\;$ &  $-a_{11}a_{22}a_{34}a_{43}=-(1/2)(-1)(0)(4)=0 $ \\\h
$(1, 3, 2, 4)$ &  $(3,2)\;$ &  $-a_{11}a_{23}a_{32}a_{44}=-(1/2)(0)(4)(3)=0 $ \\\hl
$(1, 3, 4, 2)$ &  $(3,2)\;(4,2)\;$ &  $+a_{11}a_{23}a_{34}a_{42}=+(1/2)(0)(0)(-3)=0
$(1, 4, 2, 3)$ &  $(4,2)\;(4,3)\;$ &  $+a_{11}a_{24}a_{32}a_{43}=+(1/2)(0)(4)(4)=0
$(1, 4, 3, 2)$ &  $(4,3)\;(4,2)\;(3,2)\;$ &  $-a_{11}a_{24}a_{33}a_{42}=-(1/2)(0)(5
```

```
$(2, 1, 3, 4)$ &  $(2,1)\;$ &  $-a_{12}a_{21}a_{33}a_{44}=-(0)(3)(5)(3)=0 $ \\\hlir
$(2, 1, 4, 3)$ &  $(2,1)\;(4,3)\;$ &  $+a_{12}a_{21}a_{34}a_{43}=+(0)(3)(0)(4)=0 $
$(2, 3, 1, 4)$ &  $(2,1)\;(3,1)\;$ &  $+a_{12}a_{23}a_{31}a_{44}=+(0)(0)(1)(3)=0 $
$(2, 3, 4, 1)$ &  $(2,1)\;(3,1)\;(4,1)\;$ &  $-a_{12}a_{23}a_{34}a_{41}=-(0)(0)(0)
$(2, 4, 1, 3)$ &  $(2,1)\;(4,1)\;(4,3)\;$ &  $-a_{12}a_{24}a_{31}a_{43}=-(0)(0)(1)
$(2, 4, 3, 1)$ &  $(2,1)\;(4,3)\;(4,1)\;(3,1)\;$ &  $+a_{12}a_{24}a_{33}a_{41}=+(0)
$(3, 1, 2, 4)$ &  $(3,1)\;(3,2)\;$ &  $+a_{13}a_{21}a_{32}a_{44}=+(0)(3)(4)(3)=0 $
$(3, 1, 4, 2)$ &  $(3,1)\;(3,2)\;(4,2)\;$ &  $-a_{13}a_{21}a_{34}a_{42}=-(0)(3)(0)
$(3, 2, 1, 4)$ &  $(3,2)\;(3,1)\;(2,1)\;$ &  $-a_{13}a_{22}a_{31}a_{44}=-(0)(-1)(1)
$(3, 2, 4, 1)$ &  $(3,2)\;(3,1)\;(2,1)\;(4,1)\;$ &  $+a_{13}a_{22}a_{34}a_{41}=+(0)
$(3, 4, 1, 2)$ &  $(3,1)\;(3,2)\;(4,1)\;(4,2)\;$ &  $+a_{13}a_{24}a_{31}a_{42}=+(0)
$(3, 4, 2, 1)$ &  $(3,2)\;(3,1)\;(4,2)\;(4,1)\;(2,1)\;$ &  $-a_{13}a_{24}a_{32}a_{
$(4, 1, 2, 3)$ &  $(4,1)\;(4,2)\;(4,3)\;$ &  $-a_{14}a_{21}a_{32}a_{43}=-(0)(3)(4)
$(4, 1, 3, 2)$ &  $(4,1)\;(4,3)\;(4,2)\;(3,2)\;$ &  $+a_{14}a_{21}a_{33}a_{42}=+(0)
$(4, 2, 1, 3)$ &  $(4,2)\;(4,1)\;(4,3)\;(2,1)\;$ &  $+a_{14}a_{22}a_{31}a_{43}=+(0)
$(4, 2, 3, 1)$ &  $(4,2)\;(4,3)\;(4,1)\;(2,1)\;(3,1)\;$ &  $-a_{14}a_{22}a_{33}a_{
$(4, 3, 1, 2)$ &  $(4,3)\;(4,1)\;(4,2)\;(3,1)\;(3,2)\;$ &  $-a_{14}a_{23}a_{31}a_{
$(4, 3, 2, 1)$ &  $(4,3)\;(4,2)\;(4,1)\;(3,2)\;(3,1)\;(2,1)\;$ &  $+a_{14}a_{23}a_{
&& Sum: $-15/2$\\\hline
\end{tabular}
```

In [44]:
```
A = Matrix([[1,3,2,4],
            [-2,-6,0,4],
            [2,1,4,-2],
            [3,-3,0,2]])
A
```

Out[44]:

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ -2 & -6 & 0 & 4 \\ 2 & 1 & 4 & -2 \\ 3 & -3 & 0 & 2 \end{bmatrix}$$

In [45]:
```
print(latex(A))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\-2 & -6 & 0 & 4\\2 & 1 & 4 & -2\\3 & -3 & 0 & 2\
```

In [46]:
```
A1 = rop(A, 'R1*(2)+R2=>R2', 'R1*(-2)+R3=>R3', 'R1*(-3)+R4=>R4')
A1
```

Out[46]:

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 0 & 4 & 12 \\ 0 & -5 & 0 & -10 \\ 0 & -12 & -6 & -10 \end{bmatrix}$$

In [47]:
```
print(latex(A1))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & 0 & 4 & 12\\0 & -5 & 0 & -10\\0 & -12 & -6 &
```

In [48]:
```
A2 = rop(A1, 'R2<=>R3')
A2
```

Out[48]:

$$\begin{bmatrix}1 & 3 & 2 & 4\\0 & -5 & 0 & -10\\0 & 0 & 4 & 12\\0 & -12 & -6 & -10\end{bmatrix}$$

In [49]:
```
print(latex(A2))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & -5 & 0 & -10\\0 & 0 & 4 & 12\\0 & -12 & -6 &
```

In [50]:
```
A3 = rop(A2, 'R2*(-1/5)=>R2')
A3
```

Out[50]:

$$\begin{bmatrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 4 & 12\\0 & -12 & -6 & -10\end{bmatrix}$$

In [51]:
```
print(latex(A3))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 4 & 12\\0 & -12 & -6 & -1
```

In [52]:
```
A4 = rop(A3, 'R2*(12)+R4=>R4')
A4
```

Out[52]:

$$\begin{bmatrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 4 & 12\\0 & 0 & -6 & 14\end{bmatrix}$$

In [53]:
```
print(latex(A4))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 4 & 12\\0 & 0 & -6 & 14\e
```

In [54]:
```
A5 = rop(A4, 'R3*(1/4)=>R3')
A5
```

Out[54]:

$$\begin{bmatrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\\0 & 0 & -6 & 14\end{bmatrix}$$

In [55]:
```
print(latex(A5))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\\0 & 0 & -6 & 14\en
```

In [56]:
```
A6 = rop(A5, 'R3*(6)+R4=>R4')
A6
```

Out[56]:

$$\begin{bmatrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\\0 & 0 & 0 & 32\end{bmatrix}$$

In [57]:
```
print(latex(A6))
```

```
\left[\begin{matrix}1 & 3 & 2 & 4\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\\0 & 0 & 0 & 32\enc
```

In [58]:
```
32*5*4
```

Out[58]:

$$640$$

In [59]:
```
A.det()
```

Out[59]:

$$640$$

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

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