Exercise 1. Compute the determinant of the following matrices. Do not use the `det()` method. Indicate the method you used to compute the determinant, and show all computations:

(a) $\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix}$

Exercise 2. A is a 5×5 matrix and it is known that $\det(A) = -2$. The matrix B is obtained by applying the following operations to A:

- Multiply row 3 row by 2.
- Add to column 4 the result of multiplying column 1 by -2.
- Swap columns 2 and 5.
- Add row 3 to row 4.
- Multiply column 5 by -3.
- Swap rows 1 and 3.

Find $\det(B)$, and justify your answer.

Exercise 3. Let:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 4 \\ 9 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

Let

$$V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

(a) Find a basis of V. What is the dimension of V?

(b) Determine if the vector $\mathbf{w} = \begin{bmatrix} 3 \\ -9 \\ 4 \\ 13 \end{bmatrix}$ is in V. If it is, write it as a linear combination of the vectors in the basis you found in the previous item.

(c) Find a vector in $\mathbb{R}^4$ that is not in V.

Exercise 4. Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) Show that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of $\mathbb{R}^4$.

(b) Find the change of basis matrix from basis B to basis E, the standard basis of $\mathbb{R}^4$.

(c) Find the change of basis matrix from basis E to basis B.

(d) Find the coordinates in basis B of the vector $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix}_E$.

Exercise 5. For each of the matrices below, find all eigenvalues and a basis for each eigenspace. Then, determine if the matrix is diagonalizable. If it is, find a matrix P such that $D = P^{-1}AP$ is diagonal, and compute $P^{-1}AP$ to verify that your solution is correct.

(a) $A = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 5 & -3 \\ -6 & 6 & -5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 5 & -2 \\ -3 & -8 & 2 \\ -5 & -9 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 13 & -12 & 21 \\ 15 & -14 & 21 \\ 0 & 0 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} -10 & 0 & -10 & 8 & -19 \\ 3 & 1 & 26 & -8 & 11 \\ -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 16 & -3 & 4 \\ 5 & 0 & 6 & -4 & 10 \end{bmatrix}$

(e) $A = \begin{bmatrix} -16 & -72 & 27 & 3 \\ 0 & -10 & 9 & -9 \\ -9 & -66 & 38 & -21 \\ -9 & -54 & 27 & -10 \end{bmatrix}$

Exercise 6. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}.$$

Find a basis for the subspace of all vectors $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ in $\mathbb{R}^4$ that are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

Exercise 7. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$ such that:

- $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
- $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are also orthogonal.

Given this information, what can you conclude about $||\mathbf{u}||$ and $||\mathbf{v}||$? Justify your answer.

Exercise 8. In each of the items below, use the Gram-Schmidt process to find an orthonormal basis of the subspace spanned by the given vectors.

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

(c) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Exercise 9. For each of the following items, do the following:

- Find an orthonormal basis consisting of eigenvectors of the symmetric matrix A.
- Find a matrix P such that $D = P^T AP$ is a diagonal matrix.
- Compute the product $P^T AP$ to confirm that it is equal to a diagonal matrix with the eigenvalues of A on its diagonal.

(a) $A = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} -2 & -5 & 5 \\ -5 & -2 & 5 \\ 5 & 5 & -2 \end{bmatrix}$

(c) $A = \begin{bmatrix} -4 & 3 & -2 & -5 \\ 3 & -4 & -2 & 5 \\ -2 & -2 & 1 & 0 \\ -5 & 5 & 0 & -2 \end{bmatrix}$