19-SVD-Applied-Example

January 6, 2017

```
In [1]: %matplotlib inline
        import numpy as np
        from numpy.linalg import svd
        import matplotlib.pyplot as plt
        from numpy import linalg
        from numpy import random
        from PIL import Image
```

In this notebook, we consider two applications of the singular value decomposition to image processing.

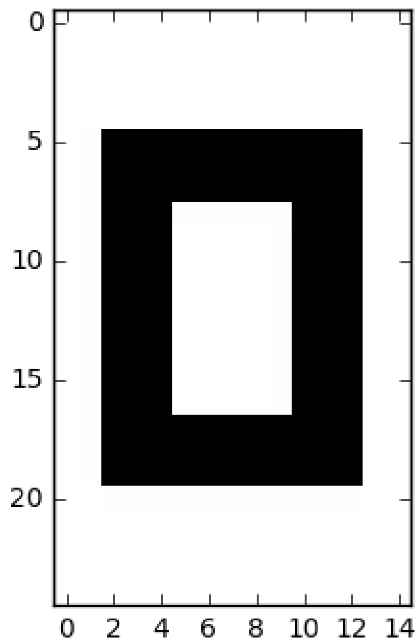
The examples are adapted from the article We Recommend a Singular Value Decomposition by David Austin. This article is a great introduction to the SVD interpretation and applications.

To demonstrate the use of the SVD in image processing, let's start by defining a greyscale image:

```
In [2]: m = 25
        n = 15
        M = np.zeros((m,n))
        M[5:8,2:-2].fill(1)
        M[8:-8,2:5].fill(1)
        M[8:-8,-5:-2].fill(1)
        M[-8:-5,2:-2].fill(1)
        M = 1 - M
```

The array `M` is a 25×15 array of zeros and ones, where 0 corresponds to black and 1 corresponds to black. The image, a black rectangular "box", is displayed with the following code:

```
In [3]: plt.gray()
        plt.imshow(255 * M, interpolation='none')
        None
```



We now consider the array `M` as being a real matrix, and compute its SVD, using the `svd` function from the module `linalg`

```
In [4]: U,s,V = linalg.svd(M, full_matrices=False)
```

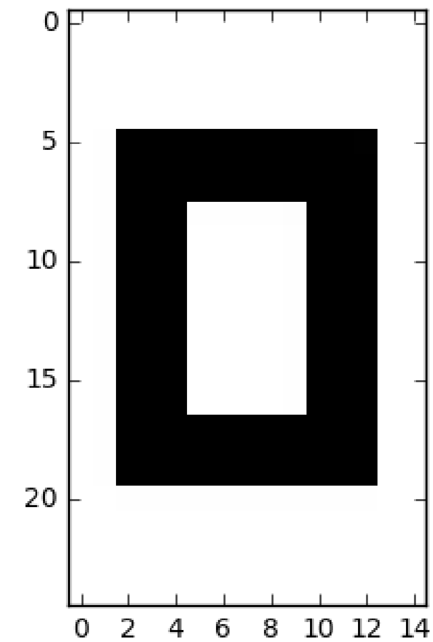
The output of the `linalg.svd()` function consists of a vector s and two matrices U and V. The vector s containing the singular values. The matrices U and V are such that:

$$M = UDV$$

where D is a matrix with the singular values on its diagonal. (In the notation we used in class, V is P and U is Q^T. The matrices U and V are orthogonal.

We can check the correctness of the decomposition by reconstructing the matrix M and redisplaying the image:

```
In [5]: S = np.diag(s)
        MR = np.dot(U, np.dot(S,V))
        plt.gray()
        plt.imshow(255 * MR, interpolation='none')
        None
```



1 Image Compression

So far, we didn't really accomplish much, since the amount of data needed to represent U, s and V is about the same needed for the full image matrix M. However, let's examine the components of vector s:

```
In [6]: s
```

```
Out[6]: array([  1.47242531e+01,   5.21662293e+00,   3.31409370e+00,
                 1.51448821e-16,   1.48952049e-16,   1.02144979e-32,
                 2.71957431e-33,   0.00000000e+00,   0.00000000e+00,
                 0.00000000e+00,   0.00000000e+00,   0.00000000e+00,
                 0.00000000e+00,   0.00000000e+00,   0.00000000e+00])
```

Notice that the values beyond the third are essentially zero. If the computations were done without roundoff errors, these would be exactly zero, because the rank of the matrix M is 3. To see why this is the case, notice that there are only three different kinds of columns in the image:

- The white margins of the image, which are represented by an array of ones (remember, 1=white, 0=black).
- The columns corresponding to the left and right sides of the box.
- The columns corresponding to the top and bottom of the box.

As a consequence, we can represent the image by keeping only the first three singular values:

```
In [7]: s1 = s[0:3]
        s1
```

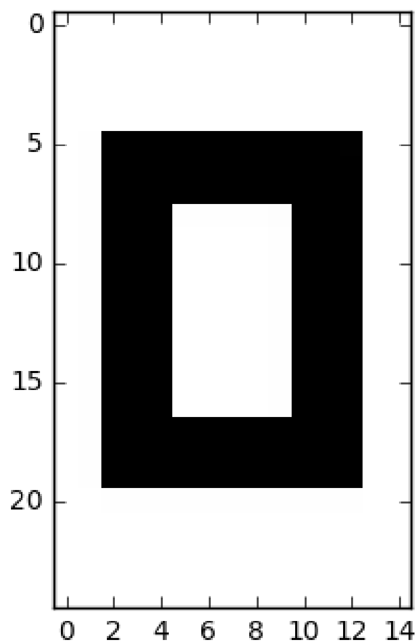
```
Out[7]: array([ 14.72425306,   5.21662293,   3.3140937 ])
```

To reconstruct the image, we need the first three columns of the matrix U and the first three rows of matrix V:

```
In [8]: U1 = U[:,0:3]
        V1 = V[0:3,:]
```

Finally, here is the reconstruction:

```
In [9]: S1 = np.diag(s1)
        M2 = np.dot(U1, np.dot(S1,V1))
        plt.gray()
        plt.imshow(255*M2, interpolation='none')
        None
```



How much compression we achieved? The original matrix, M, has $25 \times 15 = 375$ entries. If, instead of the matrix M, we transmit the matrices U_1, V_1 and the vector s_1, we have a total of $25 \times 3 + 3 \times 15 + 3 = 123$ entries, so we get a compression factor of $123/375$, or about 33%. Notice that for a more realistic image the compression rate would probably not be so good.

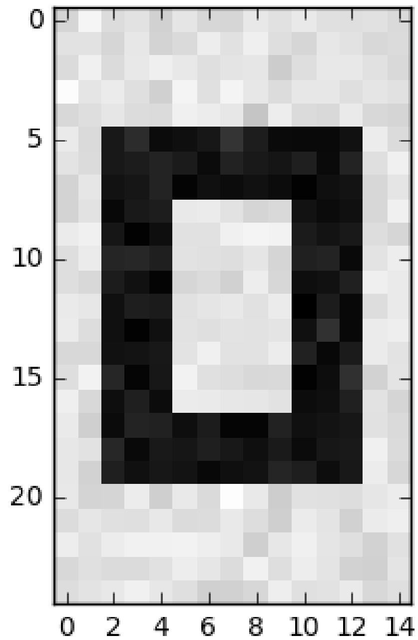
2 Noise Reduction

In the next application, let's assume that the image transmission is noisy. We can simulate this situation by adding a small random perturbation to the image data. This is done in the next cell, where we add a normally distributed random value to each entry of the matrix:

```
In [10]: sd = 0.05
         MN = M + sd * random.randn(25,15)
```

This is what the noisy image looks like:

```
In [11]: plt.gray()
         plt.imshow(255 * MN, interpolation='none')
         None
```



Let's see how we can "clean up" the noise by using a Singular Value Decomposition. First, we comopute the SVD:

```
In [12]: U,s,V = linalg.svd(MN, full_matrices=False)
```

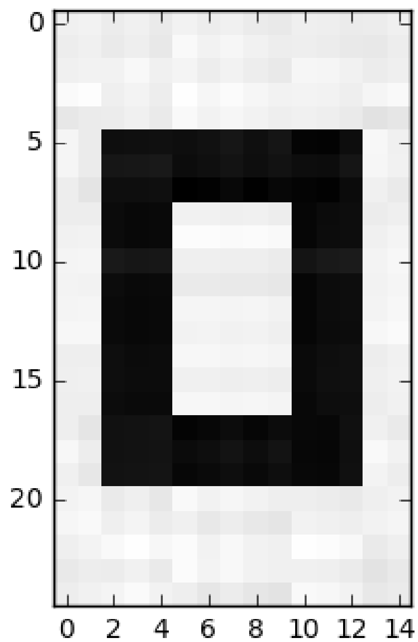
Let's now see what the singular values look like:

```
In [13]: s

Out[13]: array([ 14.7578131 ,   5.22937477,   3.33441919,   0.383975  ,
                  0.33826818,   0.31271074,   0.29881508,   0.2648192 ,
                  0.25440087,   0.19885545,   0.18167082,   0.16583319,
                  0.15139172,   0.12084663,   0.10242299])
```

Notice that there is a somewhat sharp drop in the size of the singular values after the third one. Let's see what happens if we keep only the first three singular values:

```
In [14]: nc = 3
         s1 = s[0:nc]
         U1 = U[:,0:nc]
         V1 = V[0:nc,:]
         S1 = np.diag(s1)
         M2 = np.dot(U1, np.dot(S1,V1))
         plt.gray()
         plt.imshow(255*M2, interpolation='none')
         None
```



Notice that the noise in the image was significantly decreased. In a real application, we would have to decide how many components should be kept