09-Linear-Independence

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```
In [2]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

1 Linear Independence

1.1 First strategy:

To determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent we proceed as follows:

- Solve the linear system:
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$
- If the only solution of the system is $c_1 = c_2 = \cdots c_k = 0$, the vectors are linearly independent. Otherwise, they are linearly dependent.

1.2 Example 1

Determine if the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix}$$

1.2.1 Solution:

We write the system:

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 3 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ -4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 3 & -4 & -3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since this is a homogeneous system, we don't need to keep track of the right-hand side vector, so we work with the matrix A itself, instead of the augmented matrix:

```
In [3]: A = matrix_to_rational([[ 1,  0,  1],
                                [-2,  2,  1],
                                [ 3, -4, -3],
                                [ 3,  1,  2]])
        A
```

Out[3]:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 3 & -4 & -3 \\ 3 & 1 & 2 \end{bmatrix}$$

We now use the function `reduced_row_echelon_form()` to find the RREF matrix equivalent to A:

```
In [4]: R = reduced_row_echelon_form(A)
        R
```

Out[4]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The three columns in this matrix correspond to the variables c_1, c_2 and c_3 (remember, this is *not* the augmented matrix). So, this matrix corresponds to the solution:

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Notice that the last row consists only of zeros, and can be ignored.

We conclude that, the only solution to:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

is

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$$

It follows that the vectors are *linearly independent*.

1.3 Example 2

Determine if the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 9 \\ -2 \\ -6 \end{bmatrix}$$

are linearly independent.

1.3.1 Solution:

We need to solve the linear system:

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 9 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In matrix form:

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 9 \\ 0 & 2 & 1 & -2 \\ 3 & 4 & 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The matrix of the system is (ignoring the right-hand side, since the system is homogeneous):

```
In [5]: A = matrix_to_rational([[-1, 0, 1,  2],
                                [ 2, 1, 2,  9],
                                [ 0, 2, 1, -2],
                                [ 3, 4, 3,  6]])
        A
```

```
Out[5]:
```

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 9 \\ 0 & 2 & 1 & -2 \\ 3 & 4 & 3 & 6 \end{bmatrix}$$

The RREF equivalent matrix is:

```
In [6]: R = reduced_row_echelon_form(A)
        R
```

```
Out[6]:
```

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the original system is equivalent to:

$$\begin{aligned} c_1 + 2c_4 &= 0 \\ c_2 - 3c_4 &= 0 \\ c_3 + 4c_4 &= 0 \end{aligned}$$

Since there are free variables, this system has infinitely many solutions. For example, we can let $c_4 = 1$, so that $c_1 = -2$, $c_2 = 3$ and $c_3 = -4$, and we get the relation:

$$-2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$$

We conclude that the vectors are *linearly dependent*.

Notice that, from the formula above, we can express any of the vectors that appears with a nonzero coefficient in terms of the other. For example:

$$-4\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 - \mathbf{v}_4$$

Thus:

$$\mathbf{v}_3 = -\frac{1}{2}\mathbf{v}_1 - \frac{3}{4}\mathbf{v}_2 + \frac{1}{4}\mathbf{v}_4$$

So, if we remove the vector $\mathbf{v}_3$ from the set, we are removing a "dependency", and we may ask if the set of remaining vectors:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 9 \\ -2 \\ -6 \end{bmatrix}$$

is linearly independent. Again, we can use this using the definition of linear independence and some algebra. We need to solve the system:

$$c_1 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 9 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In matrix form:

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 9 \\ 0 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution to the system is as follows:

```
In [7]: A = matrix_to_rational([[-1, 0,  2],
                                [ 2, 1,  9],
                                [ 0, 2, -2],
                                [ 3, 4,  6]])
        A
```

Out[7]:

$$\begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & 9 \\ 0 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix}$$

```
In [8]: R = reduced_row_echelon_form(A)
        R
```

Out[8]:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The system now only has the trivial solution, $c_1 = 0$, $c_2 = 0$ $c_3 = 0$. We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_4$ are *linearly independent*.

The next example demonstrates a streamlined method to obtain a linearly independent subset from a given set of vectors.

1.4 Example 3

Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ that is linearly independent, where:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -3 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 4 \\ 0 \\ 6 \\ 12 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ -11 \\ 3 \\ -9 \\ -21 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} -2 \\ 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Solution:

We first solve the system:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 + c_5\mathbf{v}_5 = \mathbf{0}$$

In matrix form this becomes:

$$\begin{bmatrix} 1 & -2 & 2 & 7 & -2 \\ -2 & 4 & -3 & -11 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ -3 & 6 & -2 & -9 & 0 \\ -6 & 12 & -5 & -21 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is in the following cells:

```
In [9]: A = matrix_to_rational([[ 1, -2,  2,   7, -2],
                                [-2,  4, -3, -11,  2],
                                [ 0,  0,  1,   3, -1],
                                [-3,  6, -2,  -9,  0],
                                [-6, 12, -5, -21,  1]])
        A
```

Out[9]:

$$\begin{bmatrix} 1 & -2 & 2 & 7 & -2 \\ -2 & 4 & -3 & -11 & 2 \\ 0 & 0 & 1 & 3 & -1 \\ -3 & 6 & -2 & -9 & 0 \\ -6 & 12 & -5 & -21 & 1 \end{bmatrix}$$

The RREF is:

```
In [10]: R = reduced_row_echelon_form(A)
         R
```

```
Out[10]:
```

$$\begin{bmatrix}1 & -2 & 0 & 1 & 0\\0 & 0 & 1 & 3 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

The original system is equivalent to:

$$\begin{aligned}c_1 - 2c_2 + c_4 &= 0\\c_3 + 3c_4 &= 0\\c_5 &= 0\end{aligned}$$

The free variables are c_2 and c_4, and the privot variables are c_1, c_3 and c_5.

Now, notice that *if we set the free variables equal to* 0*, then the pivot variables will also be zero*. This means that the only solution of:

$$c_1\mathbf{v}_1 + c_3\mathbf{v}_3 + c_5\mathbf{v}_5 = \mathbf{0}$$

is $c_1 = c_3 = c_5 = 0$, and the vectors $\mathbf{v}_1$, $\mathbf{v}_3$, $\mathbf{v}_5$ are linearly independent. In fact, this is a *maximal* linearly independent subset of the given vectors, because the solution of the system above shows that v_2 and v_4 can be expressed in terms of v_1, v_3, v_5.

This principle can be used in general to solve this kind of problem:

Suppose that a finite set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, and we want to find a *maximal linearly independent subset* of this set. Proceed as follows:

- Let A be the matrix that has the given vectors in its columns: $A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_k\end{bmatrix}$
- Find the RREF of A.
- A minimal set of linearly independent vectors is formed by the vectors that correspond to the *pivot columns* of the RREF.

1.5 Example 4

Find a subset of the vectors that is maximally linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix}2\\4\\-1\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}4\\-8\\2\end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix}0\\2\\0\end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix}2\\6\\-1\end{bmatrix},$$

Solution:

Form a matrix with the vectors on its columns:

```
In [11]: A = matrix_to_rational([[ 2,  -4, 0,  2],
                                 [ 4,  -8, 2,  6],
                                 [-1,   2, 0, -1]])
         A
```

Out[11]:

$$\begin{bmatrix} 2 & -4 & 0 & 2 \\ 4 & -8 & 2 & 6 \\ -1 & 2 & 0 & -1 \end{bmatrix}$$

Find the RREF equivalent matrix:

```
In [12]: R = reduced_row_echelon_form(A)
         R
```

Out[12]:

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 3 are pivot columns, so a maximal linearly independent set is given by:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

```
In [ ]:
```