Exercise 1. Compute the determinant of the following matrices. Do not use the `det()` method. Indicate the method you used to compute the determinant, and show all computations:

(a) $\det\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = 1 \times 1 - (-3) \times 2 = 1 + 6 = 7$ ☺

(b) $\det\begin{bmatrix} 2 & 0 & -3 \\ 1 & 2 & 0 \\ 3 & -1 & 2 \end{bmatrix} = 2 \times 2 \times 2 + 0 \times 0 \times 3 + (-3) \times 1 \times (-1) - (-3) \times 2 \times 3 - 2 \times 0 \times (-1) - 0 \times 1 \times 2$
$= 8 + 0 + 3 + 18 - 0 - 0 = 29$

Exercise 2. A is a 5×5 matrix and it is known that $\det(A) = -2$. The matrix B is obtained by applying the following operations to A:

- Multiply row 3 row by 2. The determinant is multiplied by 2, so the value after this step becomes $-2 \times 2 = 4$.
- Add to column 4 the result of multiplying column 1 by -2. The determinant does not change.
- Swap columns 2 and 5. The determinant changes sign, so the value after this step is $-(-4) = 4$.
- Add row 3 to row 4. The determinant is not change.
- Multiply column 5 by -3. The determinant is multiplied by -3, so the value after this step is $4 \times (-3) = 12$
- Swap rows 1 and 3. The determinant changes sign, so the value after this step is -12.

Find $\det(B)$, and justify your answer.

Exercise 3. Let:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 4 \\ 9 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix}$$

Let

$$V = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

(a) Find a basis of V. What is the dimension of V?

Solution: We construct a matrix with the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ on its columns and find the RREF of the matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & -7 & -1 \\ 2 & 0 & 4 & 0 \\ 3 & -1 & 9 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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The pivots in the RREF are in columns 1, 2 and 4, so $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for V. Since there are three vectors in the basis, the dimension is 3. ☺

(b) Determine if the vector $\mathbf{w} = \begin{bmatrix} 3 \\ -9 \\ 4 \\ 13 \end{bmatrix}$ is in V. If it is, write it as a linear combination of the vectors in the basis you found in the previous item.

Solution. We have to determine if there are scalars c_1, c_2, c_3 such that:

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 4 \\ 13 \end{bmatrix}$$

Equivalently, we need to solve the system of linear equations:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 4 \\ 13 \end{bmatrix}$$

Write the augmented matrix for the system and find its RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ -2 & 1 & -1 & -9 \\ 2 & 0 & 0 & 4 \\ 3 & -1 & 3 & 13 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to the RREF is:

$$\begin{aligned} c_1 &= 2 \\ c_2 &= -4 \\ c_3 &= 1 \end{aligned}$$

Since the system has a solution, the vector $\begin{bmatrix} 3 \\ -9 \\ 4 \\ 13 \end{bmatrix}$ is in V. ☺

(c) Find a vector in $\mathbb{R}^4$ that is not in V.

Solution We have to find w, x, y, z for which the system below does not have a solution c_1, c_2, c_3:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

We write the augmented matrix and find its RREF:

$$\begin{bmatrix} 1 & 0 & 1 & w \\ -2 & 1 & -1 & x \\ 2 & 0 & 0 & y \\ 3 & -1 & 3 & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \frac{y}{2} \\ 0 & 1 & 0 & w + x + \frac{y}{2} \\ 0 & 0 & 1 & w - \frac{y}{2} \\ 0 & 0 & 0 & -2w + x + \frac{y}{2} + z \end{bmatrix}$$

The system has a solution unless:

$$-2w + x + \frac{y}{2} + z \neq 0.$$

We can thus choose any values of w, x, y, z for which the expression above is not zero. For example, we can pick $w, x, y = 0$ and $z = 1$. So, the vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is not in V. ☺

Exercise 4. Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) Show that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of $\mathbb{R}^4$.

Solution: To determine if the given vectors are a basis of $\mathbb{R}^4$, compute the determinant:

$$\det \begin{bmatrix} 0 & 2 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 \\ -2 & 3 & 1 & 0 \end{bmatrix} = 18$$

The determinant was computed in Jupyter, with the function det. Since the determinant is not zero, the given vectors form a basis. ☺

(b) Find the change of basis matrix from basis B to basis E, the standard basis of $\mathbb{R}^4$.

Solution: This is the same matrix as the one used in the previous item, with the vectors of the basis B on its columns:

$$P = \begin{bmatrix} 0 & 2 & 4 & 2 \\ -1 & 0 & -1 & 0 \\ 3 & 1 & 2 & 1 \\ -2 & 3 & 1 & 0 \end{bmatrix}$$

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(c) Find the change of basis matrix from basis E to basis B.

Solution: This is the inverse of the matrix from the previous item:

$$P^{-1} = \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -1 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{5}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

The inverse was computed with the Python expression:

```
P**(-1)
```

(d) Find the coordinates in basis B of the vector $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix}_E$.

Solution: The coordinates in base B of the given vector are computed by:

$$P^{-1}\mathbf{v} = \begin{bmatrix} -\frac{1}{6} & 0 & \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -1 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{5}{3} & \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{5}{2} \\ -4 \end{bmatrix}$$

The vector was computed with the following Python code:

```
v = Matrix([1, -3, 2, 0])
P**(-1) * v
```

☺

Exercise 5. For each of the matrices below, find all eigenvalues and a basis for each eigenspace. Then, determine if the matrix is diagonalizable. If it is, find a matrix P such that $D = P^{-1}AP$ is diagonal, and compute $P^{-1}AP$ to verify that your solution is correct.

(a) $A = \begin{bmatrix} 2 & 0 & 1 \\ -3 & 5 & -3 \\ -6 & 6 & -5 \end{bmatrix}$

Solution: To find the eigenvalues we factor the characteristic polynomial:

$$\det(A - \lambda I) = -(\lambda - 2)(\lambda - 1)(\lambda + 1)$$

We conclude that the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 1$

Let's now find a basis for each of the eigenspaces.

- $\lambda_1 = 2$. We have to find solutions of the system $(A - 2I)\mathbf{v} = \mathbf{0}$. Using the computer, we find the RREF:
$$A - 2I \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$
The system corresponding to this matrix is:
$$\begin{aligned} x_1 - x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$
There is only one free variable, x_2, so the eigenspace $E(2)$ has dimension 1. To get a nonzero solution, we can let $x_2 = 1$, and get $x_1 = 1$ and $x_3 = 0$. We conclude that:
$$\text{A basis of } E(2) \text{ is: } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- $\lambda_2 = 1$. We have to find solutions of the system $(A - 1I)\mathbf{v} = \mathbf{0}$. Using the computer, we find the RREF:
$$A - 1I \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
The system corresponding to this matrix is:
$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$
There is only one free variable, x_3, so the eigenspace $E(1)$ has dimension 1. To get a nonzero solution, we can let $x_3 = 1$, and get $x_1 = -1$ and $x_2 = 0$. We conclude that:
$$\text{A basis of } E(1) \text{ is: } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\lambda_3 = -1$. We have to find solutions of the system $(A - (-1)I)\mathbf{v} = \mathbf{0}$. Using the computer, we find the RREF:
$$A - (-1)I \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$
The system corresponding to this matrix is:
$$\begin{aligned} x_1 + \frac{1}{3}x_3 &= 0 \\ x_2 - \frac{1}{3}x_3 &= 0 \end{aligned}$$

There is only one free variable, x_3, so the eigenspace $E(1)$ has dimension 1. To get a nonzero solution, we can let $x_3 = 3$, and get $x_1 = -1$ and $x_2 = 1$. We conclude that:

$$\text{A basis of } E(2) \text{ is: } \left\{ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Conclusion: Since we found a total of 3 linearly independent eigenvectors, the matrix is diagonalizable. The columns of the matrix P are the eigenvectors of A:

$$P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

We can now compute:

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The product was computed with the following Python expression:

```
P**(-1)*A*P
```

Since this is a diagonal matrix with the eigenvalues of A on its diagonal, our calculations are correct. ☺

(d) $A = \begin{bmatrix} -10 & 0 & -10 & 8 & -19 \\ 3 & 1 & 26 & -8 & 11 \\ -1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 16 & -3 & 4 \\ 5 & 0 & 6 & -4 & 10 \end{bmatrix}$

Solution: To find the eigenvalues we factor the characteristic polynomial, using the following code in Jupyter:

```
lbd = symbols('lambda')
p = det(A - lbd*eye(5))
factor(p)
```

We get:

$$\det(A - \lambda I) = -(\lambda - 1)^3 (\lambda + 1)^2$$

We conclude that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

Let's now find a basis for each of the eigenspaces.

- $\lambda_1 = 1$. We have to find solutions of the system $(A - 1I)\mathbf{v} = \mathbf{0}$. We use the following code in Jupyter:

  ```
  R = reduced_row_echelon_form(A - 1*eye(5))
  R
  ```

This results:

$$A - 1I \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to this matrix is:

$$\begin{aligned} x_1 - \frac{1}{2}x_4 + \frac{3}{2}x_5 &= 0 \\ x_3 - \frac{1}{4}x_4 + \frac{1}{4}x_5 &= 0 \end{aligned}$$

There are three free variables, x_2, x_4 and x_5 so the eigenspace $E(1)$ has dimension 3. To find a basis, we construct the table:

$$\begin{array}{c|ccc} x_2 & 1 & 0 & 0 \\ x_4 & 0 & 1 & 0 \\ x_5 & 0 & 0 & 1 \\ x_1 = \frac{1}{2}x_4 - \frac{3}{2}x_5 & 0 & \frac{1}{2} & -\frac{3}{2} \\ x_3 = \frac{1}{4}x_4 - \frac{1}{4}x_5 & 0 & \frac{1}{4} & -\frac{1}{4} \end{array}$$

So we get:

$$\text{A basis of } E(1) \text{ is: } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{4} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ 0 \\ -\frac{1}{4} \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\lambda_2 = -1$. We have to find solutions of the system $(A - (-1)I)\mathbf{v} = \mathbf{0}$. We use the following code in Jupyter:

```
R = reduced_row_echelon_form(A - (-1)*eye(5))
R
```

This results:

$$A - (-1)I \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{7}{3} \\ 0 & 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to this matrix is:

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

There is only one free variable, x_3, so the eigenspace $E(1)$ has dimension 1. To get a nonzero solution, we can let $x_3 = 1$, and get $x_1 = -1$ and $x_2 = 0$. We conclude that:

$$\text{A basis of } E(1) \text{ is: } \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\lambda_3 = -1$. We have to find solutions of the system $(A - (-1)I)\mathbf{v} = \mathbf{0}$. Using the computer, we find the RREF:

$$A - (-1)I \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to this matrix is:

$$\begin{aligned} x_1 + \frac{7}{3}x_5 &= 0 \\ x_2 + \frac{1}{3}x_5 &= 0 \\ x_3 + \frac{1}{3}x_5 &= 0 \\ x_4 + \frac{2}{3}x_5 &= 0 \end{aligned}$$

There is only one free variable, x_5, so the eigenspace $E(-1)$ has dimension 1. To get a nonzero solution, we can let $x_5 = 3$, and get $x_1 = -7$, $x_2 = -1$, $x_3 = -1$ and $x_4 = -2$. We conclude that:

$$\text{A basis of } E(-2) \text{ is: } \left\{ \begin{bmatrix} -7 \\ -1 \\ -1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Conclusion: Since we found a total of 4 linearly independent eigenvectors and $\mathbb{R}^5$ has dimension five, it is not possible to find a basis of eigenvectors, and the matrix is not diagonalizable. ☺

Exercise 6. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix}.$$

Find a basis for the subspace of all vectors $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ in $\mathbb{R}^4$ that are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$

Solution: Let $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$ an arbitrary vector in the subspace. Then $\mathbf{u} \cdot \mathbf{v}_1 = 0$ and $\mathbf{u} \cdot \mathbf{v}_2 = 0$, from which we get the system:

$$\begin{aligned} 2x - 1y + 0z + 3y &= 0 \\ 1x + 2y - 2z + 0t &= 0 \end{aligned}$$

This is a homogeneous system with matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 3 \\ 1 & 2 & -2 & 0 \end{bmatrix}$$

The RREF for this matrix is:

$$\begin{bmatrix} 1 & 0 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

Writing this back as a system we get:

$$\begin{aligned} x - \frac{2}{5}z + \frac{6}{5}t &= 0 \\ y - \frac{4}{5}z - \frac{3}{5}t &= 0 \end{aligned}$$

The system has two free variables, z and t, so the space has dimension 2. To find a basis of the solution subspace, construct the table:

$$\begin{array}{l|rr} z & 1 & 0 \\ t & 0 & 1 \\ x = \frac{2}{5}z - \frac{6}{5}t & \frac{2}{5} & -\frac{6}{5} \\ y = \frac{4}{5}z + \frac{3}{5}t & \frac{4}{5} & \frac{3}{5} \end{array}$$

We conclude that a basis for the subspace is:

$$\left\{ \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \end{bmatrix} \right\}$$

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Exercise 7. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$ such that:

- $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
- $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are also orthogonal.

Given this information, what can you conclude about $||\mathbf{u}||$ and $||\mathbf{v}||$? Justify your answer.

Solution: Since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we have $\mathbf{u}\cdot\mathbf{v} = 0$. Likewise, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$. Expanding this we get:

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = 0$$

Since $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$, the two terms in the middle cancel, and we get:

$$\mathbf{u}\cdot\mathbf{u} - \mathbf{v}\cdot\mathbf{v} = 0.$$

Now, recall that for any vector $\mathbf{x}$, $||\mathbf{x}||^2 = \mathbf{x}\cdot\mathbf{x}$. So, we can conclude that:

$$||\mathbf{u}||^2 - ||\mathbf{v}||^2 = 0,$$

that is:

$$||\mathbf{u}||^2 = ||\mathbf{v}||^2.$$

Since $||\mathbf{u}||$ and $||\mathbf{v}||$ are non-negative, the identity above implies:

$$||\mathbf{u}|| = ||\mathbf{v}||.$$

☺

Exercise 8. In each of the items below, use the Gram-Schmidt process to find an orthonormal basis of the subspace spanned by the given vectors.

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$

Solution: We will do all computations in Jupyter. We start by entering the vectors, using the code below:

```
v1 = Matrix([0, 0, 1])
v2 = Matrix([2, -2, 1])
v3 = Matrix([1,  0, 3])
```

Notice that this generates column vectors, even though each vector is specified in a single row.

Step 1: Let $\mathbf{u}_1 = \mathbf{v}_1$. In Jupyter this is coded as:

```
u1 = v1
u1
```

This produces the output:

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 2: Project $\mathbf{v}_2$ on the direction of $\mathbf{u}_1$ using the formula:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1$$

In Jupyter this is coded as:

```
u2 = v2 - v2.dot(u1)/u1.dot(u1)*u1
u2
```

We get the vector:

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

Step 3. Project $\mathbf{v}_3$ on the subspace $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$, using the formula:

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2$$

In Jupyter, this is coded as:

```
u3 = v3 - v3.dot(u1)/u1.dot(u1)*u1 - v3.dot(u2)/u2.dot(u2)*u2
u3
```

We get the vector:

$$\mathbf{u}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

So far, we got the following orthogonal basis:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right\}$$

To get an orthonormal basis we must normalize the vectors. This can be done with the following code:

- ```
  n1 = u1/u1.norm()
  n1
  ```

  Output:

  $$\mathbf{n}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
  ```

- ```
  n2 = u2/u2.norm()
  n2
  ```

  Output:

  $$\mathbf{n}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

- ```
  n3 = u3/u3.norm()
  n3
  ```

  Output:

  $$\mathbf{n}_3 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

  So, we got the following orthonormal basis:

  $$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \right\}$$

  To check that our solution is correct, we can use the following code:

  ```
  P = Matrix.hstack(n1, n2, n3)
  P.T * P
  ```

  The output is:

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  That is, $P^{-1} = P^T$ and the matrix P is orthogonal, as expected.

(c) $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

Exercise 9. For each of the following items, do the following:

- Find an orthonormal basis consisting of eigenvectors of the symmetric matrix A.
- Find a matrix P such that $D = P^T AP$ is a diagonal matrix.
- Compute the product $P^T AP$ to confirm that it is equal to a diagonal matrix with the eigenvalues of A on its diagonal.

(c) $A = \begin{bmatrix} -4 & 3 & -2 & -5 \\ 3 & -4 & -2 & 5 \\ -2 & -2 & 1 & 0 \\ -5 & 5 & 0 & -2 \end{bmatrix}$

Solution: We start by finding the eigenvalues, using the code:

```
lbd = symbols('lambda')
p = det(A - lbd*eye(4))
factor(p)
```

This outputs:

$$(\lambda - 3)^2 (\lambda + 3) (\lambda + 12)$$

We conclude that the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -3$ and $\lambda_3 = -12$.

We now need to find an orthonormal basis for each of the eigenspaces.

- $\lambda_1 = 3$. We have to find solutions of the system $(A - 3I)\mathbf{v} = \mathbf{0}$. We use the following code in Jupyter:

  ```
  R = reduced_row_echelon_form(A - 1*eye(5))
  R
  ```

  This results:

  $$A - 3I \sim \mathbb{R} = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  The system corresponding to this matrix is:

  $$\begin{aligned} x_1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 &= 0 \\ x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 &= 0 \end{aligned}$$

  There are two free variables, x_3 and x_4, so the eigenspace $E(3)$ has dimension 2. To find a basis, we construct the table:

  $$\begin{array}{r|rr} x_3 & 1 & 0 \\ x_4 & 0 & 1 \\ x_1 = -\frac{1}{2}x_3 - \frac{1}{2}x_4 & -\frac{1}{2} & -\frac{1}{2} \\ x_2 = -\frac{1}{2}x_3 + \frac{1}{2}x_4 & -\frac{1}{2} & \frac{1}{2} \end{array}$$

  So, we get the following two vectors in a basis of $E(3)$:

  $$\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

Notice that these two vectors are already orthogonal: $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, so we do not need to use Gram-Schmidt.

- $\lambda_2 = -3$. We have to find solutions of the system $(A - (-3)I)\mathbf{v} = \mathbf{0}$. We use the following code in Jupyter:

```
R = reduced_row_echelon_form(A - (-3)*eye(4))
R
```

This results:

$$A - (-3)I \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to this matrix is:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

There is only one free variable, x_3, so the eigenspace $E(-3)$ has dimension 1. To get a nonzero solution, we can let $x_3 = 1$, and get $x_1 = 1$, $x_2 = 1$ and $x_4 = 0$. We conclude that a basis of $E(-3)$ consists of the single vector:

$$\mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

- $\lambda_3 = -12$. We have to find solutions of the system $(A - (-12)I)\mathbf{v} = \mathbf{0}$. We use the following code in Jupyter:

```
R = reduced_row_echelon_form(A - (-12)*eye(4))
R
```

This results:

$$A - (-12)I \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The system corresponding to this matrix is:

$$\begin{aligned} x_1 - x_4 &= 0 \\ x_2 + x_4 &= 0 \\ x_3 &= 0 \end{aligned}$$

There is only one free variable, x_4, so the eigenspace $E(-12)$ has dimension 1. To get a nonzero solution, we can let $x_4 = 1$, and get $x_1 = 1$, $x_2 = -1$ and $x_3 = 0$. We conclude that a basis of $E(-12)$ consists of a single vector:

$$\mathbf{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We are finally done computing the eigenspaces. So far, we found an orthogonal basis of $\mathbb{R}^4$ formed by eigenvectors of A:

$$\mathbf{u}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

At this point, it is recommended that we check that the vectors are indeed orthogonal to prevent mistakes. The last step is to normalize the vectors, using the formula:

$$\mathbf{n}_j = \frac{1}{\|\mathbf{u}_j\|}\mathbf{u}_j \text{ for } j = 1, 2, 3, 4.$$

We get:

$$\mathbf{n}_1 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ 0 \\ \frac{\sqrt{6}}{3} \end{bmatrix}, \quad \mathbf{n}_3 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ 0 \end{bmatrix}, \quad \mathbf{n}_4 = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{3}}{3} \\ 0 \\ \frac{\sqrt{3}}{3} \end{bmatrix}$$

To check our computations, let P be the change of basis matrix, that is, the matrix with the orthonormal basis on its columns. We can construct the matrix P in Jupyter with the code:

```
P = Matrix.hstack(n1,n2,n3,n4)
P
```

This produces the output:

$$\begin{bmatrix} -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \end{bmatrix}$$

This matrix must be orthogonal, that is $P^TP = I$. We can verify this with the code:

```
P.T * P
```

The output is the identity matrix, as expected:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Next, let's compute $P^T AP$, which should be a diagonal matrix with the eigenvalues of A on the diagonal. We compute this with the code:

```
P.T * A * P
```

The output is:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

This confirms that the solution is correct. ☺