18-Singular-Value-Decomposition

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```
In [1]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

1 A Simple Example

As an example of the calculations needed to find the Singular Value Decomposition (SVD), let's consider the matrix:

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

Since M a 3×2 matrix, the associated linear transformation $L(\mathbf{x}) = M\mathbf{x}$ has domain $\mathbb{R}^{\not\models}$ and codomain $\mathbb{R}^2$. So, the goal of the SVD is to find:

- An orthonormal basis B of $\mathbf{R^3}$. Denote by P the matrix that has the vectors of B as its columns.
- An orthonormal basis C of $\mathbf{R^2}$. Denote by Q the matrix that has the vectors of C as ist columns.
- The bases have the property that the matrix of L from basis B to basis C has the form:

  $$D = Q^T M P$$

  where:

  $$D = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{bmatrix}$$

  λ_1 and λ_2 are the eigenvalues of the matrix $M^T M$. The singular values of M are $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$.

Let's now see the steps needed to find the SVD:

```
In [2]: M = matrix_to_rational([[3,2],
                                [2,3],
                                [2,-2]])
        M
```

Out[2]:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

This is a 6×3 matrix, so we can use the singular value decomposition.

Step 1: Compute $A = M^T M$, find its eigenvalues and an orthonormal basis of eigenvectors:

```
In [3]: A = M.T * M
        A
```

Out[3]:

$$\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

As expected, this is a symmetric 3×3 matrix. To find an orthonormal basis of $\mathbb{R}^2$ that consists of eigenvectors of A, we start by computing the eigenvalues of A:

```
In [4]: lbd = symbols('lambda')
        p = det(A - lbd * eye(2))
        p
```

Out[4]:

$$\lambda^2 - 34\lambda + 225$$

```
In [5]: factor(p)
```

Out[5]:

$$(\lambda - 25)(\lambda - 9)$$

The eigenvalues are $\lambda_1 = 25$ and $\lambda_2 = 9$. Notice that all eigenvalues are non-negative real numbers, as will always be the case.

Eigenspace associated to $\lambda_1 = 25$:

```
In [6]: R = reduced_row_echelon_form(A - 25*eye(2))
        R
```

Out[6]:

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

This yields the system:

$$x_1 - x_2 = 0$$

Letting $x_2 = 1$, we get $x_1 = 1$, and we get the eigenvector:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The first vector of our orthonormal basis is this vector, normalized to length 1, as computed in the next cell:

```
In [7]: v = Matrix([1,1])
        u1 = v / v.norm()
        u1
```

Out[7]:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Eigenspace associated to $\lambda_2 = 9$:

```
In [8]: R = reduced_row_echelon_form(A - 9*eye(2))
        R
```

Out[8]:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

This yields the system:

$$x1 + x2 = 0$$

Letting $x2 = -1$, we get $x1 = 1$, and we get the eigenvector:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The second vector of our orthonormal basis is this vector, normalized to length 1, as computed in the next cell:

```
In [9]: v = Matrix([1,-1])
        u2 = v / v.norm()
        u2
```

Out[9]:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$

We conclude that the orthonormal basis of the domain of $\mathbb{R}^2$ is $B = \{\mathbf{u}_1, \mathbf{u}_2\}$. The corresponding change of basis matrix is:

```
In [10]: P = Matrix.hstack(u1,u2)
         P
```

Out[10]:

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

We can check that this is correct by computing:

```
In [11]: P.T * P
```

Out[11]:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

```
In [12]: P.T * A * P
```

Out[12]:

$$\begin{bmatrix}25 & 0\\0 & 9\end{bmatrix}$$

We now need to find an orthonormal basis of the codomain, $\mathbb{R}^3$. The starting point is to compute:

$$\mathbf{v}_i = \frac{M\mathbf{u}_i}{\sqrt{\lambda_i}}$$

for all vectors $\mathbf{u}_i$ of the orthonormal basis B:

```
In [13]: v1 = M * u1 / sqrt(25)
         v2 = M * u2 / sqrt(9)
```

```
In [14]: v1
```

Out[14]:

$$\begin{bmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\end{bmatrix}$$

```
In [15]: v2
```

Out[15]:

$$\begin{bmatrix}\frac{\sqrt{2}}{6}\\-\frac{\sqrt{2}}{6}\\\frac{2\sqrt{2}}{3}\end{bmatrix}$$

The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal set, as verified in the following computational cells.

```
In [16]: v1.dot(v2)
```

Out[16]:

$$0$$

```
In [17]: v1.norm()
```

Out[17]:

$$1$$

```
In [18]: v2.norm()
```

Out[18]:

$$1$$

We now need to complete $\{\mathbf{v}_1, \mathbf{v}_2\}$ to an orthonormal basis of $\mathbb{R}^3$. We need to find one more vector for the basis, which can be done by using the Gram-Schmidt procedure as follows:

```
In [19]: v = Matrix([1,0,0])
         v3 = v - v.dot(v1) / v1.dot(v1) * v1 - v.dot(v2) / v2.dot(v2) * v2
         v3 = v3 / v3.norm()
         v3
```

Out[19]:

$$\begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{bmatrix}$$

The following cell finishes the check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthonormal basis of $\mathbb{R}^3$:

```
In [20]: v1.dot(v3), v2.dot(v3), v3.dot(v3)
```

Out[20]:

$$(0, \quad 0, \quad 1)$$

Let's now compute the change of basis matrix for the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$:

```
In [21]: Q = Matrix.hstack(v1,v2,v3)
         Q
```

Out[21]:

$$\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} & \frac{2}{3}\\\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} & -\frac{2}{3}\\0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3}\end{bmatrix}$$

This should be an orthogonal matrix:

```
In [22]: Q.T*Q
```

Out[22]:

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

We can now check that

$$Q^T M P = \begin{bmatrix}\sqrt{\lambda_1} & 0\\0 & \sqrt{\lambda_2}\\0 & 0\end{bmatrix}$$

In [23]:
```
Q.T * M * P
```

Out[23]:

$$\begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$$

In [ ]: