Problem 1.(30 points.) Let A be the matrix:

$$A = \begin{bmatrix} -19 & -10 & 20 \\ -5 & -14 & 10 \\ -15 & -15 & 21 \end{bmatrix}$$

(a) (15 points) Find all the eigenvalues of A, and determine the multiplicity of each eigenvalue.

Solution. First, compute the eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda - 19 & -10 & 20 \\ -5 & -\lambda - 14 & 10 \\ -15 & -15 & -\lambda + 21 \end{bmatrix} = -\lambda^3 - 12\lambda^2 + 27\lambda + 486 = -(\lambda - 6)(\lambda + 9)^2$$

We conclude that the eigenvalues are $\lambda_1 = -9$, with multiplicity 2 and $\lambda_2 = 6$, with multiplicity 1. ☺

(b) (15 points) Find a basis for the eigenspace associated to each of the eigenvalues of A.

Solution.

Eigenspace of $\lambda_1 = -9$:

$$A - (-9)I = \begin{bmatrix} -10 & -10 & 20 \\ -5 & -5 & 10 \\ -15 & -15 & 30 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From the reduced row echelon form, the equations for a vector in the kernel are:

$$x_1 + x_2 - 2x_3 = 0$$

Since this system has two free variables, the kernel has dimension 2.

Letting $x_2 = 1$, $x_3 = 0$ we get the vector

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Letting $x_2 = 0$ and $x_3 = 1$ we get the vector:

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

We conclude that a basis of the eigenspace of $\lambda_1 = -9$ is:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Eigenspace of $\lambda_2 = 6$:

$$A - 6I = \begin{bmatrix} -25 & -10 & 20 \\ -5 & -20 & 10 \\ -15 & -15 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{2}{3} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

The equations for a vector in the kernel are:

$$x_1 - \frac{2}{3}x_3 = 0, \quad x_2 - \frac{1}{3}x_3 = 0$$

There is only one free variable, so the kernel has dimension 1. Choosing $x_3 = 3$ we get the following basis for the kernel:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}$$

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Problem 2. (30 points.) Answer the following items for the following basis of $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

(a) (15 points) Convert the following vector to a coordinate vector with respect to the standard basis:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Solution. Let P be the change of basis matrix:

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then:

$$\mathbf{x} = P[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ -1 \end{bmatrix}$$

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(b) (15 points) Convert the following vector to a coordinate vector with respect to the basis $\mathcal{B}$:

$$\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Solution. We need the inverse of the matrix P:

$$P^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

Then:

$$[\mathbf{x}]_{\mathcal{B}} = P^{-1}\mathbf{x} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

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Problem 3. (30 points.) Let A be a 4×4 matrix. The following information is given about the eigenvalues and corresponding eigenspaces of the matrix A:

$$\text{Eigenvalue: } \lambda_1 = 1; \quad \text{Basis for eigenspace: } \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\text{Eigenvalue: } \lambda_2 = -2; \quad \text{Basis for eigenspace: } \left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Is it possible to determine the matrix A using only this information? If possible, find the matrix A. If not, explain why.

Solution. Since there are four linearly independent eigenvectors, there is a basis of $\mathbb{R}^4$ consisting only of eigenvectors, and the matrix A is diagonalizable. This means that, if we let:

$$P = \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

we have $P^{-1}AP = D$. So:

$$A = PDP^{-1} = \begin{bmatrix} -2 & -6 & 0 & 6 \\ 1 & 6 & -3 & 1 \\ 2 & 13 & -8 & 5 \\ 1 & 5 & -3 & 2 \end{bmatrix}$$

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Problem 4. (10 points.) Let a be a scalar and consider the matrix:

$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

Answer the following items for the matrix A. Notice that your answers will depend on the unspecified scalar a

(a) Determine the eigenvalues of the matrix A, and specify the multiplicity of each eigenvalue.

Solution.

$$\det(A - \lambda I) = \det \begin{bmatrix} a-\lambda & 1 \\ 0 & a-\lambda \end{bmatrix} = (a-\lambda)^2$$

Then only eigenvalue is $\lambda = a$, which has multiplicity 2 ☺

(b) Find a basis for the eigenspace corresponding to each of the eigenvalues you found in the previous item.

Solution.

$$A - aI = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This matrix is already in reduced row echelon form, so the equations for the kernel are:

$$x_2 = 0$$

There is only one free variable, namely x_1, so a basis for eigenspace of $\lambda = a$ is obtained by choosing $x_1 = 1$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

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(c) Determine if the matrix A is diagonalizable or not, and justify your answer.

Solution. Since there is only one linearly independent eigenvector, it is not possible to find a basis of A that consists only of eigenvectors, and the matrix is not diagonalizable. ☺