# 11-Bases

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```
In [1]: from latools import *
        from sympy import *
        init_printing(use_latex=True)
```

## 1 Bases of a Vector Space

### 1.1 Example 1

Determine if the set of vectors:

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\4\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}2\\3\\-2\end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix}1\\2\\4\end{bmatrix}$$

is a basis of $\mathbb{R}^3$.

*Solution*: We start by checking if the give set spans $\mathbb{R}^3$, that is, if we can always find $c_1$, $c_2$ and $c_3$ such that

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = c_1\mathbf{v}_1 + c_3\mathbf{v}_2 + c_3\mathbf{v}_3$$

To do this, we have to solve the system with augmented matrix:

```
In [2]: x,y,z = symbols('x,y,z')
        A = matrix_to_rational([[1,  2, -1, x],
                                [2,  3, -2, y],
                                [4, -1, -1, z]])
        A
```

Out[2]:

$$\begin{bmatrix}1 & 2 & -1 & x\\2 & 3 & -2 & y\\4 & -1 & -1 & z\end{bmatrix}$$

The RREF of the matrix is:

```
In [3]: R = reduced_row_echelon_form(A, extra_cols=1)
        R
```

Out[3]:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5x}{3} - y + \frac{z}{3} \\ 0 & 1 & 0 & 2x - y \\ 0 & 0 & 1 & \frac{14x}{3} - 3y + \frac{z}{3} \end{bmatrix}$$

From this RREF we see that we can always find $c_1$, $c_2$, $c_3$ for any given values of $x$, $y$, $z$. It follows that this set of vectors spans $\mathbb{R}^3$.

To check that the set of vectors is linearly independent, just notice that, ignoring the last column, there are no free variables in the left three columns of matrix $R$. This implies that the only solution of $c_1\mathbf{v}_1 + c_3\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$, and the vectors are linearly independent.

### 1.2 Example 2

Determine if the vectors below span $\mathbf{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

*Solution*: As above, we attempt to solve the system with augmented matrix:

In [4]:
```
x,y,z,t = symbols('x,y,z,t')
A = matrix_to_rational([[ 1,  3, 1, x],
                        [ 2, -1, 1, y],
                        [ 0,  4, 1, z],
                        [-1,  0, 2, t]])
A
```

Out[4]:

$$\begin{bmatrix} 1 & 3 & 1 & x \\ 2 & -1 & 1 & y \\ 0 & 4 & 1 & z \\ -1 & 0 & 2 & t \end{bmatrix}$$

The RREF is:

In [5]:
```
R = reduced_row_echelon_form(A, extra_cols=1)
R
```

Out[5]:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5x}{3} - \frac{y}{3} - \frac{4z}{3} \\ 0 & 1 & 0 & \frac{2x}{3} - \frac{y}{3} - \frac{z}{3} \\ 0 & 0 & 1 & -\frac{8x}{3} + \frac{4y}{3} + \frac{7z}{3} \\ 0 & 0 & 0 & t + 7x - 3y - 6z \end{bmatrix}$$

Notice the last line, which corresponds to the equation: [ 0=t+7x-3y-6z ] This equation is impossible if the expression in the right is not zero. We conclude that the given set _does not span $\mathbb{R}^4$.

The procedure above gives the same result for any set of three or fewer vectors in $\mathbb{R}^4$. This illustrates the general principle:

**Proposition**. A set with fewer than $n$ vectors cannot span $\mathbb{R}^n$

## 1.3 Example 3

Determine if the vectors below are linearly independent in $\mathbb{R}^2$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

*Solution*: We have to solve the homogeneous system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. The matrix of the system (not augmented) is:

```
In [6]: A = matrix_to_rational([[1, -1, -3],
                                [1,  2,  0]])
        A
```

Out[6]:

$$\begin{bmatrix} 1 & -1 & -3 \\ 1 & 2 & 0 \end{bmatrix}$$

This yields the RREF:

```
In [7]: R = reduced_row_echelon_form(A)
        R
```

Out[7]:

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

This has a free variable, $c_3$, so there are nonzero solutions, and the vectors are linearly dependent. It is easy to see that this will always be the case if we have 3 or more vectors in $\mathbf{R}^2$. This illustrated the following general fact:

**Proposition**. A set of more than $n$ vectors in $\mathbf{R}^n$ is always linearly dependent.

## 1.4 Example 4

Suppose we have an ordered basis of $\mathbb{R}^3$, $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let's define the three vectors in Python

```
In [8]: v1 = Matrix([2,  0, 1])
        v2 = Matrix([1, -3, 1])
        v3 = Matrix([1,  1, 1])
```

Construct the matrix $P$ by placing the vectors in its columns:

$$P = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

We can construct this matrix in Python with the following code:

```
In [9]: P = Matrix.hstack(v1,v2,v3)
        P
```

Out[9]:

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The matrix $P$ is the *change of basis* matrix from basis $B$ to the standard basis $E$. This means that:

$$[u]_E = P[u]_B$$

Where:

- $[u]_E$ are the coordinates of $u$ in the standard basis.
- $[u]_B$ are the coordinates of $u$ in the basis $B$

Then, we also have:

$$[u]_B = P^{-1}[u]_E$$

This means that $P^{-1}$ is the change of basis matrix from basis $E$ to basis $B$. For example, suppose that:

$$[\mathbf{u}]_E = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_E$$

Then to find the coordinates of $\mathbf{u}$ in basis $B$ we compute $P^{-1}[\mathbf{u}]_E$

```
In [10]: u = Matrix([2,-1,1])
         P**(-1) * u
```

Out[10]:

$$\begin{bmatrix} 1 \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

We can verify that this is correct by computing the corresponding linear combination of the vectors in the basis:

```
In [11]: 1*v1 + sympify('1/4')*v2 - sympify('1/4')*v3
```

Out[11]:

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Next suppose that we have a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}_E\right) = \begin{bmatrix} -\frac{17}{2} & \frac{1}{2} & 13 \\ -\frac{3}{2} & \frac{7}{2} & 3 \\ -\frac{13}{2} & \frac{1}{2} & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_E$$

Notice that, in the expression above, all coordinates are in the *standard basis*.

Recall that above we defined a basis $\mathbb{R}^3$, $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

We want to find the matrix that represents the linear transformation $L$ using $B$ as input basis and $E$ as output basis. This is particularly easy, all we have to do is to construct the matrix:

$$M = \begin{bmatrix} [L(\mathbf{v}_1)]_E & [L(\mathbf{v}_2)]_E & L([\mathbf{v}_3)]_E \end{bmatrix}$$

Then,

$$[L(\mathbf{u})]_E = M[\mathbf{u}]_B$$

We do this in the following cells. First, set up the matrix that defines $L$:

In [12]:
```
A = matrix_to_rational([[-17/2, 1/2, 13],
                        [ -3/2, 7/2,  3],
                        [-13/2, 1/2, 11]])
A
```

Out[12]:

$$\begin{bmatrix} -\frac{17}{2} & \frac{1}{2} & 13 \\ -\frac{3}{2} & \frac{7}{2} & 3 \\ -\frac{13}{2} & \frac{1}{2} & 11 \end{bmatrix}$$

Then compute $L$ applied to the vectors in the basis $B$:

In [13]:
```
Lv1 = A*v1
Lv1
```

Out[13]:

$$\begin{bmatrix} -4 \\ 0 \\ -2 \end{bmatrix}$$

In [14]:
```
Lv2 = A*v2
Lv2
```

Out[14]:

$$\begin{bmatrix} 3 \\ -9 \\ 3 \end{bmatrix}$$

```
In [15]: Lv3 = A*v3
         Lv3
```

```
Out[15]:
```

$$\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

The matrix of the linear transformation has the vectors we computed above on its columns:

```
In [16]: M=Matrix.hstack(Lv1,Lv2,Lv3)
         M
```

```
Out[16]:
```

$$\begin{bmatrix} -4 & 3 & 5 \\ 0 & -9 & 5 \\ -2 & 3 & 5 \end{bmatrix}$$

We are now ready to compute the matrix of the linear transformation from the input basis $B$ to the input basis $B$. We just have to put together two formulas that we saw before:

$$[L(u)]_E = M[u]_B$$

and

$$[L(u)]_B = P^{-1}[L(u)]_E$$

Putting these two formulas together we have:

$$[L(u)]_B = P^{-1}[L(u)]_E = P^{-1}M[u]_B$$

So, the matrix we seek is $P^{-1}M$:

```
In [17]: P**(-1)*M
```

```
Out[17]:
```

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

We notice the remarkable fact that this matrix is diagonal, that is, the linear transformation $L$ has a specially simple representation in the basis $B$. The next topic we will study is how to find these basis that make the representation of a linear transformation very simple.

```
In [ ]:
```