# Markov Decision Processes and Reinforcement Learning

## Day 2 — Optimal Value Functions and Classical Computational Methods

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# Outline

# Definition of Markov Decision Process

- A trajectory of an MDP consists of three stochastic sequences:
  - $S_0, S_1, S_2, \ldots$: the states visited by the environment.
  - $A_1, A_2, A_2, \ldots$ the actions chosen by the agent.
  - $R_0, R_1, R_2, \ldots$ the rewards awarded to the agent.
- Model is specified by two deterministic functions:

$$p(s' \mid s, a) \quad \text{and} \quad R(s, s', a).$$

- Model evolution:

$$\mathbb{P}(S_{t+1} = s' \mid S_t = s, A_t = a) = p(s' \mid s, a)$$

$$R_{t+1} = r(S_t, S_{t+1}, A_t)$$

# Rewards, returns and state value functions

- The **reward** received at time $t+1$ (after action $A_t$ is selected) is denoted by $R_{t+1}$.
- The **total return** (or simply return) for a run is:

$$\sum_{t=0}^{T} \gamma^t R_{t+1}$$

- The expected return of a run is:

$$\mathbb{E}\left[\sum_{t=0}^{T} \gamma^t R_{t+1}\right]$$

# Policies and value functions

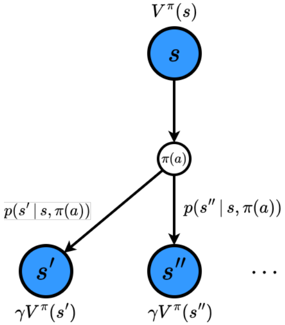
- A **deterministic policy** associates to each state $s$ and action $a = \pi(s)$, where $a$ is in the set of admissible actions $\mathscr{A}(s)$
- The state value function for policy $\pi$ is:

$$V^{\pi}(s) = \mathbb{E}_{\pi}\left[\sum_{t=0}^{T} \gamma^{t} R_{t+1} \,|\, S_0 = s\right]$$

- We assume:
  - For episodic tasks, $P(T < \infty) = 1$ and $0 < \gamma \leq 1$.
  - For continuing tasks, $T = \infty$ and $0 < \gamma < 1$.

# Bellman equation for a deterministic policy

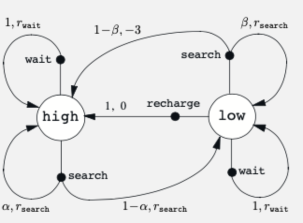
Backup diagram:




$$V^\pi(s) = \sum_{s' \in \mathscr{S}} p(s', s \mid \pi(a))(r(s, s', \pi(a)) + \gamma V^\pi(s'))$$

For episodic tasks, set $V^\pi(s) = 0$ if $s$ is a terminal state

# Value function computation example




- Policy: $\pi(\texttt{high}) = \texttt{search}, \quad \pi(\texttt{low}) = \texttt{search}$
- Bellman equations:

$$V^\pi(\texttt{h}) = \alpha(r_s + \gamma V^\pi(\texttt{h})) + (1-\alpha)(r_s + \gamma V^\pi(\texttt{l}))$$
$$V^\pi(\texttt{l}) = (1-\beta)(-3 + \gamma V^\pi(\texttt{h})) + \beta(r_s + \gamma V^\pi(\texttt{l}))$$

# Solving the linear system

$$V^\pi(\mathtt{h}) = \alpha(r_\mathtt{s} + \gamma V^\pi(\mathtt{h})) + (1-\alpha)(r_\mathtt{s} + \gamma V^\pi(\mathtt{l}))$$
$$V^\pi(\mathtt{l}) = (1-\beta)(-3 + \gamma V^\pi(\mathtt{h})) + \beta(r_\mathtt{s} + \gamma V^\pi(\mathtt{l}))$$

Rewrite in "linear algebra" style:

$$(I - \gamma P)V = b$$

Where:

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}, \quad \begin{bmatrix} r_s \\ -3(1-\beta) + r_s\beta \end{bmatrix}$$

# Numerical solution

Parameters:

$$\alpha = 0.8, \quad \beta = 0.3$$

$$r_{\texttt{search}} = 15, \quad r_{\texttt{wait}} = 10, \quad r_{\texttt{empty}} = -3.0$$

For $\gamma = 0.9$:

$$A = I - \gamma P = \begin{bmatrix} 0.28 & -0.18 \\ -0.63 & 0.73 \end{bmatrix}, \quad b = \begin{bmatrix} 15. \\ 2.4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} V(\texttt{high}) \\ V(\texttt{low}) \end{bmatrix} = \begin{bmatrix} 125.07692308 \\ 111.23076923 \end{bmatrix}$$

# Solution strategies for a linear system

$$V = b + \gamma P V$$

- **Linear algebra solution**. Write the system in the form:

  $$(I - \gamma P)V = b \quad \to \quad V = (I - \gamma P)^{-1} b$$

  *Don't* use inversion, use an LU decomposition (Gaussian elimination).
- **Gauss-Jacobi Iteration**.
  - Choose $V_0$ arbitrarily.
  - Iterate:

    $$V_{n+1} = b + \gamma P V_n$$
- This is an example of **fixed point iteration**.

# Jacobi iteration for the RR model

```
Initial approximation: [0. 0.]
Step 1: [15.   2.4]
Step 2: [26.232 12.498]
Step 3: [36.13668 22.30062]
Step 4: [45.0325212 31.1872758]
Step 5: [53.03712491 39.19105282]
................
Step 99: [125.07332253 111.22716869]
Step 100: [125.07368259 111.22752874]
```

# Gauss-Seidel Iteration

- Similar to Jacobi iteration, but *use the already updated values in each step*:
  - Initialize $V_0$ arbitrarily.
  - Iterate:

$$\begin{aligned} V_{n+1}(1) &= b_1 + \gamma[p_{11} V_n(1) + p_{12} V_n(2)] \\ V_{n+1}(2) &= b_2 + \gamma[p_{21} V_{n+1}(1) + p_{12} V_n(2) \end{aligned}$$

- Results in slightly faster convergence.

# Gauss-Seidel iteration for the RR model

```
Initial approximation: [0. 0.]
Step 1: [15.   11.85]
Step 2: [27.933   23.19729]
Step 3: [39.2872722  33.41424979]
Step 4: [49.30140095 42.48173004]
Step 5: [58.14372009 50.50061077]
........................
Step 79: [125.07002966 111.22451454]
Step 80: [125.07083397 111.22524433]
```

# Optimal Value Function

- The optimal value function is:

$$V^\star(s) = \max_\pi V^\pi(s) = \max_\pi \mathbb{E}_\pi \left[ \sum_{t=0}^{T} \gamma^t R_{t+1} \,\middle|\, S_0 = s \right]$$

where $\pi$ ranges over all admissible (randomized) policies.

- A policy $\pi^\star$ is optimal if:

$$V^{\pi^\star}(s) = V^\star(s) \quad \text{for all } s \in \mathscr{S}$$

- Under the finiteness assumptions we made, an optimal policy always exists.

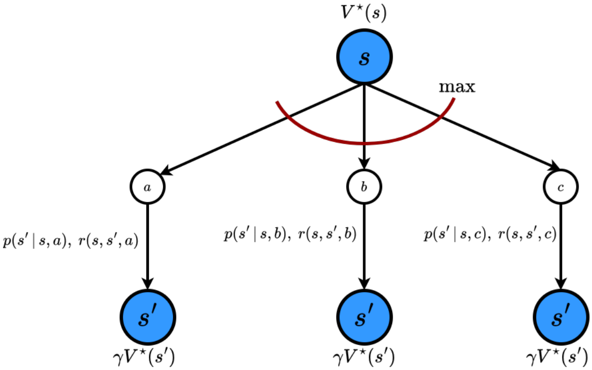
# How to live the best life

- There is an oracle that can tell us what to do to maximize our happiness. However, the oracle will only tell us what is the best we can do starting tomorrow.
- We can ask a series of questions to the oracle:
  - If I take action $a_j$ today, what is the best outcome I can expect from tomorrow onward?
- We can then find:

$$\max\{R(a_j) + \mathbb{E}[\text{rewards from tomorrow onward} \mid \text{action } a_j \text{ chosen}]\}$$

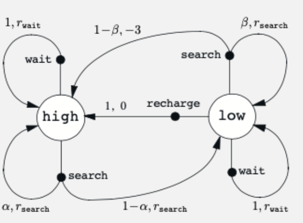
# Optimal Bellman equation

Backup diagram:




$$V^\star(s) = \max_{a \in \mathscr{A}(s)} \left\{ \sum_{s' \in \mathscr{S}} p(s, s' \mid a) \left[ r(s, s', a) + \gamma V^\star(s') \right] \right\}$$

# Example — Recycling Robot




$$\begin{aligned}V^\star(\texttt{high}) = \max \Big\{ &\alpha(r_{\texttt{search}} + \gamma V^\star(\texttt{high})) + (1-\alpha)(r_{\texttt{search}} + \gamma V^\star(\texttt{low})), \\ &r_{\texttt{wait}} + \gamma V^\star(\texttt{high}) \Big\}\end{aligned}$$

$$\begin{aligned}V^\star(\texttt{low}) = \max \Big\{ &(1-\beta)(-3 + \gamma V^\star(\texttt{high})) + \beta(r_{\texttt{search}} + \gamma V^\star(\texttt{low})), \\ &r_{\texttt{wait}} + \gamma V^\star(\texttt{low}), \\ &0 + \gamma V^\star(\texttt{high}) \Big\}\end{aligned}$$

# Verification Theorem

**Theorem**

*Suppose $V^\star$ is a solution of the optimal Bellman equation:*

$$V^\star(s) = \max_{a \in \mathscr{A}(s)} \left\{ \sum_{s' \in \mathscr{S}} p(s, s' \mid a) \left[ r(s, s', a) + \gamma V^\star(s') \right] \right\}.$$

*Then, for any randomized policy $\pi(a \mid s)$ we have:*

$$V^\star(s) \geq V^\pi(s) \quad \textit{for all } s \in \mathscr{S}$$

*Furthermore, an optimal (deterministic) policy is given by:*

$$\pi^\star(s) = \arg\max \left\{ \sum_{s' \in \mathscr{S}} p(s, s' \mid a) \left[ r(s, s', a) + \gamma V^\star(s') \right] \;\middle|\; a \in \mathscr{A}(s) \right\}$$

# What are tabular solution methods

- In **tabular solution methods**, we build *complete* tables of the optimal value function and optimal policy.
- These methods can only be used to solve relatively small problems.
- It is also assumed that we have complete information about transition probabilities and rewards (*model based RL*).
- It is important to learn them, because:
  - The theory of these methods is *very* well established, so we know very well how they behave.
  - They are essential to develop intuition.
  - They contain the main ideas used in more advanced algorithms.
  - In practice, they provide a good way to build test cases for complex algorithms.

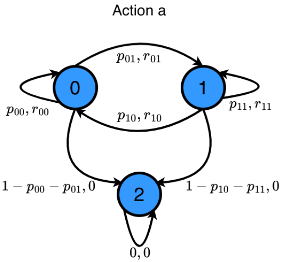
# Value Iteration algorithm

- Let $V_0(s)$ be arbitrary for $s \in \mathscr{S}$.
- Iterate:

$$V_{n+1}(s) = \max_{a \in \mathscr{A}(s)} \left\{ \sum_{s' \in \mathscr{S}} p(s, s' \mid a) \left[ r(s, s', a) + \gamma V_n(s') \right] \right\},$$

  except that we let $V_{n+1}(s) = 0$ if $s$ is a terminal state in episodic tasks.
- Assume that:
  - $0 < \gamma < 1$ for continuing tasks.
  - $0 < \gamma \leq 1$ and a terminal state is reached with probability 1 for any policy.
- Then $\lim_{n\to\infty} V_n(s) = V^\star(s)$ for all $s \in \mathscr{S}$.

# Value Iteration Example — Tri-state MDP




Optimal Bellman equation for $\gamma = 1$:

$$\begin{aligned}
V^\star(0) &= \max\{p_{00}(r_{00} + V^\star(0)) + p_{01}(r_{01} + V^\star(1)), \\
&\qquad q_{00}(s_{00} + V^\star(0)) + q_{01}(s_{01} + V^\star(1))\} \\
V^\star(1) &= \max\{p_{10}(r_{10} + V^\star(0)) + p_{11}(r_{11} + V^\star(1)), \\
&\qquad q_{10}(s_{10} + V^\star(0)) + q_{11}(s_{11} + V^\star(1))\}
\end{aligned}$$

# Tri-state MDP — Value Iteration

$$
\begin{aligned}
V_{n+1}(0) = \max\{ & p_{00}(r_{00} + V_n(0)) + p_{01}(r_{01} + V_n(1)), \\
& q_{00}(s_{00} + V_n(0)) + q_{01}(s_{01} + V_n(1))\} \\
V_{n+1}(1) = \max\{ & p_{10}(r_{10} + V_n(0)) + p_{11}(r_{11} + V_n(1)), \\
& q_{10}(s_{10} + V_n(0)) + q_{11}(s_{11} + V_n(1))\}
\end{aligned}
$$

Parameters for testing:

$$
\begin{array}{llll}
p_{00} = 0.2 & p_{01} = 0.7 & p_{10} = 0.5 & p_{11} = 0.3 \\
q_{00} = 0.1 & q_{01} = 0.6 & q_{10} = 0.4 & q_{11} = 0.3 \\
r_{00} = 10 & r_{01} = 15 & r_{10} = 8 & r_{11} = 12 \\
s_{00} = 13 & s_{01} = 13 & s_{10} = 15 & s_{11} = 20
\end{array}
$$

# Value Iteration — Results

| $n$ | $V_n(0)$ | $V_n(1)$ |
|---|---|---|
| 0 | 0.00 | 0.00 |
| 1 | 12.50 | 16.00 |
| 2 | 26.20 | 25.80 |
| $\vdots$ | $\vdots$ | $\vdots$ |
| 38 | 71.24 | 63.57 |
| 39 | 71.25 | 63.57 |
| 40 | 71.25 | 63.57 |

- Convergence is not very fast.
- Optimal policy is found by a one-step lookahead. In this case we get:

$$\pi(0) = a, \quad \pi(1) = b$$

# Gauss-Seidel Iteration

$$
\begin{aligned}
V_{n+1}(0) = \max\{ & p_{00}(r_{00} + V_n(0)) + p_{01}(r_{01} + V_n(1)), \\
& q_{00}(s_{00} + V_n(0)) + q_{01}(s_{01} + V_n(1))\} \\
V_{n+1}(1) = \max\{ & p_{10}(r_{10} + V_{n+1}(0)) + p_{11}(r_{11} + V_n(1)), \\
& q_{10}(s_{10} + V_{n+1}(0)) + q_{11}(s_{11} + V_n(1))\}
\end{aligned}
$$

| $n$ | $V_n(0)$ | $V_n(1)$ |
|---|---|---|
| 0 | 0.00 | 0.00 |
| 1 | 12.50 | 21.00 |
| 2 | 29.70 | 34.18 |
| ⋮ | ⋮ | ⋮ |
| 26 | 71.24 | 63.57 |
| 27 | 71.25 | 63.57 |
| 28 | 71.25 | 63.57 |

- Easier to code, uses less memory.
- Slightly faster convergence.