What is a System

This activity describes what is a system of ordinary differential equations.

After completing this section, students should be able to do the following:

- Recognize a system of ODEs.
- Interpret a system of ODEs as a vector field.
- Understand the representation of a vector field in the phase plane.

To introduce the concept of a system of ODEs, let's consider the *Lotka-Volterra predator-prey model*, which describes two animal species in the wild, traditionally referred to as "rabbits" (the prey) and "foxes" (the predator). Denoting by $R(t)$ and $F(t)$ the rabbit and fox populations, respectively, we use the following differential equations to model the dynamics of the two species in the wild:

$$R' = aR - bRF \tag{1}$$
$$F' = cF + dRF \tag{2}$$

How do ecologists come up with this model? This is not a simple question, since there usually are many possible ways of modeling a real-world system. As a rule of thumb, scientists try to use the simplest possible model as a first approximation. For the predator prey model, the following are reasonable hypothesis:

In the absence of predators, the prey population grows without limits. We can model this by an exponential growth model:

$$R' = aR \text{ where } a > 0.$$

In the absence of prey, the predator population decays to zero, since they don't have any available food. This can be modeled by an exponential decay model:

$$F' = -cF \text{ where } b > 0.$$

To model the interactions between the two populations, we assume that the number of possible "predation situations" is proportional to the number of possible "encounters" between individuals of the two populations, which can be approximated by the product RF. Since predation is disadvantageous to the prey population, we subtract a term proportional to RF to the prey equation:

$$R' = aR - bRF \text{ where } a, c > 0.$$

Learning outcomes:
Author(s):

Since predation is beneficial to the predator population, we add a term proportional to RF to the predator population:

$$F' = -cF + dRF \text{ where } b, d > 0.$$

Putting all together, we get the pair of equations 1, 2.

Problem 1 *Let's consider now consider two populations x and y that are in competition with each other. We make the following assumptions:*

- *In isolation, each of the populations grows exponentially. We denote the growth rates for x and y by $a > 0$ and $b > 0$, respectively.*
- *The effect of competition to each population is proportional to the product of the populations sizes, xy. We denote the corresponding proportionality constants for populations x and y by $c > 0$ and $d > 0$, respectively.*

Question 1.1 $x' = \boxed{ax - cxy}$

Question 1.2 $y' = \boxed{bx - dxy}$

Explanation. *In the absence of competition, the dynamics for each population is given by an exponential growth model:*

$$x' = ax$$
$$y' = cy.$$

Competition has a negative effect on the growth of each population, so we need to subtract a term proportional to xy to each equation, obtaining:

$$x' = ax - bxy$$
$$y' = cy - dxy.$$

Problem 2 *A more realistic assumption in the predator-prey model is that the prey population grows logistically in the absence of predators. Write the system of ODEs for this situation, using a for the intrinsic growth rate and K for the carrying capacity of the rabbit population:*

Question 2.1 $R' = \boxed{ax(1 - x/K) - bxy}$

Question 2.2 $F' = \boxed{-cx + dxy}$

Problem 3 *Another way in which system appear is in modeling higher-order differential equations. Recall the general equation for the unforced harmonic oscillator:*

$$mx'' + cx' + kx = 0,$$

where $m > 0$, $c \geq 0$ and $k > 0$. The variable x represents the displacement of the mass. Let y represent the velocity of the mass. Write below a system of ODEs for the variables x, y:

Question 3.1 $x' = \boxed{y}$

Question 3.2 $y' = \boxed{-(k/m)x - (c/m)y}$

Explanation. *The interpretation of derivative of displacement as velocity gives directly:*

$$x' = y.$$

Plugging this into the equation of the harmonic oscillator yields:

$$y' = x'' = -\frac{k}{m}x - \frac{c}{m}x' = -\frac{k}{m}x - \frac{c}{m}y.$$

We now proceed to the numerical and graphical interpretation of a system of ODEs. Let's go back to the predator-prey model, this time assuming concrete values for the parameters a, b, c and d:

$$R' = 1.2R - 0.5RF \tag{3}$$
$$F' = -F + 1.5RF \tag{4}$$

Since we have a concrete example now, it is advisable to set specific units for all variables. Let's assume that:

$$\begin{aligned} t &= \text{Time, measured in weeks;} \\ R &= \text{Rabbit population, measured in thousands of individuals;} \\ F &= \text{Fox population, measured in hundreds of individuals.} \end{aligned}$$

Notice that we use different units for the rabbit and fox populations, since the number of predators is usually much smaller that the number of prey in a realistic situation.

Next suppose that, at a certain time, we have the values:

$$R = 4, \quad F = 3$$

With this information, we can compute the rates at which each of the populations are growing, by plugging in the values of R and F into the equations 3 and 4:

$$R' = 1.2 \cdot 4 - 0.5 \cdot 4 \cdot 3 = -1.2$$
$$F' = -4 + 1.5 \cdot 4 \cdot 3 = 15.0$$

In particular, we can conclude that, for these particular sizes of the rabbit and fox population, the number of rabbits is decreasing and the number of foxes is increasing.

Problem 4 *Suppose that at a certain time, the rabbit population is* 5000 *and the fox population is* 100*. Find the rate of change of the populations, and determine if each population is increasing and decreasing.*

Hint*: Notice that you will have to scale the populations in the appropriate way!*

Question 4.1 $R' = \boxed{3.5}$

The rabbit population is

Multiple Choice:

(a) *Increasing* ✓

(b) *Decreasing*

Question 4.2 $F' = \boxed{6.5}$

The fox population is

Multiple Choice:

(a) *Increasing* ✓

(b) *Decreasing*

We can also represent the rates of change geometrically. We interpret the pair of derivatives (R', F') as the components of a vector:

$$\begin{bmatrix} R' \\ F' \end{bmatrix} = \begin{bmatrix} 1.2R - 0.5RF \\ -F + 1.5RF \end{bmatrix}$$

Then, for $R = 4$ and $F = 3$ we have:

$$\begin{bmatrix} R' \\ F' \end{bmatrix} = \begin{bmatrix} -1.2 \\ 15.0 \end{bmatrix}$$

This can be represented graphically as in the figure below: