COMBINATION OF ADAPTIVE FILTERS WITH COEFFICIENTS FEEDBACK

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ABSTRACT

In parallel combinations of adaptive filters, the component filters are usually run independently to be later on combined, leading to a stagnation phase before reaching a lower error. Conditional transfers of coefficients between the filters have been introduced in an attempt to address this issue. The present work proposes a more natural way of accelerating the convergence to steady-state, using a cyclic feedback of the overall weights to all component filters, instead of a unidirectional conditional transfer. It is shown that, depending on the cycle length, the resulting recursion is equivalent to either: (i) the independent combination, (ii) a variable step size adaptive filter, or (iii) a new hybrid algorithm. Comments on the universality of the approach are presented along with a technique to design the cycle length. Comparisons in stationary and non-stationary system identification scenarios demonstrate the superior performance of this new combination method.

Index Terms— Convex combination, adaptive filters, coefficients feedback

1. INTRODUCTION

Combinations of adaptive filters (AFs) have been introduced as a way to improve filtering performance when the accurate design of a single filter is difficult [1–4]. This approach consists of aggregating a set of AFs using a supervisor which attempts to achieve universality, i.e., making the overall system at least as good—usually in the mean-square sense—as the best filter in the set. Combinations with different step sizes, orders, adaptive algorithms and supervising rules can be found in [3–8]. In such schemes, the component filters run independently while an adaptive mixing parameter merges their weights. This structure presents a well-known convergence stagnation while the mean-square-error (MSE) of the accurate filter does not become smaller than that of the fast one. Different topologies [1] and conditional transfers of coefficients [3,9] have been proposed to overcome this problem.

In this paper the stagnation issue will be addressed by:

- Embedding a cyclic coefficients feedback in the combination (Fig. 1);

- Showing that for two least-mean-square filters (LMS), the cycle length (L) can turn the combination into either a convex LMS (CLMS) [3], a variable step size (VSS) algorithm or a hybrid of both;

- Proposing a method to design L;

- Comparing through simulations the performance of the new algorithm to existing methods in the literature [1,3,9] for different scenarios.
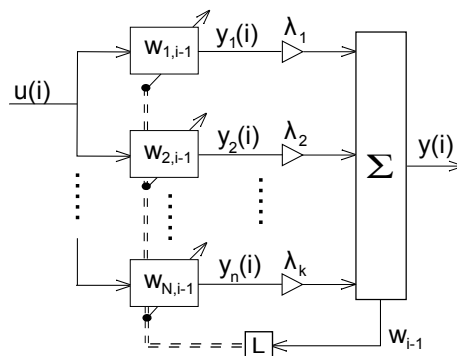

Fig. 1. Generalized parallel topology

2. PROBLEM FORMULATION

Under the widely adopted system identification scenario, the combination problem consists of finding a mixture of AFs weights which leads to an estimate at least as good as any of the component filters individually. Explicitly, given a pool of AFs

$$w_{n,i} = w_{n,i-1} + \mu_n p_n \qquad (1)$$

in which $w_{n,i}$ is the $M_n \times 1$ weights estimate at iteration i of the nth filter, μ_n is its step size and $p_n = -B_n \nabla^* J(w_{n,i-1})$, with B_n any positive definite matrix, $J(w_{n,i-1})$ the underlying cost function the filter attempts to minimize and * denoting the conjugate transpose, a combination is defined as

$$w_{i-1} = \sum_{n=1}^{N} \lambda_n(i) w_{n,i-1}. \qquad (2)$$

The mixing parameter $\lambda_n(i)$ aggregates the estimates according to supervising rules and constraints defined by the combination strategy adopted. Generally, the overall MSE $E|e(i)|^2$ is minimized, under the constrain $\sum \lambda_n(i) = 1$. The overall error $e(i) = d(i) - y(i)$ is defined over the combination output $y(i) = u_i w_{i-1}$, where u_i is the $1 \times M$ regressor vector[1] that captures samples of an input signal with variance σ_u^2, and $d(i) = u_i w^o + v(i)$ is the desired signal, in which $v(i)$ is an independent and identically distributed (i.i.d.) noise with variance σ_v^2, and w^o is an $M \times 1$ vector that models the unknown plant. For the individual filters, $e_n(i) = d(i) - y_n(i)$ and $y_n(i) = u_i w_{n,i-1}$.

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[1]To account for different filters' orders, $M = \max_n M_n$, filling up the vectors with zeros whenever necessary to match the dimensions [5,6].

For $N = 2$ and $M_1 = M_2 = M$, an approach which has gained a lot of attention in the last years is the convex combination of two independent AFs [3, 7], in which (2) reduces to

$$w_{i-1} = \lambda(i)\, w_{1,i-1} + (1 - \lambda(i))\, w_{2,i-1}, \qquad (3)$$

and $\lambda \in [0, 1]$. In order to guarantee this last constraint and reduce gradient noise when $\lambda \approx 0$ or $\lambda \approx 1$, an activation function is used[2]

$$\lambda(i) \triangleq \frac{1}{1 + e^{-a(i-1)}},$$

where the auxiliary variable $a(i)$ is adapted as [3]

$$a(i) = a(i - 1) + \mu_a e(i)[y_1(i) - y_2(i)]\lambda(i)[1 - \lambda(i)], \qquad (4)$$

and limited to $[-a^+, a^+]$ so as not to stall the adaptation.

A detailed study of the convex combination strategy using two LMS filters with different step sizes is carried out in [3]. Fig. 2 depicts the excess mean-square-error (EMSE $= \mathrm{E}|u_i(w^o - w_{i-1})|^2$) for a typical example employing $\mu_1 = 0.07$, $\mu_2 = 0.007$, $\mu_a = 1000$, $\sigma_u^2 = 1$, and $\sigma_v^2 = 10^{-3}$, averaged over 200 realizations.
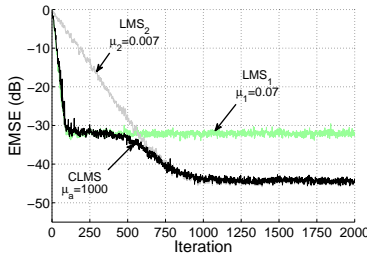

Fig. 2. EMSE of LMS filters and their convex combination

Despite the clear advantage of the CLMS over its components, the combination is most of the time ignoring one of the AFs, leading to a convergence stagnation as soon as the faster filter reaches steady-state. The combination must then hold until the accurate filter catches up to commute due to their parallel-independent operation.

Different schemes have been proposed in the literature to overcome the aforementioned effect: conditional transfer of coefficients from faster to slower filter are explored in [3, 9]; and series topology, inspired by incremental-cooperative strategies, is proposed in [1].

2.1. Conditional transfer of coefficients

Transfer of coefficients has been studied for the two LMS case in an attempt to improve the convergence of the smaller step size AF by leaking the faster filter weights into its recursion. However, this leakage degrades the steady-state performance of the combination, increasing the misadjustment of the slower filter. Therefore, it must only occur under specific conditions, usually when $\lambda(i) \geq \beta$, with β close to the maximum value of $\lambda(i)$—e.g. $\beta = 0.98$. Assuming filter 1 is faster, the second AF recursion becomes [3, 9],

$$w_{2,i} = \alpha[w_{2,i-1} + \mu_2 u_i^* e_2(i)] + (1 - \alpha)w_{1,i-1}, \qquad (5)$$

in which α is a parameter close to 1.

This method effectively addresses the stagnation problem, although it presents a few downsides from a practical implementation

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2For general activation functions and hierarchical combinations refer to [5, 6].

point of view. First, it requires a conditional test on every iteration. Second, it only implements a unidirectional transfer—i.e., from filter 1 to filter 2—, while there is no guarantee that in real scenarios filter 1 will always be the faster one—e.g., non-stationary or low signal-to-noise ratio (SNR) environments.

2.2. Parallel structure with cyclic feedback

In order to improve the results obtained with the method in section 2.1, while avoiding its implementation issues, a cyclic feedback approach is proposed as an alternative to transferring coefficients. In this approach, the overall estimate (2) is fed back to the component AFs every L iterations—*the cycle length*—, providing all of them with the best weights estimate available at that time (Fig. 1). Therefore, $w_{n,i-1}$ in (1) becomes

$$w_{n,i-1} = \delta(i - kL)w_{i-1} + (1 - \delta(i - kL))w_{n,i-1}, \qquad (6)$$

where w_{i-1} is given by (2), $\delta(i)$ is the Kronecker delta, and $k \in \mathbf{N}$.

Conceptually, this new algorithm is more natural than the one proposed in Section 2.1, given that it provides all filters with the global weights, explicitly chosen to minimize a function of the overall error—e.g. MSE. Additionally, the feedback is neither directional nor limited to any two filters, so that all components can take advantage of the overall coefficients. Still in the theoretical front, section 3 will show that this method bridges a gap between combinations of AFs of the same kind and VSS techniques. In terms of implementation, this structure also brings advantages since it depends uniquely on counters and allows efficient interruption-based algorithms. Lastly, simulations will show that it can outperform other combinations in different scenarios (Section 4).

3. A BRIEF ON ANALYSIS

In order to illustrate the idea, we restrict the sections to follow to the combination of two LMS for white real-valued Gaussian inputs. With $n = 1, 2$, $M_1 = M_2 = M$, the complete algorithm is

$$\lambda(i) = \frac{1}{1 + e^{-a(i-1)}}$$
$$a(i) = a(i - 1) + \mu_a e(i)[y_1(i) - y_2(i)]\lambda(i)[1 - \lambda(i)] \qquad (7)$$
$$w_{n,i-1} = \delta(i - kL)w_{i-1} + (1 - \delta(i - kL))w_{n,i-1}$$
$$w_{n,i} = w_{n,i-1} + \mu_n u_i^*(d(i) - u_i w_{n,i-1}).$$

Algorithm (7) shows that depending on the choice of the cycle length it is possible to (i) recover the CLMS from [3]; (ii) devise a VSS algorithm; or (iii) formulate a new algorithm:

(i) For $L \to \infty$ in (6), $\delta(i - kL) = 0$ for all finite i and the equation reduces to $w_{n,i-1} = w_{n,i-1}$. This makes the component filters operate independently and is equivalent to eliminating the feedback branch in Fig. 1, leading to the CLMS [3].

(ii) However, if $L = 1$, (6) becomes $w_{n,i-1} = w_{i-1}$, as if the feedback loop in Fig. 1 was continuously enabled. Using this result in (7) and substituting in (3) yields

$$w_i = w_{i-1} + \overline{\mu}(i)u_i^*(d(i) - u_i w_{i-1}), \qquad (8)$$

with $\overline{\mu}(i) \triangleq \lambda(i+1)\mu_1 + [1 - \lambda(i+1)]\mu_2$. Note that $\overline{\mu}(i)$ is an iteration dependent convex association of μ_1 and μ_2, leading to a VSS variant of the LMS recursion.

(iii) Finally, choosing any finite $L > 1$ gives rise to a new adaptive algorithm that will present a hybrid operation, acting as in (ii) for $i = kL$ and as in (i) for all other i.

3.1. Comments on universality

A celebrated result for the convex combination is that it may achieve (nearly) universality in steady-state [3]. Since the fCLMS of case (iii) will, at each instant, be identical to either the CLMS or the VSS algorithm (8), suffices to show that the latter is universal to make fCLMS with arbitrary L universal as well.

With no loss of generality, $\mu_1 > \mu_2$ and $\lambda(i+1) = 0$ makes (8) identical to the recursion of the smaller step size filter. Therefore, since σ_a^2 tends to be negligible [7,9], as long as $\mathrm{E}\,a(\infty) \to -\infty$ the overall MSE at steady-state will be that of the best of the component filters, which follows from similar arguments as presented in [3].

In summary, both cases (i) and (ii) can be shown to be capable of universality. Since (iii) is at each instant equivalent to any one of them, universality follows.

3.2. Design of the cycle length

Section 3.1 showed that the combination converges to the MSE of the best AF regardless the choice of L. However, due to the nature of the adaptation of λ, it may take a long time, especially for small L. This phenomenon is easier understood by looking at (4) and observing that the recurrence on a depends upon $[y_1(i) - y_2(i)] = u_i(w_{1,i-1} - w_{2,i-1})$. Although $w_{1,i-1} \neq w_{2,i-1}$, their difference can be expected to be small after a feedback, when $e_1(i) = e_2(i)$, resulting in slower updates of λ. However, it is unwise to take L to be as large as desired because then the component filters will not be able to take advantage of the overall estimation.

A method is then proposed to design the cycle length L based on the idea that, in a stationary scenario, the overall w should be fed back as soon as the faster filter stops converge, therefore supplying the slower filter with the best estimate the combination can provide at that time. To do so, a linear approximate model for the MSE convergence in dB is adopted (see Fig. 3). This model can then be used to find an approximate point where the faster filter slows down, leading to an estimate for L.

Starting from the weighted variance relation with independence for AFs of the form $w_i = w_{i-1} + \mu u_i^*[g(u_i)]^{-1}e(i)$ [10, Thm.22.4],

$$\mathrm{E}\|\tilde{w}_i\|_\Gamma^2 = \mathrm{E}\|\tilde{w}_{i-1}\|_{\Gamma'}^2 + \mu\sigma_v^2\mathrm{E}\left(\frac{u_i\Gamma u_i^*}{g^2(u_i)}\right) \tag{9}$$

$$\Gamma' = \Gamma - \mu\Gamma\mathrm{E}(\nu) - \mu\mathrm{E}(\nu)\Gamma + \mu^2\mathrm{E}(\nu\Gamma\nu)$$

where $\tilde{w}_i = w^o - w_i$, $\nu = u_i^* u_i[g(u_i)]^{-1}$, $\|x\|_A^2 = x^*Ax$ and Γ is any Hermitian positive semi-definite matrix. Under the initial assumption that u_i is white, real-valued and Gaussian, $\mathrm{E}u_i^*u_i = \sigma_u^2 I$, with I the identity matrix. Choosing $\Gamma = I$ and $g(u_i) = 1$ for LMS, $E(\nu) = \sigma_u^2 I$, $E(u_i\Gamma u_i^*) = \sigma_u^2 M$, and $E(\nu\Gamma\nu) = \sigma_u^2\mathrm{Tr}(\sigma_u^2 I)I + 2\sigma_u^4 I = \sigma_u^4(M+2)I$. Defining $\gamma = 1 - 2\mu\sigma_u^2 + \mu^2\sigma_u^4(M+2)$ and from (9), by induction one gets

$$\mathrm{E}\|\tilde{w}_{i-1}\|^2 = \gamma^i\mathrm{E}\|w^o\|^2 + \mu\sigma_v^2\sigma_u^2 M\sum_{k=0}^{i-1}\gamma^k. \tag{10}$$

Assuming small step sizes and high SNR, a coarse approximation for (10) is[3] $\mathrm{E}\|\tilde{w}_{i-1}\|^2 = \gamma^i\mathrm{E}\|w^o\|^2$. Since $e(i) = u_i\tilde{w}_{i-1} + v(i)$, $\mathrm{MSE}_{dB}(i) = 10\log(\mathrm{E}|e(i)|^2) = 10\log(\sigma_u^2\mathrm{E}\|\tilde{w}_{i-1}\|^2 + \sigma_v^2) \approx 10\log(\sigma_u^2\mathrm{E}\|\tilde{w}_{i-1}\|^2)$. Furthermore, using the data model, $\sigma_u^2\|w^o\|^2 = \sigma_d^2 - \sigma_v^2$, and the linear MSE_{dB} model becomes

$$\mathrm{MSE}_{dB}(i) = 10i\log\gamma + 10\log[\sigma_d^2 - \sigma_v^2], \tag{11}$$

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[3]For stable LMS filters [10] $0 \ll \gamma < 1$.

which takes on the form $r(i) = a.i + b$ with $a = 10\log\gamma$ and $b = 10\log[\sigma_d^2 - \sigma_v^2]$.

With (11), L is estimated via $r(L) = 10\log\mathrm{MSE}(\infty)$, where $\mathrm{MSE}(\infty) = 2\sigma_v^2(1 - \mu\sigma_u^2)/[2 - \mu(M+2)\sigma_u^2]$ [10, Ch. 16]. As it will be shown in Section 4, the algorithm is robust with respect to L, which supports the aforementioned approximations. Moreover, it means that no accurate estimates for σ_v^2 are required.

4. SIMULATIONS

This section presents simulations to compare the behavior of CLMS, CLMS with transfer of coefficients (CLMS$_2$), series topology and fCLMS. The approach taken here is to design the fCLMS once and then test it under different scenarios. All parameters were chosen as in [9]: $\sigma_u^2 = 1$, $\sigma_v^2 = 10^{-2}$ (SNR = 20dB), $\mu_1 = 0.05$, $\mu_2 = 0.005$, $\mu_a^{fCLMS} = 600$, $\mu_a^{CLMS} = 200$, $\mu_a^{CLMS_2} = 100$, $\alpha = 0.9535$, $a^+ = 4$, and $M = 7$. For the series topology, the INC-COOP$_1$ with simple design was used with the same configurations as [1], but with the step sizes above. All figures show curves averaged over 1000 runs with a normalized randomly generated w^o.

The model (11) is tested in a high SNR/small step size scenario—$\mu = 0.01$, $\sigma_v^2 = 10^{-4}$—and in the one presented above—$\mu = \mu_1$, $\sigma_v^2 = 10^{-2}$—, concurring with the simulations results (Fig. 3). For the adopted framework only, assessments of the convergence time for different cycle lengths were made (Fig. 4), showing the accuracy and robustness of the design method from section 3.2.
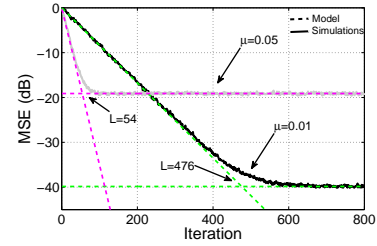

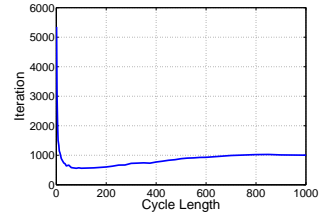**Fig. 3**. Analytical model and design of the cycle length

Fig. 4. Convergence time x Cycle length

Adopting $L = 60$, Fig. 5 depicts the filters in a stationary environment. The superior performance of series topology and fCLMS is clear, notably during the transient part, where the stagnation problem was definitely overcome. Note that the efficiency of the series strategy is due to its mixing parameter design method, adequate only for stationary scenarios [1]. When the same λ is applied to fCLMS with $L = 1$, both structures present undistinguishable results. Fig. 6 illustrates the filters behavior. Note the cyclic "stationarity" on the component filters produced by the feedback.

Following, results under non-stationary scenarios are reproduced in Fig. 7 and Fig. 8 for abrupt changes in w^o and time-varying systems, respectively. In the first figure, the series algorithm is left

out for clarity since it performs worst due to the lack of a more robust adaptation rule. In the second plot, the Mean Square Deviation ($\text{MSD}(i) = \text{E}\|\tilde{w}_{i-1}\|^2$) is showed for a random-walk model $w_i^o = w_{i-1}^o + q(i)$, with q a $M \times 1$ i.i.d. zero-mean Gaussian vector with covariance matrix $\sigma_q^2 I$ and $\sigma_q^2 = 10^{-4}$. In this case, all AFs performed equally, except for the series topology.

In summary, the key point is the fCLMS robustness: it is able to either outperform or match existing algorithms in different scenarios with one simple design. Performance could be improved in each scenario with specific tuning.

5. CONCLUSION

A novel scheme to overcome the stagnation problem of parallel-independent combinations was proposed by embedding cyclic coefficients feedbacks in the topology, a more natural way of reusing the AFs weights than conditional transfers. A case study for two LMS filters was used to illustrate this method, where the structure was showed to be equivalent to a CLMS, a VSS algorithm or a hybrid AF, depending on the cycle length L. A method to design this parameter was developed based on a transient model for LMS and validated. Simulations showed that the new algorithm can outperform CLMS, transfer of coefficients and series topology under different scenarios.

6. REFERENCES

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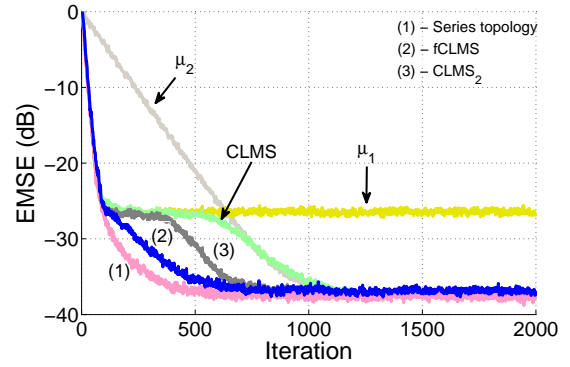[10] A.H. Sayed, *Adaptive filters*, Wiley-IEEE Press, 2008.
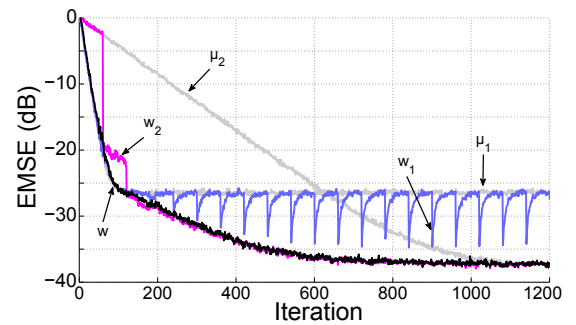
Fig. 5. Stationary environment

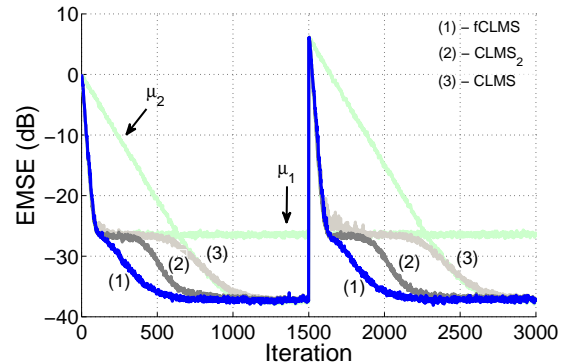**Fig. 6**. Component filters behavior for $L = 60$
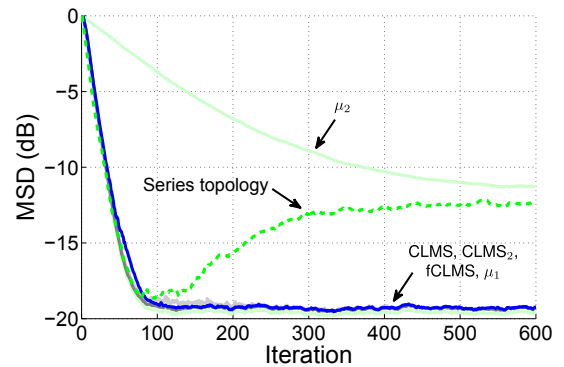

Fig. 7. Abrupt change in w^o

Fig. 8. Time varying plant (random walk with $\sigma_v^2 = 10^{-4}$)