Universität Stuttgart

IMPRS tutorial
Sep. 19, 2024

supervised and reinforcement learning under requirements

Luiz F. O. Chamon

SimTech
KI institute

Who am I?

Luiz

2025–: École Polytechnique de Paris (Professor)

2022–2024: ELLIS-SimTech (Research group leader)
IMPRS-IS faculty

2021–2022: Simons Institute, UC Berkeley (Postdoc)

2020: University of Pennsylvania (PhD)

< 2015: University of São Paulo (BSc. & MSc.)

- I speak 4.5 languages (German $=\epsilon$)

Acknowledgment: this material is based on a tutorial prepared in collaboration with Miguel Calvo-Fullana (UPF), Santiago Paternain (RPI), and Alejandro Ribeiro (Penn).

Agenda

I. Constrained supervised learning
- Constrained learning theory
- Constrained learning algorithms
- Resilient constrained learning

Break (10 min)

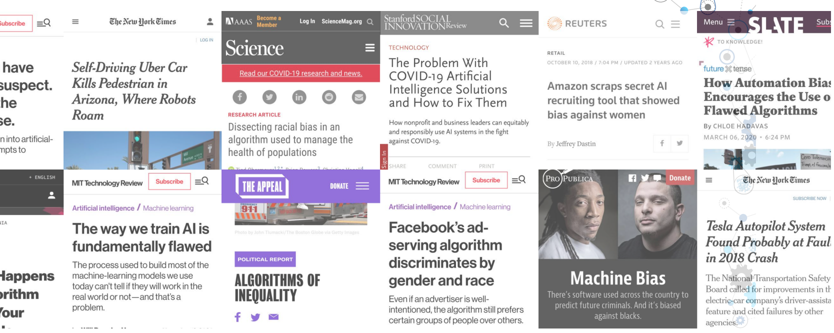
II. Constrained reinforcement learning
- Constrained RL duality
- Constrained RL algorithms

Q&A and discussions

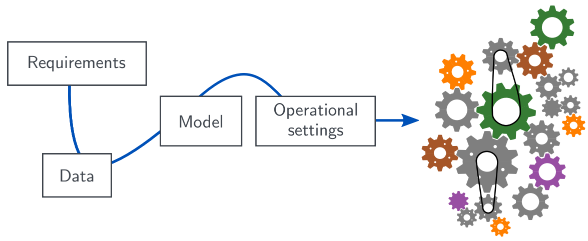
https://luizchamon.com/imprs2024

Why learning under requirements?

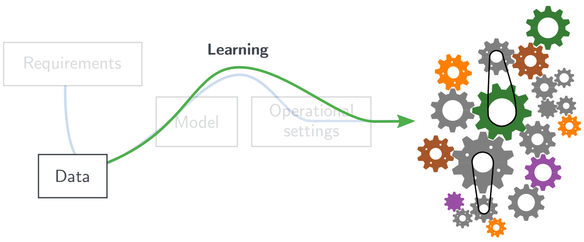
Why learning under requirements?



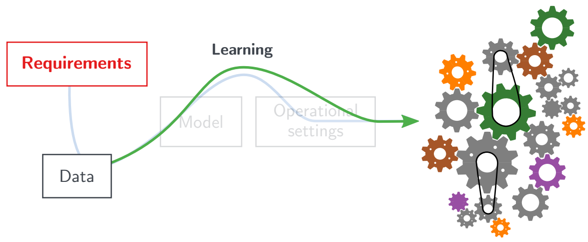
Why learning under requirements?



Why learning under requirements?

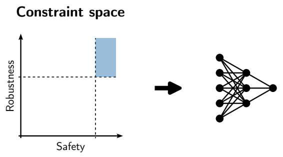


Why learning under requirements?



What is a requirements?

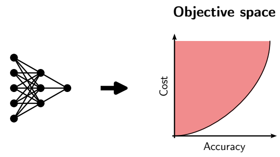
- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide



[NASA. "Systems engineering handbook," 2019]

What is a requirements?

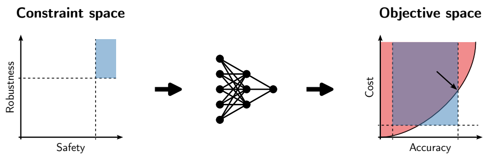
- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide
- **Goals are "should" statements:** express recommendations (once "shall" statements are satisfied)
  - *Objective space*: things the system achieves



[NASA. "Systems engineering handbook," 2019]

What is a requirements?

- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide
- **Goals are "should" statements:** express recommendations (once "shall" statements are satisfied)
  - *Objective space*: things the system achieves



[NASA. "Systems engineering handbook," 2019]

What is (un)constrained learning?

$$P_{\text{U}}^\star = \min_{\theta} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

What is (un)constrained learning?

$$\begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \leq u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

What about penalties?

$$\begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \leq u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

$$\Updownarrow$$

$$\min_{\theta} \ \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \lambda\,\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{P}}\Big[\mu(\boldsymbol{x},y)h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

What about penalties?

NON-CONVEX

$$\begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \leq u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

$$\not\Updownarrow$$

$$\min_{\theta} \ \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \lambda\,\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{P}}\Big[\mu(\boldsymbol{x},y)h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

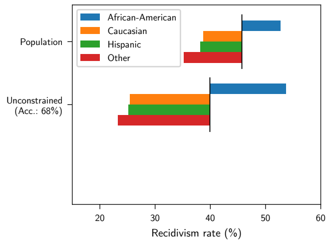
- ✗ There may not exist (λ, μ) such that the penalized solution is optimal *and* feasible
- ✗ Even if such (λ, μ) exist, they are not easy to find (hyperparameter search, cross-validation. . .)
- ✓ Constrained learning yields stronger guarantees, better performance, better trade-offs. . .

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- . . .

Fairness

Problem
Predict whether an individual will recidivate



*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.

Fairness: "Equality" of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\min_{\boldsymbol{\theta}} \quad \text{Prediction error}$$
$$\text{subject to} \quad \text{Prediction rate disparity (Race)} \leq c,$$
$$\text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\}$$

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23]

Fairness: "Equality" of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$
$$\text{subject to} \quad \text{Prediction rate disparity (Race)} \leq c,$$
$$\text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\}$$

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23]

Fairness: "Equality" of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$
$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = 1 \mid \text{Race}\right] \leq \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = 1\right] + c,$$
$$\text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\}$$

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23]

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- ...

Federated learning

Problem
Learn a common model using data from K clients

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}})$$

- k-th client loss: $\text{Loss}_k(f_{\boldsymbol{\theta}}) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

[Shen et al., ICLR'22]

Heterogeneous federated learning

Problem
Learn a common model using data from K clients

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}})$$

- k-th client loss: $\text{Loss}_k(f_{\boldsymbol{\theta}}) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

[Shen et al., ICLR'22]

Heterogeneous federated learning

Problem
Learn a common model using data from K clients that is good for all clients

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}})$$
$$\text{subject to} \quad \text{Loss}_k(f_{\boldsymbol{\theta}}) \leq \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}}) + c,$$
$$k = 1, \dots, K$$

- k-th client loss: $\text{Loss}_k(f_{\boldsymbol{\theta}}) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

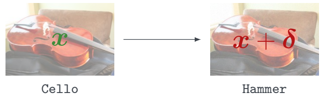
[Shen et al., ICLR'22]

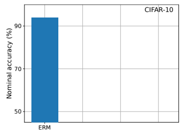
Applications

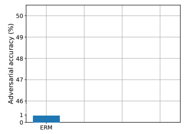
- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- ...

Robustness

Problem
Learn an accurate classifier that is robust to input perturbations







Robustness

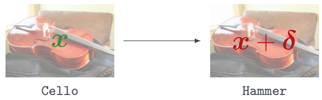
Problem
Learn an accurate classifier that is robust to input perturbations

$$\begin{aligned} \min_{\theta} \quad & \text{Nominal loss} \\ \text{subject to} \quad & \text{Robustness loss} \leq c \end{aligned}$$

[Chamon and Ribeiro, NeurIPS'20; Robey*, Chamon*, Pappas, Hassani, and Ribeiro, NeurIPS'21; Chamon, Paternain, Calvo-Fullana, and Ribeiro, IEEE TIT'23]

Robustness

Problem
Learn an accurate classifier that is robust to input perturbations



$$\begin{aligned} \min_{\theta} \quad & \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(x_n), y_n\big) \\ \text{subject to} \quad & \text{Robustness loss} \leq c \end{aligned}$$

[Chamon and Ribeiro, NeurIPS'20; Robey*, Chamon*, Pappas, Hassani, and Ribeiro, NeurIPS'21; Chamon, Paternain, Calvo-Fullana, and Ribeiro, IEEE TIT'23]

Robustness

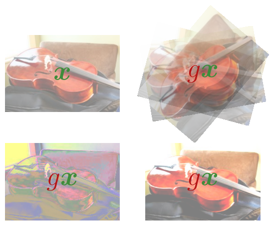
Problem
Learn an accurate classifier that is robust to input perturbations

$$\begin{aligned} \min_{\theta} \quad & \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N} \sum_{n=1}^{N} \Big[\max_{\|\delta\|_\infty \leq \epsilon} \text{Loss}\big(f_{\theta}(x_n + \delta), y_n\big) \Big] \leq c \end{aligned}$$

[Chamon and Ribeiro, NeurIPS'20; Robey*, Chamon*, Pappas, Hassani, and Ribeiro, NeurIPS'21; Chamon, Paternain, Calvo-Fullana, and Ribeiro, IEEE TIT'23]

Invariance

Problem
Learn an accurate classifier that is invariant to transformation $g \in \mathcal{G}$, e.g., $\mathcal{G} = \left\{ \begin{array}{l} \text{Rotate, TranslateX(Y),} \\ \text{ShearX(Y), Crop, Invert,} \\ \text{Solarize, Contrast,} \\ \text{Brightness, Sharpness...} \end{array} \right\}$



$$\begin{aligned} \min_{\theta} \quad & \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(x_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N} \sum_{n=1}^{N} \Big[\max_{g \in \mathcal{G}} \text{Loss}\big(f_{\theta}(gx_n), y_n\big) \Big] \leq c \end{aligned}$$

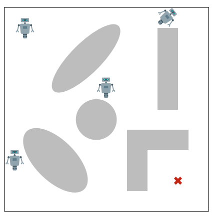
[Hounie, Chamon, Ribeiro, NeurIPS'23]

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- ...

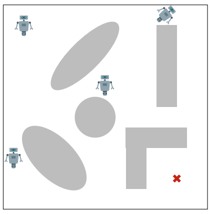
Safety

Problem
Find a control policy that navigates the environment effectively and safely



Safety

Problem
Find a control policy that navigates the environment effectively and safely

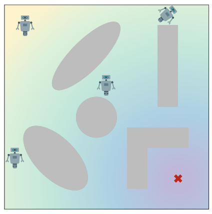


$$\begin{aligned} \underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} \quad & \text{Task reward} \\ \text{subject to} \quad & \mathbb{P}\left[\text{Colliding with } \mathcal{O}_i\right] \leq \delta, \\ & \text{for } i = 1, 2, \dots \end{aligned}$$

[Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

Safety

Problem
Find a control policy that navigates the environment effectively and safely

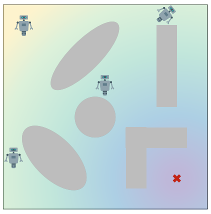


$$\begin{aligned} \underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} \quad & \mathbb{E}_{s,a \sim \pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t, a_t)\right] \\ \text{subject to} \quad & \mathbb{P}\left[\text{Colliding with } \mathcal{O}_i\right] \leq \delta, \\ & \text{for } i = 1, 2, \dots \end{aligned}$$

[Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

Safety

Problem
Find a control policy that navigates the environment effectively and safely



$$\begin{aligned} \underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} \quad & \mathbb{E}_{s,a \sim \pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t, a_t)\right] \\ \text{subject to} \quad & \mathbb{P}\left(\bigcap_{t=0}^{T-1} \{s_t \notin \mathcal{O}_i\} \,\middle|\, \pi\right) \geq 1 - \delta_i, \\ & \text{for } i = 1, 2, \dots \end{aligned}$$

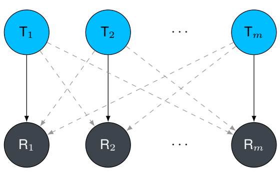
[Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
  (e.g., [Shen et al., ICLR'22; Hounie et al., NeurIPS'23])
- Adversarially robust learning
  (e.g., [Chamon et al., NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23])
- Safe learning
  (e.g., [Paternain et al., IEEE TAC'23])
- Wireless resource allocation
  (e.g., [Eisen et al., IEEE TSP'19; NaderiAlizadeh et al., IEEE TSP'22; Chowdhury et al., Asilomar'23])
- ...

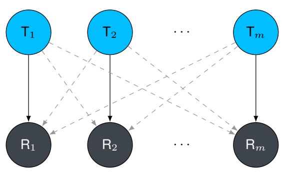
Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate



Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate



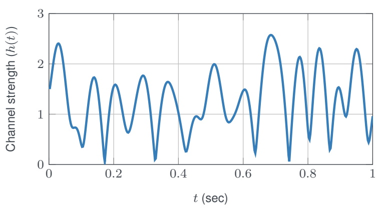
Input (p)

Channel (h)

Output $=$ Input $\times$ Channel $+$ Noise
$\Rightarrow$ Rate(p, h)

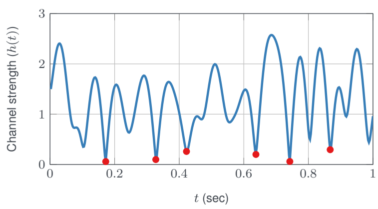
Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate



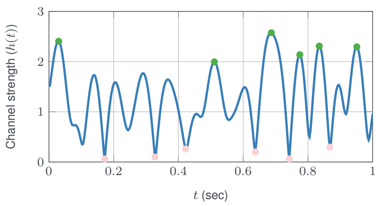
Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate



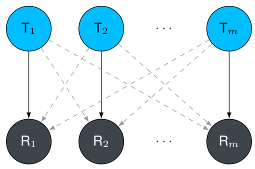
Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate



Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate

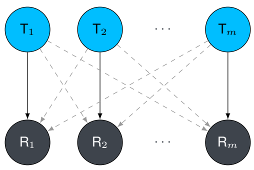


$$\min_{\pi \in \mathcal{P}(\mathcal{S})} \quad \text{Total transmit power}$$
$$\text{s. to} \quad \text{Rate } T_i \to R_i \geq c_i$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate

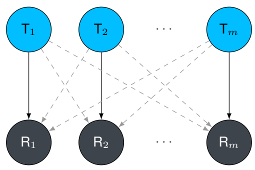


$$\max_{\pi \in \mathcal{P}(\mathcal{S})} \quad -\sum_{i=1}^{m} \mathbb{E}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h})}\left[\frac{1}{T}\sum_{t=0}^{T-1}\sum_{i=1}^{m} p_{i,t}\right]$$
$$\text{s. to} \quad \text{Rate } T_i \to R_i \geq c_i$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate the least transmit power to m devices to achieve a communication rate

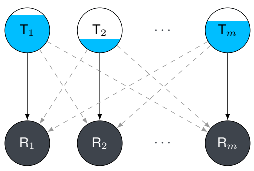


$$\max_{\pi \in \mathcal{P}(\mathcal{S})} \quad -\sum_{i=1}^{m} \mathbb{E}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h})}\left[\frac{1}{T}\sum_{t=0}^{T-1}\sum_{i=1}^{m} p_{i,t}\right]$$
$$\text{s. to} \quad \mathbb{E}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h})}\left[\frac{1}{T}\sum_{t=0}^{T-1}\text{Rate}_i(\boldsymbol{p}_t,\boldsymbol{h}_t)\right] \geq c_i$$

[Eisen, Zhang, Chamon, Lee, and Ribeiro, IEEE TSP'19]

Wireless resource allocation

Problem
Allocate power without depleting the battery of m devices to achieve a communication rate

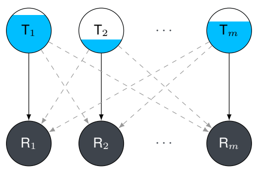


$$\min_{\pi \in \mathcal{P}(\mathcal{S})} \quad \text{Total probability of depleting battery}$$
$$\text{s. to} \quad \mathbb{E}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h},\boldsymbol{b})}\left[\frac{1}{T}\sum_{t=0}^{T-1}\text{Rate}_i(\boldsymbol{p}_t,\boldsymbol{h}_t)\right] \geq c_i$$

[Chowdhury, Paternain, Verma, Swami, Segarra, Asilomar'23]

Wireless resource allocation

Problem
Allocate power without depleting the battery of m devices to achieve a communication rate



$$\max_{\pi \in \mathcal{P}(\mathcal{S})} \quad -\sum_{i=1}^{m} \mathbb{P}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h},\boldsymbol{b})}\left[\bigcap_{t=0}^{T-1}\{b_{i,t}=0\}\right]$$
$$\text{s. to} \quad \mathbb{E}_{\boldsymbol{h},\boldsymbol{p}\sim\pi(\boldsymbol{h},\boldsymbol{b})}\left[\frac{1}{T}\sum_{t=0}^{T-1}\text{Rate}_i(\boldsymbol{p}_t,\boldsymbol{h}_t)\right] \geq c_i$$

[Chowdhury, Paternain, Verma, Swami, Segarra, Asilomar'23]

And many more...

- Precision, recall, churn (e.g., [Cotter et al., JMLR'19])
- Scientific priors (e.g., [Lu et al., SIAM J. Sci. Comp.'21; Moro and Chamon, arXiv'24])
- Continual learning (e.g., [Peng et al., ICML'23])
- Active learning (e.g., [Elenter et al., NeurIPS'22])
- Semi-supervised learning (e.g., [Cerviño et al., ICML'23])
- Minimum norm interpolation, SVM...

Constrained supervised learning

What is (un)constrained learning?

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \le u, \quad r = 1, \dots, N$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $(\boldsymbol{x}_n, y_n) \sim \mathfrak{D}$, $(\boldsymbol{x}_m, y_m) \sim \mathfrak{A}$, $(\boldsymbol{x}_r, y_r) \sim \mathfrak{P}$ (i.i.d.)

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c \quad \xrightarrow{?} \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \le u \qquad h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c \quad \xrightarrow{?} \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \le u \qquad h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c \quad \xrightarrow{?} \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \le u \qquad h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c \quad \xrightarrow{?} \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \le c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \le u \qquad h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \le u \text{ a.e.}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Agenda

Constrained learning theory

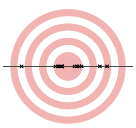
Constrained learning algorithms

Resilient constrained learning

What classical learning theory says?

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \quad \xrightarrow{\text{"LLN"}} \quad \min_{\boldsymbol{\theta}} \quad \mathbb{E}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

✓ $f_{\boldsymbol{\theta}}$ is *probably approximately correct (PAC)* learnable
e.g., linear functions, smooth functions (finite RKHS norm, bandlimited), NNs...
$(N \approx 1/\epsilon^2)$



[Rostamizadeh, Talwalkar, Mohri. Foundations of machine learning, 2012]; [Ben-David, Shalev-Shwartz. Understanding machine learning..., 2014]

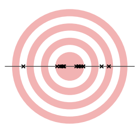
What's in a solution?

Definition (PAC learnability)

$f_{\boldsymbol{\theta}}$ is a *probably approximately correct (PAC)* learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\boldsymbol{\theta}^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal

$$P^\star - \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\Big] \le \epsilon$$



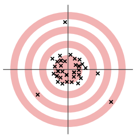
[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

What's in a solution?

Definition (PACC learnability)

$f_{\boldsymbol{\theta}}$ is a *probably approximately correct constrained (PACC)* learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\boldsymbol{\theta}^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal $$\left| P^\star - \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\left[\ell\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \right| \leq \epsilon$$
- approximately feasible $$\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \leq c + \epsilon$$



[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

When is constrained learning possible?

$$\begin{aligned} \hat{P}^\star = \min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c \end{aligned} \quad \xrightarrow{?} \quad \begin{aligned} P^\star = \min_{\boldsymbol{\theta}} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\left[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\left[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \leq c \end{aligned}$$

Proposition

$f_{\boldsymbol{\theta}}$ is PAC learnable $\not\Rightarrow$ $f_{\boldsymbol{\theta}}$ is PACC learnable

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

ECRM is not a PACC learner

Counter-example

$$\begin{aligned} P^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) \\ \text{subject to} \quad & \theta_2\, \mathbb{E}_\tau[\tau] \leq \theta_1 - 1 \\ & -\theta_1\, \mathbb{E}_\tau[\tau] \leq \theta_2 - 1 \end{aligned} \qquad J(\boldsymbol{\theta}) = \begin{cases} 1/16, & \boldsymbol{\theta} = [1/2, 1/2] \\ 1/8, & \boldsymbol{\theta} = [1, 1] \\ 1/4, & \boldsymbol{\theta} = [1, 0] \end{cases}$$

- $\tau \sim$ Uniform$\big(-1/2, 1/2\big)$

ECRM is not a PACC learner

Counter-example

$$\begin{aligned} P^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) = \frac{1}{8} \\ \text{subject to} \quad & \theta_2\, \mathbb{E}_\tau[\tau] \leq \theta_1 - 1 \Rightarrow \theta_1 \geq 1 \\ & -\theta_1\, \mathbb{E}_\tau[\tau] \leq \theta_2 - 1 \Rightarrow \theta_2 \leq 1 \end{aligned} \qquad J(\boldsymbol{\theta}) = \begin{cases} 1/16, & \boldsymbol{\theta} = [1/2, 1/2] \\ 1/8, & \boldsymbol{\theta} = [1, 1] \\ 1/4, & \boldsymbol{\theta} = [1, 0] \end{cases}$$

- $\tau \sim$ Uniform$\big(-1/2, 1/2\big)$

ECRM is not a PACC learner

Counter-example

$$\begin{aligned} P^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) = \frac{1}{8} \\ \text{subject to} \quad & \theta_2\, \mathbb{E}_\tau[\tau] \leq \theta_1 - 1 \Rightarrow \theta_1 \geq 1 \\ & -\theta_1\, \mathbb{E}_\tau[\tau] \leq \theta_2 - 1 \Rightarrow \theta_2 \leq 1 \end{aligned} \qquad J(\boldsymbol{\theta}) = \begin{cases} 1/16, & \boldsymbol{\theta} = [1/2, 1/2] \\ 1/8, & \boldsymbol{\theta} = [1, 1] \\ 1/4, & \boldsymbol{\theta} = [1, 0] \end{cases}$$

$$\begin{aligned} \hat{P}^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) \\ \text{subject to} \quad & \theta_2 \bar{\tau}_N \leq \theta_1 - 1 \\ & -\theta_1 \bar{\tau}_N \leq 1 - \theta_2 \end{aligned} \qquad \mathbb{P}\left[|\hat{P}^\star - P^\star| \leq 1/32\right] = \mathbb{P}\left[\bar{\tau}_N = 0\right] = 0$$

- $\tau \sim$ Uniform$\big(-1/2, 1/2\big)$ $\rightarrow \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^{N} \tau_n$

ECRM is not a PACC learner

Counter-example

$$\begin{aligned} P^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) = \frac{1}{8} \\ \text{subject to} \quad & \theta_2\, \mathbb{E}_\tau[\tau] \leq \theta_1 - 1 \Rightarrow \theta_1 \geq 1 \\ & -\theta_1\, \mathbb{E}_\tau[\tau] \leq \theta_2 - 1 \Rightarrow \theta_2 \leq 1 \end{aligned} \qquad J(\boldsymbol{\theta}) = \begin{cases} 1/16, & \boldsymbol{\theta} = [1/2, 1/2] \\ 1/8, & \boldsymbol{\theta} = [1, 1] \\ 1/4, & \boldsymbol{\theta} = [1, 0] \end{cases}$$

$$\begin{aligned} \hat{P}_r^\star = \min_{\boldsymbol{\theta}\in\Theta} \quad & J(\boldsymbol{\theta}) \\ \text{subject to} \quad & \theta_2 \bar{\tau}_N \leq \theta_1 - 1 + r_1 \\ & -\theta_1 \bar{\tau}_N \leq 1 - \theta_2 + r_2 \end{aligned} \qquad \mathbb{P}\left[|\hat{P}_r^\star - P^\star| \leq 1/32\right] \leq 4e^{-0.001N},$$

unless $\bar{\tau}_N \leq r_1 < \dfrac{\bar{\tau}_N + 1}{2}$ and $r_2 \geq \bar{\tau}_N$

- $\tau \sim$ Uniform$\big(-1/2, 1/2\big)$ $\rightarrow \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^{N} \tau_n$

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{\text{PAC}} \quad \begin{aligned} P^\star = \min_{\boldsymbol{\theta}} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\left[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\left[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \leq u \text{ a.e.} \end{aligned}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\begin{aligned} \hat{P}^\star = \min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ \text{subject to} \quad & \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \leq u \end{aligned} \quad \xrightarrow{\text{PAC}} \quad \begin{aligned} P^\star = \min_{\boldsymbol{\theta}} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\left[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\left[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\right] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \leq u \text{ a.e.} \end{aligned}$$

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Duality

PRIMAL

$\updownarrow$

DUAL

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

PRIMAL

$\updownarrow$

DUAL

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

PRIMAL

$\updownarrow$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

DUAL

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

PRIMAL

$\updownarrow$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

DUAL

- In general, $\hat{D}^\star \leq \hat{P}^\star$
- But in some cases, $\hat{D}^\star = \hat{P}^\star$ (strong duality) [e.g., convex optimization]

Duality

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

PRIMAL

$\not\updownarrow$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

DUAL

- In general, $\hat{D}^\star \leq \hat{P}^\star$
- ~~But in some cases, $\hat{D}^\star = \hat{P}^\star$ (strong duality) [e.g., convex optimization]~~

An alternative path

$$\hat{P}^\star = \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to} \quad \frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \leq c$$

PRIMAL $\not\longleftrightarrow$ DUAL

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\Big(\frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\Big)$$

$\not\updownarrow$ PAC

$$P^\star = \min_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{\boldsymbol{z}}\big[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\big] \quad \text{s. to} \quad \mathbb{E}_{\boldsymbol{z}}\big[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\big] \leq c$$

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

An alternative path

$$\hat{P}^\star = \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to} \quad \frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \leq c$$

PRIMAL $\not\longleftrightarrow$ DUAL

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta}\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\Big(\frac{1}{N}\sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\Big)$$

$\not\updownarrow$ PAC

$$P^\star = \min_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{\boldsymbol{z}}\big[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\big] \quad \text{s. to} \quad \mathbb{E}_{\boldsymbol{z}}\big[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\big] \leq c$$

$$\Big\downarrow \mathcal{H}_\theta \subset \mathcal{H}$$

$$\tilde{P}^\star = \min_{\phi\in\mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\big[\ell(\phi, \boldsymbol{z})\big] \quad \text{s. to} \quad \mathbb{E}_{\boldsymbol{z}}\big[g(\phi, \boldsymbol{z})\big] \leq c \quad \overset{?}{\longleftrightarrow} \quad \tilde{D}^\star = \max_{\lambda \geq 0} \min_{\phi\in\mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\big[\ell(\phi, \boldsymbol{z})\big] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\big[g(\phi, \boldsymbol{z})\big] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Non-convex variational duality

Convex optimization: Primal $\longleftrightarrow$ Dual

Non-convex, finite dimensional optimization: Primal $\not\longleftrightarrow$ Dual

Non-convex variational duality

Convex optimization: Primal $\longleftrightarrow$ Dual

Non-convex, finite dimensional optimization: Primal $\not\longleftrightarrow$ Dual

Non-convex, **infinite** dimensional optimization: Primal $\longleftrightarrow$ Dual

[Chamon, Eldar, Ribeiro, IEEE TSP'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Sparse logistic regression

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \boldsymbol{\theta}^T \boldsymbol{x}_n\right)\right]$$
$$\text{s. to } \|\boldsymbol{\theta}\|_0 = \sum_{t=1}^{p} \mathbb{I}\left[\boldsymbol{\theta}_t \neq 0\right] \leq k$$

Discrete, non-convex

[Chen et al., JMLR'19]: NP-hard

Sparse logistic regression

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \boldsymbol{\theta}^T \boldsymbol{x}_n\right)\right]$$
$$\text{s. to } \|\boldsymbol{\theta}\|_0 = \sum_{t=1}^{p} \mathbb{I}\left[\boldsymbol{\theta}_t \neq 0\right] \leq k$$

Discrete, non-convex

[Chen et al., JMLR'19]: NP-hard

$$\min_{\theta \in L_2} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \int \theta(t) x_n(t) dt\right)\right]$$
$$\text{s. to } \|\theta\|_{L_0} = \int \mathbb{I}\left[\theta(t) \neq 0\right] dt \leq \frac{k}{p}$$

Continuous, non-convex

[Chamon et al., IEEE TSP'20]: tractable

Sparse logistic regression

$$\min_{\boldsymbol{\theta} \in \mathbb{R}^p} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \boldsymbol{\theta}^T \boldsymbol{x}_n\right)\right]$$
$$\text{s. to } \|\boldsymbol{\theta}\|_0 = \sum_{t=1}^{p} \mathbb{I}\left[\boldsymbol{\theta}_t \neq 0\right] \leq k$$

Discrete, non-convex

[Chen et al., JMLR'19]: NP-hard

$$\min_{\theta \in L_2} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \int \theta(t) x_n(t) dt\right)\right]$$
$$\text{s. to } \|\theta\|_{L_0} = \int \mathbb{I}\left[\theta(t) \neq 0\right] dt \leq \frac{k}{p}$$

Continuous, non-convex

[Chamon et al., IEEE TSP'20]: tractable

An alternative path

PRIMAL

$$\hat{P}^\star = \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to } \frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \leq c$$

DUAL

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\left(\frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\right)$$

PAC

$$P^\star = \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \leq c$$

$$\tilde{P}^\star = \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] \leq c \quad \overset{=}{\longleftrightarrow} \quad \tilde{D}^\star = \max_{\lambda \geq 0} \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

An alternative path

PRIMAL

$$\hat{P}^\star = \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to } \frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \leq c$$

DUAL

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\left(\frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\right)$$

PAC

$$P^\star = \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \leq c \quad \overset{\epsilon_0}{\longleftrightarrow} \quad D^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] - c\right)$$

ϵ_0 $\qquad$ ϵ_0

$$\tilde{P}^\star = \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] \leq c \quad \overset{=}{\longleftrightarrow} \quad \tilde{D}^\star = \max_{\lambda \geq 0} \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

An alternative path

PRIMAL

$$\hat{P}^\star = \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \quad \text{s. to } \frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) \leq c$$

DUAL

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \frac{1}{N} \sum_{n=1}^{N} \ell(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) + \lambda\left(\frac{1}{N} \sum_{n=1}^{N} g(f_{\boldsymbol{\theta}}, \boldsymbol{z}_n) - c\right)$$

$O(\epsilon)$

PAC

$$P^\star = \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] \leq c \quad \overset{\epsilon_0}{\longleftrightarrow} \quad D^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \Theta} \mathbb{E}_{\boldsymbol{z}}\left[\ell(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\left[g(f_{\boldsymbol{\theta}}, \boldsymbol{z})\right] - c\right)$$

ϵ_0 $\qquad$ ϵ_0

$$\tilde{P}^\star = \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] \quad \text{s. to } \mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] \leq c \quad \overset{=}{\longleftrightarrow} \quad \tilde{D}^\star = \max_{\lambda \geq 0} \min_{\phi \in \mathcal{H}} \mathbb{E}_{\boldsymbol{z}}\left[\ell(\phi, \boldsymbol{z})\right] + \lambda\left(\mathbb{E}_{\boldsymbol{z}}\left[g(\phi, \boldsymbol{z})\right] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Dual (near-)PACC learning

Theorem

Let f be ν-universal, i.e., for each $\boldsymbol{\theta}_1$, $\boldsymbol{\theta}_2$, and $\gamma \in [0,1]$ there exists $\boldsymbol{\theta}$ such that

$$\mathbb{E}\left[\left|\gamma f_{\boldsymbol{\theta}_1}(\boldsymbol{x}) + (1-\gamma) f_{\boldsymbol{\theta}_2}(\boldsymbol{x}) - f_{\boldsymbol{\theta}}(\boldsymbol{x})\right|\right] \leq \nu$$

[$\{f_\theta\}$ is a good covering of $\overline{\operatorname{conv}}(\{f_\theta\})$]

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal, i.e., for each θ_1, θ_2, and $\gamma \in [0,1]$ there exists θ such that

$$\mathbb{E}\left[\left|\gamma f_{\theta_1}(\boldsymbol{x}) + (1-\gamma) f_{\theta_2}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})\right|\right] \le \nu$$

Then $\hat{D}^\star$ is a (near-)PACC learner, i.e., with probability $1-\delta$,

Near-optimal: $\left|P^\star - \hat{D}^\star\right| \le \tilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

(mild additional conditions apply)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal, i.e., for each θ_1, θ_2, and $\gamma \in [0,1]$ there exists θ such that

$$\mathbb{E}\left[\left|\gamma f_{\theta_1}(\boldsymbol{x}) + (1-\gamma) f_{\theta_2}(\boldsymbol{x}) - f_{\theta}(\boldsymbol{x})\right|\right] \le \nu$$

Then $\hat{D}^\star$ is a (near-)PACC learner, i.e., for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, with probability $1-\delta$,

Near-optimal: $\left|P^\star - \hat{D}^\star\right| \le \tilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

Approximately feasible: $\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + \tilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

(ℓ_0 strongly convex and g, h convex) $h\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big) \le r$, with $\mathfrak{P}$-prob. $1 - \tilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

(mild additional conditions apply)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$, ℓ_0 strongly convex, and g convex. Then, $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL) for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + (1+\Delta)^{3/2}\left(M\sqrt{\epsilon_0} + \epsilon\right)$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\boldsymbol{\lambda}^\star\|_1, \|\hat{\boldsymbol{\lambda}}^\star\|_1, \|\tilde{\boldsymbol{\lambda}}^\star\|_1\right)$$

Sources of error
parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$, ℓ_0 strongly convex, and g convex. Then, $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL) for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + (1+\Delta)^{3/2}\left(M\sqrt{\epsilon_0} + \epsilon\right)$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\boldsymbol{\lambda}^\star\|_1, \|\hat{\boldsymbol{\lambda}}^\star\|_1, \|\tilde{\boldsymbol{\lambda}}^\star\|_1\right)$$

Sources of error
parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$, ℓ_0 strongly convex, and g convex. Then, $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL) for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + (1+\Delta)^{3/2}\left(M\sqrt{\epsilon_0} + \epsilon\right)$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\boldsymbol{\lambda}^\star\|_1, \|\hat{\boldsymbol{\lambda}}^\star\|_1, \|\tilde{\boldsymbol{\lambda}}^\star\|_1\right)$$

Sources of error
parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$, ℓ_0 strongly convex, and g convex. Then, $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL) for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + (1+\Delta)^{3/2}\left(M\sqrt{\epsilon_0} + \epsilon\right)$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\boldsymbol{\lambda}^\star\|_1, \|\hat{\boldsymbol{\lambda}}^\star\|_1, \|\tilde{\boldsymbol{\lambda}}^\star\|_1\right)$$

Sources of error
parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$, ℓ_0 strongly convex, and g convex. Then, $f_{\boldsymbol{\theta}^\dagger}$ is a (near-)PACC solution of (P-CSL) for all $(\boldsymbol{\theta}^\dagger, \boldsymbol{\lambda}^\dagger)$ that achieve $\hat{D}^\star$, i.e., with probability at least $1-\delta$,

$$\left|P^\star - \hat{D}^\star\right| \le (1+\Delta)(\epsilon_0 + \epsilon)$$

$$\mathbb{E}\left[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\right] \le c + (1+\Delta)^{3/2}\left(M\sqrt{\epsilon_0} + \epsilon\right)$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\boldsymbol{\lambda}^\star\|_1, \|\hat{\boldsymbol{\lambda}}^\star\|_1, \|\tilde{\boldsymbol{\lambda}}^\star\|_1\right)$$

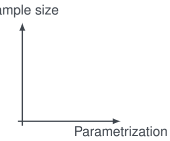
Sources of error
parametrization richness (ν) sample size (N) requirements difficulty ($\boldsymbol{\lambda}^\star$)

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23; Elenter, Chamon, Ribeiro, ICLR'24]

Dual learning trade-offs

- Unconstrained learning

  parametrization $\times$ sample size



[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

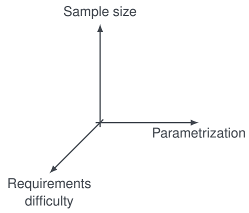
Dual learning trade-offs

- Unconstrained learning

  parametrization × sample size

- Constrained learning

  parametrization × sample size × requirements



[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

When is constrained learning possible?

Corollary

f_θ is PAC learnable $\approx^*$ f_θ is PACC learnable

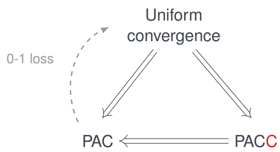
Constrained learning is **essentially as hard as** unconstrained learning

[mild conditions apply]

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

When is constrained learning possible?

Corollary



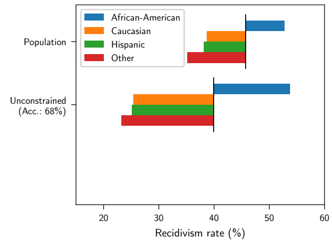
[mild conditions apply]

[Chamon and Ribeiro, NeurIPS'20; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness

Problem

Predict whether an individual will recidivate



*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.

Fairness: "Equality" of odds

Problem

Predict whether an individual will recidivate at the same rate across races

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \ \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(\boldsymbol{x}_n) = 1 \mid \text{Race}\right] \leq \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(\boldsymbol{x}_n) = 1\right] + c,$$

for Race $\in$ {African-American, Caucasian, Hispanic, Other}

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Cotter et al., JMLR'19; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: "Equality" of odds

Problem

Predict whether an individual will recidivate at the same rate across races

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \ \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(\boldsymbol{x}_n) = 1 \mid \text{Race}\right] \leq \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(\boldsymbol{x}_n) = 1\right] + c,$$

for Race $\in$ {African-American, Caucasian, Hispanic, Other}

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Cotter et al., JMLR'19; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: "Equality" of odds

Problem

Predict whether an individual will recidivate at the same rate across races

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \ \frac{1}{N}\sum_{n=1}^{N} \sigma\big(f_\theta(\boldsymbol{x}_n) - 0.5\big)\, \mathbb{I}\left[\boldsymbol{x}_n \in \text{Race}\right] \leq \frac{1}{N}\sum_{n=1}^{N} \sigma\big(f_\theta(\boldsymbol{x}_n) - 0.5\big) + c,$$

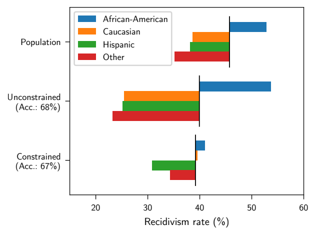
for Race $\in$ {African-American, Caucasian, Hispanic, Other}

*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Cotter et al., JMLR'19; Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: "Equality" of odds

Problem

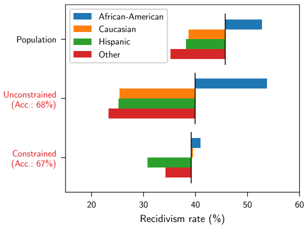
Predict whether an individual will recidivate at the same rate across races



*We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: “Equality” of odds

Problem
Predict whether an individual will recidivate at the same rate across races



*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: “Equality” of odds

Group	True label	Unconstrained: Prediction 0	Unconstrained: Prediction 1	Constrained: Prediction 0	Constrained: Prediction 1
African-American	0	31%	16%	36%	11%
African-American	1	16%	37%	23%	30%
Caucasian	0	52%	9%	44%	17%
Caucasian	1	23%	16%	16%	23%

*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: “Equality” of odds

Group	True label	Unconstrained: Prediction 0	Unconstrained: Prediction 1	Constrained: Prediction 0	Constrained: Prediction 1
African-American	0	31%	16%	36%	11%
African-American	1	16%	37%	23%	30%
Caucasian	0	52%	9%	44%	17%
Caucasian	1	23%	16%	16%	23%

*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Fairness: “Equality” of odds

Group	True label	Unconstrained: Prediction 0	Unconstrained: Prediction 1	Constrained: Prediction 0	Constrained: Prediction 1
African-American	0	31%	16%	36%	11%
African-American	1	16%	37%	23%	30%
Caucasian	0	52%	9%	44%	17%
Caucasian	1	23%	16%	16%	23%

*We say “Race” to follow the terminology used during the data collection of the COMPAS dataset.
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Constrained optimization methods

$$\hat{P}^\star = \min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \le c$$

$$h\big(f_\theta(x_r), y_r\big) \le u$$

Constrained optimization methods

$$\hat{P}^\star = \min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \le c$$

$$h\big(f_\theta(x_r), y_r\big) \le u$$

- Feasible update methods
  e.g., conditional gradients (Frank-Wolfe)
  - ✗ Tractability [non-convex constraints]
  - ✓ Feasible candidate solution
- Interior point methods
  e.g., barriers, projection, polyhedral approx.
  - ✗ Tractability [non-convex constraints]
  - ✓ Feasible candidate solution

Constrained optimization methods

$$\hat{P}^\star = \min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_\theta(x_m), y_m\big) \le c$$

$$h\big(f_\theta(x_r), y_r\big) \le u$$

- Feasible update methods
  e.g., conditional gradients (Frank-Wolfe)
  - ✗ Tractability [non-convex constraints]
  - ✓ Feasible candidate solution
- Interior point methods
  e.g., barriers, projection, polyhedral approx.
  - ✗ Tractability [non-convex constraints]
  - ✓ Feasible candidate solution
- Duality
  e.g., (augmented) Lagrangian
  - ✓ Tractability
  - ✓ (near-)feasible solution [small duality gap]

Dual learning algorithm

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Dual learning algorithm

- Minimize the primal ($\equiv$ **ERM**)

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \left[\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\right]$$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Dual learning algorithm

- Minimize the primal ($\equiv$ **ERM**)

$$\boldsymbol{\theta}^+ \approx \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)\Big], \quad n = 1, 2, \dots$$

[Haeffele et al., CVPR'17; Ge et al., ICLR'18; Mei et al., PNAS'18; Kawaguchi et al., AISTATS'20...]

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Dual learning algorithm

- Minimize the primal ($\equiv$ **ERM**)

$$\boldsymbol{\theta}^+ \approx \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)\Big], \quad n = 1, 2, \dots$$

- Update the dual

$$\lambda^+ = \left[\lambda + \eta\left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}^+}(\boldsymbol{x}_m), y_m\big) - c\right)\right]_+$$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

A (near-)PACC learner

Theorem
Suppose $\boldsymbol{\theta}^\dagger$ is a ρ-approximate solution of the regularized ERM:

$$\boldsymbol{\theta}^\dagger \approx \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \Big(\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\Big).$$

Then, after $T = \left\lceil \frac{\|\boldsymbol{\lambda}^\star\|^2}{2\eta M \nu} \right\rceil + 1$ dual iterations with step size $\eta \leq \frac{2\epsilon}{mB^2}$,

the iterates $\big(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\big)$ are such that

$$\Big|P^\star - L\Big(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\Big)\Big| \leq (2 + \Delta)(\epsilon_0 + \epsilon) + \rho$$

with probability $1 - \delta$ over sample sets.

[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

In practice...

- Minimize the primal ($\equiv$ **ERM**)

$$\boldsymbol{\theta}^+ \approx \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)\Big], \quad n = 1, 2, \dots$$

- Update the dual

$$\lambda^+ = \left[\lambda + \eta\left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}^+}(\boldsymbol{x}_m), y_m\big) - c\right)\right]_+$$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

In practice...

- Minimize the primal ($\equiv$ **ERM**)

$$\boldsymbol{\theta}^+ = \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)\Big], \quad n = 1, 2, \dots, N$$

- Update the dual

$$\lambda^+ = \left[\lambda + \eta\left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}^+}(\boldsymbol{x}_m), y_m\big) - c\right)\right]_+$$

$$\hat{D}^\star = \max_{\lambda \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda \left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_{\boldsymbol{\beta}} \Big[\ell\big(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\big) + \lambda_{t-1} g\big(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\big)\Big]$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_n\big) - c\right)\right]_+$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$

SGD

Dual update

PyTorch
https://github.com/lfochamon/csl

In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_\beta \left[\ell\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right) + \lambda_{t-1} g\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right)\right]$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_m\right) - c\right)\right]_+$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$

Use adaptive method (e.g., ADAM)

PyTorch
`https://github.com/lfochamon/csl`

In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_\beta \left[\ell\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right) + \lambda_{t-1} g\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right)\right]$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_m\right) - c\right)\right]_+$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$

Use adaptive method (e.g., ADAM)
Use different time-scales ($\eta_\lambda = 0.1\eta_\theta$)

PyTorch
`https://github.com/lfochamon/csl`

In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_\beta \left[\ell\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right) + \lambda_{t-1} g\left(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\right)\right]$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_m\right) - c\right)\right]_+$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$

Check slack:
- feasibility: $s_t \leq 0$
- "duality gap": $\lambda_t s_t$

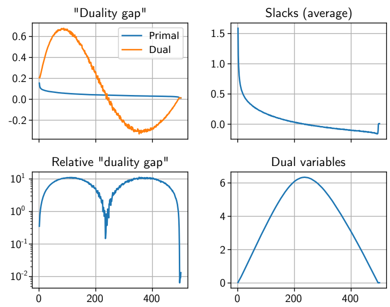
$$s_t = \frac{1}{N}\sum_{n=1}^{N} g\left(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_n), y_n\right) - c$$

Use adaptive method (e.g., ADAM)
Use different time-scales ($\eta_\lambda = 0.1\eta_\theta$)

PyTorch
`https://github.com/lfochamon/csl`

In practice...

Initialize: $\boldsymbol{\theta}_0, \lambda_0$
for $t = 1, \dots, T$
  $\boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
  for $n = 1, \dots, N$
    $\boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n$
  end
  $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
  $\lambda_t = \left[\lambda_{t-1} + \right.$
end
Output: $\boldsymbol{\theta}_T, \lambda_T$



ethod (e.g., ADAM)
ne-scales ($\eta_\lambda = 0.1\eta_\theta$)

rch
`https://github.com/lfochamon/csl`

Penalty-based vs. dual learning

Penalty-based learning

$$\boldsymbol{\theta}^\dagger \in \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\ \text{Loss}(\boldsymbol{\theta}) + \lambda \cdot \text{Penalty}(\boldsymbol{\theta})$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss + λPenalty

Dual learning

$$\boldsymbol{\theta}^\dagger \in \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\ \text{Loss}(\boldsymbol{\theta}) + \lambda \cdot \text{Penalty}(\boldsymbol{\theta})$$

$$\lambda^+ = \left[\lambda + \eta\left(\text{Penalty}(\boldsymbol{\theta}^\dagger) - c\right)\right]_+$$

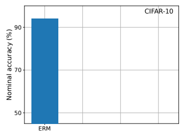
- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\leq c$

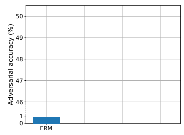
Robust learning

Problem
Learn an accurate classifier that is robust to input perturbations

Cello

Hammer





Adversarial training

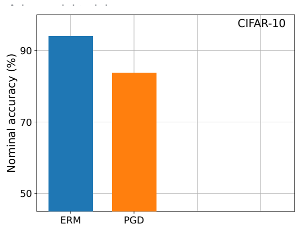
Problem
Learn an accurate classifier that is robust to input perturbations

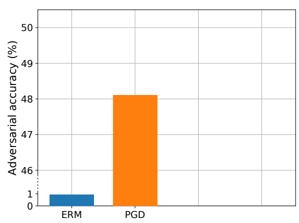
- Adversarial training [Szegedy et al., ICLR'14; Goodfellow et al., ICLR'15; Madry et al., ICLR'18; ...]

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) \longrightarrow \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \leq \epsilon} \text{Loss}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\right)\right]$$

Adversarial training

Problem
Learn an accurate classifier that is robust to input perturbations





[Robey*, Chamon*, Pappas, Hassani, Ribeiro, NeurIPS'21]

Adversarial training

Problem
Learn an accurate classifier that is robust to input perturbations

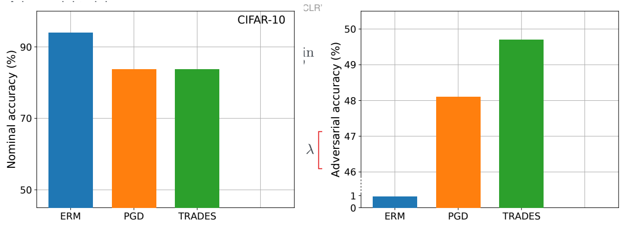
- Adversarial training [Szegedy et al., ICLR'14; Goodfellow et al., ICLR'15; Madry et al., ICLR'18; ...]

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \leq \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda \left[\max_{\|\boldsymbol{\delta}\|_\infty \leq \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

Adversarial training

Problem
Learn an accurate classifier that is robust to input perturbations



[Zhang et al., ICML'19]

Constrained learning for robustness

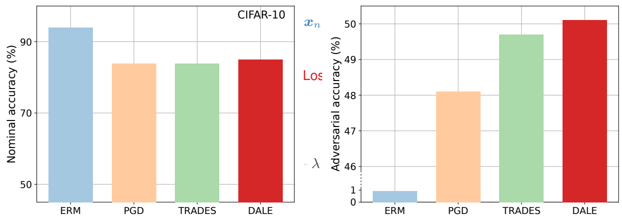
Problem
Learn an accurate classifier that is robust to input perturbations

$$\begin{aligned} &\min_{\boldsymbol{\theta}} && \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \\ &\text{subject to} && \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\boldsymbol{\delta}\|_\infty \leq \epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right] \leq c \end{aligned}$$

[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Constrained learning for robustness

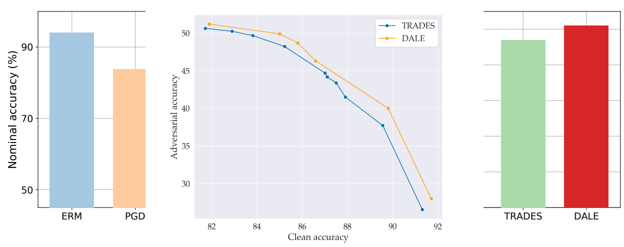
Problem
Learn an accurate classifier that is robust to input perturbations



[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

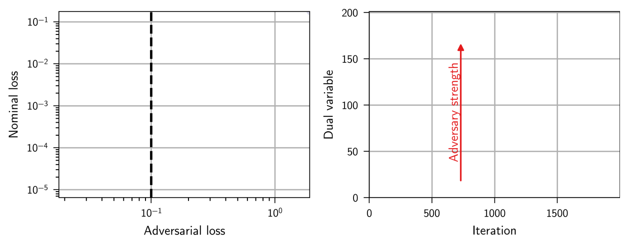
Constrained learning for robustness

Problem
Learn an accurate classifier



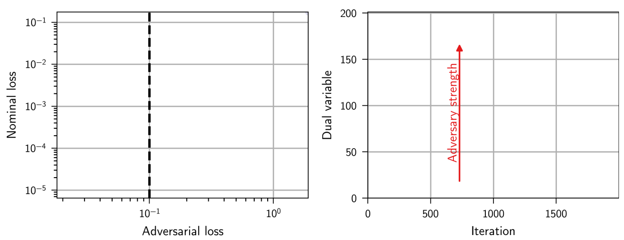
[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Constrained learning for robustness



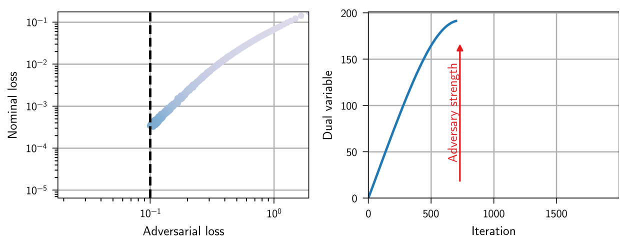
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Constrained learning for robustness



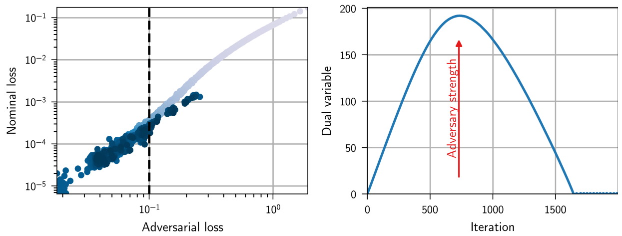
[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

Constrained learning for robustness

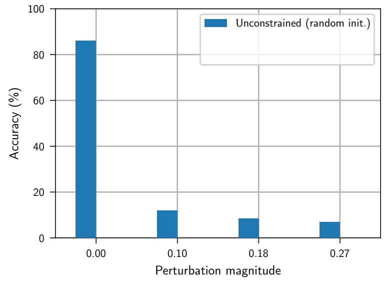


[Chamon, Paternain, Calvo-Fullana, Ribeiro, IEEE TIT'23]

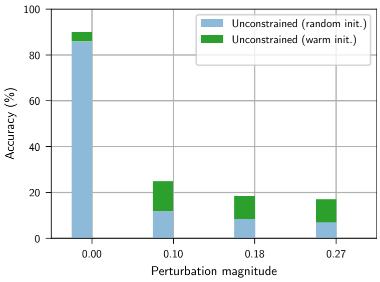
Constrained learning for robustness



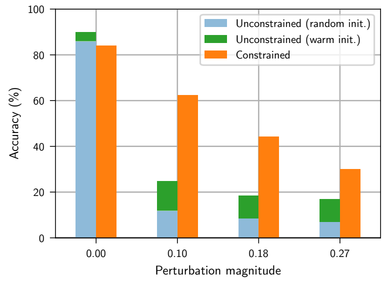
Constrained learning for robustness



Constrained learning for robustness



Constrained learning for robustness



Penalty-based vs. dual learning

Penalty-based learning

$$\theta^\dagger \in \operatorname*{argmin}_{\theta} \ \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss $+\ \lambda$Penalty

Dual learning

$$\theta^\dagger \in \operatorname*{argmin}_{\theta} \ \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

$$\lambda^+ = \Big[\lambda + \eta\Big(\text{Penalty}(\theta^\dagger) - c\Big)\Big]_+$$

- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\leq c$

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Heterogeneous federated learning

Problem
Learn a common model using data from K clients that is good for all clients

$$\min_{\theta} \ \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta)$$
$$\text{subject to} \ \ \text{Loss}_k(f_\theta) \leq \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta) + c$$
$$k = 1, \dots, K$$

- k-th client loss: $\text{Loss}_k(\phi) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_\theta(x_{n_k}), y_{n_k}\big)$

Heterogeneous federated learning

Problem
Learn a common model using data from K clients that is good for all clients

$$\min_{\theta} \ \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta)$$
$$\text{subject to} \ \ \text{Loss}_k(f_\theta) \leq \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta) + c_k$$
$$k = 1, \dots, K$$

- k-th client loss: $\text{Loss}_k(\phi) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_\theta(x_{n_k}), y_{n_k}\big)$

Resilient constrained learning

Definition (Resilience)
(ecology) ability of an ecosystem to adapt its function to accommodate operating conditions

Resilient constrained learning

Definition (Resilience)
~~(ecology)~~ ability of an ~~ecosystem~~ to adapt its ~~function~~ to accommodate ~~operating conditions~~
(learning) learning system specification data properties

Resilient constrained learning

Definition (Resilience)
~~(ecology)~~ ability of an ~~ecosystem~~ to adapt its ~~function~~ to accommodate ~~operating conditions~~
(learning) learning system specification data properties

$$\begin{aligned} P^\star = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\mathsf{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i \end{aligned}$$

Resilient constrained learning

Definition (Resilience)
~~(ecology)~~ ability of an ~~ecosystem~~ to adapt its ~~function~~ to accommodate ~~operating conditions~~
(learning) learning system specification data properties

$$\begin{aligned} P^\star(\boldsymbol{r}) = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\mathsf{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i + r_i \end{aligned}$$

Resilient constrained learning

Definition (Resilience)
~~(ecology)~~ ability of an ~~ecosystem~~ to adapt its ~~function~~ to accommodate ~~operating conditions~~
(learning) learning system specification data properties

$$\begin{aligned} P^\star(\boldsymbol{r}) = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\mathsf{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i + r_i \end{aligned}$$

- Larger relaxations $\boldsymbol{r}$ decrease the objective $P^\star(\boldsymbol{r})$ (benefit),
  but increase specification violation $c_i + r_i$ (cost)

Resilient constrained learning

Definition (Resilience)
~~(ecology)~~ ability of an ~~ecosystem~~ to adapt its ~~function~~ to accommodate ~~operating conditions~~
(learning) learning system specification data properties

$$\begin{aligned} P^\star(\boldsymbol{r}) = \min_{\theta} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\mathsf{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i + r_i \end{aligned}$$

- Larger relaxations r decrease the objective $P^\star(\boldsymbol{r})$ (benefit),
  but increase specification violation $c_i + r_i$ (cost)
- Resilience is a compromise!

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \qquad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

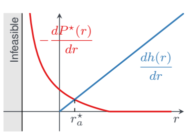
[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \qquad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing



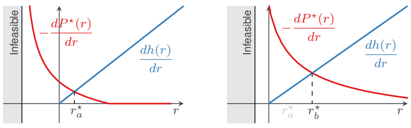
[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \quad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing



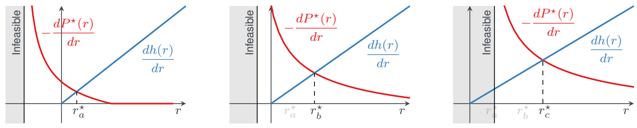
[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \quad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing



[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) = \boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

- After relaxing, $\boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$ is *smaller* than $\boldsymbol{\lambda}^\star(\mathbf{0})$
  $\Rightarrow$ Resilient constrained learning "generalizes better" (lower sample complexity)

[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) = \boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

- After relaxing, $\boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$ is *smaller* than $\boldsymbol{\lambda}^\star(\mathbf{0})$
  $\Rightarrow$ Resilient constrained learning "generalizes better" (lower sample complexity)
- The resilient equilibrium *exists and is unique* (because h is strictly convex)

[Hounie, Chamon, Ribeiro, NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)
For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

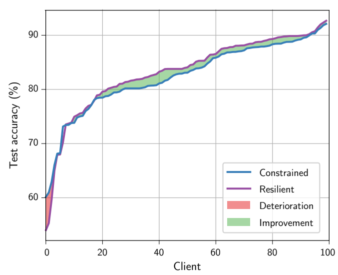
$$\begin{aligned} P^\star(\boldsymbol{r}^\star) = \min_{\boldsymbol{\theta},\boldsymbol{r}} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + h(\boldsymbol{r}) \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i + r_i \end{aligned}$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

- After relaxing, $\boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$ is *smaller* than $\boldsymbol{\lambda}^\star(\mathbf{0})$
  $\Rightarrow$ Resilient constrained learning "generalizes better" (lower sample complexity)
- The resilient equilibrium *exists and is unique* (because h is strictly convex)

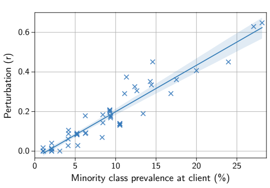
[Hounie, Chamon, Ribeiro, NeurIPS'23]

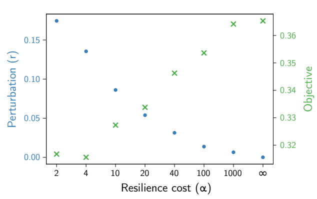
Heterogeneous federated learning



[Hounie, Chamon, Ribeiro, NeurIPS'23]

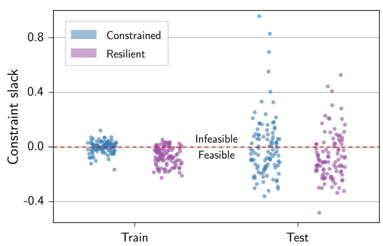
Heterogeneous federated learning





[Hounie, Chamon, Ribeiro, NeurIPS'23]

Heterogeneous federated learning



[Hounie, Chamon, Ribeiro, NeurIPS'23]

Summary

- **Constrained learning is the a tool to learn under requirements**

- **Constrained learning is hard. . .**

- **. . . but possible. How?**

Summary

- **Constrained learning is the a tool to learn under requirements**
  Constrained learning imposes generalizable requirements organically during training, e.g., fairness [Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23], heterogeneity [Shen et al., ICRL'22]. . .

- **Constrained learning is hard. . .**

- **. . . but possible. How?**

Summary

- **Constrained learning is the a tool to learn under requirements**
  Constrained learning imposes generalizable requirements organically during training, e.g., fairness [Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23], heterogeneity [Shen et al., ICRL'22]. . .

- **Constrained learning is hard. . .**
  Constrained, non-convex, statistical optimization problem

- **. . . but possible. How?**

Summary

- **Constrained learning is the a tool to learn under requirements**
  Constrained learning imposes generalizable requirements organically during training, e.g., fairness [Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23], heterogeneity [Shen et al., ICRL'22]. . .

- **Constrained learning is hard. . .**
  Constrained, non-convex, statistical optimization problem

- **. . . but possible. How?**
  We can learn under requirements (essentially) whenever we can learn at all by solving *(penalized) ERM problems.* Resilient learning can then be used to adapt the requirements to the task difficulty [Hounie et al., NeurIPS'23]

Agenda

I. Constrained supervised learning
   - Constrained learning theory
   - Constrained learning algorithms
   - Resilient constrained learning

   Break (10 min)

II. Constrained reinforcement learning
   - Constrained RL duality
   - Constrained RL algorithms

   Q&A and discussions

`https://luizchamon.com/imprs2024`