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AAAI tutorial
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supervised and reinforcement learning under requirements

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Constrained reinforcement learning

Agenda

Constrained reinforcement learning

CMDP duality

Primal-Dual algorithms, state augmentation, guarantees

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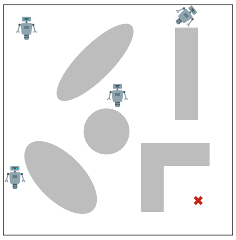
CMDP duality

Primal-Dual algorithms, state augmentation, guarantees

Safe navigation

Problem
Find a control policy that navigates the environment effectively and safely



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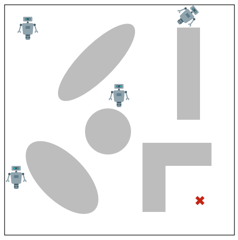

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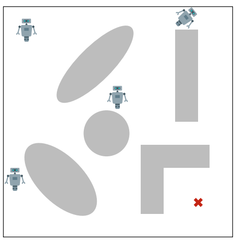

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Find a control policy that navigates the environment effectively and safely

- CBFs, artificial potentials, MPC
  [Koditschek et al., AAM'90; Mayne et al., Autom.'00; Wieland et al., IFAC'07. . .]
  - ✖ knowledge of dynamical system
- System identification
  [Deistler et al., Autom.'95; Tsiamis et al., CDC'19; Dean et al., FCM'19. . .]
  - ✖ "consistency" guarantees for linear systems

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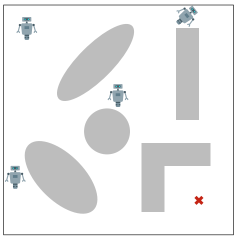
- RL
  [Bertsekas & Tsitsiklis'96; Sutton & Barto'18; Bertsekas'19...]



Reinforcement learning

- Model-free framework for decision-making in Markovian settings

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$$\Pr\left(s_{t+1} \mid \{s_u, a_u\}_{u \leq t}\right) = \Pr\left(s_{t+1} \mid s_t, a_t\right) = p(s_{t+1} \mid s_t, a_t)$$

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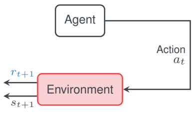

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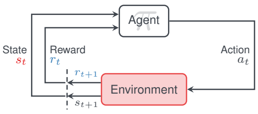

$$\underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} \ V(\pi) \triangleq \mathbb{E}_{s,a \sim \pi}\left[\frac{1}{T}\sum_{t=0}^{T} \gamma^t r(s_t, a_t)\right] \qquad \text{(P-RL)}$$

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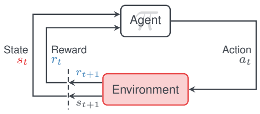

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- (P-RL) can be solved using policy gradient and/or Q-learning type algorithms
  [W'92, WD'92, BT'96, KT'00, JFEPF'14, HKSC'15, NFPIY'15, AJFR'17, PP'18, SB'18, B'19, KCP'19...]

Constrained RL

$$\begin{aligned} \underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} \quad & V_0(\pi) \triangleq \mathbb{E}_{s,a \sim \pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t, a_t)\right] \\ \text{subject to} \quad & V_i(\pi) \triangleq \mathbb{E}_{s,a \sim \pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_i(s_t, a_t)\right] \geq c_i, \quad i = 1, \ldots, m \end{aligned} \qquad \text{(P-CRL)}$$

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[Altman'99; Achiam et al., ICML'17; Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19; Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23...]

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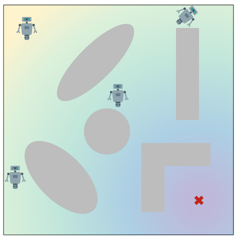

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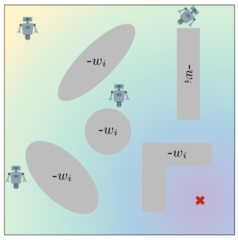
$$r(s,a) = \underbrace{-\|s - s_{\text{goal}}\|^2}_{r_0} + \sum_{i=1}^{5} \underbrace{w_i \mathbb{I}(s_t \in \mathcal{O}_i)}_{r_i}$$



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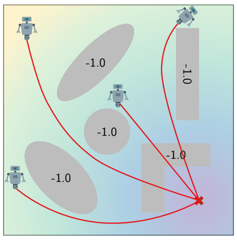

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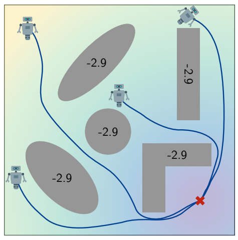

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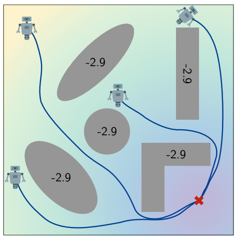
  ✖ "consistency" guarantees for linear systems
- RL with reward shaping
  [Bertsekas & Tsitsiklis'96; Sutton & Barto'18; Bertsekas'19...]
  ✖ weak guarantee



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$$\begin{aligned} &\underset{\pi \in \mathcal{P}(\mathcal{S})}{\text{maximize}} && \text{Task reward} \\ &\text{subject to} && \Pr\left(\text{Not colliding with } \mathcal{O}_i\right) \geq 1 - \delta, \quad i = 1, 2, \dots \end{aligned}$$

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- Probabilistic version of control invariant sets

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\end{aligned}$$

- Probabilistic version of control invariant sets
- Constraint tightening: $\Pr\left(\bigcap_{t=0}^{T-1} \mathcal{E}_t\right) \geq 1 - \delta \Longleftarrow \sum_{t=0}^{T-1} \Pr(\mathcal{E}_t) \geq T - \delta$

[Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

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CRL methods

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\end{aligned}$$

- Reward shaping ≈ penalty methods
  - ✗ Manual, time-consuming, domain-dependent
  - ✗ Trade-offs, training plateaux
- Prior knowledge ≈ projection methods
  e.g., safe exploration [Berkenkamp et al., NeurIPS'17, Dalal et al., arXiv'18]
  - ✗ Requires set of safe actions or safe policies
  - ✗ Intractable projections
- Linearization and convex surrogates
  e.g., CPO [Achiam et al., ICML'17]
  - ✗ No approximation guarantee
  - ✗ Approximate problem may be infeasible

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- Linearization and convex surrogates
  e.g., CPO [Achiam et al., ICML'17]
- Duality
  [Bhatnagar et al., JOTA'12; Tesler et al., ICRL'19; PCCR, NeurIPS'19; Ding et al., NeurIPS'20; PCCR, IEEE TAC'23 . . .]
  - ✓ Domain independent
  - ✓ Tractable
  - ✗ Approximation guarantee [non-convexity]

Agenda

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CMDP duality

Primal-Dual algorithms, state augmentation, guarantees

Duality

DUAL

↕

PRIMAL

Duality

DUAL

↕

PRIMAL

$$P^\star = \max_{\pi \in \mathcal{P}(\mathcal{S})} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t,a_t)\right] \text{ subject to } \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_1(s_t,a_t)\right] \geq 0$$

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$$D^\star = \min_{\lambda \geq 0} \overbrace{\max_{\pi \in \mathcal{P}(\mathcal{S})} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t,a_t)\right] + \lambda\, \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_1(s_t,a_t)\right]}^{L(\pi,\lambda)}$$

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- No hyperparameters to be tuned in the problem $\Rightarrow$ Domain Independent
- Equivalent to solving a sequence a unconstrained RL problems $\Rightarrow$ Tractable

Duality

$$D^\star = \min_{\lambda\succeq 0}\max_{\pi\in\mathcal{P}(\mathcal{S})} \overbrace{\mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_0(s_t,a_t)\right] + \lambda\,\mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right]}^{L(\pi,\lambda)}$$

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Strong duality of CRL

Theorem (Paternain, Chamon, Calvo-Fullana, Ribeiro'19)
If there exists $\pi^\dagger \in \mathcal{P}(\mathcal{S})$ such that $V_i(\pi^\dagger) > c_i$ for all $i = 1,\dots,m$, then $D^\star = P^\star$.

[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19; Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

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- **Non-proof:** There is an equivalent linear program

(P-CRL) $\equiv$ LP : $\rho_\pi(s,a) = \dfrac{1-\gamma}{1-\gamma^T}\displaystyle\sum_{t=0}^{T-1}\gamma^t \Pr_\pi\big(s_t = s, a_t = a\big) \longleftrightarrow \pi(a|s) = \dfrac{\rho_\pi(s,a)}{\int_{\mathcal{A}}\rho_\pi(s,a)da}$

$$V(\pi) = \mathbb{E}_{s,a\sim\pi}\left[\sum_{t=0}^{T-1}\gamma^t r(s_t,a_t)\right] \quad\propto\quad \mathbb{E}_{(s,a)\sim\rho_\pi}\big[r(s,a)\big] = \int_{\mathcal{S}\times\mathcal{A}} r(s,a)\rho_\pi(s,a)\,dsda$$

$$\begin{array}{ll} \underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} & V_0(\pi) \\ \text{subject to} & V_i(\pi) \geq c_i \end{array} \quad\equiv\quad \begin{array}{ll} \underset{\rho\in\mathcal{P}}{\text{maximize}} & \mathbb{E}_{(s,a)\sim\rho}\big[r_0(s,a)\big] \\ \text{subject to} & \mathbb{E}_{(s,a)\sim\rho}\big[r_i(s,a)\big] \geq \bar{c}_i \end{array}$$

(strongly dual)

[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19; Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

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If there exists $\pi^\dagger \in \mathcal{P}(\mathcal{S})$ such that $V_i(\pi^\dagger) > c_i$ for all $i = 1,\dots,m$, then $D^\star = P^\star$.

- **Non-proof:** There is an equivalent linear program

$\not\Leftarrow$ (P-CRL) $\equiv$ LP : $\rho_\pi(s,a) = \dfrac{1-\gamma}{1-\gamma^T}\displaystyle\sum_{t=0}^{T-1}\gamma^t \Pr_\pi\big(s_t = s, a_t = a\big) \longleftrightarrow \pi(a|s) = \dfrac{\rho_\pi(s,a)}{\int_{\mathcal{A}}\rho_\pi(s,a)da}$

$$V(\pi) = \mathbb{E}_{s,a\sim\pi}\left[\sum_{t=0}^{T-1}\gamma^t r(s_t,a_t)\right] \quad\propto\quad \mathbb{E}_{(s,a)\sim\rho_\pi}\big[r(s,a)\big] = \int_{\mathcal{S}\times\mathcal{A}} r(s,a)\rho_\pi(s,a)\,dsda$$

$$\begin{array}{ll} \underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} & V_0(\pi) \\ \text{subject to} & V_i(\pi) \geq c_i \end{array} \quad\equiv\quad \begin{array}{ll} \underset{\rho\in\mathcal{P}}{\text{maximize}} & \mathbb{E}_{(s,a)\sim\rho}\big[r_0(s,a)\big] \\ \text{subject to} & \mathbb{E}_{(s,a)\sim\rho}\big[r_i(s,a)\big] \geq \bar{c}_i \end{array}$$

~~(strongly dual)~~ $\not\Leftarrow$ (strongly dual)

[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19; Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

Counterexample (1)

- Consider the following equivalent optimization problems

$$\begin{array}{rl} P^\star = \max\limits_{x} & -x \\ \text{subject to} & x^2 - 1 \geq 0 \\ & x \geq 0 \end{array} \quad\equiv\quad \begin{array}{rl} P^\star_{LP} = \max\limits_{x} & -x \\ \text{subject to} & x - 1 \geq 0 \end{array}$$

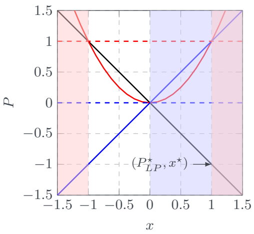
- They have the same objective and the same feasible set $x \geq 1 \Rightarrow$ Equivalent problems

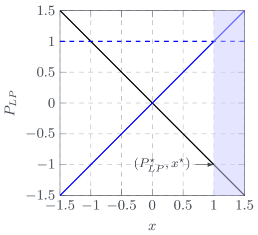
$$x^\star = 1, \qquad P^\star = P^\star_{LP} = -1$$

- Problem P_{LP} is convex (Linear Program) $\Rightarrow$ Zero duality gap
- Problem P is not convex $\Rightarrow$ Zero duality gap?

Counterexample (2)

$$P^\star = \max_x \; -x \quad \text{subject to} \quad x^2 - 1 \geq 0, \quad x \geq 0 \qquad \equiv \qquad P_{LP}^\star = \max_x \; -x \quad \text{subject to} \quad x - 1 \geq 0$$





Counterexample (3)

- Let us solve the dual problem of the LP first

$$P_{LP}^\star = \max_x \; -x \qquad = -1 \qquad \text{subject to} \quad x - 1 \geq 0$$

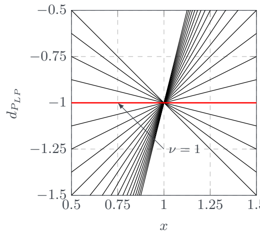
- The dual function is ($\nu \geq 0$)

$$d_{P_{LP}}(\nu) = \max_x -x + \nu(x-1) = \begin{cases} -1 & \nu = 1 \\ \infty & \text{if } \nu \neq 1 \end{cases}$$

- The solution to the dual problem is

$$D_{LP}^\star = \min_{\nu \geq 0} d_{P_{LP}}(\nu) = -1$$

- We have $D_{LP}^\star = P_{LP}^\star \Rightarrow$ no duality gap



Counterexample (4)

- Let us solve the dual problem of the non-convex problem

$$P^\star = \max_x \; -x = \qquad -1 \qquad \text{subject to} \quad x^2 - 1 \geq 0, \quad x \geq 0$$

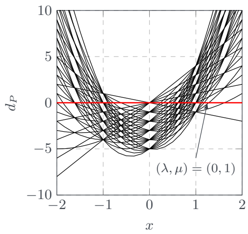
- The dual function is ($\lambda, \mu \geq 0$)

$$d_P(\lambda, \mu) = \max_x -x + \lambda(x^2 - 1) + \mu x = \begin{cases} 0 & \text{if } \lambda = 0, \mu = 1 \\ \infty & \text{otherwise} \end{cases}$$

- The solution to the dual problem is

$$D_P^\star = \min_{\lambda, \mu \geq 0} d_P(\lambda, \mu) = 0$$

- We have $D_{LP}^\star \neq P_{LP}^\star \Rightarrow$ There is duality gap



Proof outline

- The proof of the result is based on geometric arguments

$$P^\star \triangleq \max_{\pi \in \mathcal{P}(\mathcal{S})} \; V_0(\pi) \triangleq \mathbb{E}_{s,a\sim\pi}\left[\sum_{t=0}^{\infty} \gamma^t r_0(s_t, a_t)\right]$$
$$\text{subject to} \quad V_i(\pi) \triangleq \mathbb{E}_{s,a\sim\pi}\left[\sum_{t=0}^{\infty} \gamma^t r_i(s_t, a_t)\right] \geq c_i, i = 1, \ldots, m.$$

- Define the epigraph set

$$\mathcal{C} = \left\{\xi \in \mathbb{R}^{m+1} \mid \exists \pi \text{ s.t.} V_i(\pi) \geq \xi_i \text{ for all } i = 0, \ldots, m\right\}$$

  - The set $\mathcal{C}$ is convex $\Rightarrow$ Zero duality gap follows the same arguments as in convex optimization
  - The supporting hyper-plane at $(P^\star, \mathbf{0})$ is defined by the optimal Lagrange multipliers

$$P^\star + \sum_{i=1}^m \lambda_i \mathbf{0} \geq \xi_0 + \sum_{i=1}^m \lambda_i \xi_i \geq V_0(\pi) + \sum_{i=1}^m \lambda_i V_i(\pi)$$

  - This implies strong duality $P^\star \geq D^\star$

Dual CRL

$$D^\star = \min_{\lambda \succeq 0} \max_{\pi \in \mathcal{P}(\mathcal{S})} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t, a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_1(s_t, a_t)\right]\right)$$

DUAL

- $D^\star = P^\star$ (strong duality) [despite non-convexity]

Dual CRL

$$D^\star = \min_{\lambda \succeq 0} \max_{\pi \in \mathcal{P}(\mathcal{S})} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t, a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_1(s_t, a_t)\right]\right)$$

DUAL

- $D^\star = P^\star$ (strong duality) [despite non-convexity]
- Infinite dimensionality of $\mathcal{P}(\mathcal{S})$

Dual CRL

$$D_\theta^\star = \min_{\lambda \succeq 0} \max_{\theta \in \Theta} \mathbb{E}_{s,a\sim\pi_\theta}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_0(s_t, a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi_\theta}\left[\frac{1}{T}\sum_{t=0}^{T-1} \gamma^t r_1(s_t, a_t)\right]\right)$$

DUAL

- $D^\star = P^\star$ (strong duality) [despite non-convexity]
- ~~Infinite dimensionality of $\mathcal{P}(\mathcal{S})$~~ Finite dimensional parametrization π_θ

Dual CRL

Theorem (Paternain, Chamon, Calvo-Fullana, Ribeiro'19)

Let π_θ be ν-universal, i.e.,

$$\min_{\theta \in \Theta} \max_{s \in \mathcal{S}} \int_{\mathcal{A}} \left|\pi(a|s) - \pi_\theta(a|s)\right| da \leq \nu, \text{ for all } \pi \in \mathcal{P}(\mathcal{S}).$$

Then,

$$\left|P^\star - D_\theta^\star\right| \leq \frac{1 + \|\lambda_\nu^\star\|_1}{1 - \gamma} B\nu$$

Alternative: $|P_\theta^\star - D_\theta^\star|$ can be bounded using ν-universality only over $\pi \in \overline{\text{conv}}\left(\{\pi_\theta | \theta \in \Theta\}\right)$

[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19; Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

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Sources of error

parametrization richness (ν) requirements difficulty ($\lambda_\nu^\star$) horizon (γ)

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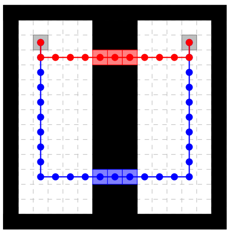
Grid world example

- Consider a grid world with a safe and an unsafe bridge
  - Only two potentially optimal policies depending on the cost of crossing each bridge



[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19]

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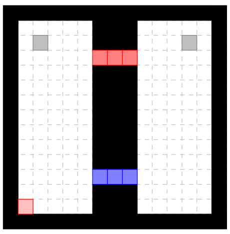

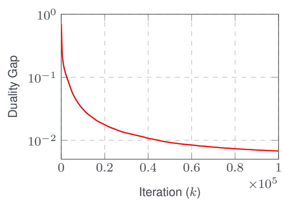

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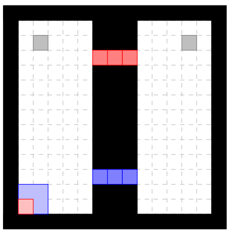

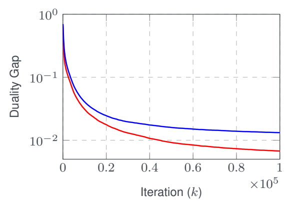

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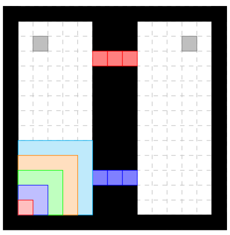

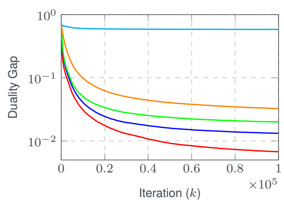

[Paternain, Chamon, Calvo-Fullana, Ribeiro, NeurIPS'19]

Dual CRL

$$D_\theta^\star = \min_{\lambda\succeq 0} \max_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_0(s_t,a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right]\right)$$

- $D^\star = P^\star$ (strong duality) [despite non-convexity]

- Infinite dimensionality of $\mathcal{P}(\mathcal{S})$ Finite dimensional parametrization $\pi_{\boldsymbol{\theta}}$

  $\pi_{\boldsymbol{\theta}}$ is ν-universal $\Rightarrow \left|P^\star - D_\theta^\star\right| \leq O(\nu)$

Agenda

Constrained reinforcement learning

CMDP duality

Primal-Dual algorithms, state augmentation, guarantees

Primal-dual algorithm

$$D_\theta^\star = \min_{\lambda\succeq 0}\max_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_0(s_t,a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right] - c_1\right)$$

Primal-dual algorithm

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$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_\lambda(s_t,a_t)\right]$$

$$r_\lambda(s,a) = r_0(s,a) + \lambda r_1(s,a)$$

Primal-dual algorithm

$$D_\theta^\star = \min_{\lambda\succeq 0}\max_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_0(s_t,a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right] - c_1\right)$$

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$$r_\lambda(s,a) = r_0(s,a) + \lambda r_1(s,a)$$

- Update the dual ($\equiv$ policy evaluation)

$$\lambda^+ = \left[\lambda - \eta\left(\mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}^\dagger}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right] - c_1\right)\right]_+$$

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In practice...

$$D_\theta^\star = \min_{\lambda\succeq 0}\max_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_0(s_t,a_t)\right] + \lambda\left(\mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t)\right] - c_1\right)$$

- Maximize the primal ($\equiv$ vanilla RL): $\{s_t, a_t\} \sim \pi_{\boldsymbol{\theta}_k}$

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \eta\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_\lambda(s_t,a_t)\right]\nabla_{\boldsymbol{\theta}}\log\left(\pi_{\boldsymbol{\theta}}(a_0|s_0)\right)$$

- Update the dual ($\equiv$ policy evaluation): $\{s_t, a_t\} \sim \pi_{\boldsymbol{\theta}_{k+1}}$

$$\lambda^+ = \left[\lambda - \eta\left(\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_1(s_t,a_t) - c_1\right)\right]_+$$

Dual CRL

Theorem

Suppose $\boldsymbol{\theta}^\dagger$ is a ρ-approximate solution of the regularized RL problem:

$$\boldsymbol{\theta}^\dagger \approx \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} \mathbb{E}_{s,a\sim\pi_{\boldsymbol{\theta}}}\left[\frac{1}{T}\sum_{t=0}^{T-1}\gamma^t r_\lambda(s_t,a_t)\right].$$

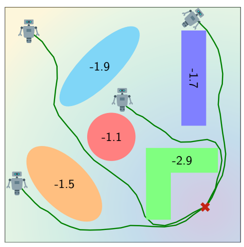
Then, after $K = \left\lceil \frac{\|\boldsymbol{\lambda}^\star\|^2}{2\eta\nu} \right\rceil + 1$ dual iterations with step size $\eta \leq \frac{1-\gamma}{mB}$,

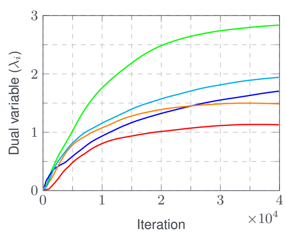
the iterates $\left(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\right)$ are such that

$$\left|P^\star - L\left(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\right)\right| \leq \frac{1 + \|\boldsymbol{\lambda}_\nu^\star\|_1}{1-\gamma} B\nu + \rho$$

[Paternain, C., Calvo-Fullana, and Ribeiro, NeurIPS'19; C. and Ribeiro, NeurIPS'20; C., Paternain, Calvo-Fullana, and Ribeiro, IEEE TIT'23]

Safe navigation





[Paternain, Calvo-Fullana, Chamon, Ribeiro, IEEE TAC'23]

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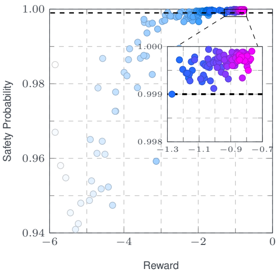

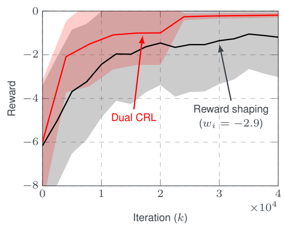

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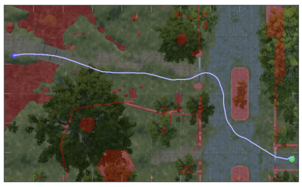

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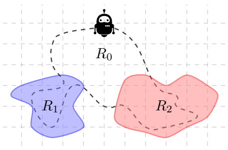
Monitoring task

Problem
Find a control policy that maximizes the time in R_0
while monitoring R_1 and R_2 at least $1/3$ of the time each



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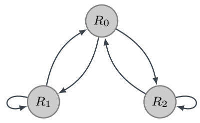

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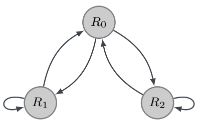

- ✓ $\pi^\star =$ draw actions uniformly at random

[Calvo-Fullana, Paternain, Chamon, Ribeiro, IEEE TAC'23]

Monitoring task

Problem
Find a control policy that maximizes the time in R_0
while monitoring R_1 and R_2 at least $1/3$ of the time each

$$\max_{\pi \in \mathcal{P}(\mathcal{S})} \lim_{T\to\infty} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} \mathbb{I}\big(s_t \in R_0\big)\right]$$
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$$\max_{\pi \in \mathcal{P}(\mathcal{S})} \lim_{T\to\infty} \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_{\boldsymbol{\lambda}}(s_t)\right]$$
$$r_{\boldsymbol{\lambda}}(s) = \mathbb{I}\big(s \in R_0\big) + \lambda_1 \mathbb{I}\big(s \in R_1\big) + \lambda_2 \mathbb{I}\big(s \in R_2\big)$$

- ✓ $\pi^\star =$ draw actions uniformly at random

- ✓ $\lambda_1 = \lambda_2 = 1$: all $\pi \in \mathcal{P}(\mathcal{S})$ are optimal
- ✗ $\lambda_1, \lambda_2 < 1$: $\pi^\star$ s.t. $\Pr\left[s \in R_0\right] = 1/2$
- ✗ $\lambda_i > 1$ and $\lambda_i > \lambda_j$: $\pi^\star$ s.t. $\Pr\left[s \in R_i\right] = 1$

[Calvo-Fullana, Paternain, Chamon, Ribeiro, IEEE TAC'23]

So CRL is hard?

- There are tasks that CRL can tackle and RL cannot

$$\begin{array}{ll} \max\limits_{\pi \in \mathcal{P}(\mathcal{S})} & V_0(\pi) \\ \text{subject to} & V_i(\pi) \geq c_i \end{array} \quad \supsetneq \quad \max_{\pi \in \mathcal{P}(\mathcal{S})} V(\pi)$$

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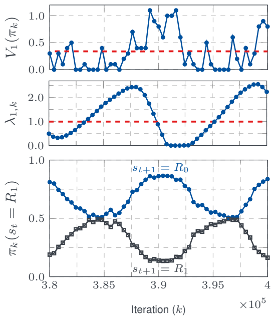
- Dual CRL cannot solve all CRL problems

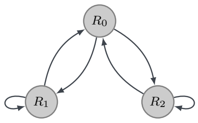
Theorem (Paternain, Chamon, Calvo-Fullana, Ribeiro'19)
If $\pi_{\boldsymbol{\theta}}$ is ν-universal, then $|P^\star - D_\theta^\star| \leq O(\nu)$.

$$\implies \exists\, \boldsymbol{\theta}^\dagger \in \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmax}}\ V_0(\pi_{\boldsymbol{\theta}}) + \sum_{i=1}^m \lambda_i^\star V_i(\pi_{\boldsymbol{\theta}})$$ that is approximately feasible.

[Calvo-Fullana, Paternain, Chamon, Ribeiro, IEEE TAC'23]

So CRL is hard?





Primal recovery

- General issue with duality
  - (Primal-)dual methods: $f(\boldsymbol{\theta}_k) \not\to f(\boldsymbol{\theta}^\star)$ but $\frac{1}{K}\sum_{k=0}^{K-1} f(\boldsymbol{\theta}_k) \to f(\boldsymbol{\theta}^\star)$

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- Convex optimization ⇒ dual averaging
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- Non-convex optimization ⇒ randomization
  - $\boldsymbol{\theta}^\dagger \sim$ Uniform$(\boldsymbol{\theta}_k) \Rightarrow \mathbb{E}\left[f(\boldsymbol{\theta}^\dagger)\right] = \frac{1}{K}\sum_{k=0}^{K-1} f(\boldsymbol{\theta}_k) \to f(\boldsymbol{\theta}^\star)$

Intuition

$$\begin{cases} \boldsymbol{\theta}_k \in \operatorname*{argmax}_{\boldsymbol{\theta}\in\Theta} V_{\lambda_k}(\pi_{\boldsymbol{\theta}}), \quad V_\lambda(\pi) = V_0(\pi) + \lambda V_1(\pi) \\ \lambda_{k+1} = \left[\lambda_k - \eta\big(V_1(\pi_{\boldsymbol{\theta}_k}) - c_1\big)\right]_+ \end{cases}$$

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⇒ ~~Randomization:~~ $\theta^\dagger \sim$ Uniform$(\boldsymbol{\theta}_k) \Rightarrow \mathbb{E}\left[V_i(\pi_{\boldsymbol{\theta}^\dagger})\right] = \frac{1}{K}\sum_{k=0}^{K-1} V_i(\pi_{\boldsymbol{\theta}^\dagger}) \to V_i(\pi_{\boldsymbol{\theta}^\star})$

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State augmentation

- Construct a new MDP based on *known* state space $\mathcal{M}$ and transition kernel q:

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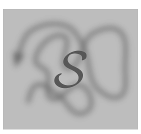

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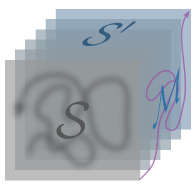

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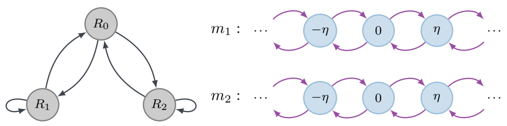
  - e.g., $\mathcal{M} = \mathbb{R}^2$ and $m_{i,t+1} = m_{i,t} + \eta\big[\mathbb{I}(s_t = R_i) - \mathbb{I}(s_t \neq R_i)\big]$



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- In general, it is not clear…
  - …*how many* and *which* states to augment ($\mathcal{M}$)
  - …*what dynamics* these states should follow (q)

…to guarantee optimality and feasibility

Intuition: State-augmented CRL

$$\begin{cases} \boldsymbol{\theta}_k \in \underset{\boldsymbol{\theta}\in\Theta}{\operatorname{argmax}}\ V_{\lambda_k}(\pi_{\boldsymbol{\theta}}), \quad V_\lambda(\pi) = V_0(\pi) + \lambda V_1(\pi) \\ \lambda_{k+1} = \Big[\lambda_k - \eta\big(V_1(\pi_{\boldsymbol{\theta}_k}) - c_1\big)\Big]_+ \end{cases}$$

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Online $\quad \lambda_{k+1} = \Big[\lambda_k - \eta\big(V_1(\pi_{\boldsymbol{\theta}_k}) - c_1\big)\Big]_+$

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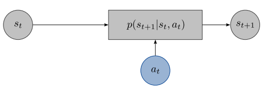
State-augmented CRL



[Calvo-Fullana, Paternain, Chamon, Ribeiro, IEEE TAC'23]

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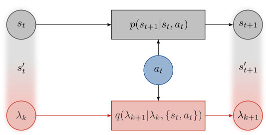

State space: $\mathcal{M} = \{\lambda\} \Rightarrow s' = (s, \lambda)$

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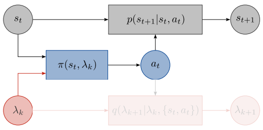

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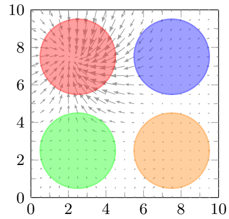
Dynamics: $\lambda_{i,k+1} = \left[\lambda_{i,k} - \eta\left(V_i(\pi) - c_i\right)\right]_+$

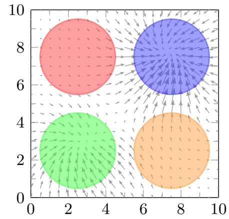
- **Training** (offline)
  - Train policy against $r(s',a) = r_0(s,a) + \sum_{i=1}^m \lambda_i r_1(s,a)$ with static λ (no dynamics)

$$\equiv \pi^\dagger(\lambda) \in \underset{\pi \in \mathcal{P}(\mathcal{S})}{\operatorname{argmax}}\ V_0(\pi) + \sum_{i=1}^m \lambda_i \left(V_i(\pi) - c_i\right)$$

Monitoring task



Policy learned for
$\lambda = [5, 0, 0, 0]$



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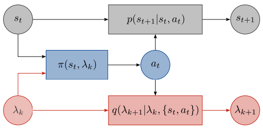

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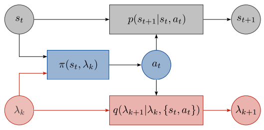
- **Training** (offline) $\Rightarrow \pi^\dagger(\lambda) \approx \underset{\pi \in \mathcal{P}(\mathcal{S})}{\operatorname{argmax}}\ V_0(\pi) + \sum_{i=1}^m \lambda_i V_i(\pi)$
- **Execution** (online)
  - Execute $\pi^\dagger(\cdot|s,\lambda_k)$ for fixed horizon T_0 and use stochastic approximation of λ-dynamics

$$\lambda_{i,k+1} = \left[\lambda_{i,k} - \eta\left(\frac{1}{T_0}\sum_{\tau=0}^{T_0-1} r_{i,\tau} - c_i\right)\right]_+$$

State-augmented CRL



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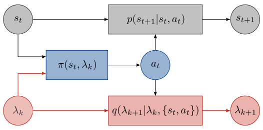

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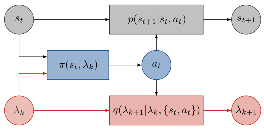
State-augmented CRL



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- It is systematic: no *ad hoc* state augmentation
- Accommodates online modifications of requirements: trained policy does not depend on c
- It works

State-augmented CRL

Theorem (Calvo-Fullana, Paternain, Chamon, Ribeiro'23)

State-augmented CRL generates *state-action sequences* $\{(s_t,a_t)\}$ that are almost surely feasible

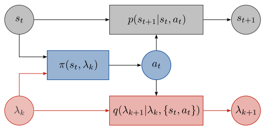
$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} r_i(s_t,a_t) \geq c_i \text{ a.s., for all } i,$$

and near-optimal

$$\lim_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t,a_t)\right] \geq P^\star - \frac{\eta B^2}{2}$$

(mild conditions apply)

State-augmented CRL



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- It is systematic: no *ad hoc* state augmentation
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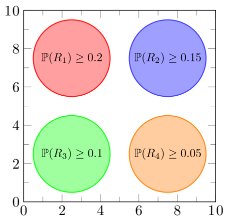
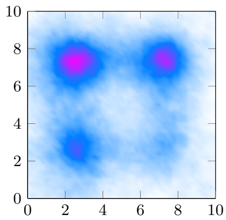
- It works
  - Does not find a policy $\Rightarrow$ generates trajectories during execution that solve (P-CRL)

Monitoring task





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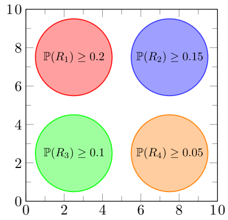

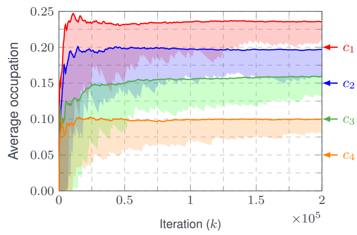

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Summary

- **Constrained RL is** the **a tool for decision making under requirements**
- **Constrained RL is hard...**
- **...but possible. How?**

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  CRL is a natural way of specifying complex behaviors that precludes fine tuning of rewards, e.g., safety
  [Paternain et al., IEEE TAC'23]
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  Although strong duality holds for CRL (despite non-convexity),
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  that is not always enough to obtain feasible solutions $\Rightarrow$ (P-RL) $\subsetneq$ (P-CRL)
- **...but possible. How?**
  When combined with a *systematic state augmentation* technique,
  we can use policies that solve (P-RL) to solve (P-CRL)

Agenda

I. Constrained supervised learning

II. Robustness-constrained learning

Break (30 min)

III. Constrained reinforcement learning

`https://luizchamon.com/aaai`

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AAAI tutorial
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supervised and reinforcement learning under requirements