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AAAI tutorial
Feb. 20, 2023

supervised and reinforcement learning under requirements

Agenda

I. Constrained supervised learning

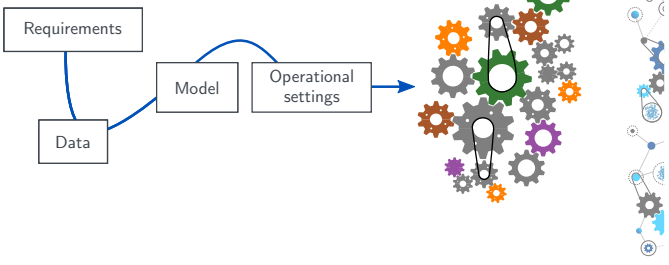
II. Robustness-constrained learning

Break (30 min)

III. Constrained reinforcement learning

https://luizchamon.com/aaai

Why requirements?



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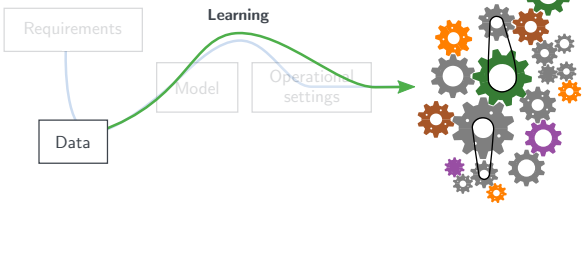

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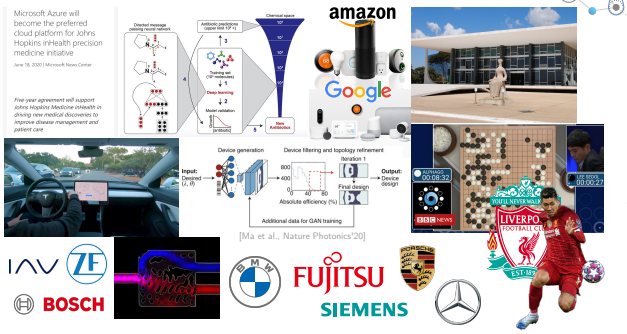

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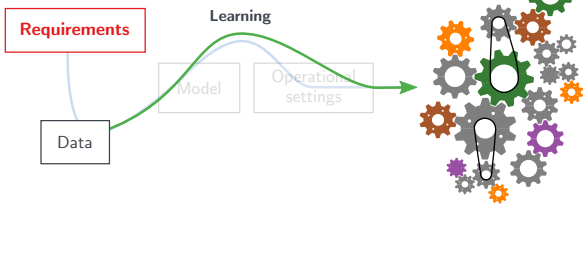

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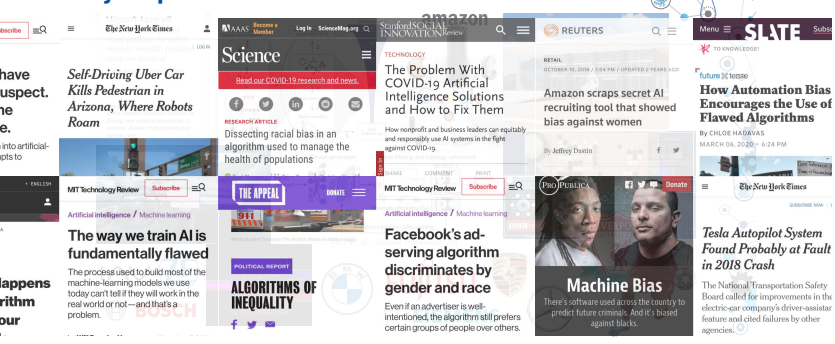
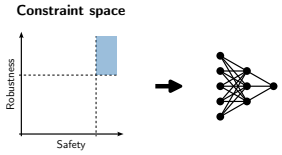


What is a requirements?

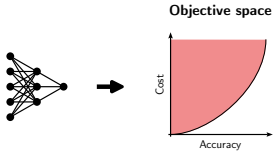
- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide



[NASA. "Systems engineering handbook," 2019]

What is a requirements?

- **Requirements are "shall" statements:** describe *necessary* features subject to verification
  - *Constraint space*: things we decide
- **Goals are "should" statements:** express recommendations (once "shall" statements are satisfied)
  - *Objective space*: things the system achieves



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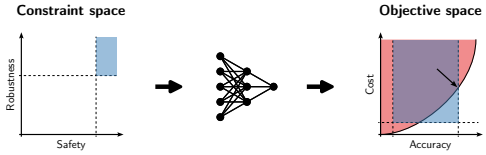

[NASA. "Systems engineering handbook," 2019]

What is (un)constrained learning?

$$P_{\mathrm{U}}^{\star} = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}$, $\mathfrak{A}$, $\mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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What about penalties?

$$\begin{aligned} P^{\star} = \min_{\boldsymbol{\theta}} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \\ \text{subject to} \quad & \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] \leq c \\ & h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big) \leq u, \quad \mathfrak{P}\text{-a.e.} \end{aligned}$$

$$\Updownarrow$$

$$\min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \lambda\,\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{P}}\Big[\mu(\boldsymbol{x},y)h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

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NON-CONVEX

$$\Updownarrow$$

$$\min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \lambda\,\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{P}}\Big[\mu(\boldsymbol{x},y)h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big]$$

- There need not exist (λ,μ) for which the penalized solution is optimal *and* feasible
- Even if such (λ,μ) exist, they are not easy to find (hyperparameter search, cross-validation...)
- Constrained learning yields better guarantees, better performance, better trade-offs...

Applications

- Fairness
  (e.g., [Goh et al., NeurIPS'16; Kearns et al., ICML'18; Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23])
- Federated learning
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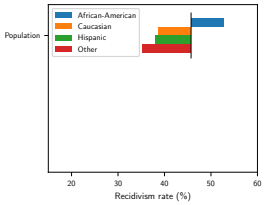
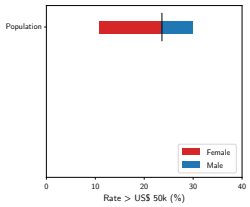
- ...

Fairness

Problem
Predict whether an individual will recidivate



Problem
Predict whether an individual makes > $50k



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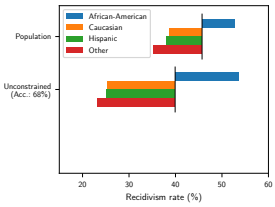

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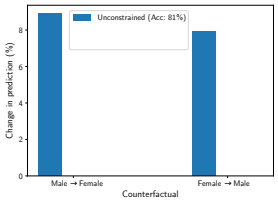

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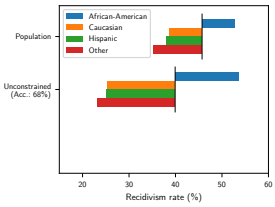

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Fairness: "Equality" of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\begin{aligned} \min_{\theta} \quad & \text{Prediction error} \\ \text{subject to} \quad & \text{Prediction rate disparity (Race)} \leq c, \\ & \text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\} \end{aligned}$$

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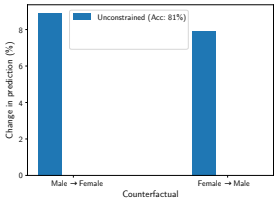

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Counterfactual fairness

Problem
Predict whether an individual makes > $50k while being invariant to gender

$$\begin{aligned} \min_{\theta} \quad & \text{Prediction error} \\ \text{subject to} \quad & \text{Change in prediction } (\rho x) \leq c \quad \text{a.e.} \\ & (\rho : \text{Male} \leftrightarrow \text{Female}) \end{aligned}$$

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Federated learning

Problem
Learn a common model using data using data distributed among K clients

$$\min_{\theta} \quad \text{Average loss across clients}$$

- k-th client loss: $\text{Loss}_k(\phi) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_\theta(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

[Shen et al., ICRL'22]

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$$\min_{\theta} \quad \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_\theta)$$

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Heterogeneous federated learning

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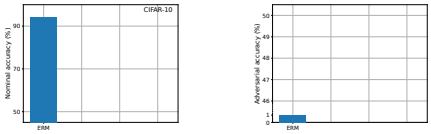
Robustness

Problem
Learn a classifier that is robust to input perturbations



Robustness

Problem
Learn a classifier that is robust to input perturbations

$$\min_{\theta} \quad \text{Nominal loss}$$
$$\text{subject to} \quad \text{Adversarial loss} \leq c$$

Robustness

Problem
Learn a classifier that is robust to input perturbations

$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)$$
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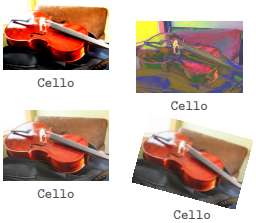
Invariance

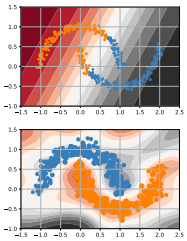
Problem
Learn a classifier that is invariant to transformation $g \in \mathcal{G}$



$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)$$
$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \left[\max_{g \in \mathcal{G}} \text{Loss}\big(f_\theta(g x_n), y_n\big)\right] \leq c$$

Smoothness

Problem
Learn a classifier that is smooth, i.e., Lipschitz continuous (on a manifold)



$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)$$
$$\text{subject to} \ \max_{x\in\mathcal{M}} \|\nabla_{\mathcal{M}} f_\theta(x)\|^2 \le L$$

[Cerviño et al., ICML'23]

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Safety

Problem
Find a control policy that navigates the environment effectively and safely

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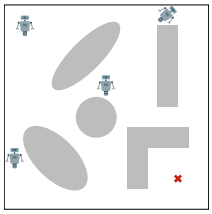
Find a control policy that navigates the environment effectively and safely

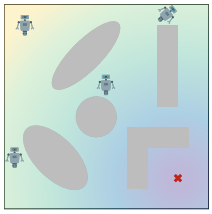


$$\underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} \quad \text{Task reward}$$
$$\text{subject to} \quad \Pr\left[\text{Colliding with } \mathcal{O}_i\right] \le \delta_i, \quad \text{for } i = 1, 2, \dots$$

[Paternain et al., IEEE TAC'23]

Safety

Problem
Find a control policy that navigates the environment effectively and safely



$$\underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} \quad \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t, a_t)\right]$$
$$\text{subject to} \quad \Pr\left[\text{Colliding with } \mathcal{O}_i\right] \le \delta_i, \quad \text{for } i = 1, 2, \dots$$

[Paternain et al., IEEE TAC'23]

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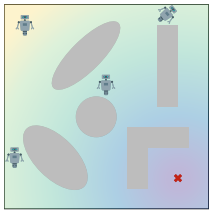

$$\underset{\pi\in\mathcal{P}(\mathcal{S})}{\text{maximize}} \quad \mathbb{E}_{s,a\sim\pi}\left[\frac{1}{T}\sum_{t=0}^{T-1} r_0(s_t, a_t)\right]$$
$$\text{subject to} \quad \Pr\left(\bigcap_{t=0}^{T-1} \{s_t \notin \mathcal{O}_i\} \,\middle|\, \pi\right) \ge 1 - \delta_i, \quad \text{for } i = 1, 2, \dots$$

[Paternain et al., IEEE TAC'23]

And many more...

- Precision, recall, churn (e.g., [Cotter et al., JMLR'19])
- Scientific priors (e.g., [Lu et al., SIAM J. Sci. Comp.'21])
- Wireless resource allocation (e.g., [Eisen et al., IEEE TSP'19])
- Continual learning (e.g., [Peng et al., ICML'23])
- Active learning (e.g., [Elenter et al., NeurIPS'22])
- Semi-supervised learning (e.g., [Cerviño et al., ICML'23])
- Minimum norm interpolation, SVM...

Constrained supervised learning

What is (un)constrained learning?

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$
$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$
$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \leq u, \quad r = 1, \dots, N$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $(\boldsymbol{x}_n, y_n) \sim \mathfrak{D}$, $(\boldsymbol{x}_m, y_m) \sim \mathfrak{A}$, $(\boldsymbol{x}_r, y_r) \sim \mathfrak{P}$ (i.i.d.)

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

What is (un)constrained learning?

$$P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$
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$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big) \leq u, \quad \mathfrak{P}\text{-a.e.}$$

- ℓ, g are bounded, Lipschitz continuous (possibly non-convex) functions
- $f_{\boldsymbol{\theta}}$ is a (possibly nonlinear) parametrization [e.g., logistic classifier, (G)(C)NN]
- $\mathfrak{D}, \mathfrak{A}, \mathfrak{P}$ unknown

[Chamon et al., IEEE ICASSP'20 (best student paper); Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

Constrained learning challenges

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \qquad P^\star = \min_{\boldsymbol{\theta}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$
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Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?

Constrained learning challenges

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Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

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1) *Statistical*: does the solution of the constrained empirical problem generalize?
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Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Constrained learning challenges

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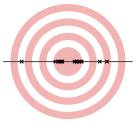
Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

What classical learning theory says?

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \xrightarrow{\text{"LLN"}} \min_{\boldsymbol{\theta}} \mathbb{E}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big]$$

- $f_{\boldsymbol{\theta}}$ is *probably approximately correct (PAC)* learnable
  e.g., linear functions, smooth functions (finite RKHS norm, bandlimited), NNs... ($N \approx 1/\epsilon^2$)



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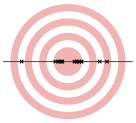
- Requirements?

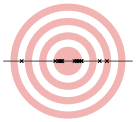


What's in a solution?

Definition (PAC learnability)

$f_{\boldsymbol{\theta}}$ is a *probably approximately correct (PAC)* learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\boldsymbol{\theta}^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal
$$P^\star - \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\Big] \leq \epsilon$$

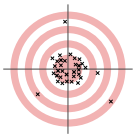


What's in a solution?

Definition (PACC learnability)

$f_{\boldsymbol{\theta}}$ is a *probably approximately correct constrained (PACC)* learnable if for every ϵ, δ and every distributions $\mathfrak{D}, \mathfrak{A}$, we can obtain $f_{\boldsymbol{\theta}^\dagger}$ from $N_f(\epsilon, \delta)$ samples such that, with prob. $1-\delta$,

- near-optimal
$$\Big|P^\star - \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\Big]\Big| \leq \epsilon$$
- approximately feasible
$$\mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}^\dagger}(\boldsymbol{x}), y\big)\Big] \leq c + \epsilon$$



When is constrained learning possible?

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \quad \text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

$$\xrightarrow{\ ?\ }$$

$$P^\star = \min_{\boldsymbol{\theta}} \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{D}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \quad \text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathfrak{A}}\Big[g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}), y\big)\Big] \leq c$$

Proposition

$f_{\boldsymbol{\theta}}$ is PAC learnable $\not\Rightarrow$ $f_{\boldsymbol{\theta}}$ is PACC learnable

ECRM is not a PACC learner

Counter-example

$$P^\star = \min_{\boldsymbol{\theta}\in\Theta} J(\boldsymbol{\theta}) \quad \text{subject to} \quad \begin{aligned} \theta_2\,\mathbb{E}_\tau[\tau] &\leq \theta_1 - 1 \\ -\theta_1\,\mathbb{E}_\tau[\tau] &\leq \theta_2 - 1 \end{aligned}$$

$$J(\boldsymbol{\theta}) = \begin{cases} 1/16, & \boldsymbol{\theta} = [1/2, 1/2] \\ 1/8, & \boldsymbol{\theta} = [1, 1] \\ 1/4, & \boldsymbol{\theta} = [1, 0] \end{cases}$$

- $\tau \sim \text{Uniform}\big(-1/2, 1/2\big)$

ECRM is not a PACC learner

Counter-example

$$P^\star = \min_{\boldsymbol{\theta}\in\Theta} J(\boldsymbol{\theta}) = \frac{1}{8} \quad \text{subject to} \quad \begin{aligned} \theta_2\,\mathbb{E}_\tau[\tau] &\leq \theta_1 - 1 \Rightarrow \theta_1 \geq 1 \\ -\theta_1\,\mathbb{E}_\tau[\tau] &\leq \theta_2 - 1 \Rightarrow \theta_2 \leq 1 \end{aligned}$$

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$$\hat{P}^\star = \min_{\boldsymbol{\theta}\in\Theta} J(\boldsymbol{\theta}) \quad \text{subject to} \quad \begin{aligned} \theta_2\bar{\tau}_N &\leq \theta_1 - 1 \\ -\theta_1\bar{\tau}_N &\leq 1 - \theta_2 \end{aligned}$$

$$\Pr\big[|\hat{P}^\star - P^\star| \leq 1/32\big] = \Pr\big[\bar{\tau}_N = 0\big] = 0$$

- $\tau \sim \text{Uniform}\big(-1/2, 1/2\big) \to \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^{N} \tau_n$

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$$\hat{P}^\star_r = \min_{\boldsymbol{\theta}\in\Theta} \quad J(\boldsymbol{\theta})$$
$$\text{subject to} \quad \theta_2\bar{\tau}_N \le \theta_1 - 1 + r_1$$
$$-\theta_1\bar{\tau}_N \le 1 - \theta_2 + r_2$$

$$\Pr\left[|\hat{P}^\star_r - P^\star| \le 1/32\right] \le 4e^{-0.001N},$$

unless $\bar{\tau}_N \le r_1 < \frac{\bar{\tau}_N + 1}{2}$ and $r_2 \ge \bar{\tau}_N$

- $\tau \sim \text{Uniform}(-1/2, 1/2) \;\to \bar{\tau}_N = \frac{1}{N}\sum_{n=1}^N \tau_n$

Constrained learning challenges

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1) *Statistical*: does the solution of the constrained empirical problem generalize?
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Duality

PRIMAL

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^N \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^N g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

DUAL

Duality

PRIMAL

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DUAL

$$\hat{D}^\star = \max_{\lambda\ge 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^N \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda\left[\frac{1}{N}\sum_{m=1}^N g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) - c\right]$$

Duality

PRIMAL

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^N \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) \text{ subject to } \frac{1}{N}\sum_{m=1}^N g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \le c$$

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- In general, $\hat{D}^\star \le \hat{P}^\star$
- But in some cases, $\hat{D}^\star = \hat{P}^\star$ (strong duality) [e.g., convex optimization]

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An alternative path

$$\hat{P}^\star = \min_{\theta\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_\theta, z_n) \quad \text{s. to } \frac{1}{N}\sum_{n=1}^{N} g(f_\theta, z_n) \le c \qquad \not\longleftrightarrow \qquad \hat{D}^\star = \max_{\lambda\ge 0} \min_{\theta\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_\theta, z_n) + \lambda\left(\frac{1}{N}\sum_{n=1}^{N} g(f_\theta, z_n) - c\right)$$

PRIMAL DUAL

PAC

$$P^\star = \min_{\theta\in\Theta} \mathbb{E}_z\left[\ell(f_\theta, z)\right] \quad \text{s. to } \mathbb{E}_z\left[g(f_\theta, z)\right] \le c$$

[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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PRIMAL DUAL

PAC

$$P^\star = \min_{\theta\in\Theta} \mathbb{E}_z\left[\ell(f_\theta, z)\right] \quad \text{s. to } \mathbb{E}_z\left[g(f_\theta, z)\right] \le c$$

$$\downarrow \mathcal{H}_\theta \subset \mathcal{H}$$

$$\tilde{P}^\star = \min_{\phi\in\mathcal{H}} \mathbb{E}_z\left[\ell(\phi, z)\right] \quad \text{s. to } \mathbb{E}_z\left[g(\phi, z)\right] \le c \qquad \overset{?}{\longleftrightarrow} \qquad \tilde{D}^\star = \max_{\lambda\ge 0} \min_{\phi\in\mathcal{H}} \mathbb{E}_z\left[\ell(\phi, z)\right] + \lambda\left(\mathbb{E}_z\left[g(\phi, z)\right] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

Non-convex variational duality

Convex optimization: Primal ⟷ Dual

Non-convex, finite dimensional optimization: Primal ⟷ Dual

Non-convex variational duality

Convex optimization: Primal ⟷ Dual

Non-convex, finite dimensional optimization: Primal ⟷ Dual

Non-convex, **infinite** dimensional optimization: Primal ⟷ Dual

[Chamon et al., IEEE TSP'20]

Sparse logistic regression

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^p} -\sum_{n=1}^{N} \log\left[1 + \exp\left(y_n \cdot \boldsymbol{\theta}^T \boldsymbol{x}_n\right)\right]$$

$$\text{s. to } \|\boldsymbol{\theta}\|_0 = \sum_{t=1}^{p} \mathbb{I}\left[\boldsymbol{\theta}_t \neq 0\right] \le k$$

Discrete, non-convex

[Chen et al., JMLR'19]: NP-hard

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An alternative path

$$\hat{P}^\star = \min_{\theta\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_\theta, z_n) \quad \text{s. to } \frac{1}{N}\sum_{n=1}^{N} g(f_\theta, z_n) \le c \qquad \not\longleftrightarrow \qquad \hat{D}^\star = \max_{\lambda\ge 0} \min_{\theta\in\Theta} \frac{1}{N}\sum_{n=1}^{N} \ell(f_\theta, z_n) + \lambda\left(\frac{1}{N}\sum_{n=1}^{N} g(f_\theta, z_n) - c\right)$$

PRIMAL DUAL

PAC

$$P^\star = \min_{\theta\in\Theta} \mathbb{E}_z\left[\ell(f_\theta, z)\right] \quad \text{s. to } \mathbb{E}_z\left[g(f_\theta, z)\right] \le c$$

$$\tilde{P}^\star = \min_{\phi\in\mathcal{H}} \mathbb{E}_z\left[\ell(\phi, z)\right] \quad \text{s. to } \mathbb{E}_z\left[g(\phi, z)\right] \le c \qquad \overset{=}{\longleftrightarrow} \qquad \tilde{D}^\star = \max_{\lambda\ge 0} \min_{\phi\in\mathcal{H}} \mathbb{E}_z\left[\ell(\phi, z)\right] + \lambda\left(\mathbb{E}_z\left[g(\phi, z)\right] - c\right)$$

[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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PAC $\qquad$ $O(\epsilon)$

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Dual (near-)PACC learning

Theorem
Let f be ν-universal, i.e., for each θ_1, θ_2, and $\gamma \in [0,1]$ there exists θ such that

$$\mathbb{E}\left[\left|\gamma f_{\theta_1}(x) + (1-\gamma) f_{\theta_2}(x) - f_\theta(x)\right|\right] \le \nu$$

$\left[\{f_\theta\} \text{ is a good covering of } \overline{\text{conv}}(\{f_\theta\})\right]$

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Then $\hat{D}^\star$ is a (near-)PACC learner, i.e., there exists a solution $\theta^\dagger$ that, with probability $1-\delta$,

Near-optimal: $\quad \left|P^\star - \hat{D}^\star\right| \le \widetilde{O}\left(\nu + \frac{1}{\sqrt{N}}\right)$

Approximately feasible: $\quad \mathbb{E}\left[g\left(f_{\theta^\dagger}(x), y\right)\right] \le c + \widetilde{O}\left(\frac{1}{\sqrt{N}}\right)$

(mild conditions apply)

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(if losses are convex) $\quad h\left(f_{\theta^\dagger}(x), y\right) \le r$, with $\mathfrak{P}$-prob. $1 - \widetilde{O}\left(\frac{1}{\sqrt{N}}\right)$

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Dual (near-)PACC learning

Theorem
Let f be ν-universal with VC dimension $d_{\text{VC}} < \infty$. There exists $(\theta^\dagger, \lambda^\dagger)$ achieving $\hat{D}^\star$ such that $f_{\theta^\dagger}$ is a (near-)PACC solution of (P-CSL), i.e., with probability at least $1-\delta$,

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$$\mathbb{E}\left[g\left(f_{\theta^\dagger}(x), y\right)\right] \le c + \epsilon$$

$$\epsilon_0 = M\nu \qquad \epsilon = B\sqrt{\frac{1}{N}\left[1 + \log\left(\frac{4m(2N)^{d_{\text{VC}}}}{\delta}\right)\right]} \qquad \Delta = \max\left(\|\lambda^\star\|_1, \|\tilde{\lambda}^\star\|_1, \|\bar{\lambda}^\star\|_1\right)$$

Sources of error

parametrization richness (ν) $\qquad$ sample size (N) $\qquad$ requirements difficulty ($\lambda^\star$)

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[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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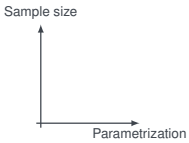
Dual learning trade-offs

- Unconstrained learning

  parametrization $\times$ sample size



[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

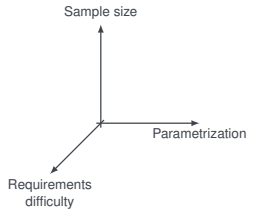
Dual learning trade-offs

- Unconstrained learning

  parametrization $\times$ sample size

- Constrained learning

  parametrization $\times$ sample size $\times$ requirements



[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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When is constrained learning possible?

Corollary

$f_{\boldsymbol{\theta}}$ is PAC learnable $\approx^*$ $f_{\boldsymbol{\theta}}$ is PACC learnable

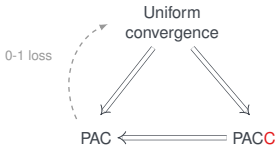
Constrained learning is **essentially as hard as** unconstrained learning

[mild conditions apply]

[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

When is constrained learning possible?

Corollary



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[Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23]

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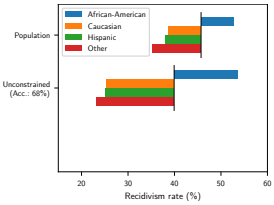
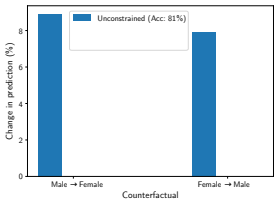
Fairness

Problem
Predict whether an individual will recidivate



Problem
Predict whether an individual makes > $50k



* We say "Race" to follow the terminology used during the data collection of the COMPAS dataset.

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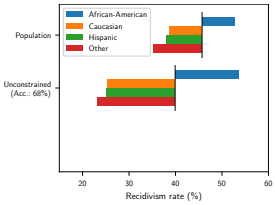

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Fairness: “Equality” of odds

Problem
Predict whether an individual will recidivate at the same rate across races

$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(x_n) = 1 \mid \text{Race}\right] \leq \frac{1}{N}\sum_{n=1}^{N} \mathbb{I}\left[f_\theta(x_n) = 1\right] + c,$$

$$\text{for Race} \in \{\text{African-American, Caucasian, Hispanic, Other}\}$$

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[Cotter et al., JMLR'19; Chamon et al., IEEE TIT'23]

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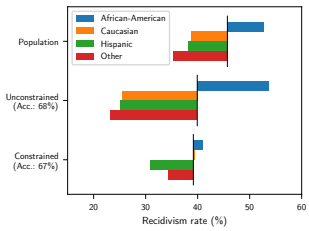

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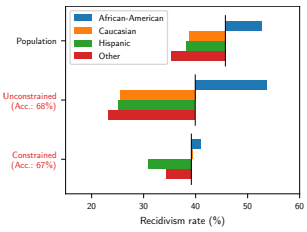

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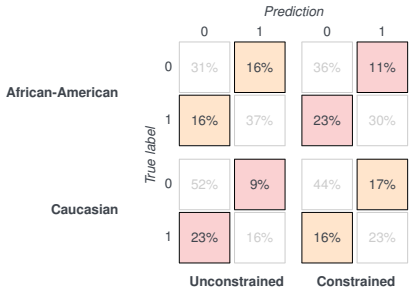

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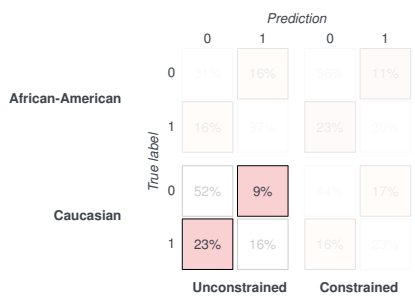

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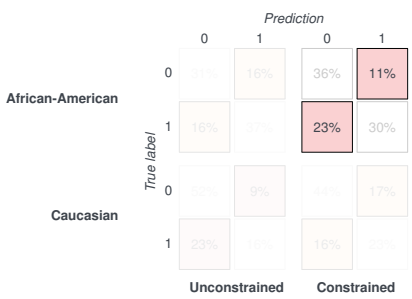

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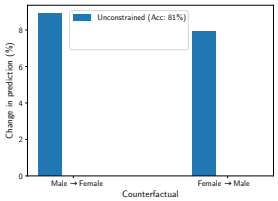

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Counterfactual fairness

Problem

Predict whether an individual makes > $50k while being invariant to gender

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \quad \mathrm{D}_{\text{KL}}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \| f_{\boldsymbol{\theta}}(\rho \boldsymbol{x}_n)\right) \leq c, \quad \text{for all } n$$

(ρ : Male $\leftrightarrow$ Female)

[Chamon and Ribeiro, NeurIPS'20]

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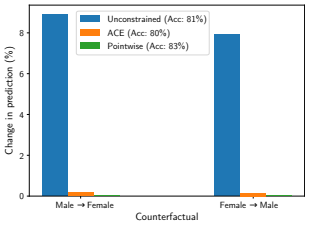

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$$\max_{\lambda_n \geq 0} \min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \sum_{n=1}^{N} \lambda_n \left[\mathrm{D}_{\text{KL}}\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \| f_{\boldsymbol{\theta}}(\rho \boldsymbol{x}_n)\right) - c\right]$$

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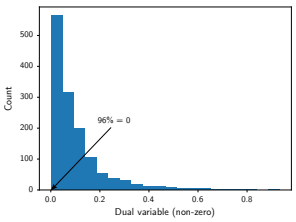
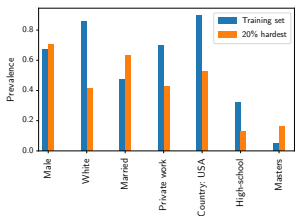
Problem

Predict whether an individual makes > $50k while being invariant to gender





[Chamon and Ribeiro, NeurIPS'20]

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Constrained optimization methods

$$\hat{P}^\star = \min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\big) \leq c$$

$$h\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_r), y_r\big) \leq u$$

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- Feasible update methods
  e.g., conditional gradients (Frank-Wolfe)
- Interior point methods
  e.g., barriers, projection, polyhedral approx.

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  - Tractability [non-convex constraints]
  - Feasible candidate solution
- Duality
  e.g., (augmented) Lagrangian
  - Tractability
  - (near-)feasible solution [small duality gap]

Dual learning algorithm

$$\hat{D}^\star = \max_{\lambda \geq 0} \ \min_{\boldsymbol{\theta} \in \mathbb{R}^p} \ \ \frac{1}{N}\sum_{n=1}^{N} \ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda\left[\frac{1}{N}\sum_{m=1}^{N} g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m), y_m\right) - c\right]$$

Dual learning algorithm

- Minimize the primal (≡ **ERM**)

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \ \frac{1}{N}\sum_{n=1}^{N}\left[\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\right]$$

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$$\boldsymbol{\theta}^+ \approx \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}}\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda g\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)\Big], \quad n = 1, 2, \ldots$$

[Haeffele et al., CVPR'17; Ge et al., ICLR'18; Mei et al., PNAS'18; Kawaguchi et al., AISTATS'20…]

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- Update the dual

$$\lambda^+ = \left[\lambda + \eta\left(\frac{1}{N}\sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}^+}(\boldsymbol{x}_m), y_m\big) - c\right)\right]_+$$

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A (near-)PACC learner

Theorem
Suppose $\boldsymbol{\theta}^\dagger$ is a ρ-approximate solution of the regularized ERM:

$$\boldsymbol{\theta}^\dagger \approx \operatorname*{argmin}_{\boldsymbol{\theta} \in \mathbb{R}^p} \ \frac{1}{N}\sum_{n=1}^{N}\Big(\ell\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right) + \lambda g\left(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\right)\Big).$$

Then, after $T = \left\lceil \frac{\|\boldsymbol{\lambda}^\star\|^2}{2\eta M \nu} \right\rceil + 1$ dual iterations with step size $\eta \leq \frac{2\epsilon}{mB^2}$,

the iterates $\left(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\right)$ are such that

$$\left|P^\star - L\left(\boldsymbol{\theta}^{(T)}, \boldsymbol{\lambda}^{(T)}\right)\right| \leq (2 + \Delta)(\epsilon_0 + \epsilon) + \rho$$

with probability $1 - \delta$ over sample sets.

[Chamon et al., IEEE TIT'23]

In practice...

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In practice...

1: Initialize: $\boldsymbol{\theta}_0$, λ_0
2: **for** $t = 1, \dots, T$
3: $\quad \boldsymbol{\beta}_1 \leftarrow \boldsymbol{\theta}_{t-1}$
4: $\quad$ **for** $n = 1, \dots, N$
5: $\quad\quad \boldsymbol{\beta}_{n+1} \leftarrow \boldsymbol{\beta}_n - \eta_\theta \nabla_{\boldsymbol{\beta}} \big[\ell\big(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\big) + \lambda_{t-1} g\big(f_{\boldsymbol{\beta}_n}(\boldsymbol{x}_n), y_n\big) \big]$ — SGD
6: $\quad$ **end**
7: $\quad \boldsymbol{\theta}_t \leftarrow \boldsymbol{\beta}_{N+1}$
8: $\quad \lambda_t = \left[\lambda_{t-1} + \eta_\lambda \left(\frac{1}{N} \sum_{m=1}^{N} g\big(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_m), y_n\big) - c \right) \right]_+$ — Dual update
9: **end**
10: Output: $\boldsymbol{\theta}_T, \lambda_T$

PyTorch
https://github.com/lfochamon/csl

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Check slack:
- feasibility: $s_t \leq 0$
- "duality gap": $\lambda_t s_t$

$$s_t = \frac{1}{N} \sum_{n=1}^{N} g\big(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_n), y_n\big) - c$$

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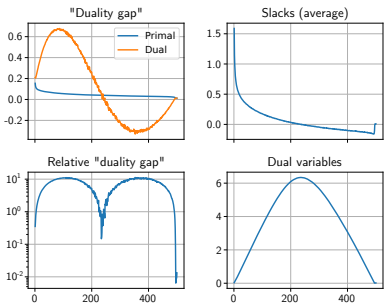

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Penalty-based vs. dual learning

Penalty-based learning

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta}} \; \text{Loss}(\boldsymbol{\theta}) + \lambda \cdot \text{Penalty}(\boldsymbol{\theta})$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss + λPenalty

Dual learning

$$\boldsymbol{\theta}^\dagger \in \operatorname*{argmin}_{\boldsymbol{\theta}} \; \text{Loss}(\boldsymbol{\theta}) + \lambda \cdot \text{Penalty}(\boldsymbol{\theta})$$

$$\lambda^+ = \Big[\lambda + \eta \big(\text{Penalty}(\boldsymbol{\theta}^\dagger) - c \big) \Big]_+$$

- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\leq c$

Agenda

Constrained learning theory

Constrained learning algorithms

Resilient constrained learning

Heterogeneous federated learning

Problem
Learn a common model using data using data distributed among K clients

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}})$$
$$\text{subject to} \quad \text{Loss}_k(f_{\boldsymbol{\theta}}) \leq \frac{1}{K}\sum_{k=1}^{K} \text{Loss}_k(f_{\boldsymbol{\theta}}) + c, \quad k = 1, \dots, K$$

- k-th client loss: $\text{Loss}_k(\phi) = \frac{1}{N_k}\sum_{n_k=1}^{N_k} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_{n_k}), y_{n_k}\big)$

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Resilient constrained learning

Definition (Resilience)
(ecology) ability of an ecosystem to adapt its function to accommodate operating conditions

Resilient constrained learning

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- Resilience is a compromise!

Resilient constrained learning

Definition (Resilient equilibrium)

For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) \quad \leftarrow (\partial\text{: subdifferential})$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

[Hounie et al., NeurIPS'23]

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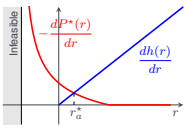

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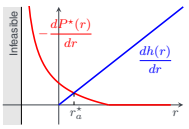

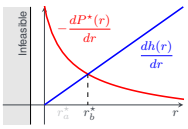

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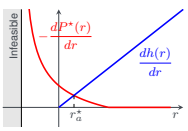

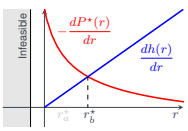

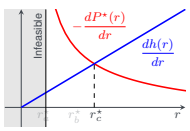

[Hounie et al., NeurIPS'23]

Resilient constrained learning

Definition (Resilient equilibrium)

For a strictly convex function $h(\boldsymbol{r})$, we say the relaxation $\boldsymbol{r}^\star$ achieves the resilient equilibrium if

$$\nabla h(\boldsymbol{r}^\star) \in -\partial P^\star(\boldsymbol{r}^\star) = \boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

- After relaxing, $\boldsymbol{\lambda}^\star(\boldsymbol{r}^\star)$ is *smaller* than $\boldsymbol{\lambda}^\star(\boldsymbol{0})$
  $\Rightarrow$ Resilient constrained learning "generalizes better" (lower sample complexity)

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$$P^\star(\boldsymbol{r}^\star) = \min_{\boldsymbol{\theta},\boldsymbol{r}} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{D}}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}),y\big)\Big] + h(\boldsymbol{r})$$
$$\text{subject to} \quad \mathbb{E}_{(\boldsymbol{x},y)\sim\mathcal{D}_i}\Big[g_i\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_m),y_m\big)\Big] \leq c_i + r_i$$

In words: at the resilient equilibrium the marginal cost of relaxing equals the marginal gain of relaxing

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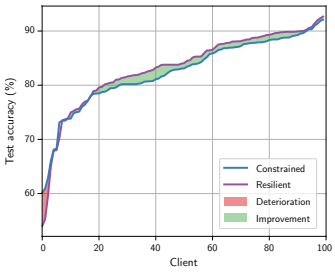
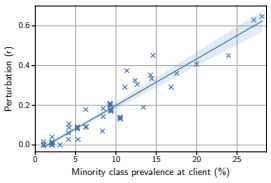
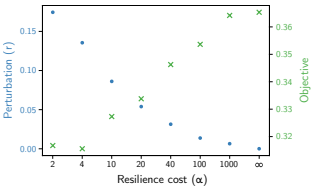
Heterogeneous federated learning



[Hounie et al., NeurIPS'23]

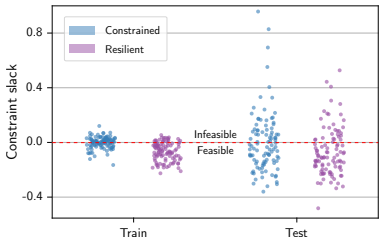
Heterogeneous federated learning





[Hounie et al., NeurIPS'23]

Heterogeneous federated learning



[Hounie et al., NeurIPS'23]

Summary

- **Constrained learning is the a tool to learn under requirements**
- **Constrained learning is hard...**
- **...but possible. How?**

Summary

- **Constrained learning is the a tool to learn under requirements**
  Constrained learning imposes generalizable requirements organically during training, e.g., fairness [Chamon and Ribeiro, NeurIPS'20; Chamon et al., IEEE TIT'23], heterogeneity [Shen et al., ICRL22]...
- **Constrained learning is hard...**
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Summary

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  Constrained, non-convex, statistical optimization problem
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Summary

- **Constrained learning is the a tool to learn under requirements**
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- **Constrained learning is hard...**
  Constrained, non-convex, statistical optimization problem
- **...but possible. How?**
  We can learn under requirements (essentially) whenever we can learn at all by solving *(penalized) ERM problems*. Resilient learning can then be used to adapt the requirements to the task difficulty [Hounie et al., NeurIPS'23]

Robustness constraints

Agenda

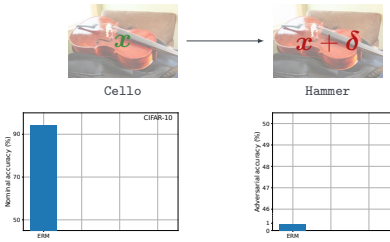
Adversarially robust learning

Semi-infinite learning

Probabilistic robustness

Robust learning

Problem
Learn an image classifier that is robust to input perturbations



Adversarial training

Problem
Learn an image classifier that is robust to input perturbations

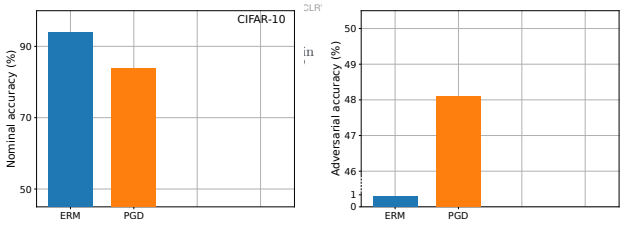
- Adversarial training [Szegedy et al., ICLR'14; Goodfellow et al., ICLR'15; Madry et al., ICLR'18; ...]

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big) \longrightarrow \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\delta\|_\infty \leq \epsilon} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

Adversarial training

Problem
Learn an image classifier that is robust to input perturbations

-



[Robey et al., NeurIPS'21]

Adversarial training

Problem
Learn an image classifier that is robust to input perturbations

- Adversarial training [Szegedy et al., ICLR'14; Goodfellow et al., ICLR'15; Madry et al., ICLR'18; ...]

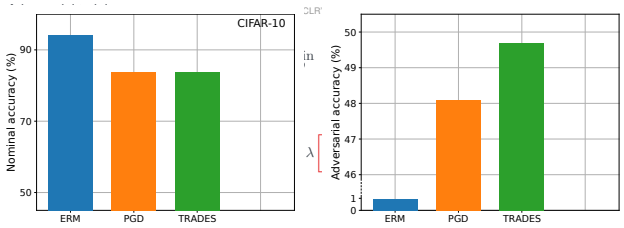
$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big) \qquad \min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\delta\|_\infty \leq \epsilon} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(\boldsymbol{x}_n), y_n\big) + \lambda \left[\max_{\|\delta\|_\infty \leq \epsilon} \text{Loss}\big(f_\theta(\boldsymbol{x}_n + \boldsymbol{\delta}), y_n\big)\right]$$

Adversarial training

Problem
Learn an image classifier that is robust to input perturbations

-



[Zhang et al., ICML'19]

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

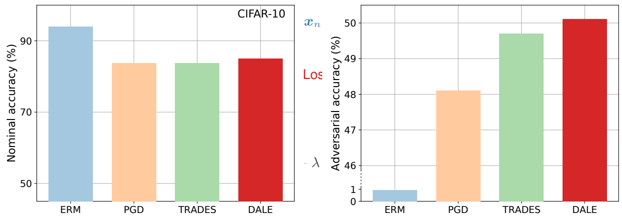
$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_{\theta}(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \left[\max_{\|\delta\|_\infty \le \epsilon} \text{Loss}\big(f_{\theta}(x_n + \delta), y_n\big) \right] \le c$$

[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations



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Constrained learning for robustness

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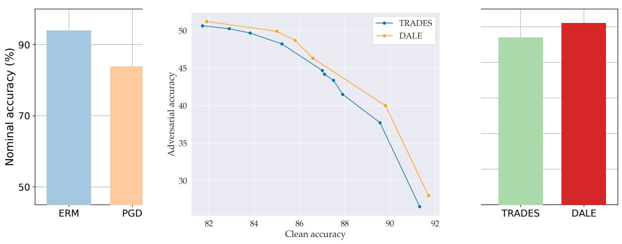
Learn an image classifier th



[Chamon and Ribeiro, NeurIPS'20; Robey et al., NeurIPS'21; Chamon et al., IEEE TIT'23]

Penalty-based vs. dual learning

Penalty-based learning

$$\theta^\dagger \in \operatorname*{argmin}_{\theta} \ \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

- Parameter: λ (data-dependent)
- Generalizes with respect to Loss $+$ λPenalty

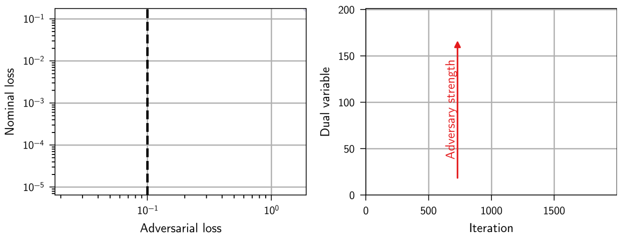
Dual learning

$$\theta^\dagger \in \operatorname*{argmin}_{\theta} \ \text{Loss}(\theta) + \lambda \cdot \text{Penalty}(\theta)$$

$$\lambda^+ = \Big[\lambda + \eta\Big(\text{Penalty}(\theta^\dagger) - c\Big)\Big]_+$$

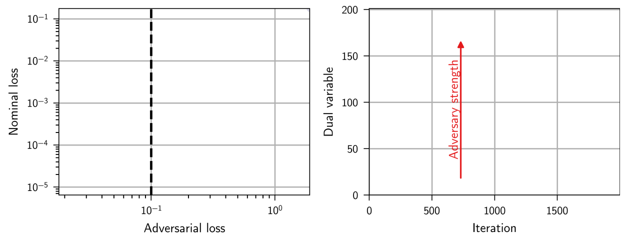
- Parameter: c (requirement-dependent)
- Generalizes with respect to Loss and Penalty $\le c$

Constrained learning for robustness



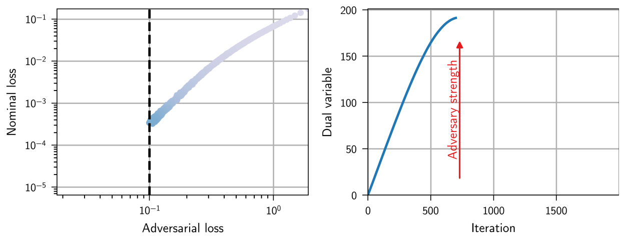
[Chamon et al., IEEE TIT'23]

Constrained learning for robustness



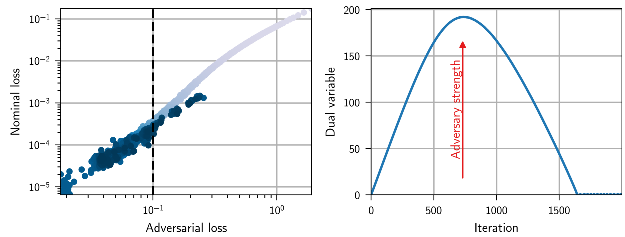
[Chamon et al., IEEE TIT'23]

Constrained learning for robustness



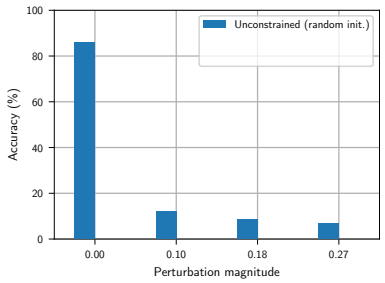
[Chamon et al., IEEE TIT'23]

Constrained learning for robustness



Empirical observations: [Zhang et al., ICML'20; Sitawarin, arXiv'20]

Constrained learning for robustness



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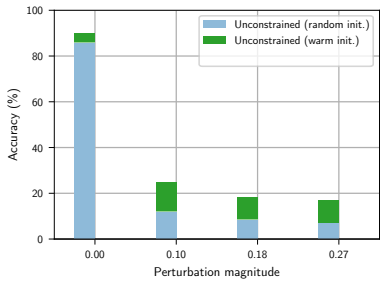
Constrained learning for robustness



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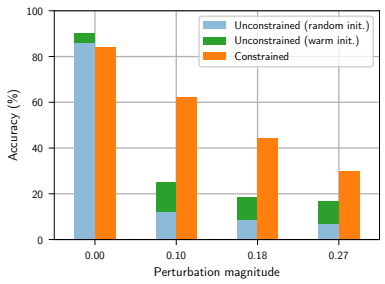
Constrained learning for robustness



[Chamon et al., IEEE TIT'23]

Constrained learning for robustness

Problem

Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\ \min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(\boldsymbol{x}_n),y_n\big)+\lambda\left[\max_{\delta\in\Delta}\ \text{Loss}\big(f_\theta(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\right]$$

- Balancing nominal accuracy and robustness ⇒ Dual constrained learning

Constrained learning for robustness

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- Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- Computing the worst-case perturbations

Adversarial training

$$\min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\delta\in\Delta}\ \text{Loss}\big(f_\theta(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\right]$$

- "PGD" [Mądry et al., ICLR'18]

1: $\boldsymbol{\delta}^1\leftarrow\boldsymbol{\delta}_{t-1}$
2: **for** $k=1,\dots,K$
3: $\quad\boldsymbol{\delta}^{k+1}\leftarrow\operatorname{proj}_{\Delta}\left[\boldsymbol{\delta}^k+\eta\operatorname{sign}\left(\nabla_{\boldsymbol{\delta}}\,\text{Loss}\big(f_{\boldsymbol{\theta}^k}(\boldsymbol{x}+\boldsymbol{\delta}^k),y\big)\right)\right]$
4: **end**
5: $\boldsymbol{\delta}_t\leftarrow\boldsymbol{\delta}^{K+1}$
6: $\boldsymbol{\theta}_{t+1}=\boldsymbol{\theta}_t-\eta\nabla_{\boldsymbol{\theta}}\,\text{Loss}\Big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}_t),y\Big)$

Adversarial training

$$\min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\left[\max_{\delta\in\Delta}\ \text{Loss}\big(f_\theta(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\right]$$

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- Random initialization
- Restarts
- Pruning
- Adaptive step size

[Dhillon et al., ICLR'18; Carmon et al., NeurIPS'19; Wu et al., NeurIPS'20; Cheng et al., IJCAI'22]

Constrained learning for robustness

Problem

Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\ \min_{\theta}\ \frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(\boldsymbol{x}_n),y_n\big)+\lambda\left[\max_{\delta\in\Delta}\ \text{Loss}\big(f_\theta(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\right]$$

- Balancing nominal accuracy and robustness ⇒ Dual constrained learning
- Computing the worst-case perturbations
  - ~~gradient ascent → non-convex, underparametrized~~

Agenda

Adversarially robust learning

Semi-infinite learning

Probabilistic robustness

Semi-infinite constrained learning

$$\min_{\boldsymbol{\theta}} \quad \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]$$

Semi-infinite constrained learning

$$\begin{aligned}\min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N}\big[t(\boldsymbol{x}_n,y_n)\big]\\ \text{subject to} \quad & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\le t(\boldsymbol{x}_n,y_n),\\ & \text{for all } (\boldsymbol{x}_n,y_n) \text{ and } \boldsymbol{\delta}\in\Delta\end{aligned}$$

- Epigraph formulation:

$$\max_{\|\boldsymbol{\delta}\|_\infty\le\epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)\le t \iff \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)\le t, \text{ for all } \|\boldsymbol{\delta}\|_\infty\le\epsilon$$

Semi-infinite constrained learning

$$\begin{aligned}\min_{\boldsymbol{\theta}} \quad & \frac{1}{N}\sum_{n=1}^{N}\big[t(\boldsymbol{x}_n,y_n)\big]\\ \text{subject to} \quad & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_0),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{\sqrt{2}}),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_e),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_\pi),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{\sqrt{3}}),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{e^\pi}),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{\pi^e}),y_n\big)\le t(\boldsymbol{x}_n,y_n)\\ & \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}_{23}),y_n\big)\le t(\boldsymbol{x}_n,y_n)\end{aligned}$$

- Epigraph formulation:

$$\max_{\|\boldsymbol{\delta}\|_\infty\le\epsilon} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)\le t \iff \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)\le t, \text{ for all } \|\boldsymbol{\delta}\|_\infty\le\epsilon$$

- Semi-infinite program

Duality

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]$$

$$\updownarrow =$$

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\big[t(\boldsymbol{x}_n,y_n)\big] \text{ s. to } \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\le t(\boldsymbol{x}_n,y_n), \forall(\boldsymbol{x}_n,y_n,\boldsymbol{\delta})$$

$$\updownarrow =$$

$$\min_{\boldsymbol{\theta}} \sup_{\mu\in\mathcal{P}} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\int_{\Delta}\mu_n(\boldsymbol{\delta})\,\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)d\boldsymbol{\delta}}_{L(\theta,\mu_n)}$$

Duality

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]$$

$$\updownarrow =$$

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\big[t(\boldsymbol{x}_n,y_n)\big] \text{ s. to } \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\le t(\boldsymbol{x}_n,y_n), \forall(\boldsymbol{x}_n,y_n,\boldsymbol{\delta})$$

$$\updownarrow =$$

$$\min_{\boldsymbol{\theta}} \sup_{\mu\in\mathcal{P}} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\boldsymbol{\delta}\sim\mu(\cdot|\boldsymbol{x}_n,y_n)}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]}_{L(\theta,\mu_n)}$$

From optimization to sampling

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]$$

$$\updownarrow \approx$$

$$\min_{\boldsymbol{\theta}} \sup_{\mu\in\mathcal{P}^2} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\boldsymbol{\delta}\sim\mu_\gamma(\cdot|\boldsymbol{x}_n,y_n)}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]}_{L(\theta,\mu)}$$

Proposition

For all $\epsilon>0$, there exists $\gamma(\boldsymbol{x},y)<\max_{\boldsymbol{\delta}\in\Delta}\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)$ s.t. $L(\boldsymbol{\theta},\mu_\gamma)\ge\sup_{\mu\in\mathcal{P}^2}L(\boldsymbol{\theta},\mu)-\xi$ for

$$\mu_\gamma(\boldsymbol{\delta}|\boldsymbol{x},y)\propto\Big[\ell\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)-\gamma(\boldsymbol{x},y)\Big]_+$$

From optimization to sampling

$$\min_{\boldsymbol{\theta}} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\boldsymbol{\delta}\in\Delta} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]$$

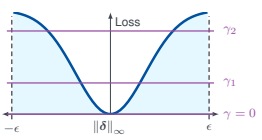
$$\updownarrow \approx$$

$$\min_{\boldsymbol{\theta}} \sup_{\mu\in\mathcal{P}^2} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\boldsymbol{\delta}\sim\mu_\gamma(\cdot|\boldsymbol{x}_n,y_n)}\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n+\boldsymbol{\delta}),y_n\big)\Big]}_{L(\theta,\mu)}$$

Proposition

For any approximation error, $\exists\,\gamma(\boldsymbol{x},y)$ such that

$$\mu_\gamma(\boldsymbol{\delta}|\boldsymbol{x},y)\propto\Big[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}+\boldsymbol{\delta}),y\big)-\gamma(\boldsymbol{x},y)\Big]_+$$



[Robey et al., NeurIPS'21]

From optimization to sampling

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]$$

$$\Updownarrow \approx$$

$$\min_{\theta}\sup_{\mu\in\mathcal{P}^2} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\delta\sim\mu_\gamma(\cdot|x_n,y_n)}\Big[\text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]}_{L(\theta,\mu)}$$

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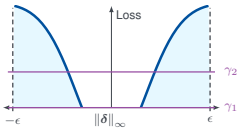

[Robey et al., NeurIPS'21]

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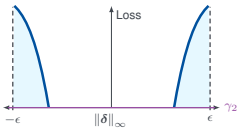

[Robey et al., NeurIPS'21]

From optimization to sampling

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]$$

$$\Updownarrow =$$

$$\min_{\theta}\sup_{\mu\in\mathcal{P}^2} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\delta\sim\mu_\gamma(\cdot|x_n,y_n)}\Big[\text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]}_{L(\theta,\mu)}$$

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For any approximation error, $\exists\,\gamma(x,y)$ such that

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[Robey et al., NeurIPS'21]

From optimization to sampling

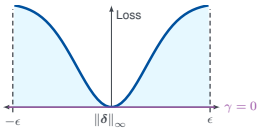
$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]$$

$$\Updownarrow \approx$$

$$\min_{\theta}\sup_{\mu\in\mathcal{P}^2} \underbrace{\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_{\delta\sim\mu_\gamma(\cdot|x_n,y_n)}\Big[\text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]}_{L(\theta,\mu)}$$

Proposition
For any approximation error, $\exists\,\gamma(x,y)$ such that

$$\mu_0(\delta|x,y) \propto \text{Loss}\big(f_\theta(x+\delta),y\big)$$



[Robey et al., NeurIPS'21]

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\min_{\theta}\frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(x_n),y_n\big)+\lambda\Big[\max_{\delta\in\Delta}\text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]$$

- Balancing nominal accuracy and robustness ⇒ Dual constrained learning

- Computing the worst-case perturbations
  - gradient ascent → non-convex, underparametrized

Constrained learning for robustness

Problem
Learn an image classifier that is robust to input perturbations

$$\max_{\lambda\geq 0}\min_{\theta}\frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(x_n),y_n\big)+\lambda\Big[\max_{\delta\in\Delta}\text{Loss}\big(f_\theta(x_n+\delta),y_n\big)\Big]$$

$\mathbb{E}_{\delta\sim\mu_0(\cdot|x_n,y_n)}$

- Balancing nominal accuracy and robustness ⇒ Dual constrained learning

- Computing the worst-case perturbations
  - ~~gradient ascent → non-convex, underparametrized~~ ⇒ sampling

Dual Adversarial LEarning

1: **for** $n = 1,\dots,N$:
2: $\quad\delta_n \sim \text{Random}(\Delta)$
3: $\quad$**for** $k = 1,\dots,K$:
4: $\quad\quad\zeta \sim \text{Laplace}(0, I)$
5: $\quad\quad\delta_n \leftarrow \underset{\Delta}{\text{proj}}\Big[\delta_n + \eta\,\text{sign}\Big[\nabla_\delta \log\Big(\text{Loss}\big(f_{\theta_t}(x_n+\delta_n),y_n\big)\Big)\Big] + \sqrt{2\eta T}\zeta\Big]$ — HMC sampling: $\delta \sim \mu_0(\cdot|x_n,y_n)$
6: $\quad$**end**
7: $\quad\theta \leftarrow \theta - \eta\nabla_\theta\Big[\text{Loss}\big(f_\theta(x_n),y_n\big)+\lambda\text{Loss}\big(f_\theta(x_n+\delta_n),y_n\big)\Big]$ — SGD
8: **end**
9: $\lambda \leftarrow \Big[\lambda+\eta\Big(\frac{1}{N}\sum_{n=1}^{N}\text{Loss}\big(f_\theta(x_n+\delta_n),y_n\big)-c\Big)\Big]_+$ — GA

[Robey et al., NeurIPS'21]

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6: $\quad$ **end**

7: $\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \left[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}_n), y_n\big) \right]$ — SGD

8: **end**

9: $\lambda \leftarrow \left[\lambda + \eta \left(\frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}_n), y_n\big) - c \right) \right]_+$ — GA

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[Robey et al., NeurIPS'21] 91

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6: $\quad$ **end**

7: $\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \left[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}_n), y_n\big) \right]$ — SGD

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[Robey et al., NeurIPS'21] 92

Dual Adversarial LEarning

1: **for** $n = 1, \dots, N$:

2: $\quad \boldsymbol{\delta}_n \sim \text{Random}(\Delta)$ — Gaussian [Lopes et al., arXiv'19] [Rusak et al., ECCV'20]; Patches [Zhong et al., AAAI'20] [Yun et al., ICCV'19]; ...

3: $\quad$ **for** $k = 1, \dots, K$:

4: $\quad\quad \boldsymbol{\zeta} \sim \text{Laplace}(0, I)$

5: $\quad\quad \boldsymbol{\delta}_n \leftarrow \underset{\Delta}{\text{proj}} \left[\boldsymbol{\delta}_n + \eta \, \text{sign} \left[\nabla_{\boldsymbol{\delta}} \log \left(\text{Loss}\big(f_{\boldsymbol{\theta}_t}(\boldsymbol{x}_n + \boldsymbol{\delta}_n), y_n\big) \right) \right] + \sqrt{2\eta T}\boldsymbol{\zeta} \right]$

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8: **end**

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[Robey et al., NeurIPS'21] 92

Dual Adversarial LEarning

1: **for** $n = 1, \dots, N$:

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6: $\quad$ **end**

7: $\quad \boldsymbol{\theta} \leftarrow \boldsymbol{\theta} - \eta \nabla_{\boldsymbol{\theta}} \left[\text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big) + \lambda \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n + \boldsymbol{\delta}_n), y_n\big) \right]$ — SGD

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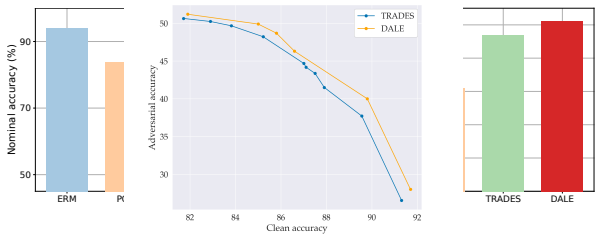
[Robey et al., NeurIPS'21] 92

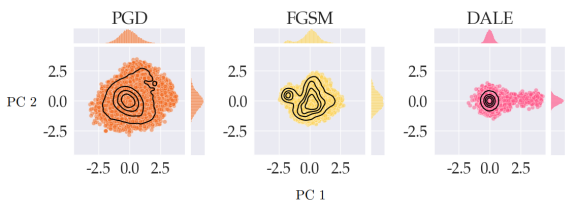
Dual Adversarial LEarning

Problem
Learn an image classifier th



Dual Adversarial LEarning

Problem
Learn an image classifier that is robust to input perturbations



Invariance

Problem
Learn a classifier that is invariant to transformation $g \in \mathcal{G}$

$$\min_{\boldsymbol{\theta}} \ \frac{1}{N} \sum_{n=1}^{N} \text{Loss}\big(f_{\boldsymbol{\theta}}(\boldsymbol{x}_n), y_n\big)$$

$$\text{subject to} \ \ \frac{1}{N} \sum_{n=1}^{N} \left[\max_{g \in \mathcal{G}} \text{Loss}\big(f_{\boldsymbol{\theta}}(g\boldsymbol{x}_n), y_n\big) \right] \leq c$$

Invariance

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Learn a classifier that is invariant to transformation $g \in \mathcal{G}$

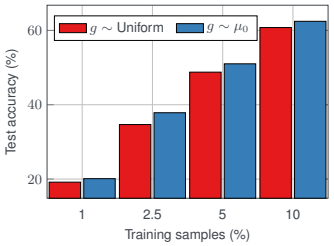
$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \text{Loss}\big(f_\theta(x_n), y_n\big)$$

$$\text{subject to} \quad \frac{1}{N}\sum_{n=1}^{N} \Big[\mathbb{E}_{g\sim\mu_0(\cdot|x_n,y_n)} \text{Loss}\big(f_\theta(gx_n), y_n\big)\Big] \leq c$$

- No differentiability required (e.g., Metropolis-Hastings)

[Hounie et al., ICML'23]

Training on a subset of ImageNet-100



- Transformations ($\mathcal{G}$)
  - ShearX, ShearY, Flips, Rotate, TranslateX, TranslateY
  - Cutout, Crop
  - AutoContrast, Invert, Equalize, Solarize, Posterize, Contrast, Color, Brightness, Sharpness

[Hounie et al., ICML'23]

"Identifying" invariances

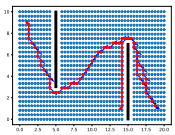
		Synthetic Invariance		
Dataset	Dual variable (λ)	Rotation	Translation	Scale
MNIST	Rotation	**0.000**	2.724	0.012
	Translation	1.218	**0.439**	0.006
	Scale	2.026	4.029	**0.003**
F-MNIST	Rotation	**0.000**	3.301	1.352
	Translation	3.572	**0.515**	**0.441**
	Scale	4.144	2.725	0.904

[Hounie et al., ICML'23]

(Manifold) smoothness

Problem
Leveraging unlabeled data

- Labeled data ({Position, Action})



Dataset

[Cerviño et al., ICML'23]

(Manifold) smoothness

Problem
Leveraging unlabeled data

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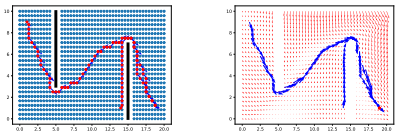


Dataset

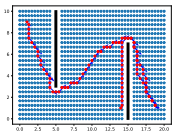
ERM
(85% success)

[Cerviño et al., ICML'23]

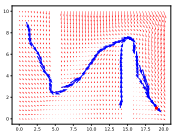
(Manifold) smoothness

Problem
Leveraging unlabeled data

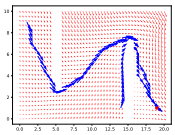
- Labeled data ({Position, Action})



Dataset



ERM
(85% success)



Lipschitz regularization
(66% success)

[Cerviño et al., ICML'23]

(Manifold) smoothness

Problem
Leveraging unlabeled data

- Labeled data ({Position, Action}) and unlabeled data ({Position})
- Use {Position} to estimate a data manifold $\mathcal{M}$

$$\min_{\theta} \quad \frac{1}{N}\sum_{n=1}^{N} \|f_\theta(\text{Position}_n) - \text{Action}_n\|^2$$

$$\text{subject to} \quad \max_{x} \|\nabla_{\mathcal{M}} f_\theta(x)\|^2 \leq c$$

[Cerviño et al., ICML'23]

(Manifold) smoothness

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Leveraging unlabeled data

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- Use {Position} to estimate a data manifold $\mathcal{M}$

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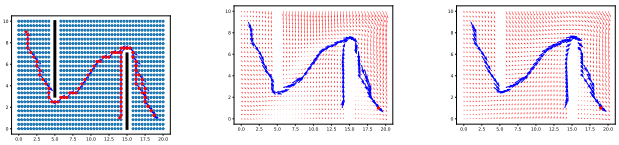
$$\text{subject to} \quad \mathbb{E}_{x\sim\mu_0} \|\nabla_{\mathcal{M}} f_\theta(x)\|^2 \leq c$$

[Cerviño et al., ICML'23]

(Manifold) smoothness

Problem
Leveraging unlabeled data

- Labeled data ({Position, Action}) and unlabeled data ({Position})



Dataset | ERM (85% success) | Manifold smoothness (94% success)

[Cerviño et al., ICML'23]

Agenda

Adversarially robust learning

Semi-infinite learning

Probabilistic robustness

Constrained learning challenges

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \xrightarrow{\text{PAC}} \min_{\theta} \mathbb{E}_{(x,y)\sim\mathcal{D}}\Big[\ell\big(f_\theta(x_n), y_n\big)\Big]$$

$$\text{s. to } \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \ell\big(f_\theta(x_n+\delta), y_n\big)\Big] \le c \xrightarrow{\text{PAC}} \text{s. to } \mathbb{E}_{(x,y)\sim\mathcal{D}}\Big[\max_{\delta\in\Delta} \ell\big(f_\theta(x_n+\delta), y_n\big)\Big] \le c$$

PACC

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

Constrained learning challenges

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N} \ell\big(f_\theta(x_n), y_n\big) \xrightarrow{\text{PAC}} \min_{\theta} \mathbb{E}_{(x,y)\sim\mathcal{D}}\Big[\ell\big(f_\theta(x_n), y_n\big)\Big]$$

$$\text{s. to } \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \ell\big(f_\theta(x_n+\delta), y_n\big)\Big] \le c \xrightarrow{\text{PAC?}} \text{s. to } \mathbb{E}_{(x,y)\sim\mathcal{D}}\Big[\max_{\delta\in\Delta} \ell\big(f_\theta(x_n+\delta), y_n\big)\Big] \le c$$

PACC

Challenges

1) *Statistical*: does the solution of the constrained empirical problem generalize?
2) *Computational*: can we solve the constrained empirical problem?

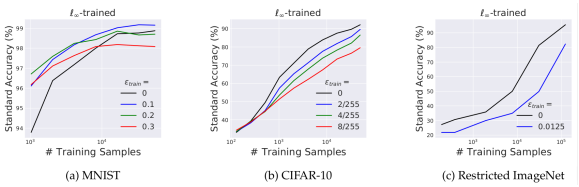
Statistical complexity

$$\min_{\theta} \frac{1}{N}\sum_{n=1}^{N}\Big[\max_{\delta\in\Delta} \text{Loss}\Big(f_\theta(x_n+\delta), y_n\Big)\Big] \xrightarrow{?} \min_{\theta} \mathbb{E}_{(x,y)}\Big[\max_{\delta\in\Delta} \text{Loss}\Big(f_\theta(x+\delta), y\Big)\Big]$$

- Is robust learning harder than non-robust learning? Do we need more samples?
  A: YES *and* NO

[Cullina, Bhagoji, Mittal. PAC-learning in the presence of evasion adversaries, NeurIPS'18]
[Yin, Ramchandran, Bartlett. Rademacher Complexity for Adversarially Robust Generalization, ICML'19]
[Montasser, Hanneke, Srebro. VC Classes are Adversarially Robustly Learnable, but Only Improperly, COLT'19]
[Awasthi, Frank, Mohri. Adversarial Learning Guarantees for Linear Hypotheses and Neural Networks, ICML'20]
[Montasser, Hanneke, Srebro. Adversarially robust learning: A generic minimax optimal learner & characterization, NeurIPS'22]

Nominal performance of robust models



(a) MNIST | (b) CIFAR-10 | (c) Restricted ImageNet

[Tsipras et al., ICLR'19]

"Softer" robustness

- Softmax or *log-sum-exp* [Li et al., ICLR'21]

$$\min_{\theta} \mathbb{E}_{(x,y)}\left[\frac{1}{\tau}\log\left(\mathbb{E}_{\delta\sim\mathrm{m}}\left[e^{\tau\cdot\text{Loss}\left(f_\theta(x+\delta),y\right)}\right]\right)\right]$$

  - $\tau \to 0$: classical learning (with randomized data augmentation)
  - $\tau \to \infty$: adversarial robustness (ess sup)

- L_p norms [Rice et al., NeurIPS'21]

$$\min_{\theta} \mathbb{E}_{(x,y)}\left[\mathbb{E}_{\delta\sim\mathrm{m}}\left[\left|\text{Loss}\left(f_\theta(x+\delta),y\right)\right|^{\tau}\right]^{1/\tau}\right]$$

  - $\tau = 1$: classical learning (with randomized data augmentation)
  - $\tau \to \infty$: adversarial robustness (ess sup)

"Softer" robustness

- Softmax or *log-sum-exp* [Li et al., ICLR'21]

$$\min_{\theta} \mathbb{E}_{(x,y)}\left[\frac{1}{\tau}\log\left(\mathbb{E}_{\delta\sim\mathrm{m}}\left[e^{\tau\cdot\text{Loss}\left(f_\theta(x+\delta),y\right)}\right]\right)\right]$$

- L_p norms [Rice et al., NeurIPS'21]

$$\min_{\theta} \mathbb{E}_{(x,y)}\left[\mathbb{E}_{\delta\sim\mathrm{m}}\left[\left|\text{Loss}\left(f_\theta(x+\delta),y\right)\right|^{\tau}\right]^{1/\tau}\right]$$

  - Computationally challenging (especially as $\tau \to \infty$, i.e., stronger robustness)
  - No guaranteed advantages (lower sample complexity? improved trade-offs?)

Towards probabilistic robustness

$$\min_{\theta} \ \frac{1}{N}\sum_{n=1}^{N}\left[t(x_n, y_n)\right]$$

$$\text{subject to} \quad \text{Loss}\left(f_\theta(x_n+\delta_0), y_n\right) \le t(x_n, y_n)$$
$$\text{Loss}\left(f_\theta(x_n+\delta_1), y_n\right) \le t(x_n, y_n)$$
$$\text{Loss}\left(f_\theta(x_n+\delta_{\sqrt{2}}), y_n\right) \le t(x_n, y_n)$$
$$\text{Loss}\left(f_\theta(x_n+\delta_\epsilon), y_n\right) \le t(x_n, y_n)$$
$$\text{Loss}\left(f_\theta(x_n+\delta_\pi), y_n\right) \le t(x_n, y_n)$$
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$$\text{Loss}\left(f_\theta(x_n+\delta_{25}), y_n\right) \le t(x_n, y_n)$$

- Epigraph formulation:

$$\max_{\|\delta\|_\infty \le \epsilon} \text{Loss}\left(f_\theta(x+\delta), y\right) \le t \iff \text{Loss}\left(f_\theta(x+\delta), y\right) \le t, \text{ for all } \|\delta\|_\infty \le \epsilon$$

- Semi-infinite program

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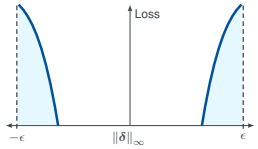
$$\min_{\theta} \ \mathbb{E}_{(x,y)}\left[\underset{\delta\in\Delta}{\approx \mathbb{E}_{\delta\sim\mu_0(\cdot|x,y)}} \text{Loss}\left(f_\theta(x+\delta), y\right)\right]$$



[Robey et al., ICML'22 (spotlight)]

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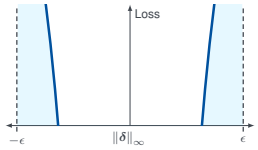


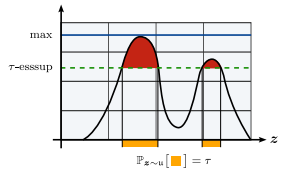
[Robey et al., ICML'22 (spotlight)]

Probabilistic robustness

- Probabilistic robustness

$$\min_{\theta} \ \mathbb{E}_{(x,y)}\left[\underset{\delta\in\Delta}{\tau\text{-esssup}} \text{Loss}\left(f_\theta(x+\delta), y\right)\right]$$

  - $\tau = 1/2$: classical learning (for symmetric m)
  - $\tau = 0$: adversarial robustness (ess sup)



[Robey et al., ICML'22 (spotlight)]

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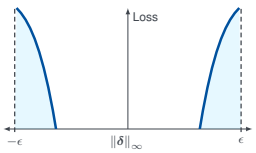

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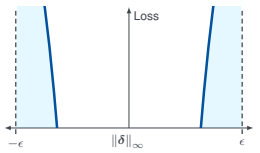


[Robey et al., ICML'22 (spotlight)]

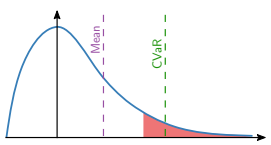
Probabilistic robustness and Risk

- Conditional value at risk:

$$\text{CVaR}_\rho(f) = \mathbb{E}_z\left[f(z) \mid f(z) \ge F_z^{-1}(\rho)\right]$$

$$= \inf_{\alpha\in\mathbb{R}} \ \alpha + \frac{\mathbb{E}_z\left[[f(z)-\alpha]_+\right]}{1-\rho}$$

  - $\text{CVaR}_0(f) = \mathbb{E}_z[f(z)]$
  - $\text{CVaR}_1(f) = \text{ess sup}_z\, f(z)$



Proposition

CVaR is the tightest *convex* upper bound of τ-esssup, i.e.,
$\tau\text{-esssup}_z\, f(z) \le \text{CVaR}_{1-\tau}(f)$ with equality when $\rho = 0$ or $\rho = 1$.

[Shapiro et al. Lectures on Stochastic Programming, 2014; Kalogerias et al., IEEE ICASSP'20]

Probabilistically robust learning

1: **for** $n = 1, \ldots, N$:
2: $\quad \alpha_0 = 0$
3: $\quad$ **for** $t = 1, \ldots, T$:
4: $\quad\quad \delta_t \sim \text{Random}(\Delta)$
5: $\quad\quad \alpha \leftarrow \alpha - \frac{\eta}{\tau}\left(\tau - \mathbb{I}\left[\text{Loss}\left(f_\theta(x_n+\delta_t), y_n\right) \ge \alpha\right]\right)$ — SGD (CVaR)
6: $\quad$ **end**
7: $\quad \theta \leftarrow \theta - \eta\nabla_\theta \underbrace{\left[\text{Loss}\left(f_\theta(x_n+\delta_T), y_n\right) - \alpha\right]_+}_{\approx \text{CVaR}_{1-\tau}\left[\text{Loss}\left(f_\theta(x_n+\delta), y_n\right)\right]}$ — SGD (θ)
8: **end**

[Robey et al., ICML'22 (spotlight)]

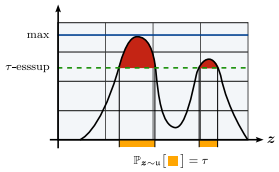
Probabilistic robustness

- Probabilistic robustness

$$\min_{\theta} \mathbb{E}_{(x,y)} \left[\underset{\delta \in \Delta}{\tau\text{-esssup}} \operatorname{Loss}\big(f_\theta(x+\delta), y\big) \right]$$

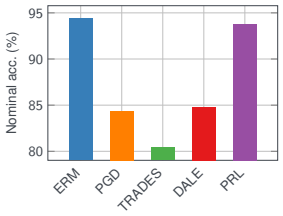
  - $\tau = 1/2$: classical learning (for symmetric m)
  - $\tau = 0$: adversarial robustness ($\operatorname{ess\,sup}$)

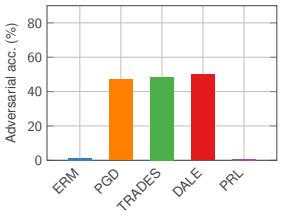
  - Potentially better sample complexity
    [Robey et al., ICML'22 (spotlight)]
    [Raman et al., NeurIPS ML Safety Workshop'22]
  - Better performance trade-off
    [Robey et al., ICML'22 (spotlight)]



[Robey et al., ICML'22 (spotlight)]

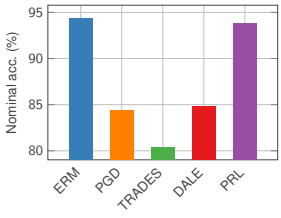
Probabilistically robust learning

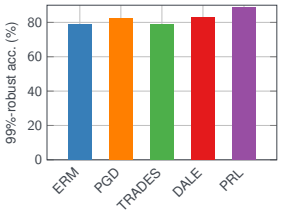




[Robey et al., ICML'22 (spotlight)]

Probabilistically robust learning





[Robey et al., ICML'22 (spotlight)]

Summary

- **Semi-infinite constrained learning is the a tool to enforce worst-case requirements**
- **Semi-infinite constrained learning...**
- **...but possible. How?**

Summary

- **Semi-infinite constrained learning is the a tool to enforce worst-case requirements**
  e.g., robustness [Robey et al., NeurIPS'21], invariance [Hounie et al., ICML'23], smoothness [Cerviño et al., ICML'23]...
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- **Semi-infinite constrained learning...**
  Learning problem with an infinite number of constraints
- **...but possible. How?**
  Using a hybrid sampling–optimization algorithm or, in the case of probabilistic robustness, a *tight* convex relaxation (CVaR) [Robey et al., ICML'22]

Break