第 12 讲: 偏序关系与格

姓名： 林凡琪　　**学号：** 211240042

评分： ______　　**评阅：** ______

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请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

- Lattice theory draws on both order theory and universal algebra.

1　作业（必做部分）

题目 1 (SM Problem 14.44)

解答：
见附图。

题目 2 (SM Problem 14.58)

证明：
(a) Define a one-to -one function $f : A \to A$, by $f(x) = x$. (a and a' are a pair)
(1) If $a \precsim a'$ then $f(a) = a \precsim f(a') = a'$.
(2)If $a||a'$, then $f(a) = a||f(a') = a'$.

(b)If $A \precsim B$ then there exists at least a function $f : A \to B$, a and a' in X, $f(a)$ and $f(a')$ in Y.
We can get that:
If $a \precsim a'$ then $f(a) \precsim f(a')$.
If $a||a'$, then $f(a)||f(a')$.
So for the function $f^{-1} : B \to A$
If $f(a) \precsim f(a')$, then $f^{-1}(f(a)) = a \precsim f^{-1}(f(a')) = a'$.

(c)If for function $f : A \to B$,$g : B \to C$, $h = g \circ f$.
We can know that:
(condition 1)If $a \precsim a'$, then $f(a) \precsim f(a')$, then $g(f(a)) \precsim g(f(a'))$
That is, if $a \precsim a'$, then $g(f(a)) \precsim g(f(a'))$

(condition 2)If $a||a^{'}$, then $f(a)||f(a^{'})$, then $g(f(a))||g(f(a^{'}))$.
That is, if $a||a^{'}$, then $g(f(a))||g(f(a^{'}))$.

题目 3 (SM Problem 14.62)
Suppose A and B are well-ordered isomorphic sets. Show that there is only one *isomorphic* mapping $f : A \to B$

解答:
Let a be the first element of A, and b be the first element of B.
By the definition, a is the only element satisfied $\forall x, (x \in A \to a \precsim x)$.
By the definition, b is the only element satisfied $\forall x, (x \in B \to b \precsim x)$.
$\forall x, (x \in A \to a \precsim x) \to \exists y_1, (y_1 = f(a) \land \forall y, (y \in B \to y_1 \precsim y))$
We can conclude $f(a) = b$.
Let A be $A \backslash a, B$ be $B \backslash b$, repeat the step above, until $A = B = \emptyset$
We can see that each element in A can only be mapped into a certain element in B.
So there is only one insomorphic mapping $f : A \to B$.

题目 4 (SM Problem 14.71)

解答:
(a)1 or p^k, p is prime and $k \in \mathbb{N}^+$.
(b)All prime numbers.

题目 5 (SM Problem 14.72)

解答:
(a)
Since $b \land c <= b$ and $b <= a \lor b$, we can conclude that $b \land c <= a \lor b$.
Since $b \land c <= c$ and $c <= a \lor c$, we can conclude that $b \land c <= a \lor c$.
So we can conclude $b \land c$ is a lower bound of $\{(a \lor b), (a \lor c)\}$, and $b \land c <= (a \lor b), \land (a \lor c)$.
Since $a <= a \lor b$ and $a <=$a is a low bound of $\{(a \lor b), \land (a \lor c)\}$,
so $a <= (a \lor b) \land (a \lor c)$.
Since $b \land c <= (a \lor b) \land (a \lor c)$ and $a <= (a \lor b) \land (a \lor c)$, we can conclude $(a \lor b) \land (a \lor c)$ is an upper bouond of $\{a, (b \land c)\}$, so $a \lor (b \land c) <= (a \lor b) \land (a \lor c)$.
(b)
Since $a \land b <= b$ and $b <= b \lor c$, we can conclude $a \land b <= b \lor c$.
Since $a \land c <= c$ and $c <= b \lor c$, we can conclude $a \land c <= b \lor c$.
So we can conclude $b \lor c >= (a \land b) \lor (a \land c)$.
Since $b \land c >= (a \land b) \lor (a \land c)$ and $a >= (a \land b) \lor (a \land c)$, we can conclude $(a \land b) \lor (a \land c)$ is a lower bound of $\{a, (b \lor c)\}$, so $a \land (b \lor c) >= (a \land b) \lor (a \lor c)$

题目 6 (SM Problem 14.75)

解答：
(a)Since $a <= c$, $a \vee c = c$.
$a \vee (b \wedge c)$
$= (a \vee b) \wedge (a \vee c)$
$= (a \vee b) \wedge c$
(b)Let $f(x, y, z) = x\ lor(y \wedge z), g(x, y, z) = (x \vee y) \wedge z$.
$\forall y, z \in (b), f(0, y, z) = y \wedge z = g(0, y, z)$.
$\forall x, y \in (b), f(x, y, 1) = x \wedge y = g(x, y, 1)$.
Consider other cases, we have that:
$f(a, 0, a) = a = g(a, 0, a), f(a, a, a) = a = g(a, a, a), f(a, 1, a) = a = g(a, 1, a), f(a, b, a) = a = g(a, b, a), f(a, c, a) = a = g(a, c, a)$
$f(b, 0, b) = b = g(b, 0, b), f(b, b, b) = b = g(b, b, b), f(b, 1, b) = b = g(b, 1, b), f(b, a, b) = b = g(b, a, b), f(b, c, b) = b = g(b, c, b)$
$f(c, 0, c) = c = g(c, 0, c), f(c, c, c) = c = g(c, c, c), f(c, 1, c) = c = g(c, 1, c), f(c, b, c) = c = g(c, b, c), f(c, a, c) = c = g(c, a, c)$
Since $\forall x, y, z \in (b)$, x <= z $\rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$, the lattice is modelar.
(c)
$a <= c$ in Fig.(a).
$a \vee (b \wedge c) = a$ but $(a \vee b) \wedge c = c$.
So $a \vee (b \wedge c) \neq (a \vee b) \wedge c$.
It is non-modular.

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2 作业（选做部分）

3 Open Topics

Open Topics 1 (Dilworth's Theorem)
介绍 Dilworth's theorem，如 (不限于):

- 定理
- 证明
- 应用

参考资料:

- Dilworth's theorem @ wiki
- Chapter 6 of Book "A Course in Combinatorics" (2nd Edition) by J.H. van Lint and R.M. Wilson

Open Topics 2 (Lattice of Stable Matchings)
请从 Distributive Lattice 的角度介绍 Stable Matching 问题, 如 (不限于):

- Stable Matching 问题
- Stable Matching 算法
- 与 Distributive Lattice 的关系

参考资料:

- Lattice of stable matchings @ wiki
- Stable Marriage Problem @ Numberphile

4 订正

5 反馈