第 3-6 讲: 树

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评分： ______ **评阅：** ______

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请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

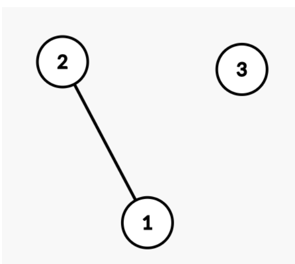
1 作业 (必做部分)

题目 1 (CZ 4.4)
Let G be a connected graph and let $e1$ and $e2$ be two edges of G. Prove that $G-e1-e2$ has three components if and only if both $e1$ and $e2$ are bridges in G.

解答:
若 $e1$ 和 $e2$ 为割边，$G-e1$ 为有两个连通分支 $G1$ 和 $G2$。假设边 $e2$ 在连通分支 $G1$ 中。在这个连通分支中将 $e2$ 去掉，则该连通分支会新成两个连通分支 $G3, G4$。此时即为 $G-e1-e2$，共有 $G2, G3, G4$ 三个连通分支。
若 $G-e1-e2$ 有三个连通分量，由于 G 连通，破坏连通性的方法为去掉割边，且去掉一条割边，只会多生成一个连通分支。因此，$e1, e2$ 必为割边。
综上，得证。

题目 2 (CZ 4.9)
Show that a graph of order n and size n − 1 need not be a tree.

解答:



n = 3

题目 3 (CZ 4.14)
A certain tree T of order 35 is known to have 25 vertices of degree 1, two vertices of degree 2, three vertices of degree 4, one vertex of degree 5 and two vertices of degree 6. It also contains two vertices of the same (unknown) degree x. What is x?

解答:
由图论第一定理，25 · 1+2 · 2+3 · 4+1 · 5+2 · 6+2 · x=2 · 34
解得 $x = 5$

题目 4 (CZ 4.22)
Let T be a tree of order n. Show that the size of the complement of $\overline{T}$ is the same as the size of K_{n-1}.

解答:
K_{n-1} 的边数是 $\frac{(n-1)(n-2)}{2}$
$\overline{T}$ 的边数是 $\frac{n(n-1)}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$

题目 5 (CZ 4.26)
Prove that an edge e of a connected graph is a bridge if and only if e belongs to every spanning tree of G.

解答:
充分:
假设最后的生成树中不含这条割边，则根据定义，去掉这条割边后的图不连通。
这与生成树的连通性矛盾。
故生成树上一定有这条割边
充分:
若其不为割边，则可在其所在环上找一条边替代他，得到一个没有该边的生成树，故矛盾

题目 6 (CZ 4.28)

Apply both Kruskal's and Prim's Algorithms to find a minimum spanning tree in the weighted graph in Figure 4.12. In each case, show how this tree is constructed, as in Figures 4.8 and 4.9.

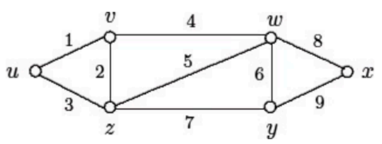


Figure 4.12: The weighted graph in Exercise 4.28

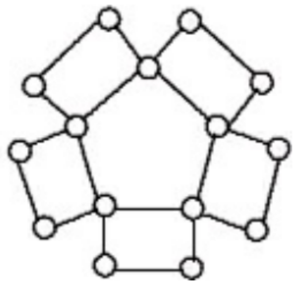
解答:
Kruscal 的加边顺序：$(u, v, 1), (v, z, 2), (v, w, 4), (w, y, 6), (w, x, 8)$
Kruscal 的加边顺序（假设初使点为 u）：$(u, v, 1), (v, z, 2), (v, w, 4), (w, y, 6), (w, x, 8)$

题目 7 (CZ 4.30)
Let G be a connected weighted graph and T a minimum spanning tree of G. Show that T is a unique minimum spanning tree of G if and only if the weight of each edge e of G that is not in T exceeds the weight of every other edge on the cycle in $T+e$.

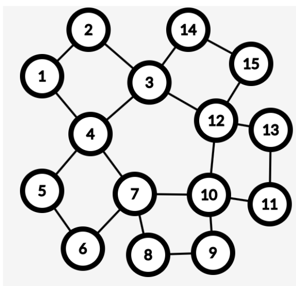
解答:
必要性:
若 T 是唯一一棵最小生成树，则若 G 中不属于 T 的任一边 e 的权值都小于等于 $T+e$ 的圈上某边的权值，则将该边去掉，加上边 e，则可得到一棵权值相等或更小的生成树，与假设违背。故 G 中不属于 T 的任一边 e 的权值都大于 $T+e$ 的圈上某边的权值
充分性:
若 G 中不属于 T 的任一边 e 的权值都大于 $T+e$ 的圈上某边的权值，假设存在另一棵最小生成树 $T1$，设 $e=(u,v)\in T$ 且 $e\notin T1$, 则在 $T1$ 中 $u\to v$ 的路径上，必存在一条边 $e2$ 不在 T 中。由前提可知 $e2$ 的权值大于 e，显然将 $e2$ 替换为 e 更优。故不存在 $T2$, 最小生成树唯一。

题目 8 (CZ 4.36)
Find the number of spanning trees in the graph G

$G:$



解答:



先将其进行点标号。

其矩阵 G 如上。

其矩阵 C 如上。

```
0 1 0 1 0 0 0 0 0 0 0 0 0 0 0
1 0 1 0 0 0 0 0 0 0 0 0 0 0 0
0 1 0 1 0 0 0 0 0 0 0 1 0 1 0
1 0 1 0 1 0 1 0 0 0 0 0 0 0 0
0 0 0 1 0 1 0 0 0 0 0 0 0 0 0
0 0 0 0 1 0 1 0 0 0 0 0 0 0 0
0 0 0 1 0 1 0 1 0 1 0 0 0 0 0
0 0 0 0 0 0 1 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 1 0 1 0 0 0 0 0
0 0 0 0 0 0 1 0 1 0 1 1 0 0 0
0 0 0 0 0 0 0 0 0 1 0 0 1 0 0
0 0 1 0 0 0 0 0 0 1 0 0 1 0 1
0 0 0 0 0 0 0 0 0 0 1 1 0 0 0
0 0 1 0 0 0 0 0 0 0 0 0 0 0 1
0 0 0 0 0 0 0 0 0 0 0 1 0 1 0
```

```
2 -1 0 -1 0 0 0 0 0 0 0 0 0 0 0
-1 2 -1 0 0 0 0 0 0 0 0 0 0 0 0
0 -1 4 -1 0 0 0 0 0 0 0 -1 0 -1 0
-1 0 -1 4 -1 0 -1 0 0 0 0 0 0 0 0
0 0 0 -1 2 -1 0 0 0 0 0 0 0 0 0
0 0 0 0 -1 2 -1 0 0 0 0 0 0 0 0
0 0 0 -1 0 -1 4 -1 0 -1 0 0 0 0 0
0 0 0 0 0 0 -1 2 -1 0 0 0 0 0 0
0 0 0 0 0 0 0 -1 2 -1 0 0 0 0 0
0 0 0 0 0 0 -1 0 -1 4 -1 -1 0 0 0
0 0 0 0 0 0 0 0 0 -1 2 0 -1 0 0
0 0 -1 0 0 0 0 0 0 -1 0 4 -1 0 -1
0 0 0 0 0 0 0 0 0 0 -1 -1 2 0 0
0 0 -1 0 0 0 0 0 0 0 0 0 0 2 -1
0 0 0 0 0 0 0 0 0 0 0 -1 0 -1 2
```

经过计算，其共有 3840 个生成树。

2　作业（选做部分）

3　Open Topics

Open Topics 1 (Chu–Liu/Edmonds algorithm)

- In graph theory, an arborescence is a directed graph in which, for a vertex u called the root and any other vertex v, there is exactly one directed path from u to v.

- In graph theory, Edmonds' algorithm or Chu–Liu/Edmonds' algorithm is an algorithm for finding a spanning arborescence of minimum weight (sometimes called an optimum branching).

  参考资料：https://en.wikipedia.org/wiki/Edmonds%27_algorithm

Open Topics 2 (Minimum bottleneck spanning tree)

In mathematics, a minimum bottleneck spanning tree (MBST) in an undirected graph is a spanning tree in which the most expensive edge is as cheap as possible.

参考资料：https://en.wikipedia.org/wiki/Minimum_bottleneck_spanning_tree

4　反馈