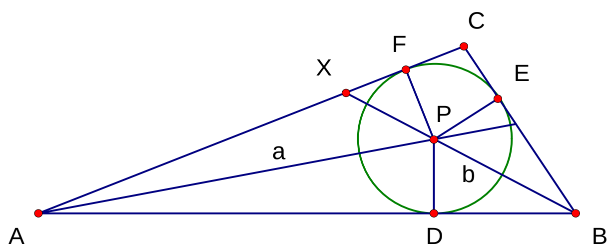
PS#1 Solutions
Math 355 Spring 2011

p.6 #7. *Prove that the inverse of an isometry is an isometry.*
Proof: Let α be an isometry. Then α is bijective by Proposition 15 and Theorem 19. Thus, given points $P, Q \in \mathbb{R}^2$, there exist unique points $P', Q' \in \mathbb{R}^2$ such that $\alpha(P') = P$ and $\alpha(Q') = Q$. By Definition 8, the inverse of α is the function $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ such that $P' = \beta(P)$ and $Q' = \beta(Q)$. But $QP = Q'P'$ since α is an isometry, so β is an isometry as well.

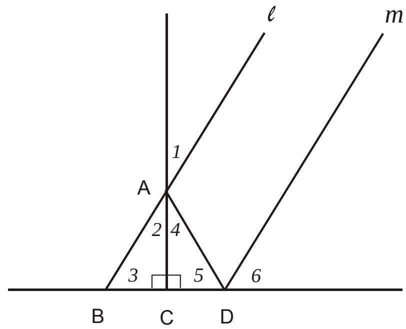
p.6 #10. *Prove that the interior angle bisectors of a non-degenerate triangle are concurrent at some point P in the interior of the triangle.*
Proof: *Let* A, B, and C be distinct non-collinear points and consider $\triangle ABC$. Let a and b be the respective interior angle bisectors of $\angle A$ and $\angle B$, and let $X = b \cap \overline{AC}$ (see diagram below). Since a is also the interior angle bisector of $\angle A$ in $\triangle ABX$, the point $P = a \cap \overline{BX}$ lies between B and X in the interior of $\triangle ABC$. Let D, E, and F be the feet of the perpendiculars from P to $\overline{AB}$, from P to $\overline{BC}$, and from P to $\overline{AC}$, respectively. Then $\triangle APD \cong \triangle APF$ and $\triangle BPD \cong \triangle BPE$ by AAS, since each pair of right triangles has a shared hypotenuse and a pair of congruent interior angles determined by a or b, and $PD = PE = PF$ by CPCTC. Now consider ray $c = \overrightarrow{CP}$; note that right triangles $\triangle CPE$ and $\triangle CPF$ are congruent by HL, and c is the interior angle bisector of $\angle C$ by CPCTC. Thus all three interior angle bisectors a, b, and c are concurrent at P.



p.6 #17. *Prove that an isometry preserves betweenness.*
Proof: Let α be an isometry and let A, B, and C be distinct collinear points with $A - B - C$. Then $AB + BC = AC$, by the degenerate case of the triangle inequality. Let $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. Since α is an isometry, $A'B' = AB$, $B'C' = BC$, and $A'C' = AC$. By substitution we have $A'B' + B'C' = A'C'$. Therefore A', B', and C' are collinear by the degenerate case of the triangle inequality, and $A' - B' - C'$.

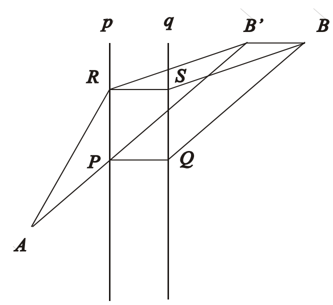
p.6 #18. *Prove that an isometry preserves angle.*
Proof: Let α be an isometry, let A, B, and C be distinct points, and let $A' = \alpha(A)$, $B' = \alpha(B)$, and $C' = \alpha(C)$. I claim that $\angle ABC \cong \angle A'B'C'$. Since α is an isometry, $A'B' = AB$, $B'C' = BC$, and $A'C' = AC$, so that $\triangle ABC \cong \triangle A'B'C'$ by SSS and $\angle ABC \cong \angle A'B'C'$ by CPCTC (triangles $\triangle ABC$ and $\triangle A'B'C'$are degenerate when A, B, and C are collinear, in which case $m\angle ABC = m\angle A'B'C' = 0°$).

p.12 #7. *Light is reflected by two perpendicular mirrors. Show that the emerging ray is parallel to the initial ray.*
Proof: Let l be the line containing the incoming ray; let m be the line containing the emerging ray; let C be the point at which the two mirrors intersect. Assume l intersects the first mirror at A and m intersects the second mirror at D. Let $B = \overleftrightarrow{CD} \cap l$ (see diagram below). By a previous problem, the angle of incidence equals the angle of reflection, so referring to the diagram below we have $\angle 1 \cong \angle 4$ and $\angle 5 \cong \angle 6$. Since vertical angles are equal, $\angle 2 \cong \angle 1$ and side $\overline{AC}$ is shared by the right triangles $\triangle ABC$ and $\triangle ADC$. So $\triangle ABC \cong \triangle ADC$ by ASA. Hence $\angle 3 \cong \angle 5$ since CPCTC. But line $\overleftrightarrow{CD}$ is transverse to ℓ and m and corresponding angles $\angle 3$ and $\angle 6$ are congruent. Therefore $l \parallel m$.



p.12 #17. *Let l and m be lines. Prove that if $\sigma_m(l) = l$, then either $l \perp m$ or $l = m$.*
Proof: Assume $l \neq m$ and choose a point P on l and off m. Let $P' = \sigma_m(P)$; then $P \neq P'$ and m is the perpendicular bisector of $\overline{PP'}$, by definition of σ_m. But by assumption, $\sigma_m(l) = l$, hence P' is also on l and it follows that $l = \overleftrightarrow{PP'} \perp m$.

p.20 #1b. *A river with parallel banks p and q is to be spanned by a bridge at right angles to p and q. Cities A and B are on opposite sides of the river. Where should the bridge be located so that the distance from city A to city B is a minimum?*
Solution: Let ℓ be any line perpendicular to p, let $R = \ell \cap p$, let $S = \ell \cap q$, let $B' = \tau_{\mathbf{SR}}(B)$, let $P = p \cap \overleftrightarrow{AB'}$ and let Q be the point on q such that $\overleftrightarrow{PQ} \perp q$ (see diagram below). Note that $AB' = AP + PB'$. I Claim that $AP + PQ + QB$ is the minimum distance from city A to city B via the bridge $\overline{PQ}$ perpendicular to the river banks, i.e., if $\overline{RS}$ is distinct from $\overline{PQ}$, we must show that $AP + PQ + QB < AR + RS + SB$. Since $\mathbf{SR} = \mathbf{QP}$ we have $B' = \tau_{\mathbf{SR}}(B) = \tau_{\mathbf{QP}}(B)$ so that $\square BB'RS$ and $\square BB'PQ$ are parallelograms by definition of a translation. Thus $AR + RS + SB = AR + RB' + B'B$ and $AP + PQ + QB = AP + PB' + B'B$. But by the triangle inequality, $AB' < AR + RB'$ so that $AP + PQ + QB = AP + PB' + B'B = AB' + B'B < AR + RB' + BB' = AR + RS + SB$ as claimed.



p.20 #6. *Prove that $\tau_{\mathbf{AB}}^{-1} = \tau_{\mathbf{BA}}$.*
Proof: By Proposition 40 we have $\tau_{\mathbf{AB}}^{-1} = \tau_{-\mathbf{AB}}$. Since $-\mathbf{AB} = \mathbf{BA}$ we have $\tau_{\mathbf{AB}}^{-1} = \tau_{\mathbf{BA}}$.

p.24 #8. *Let $P = \begin{bmatrix} a \\ b \end{bmatrix}$ and $Q = \begin{bmatrix} c \\ d \end{bmatrix}$ be distinct points. Find the equations of $\varphi_B \circ \varphi_A$ and prove that the composition $\varphi_B \circ \varphi_A$ is a translation τ. Find the vector of τ.*
Proof: The equations of φ_P are $x' = 2a - x$ and $y' = 2b - y$; the equations of φ_Q are $x' = 2c - x$ and $y' = 2d - y$. Composing these equations gives: $x' = 2c - (2a - x) = x + 2(c - a)$ and $y' = 2d - (2b - y) = y + 2(d - b)$. Thus by Proposition 37, $\varphi_Q \circ \varphi_P$ is a translation τ with vector $\begin{bmatrix} 2(c-a) \\ 2(d-b) \end{bmatrix}$.