第 4-3 讲: 群同态基本定理与正规子群

姓名: 林凡琪　　**学号:** 211240042

评分: ______　　**评阅:** ______

2023 年 3 月 15 日

请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

1 作业 (必做部分)

题目 1 (TJ 9-11)

解答:
$Z/8Z, (Z/4Z)\times(Z/2Z), (Z/2Z)\times(Z/2Z)\times(Z/2Z)$, $D4$ (or $D8$) and the quaternion group $Q8$.

题目 2 (TJ 9-16)

解答:
The order of an element in a direct product of groups is the least common multiple of the orders of its components.
(a)the order of (3, 4) in $Z_4 \times Z_6$ is lcm(4, 6) = 12.
(b) The order of (6, 15, 4) in $Z_{30} \times Z_{45} \times Z_{24}$ is lcm(5, 3, 6) = 30.
(c)The order of (5, 10, 15) in $Z_{25} \times Z_{25} \times Z_{25}$ is lcm(5, 5, 5) = 5.
(d) The order of (8, 8, 8) in $Z_{10} \times Z_{24} \times Z_{80}$ is lcm(5, 3, 10) = 30.

题目 3 (TJ 9-23)

解答:
The assertion is false.
Disproof:
A counterexample is given by $G = Z_2\times Z_4, H = Z_4\times Z_2$, and $K = Z_2$. Then $G\times K \cong H\times K \cong Z_2\times Z_4\times Z_2$, but $G \cong H$. This is because the direct product of groups is commutative and associative up to isomorphism, but the order of factors matters for the isomorphism type of the group.

题目 4 (TJ 10-1(a,c))

解答:

A subgroup H of a group G is normal if and only if $gH = Hg$ for all $g \in G$. This means that every left coset of H is equal to a right coset of H. Equivalently, H is normal if and only if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. This means that aevery element of H is conjugate to itself by any element of G.

(a) The subgroup A_4 of S_4 is normal because it is the kernel of the sign homomorphism from S_4 to Z_2. Alternatively, we can check that every element of A_4 is conjugate to itself by any element of S_4. For example, $(123)(14)(123)^{-1} = (134)$ is in A_4. The factor group S_4/A_4 has order 2 and consists of the cosets A_4 and $(12)A_4$. The Cayley table for S_4/A_4 is:

	A_4	$(12)A_4$
A_4	A_4	$(12)A_4$
$(12)A_4$	$(12)A_4$	A_4

(c) The subgroup D_4 of S_4 is not normal because it is not invariant under conjugation by some elements of S_4. For example, $(13)(12)(13)^{-1} = (23)$ is not in D_4. Therefore, we cannot form a factor group G/H in this case.

题目 5 (TJ 10-11)

解答:

Let g be any element of G and let h be any element of H. Then $ghg^(-1)$ is also an element of G with order k, since $(ghg(-1))k = ghkg(-1) = g1g^(-1) = 1$. Therefore, $ghg^(-1)$ must belong to H, since H is the only subgroup of G with order k. This shows that H is normal in G.

题目 6 (TJ 10-12)

解答:

The centralizer of an element g in a group G is the set of elements of G that commute with g, or in other words, that satisfy $xg = gx$ for any x in $C(g)$. To show that $C(g)$ is a subgroup of G, we need to check three conditions:

- $C(g)$ is non-empty. This is true because g belongs to $C(g)$, since $gg = gg$.

- $C(g)$ is closed under the group operation. This means that if x and y belong to $C(g)$, then xy also belongs to $C(g)$. To see this, note that $(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy)$, using the fact that x and y commute with g.

- $C(g)$ is closed under taking inverses. This means that if x belongs to $C(g)$, then x^{-1} also belongs to $C(g)$. To see this, note that $x^{-1}g = (xg)^{-1} = (gx)^{-1} = g^{-1}x^{-1} = x^{-1}g^{-1}$, using the fact that x commutes with g and the inverse property.

Therefore, $C(g)$ is a subgroup of G. If g generates a normal subgroup of G, then we need to show that $C(g)$ is normal in G. This means that for any x in G and

y in $C(g)$, we have xyx^{-1} in $C(g)$. To see this, note that $(xyx^{-1})g = xygx^{-1} = xyx^{-1}xg = yg$, using the fact that y commutes with g and the inverse property. Similarly, $(gxyx^{-1}) = gyx^{-1} = yx^{-1}gx^{-1} = xyx^{-1}g$. Therefore, (xyx^{-1}) commutes with g and belongs to C(g).

题目 7 (TJ 11-5)

解答:

A homomorphism from Z_{24} to Z_{18} is a function that preserves the group operation, that is, $f(x+y) = f(x) + f(y)$ for any $x, y \in Z_{24}$. Such a function is completely determined by the value of $f(1)$, since $f(n) = nf(1)$ for any $n \in Z_{24}$. Moreover, the value of $f(1)$ must satisfy two conditions:

-It must be an element of Z_{18}, that is, an integer between 0 and 17.

-It must divide 24, that is, it must be a multiple of 3.

The second condition follows from the fact that the order of $f(1)$ in Z_{18} must divide the order of 1 in Z_{24}, which is 24. Therefore, there are only six possible values for $f(1)$: 0, 3, 6, 9, 12, and 15. For each of these values, we can define a homomorphism by

$$f_k(n) = 3kn \mod 18$$

where k is the value of $f(1)$. For example, if $f(1) = 9$, then

$$f_9(n) = 27n \mod 18 = 9n \mod 18$$

These are all the homomorphisms from Z_{24} to Z_{18}.

题目 8 (TJ 11-2(b,d,e))

Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel? (b)$\phi : \mathbb{R} \to GL_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

(d)$\phi : GL_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = b,$$

where $\mathbb{M}_2(\mathbb{R})$ is the additive group of 2×2 matrices with entries in $\mathbb{R}$.

解答:

一个同态映射是一个保持群运算的映射，即 $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$，其中 g_1, g_2 是群的元素。一个同态映射的核是所有映到群的单位元的元素组成的集合，即 $\ker(\phi) = g \in G|\phi(g) = e$，其中 e 是群的单位元。核是一个群的子群。

(b) $\phi:\mathbb{R}\to GL_2(\mathbb{R})$ 是一个同态映射.
因为

$$\begin{aligned}\phi(a+b)&=\begin{pmatrix}1&0\\a+b&1\end{pmatrix}\\&=\begin{pmatrix}1&0\\a&1\end{pmatrix}\begin{pmatrix}1&0\\b&1\end{pmatrix}\quad=\phi(a)\phi(b)\end{aligned}$$

它的核是 $\ker(\phi)=0$，因为只有当 $a=0$ 时，$\phi(a)$ 才是 $GL_2(\mathbb{R})$ 的单位元。

(d) $\phi:GL_2(\mathbb{R})\to\mathbb{R}^*$ 是一个同态映射，因为

$$\begin{aligned}&\phi\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\right)\\=&\phi\left(\begin{pmatrix}a_1a_2+b_1c_2&a_1b_2+b_1d_2\\c_1a_2+d_1c_2&c_1b_2+d_1d_2\end{pmatrix}\right)\\=&(a_1a_2+b_1c_2)(c_1b_2+d_1d_2)-(a_1b_2+b_1d_2)(c_1a_2+d_1c_2)\\=&(a_1d_1-b_1c_1)(a_2d_2-b_2c_2)\\=&\phi\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}\right)\phi\left(\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\right)\end{aligned}$$

它的核是 $\ker(\phi)=SL_2(\mathbb{R})$，即所有行列式为 1 的矩阵组成的集合。

(e) $\phi:\mathbb{M}_2(\mathbb{R})\to\mathbb{R}$ 不是一个同态映射。要看到这一点，让 $A,B\in\mathbb{M}_2(\mathbb{R})$ 并计算：

$$\phi(A+B)=\phi\left(\begin{pmatrix}a_1&b_1\\c_1&d_1\end{pmatrix}+\begin{pmatrix}a_2&b_2\\c_2&d_2\end{pmatrix}\right)=\phi\left(\begin{pmatrix}a_1+a_2&b_1+b_2\\c_1+c_2&d_1+d_2\end{pmatrix}\right)$$

$$=b_1+b_2$$

但是

$$\phi(A)+\phi(B)=b_1+b_2$$

所以，$\phi(A+B)=\phi(A)+\phi(B)$ 只有当 $b_1=b_2=0$ 时才成立，这对于 $\mathbb{M}_2(\mathbb{R})$ 中的所有矩阵都不成立。因此，ϕ 不是同态。

由于 ϕ 不是同态，ϕ 的核不是良定义的。然而，如果我们仍然想要找到被 ϕ 映射到零的矩阵的集合，我们需要解 $\phi(A)=0$，其中 A 是一个 2×2 矩阵。这给我们 $b=0$，所以集合是形式为 $\begin{pmatrix}a&0\\c&d\end{pmatrix}$ 的矩阵的集合，其中 $a,c,d\in\mathbb{R}$。

2　作业 (选做部分)

题目 1 (SageMath 学习)
学习 TJ 第 9、10/11 章关于 SageMath 的内容

解答:

题目 2 (TJ 11-17)

解答:

题目 3 (6、8 阶群)
请给出同构意义下的所有 6 阶、8 阶群。

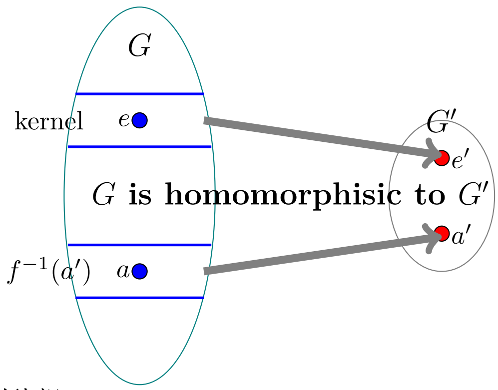
解答:
同构意义下的所有 6 阶群, 两种: 循环群和三元对称群。同构意义下的所有 8 阶群, 五种: 循环群、四元数群、二阶循环群的直积、二阶循环群的半直积和二阶对称群。

---

3 Open Topics

Open Topics 1 (群同态第二定理)
请证明群同态第二定理。

Open Topics 2 (同态猜想)



请证明或证否下列猜想

- Kernel 和任意的 G' 中非单位元元素的逆像不相交
- Kernel 和任意的 G' 中非单位元元素的逆像同势
- 任意的 G' 中元素的逆像不相交且同势
- 任意的 G' 中元素的逆像必定是 kernel 的某个陪集

4 反馈