第 9 讲: 关系及其性质

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评分： ______　　**评阅：** ______

2019 年 12 月 1 日

请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

- “关系”至关重要

1 作业（必做部分）

题目 1 (UD Problem 10.9)
Define a relation $\sim$ on $\mathbb{R}^2$ as follows: For $(x1, x2)$,$(y1, y2) \in \mathbb{R}^2$, we say that $(x1, x2) \sim (y1, y2)$ if and only if both $x1-y1$ and $x2-y2$ are even integers. Is this relation an equivalence relation? Why or why not?

解答：
This relation is an equivalence relation.
reflexive:
let $x1, x2 \in \mathbb{R}$.
$x1 - x1 \equiv 0 \pmod 2 \wedge x2 - x2 \equiv 0 \pmod 2$
$\rightarrow (x1, x2) \sim (x1, x2)$
symmetric:
let $x1, x2, y1, y2 \in \mathbb{R}$.
$(x1, x2) \sim (y1, y2)$
$\rightarrow x1 - y1 \equiv 0 \pmod 2 \wedge x2 - y2 \equiv 0 \pmod 2$
$\rightarrow y1 - x1 \equiv 0 \pmod 2 \wedge y2 - x2 \equiv 0 \pmod 2$
$\rightarrow (y1, y2) \sim (x1, x2)$
transitive :
let $x1, x2, y1, y2, z1, z2 \in \mathbb{R}$.
$(x1, x2) \sim (y1, y2) \wedge (y1, y2) \sim (z1, z2)$
$\rightarrow x1 - y1 \equiv 0 \pmod 2 \wedge x2 - y2 \equiv 0 \pmod 2 \wedge y1 - z1 \equiv 0 \pmod 2 \wedge y2 - z2 \equiv 0 \pmod 2$
$\rightarrow x1 - z1 \equiv 0 \pmod 2 \wedge x2 - z2 \equiv 0 \pmod 2$
$\rightarrow (x1, x2) \sim (z1, z2)$

题目 2 (UD Problem 10.10)
Let X be a nonempty set with an equivalence relation $\sim$ on it. Prove that for all elements x and y in X, the equality $E_x = E_y$ holds if and only if $x \sim y$.

解答:
Let $x, y \in X$.
$E_x = E_y$
$\to E_x = E_y \wedge x \in E_x$(reflexive)
$\to x \in E_y$
$\to y \sim x.$
$\to x \sim y.$

Let $x, y, z \in X$.
$x \sim y.$
$\to \forall z, (x \sim z \to y \sim z)$
$\to \forall z, (z \in E_x \to z \in E_y)$
$\to E_x \subseteq E_y.$
Similarly, we have that $E_y \subseteq E_x$.
So $E_x = E_y$.

题目 3 (UD Problem 10.13)
Recall that a **polynomial** p over $\mathbb{R}$ is an expression of the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0$ where each $a_j \in \mathbb{R}$ and $n \in \mathbb{N}$. The largest integer j such that $a_j \neq 0$ is the **degree** of p. We define the degree of the constant polynomial $p = 0$ to be $-\infty$. (A polynomial over $\mathbb{R}$ defines a function $p : \mathbb{R} \to \mathbb{R}$.

(a) Define a relation on the set of polynomials by $p \sim q$ if and only if $p(0) = q(0)$. Is this an equivalence relation? If so, what is the equivalence class of the polynomial given by $p(x) = x$?

(b)Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is the same as the degree of q. Is this an equivalence relation? If so, what is E_r if $r(x) = 3x + 5$?

(c)Define a relation on the set of polynomials by $p \sim q$ if and only if the degree of p is less than or equal to the degree of q. Is this an equivalence relation? If so, what is E_r, where $r(x) = x^2$?

解答:
(a)This is an equivalence relation.
reflexive:
$p(0) = p(0) \to p \sim p.$
symmetric:
$p \sim q \to p(0) = q(0) \to q(0) = p(0) \to q \sim p.$
transitive:
Let f be a polynomial over $\mathbb{R}$.
$p \sim q \wedge q \sim f$
$\to p(0) = q(0) \wedge q(0) = f(0)$
$\to p(0) = f(0)$
$\to p \sim f.$
$E_p = \{y \in \text{polynomial over } \mathbb{R} : y(0) = 0\}.$

(b)This is an equivalence relation.
Let $D(x)$ means the degree of the polynomial x.
reflexive:
$D(p) = D(p) \rightarrow p \sim p$.
symmetric:
$p \sim q \rightarrow D(p) = D(q) \rightarrow D(q) = D(p) \rightarrow q \sim p$.
transitive:
Let f be a polynomial.
$p \sim q \wedge q \sim f$
$\rightarrow D(p) = D(q) \wedge D(q) = D(f)$
$\rightarrow D(p) = D(f)$
$\rightarrow p \sim f$.
$E_r = \{y \in$polynomial over $\mathbb{R}$:$D(y) = 1\}$.

(c)This is not an equivalence relation.
It is not symmetric.

题目 4 (UD Problem 11.4)
(a) For each $r \in \mathbb{R}$, let $A_r = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = r\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.
(b) For each $r \in \mathbb{R}$, let $A_r = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$. Is this a partition of $\mathbb{R}^3$? If so, give a geometric description of the partitioning sets.

解答:
(a)This is a partition of $\mathbb{R}^3$.
(i)$\forall r, (r \in \mathbb{R} \rightarrow (r, 0, 0) \in A_r) \rightarrow A_r \neq \emptyset$.
(ii)$\forall (x, y, z), ((x, y, z) \in \bigcup_{r \in \mathbb{R}} A_r \rightarrow (x, y, z) \in \mathbb{R}^3)$.
$\rightarrow \bigcup_{r \in \mathbb{R}} A_r \subseteq \mathbb{R}^3$ (1)
$(x, y, z) \in \mathbb{R}^3$
$\rightarrow \exists p, (p \in \mathbb{R} \wedge p = x + y + z)$
$\rightarrow (x, y, z) \in A_p$
$\rightarrow (x, y, z) \in \bigcup_{r \in \mathbb{R}} A_r$.
So $\mathbb{R}^3 \subseteq \bigcup_{r \in \mathbb{R}} A_r$ (2)
$(1) \wedge (2) \rightarrow \mathbb{R}^3 = \bigcup_{r \in \mathbb{R}} A_r$
(iii)$A_s \cap A_r \neq 0$
$\rightarrow \exists (x, y, z), ((x, y, z) \in \mathbb{R}^3 \wedge (x, y, z) \in A_s \wedge (x, y, z) \in A_r)$
$\rightarrow s = x + y + z \wedge r = x + y + z$
$\rightarrow s = r$
$\rightarrow A_s = A_r$
description:The partitions are planes.

(b)This is a partition of $\mathbb{R}^3$.
(i)$\forall r$,$(r \in \mathbb{R} \rightarrow (r, 0, 0) \in A_r) \rightarrow A_r \neq \emptyset$
(ii)$\forall (x, y, z), ((x, y, z) \in \bigcup_{r \in \mathbb{R}} A_r \rightarrow (x, y, z) \in \mathbb{R}^3)$.
$\rightarrow \bigcup_{r \in \mathbb{R}} A_r \subseteq \mathbb{R}^3$ (1)
$(x, y, z) \in \mathbb{R}^3$
$\rightarrow \exists s, (s \in \mathbb{R} \wedge s^2 = x^2 + y^2 + z^2)$
$\rightarrow (x, y, z) \in A_s$
$\rightarrow (x, y, z) \in \bigcup_{r \in \mathbb{R}} A_r$
So $\mathbb{R}^3 \subseteq \bigcup_{r \in \mathbb{R}} A_r$ (2)
$(1) \wedge (2) \rightarrow \bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}^3$

(iii)$A_s \cap A_r \neq \emptyset$
$\rightarrow \exists(x,y,z),((x,y,z) \in \mathbb{R}^3 \wedge (x,y,z) \in A_s \wedge (x,y,z) \in A_r)$
$\rightarrow s^2 = x^2+y^2+z^2 \wedge r^2 = x^2+y^2+z^2$
$\rightarrow s^2 = r^2$
$\rightarrow A_s = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 = s^2\} = \{(x,y,z) \in \mathbb{R}^3, x^2+y^2+z^2 = r^2\} = A_r$.
description:The discriptions are the spheres.

题目 5 (UD Problem 11.8)
Consider the set P of polynomials with real coefficients. Decide whether or not each of the following collection of sets determines a partition of P. If you decide that it does determine a partition, show it carefully. If you decide that it does not determine a partition, list the part(s) of the definition that is (are) not satisfied and justify your claim with an example. (See Problem 10.13 for more information about polynomials.)

(a) For $m \in \mathbb{N}$, let A_m denote the set of polynomials of degree m. The collection of sets is $\{A_m : m \in \mathbb{N}\}$.
(b) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(0) = c$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.
(c) For a polynomial q, let A_q denote the set of all polynomials p such that q is a factor of p; that is, there is a polynomial r such that $p = qr$. The collection of sets is $\{A_q : q \in P\}$.
(d) For $c \in \mathbb{R}$, let A_c denote the set of polynomials p such that $p(c) = 0$. The collection of sets is $\{A_c : c \in \mathbb{R}\}$.

解答:
(a)This is a partition of P.
(i)$\forall m,(m \in \mathbb{Z} \rightarrow x^m \in A(m)) \rightarrow A_m \neq \emptyset$.
(ii)$\forall p,(p \in \bigcup_{m\in\mathbb{Z}} A_m \rightarrow p \in P)$
$$\rightarrow \bigcup_{m\in\mathbb{Z}} A_m \subseteq P \tag{1}$$
$p \in P$
$\rightarrow \exists m,(m \in \mathbb{Z} \wedge \text{the degree of } p \text{ is m})$
$\rightarrow p \in A_m$
$\rightarrow p \in \bigcup_{m\in\mathbb{Z}} A_m$
$$\text{So } P \subseteq \bigcup_{m\in\mathbb{Z}} A_m \tag{2}$$
$(1) \wedge (2) \rightarrow \bigcup_{m\in\mathbb{Z}} A_m = P$.
(iii)$A_m \cap A_n \neq \emptyset$
$\rightarrow \exists p,(p \in P \wedge p \in A_m \wedge p \in A_n)$.
$\rightarrow$ the degree of p is $m \wedge$ the degree of p is n.
$\rightarrow m = n$
$\rightarrow A_m = A_n$.

(b)This is a partition of P.
(i)$\forall c,(c \in \mathbb{R} \rightarrow x + c \in A_c) \rightarrow A_c \neq \emptyset$.
(ii)$\forall p,(p \in \bigcup_{c\in\mathbb{R}} A_c \rightarrow p \in P)$
$$\rightarrow \bigcup_{c\in\mathbb{R}} A_c \subseteq P \tag{1}$$
$p \in P$
$\rightarrow \exists c,(c \in \mathbb{R} \wedge c = p(0))$
$\rightarrow p \in A_c$
$\rightarrow p \in \bigcup_{c\in\mathbb{R}} A_c$
$$\text{So } P \subseteq \bigcup_{c\in\mathbb{R}} A_c \tag{2}$$
$(1) \wedge (2) \rightarrow \bigcup_{c\in\mathbb{R}} A_c = P$.

(iii)$A_c \cap A_n \neq \emptyset$
$\rightarrow \exists p, (p \in P \wedge p \in A_c \wedge p \in A_n)$
$\rightarrow p(0) = c \wedge p(0) = n$
$\rightarrow c = n$
$\rightarrow A_c = A_n$.

(c)This is not a partition of P.
It does not meet the condition (iii).
Example:
Let q be 1 and s be 2.
We have that $2x \in A_q \cap A_s$ but $A_q \neq A_s$.

(c)This is not a partition of P.
It does not meet the condition (iii).
Example:
Let p be $x^2 - 6x + 5$
We have that $p \in A_1 \cap A_5$ but $A_1 \neq A_5$.

题目 6 (UD Problem 11.10)
Let X be a nonempty set and $\{A_\alpha : \alpha \in I\}$ be a partition of X.

(a) Let B be a subset of X such that $A_\alpha \cap B \neq \emptyset$ for every $\alpha \in I$. Is $\{A_\alpha \cap B : \alpha \in I\}$ a partition of B? Prove it or give a counterexample.

(b) Suppose further that $A_\alpha \neq X$ for every $\alpha \in I$. Is $\{X \backslash A_\alpha : \alpha \in I\}$ a partition of X? Prove it or show that it is not a partition. (Make sure you consider each of the following cases: the partition $\{A_\alpha : \alpha \in I\}$ has zero, one, two, or at least three elements.)

解答：
(a)This is a partition of B.
(i)We can get $\forall \alpha \in I, A_\alpha \cap B \neq \emptyset$ by definition.
(ii)$\forall x, (x \in \bigcup_{\alpha \in I}(A_\alpha \cap B) \rightarrow x \in B)$
$$\rightarrow \bigcup_{\alpha \in I}(A_\alpha \cap B) \subseteq B \tag{1}$$
$\bigcup_{\alpha \in I} A_\alpha = X \wedge B \subseteq X$
$\rightarrow B \subseteq \bigcup_{\alpha \in I} A_\alpha$
$$\rightarrow B \subseteq \bigcup_{\alpha \in I}(A_\alpha \cap B) \tag{2}$$
$(1) \wedge (2) \rightarrow \bigcup_{\alpha \in I}(A_\alpha \cap B) = B$.
(iii) $(A_\alpha \cap B) \cap (A_\beta \cap B) \neq \emptyset$
$\rightarrow A_\alpha \cap A_\beta \neq \emptyset$
$\rightarrow A_\alpha = A_\beta$(condition(iii) in X)
$\rightarrow A_\alpha \cap B = A_\beta \cap B$

(b)This is not a partition of X.
Let X={1,2,3} and $\bigcup_{\alpha \in I} A_\alpha = \{\{1\}, \{2\}, \{3\}\}$.
So $\bigcup_{\alpha \in I} X \backslash A_\alpha = \{\{2, 3\}, \{1, 3\}, \{1, 2\}\}$.
$\{2, 3\} \cap \{1, 3\} \neq \emptyset$ but $\{2, 3\} \neq \{1, 3\}$

题目 7 (UD Problem 12.11 (a, b))
Let S and T be nonempty bounded subsets of $\mathbb{R}$.
(a) Show that $\sup(S \cup T) \geq \sup S$, and $\sup(S \cup T) \geq \sup T$.
(b) Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

解答:
(a)
$\forall x, (x \in S \cup T \to x \leq \sup(S \cup T))$
$\to \forall x, (x \in S \to x \leq \sup(S \cup T))$
$\to \sup(S \cup T) \geq \sup S$.

$\forall x, (x \in S \cup T \to x \leq \sup(S \cup T))$
$\to \forall x, (x \in T \to x \leq \sup(S \cup T))$
$\to \sup(S \cup T) \geq \sup T$.

(b)
$\forall y(y \in S \to y \leq \sup S) \wedge \forall z(z \in T \to z \leq \sup T)$
$\to \forall x(x \in S \cup T \to x \leq \max\{\sup S, \sup T\})$ (1)

$(\forall M_1, (M_1 \in \mathbb{R} \wedge \forall y(y \in S \to M_1 \geq y) \to M_1 \geq \sup S)) \wedge (\forall M_2, (M_2 \in \mathbb{R} \wedge \forall z(z \in T \to M_2 \geq z) \to M_2 \geq \sup T))$
$\to \forall M, (M \in \mathbb{R} \wedge \forall x(x \in S \cup T \to M \geq x) \to M \geq \max\{\sup S, \sup T\})$ (2)

$(1) \wedge (2) \to \sup(S \cup T) = \max\{\sup S, \sup T\}$.

题目 8 (UD Problem 12.12)
Let $x \in \mathbb{R}$ and let S be a nonempty subset of $\mathbb{R}$ that is bounded above. We define a new set, $x + S$, by $x + S = \{x + s : s \in S\}$.

(a) Prove that $x + S$ is bounded above.
(b) Prove that $x + \sup S$ is an upper bound of $x + S$. Using this result, conclude that $\sup(x + S) \leq x + \sup S$.
(c) Prove that $x + \sup S = \sup(x + S)$

解答:
(a)
$\exists M, (M \in \mathbb{R} \wedge \forall y, (y \in S \to y \leq M))$
$\to \exists N, (N = M + x + 1 \wedge \forall y, (y \in x + S \to y \leq N))$

(b)
$\forall y, (y \in S \to y \leq \sup S)$
$\to \forall y, (y \in x + S \to y \leq x + \sup S)$

(c)
$\forall y, (y \in x + S \to y \leq x + \sup S)$ (1)

$\forall M, (M \in \mathbb{R} \wedge \forall y, (y \in S \to y \leq M) \to M \geq \sup S)$
$\to \forall N, (N \in \mathbb{R} \wedge \forall y, (y \in x + S \to y \leq N) \to N \geq x + \sup S)$ (2)

$(1) \wedge (2) \to x + \sup S = \sup(x + S)$.

题目 9 (UD Problem 13.14)
This problem uses the definitions introduced in Problem 13.13.
Let A be a set containing at least two elements. We define an order on $\mathcal{P}(A)$).
using set inclusion $\subseteq$. Show that $\subseteq$ is a partial order, but not a total order on $\mathcal{P}(A)$.

解答:
Reflexive property : $\forall x, (x \in \mathcal{P}(A) \to x \subseteq x)$.
Transitive property : $\forall x, y, z \in \mathcal{P}(A), (x \subseteq y \land y \subseteq z \to \exists n, (n \in x \to n \in y \to n \in z))$.
$\to \forall x, y, z \in \mathcal{P}(A), (x \subseteq y \land y \subseteq z \to x \subseteq z)$.
Antisymmetric property : $\forall x, y \in \mathcal{P}(A), (x \subseteq y \land y \subseteq x \to x = y)$.
So the $\subseteq$ is a partial order in $\mathcal{P}(A)$.
Not a tot order: $\exists x, y \in \mathcal{P}(A), (A \subsetneq B \land B \subsetneq A)$

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2 作业 (选做部分)

题目 1 (关系的复合)
定义二元关系 R 与 S 的复合为:

$$S \circ R = \{(x, z) \mid \exists y\big((x, y) \in R \land (y, z) \in S\big)\}.$$

图 1: “舅姥爷” 是什么关系复合而成的?

请证明复合操作满足结合律:

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

证明:
$T \circ (S \circ R)$
$= \{(x, z) \mid \exists y((x, y) \in S \circ R \land (y, z) \in T)\}$
$= \{(x, z) \mid \exists y(\exists p((x, p) \in R \land (p, y) \in S)) \land (y, z) \in T\}$
$= \{(x, z) \mid \exists y \exists p((x, p) \in R \land (p, y) \in S \land (y, z) \in T)\}$
$= \{(x, z) \mid \exists y \exists p((x, y) \in R \land (y, p) \in S \land (p, z) \in T)\}$
$= \{(x, z) \mid \exists y((x, y) \in R \land (\exists p((y, p) \in S \land (p, z) \in T)))\}$
$= \{(x, z) \mid \exists y((x, y) \in R \land (y, z) \in T \circ S)\}$
$= (T \circ S) \circ R$ $\square$

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3 Open Topics

Open Topics 1 (二元关系)
介绍花样繁多的 “二元关系”, 如 (不限于):

- Preorder
- Strict weak order
- Strict partial order
- $\cdots$

基本要求:

- 举例说明每种二元关系的应用

参考资料:

- Binary relation @ wiki

Open Topics 2 (实数)
介绍实数的完备性 (Completeness), 如 (不限于):

- 概念
- 等价形式
- 实数的构造方式

参考资料:

- Completeness of the real numbers @ wiki

4 订正

5 反馈