第 7 讲: 离散概率基础

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评分： ______　　**评阅：** ______

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请独立完成作业，不得抄袭。 若得到他人帮助, 请致谢。 若参考了其它资料，请给出引用。 鼓励讨论，但需独立书写解题过程。

1　作业（必做部分）

题目 1 (CS 5.1-10)
How many five-card hands chosen from a standard deck of playing cards consist of five cards in a row (such as the nine of diamonds, ten of clubs, jack of clubs, queen of hearts, and king of spades)? Such a hand is called a straight. What is the probability that a five-card hand is a straight? Explore whether you get the same answer by using five-element sets as your model of hands or five-element permutations as your model of hands.

解答：
顺子：A2345，23456，...，10JQKA
则顺子一共有 10 种 (不区分花色)。
在一套扑克牌中，若区分花色，则顺子有 $4^5 \times 10 = 10240$ 种
Assume that the standard deck of playing cards has 52 cards.
S=$\binom{52}{5}$=2598960
则 $P = \frac{10240}{2598960}$(set 的情况)
若考虑排列，则 $P = \frac{10240\times 5!}{2598960\times 5!} = \frac{10240}{2598960}$.
两者概率相等

题目 2 (CS 5.1-12)
A die is made of a cube with a square painted on one side, a circle on two sides, and a triangle on three sides. If the die is rolled twice, what is the probability that the two shapes you see on top are the same?

解答：
对于 a die is made of a cube:
(square,square)+(circle,circle)+(triangle,triangle)
则 $P = (\frac{1}{6})^2 + (\frac{1}{3})^2 + (\frac{1}{2})^2 = \frac{7}{18}$

题目 3 (CS 5.2-4)
A bowl contains two red, two white, and two blue balls. If you remove two balls, what is the probability that at least one is red or white? Compute the probability that at least one is red

解答:
(a)
At least one is red or white:
1-(blue,blue)
$P = 1 - \frac{2}{6} \times \frac{1}{5} = \frac{14}{15}$
(b)
At least one is red:
(red,red)+(red,not_red)+(not_red,red)
$P = \frac{2}{6} \times \frac{1}{5} + \frac{2}{6} \times (1 - \frac{1}{5}) + (1 - \frac{2}{6}) \times \frac{2}{5} = \frac{3}{5}$

题目 4 (CS 5.2-10)
If you are hashing n keys into a hash table with k locations, what is the probability that every location gets at least one key?

解答:
设 E_i 表示至少有 i 个 location 没有 key 的概率
$E_i = \binom{k}{i}\frac{(k-i)^n}{k^n}$
$P(\bigcup_{i=1}^{k} E_i) = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{(k-i)^n}{k^n}$
则每个 location 都有 key 的概率为: $1 - \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{(k-i)^n}{k^n}$
即 $\sum_{i=0}^{k}(-1)^{i}\binom{k}{i}\frac{(k-i)^n}{k^n}$

题目 5 (CS 5.3-2)
In three flips of a coin, is the event that two flips in a row are heads independent of the event that there is an even number of heads?

解答:
设事件 E 为连续两次翻装转都为正面 (包括连续三次的情况), 事件 F 为正面次数为偶数。
$P(E) = 2 \times (\frac{1}{2})^2 - (\frac{1}{2})^3 = \frac{3}{8}$
$P(F) = 4 \times (\frac{1}{2})^3 = \frac{1}{2}$
$P(E \cap F) = 2 \times (\frac{1}{2})^3 = \frac{1}{4}$
Since $P(E \cap F) \neq P(E)P(F)$, E is not independent of F.

题目 6 (CS 5.3-12)
In a family consisting of a mother, father, and two children of different ages, what is the probability that the family has two girls, given that one of the children is a girl? What is the probability that the children are both boys, given that the older child is a boy?

解答:
(a)
设事件 E 为 the family has two girls , 事件 F 为 one of the children is a girl.

$P(E)=\frac{1}{4}$, $P(F)=2\times\frac{1}{2}-(\frac{1}{3})^2=\frac{3}{4}$, $P(E\cap F)=\frac{1}{4}$
$P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{4}}{\frac{3}{4}}=\frac{1}{3}$
(b)
设事件 E 为 the family has two boys , 事件 F 为 the older child is a boy.
$P(E)=\frac{1}{4}$, $P(F)=\frac{1}{2}$, $P(E\cap F)=\frac{1}{4}$
$P(E|F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}$

题目 7 (CS 5.4-10)
What is the expected value of the constant random variable X that has X(s) = c for every member s of the sample space? (We frequently use c to stand for this random variable. Thus, this question is asking for E(c).)

解答:
$$\begin{aligned}E(X) &= \sum_{s:s\in S} X(s)P(s)\\ &= c\sum_{s:s\in S} P(s)\\ &= c\end{aligned}$$

题目 8 (CS 5.4-15)
Prove Theorem 5.11.
Theorem 5.11: Suppose X is a random variable on a sample space S. Then for any number c, we have
$$E(cX)=cE(X).$$

解答:
$$\begin{aligned}E(cX) &= \sum_{s:s\in S} cX(s)P(s)\\ &= c\sum_{s:s\in S} X(s)P(s)\\ &= cE(x)\end{aligned}$$

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2 作业 (选做部分)

题目 1 (The Ballot Problem)
In an election, candidate A receives n votes, and candidate B receives m votes where $n>m$. Assuming that all orderings are equally likely, what is the probability that A is always ahead in the count of votes?

解答:
问题转换为从 $(0,0)$ 走到 (n,m) 且不越过 y=x 的方案数。
其中不合法方案数等同于从 $(-1,1)$ 走到 (n,m) 的方案数。
即方案数为 A=$\binom{n+m}{n}$-$\binom{n+m}{m+1}$.
总方案数为 B=$\binom{n+m}{n}$
则 $P=\frac{A}{B}$

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3 Open Topics

Open Topics 1 (Monty Hall Problem)
请介绍 Monty Hall Problem, 尽量讲清楚各种版本背后的概率解释。
参考资料:

- Monty Hall problem @ wiki
- "21" Movie @ Youtube

Open Topics 2 (Shuffling Cards)
请参考下列资料介绍 "洗牌" 中的数学。(不必追求严格推导, 主要介绍基本思想。)

> "How often does one have to shuffle a deck of cards until it is random?"

参考资料:

- Section "Top-in-at-random shuffles" of Chapter 30 of Book: "Proofs from THE BOOK" (见课程网站)

4 反馈