第 4-1 讲: 群论初步

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评分: ______　　**评阅:** ______

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请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

1 作业 (必做部分)

题目 1 (TJ 3-3)

解答:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\circ$	identity	180 rotation	vertical axis	horizontal axis
identity	identity	180 rotation	vertical axis	horizontal axis
180 rotation	180 rotation	identity	horizontal axis	vertical axis
vertical axis	vertical axis	horizontal axis	identity	180 rotation
horizontal axis	horizontal axis	vertical axis	180 rotation	identity

$\mathbb{Z}_4$ has 1 nontrivial proper subset $\{0, 2\}$. The number of other group's nontrivial proper is more than one. So they are not same group.

题目 2 (TJ-3-7)

解答:
(1)associative law:

$$(a*b)*c = (ab+a+b)*c = a+b+c+ab+ac+bc+ab+abc = a*(bc+b+c) = a*(b*c)$$

(2)identity element:

$$\forall x \in R\backslash\{-1\}, x * 0 = x$$

(3)inverse element:

$$\forall x \in R\backslash\{-1\}, \exists y = -\frac{a}{a+1} \in R\backslash\{-1\}, st.a * b = 0$$

(4)abelian:

$$a * b = a + b + ab = b * a$$

题目 3 (TJ 3-39)

解答:

$$\forall a = x + yi, b = m + ni \in \mathbb{T}, ab^{-1} = (x + yi)(m - ni) = xm + yn + (my - nx)i \in \mathbb{T}$$

So $\mathbb{T}$ is a subgroup of $\mathbb{C}^*$

题目 4 (TJ 3-42)

解答:
$\forall g, h \in H$, Let

$$g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \qquad h = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

We have that

$$h^{-1} = \begin{pmatrix} -a_2 & -b_2 \\ -c_2 & -d_2 \end{pmatrix} \qquad g \circ h^{-1} = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{pmatrix}$$

So $g \circ h^{-1} \in H$, H is a subgroup of G.

题目 5 (TJ 3-49)

解答:

$$a^4 b = a^3 ab = eab = ab = ba$$

题目 6 (TJ 3-51)

解答:

$$\forall x \in G, xe = x = x^{-1}e^{-1} = x^{-1}$$

$$\to \forall x, y \in G, xy \in G, xy = (xy)^{-1} = (y^{-1}x^{-1}) = yx$$

So the group G is abelian.

题目 7 (TJ 4-1)

解答:
(a)
False. 49 is a generator of $\mathbb{Z}_{60}$, but it is not prime.
(b)
False. $1, 3, 5, 7$ are all not generator of $U(8)$.
(c)
False. Assume that g is a generator of $\mathbb{Q}$, but g can not generate $\frac{g}{2}$.
(d)
False. The symmetry group of an equilateral triangle $S3$ is not cyclic, but the subgroup of $S3$ are all cyclic.
(e)
True.
Suppose, to the contrary, an infinite group G has finite number of subgroup.
(1) If G has an infinite order generator g, then $< g >, < g^2 >, < g^3 >, < g^4 >, ..., < g^k >$ are all the subgroup of G, so it has infinite number of subgroup. It is contradict with the assumption.
(2) If the order of generators are all finite, then we let S $= \{< x > | x \in G\}$. We have that S is a finite set. The group G is the union of the finite set S, so G is finite, it is contradict with the assumption.
Therefore, a group with a finite number of subgroups is finite.

题目 8 (TJ 4-24)

解答:
$\phi(pq) = \phi(p)\phi(q) = (p-1)(q-1) = pq - p - q + 1$(p and q are different primes)

题目 9 (TJ 4-12)

解答:
one generator: $\mathbb{Z}_2$
Two generators: $\mathbb{Z}_4$
Four generators: $\mathbb{Z}_8$
n generators: $\exists x, st.\phi(x) = n$. $\mathbb{Z}_x$ has n generators.

题目 10 (TJ 4-32)

解答:
Due to Theorem4.13 in TJ, $y = x^k$, the order of y is $\frac{n}{gcd(k,n)} = \frac{n}{1} = n$. Therefore, y is a generator of G.

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2 作业 (选做部分)

题目 1 (Z_p)
证明: 设 p 为素数, 则 $Z_p = \{1, 2, ..., p-1\}$ 关于 p **乘法**构成的 $p-1$ 阶循环群。(此处的 $1, 2, ..., p-1$ 是模 p 等价类的代表元)

解答:

题目 2 (SageMath 学习)
安装 SageMath, 并学习 TJ 第三章 3.6 节、3.7 节; 第四章 4.6 节、4.7 节关于 SageMath 的内容

解答:

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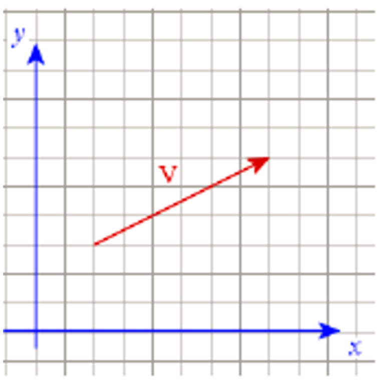
3 Open Topics



在二维平面上的"移动"(例如向东北 30 度移动 9 公里)。你能够以这些"移动"为元素构建一个群吗?

Open Topics 1 (“移动” 群-1)

- 它的几何元素和运算分别是什么?
- 它为什么符合群的定义?
- 它是阿贝尔群吗？为什么?

Open Topics 2 (“移动” 群-2)

- 你能找出它的一些子群吗？并说明为什么找到的是子群
- 它是循环群吗? 如果是，生成元是什么？生成元唯一吗？如果不是，如何改造出一个循环群?
- 你能找出这个（改造后的）循环群的一些子群么？它们是循环群么?

4 反馈