第 10 讲: 函数

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评分: ______　　**评阅:** ______

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请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

- 有了 functions，(大部分) 数学就 functions 了。

1 作业（必做部分）

题目 1 (UD Problem 14.3 (b, d, g))

解答:
(b) The relation in (b) is not function, because when $x = 1$ and $x \in \mathbb{R}$, The denominator in f (x) is 0, there is no element to match with it.So it violates condition(i).
(d)The relation in (d) is function.$\forall x \in [a, b], f(x) = a, a \in \mathbb{R}$;and $\forall x \in [a, b]$, there is only $a \in \mathbb{R}$ can satisfy $(x, a) \in f$
(g)The relation in (g) is not function.$\exists x = 6k(k \in \mathbb{Z}) s.t. x \in 2\mathbb{Z} and x \in 3\mathbb{Z}$such that $f(x) = x + 1 = x - 1$, that means -1 = 1, which is obviously wrong.

题目 2 (UD Problem 14.5)

证明:
for condition(i): let there be a set $A \subseteq \mathbb{R}$ there doesn't exist $b \in \mathbb{Z}$ such that $(A, b) \in f$. That means $A \cap \mathbb{N} = \emptyset$ is false and $A \cap \mathbb{N} \neq \emptyset$ is also false, in other words $A \cap \mathbb{N} = \emptyset$ and $A \cap \mathbb{N} \neq \emptyset$, which is obviously wrong.So there is no such A. This assumption is wrong.
for condition(ii): if and only if $A \cap \mathbb{N} = \emptyset$ and $A \cap \mathbb{N} \neq \emptyset$, there will be two different values corresponding to the same A. There is no such A. $\square$

题目 3 (UD Problem 14.23)

解答:
x = x

题目 4 (UD Problem 15.10 (f, g, h))

解答:
(f) The function is one-to-one.
$\forall a_1, a_2, f(a_1) = f(a_2) \rightarrow \exists b \in B$, and $(a_1, b) = (a_2, b) \rightarrow a_1 = a_2$;
The function is onto.

(g)The function is one-to-one.
$\forall A, B \subseteq P(X)$, $f(A) = f(B) \rightarrow X \backslash A = X \backslash B \rightarrow A = B$
The functionn is onto.
(h)The function is not one-to-one.
Let $B \subseteq X, A \subseteq X \backslash B$ such that $A \cap B = \emptyset$. Obviously, A can be $\emptyset$ and for $B! = X$ there must exist another C != $emptyset$ and $C \subseteq X \backslash B$ such that $C \cap B = \emptyset$.
example:$X = \{1, 2, 3, 4, 5\}; B = 1, 2; A = \emptyset; C = 3$, $C \cap B = A \cap B = \emptyset$ and $A! = C$
The function is not onto. The range is B.

题目 5 (UD Problem 15.14)

解答:
Let $x \in [a, b], y \in [c, d]$
Define $f : [a, b] \rightarrow [c, d]$ by $f(x) = ((x - a)/(b - a)) * (d - c)$;
$proof$.
one-to-one:f(q) = f(p) $\rightarrow ((q - a)/(b - a)) * (d - c) = ((p - a)/(b - a)) * (d - c)$, it's easy to get that $p = q$.
onto:Let $y \in [c, d]$ and let $x = (y/(d - c)) * (b - a) + a$ Then $x \in [a, b] = dom(f)$ and $f(x) = (((y/(d - c)) * (b - a) + a) - a)/(b - a)) * (d - c) = y$.

题目 6 (UD Problem 15.15)

解答:
ϕ is a function from $F([0, 1])$ to $\mathbb{R}$.
$\forall f \in F([0, 1]), \exists y = f(0) = \phi(f)$
$\forall f \in F([0, 1]), \forall y_1, y_2 \in \mathbb{R}$, and $\phi(f) = f(0) = y_1 = y_2$
ϕ is not one to one. $f_1(x) = x, f_2(x) = x^2$, $\phi(f_1) = f_1(0) = 0, \phi(f_2) = f_2(0) = 0$, we can get that $\phi(f_1) = \phi(f_2), f_1 \neq f_2$.
ϕ is onto.$\forall y \in \mathbb{R}, \exists f \in dom(\phi), \phi(f) = f(0) = y$
example:Let $f_q = x + q, q \in \mathbb{R}$, $f(0) = q \in \mathbb{R}$.

题目 7 (UD Problem 16.6)

解答:
(a) $f \circ g = \frac{\frac{3+2x}{1-x}-3}{\frac{3+2x}{1-x}+2} = x(x \neq 1)$
$g \circ f = \frac{3+2*\frac{x-3}{x+2}}{1-\frac{x-3}{x+2}} = x, (x \neq -2)$
(b) $g = f^{-1}$
Theorem 16.4. Let $f : A \Rightarrow B$ be a bijective function. Then (iv) $Ifg : B \to A$ is a functioni satisfying $f \circ g = i_B org \circ f = i_A, theng = f^{-1}$.

题目 8 (UD Problem 16.14)

解答:
$\forall y \in B, \exists x, f(g_1(x)) = f(g_2(x)) = y$ so $g_1(x) = g_2(x)$.
$\forall y \in A, \exists x, g_1(f(x)) = g_2(f(x)) = y$, that is $g_1 = g_2$.

题目 9 (UD Problem 16.17)

解答:
(a)
one-to-one:
let $H(a_1, c_1) = H(a_2, c_2) = (f(a_1), f(c_1)) = (f(a_2), f(c_2))$. That is $f(a_1) = f(a_2)$ and $f(c_1) = f(c_2)$. So $a_1 = a_2, c_1 = c_2$

function:
$\forall x_1 \in A, x_2 \in C$ there exists $f(x_1) \in B$ and $f(x_2) \in D$, that is $\forall (x_1, x_2) \in A \times C$ there exists $(f(x_1), f(x_2)) \in B \times D$.
(b)
$\forall y_1 \in B$, $\exists x_1 \in A, f(x_1) = y_1$,and $\forall y_2 \in B$, $\exists x_2 \in A, g(x_2) = y_2$
So $\forall (y_1, y_2) \in B \times D$, $\exists (x_1, x_2) \in A \times C$ $(f(x_1), g(x_2)) = (y_1, y_2)$

题目 10 (UD Problem 16.22)

证明:
$f(f(x)) = x$ $f(x_1) = f(x_2) \to f(f(x_1)) = f(f(x_2)) \to x_1 = x_2$
So f is bijective. □

题目 11 (UD Problem 17.22)

解答:
(a)Needn't.
Let $f(x) = x^2$, $A_1 = \{1, 2\}, A_2 = \{-1, -2\}$, $f(A_1) = f(A_2) = \{1, 4\}$

(b) $\forall x_1 \in A_1, \forall x_2 \in A_2$, for f is a bijective function, $f(x_1) = f(x_2) \to x_1 = x_2$.
So if $f(A_1) = f(A_2)$, then $\forall x_1 \in A_1 \Leftrightarrow \forall x_2 \in A_2, \Rightarrow A_1 = A_2$.

题目 12 (UD Problem 17.23)

解答:
(a)Needn't.
Let $f(x) = x^2$, $B_1 = \{-1, 0\}, B_2 = \{0\}$, obviously $f^{-1}(B_1) = f^{-1}(B_2)$ and $B_1 \neq B_2$.
(b)Because f is onto, $\forall y_1 \in B_1, \forall y_2 \in B_2, \exists x_1 \in f^{-1}(B_1), \exists x_2 \in f^{-1}(B_2), f(x_1) = y_1, f(x_2) = y_2$(onto)
If $f^{-1}(y_1) = f^{-1}(y_2) \to y_1 = y_2$.(one-to-one)
So if $f^{-1}(B_1) = f^{-1}(B_2) \to B_1 = B_2$

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2 作业 (选做部分)

题目 1 (Monotonicity)
Assume that $F : \mathcal{P}(A) \to \mathcal{P}(A)$ and that F has the monotonicity property:

$$X \subseteq Y \subseteq A \implies F(X) \subseteq F(Y).$$

Define

$$B = \bigcap \{X \subseteq A \mid F(X) \subseteq X\}$$

$$C = \bigcup \{X \subseteq A \mid X \subseteq F(X)\}.$$

(a) Show that $F(B) = B$ and $F(C) = C$.

(b) Show that if $F(X) = X$, then $B \subseteq X \subseteq C$.

解答:

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3 Open Topics

Open Topics 1 (自然数)
介绍如何使用集合定义 (不限于):

- 自然数
- 自然数上的大小关系
- 自然数上的运算

参考资料:

- Natural number @ wiki

Open Topics 2 (选择公理)
介绍选择公理 (Axiom of Choice), 如 (不限于):

- 不同定义形式
- 怎么理解 (怎么也不理解)
- 有什么用

参考资料:

- Axiom of choice @ wiki
- The Axiom of Choice @ Stanford Encyclopedia of Philosophy

4 订正

5 反馈