第 3 讲: 组合与计数

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评分: ______ **评阅:** ______

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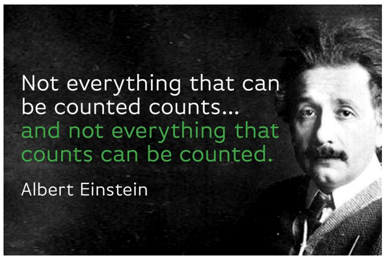
请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。



1 作业 (必做部分)

题目 1 (CS 1.2-1)
In how many ways can we pass out k distinct pieces of fruit to n children (with no restriction on how many pieces of fruit a child may get)?

解答:
According to the method learned in the high school, we can get that:
$C(k+1, n-1) = \frac{(k+1)!}{(n-1)! * (k+1-(n-1))!}$

题目 2 (CS 1.2-5)
Assuming k n, in how many ways can we pass out k distinct pieces of fruit to n children if each child may get at most one piece? What if k>n? Assume for both questions that we pass out all the fruit.

解答:
(a)$k <= n$
There are at most k children who can get one piece of fruit. So we should choose k children from n children. And for that the fruit pieces are distinct, so in different cases even the chosen children are the same, the kind of fruit pieces are still different.
$C(n, k) * A(k, k) = \frac{n!}{k!(n-k)!} * k! = \frac{n!}{(n-k)!} = n * (n-1) * ... * (n-k+1)$
So there are $\frac{n!}{(n-k)!}$ ways we can pass out k distinct pieces of fruit to n children if each child may get at most one piece.
(b)$k > n$
There is no way to give out all the fruit, when each child may get at most one piece. So the answer is 0.

题目 3 (CS 1.2-6)
Assuming k n, in how many ways can we pass out k identical pieces of fruit to n children if each child may get at most one? What if k>n? Assume for both questions that we pass out all the fruit.

解答:
(a)$k <= n$
What we need to do is choose k children who can get the fruit piece from n children. So there are $C(k, n)$ ways to artribute the fruit pieces.
(b)$k > n$
If there can be no fruit piece left, then there is no way to pass out them. So accordingly, the answer is 0.

题目 4 (CS 1.2-15)
A tennis club has 2n members. We want to pair up the members by twos for singles matches. In how many ways can we pair up all the members of the club? Suppose that in addition to specifying who plays whom, we also determine who serves first for each pairing. Now in how many ways can we specify our pairs?

解答:
(a)$\sum C_i^2, i = 2k(k = 1, 2, 3, ..., n)$
(b)A_{2n}^{2n}/A_n^n

题目 5 (CS 1.5-4)
Use multisets to determine the number of ways to pass out k identical apples to n children. Assume that a child may get more than one apple.

解答:
The number of k-element multisets chosen from an n-element set is$frac(n + k - 1)!k!(n - 1)! = \binom{n+k-1}{k}$
So there are $frac(n + k - 1)!k!(n - 1)! = \binom{n+k-1}{k}$ ways to pass out the apples.

题目 6 (CS 1.5-12)
A standard notation for the number of partitions of an n-element set into k classes is S(n, k). Because the empty family of subsets of the empty set is a partition of the empty set, S(0, 0) is 1. In addition, S(n, 0) is 0 for n > 0, because there are no partitions of a nonempty set into no parts. S(1, 1) is 1.
a. Explain why S(n, n) is 1 for all n > 0. Explain why S(n, 1) is 1 for all n > 0.
b. Explain why S(n, k) = S(n − 1, k − 1) + kS(n − 1, k) for $1 < k < n$.
c. Make a table like Table 1.1 that shows the values of S(n, k) for values of n and k ranging from 1 to 6.

解答:
a.
(1)Why S(n, n) is 1 for all n > 0?
There is an n-element set. We should attribute the n elements into n classes, and every class can't be empty. It's to say, there is at least 1 element in each class. So in the first step, we attribute 1 element to the n classes, then the n elements have been all passed out. So it is the only way to get S(n, n).
It's like $\frac{n!}{n!} = 1$
(2)Why S(n, 1) is 1 for all n > 0?

k\n	1	2	3	4	5	6
1	1	1	1	1	1	1
2	0	1	3	7	15	31
3	0	0	1	6	25	90
4	0	0	0	1	11	69
5	0	0	0	0	1	16
6	0	0	0	0	0	1

There is n elements, and we need to artribute them into 1 class. That is, except from the class, there are no other class, where can contain any elements. So we should put all n elements into the only 1 class. That is like $C_n^n = 1$.
b.
We distribute the n elements into k classes. We choose an arbitary element which is noted as a_1. There are two cases.
Case 1: a_1 is chosen as a separated class, then there are $n-1$ elements to be chosen into $k-1$ classes. So there are $S(n-1, k-1)$ possibilities.
Case 2:a_1 is not chosen as a separated class. Then the class is different from other ones. So we have to multiply k in the formula. Therefore, the answer is $k * S(n-1, k)$.
And the key to the original problem is $S(n, k)$, so we can get that $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ for $1 < k < n$.
c. 答案在上面

2 作业 (选做部分)

题目 1 (Summation)
请计算如下代码段的返回值 r。

```
procedure CONUNDRUM(n)
    r ← 0
    for i ← 1 to n do
        for j ← i + 1 to n do
            for k ← i + j − 1 to n do
                r ← r + 1
            end for
        end for
    end for
    return r
end procedure
```

解答:
$\sum \frac{(n-i)(n-i-1)}{2} (i = 2t <= n; t = 0, 1, 2, 3...)$

3 Open Topics

本周两个 OT 的目的是向大家介绍在算法分析中常用的数学基础。阅读书籍 [1]： [1]

Open Topics 1 (Sums)
第二章关于 "Sums" 的内容 (如前五节), 介绍你认为有用、有意思的求和技巧。

Open Topics 2 (Binomial Coefficients)
第五章关于 “Binomial Coefficients” 的内容 (如前两节或前三节), 介绍你认为有用、有意思的公式与技巧。

4 反馈