Solutions to PS#2
Math 355 Summer 1, 2009

p.37 #5. Let ℓ and m be the lines with respective equations $X+Y-2=0$ and $Y=3$.
a. Compose the equations of σ_m and σ_ℓ and show that the composition $\sigma_m \circ \sigma_\ell$ is a rotation.
b. Find the center C and the angle of rotation Θ and compare Θ with the directed angle from ℓ to m.
Proof: The equations of σ_ℓ and σ_m are

$$\sigma_\ell : \begin{cases} x' = x - (x+y-2) = 2-y \\ y' = y - (x+y-2) = 2-x \end{cases} \text{ and } \sigma_m : \begin{cases} x' = x \\ y' = y - 2(y-3) = 6-y. \end{cases}$$

The equations of the composition $\sigma_m \circ \sigma_\ell$ are

$$\sigma_m \circ \sigma_\ell : \begin{cases} x'' = x' = -y+2 \\ y'' = 6-y' = 6-(2-x) = x+4. \end{cases}$$

Line ℓ and m intersect at $C = \left[\begin{smallmatrix} -1 \\ 3 \end{smallmatrix}\right]$ with respective slopes -1 and 0, and the undirected angle from ℓ to m measures $45°$. The equations for $\rho_{C,90}$ are

$$\rho_{C,90} : \begin{cases} x' = (x+1)\cos 90 - (y-3)\sin 90 - 1 = -y+2 \\ y' = (x+1)\sin 90 + (y-3)\cos 90 + 3 = x+4. \end{cases}$$

Therefore $\sigma_m \circ \sigma_\ell = \rho_{C,90}$.

p.37 #6. Let P and R be distinct points and let Q be the midpoint of $\overline{PR}$. Let c be a line such that $\tau_{\mathbf{PR}}(c) = c$. Then $\sigma_c \circ \tau_{\mathbf{PR}} = \tau_{\mathbf{PR}} \circ \sigma_c$.
Proof: Note that $\overleftrightarrow{PQ} \| c$ since Q is the midpoint of $\overline{PR}$ and $\tau_{\mathbf{PR}}(c) = c$. Let p and q be the lines perpendicular to c at P and Q, respectively. Then $\sigma_c \circ \tau_{\mathbf{PR}} = \sigma_c \circ \varphi_Q \circ \varphi_P = \sigma_c \circ (\sigma_c \circ \sigma_q) \circ (\sigma_p \circ \sigma_c) = \sigma_q \circ (\sigma_c \circ \sigma_c) \circ \sigma_p \circ \sigma_c = \varphi_Q \circ \varphi_P \circ \sigma_c = \tau_{\mathbf{PR}} \circ \sigma_c$.

p.48 #6. Let C be a point on line ℓ. Prove that $\sigma_\ell \circ \rho_{C,\Theta} \circ \sigma_\ell = \rho_{C,-\Theta}$.
Proof: By Corollary 85, there is a unique line m passing through C such that $\rho_{C,\Theta} = \sigma_m \circ \sigma_\ell$. Hence $\sigma_\ell \circ \rho_{C,\Theta} \circ \sigma_\ell = \sigma_\ell \circ (\sigma_m \circ \sigma_\ell) \circ \sigma_\ell = \sigma_\ell \circ \sigma_m = (\sigma_m \circ \sigma_\ell)^{-1} = \rho_{C,\Theta}^{-1} = \rho_{C,-\Theta}$.

p.48 #7. If $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$, prove that lines ℓ, m and n are either concurrent or mutually parallel.
Proof: By assumption $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$; multiplication on the left by σ_n gives $\sigma_n \circ \sigma_p = \sigma_m \circ \sigma_\ell$. Lines ℓ and m are either parallel or intersect at some point C. If $\ell \cap m = C$, then $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p$ is a rotation by Theorem 83, in which case $p \cap n = C$ and ℓ, m, n and p are concurrent at C. On the other hand, if $\ell \| m$, then $\sigma_m \circ \sigma_\ell = \sigma_n \circ \sigma_p$ is a translation by Theorem 78 and ℓ, m, n and p are mutually parallel.

p. 60 #3. Let $A = \left[\begin{smallmatrix} 4 \\ 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right]$.
a. Find equations of lines ℓ, m, and n such that $\rho_{A,90} = \sigma_m \circ \sigma_\ell$ and $\rho_{B,120} = \sigma_n \circ \sigma_m$.
b. Find xy coordinates for the point C and the angle of rotation Θ such that $\rho = \rho_{B,120} \circ \rho_{A,90}$.
c. Find xy coordinates for the point D and the angle of rotation Φ such that $\rho = \rho_{A,90} \circ \rho_{B,120}$.
Solutions:
a) $\ell : X = 4$; $m : X+Y-4 = 0$; $n : Y = \left(2-\sqrt{3}\right)X + 4$.
b) $C = \ell \cap n = \left[\begin{smallmatrix} 4 \\ 12-4\sqrt{3} \end{smallmatrix}\right]$; $\Theta = 210°$.
c) Let $\ell : Y = \left(2+\sqrt{3}\right)X + 4$; $m : Y = -X+4$; and $n : Y = 0$. Then $D = \ell \cap n = \left[\begin{smallmatrix} 4\sqrt{3}-12 \\ 0 \end{smallmatrix}\right]$; $\Phi = 210°$.

p. 60 #4. Given $\triangle ABC$, let ℓ, m, and n be the respective angle bisectors of $\angle A$, $\angle B$, and *$\angle C$, and let p be the unique line such that $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$. Then $p \perp \overline{AC}$.*
Proof: Let $\Theta° = m\angle A$, $\Phi° = m\angle B$, and $\Psi° = m\angle C$. Multiply both sides of $\sigma_p = \sigma_n \circ \sigma_m \circ \sigma_\ell$ on the left by σ_n. Then for $\Theta \in \Theta°$ and $\Phi \in \Phi°$ we have $\sigma_n \circ \sigma_p = \sigma_m \circ \sigma_\ell = \sigma_m \circ \sigma_c \circ \sigma_c \circ \sigma_\ell = \rho_{B,\Phi} \circ \rho_{A,\Theta} = \rho_{P,\Theta+\Phi}$ by the Angle Addition Theorem, and the directed angle from ℓ to m and from p to n is $\frac{1}{2}(\Theta+\Phi)$. Note that m passes through the interior of $\triangle APC$ and cuts side $\overline{AC}$. Furthermore, side $\overline{AP} \subset \ell$ and side $\overline{PC} \subset n$. Thus the directed angle measure from ℓ to n is the sum of the directed angle measures from ℓ to m and from m to n, and it follows that p cuts $\overline{AC}$ at some point D. Finally, in $\triangle CDP$ we have $m\angle PDC = 180° - m\angle CPD - m\angle DCP = 180° - \frac{1}{2}(\Theta+\Phi)° - \frac{1}{2}\Psi° = 90°$.

p.66 #9. Prove that an even isometry fixing two distinct points is the identity.

Proof: Theorem 101 tells that an even isometry α can be expressed as a product of two reflections. Thus α is either the identity, a non-identity rotation or a non-identity translation by Theorem 88. But neither of the later two possibilities fix more than one fixed point, so $\alpha = \iota$.

p.66 #10. Prove that a translation preserves orientation.

Proof: Choose an ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, i.e., an orientation, then linear independence implies $\det [\mathbf{v}_1 \mid \mathbf{v}_2] \neq 0$. Position $\mathbf{v}_1 = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ and $\mathbf{v}_2 = \left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]$ in standard position and consider a translation $\tau_{\mathbf{w}}$ with $\mathbf{w} = \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$. Then $\tau_{\mathbf{w}}\left(\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right]$, $\tau_{\mathbf{w}}\left(\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} a+e \\ b+f \end{smallmatrix}\right]$ and $\tau_{\mathbf{w}}\left(\left[\begin{smallmatrix} c \\ d \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} c+e \\ d+f \end{smallmatrix}\right]$ so that $\tau_{\mathbf{w}}(\mathbf{v}_1) = \left[\begin{smallmatrix} a+e \\ b+f \end{smallmatrix}\right] - \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right] = \mathbf{v}_1$ and $\tau_{\mathbf{w}}(\mathbf{v}_2) = \left[\begin{smallmatrix} c+e \\ d+f \end{smallmatrix}\right] - \left[\begin{smallmatrix} e \\ f \end{smallmatrix}\right] = \mathbf{v}_2$. Thus $\det [\tau_{\mathbf{w}}(\mathbf{v}_1) \mid \tau_{\mathbf{w}}(\mathbf{v}_2)] = \det [\mathbf{v}_1 \mid \mathbf{v}_2]$ and sign of the determinant is preserved.

p.66 #11. Prove that a reflection in a line through the origin reverses orientation.

Proof: Choose an ordered basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, i.e., an orientation, then $\det [\mathbf{v}_1 \mid \mathbf{v}_2] \neq 0$ by linear independence. A general line ℓ through the origin has equation $aX + bY = 0$ with $a^2 + b^2 > 0$. The equations for σ_ℓ are

$$\sigma_\ell : \begin{cases} x' = x - \frac{2a}{a^2+b^2}(ax+by) = \frac{1}{a^2+b^2}\left[\left(b^2-a^2\right)x - 2aby\right] \\ y' = y - \frac{2b}{a^2+b^2}(ax+by) = \frac{1}{a^2+b^2}\left[-2abx + \left(a^2-b^2\right)y\right] \end{cases}.$$

Since σ_ℓ is a linear map, we may express σ_ℓ in the matrix form $\mathbf{x}' = A\mathbf{x}$, where

$$A = \frac{1}{a^2+b^2}\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix}.$$

Note that $\det A = \frac{1}{(a^2+b^2)^2}\det\begin{bmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{bmatrix} = \frac{1}{(a^2+b^2)^2}\left(-\left(a^2-b^2\right)^2 - 4a^2b^2\right)$

$$= \frac{-1}{\left(a^2+b^2\right)^2}\left(a^4 - 2a^2b^2 + b^4 + 4a^2b^2\right) = \frac{-1}{\left(a^2+b^2\right)^2}\left(a^2+b^2\right)^2 = -1.$$

Therefore $\det [\sigma_\ell(\mathbf{v}_1) \mid \sigma_\ell(\mathbf{v}_2)] = \det [A\mathbf{v}_1 \mid A\mathbf{v}_2] = \det (A \cdot [\mathbf{v}_1 \mid \mathbf{v}_2]) = (\det A)(\det [\mathbf{v}_1 \mid \mathbf{v}_2]) = -\det [\mathbf{v}_1 \mid \mathbf{v}_2].$ Since the determinant reverses sign, σ_ℓ reverses orientation.