第 4-2 讲: 置换群与拉格朗日定理

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评分： ______　　**评阅：** ______

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请独立完成作业，不得抄袭。 若得到他人帮助, 请致谢。 若参考了其它资料，请给出引用。 鼓励讨论，但需独立书写解题过程。

1　作业 (必做部分)

题目 1 (TJ 5-3(d))

解答：
(17254)(1423)(154632)=(17254)(24615)=(14672)=(12)(17)(16)(14)
It is an even permutation.

题目 2 (TJ 5-5 (注：只需列出 S4 的所有子群, 无需解 (a)、(b)、(c)))

解答：
The subgroup of $S4$:
$N_1 = \{1\}$
$N_2 = \{1), (12)\}$
$N_3 = \{1), (13)\}$
$N_4 = \{1), (23)\}$
$N_5 = \{1), (24)\}$
$N_6 = \{1), (14)\}$
$N_7 = \{1), (34)\}$
$N_8 = \{1), (12), (34)\}$
$N_9 = \{1), (13), (24)\}$
$N_{10} = \{1), (14), (23)\}$
$N_{11} = \{1), (123), (132)\}$
$N_{12} = \{1), (134), (143)\}$
$N_{13} = \{1), (124), (142)\}$
$N_{14} = \{1), (234), (243)\}$
$N_{15} = \{1), (1234), (13)(24), (1432)\}$
$N_{16} = \{1), (1234), (12)(34), (1432)\}$

$N_{17} = \{1), (1243), (14)(23), (1342)\}$
$N_{18} = \{1), (12), (34), (12)(34)\}$
$N_{19} = \{1), (13), (24), (13)(24)\}$
$N_{20} = \{1), (14), (23), (14)(23)\}$
$N_{21} = \{1), (12)(34), (13)(24), (14)(23)\}$
$N_{22} = \{(1), (1234), (13)(24), (1432), (13), (12)(34), (24), (14)(23)\}$
$N_{23} = \{(1), (1324), (12)(34), (1423), (12), (13)(24), (34), (14)(32)\}$
$N_{24} = \{(1, (1243), (14)(23), (1342), (14), (12)(43), (34), (14)(32)\}$
$N_{25} = S_4$
$N_{26} = \{(1), (12), (13), (23), (123), (132)\}$
$N_{27} = \{(1), (12), (24), (14), (124), (142)\}$
$N_{28} = \{(1), (34), (13), (14), (143), (134)\}$
$N_{29} = \{(1), (34), (24), (23), (234), (243)\}$
$N_{30} = \{(1), (123), (132), (134), (143), (124), (142), (234), (243), (12)(34), (13)(24), (14)(23)\}$

题目 3 (TJ 5-16)

解答:
For a tetrahedron, we mark its vertex as A, B, C and D.
Then $\{(id), (ABC), (ACB), (ABD), (ADB), (ACD), (ADC), (BCD), (BDC), (AB)(CD), (AC)(BD), (AD)(BC)\}$ form the sports group.
There is a bijective function f between A_4 and tetrahedron apparently by $\sigma(A) \to 1, \sigma(B) \to 2, \sigma(C) \to 3, \sigma(D) \to 4$.

题目 4 (TJ 5-26(b))

解答:
$\forall (a_1, a_2, ..., a_n) \in A_4$

$$(a_1, a_2, ..., a_n) = (a_1 a_n)(a_1 a_{n-1})\ (a_1 a_3)(a_1 a_2)$$

For (a_1, a_2), we assume that $a_1 \le a_2$.
We have that $(a_1, a_2) = (a_1, a_1 + 1)(a_1 + 1, a_1 + 2)...(a_2 - 2, a_2 - 1)(a_2 - 1, a_2)(a_2 - 2, a_2 - 1)...(a_1 + 1, a_1 + 2)(a_1, a_1 + 1)$
It is the same as $(a_1, a_3), ..., (a_1, a_n)$.
Therefore, any element in S_4 can be written as a finite product of $(12), (23), ..., (n - 1, n)$

题目 5 (TJ 5-29)

解答:
$Z(D_8) = \{1, r^4\}, Z(D_{10}) = \{1, r^5\}$
$Z(D_n) = \{1, \frac{n}{2}\}$(n is even), $Z(D_n) = \{1\}$(n is odd).

题目 6 (TJ 5-36)

解答:
(a)

$$s^2 = 1 \to s = s^{-1}$$
$$srs = r^{-1} \Leftrightarrow rsr = s$$

So we only need to show $rsr = s$.
We assume that $s = s_1$. Observe that the graph after operations rsr.
At the beginning, the first vertex is 1, and the second vertex is 2.
After the operation r, the first vertex is 2, and the second vertex is 3.
After the operation s, the first vertex is 2, and the second vertex is 1.
At the end, after the operation r, the first vertex is 1, and the second vertex is n.
It is the same with the single operation s. So $rsr = s$ and $srs = r^{-1}$
(b)

$$srs = r^{-1} \Leftrightarrow (srs)^k = r^{-k} \Leftrightarrow srss^{-1}rs...s^{-1}rs = r^{-1} \Leftrightarrow sr^k s = r^{-k} \Leftrightarrow r^k s = sr^{-k}$$

(c)
Let $C_n = \{r^k | r^k \in D_n\}$. C_n is a subgroup of D_n.
Obviously, C_n is a cyclic group with the generator r. The order of r is n.
Due to **Theorem 4.13** in TJ, the order of r_k is $\frac{n}{gcd(n,k)}$ in the group C_n.
Since C_n is a cyclic subgroup of D_n, the order of r^k is also $\frac{n}{gcd(n,k)}$.

题目 7 (TJ 6-11 (注意：(c) 中 $\subset$ 表示 $\subseteq$))

解答:
$1.(a) \to (c):$
$g_1 H = g_2 H$
$\to (x \in g_1 H \to x \in g_2 H)$
$\to g_1 H \subset g_2 H$

$2.(c) \to (e):$
$g_1 H \subset g_2 H$
$\to (x \in g_1 H \to x \in g_2 H)$
$\to \forall h_1 \in H, (\exists h_2 \in H, g_1 h_1 = g_2 h_2)$
$\to \forall h_1 \in H, (\exists h_2 \in H, h_1 h_2^{-1} = g_1^{-1} g_2)$
$\to g_1^{-1} g_2 \in H$

$3.(e) \to (d):$
$g_1^{-1} g_2 \in H$
$\to \exists h \in H, g_1^{-1} g_2 = h$
$\to \exists h \in H, g_2 = g_1 h$
$g_2 \in g_1 H$

$4.(d) \to (b):$
$g_2 \in g_1 H$
$\to \exists h \in H, g_2 = g_1 h$
$\to \exists h \in H, g_1^{-1} = h g_2^{-1}$
$\to \exists h \in H, (\forall h_1 \in H, h_1 g_1^{-1} = h_1 h g_2^{-1} \in H g_2^{-1})$
$\to H g_1^{-1} \subset H g_2^{-1}$
similarly, we have $H g_2^{-1} \subset H g_1^{-1}$
So $H g_1^{-1} = H g_2^{-1}$

$5.(b) \to (a):$
$H g_1^{-1} = H g_2^{-1}$
$\to H g_1^{-1} \subset H g_2^{-1}$
$\to \forall h_1 \in H, \exists h_2 \in H, h_1 g_1^{-1} = h_2 g_2^{-1}$
$\to \exists h_3 \in H, g_1 = g_2 h_3$
$\to \forall h_4 \in H, \exists h_3 \in H, g_1 h_4 = g_2 h_3 h_4 \in g_2 H$
$\to g_1 H \subset g_2 H$
similarly, we have $g_2 H \subset g_1 H$
So $g_1 H = g_2 H$

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2 作业 (选做部分)

题目 1 (Z_p)
证明：A_n 中的每个置换皆可表成形如 $(k\ k+1\ k+2)$ 的 3-cycle 的乘积。

解答:

题目 2 (SageMath 学习)
学习 TJ 第五章, 第六章关于 SageMath 的内容

解答:

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3 Open Topics

Open Topics 1 (二阶魔方)
请构造出二阶魔方相关的置换群，你能设计一种算法来解二阶魔方复原吗?

Open Topics 2 (transpositions)
证明：Show that any cycle can be written as the product of transpositions:

$$(a_1, a_2, ..., a_n) = (a_1 a_n)(a_1 a_{n-1})\ (a_1 a_3)(a_1 a_2)$$

4 反馈