PS#1 Solutions
Math 355 Summer 1, 2009

p.11 #6. *Let ℓ and m be distinct lines intersecting at the point Q. Let $\ell' = \sigma_m(\ell)$, let P and R be points on ℓ and ℓ' on the same side of m, respectively, and let S and T be the feet of the perpendiculars from P and R to m, respectively. Then $m\angle PQS = m\angle RQT$.*
Proof: Let $P' = \sigma_m(P)$; then S is the midpoint of $\overline{PP'}$, by definition of σ_m. Thus $\triangle PQS \cong \triangle P'QS$ by SAS and $m\angle PQS = m\angle P'QS$ (CPCTC). But $m\angle P'QS = m\angle RQT$ since $\angle P'QS$ and $\angle RQT$ are vertical angles, and the conclusion follows.

p.13 #16. **Theorem 21.** *Let ℓ be a line, let P and Q be distinct points, let $P' = \sigma_\ell(P)$* and let $Q' = \sigma_\ell(Q)$, *then $PQ = P'Q'$.*
Proof of Case 4: *Assume P and Q are off of and on opposite sides of ℓ.*
(a) Subcase 4a. *Assume $\overleftrightarrow{PQ} \perp \ell$.* If $Q = P'$, then $P = Q'$ by definition of σ_ℓ, and $PQ = PP' = Q'P' = P'Q'$ by substitution. If $Q \neq P'$, let $R = \ell \cap \overline{PQ}$. Then P' and Q' are on $\overleftrightarrow{PQ}$ since $\overleftrightarrow{PQ} \perp \ell$ by assumption, $PR = RP'$ and $QR = RQ'$, by definition of σ_ℓ. Assume $PR > QR$ (if not, interchange labels P and Q). Then $PQ = PR + RQ = RP' + QR = RP' + RQ' = P'Q'$.
(b) Subcase 4b. *Assume $\overleftrightarrow{PQ} \not\perp \ell$.* Let $R = \overline{PP'} \cap \ell$, $S = \overline{QQ'} \cap \ell$, and $T = \overline{PQ} \cap \ell$. Then $\triangle PRT \cong \triangle P'RT$ and $\triangle QST \cong \triangle Q'ST$ by SAS so that $PT = TP'$ and $QT = TQ'$ (CPCTC). Therefore $PQ = PT + TQ = TP' + QT = P'T + TQ' = P'Q'$.

p.13 #17. *For any line ℓ, $\sigma_\ell^{-1} = \sigma_\ell$.*
Proof: If P is on ℓ, then $\sigma_\ell(P) = P$ by definition of σ_ℓ, and $(\sigma_\ell \circ \sigma_\ell)(P) = P$. If P is off ℓ, let $P' = \sigma_\ell(P)$. Then ℓ is the perpendicular bisector of $\overline{PP'}$ and $\sigma_\ell(P') = P$ by definition of σ_ℓ, so that $(\sigma_\ell \circ \sigma_\ell)(P) = \sigma_\ell(P') = P$. Thus $(\sigma_\ell \circ \sigma_\ell)(P) = P$ for all P and $\sigma_\ell \circ \sigma_\ell = \iota$. Therefore $\sigma_\ell^{-1} = \sigma_\ell$, by definition of σ_ℓ^{-1}.

p.17 #5. *Let τ be a translation, let P and Q be points, and let $P' = \tau(P)$ and $Q' = \tau(Q)$. If P, Q, P', and Q' are collinear, then $PQ = P'Q'$.*
Proof: (This proof does not assume collinearity of P, Q, P', *and* Q'.) Note that $\mathbf{PP'} = \mathbf{QQ'}$ since $\tau = \tau_{\mathbf{PP'}} = \tau_{\mathbf{QQ'}}$ by Corollary 29. Therefore $\mathbf{PQ} = \mathbf{PP'} + \mathbf{P'Q} = \mathbf{QQ'} + \mathbf{P'Q} = \mathbf{P'Q} + \mathbf{QQ'} = \mathbf{P'Q'}$ and it follows that $PQ = P'Q'$.

p.17 #9. *Let A and B be points. Then $\tau_{\mathbf{AB}}^{-1} = \tau_{\mathbf{BA}}$.*
We first prove **Proposition 33, part 1**: *Let P, Q, R, and S be points. Then $\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}+\mathbf{RS}}$.*
Proof of Proposition 33, part 1: Let A be any point. Then by Definition 27, $\tau_{\mathbf{PQ}}(A) = \mathbf{OA} + \mathbf{PQ}$. Let A' be the unique point such that $\mathbf{OA'} = \mathbf{OA} + \mathbf{PQ}$. Then $(\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}})(A) = \tau_{\mathbf{RS}}(A') = \mathbf{OA'} + \mathbf{RS} = \mathbf{OA} + \mathbf{PQ} + \mathbf{RS} = \tau_{\mathbf{PQ}+\mathbf{RS}}(A)$. Therefore $\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}+\mathbf{RS}}$.
Proof: By Proposition 33, part 1, $\tau_{\mathbf{AB}} \circ \tau_{\mathbf{BA}} = \tau_{\mathbf{AB}+\mathbf{BA}} = \tau_{\mathbf{AA}} = \iota$. Therefore $\tau_{\mathbf{AB}}^{-1} = \tau_{\mathbf{BA}}$ by definition of $\tau_{\mathbf{AB}}^{-1}$.

p.21 #4. *If $\gamma = \sigma_c \circ \tau$ is a glide reflection with axis c, then*
(a) $\gamma^{-1} = \sigma_c \circ \tau^{-1}$.
(b) γ^{-1} *is a glide reflection with axis c.*
We first prove **Proposition 39:** *Let τ be a non-identity translation that fixes line c. Then $\sigma_c \circ \tau = \sigma_c \circ \tau$.*
Proof of Proposition 39: Let P be a point and let $P' = \tau(P)$. If P is on c, so is P' since $\tau(c) = c$, and $(\sigma_c \circ \tau)(P) = \sigma_c(P') = P' = \tau(P) = (\tau \circ \sigma_c)(P)$. If P is off c, let $Q = \sigma_c(P)$ and let $Q' = \tau(Q)$. Then $(\tau \circ \sigma_c)(P) = Q'$ and $\tau = \tau_{\mathbf{PP'}} = \tau_{\mathbf{QQ'}}$ so that $\mathbf{PP'} = \mathbf{QQ'}$ and $\square PP'Q'Q$ is a parallelogram. But $\overleftrightarrow{PP'} \| c$ since $\tau_{\mathbf{PP'}}(c) = c$, and $\overleftrightarrow{PQ} \perp c$ by definition of σ_c. Therefore $\square PP'Q'Q$ is a rectangle, and it follows that $(\sigma_c \circ \tau)(P) = \sigma_c(P') = Q'$. Thus for all P, $(\sigma_c \circ \tau)(P) = (\tau \circ \sigma_c)(P)$.
Proof: (a) First, $\gamma^{-1} = (\sigma_c \circ \tau)^{-1} = \tau^{-1} \circ \sigma_c^{-1} = \tau^{-1} \circ \sigma_c$ by Exercises 1.1.8 and 1.3.17. Let $\tau = \tau_{\mathbf{v}}$; then $\tau^{-1} = \tau_{-\mathbf{v}}$ by Exercise 1.3.17. Since $\tau(c) = c$ by assumption, $\tau^{-1}(c) = c$ and we may apply Proposition 39 to τ^{-1} and conclude that $\gamma^{-1} = \tau^{-1} \circ \sigma_c = \sigma_c \circ \tau^{-1}$.
(b) We observed in part (a) that τ^{-1} is a translation such that $\tau^{-1}(c) = c$. Therefore $\gamma^{-1} = \sigma_c \circ \tau^{-1}$ is a glide reflection by definition.

p.21 #5. *Let γ be a glide reflection with axis c and τ be a translation that fixes c. Then $\tau \circ \gamma = \gamma \circ \tau$.*

We first prove **Proposition 33, part 2**: *Let $P, Q, R,$ and S be points. Then* $\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}} \circ \tau_{\mathbf{RS}}$.
Proof of Proposition 33, part 2: By part 1 and commutativity of vector addition, $\tau_{\mathbf{RS}} \circ \tau_{\mathbf{PQ}} = \tau_{\mathbf{PQ}+\mathbf{RS}} = \tau_{\mathbf{RS}+\mathbf{PQ}} = \tau_{\mathbf{PQ}} \circ \tau_{\mathbf{RS}}$.
Proof: By definition, $\gamma = \sigma_c \circ \tau'$ for some translation τ' that fixes c. By Proposition 33, part 2, Proposition 39, and Exercise 1.1.4 we have $\tau \circ \gamma = \tau \circ (\sigma_c \circ \tau') = (\tau \circ \sigma_c) \circ \tau' = (\sigma_c \circ \tau) \circ \tau' = \sigma_c \circ (\tau \circ \tau') = \sigma_c \circ (\tau' \circ \tau) = (\sigma_c \circ \tau') \circ \tau = \gamma \circ \tau$.

p.25 #9. *Let $P = \begin{bmatrix} a \\ b \end{bmatrix}$ and $Q = \begin{bmatrix} c \\ d \end{bmatrix}$ be distinct points. Then* $\varphi_Q \circ \varphi_P = \tau_{2\mathbf{PQ}}$.
Proof: The equations for φ_P are $x' = 2a - x$ and $y' = 2b - y$; and the equations for φ_Q are $x' = 2c - x$ and $y' = 2d - y$. Composing these equations gives $x' = 2c - (2a - x) = x + 2(c - a)$ and $y' = 2d - (2b - y) = y + 2(d - b)$, which are the equations for $\tau_{2\mathbf{PQ}}$ with $2\mathbf{PQ} = \begin{bmatrix} 2(c-a) \\ 2(d-b) \end{bmatrix}$.

p.25 #12. *If* $\varphi_A \circ \varphi_B = \varphi_B \circ \varphi_A$, then $A = B$.
Proof: $\tau_{2\mathbf{AB}} = \varphi_A \circ \varphi_B = \varphi_B \circ \varphi_A = \tau_{2\mathbf{BA}}$ implies $\mathbf{AB} = \mathbf{BA} = -\mathbf{AB}$. Thus $\mathbf{AB} = \mathbf{0}$ and $A = B$.

p.31 #6. *Let τ be a translation, let C be a point, let $C' = \tau(C)$, and let D be the midpoint of $\overline{CC'}$. Then*
(a) $\tau \circ \varphi_C = \varphi_D$ *and*
(b) $\varphi_C \circ \tau = \varphi_E$, *where* $E = \tau^{-1}(D)$.
Proof: Note that $\tau = \tau_{2\mathbf{CD}}$, since D is the midpoint of $\overline{CC'}$.
(a) By Theorem 56**,** $\tau \circ \varphi_C = \tau_{2\mathbf{CD}} \circ \varphi_C = (\varphi_D \circ \varphi_C) \circ \varphi_C = \varphi_D$.
(b) By Exercise 1.3.9, $\tau^{-1} = \tau_{2\mathbf{CD}}^{-1} = \tau_{2\mathbf{DC}}$, and $E = \tau^{-1}(D) = \tau_{2\mathbf{DC}}(D)$ implies $2\mathbf{DC} = \mathbf{DE}$. Furthermore, $\mathbf{DE} = \mathbf{DC} + \mathbf{CE}$ so that $2\mathbf{DC} = \mathbf{DC} + \mathbf{CE}$ and $\mathbf{DC} = \mathbf{CE}$ by cancellation. Equivalently, $\mathbf{CD} = \mathbf{EC}$ and we have $\varphi_C \circ \tau = \varphi_C \circ \tau_{2\mathbf{CD}} = \varphi_C \circ \tau_{2\mathbf{EC}} = \varphi_C \circ \varphi_C \circ \varphi_E = \varphi_E$.