# 第 11 讲: 有穷与无穷

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**评分：** ______　　**评阅：** ______

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请独立完成作业，不得抄袭。
若得到他人帮助, 请致谢。
若参考了其它资料，请给出引用。
鼓励讨论，但需独立书写解题过程。

- “Veniet tempus, quo ista que nunc latent, in lucem dies extrahat et longioris avi diligentia.”

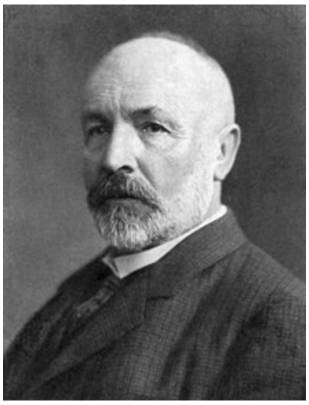

图 1: Georg Cantor (1845 ~ 1918)

## 1　作业（必做部分）

**题目 1 (UD Problem 21.6)**

**解答:**

(a)Let $f(x) = tan(\pi x - \frac{\pi}{2}), x \in (0,1), ran(f) = \mathbb{R}$.

(b)Let $f(x) = |x|, x \in (\mathbb{R}/0)$,

$f(0) = 1$

then $f$ is the bijection from $\mathbb{R}$ to $\mathbb{R}^+$.

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**题目 2 (UD Problem 21.10)**

**证明:**

According to Problem 16.17, let $f : A \to B$ and $g : C \to D$ be functions. Then $H : A \times C \to B \times D$ is a one-to-one function.

Similarly, $A \approx C$ and $B \approx D$(it means there exists bijections, $f : A \to C$, $g : B \to D$),so it is easy to know that $H : A \times B \to C \times D$, that is $A \times B \approx C \times D$.

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**题目 3 (UD Problem 22.10)**

**解答:**

According to Corollary 21.10, let S be a finite set. Then every subset of S is finite.

B is an infinite subset of A, so A can’t be a finite set, that is A is a infinite set.

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**题目 4 (UD Problem 22.11)**

**解答:**
If there doesn't exist $b \in B$ such that $A_b = f^{-1}(\{b\})$ is infinite, then $ran(f^{-1}) = \cup A_b$ is finite, while actually $ran(f^{-1}) = A$ is infinite.
Thus there exists $b \in B$ such that $f^{-1}(\{b\})$ is infinite.

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**题目 5 (UD Problem 22.18)**

**解答:**
(a)
Let the cardinality of A to be n, the cardinality of B be m.
Suppose, to the contrary, |B| > |A|, which means m > n.
According to the pigeonhole principle, we can get that $f : B \to A$ is not a one-to-one function.
But actually $B \subseteq A$, we can easily get that $g : B \to A$ by $g(x) = x$.It is contradict with our assumption.
So $|B| <= |A|$.

(b)
According to (a), A is a finite set and $B \subseteq A$, then $|B| <= |A|$.We suppose that |B| = n, |A| = m.
If $|B| = |A|$. And $B \subseteq A$, then $\forall b \in B, \exists a \in A$, that $a = b$.
Because there is m elements in B, so there is at least n elements which satisfy $a = b$ in A. And m = n, so all elements in A satisfy $a = b$, that is A = B, which is contradict with $B \neq A$.
So $|B| = |A|$ is wrong.
Since $|B| <= |A|$ and $|B| \neq |A|$, we can know $|B| < |A|$.

(c)
According to (a) if $B \subseteq A$ then $|B| <= |A|$.
So if in the same time $|B| >= |A|$ then $|B| = |A|$.
According to (b), if $|B| = |A|$ then A = B.

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**题目 6 (UD Problem 22.21)**

**解答:**
1)Suppose to the contrary that if $f$ is one-to-one then $f$ is not onto.
Let $f(x) = x, x \in A$. We can easily know $f$ is a one-to-one function.
Meanwhile $f$ is onto.
So the assumption is false.

2)Suppose to the contrary that if $f$ is onto then $f$ is not one-to-one.

Put forward the same example, $f(x) = x, x \in A$, Since $f$ is a bijection, which means $f$ is onto and $f$ is one-to-one.
So the assumption is false.
Judging from (1) and (2), we can know $f : A \to A$ is one-to-one if and only if it is onto.

Using the same example $f(x) = x, x \in A$, we can also reach the same conclusion when A is infinite.

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**题目 7 (UD Problem 23.1)**

**解答:**
(a)$\{k\mathbb{N}\}$, k is a prime number.
(b)Not possible.
(c)Not possible.

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**题目 8 (UD Problem 23.3 (a, d))**

**解答:**
(a)It's countable.
Define the set of all lines with rational slopes to be denoted by A.
Let $f : A \to \mathbb{Q}$, for the line $l : y = kx + b, f(l) = k \in \mathbb{Q}$. So $A \approx \mathbb{Q}$.
Since $\mathbb{Q}$ is countable, $A$ is also countable.
(d)It's uncountable. Let $f : \mathbb{R} \to \mathbb{R}, y = f(x) = 1 - x$. Since $fx$ is a rigidly monotonically increasing function, $f$ is bijection.
So$\{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\} \approx \mathbb{R}$.Since $\mathbb{R}$ is infinity, $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y = 1\}$ is also infinity.

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**题目 9 (UD Problem 23.4)**

**解答:**
It's uncountable.
We will suppose, to the contrary, that the sequences are countable.Let the sequences be denoted by A. We can know that $\{0, 1\}$ is finite, there exists a bijection function $f : \{0, 1\} \to A$. We will list the values of $f$. So
$f(1) = a_{11}a_{12}a_{13}...$
$f(2) = a_{21}a_{22}a_{23}...$
$f(3) = a_{31}a_{32}a_{33}...$
.
.
.
where each $a_{i,j}$ represents 0 or 1. Since $f$ is onto, each sequence in A appears in this list.

The odd thing is this: we can construct a number $b = b_1 b_2...$ not in this list (hence showing that our function cannot possibly be onto) by describing it as follows.
We constructed $b$ so that $b_n \neq a_{nn}$ and therefore $b \neq f(n)$ for every n. Then $b$ can not be in our list. So this contradiction must mean that we have assunmed falsely that the sequence are countable.

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**题目 10 (UD Problem 23.9)**

**证明:**
First, if the set A is finite, which means it's countable, then its subset is also finite, so its subset is also countable.
If the set A is countably infinite. There can be a bijection $f : A \to \mathbb{N}$.
Define a restriction$f|_B : B \to \mathbb{N}$. Then $f|_B$ is a one-to-one function.
And according to Exercise 23.5, B is countable.

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**题目 11 (UD Problem 23.12)**

**解答:**
$A_i = \{\frac{p}{i} | i \in \mathbb{N}^+\}$ Define a function $f_i : A_i \to \mathbb{N}^+$ ny $f_i(x) = x \times i$.
$\forall x_1, x_2 \in A_i$,if $f_i(x_1) = f_i(x_2)$ ,then $x_1 = x_2 = \frac{n}{i}$.
So $f_i$ is one-to-one.
$\Rightarrow A_i$ is countable. We will prove $\cup_{i \in \mathbb{N}^+}^{n} A_i$ is countable by introduction on $n$.
$(i)$ (The base step) (n = 1), $\cup_{i \in \mathbb{N}^+}^{1} A_i = A_1$ is obviously countable, and
$(ii)$ (The induction step)for the positive integer $n$, $\cup_{i \in \mathbb{N}^+}^{n} A_i$ is countable.
Then $\cup_{i \in \mathbb{N}^+}^{n+1} A_i = (\cup_{i \in \mathbb{N}^+}^{n} A_i) \cup A_{m+1}$.
According to Theorem 23.6, $\cup_{i \in \mathbb{N}^+}^{n+1} A_i$ is countable.
So, $\cup_{i \in \mathbb{N}_+} A_i$ is countable. So $\mathbb{Q}_+$ is countable.
So $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$ is countble.

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**题目 12 (UD Problem 24.16)**

**解答:**
In the proof of Theorem 23.12. We got that $(0, 1) \approx \mathbb{R}$.
According to Theorem 21.13, $(0, 1) \times (0, 1) \approx \mathbb{R} \times \mathbb{R}$
Define $f : (0, 1) \to (0, 1) \times (0, 1)$, $f$ is bijection. So $(0, 1) \approx (0, 1) \times (0, 1)$ Since $(0, 1) \times (0, 1) \approx \mathbb{R} \times \mathbb{R}$, $(0, 1) \approx \mathbb{R} \times \mathbb{R}$
So $\mathbb{R} \approx \mathbb{R} \times \mathbb{R}$
Define $g : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by $g(a, b) = a + bi, a, b \in \mathbb{R}$.
Since $g$ is a bijection, $\mathbb{R} \times \mathbb{R} \approx \mathbb{C}$
Since $\mathbb{R} \times \mathbb{R} \approx \mathbb{R}$, $\mathbb{R} \approx \mathbb{C}$.
So $|\mathbb{R}| = |\mathbb{C}|$

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# 2 作业（选做部分）

**题目 1 (UD Problem 24.15)**

**解答:**

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# 3 Open Topics

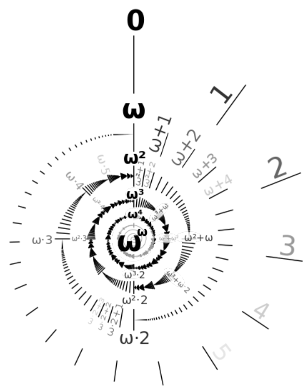
注: 基数与序数比较难以理解。你可以选择介绍一些相对容易的部分。

**Open Topics 1 (基数)**
请介绍基数 (Cardinal number)，如 (不限于):

- 定义
- 运算
- 你认为有意思的相关内容

参考资料:

- Cardinal number @ wiki




**Open Topics 2 (序数)**
请介绍序数 (Ordinal number)，如 (不限于):

- 定义
- 运算
- 你认为有意思的相关内容

参考资料:

- Ordinal number@ wiki

# 4 订正

8-1
(f)$(B \cap C) \backslash A$
(g)$(A \cup B \cup C) \backslash (A \cap B \cap C)$

8-2
(d)
$(i)$
If $A \subseteq B$ then $\forall x \in A, \rightarrow x \in B$, so $\forall x \notin B \rightarrow x \notin A$. That is $\forall x \in (X \backslash B) \rightarrow x \in (X \backslash A)$.
So $(X \backslash B) \subseteq (X \backslash A)$
$(ii)$

If $(X \backslash B) \subseteq (X \backslash A)$, then $\forall x \in (X \backslash B) \to x \in (X \backslash A)$,that is $\forall x \notin B \to x \notin A$. So $\forall x \in A, \to x \in B$.
So $A \subseteq B$.

(f)
$(i)$
If $A \cap B = B$
$\Rightarrow \forall x \in B, \to x \in A \cap B$, that is $x \in A$.
So $B \subseteq A$
$(ii)$
If $B \subseteq A$, then $\forall x \in B \to x \in A$. So $x \in A \cap B$.
In other words, $\forall x \in B \to x \in A \cap B$.
So $B = A \cap B$.

8-7
(a)
$\bigcup_{n=1}^{\infty} A_n = [0, 1)$
$\bigcup_{n=1}^{\infty} B_n = [0, 1]$
$\bigcup_{n=1}^{\infty} C_n = (0, 1)$
(b)
$\bigcap_{n=1}^{\infty} A_n = \{0\}$
$\bigcap_{n=1}^{\infty} B_n = \{0\}$
$\bigcap_{n=1}^{\infty} C_n = \emptyset$

8-11
Let $E \in \mathcal{P}(A_\alpha)$
then $E \in \mathcal{P}(A_{\alpha_1}) \cup \mathcal{P}(A_{\alpha_2}) \cup \mathcal{P}(A_{\alpha_3}) ... \cup \mathcal{P}(A_{\alpha_n}), \alpha_1, \alpha_2 ... \in I$
then $E \in \mathcal{P}(A_{\alpha_1}) \cup \mathcal{P}(A_{\alpha_2}) ... \cup \mathcal{P}(A_{\alpha_n})$
So, $E \in \mathcal{P}(A_{\alpha_1} \cup A_{\alpha_2} ... \cup A_{\alpha_n})$

8-12
(i)prove $\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) \subseteq \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha):$
$E \in \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha)$
then $E \in \mathcal{P}(A_{\alpha_1}) \cap \mathcal{P}(A_{\alpha_2}) ... \cap \mathcal{P}(A_{\alpha_n})$
then $E \in \mathcal{P}(A_{\alpha_1} \cap A_{\alpha_2} ... \cap A_{\alpha_n})$
so $E \in \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$
(ii)prove $\mathcal{P}(\bigcap_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in I} \mathcal{P}(A_\alpha)$
$E \in \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$
then $E \in \mathcal{P}(A_{\alpha_1} \cap A_{\alpha_2} ... \cap A_{\alpha_n})$
then $E \in \mathcal{P}(A_{\alpha_1}) \cap \mathcal{P}(A_{\alpha_2}) ... \cap \mathcal{P}(A_{\alpha_n})$
then $E \in \mathcal{P}(A_\alpha)$

SO $\bigcap_{\alpha \in I} \mathcal{P}(A_\alpha) = \mathcal{P}(\bigcap_{\alpha \in I} A_\alpha)$

3-4 第一类数学归纳法证明第二类数学归纳法
要证第二类数学归纳法，也即任给一个命题 $F$，若满足 $F(1)$ 及 $(F(1) \wedge F(2) \wedge ... \wedge F(n)) \Rightarrow F(n+1)$ ，则有 $\forall k \in \mathbb{N}, F(k)$ 成立。
构造命题 $G(n) = F(1) \wedge F(2) \wedge ... \wedge F(n)$

显然，$G(n) \Rightarrow F(n+1)$，又 $G(n) \Rightarrow G(n)$ 所以 $G(n) \Rightarrow G(n) \wedge F(n+1) = G(n+1)$ 所以 G 满足第一类数学归纳法的条件，所以 $\forall k \in \mathbb{N}, G(k)$ 成立。而 $G(n) \Rightarrow F(n)$，故 $\forall k \in \mathbb{N}, F(k)$ 成立，也即第二类数学归纳法成立。

第二类数学归纳法证明第一类数学归纳法要证第一类数学归纳法，也即任给一个命题 $F$，若满足 $F(1)$ 及 $F(n) \rightarrow F(n+1)$，则有 $\forall k \in \mathbb{N}, F(k)$ 成立。
因为 1 的条件比 2 强, 所以 F 一定满足第二类数学归纳法. 故根据第二类数学归纳法 $F(k)$ 对所有正整数 $k$ 都成立, 也即第一类数学归纳法成立.

显然，[公式] 是满足第二类数学归纳法的条件的（因为 1 的条件比 2 强），故根据第二类数学归纳法，[公式] 对所有正整数 [公式] 成立，也即第一类数学归纳法成立.

致谢: 鄢振宇学长 (zhihu)

# 5 反馈