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Information theory
Luis Felipe Ramos Ferreira - 2019022553
Problem Set - Discrete Probability

1- a) Two conditions should be met:

i) Any outcome should have a probability between 0 and 1, i.e., for any outcome σ, it's probability P_σ should respect $0 \leq P_\sigma \leq 1$.

ii) The sum of the probabilities of every outcome in the finite sample space should equal 1,

$$\sum_{i \in S} P_i = 1$$

b) We know that: $P(heads) + P(tails) = 1$ and $P(heads) = 3P(tails)$.

$3P(t) + P(t) = 1 \rightarrow 4P(t) = 1 \rightarrow P(t) = 1/4$

Therefore, $P(h) = 3(P(t)) = 3.(1/4) = 3/4$

$P(heads) = 3/4$ and $P(tails) = 1/4$

2- a) It is the probability of the event E inside the sample space where the event F has already happened/been satisfied.

b) $P(E) = 1/2 \quad P(F) = 1/2$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/6}{1/2} = 1/3$$

3- a) When $P(E \cap F) = P(E).P(F)$

b) $P(E) = 1/2 \quad P(F) = 1/3 \qquad P(E \cap F) = 1/6$

$P(E \cap F) = P(E).P(F) = \frac{1}{2}.\frac{1}{3} = \frac{1}{6}$

Yes, the events E and F are independent since $P(E \cap F) = P(E).P(F)$

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4- a) A random variable is a special function that maps, for each outcome of a sample space, a real value.

b) $X(1,1) = 1$ $X(2,2) = 2$ $X(3,3) = 3$

$X(1,2) = 2$ $X(2,3) = 3$ $X(3,4) = 4$

$X(1,3) = 3$ $X(2,4) = 4$ $X(3,5) = 5$

$X(1,4) = 4$ $X(2,5) = 5$ $X(3,6) = 6$

$X(1,5) = 5$ $X(2,6) = 6$

$X(1,6) = 6$

$X(4,4) = 4$ $X(5,5) = 5$ $X(6,6) = 6$

$X(4,5) = 5$ $X(5,6) = 6$

$X(4,6) = 6$

Since the order of the two dice doesn't matter, the other permutations were omitted to save space.

5- a) The expected value of a random variable X, $E(x)$, is defined as:

$$E(x) = \sum_{e \in S} p(e) \cdot X(e).$$

It can be seen as the weighted average of the values the variable can assume.

b) $E(x) = \frac{1}{36} \cdot 1 + \frac{3}{36} \cdot 2 + \frac{5}{36} \cdot 3 + \frac{7}{36} \cdot 4 + \frac{9}{36} \cdot 5 + \frac{11}{36} \cdot 6$

$E(x) \cong 4{,}472$

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6-a) A "bernoulli trial" is a experiment with only two possible results, success, with probability p, and failure, with probability 1-p.

b) Assuming the probability of success is p, we have:

$$P = C(N,k) \cdot p^k \cdot (1-p)^{N-k}$$

c) E(Nº of successes) = Np.

7- It means that, given N random variables X_i, $i \leq N$, from a sample space, then:

1. $E(X_1 + \dots + X_N) = E(X_1) + \dots + E(X_N)$
2. $E(aX_i + b) = a\,E(X_i) + b$

8- Bayes theorem states: $P(E|F) = \dfrac{P(E)\,P(F|E)}{P(F)}$.

$$P(F|E) = \frac{P(F)\,P(E|F)}{p(F)\,p(E|F) + p(\bar{F})\,p(E|\bar{F})} = \frac{(2/3)\cdot(1/3)}{(2/3)(1/3) + (1/3)(1/4)}$$

$$P(F|E) = \frac{2/9}{2/9 + 1/12} = \frac{2/9}{8+3/36} = \frac{2/9}{11/36} = \frac{2}{\cancel{9}} \cdot \frac{\cancel{36}^{4}}{11} = \frac{8}{11}$$

9- The variance of a random variable is a measure of how scattered the values of this variable are from it's expected value.

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10- Let $p(E)$ and $p(F)$ be two different probabilities of a sample space, where $p(E) = 0{,}7$ and $p(F) = 0{,}5$

We know that $p(E \cup F) = P(E) + P(F) - P(E \cap F)$, then we have

$p(E \cup F) = 0{,}5 + 0{,}7 - P(E \cap F) = 1{,}2 - P(E \cap F)$

Since $p(E \cup F) \leq 1$, $P(E \cap F)$ must be $\geq 0{,}2$.

Also, $P(E \cup F)$ must be $\geqslant 0{,}7$ since the probability of the union is always at least the probability of each individual event.

11- $P(E).P(F) = P(E \cap F)$

$P(E \cap \bar{F}) = P(E) - P(E \cap F) = P(E) - P(E).P(F)$

$\Rightarrow P(E)(1 - P(F)) = P(E)P(\bar{F})$

Since $P(E \cap \bar{F}) = P(E).P(\bar{F})$, the events E and $\bar{F}$ are independent.

12- A -> At least Two consecutive zeros B -> First bit is one

$P(B) = 1/2$ $P(A|B) = \frac{P(A).P(B|A)}{P(B)} = \frac{1/2 \cdot 3/8}{1/2} = 3/8$ //

P(A) →	0000 e	0100 e	1000 e	1100 e ~> $P(A) = 1/2$
	0001 e	0101	1001 e	1101
	0010 e	0110	1010	1110
	0011 e	0111	1011	1111

13- $P(E) = 14/16$ -> We exclude the cases with only boys or girls

$P(F) = 5/16$

No, the events are not independent.

$(P(E).P(F) = 70/256 = 35/128) \neq (P(E \cap F) = 4/16 = 1/4)$

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14- $P(HIV) = 8/100$ $P(\overline{HIV}) = 92/100$ $P(+) = 10{,}6/100$

$P(+|HIV) = 98/100$ $P(+|\overline{HIV}) = 3/100$ $P(\mp) = 89{,}4/100$

a) $P(HIV|+) = \dfrac{P(HIV) \cdot P(+|HIV)}{P(HIV) \cdot P(+|HIV) + P(\overline{HIV}) \cdot P(+|\overline{HIV})}$

$P(HIV|+) = \dfrac{8/100 \cdot 98/100}{8/100 \cdot 98/100 + 92/100 \cdot 3/100} = \dfrac{8.98}{8.98 + 92.3} = \dfrac{784}{1060} \cong \dfrac{73{,}96}{100}$

b) $P(\overline{HIV}|+) = \dfrac{P(\overline{HIV}) \cdot P(+|\overline{HIV})}{P(+)} = \dfrac{92/100 \cdot 3/100}{1060/10000} = \dfrac{276}{1060} \cong \dfrac{26{,}03}{100}$

c) $P(HIV|\mp) = \dfrac{P(HIV) \cdot P(\mp|HIV)}{P(\mp)} = \dfrac{8/100 \cdot 2/\cancel{100}}{89{,}4/\cancel{100}} = \dfrac{16}{100} \cdot \dfrac{1}{89{,}4} \cong \dfrac{0{,}179}{100}$

d) $P(\overline{HIV}|\mp) = \dfrac{P(\overline{HIV}) \cdot P(\mp|\overline{HIV})}{P(\mp)} = \dfrac{92/100 \cdot 97/\cancel{100}}{89{,}4/\cancel{100}} \cong \dfrac{99{,}82}{100}$