# Information Theory

Problem Set 06 - Dependent Random Variables


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1. (a) $H(X, Y)$ is the joint entropy of $X$ and $Y$. It means how much information, on average, each of the joint outcomes carries. On other words, is the expected value of information of the joint outcomes from ensembles $X$ and $Y$.

$$H(X, Y) = \sum_{xy \in \mathcal{A}_x \mathcal{A}_y} P(x, y) log \frac{1}{P(x, y)}$$

   (b) $H(X|Y)$ is the conditional entropy of $X$ given $Y$. It represents the average information infomration content of $X$ given each $y \in \mathcal{A}_y$.

$$H(X|Y) = \sum_{xy \in \mathcal{A}_x \mathcal{A}_y} P(x, y) log \frac{1}{P(x|y)}$$

   (c) $I(X, Y)$ is the mutual information between $X$ and $Y$. It is the average reduction of uncertainty gain of information about $X$ when learning the values of $Y$, or vice-versa, since the mutual information is symmetric.

$$I(X; Y) = H(X) - H(X|Y)$$

   (d) $I(X : Y|Z)$ is the conditional mutual information between $X$ and $Y$ given $Z$. It is the amount of information you gain about $X$ when you learn about $Y$ given that the ensemble $Z$ is known.

$$I(X; Y|Z) = H(X|Z) - H(X|Y, Z)$$

2. (a) The chain rule for entropy states that:

$$H(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} H(X_i|X_1, \ldots, X_{i-1})$$

   (b) The chain rule for mutual information states that:

$$I(X_1, X_2, \ldots, X_n; Y) = \sum_{i=1}^{n} I(X_i; Y|X_1, \ldots, X_{i-1})$$

(c) The data-processing inequality (DPI) staets that, if $X \to Y \to Z$ is a Markov Chain, i. e. $p(x, y, z) = p(x)p(y|x)p(z|y)$ for all values in the ensembles, then $I(X; Z) \leq I(Y; Z)$. In general words, the inequality states that post-processing cannot create information, i.e, the amount of information you have after post-processing data is at most equal to the amount of information before the processing.

3. • $H(X, Y)$

$$H(X, Y) = H(U, V, V, W) = H(U) + H(V|U) + H(V|U, V) + H(W|U, V, V)$$

$$H(X, Y) = H(U) + H(V) + 0 + H(W) = H_u + H_v + H_w$$

$H(V|U) = 0$ since $U$ and $V$ are independent, and $H(V|U, V)$ is obviously zero cause no information is gained.

• $H(X|Y)$

$$H(X|Y) = H(U, V|V, W) = H(U|V, W) + H(V|V, W, U) = H(U) + 0 = H(U) = H_u$$

$H(U|V, W) = 0$ since $U, V, W$ are independent and $H(V|V, W, U)$ is obviously zero.

• $I(X; Y)$

$$I(X; Y) = H(X) - H(X|Y) = H(U, V) - H(U) = H(U) + H(V) - H(U) = H(V) = H_v$$

4. We can see and example in question 6. In that case, $H(X) = \frac{7}{3}$ bits but $H(X|Y = 3) = 2$ bits. To prove $H(X|Y) \leq H(X)$:

$$H(X|Y) \equiv \sum_{y \in A_Y} P(y) \left[ \sum_{x \in A_X} P(x|y) \log \frac{1}{P(x|y)} \right] = \sum_{xy \in A_X A_Y} P(x, y) \log \frac{1}{P(x|y)}$$

$$= \sum_{xy} P(x)P(y|x) \log \frac{P(y)}{P(y|x)P(x)}$$

$$= \sum_{x} P(x) \log \frac{1}{P(x)} + \sum_{x} P(x) \sum_{y} P(y|x) \log \frac{P(y)}{P(y|x)}$$

This last sum, as stated by [1] in his book, is a sum of relative entropies between the distribution $P(y|x)$ and $P(y)$, so $H(X|Y) \leq H(X) + 0$.

5. $D_H(X, Y) \equiv H(X, Y) - I(X; Y)$

• $D_H(X, Y) \geq 0$

$$H(X, Y) - I(X; Y) = H(X) + H(Y|X) - (H(X) - H(X|Y))$$

$$H(X, Y) - I(X; Y) = H(Y|X) + H(X|Y) \geq 0$$

We can afirm that given that conditional entropy is always greater or equal to 0.

- $D_H(X, X) = 0$

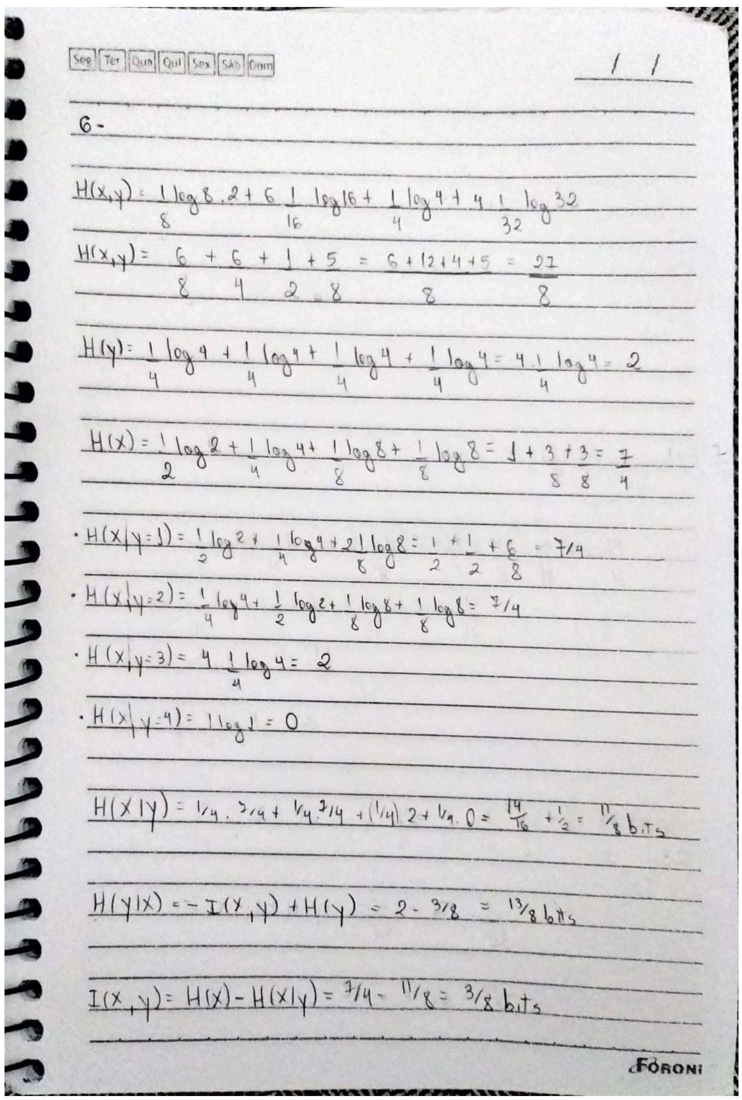
  $H(X,X) - I(X;X) = H(X) + H(X|X) - (H(X) - H(X|X)) = H(X) + 0 - H(X) + 0 = 0$

- $D_H(X, Y) = D_H(Y, X)$ It is easy to proove since conditional entropy and mutual information are symmetric.

$$H(X,Y) - I(X;Y) = H(Y,X) - I(Y;X) = D_H(Y,X)$$

- $D_H(X, Z) \leq D_H(X, Y) + D_H(Y, Z)$

6. Handmade exercise.



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$H(x,y) = \frac{1}{8}\log 8 \cdot 2 + 6\,\frac{1}{16}\log 16 + \frac{1}{4}\log 4 + 4\,\frac{1}{32}\log 32$

$H(x,y) = \frac{6}{8} + \frac{6}{4} + \frac{1}{2} + \frac{5}{8} = \frac{6+12+4+5}{8} = \frac{27}{8}$

$H(y) = \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4 = 4\,\frac{1}{4}\log 4 = 2$

$H(x) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 = 1 + \frac{3}{8} + \frac{3}{8} = \frac{7}{4}$

- $H(x|y=1) = \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + 2\,\frac{1}{8}\log 8 = \frac{1}{2} + \frac{1}{2} + \frac{6}{8} = 7/4$
- $H(x|y=2) = \frac{1}{4}\log 4 + \frac{1}{2}\log 2 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 = 7/4$
- $H(x|y=3) = 4\,\frac{1}{4}\log 4 = 2$
- $H(x|y=4) = 1\log 1 = 0$

$H(x|y) = 1/4 \cdot 7/4 + 1/4 \cdot 7/4 + (1/4)\,2 + 1/4 \cdot 0 = \frac{14}{16} + \frac{1}{2} = 11/8$ bits

$H(y|x) = -I(x,y) + H(y) = 2 - 3/8 = 13/8$ bits

$I(x,y) = H(x) - H(x|y) = 7/4 - 11/8 = 3/8$ bits

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7. (a) $\mathcal{P}_z = \{1/2, 1/2\}$

$$\mathcal{P}(z=0) = \mathcal{P}(x=0)\mathcal{P}(y=0) + \mathcal{P}(x=1)\mathcal{P}(y=1)$$

$$\mathcal{P}(z=0) = \mathcal{P}(x=0)\frac{1}{2} + \mathcal{P}(x=1)\frac{1}{2}$$

$$\mathcal{P}(z=0) = \frac{p+1-p}{2} = \frac{1}{2}$$

$$I(Z;X) = H(Z) - H(Z|X) = 1 - 1 = 0$$

(b) For genral $p$ and $q$ we have

$$\mathcal{P}(z=0) = \mathcal{P}(x=0)\mathcal{P}(y=0) + \mathcal{P}(x=1)\mathcal{P}(y=1)$$

$$\mathcal{P}(z=0) = pq + (1-p)(1-q)$$

Consequently,

$$\mathcal{P}(z=1) = 1 - pq - (1-p)(1-q) = p(1-q) + q(1-p)$$

Therefora, for general $p$ and $q$, we have $\mathcal{P}_z = \{pq+(1-p)(1-q), p(1-q)+q(1-p)\}$.
For the mutual information, we have:

$$I(Z;Y) = H(Z) - H(Z|X) = H(pq + (1-p)(1-q)) - H(q)$$

8. • Handmade exercise.

8-a) $P(\text{lower black} \mid \text{upper black}) = \dfrac{P(\text{lower black, upper black})}{P(\text{upper black})}$

$= \dfrac{1/3}{1/2} = \dfrac{2}{3}$

$P(\text{lower white} \mid \text{upper black}) = 1 - \dfrac{2}{3} = \dfrac{1}{3}$

As we can see, the probability that the lower face of the card is white is 1/3 and the probability that the lower face of the card is black is 2/3.

- Yes, it certainly does. Since the probability distribution over the color of the upper or lower face of a random selected card is uniform, we can say that both $H(U)$ and $H(L)$ are qual to 1 bit. If we know the color of the upper side of the card, we have:

$$H(L|U = black) = H(L|U = white) = H_2(1/3, 2/3) = -\frac{1}{3}log\frac{1}{3} - \frac{2}{3}log\frac{2}{3}$$

$$H(L|U) = P(U = black)H(L|U = black) + P(U = white)H(L|U = white)$$

$$H(L|U) = \frac{1}{2}(log3 - \frac{2}{3})2$$

$$H(L|U) = (log3 - \frac{2}{3})$$

With this value, we can finally compute the mutual information $I(L;U) = H(L) - H(U|L) = 1 - log3 + \frac{2}{3} = \frac{5}{3} - log3$. So we can say that learning the color of the upper side of the card gives a non zero amount of bits of information about the color of the lower side of the card.

## References

[1] David J. C. MacKay. *Information Theory, Inference and Learning Algorithms.* 7th edition, 2005.