Chapter 5
Model-Reference Adaptive Control

Abstract This chapter presents the fundamental theory of model-reference adaptive control. Various types of uncertainty are defined. The composition of a model-reference adaptive control system is presented. Adaptive control theory for first-order single-input single-output (SISO) systems, second-order SISO systems, and multiple-input multiple-output (MIMO) systems is presented. Both direct and indirect adaptive control methods are discussed. The direct adaptive control methods adjust the control gains online directly, whereas the indirect adaptive control methods estimate unknown system parameters for use in the update of the control gains. Asymptotic tracking is the fundamental property of model-reference adaptive control which guarantees that the tracking error tends to zero in the limit. On the other hand, adaptive parameters are only bounded in the model-reference adaptive control setting.

When designing a controller for a system, a control designer typically would like to know how the system behaves physically. This knowledge is usually captured in the form of a mathematical model. For many real-world applications, modeling of physical systems can never be perfect as systems may have parameter variations due to nonlinearity, parameter uncertainty due to modeling inaccuracy or imprecise measurements, uncertainty in exogenous disturbances coming from the operating environment, or other sources of uncertainty. The role of a modeling specialist is to reduce the system uncertainty as much as practicable. The control designer then uses a mathematical model of the system to design a controller which may incorporate performance measures and stability margins to account for any residual system uncertainty that cannot be completely accounted for.

In situations when the system uncertainty may become significant beyond a level of desired tolerance that can adversely affect the performance of a controller, adaptive control can play an important role in reducing the effects of the system uncertainty on the controller performance. Situations that may warrant the use of adaptive control could include unintended consequences of off-nominal modes of operation such as system failures or highly uncertain operating conditions, and complex system behaviors that can result in an increase in the complexity and hence cost of the modeling efforts. In this chapter, the learning objectives are as follows:

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- To develop a basic understanding of system uncertainty and the composition of a typical model-reference adaptive control system and its functionality;
- To be able to apply various model-reference adaptive control techniques for direct and indirect adaptation for first-order, second-order, and MIMO systems;
- To be able to perform a Lyapunov stability proof of model-reference adaptive control using the Lyapunov's direct method and Barbalat's lemma; and
- To recognize that model-reference adaptive control achieves asymptotic tracking but only boundedness of adaptive parameters.

■

A typical model-reference adaptive control (MRAC) system block diagram is shown in Fig. 5.1.

There are generally two classes of adaptive control: (1) direct adaptive control and (2) indirect adaptive control [1, 2]. Adaptive control architectures that combine both types of adaptive control are also frequently used and are referred to as composite [2, 3], combined, or hybrid direct–indirect adaptive control [4]. A typical direct adaptive controller may be expressed as

$$u = k_x(t)\,x + k_r(t)\,r \tag{5.1}$$

where $k_x(t)$ and $k_r(t)$ are adjustable control gains. The mechanism to adjust these control gains is via an adaptive law. Thus, a direct adaptive control in effect adjusts a feedback control mechanism of a control system directly to cancel out any unwanted system uncertainty so that the performance of the control system can be regained in the presence of significant system uncertainty.

In contrast, an indirect adaptive controller achieves the same objective by adjusting the control gains in an indirect way, which may be expressed as

$$u = k_x(p(t))\,x + k_r(p(t))\,r \tag{5.2}$$

where $p(t)$ are system parameters that are estimated online to update the control gains.

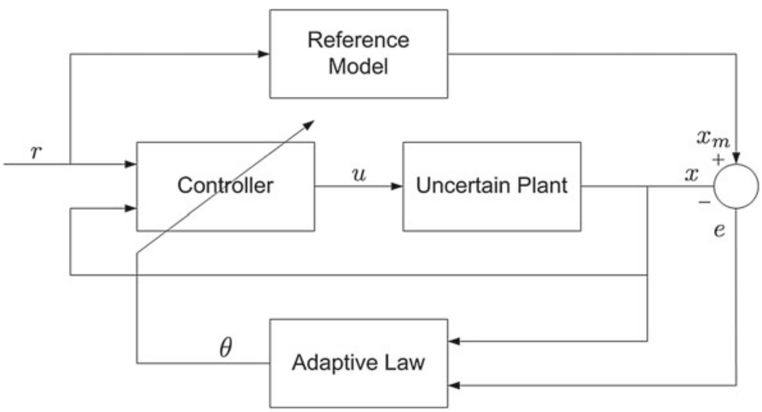

Fig. 5.1 A model-reference adaptive control system

Example 5.1 Consider a second-order LTI system

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u$$

A controller is designed to enable the output $x(t)$ to track a constant command $r(t)$ and meet performance specifications for a closed-loop damping of ζ_m and a bandwidth frequency of ω_m. A PD (proportional-derivative) controller is designed as

$$u = k_p e + k_d \dot{e} + k_r r$$

where $e(t) = r(t) - x(t)$, to achieve the following closed-loop characteristic:

$$\ddot{x} + (2\zeta\omega_n + k_d)\dot{x} + \left(\omega_n^2 + k_p\right)x = (k_r + k_p)r$$

Choose

$$\omega_n^2 + k_p = \omega_m^2 \Rightarrow k_p = \omega_m^2 - \omega_n^2$$

$$2\zeta\omega_n + k_d = 2\zeta_m\omega_m \Rightarrow k_d = 2\left(\zeta_m\omega_m - \zeta\omega_n\right)$$

$$k_r = \omega_n^2$$

Then, the output $x(t)$ tracks the command $r(t)$ as $t \to \infty$.

Suppose the open-loop system natural frequency ω_n suddenly changes to a new value ω_n^* which may be unknown. The closed-loop system can now have a drastically different performance if the original control gains are used. To maintain the same performance specifications, the ideal control gains must be

$$k_p^* = \omega_m^2 - \omega_n^{*2} = k_p + \omega_n^2 - \omega_n^{*2}$$

$$k_d^* = 2\left(\zeta_m\omega_m - \zeta\omega_n^*\right) = k_d + 2\zeta\left(\omega_n - \omega_n^*\right)$$

$$k_r^* = \omega_n^{*2} = k_r - \omega_n^2 + \omega_n^{*2}$$

A direct adaptive controller seeks to adjust the original control gains k_p, k_d, and k_r toward the ideal control gains k_p^*, k_d^*, and k_r^*, respectively, directly without knowing the value of the uncertain parameter ω_n^*. On the other hand, an indirect adaptive controller seeks to adjust the control gains by estimating online the uncertain parameter ω_n^* and use the estimate of ω_n^* to re-compute the control gains as if the parameter estimate is the true value. This approach is often referred to as the Certainty Equivalence Principle.

5.1 Composition of a Model-Reference Adaptive Control System

5.1.1 Uncertain Plant

Adaptive control can deal with either linear or nonlinear plants with various types of uncertainty which can be structured uncertainty, unstructured uncertainty, or unmodeled dynamics.

1. Structured uncertainty is a source of uncertainty with uncertain parameters but known functional characteristics. It is also often referred to as parametric uncertainty.

Example 5.2 A linear spring-mass-damper system with an uncertain spring constant

$$m\ddot{x} + c\dot{x} + k^* x = u$$

where k^* is an uncertain parameter, is a system with structured or parametric uncertainty. The function $x(t)$ associated with k^* is a known characteristic that appears in the structured uncertainty.

2. Unstructured uncertainty is a source of uncertainty for which neither parameters or functional characteristics are certain.

Example 5.3 A spring-mass-damper system with an uncertain spring characteristic

$$m\ddot{x} + c\dot{x} + f^*\left(x, k^*\right) = u$$

where $f^*()$ is an uncertain function, is a system with unstructured uncertainty.

3. Unmodeled dynamics is a source of uncertainty that represents system internal or external dynamics that are not included in a plant model because they may be unmeasurable, unobservable, or assumed incorrectly to be negligible.

Example 5.4 The following linear spring-mass-damper system has unmodeled dynamics:

$$m\ddot{x} + c\dot{x} + k^* x = c_1 y + u$$

$$\dot{y} = c_2 x + c_3 y$$

where $y(t)$ is an internal system state that is not modeled, and c_i, $i = 1, 2, 3$ are parameters.

4. Matched uncertainty is a type of structured uncertainty that can be matched by the control input for a class of MIMO linear affine-in-control systems of the form

$$\dot{x} = f(x) + B\left[u + \Theta^{*\top}\Phi(x)\right] \tag{5.3}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $B \in \mathbb{R}^n \times \mathbb{R}^m$ is a control input matrix, $\Theta^* \in \mathbb{R}^p \times \mathbb{R}^m$ is a matrix of uncertain parameters, and $\Phi(x) \in \mathbb{R}^p$ is a known bounded regressor function.
The quantity $\Theta^{*\top}\Phi(x)$ is called a parametric matched uncertainty since it appears in the range space of the control input matrix B. Recall from linear algebra that a range or column space of a matrix B consists of all possible products Bu. When a parametric uncertainty is matched, the control input can cancel out the uncertainty completely when the adaptation is perfect.

Example 5.5 The parametric uncertainty in the following LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(u + \begin{bmatrix} \delta_1 & \delta_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

is a matched uncertainty where δ_1 and δ_2 are uncertain parameters.

5. Unmatched uncertainty is a type of uncertainty that cannot be matched by the control input for a class of MIMO linear affine-in-control systems of the form

$$\dot{x} = f(x) + Bu + \Theta^{*\top}\Phi(x) \tag{5.4}$$

A parametric uncertainty cannot be matched if the control input matrix $B \in \mathbb{R}^n \times \mathbb{R}^m$ is a non-square "tall" matrix, i.e., $n > m$, or if $B \in \mathbb{R}^n \times \mathbb{R}^n$ is a rank-deficient square matrix such that its inverse does not exist. In such a case, the control input cannot completely cancel out the uncertainty by adaptive control. Otherwise, the uncertainty may be cast as a matched uncertainty by the following pseudo-inverse transformation:

$$\dot{x} = f(x) + B\left[u + B^\top \left(BB^\top\right)^{-1} \Theta^{*\top}\Phi(x)\right] \tag{5.5}$$

where $B^\top \left(BB^\top\right)^{-1}$ is the right pseudo-inverse of a full-rank non-square "wide" matrix $B \in \mathbb{R}^n \times \mathbb{R}^m$ with $n < m$ and $\operatorname{rank}(B) = n$, or where $B^\top \left(BB^\top\right)^{-1} = B^{-1}$ for a full-rank square matrix B.

Example 5.6 The parametric uncertainty in the following LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} \delta_{11}x_1 + \delta_{12}x_2 \\ \delta_{21}x_1 + \delta_{22}x_2 \end{bmatrix}$$

is an unmatched uncertainty since B is a "tall" matrix. Intuitively, a tall matrix B implies the number of control inputs is less than the number state variables. Therefore, it would be difficult for the control inputs to cancel out the uncertainty in all the state variables.

Example 5.7 The parametric uncertainty in the following LTI system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is actually a matched uncertainty since B is a full-rank "wide" matrix whose pseudo-inverse exists

$$B^\top \left(B B^\top \right)^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So, the system can be cast as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} \delta_{11} + \delta_{21} & \delta_{12} + \delta_{22} \\ \delta_{21} & \delta_{22} \\ \delta_{11} & \delta_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)$$

6. Control input uncertainty is a type of uncertainty that exists in the control input matrix for a class of MIMO linear affine-in-control systems of the form

$$\dot{x} = f(x) + B \Lambda u \tag{5.6}$$

where Λ is a positive diagonal matrix whose diagonal elements represent the control input effectiveness uncertainty which can be in the amplitude or in the sign or both. When the uncertainty is in the amplitude, a control saturation can occur and may worsen the performance of a controller. When the uncertainty is in the sign, a control reversal can occur and potentially can cause instability. An alternative form of a control input uncertainty is given by

$$\dot{x} = f(x) + (B + \Delta B) u \tag{5.7}$$

which is less common in adaptive control.

5.1.2 Reference Model

A reference model is used to specify a desired response of an adaptive control system to a command input. It is essentially a command shaping filter to achieve a desired command following. Since adaptive control is formulated as a command following or tracking control, the adaptation is operated on the tracking error between the reference model and the system output. A reference model must be designed properly for an adaptive control system to be able to follow. Typically, a reference model is formulated as a LTI model, but a nonlinear reference model can be used although a nonlinear design always brings up many complex issues. A LTI reference model should capture all important performance specifications such as rise time and settling time, as well as robustness specifications such as phase and gain stability margins.

Example 5.8 For Example 5.1, the reference model for an adaptive control system could be selected to be a second-order system as

$$\ddot{x}_m + 2\zeta_m \omega_m \dot{x}_m + \omega_m^2 x_m = \omega_m^2 r$$

where $x_m(t)$ is a model-reference signal that only depends on the reference command input $r(t)$. ■

The tracking error is defined as

$$e = x_m - x \tag{5.8}$$

The objective of an adaptive control system is to adapt to system uncertainty so as to keep the tracking error as small as possible. In an ideal case when $e(t) \to 0$, then the system state follows the model-reference signal perfectly, i.e., $x(t) \to x_m(t)$.

5.1.3 Controller

A controller must be designed to provide overall system performance and stability for a nominal plant without uncertainty. Thus, it can be thought of as a baseline or nominal controller. The type of controllers is dictated by the objective of a control design. A controller can be linear or nonlinear but as always nonlinear controllers are much more difficult to design, analyze, and ultimately certify for operation in real systems. The controller can be a nominal controller augmented with an adaptive controller or a fully adaptive controller. The adaptive augmentation control design is more prevalent and generally should be more robust than a fully adaptive control design.

5.1.4 Adaptive Law

An adaptive law is a mathematical relationship that expresses explicitly how adaptive parameters should be adjusted to keep the tracking error as small as possible. An adaptive law can be either linear time-varying or nonlinear. In any case, stability of an adaptive control system usually must be analyzed using Lyapunov stability theory. Many different adaptive laws have been developed, and each has its own advantages as well as disadvantages. Ultimately, designing an adaptive control system comes down to a trade-off between performance and robustness. This trade-off can be made by a suitable selection of an adaptive law and a set of tuning parameters that are built into an adaptive law.

5.2 Direct MRAC for First-Order SISO Systems

Consider a first-order nonlinear SISO system

$$\dot{x} = ax + b\left[u + f(x)\right] \tag{5.9}$$

subject to $x(0) = x_0$, where $f(x)$ is a structured matched uncertainty that can be linearly parametrized as

$$f(x) = \sum_{i=1}^{p} \theta_i^* \phi_i(x) = \Theta^{*\top} \Phi(x) \tag{5.10}$$

where $\Theta^* = \left[\theta_1\ \theta_2\ \ldots\ \theta_p\right]^\top \in \mathbb{R}^p$ is an unknown constant vector, and $\Phi(x) = \left[\phi_1(x)\ \phi_2(x)\ \ldots\ \phi_p(x)\right]^\top \in \mathbb{R}^p$ is a vector of known bounded basis functions.

5.2.1 *Case I: a and b Unknown but Sign of b Known*

A reference model is specified as

$$\dot{x}_m = a_m x_m + b_m r \tag{5.11}$$

subject to $x_m(0) = x_{m_0}$, where $a_m < 0$ and $r(t) \in \mathscr{L}_\infty$ is a piecewise continuous bounded reference command signal, so that $x_m(t)$ is a uniformly bounded model-reference signal.

Firstly, define an ideal controller that perfectly cancels out the uncertainty and enables $x(t)$ to follow $x_m(t)$ as

$$u^* = k_x^* x + k_r^* r(t) - \Theta^{*\top} \Phi(x) \tag{5.12}$$

where the superscript $*$ denotes ideal constant values which are unknown.

Upon substituting into the plant model, we get the ideal closed-loop plant

$$\dot{x} = \left(a + bk^*\right) x + bk_r^* r \tag{5.13}$$

Comparing the ideal closed-loop plant to the reference model, the ideal gains k_x^* and k_r^* can be determined by the following model matching conditions:

$$a + bk_x^* = a_m \tag{5.14}$$

$$bk_r^* = b_m \tag{5.15}$$

It turns out that the solutions for k_x^* and k_r^* always exist since there are two independent equations with two unknowns.

The actual adaptive controller is an estimate of the ideal controller with a goal that in the limit the adaptive controller approaches the ideal controller. Let

$$u = k_x(t)\,x + k_r(t)\,r - \Theta^\top(t)\,\Phi(x) \tag{5.16}$$

be the adaptive controller, where $k_x(t)$, $k_r(t)$, and $\Theta(t)$ are the estimates of k_x^*, k_r^*, and Θ^*, respectively.

The adaptive controller is a direct adaptive controller since $k_x(t)$, $k_r(t)$, and $\Theta(t)$ are estimated directly without the knowledge of the unknown system parameters a, b, and Θ^*.

Now, define the estimation errors as

$$\tilde{k}_x(t) = k_x(t) - k_x^* \tag{5.17}$$

$$\tilde{k}_r(t) = k_r(t) - k_r^* \tag{5.18}$$

$$\tilde{\Theta}(t) = \Theta(t) - \Theta^* \tag{5.19}$$

Substituting these into the plant model gives

$$\dot{x} = \left(\underbrace{a + bk_x^*}_{a_m} + b\tilde{k}_x\right)x + \left(\underbrace{bk_r^*}_{b_m} + b\tilde{k}_r\right)r - b\tilde{\Theta}^\top\Phi(x) \tag{5.20}$$

Let $e(t) = x_m(t) - x(t)$ be the tracking error. Then, the closed-loop tracking error equation is established as

$$\dot{e} = \dot{x}_m - \dot{x} = a_m e - b\tilde{k}_x x - b\tilde{k}_r r + b\tilde{\Theta}^\top\Phi(x) \tag{5.21}$$

Note that the tracking error equation is non-autonomous due to $r(t)$.

Now, the task of defining the adaptive laws to adjust $k_x(t)$, $k_r(t)$, and $\Theta(t)$ is considered next. This can be accomplished by conducting a Lyapunov stability proof as follows:

Proof Choose a Lyapunov candidate function

$$V\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) = e^2 + |b|\left(\frac{\tilde{k}_x^2}{\gamma_x} + \frac{\tilde{k}_r^2}{\gamma_r} + \tilde{\Theta}^\top\Gamma^{-1}\tilde{\Theta}\right) > 0 \tag{5.22}$$

where $\gamma_x > 0$ and $\gamma_r > 0$ are called the adaptation (or learning) rates for $k_x(t)$ and $k_r(t)$, and $\Gamma = \Gamma^\top > 0 \in \mathbb{R}^p \times \mathbb{R}^p$ is a positive-definite adaptation rate matrix for $\Theta(t)$.

$\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is evaluated as

$$\begin{aligned}\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) &= 2e\dot{e} + |b|\left(\frac{2\tilde{k}_x\dot{\tilde{k}}_x}{\gamma_x} + \frac{2\tilde{k}_r\dot{\tilde{k}}_r}{\gamma_r} + 2\tilde{\Theta}^{\top}\Gamma^{-1}\dot{\tilde{\Theta}}\right) \\ &= 2a_m e^2 + 2\tilde{k}_x\left(-ebx + |b|\frac{\dot{\tilde{k}}_x}{\gamma_x}\right) + 2\tilde{k}_r\left(-ebr + |b|\frac{\dot{\tilde{k}}_r}{\gamma_r}\right) \\ &\quad + 2\tilde{\Theta}^{\top}\left[eb\Phi(x) + |b|\Gamma^{-1}\dot{\tilde{\Theta}}\right]\end{aligned} \tag{5.23}$$

Since $b = |b|\,\mathrm{sgn}b$, then $\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) \le 0$ if

$$-ex\,\mathrm{sgn}b + \frac{\dot{\tilde{k}}_x}{\gamma_x} = 0 \tag{5.24}$$

$$-er\,\mathrm{sgn}b + \frac{\dot{\tilde{k}}_r}{\gamma_r} = 0 \tag{5.25}$$

$$e\Phi(x)\,\mathrm{sgn}b + \Gamma^{-1}\dot{\tilde{\Theta}} = 0 \tag{5.26}$$

Because k_x^*, k_r^*, and Θ^* are constant, therefore from Eqs. (5.17)–(5.19) $\dot{\tilde{k}}_x = \dot{k}_x$, $\dot{\tilde{k}}_r = \dot{k}_r$, and $\dot{\tilde{\Theta}} = \dot{\Theta}$. Thus, the following adaptive laws are obtained:

$$\dot{k}_x = \gamma_x x e\,\mathrm{sgn}b \tag{5.27}$$

$$\dot{k}_r = \gamma_r r e\,\mathrm{sgn}b \tag{5.28}$$

$$\dot{\Theta} = -\Gamma\Phi(x)\,e\,\mathrm{sgn}b \tag{5.29}$$

Then,

$$\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) = 2a_m e^2 \le 0 \tag{5.30}$$

Since $\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) \le 0$, then $e(t)$, $k_x(t)$, $k_r(t)$, and $\Theta(t)$ are bounded. Then,

$$\lim_{t\to\infty} V\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) = V\left(e_0, \tilde{k}_{x_0}, \tilde{k}_{r_0}, \tilde{\Theta}_0\right) + 2a_m\|e\|_2^2 \tag{5.31}$$

where $e(0) = e_0$, $\tilde{k}_x(0) = \tilde{k}_{x_0}$, $\tilde{k}_r(0) = \tilde{k}_{r_0}$, and $\tilde{\Theta}(0) = \tilde{\Theta}_0$.

So, $V\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|_2$ exists, therefore $e(t) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$, but $\|\dot{e}\| \in \mathscr{L}_\infty$.

$\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ can be shown to be uniformly continuous by examining its derivative to see whether it is bounded. $\ddot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is computed as

$$\ddot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) = 4a_m e\dot{e} = 4a_m e\left[a_m e - b\tilde{k}_x x - b\tilde{k}_r r + b\tilde{\Theta}^\top \Phi\left(x\right)\right] \tag{5.32}$$

Since $e\left(t\right), k_x\left(t\right), k_r\left(t\right)$, and $\Theta\left(t\right)$ are bounded by the virtue that $\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) \leq 0$, $x\left(t\right)$ is bounded because $e\left(t\right)$ and $x_m\left(t\right)$ are bounded, $r\left(t\right)$ is a bounded reference command signal, and $\Phi\left(x\right)$ is bounded because $x\left(t\right)$ is bounded; therefore, $\ddot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is bounded. Thus, $\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) \to 0$, hence $e\left(t\right) \to 0$ as $t \to \infty$. The tracking error is asymptotically stable, but the whole adaptive control system is not asymptotically stable since $k_x\left(t\right)$, $k_r\left(t\right)$ and $\Theta\left(t\right)$ can only be shown to be bounded.

5.2.2 *Case II: a and b Known*

If a and b are known, then the gains k_x and k_r need not be estimated since they are known and can be found from the model matching conditions

$$k_x = \frac{a_m - a}{b} \tag{5.33}$$

$$k_r = \frac{b_m}{b} \tag{5.34}$$

Then, the adaptive controller is given by

$$u = k_x x + k_r r - \Theta^\top\left(t\right)\Phi\left(x\right) \tag{5.35}$$

where only $\Theta\left(t\right)$ needs to be adjusted.

The tracking error equation is then obtained as

$$\dot{e} = a_m e + b\tilde{\Theta}^\top \Phi\left(x\right) \tag{5.36}$$

The adaptive law can be found to be

$$\dot{\Theta} = -\Gamma\Phi\left(x\right)eb \tag{5.37}$$

Proof To show that the adaptive law is stable, choose a Lyapunov candidate function

$$V\left(e, \tilde{\Theta}\right) = e^2 + \tilde{\Theta}^\top \Gamma^{-1} \tilde{\Theta} > 0 \tag{5.38}$$

Then,

$$\dot{V}\left(e, \tilde{\Theta}\right) = 2e\dot{e} + 2\tilde{\Theta}^\top \Gamma^{-1} \dot{\tilde{\Theta}} = 2a_m e^2 + 2eb\tilde{\Theta}^\top \Phi\left(x\right) - 2\tilde{\Theta}^\top eb\Phi\left(x\right) = 2a_m e^2 \leq 0 \tag{5.39}$$

The Barbalat's lemma can be used to show that the tracking error is asymptotically stable, i.e., $e\left(t\right) \to 0$ as $t \to \infty$.

Example 5.9 Let $a = 1$, $b = 1$, $a_m = -1$, $b_m = 1$, $r\left(t\right) = \sin t$, and $f\left(x\right) = \theta^* x\left(t\right)$, where θ^* is an unknown constant but for simulation purposes is taken to be $\theta^* = 0.1$. Then, the control gains are computed as

$$k_x = \frac{a_m - a}{b} = -2$$

$$k_r = \frac{b_m}{b} = 1$$

The adaptive controller is given by

$$u = -2x + r - \theta x$$

$$\dot{\theta} = -\gamma xeb$$

where $\gamma = 1$ is chosen as the adaptation rate for $\theta\left(t\right)$.

Note that the controller is nonlinear even though the plant is linear. Figure 5.2 illustrates a block diagram of the adaptive controller.

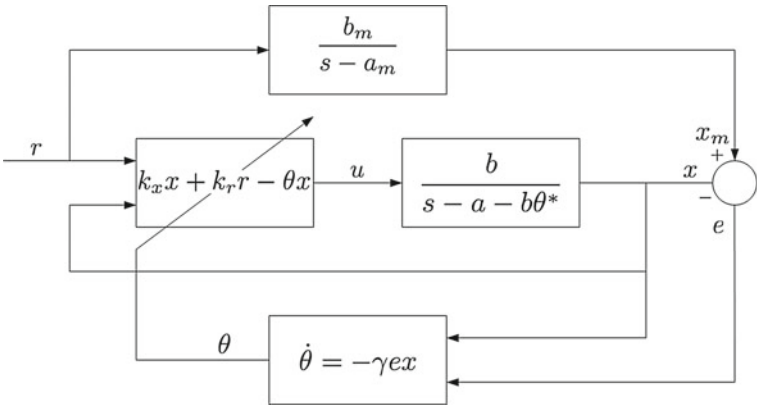

Fig. 5.2 Adaptive control block diagram

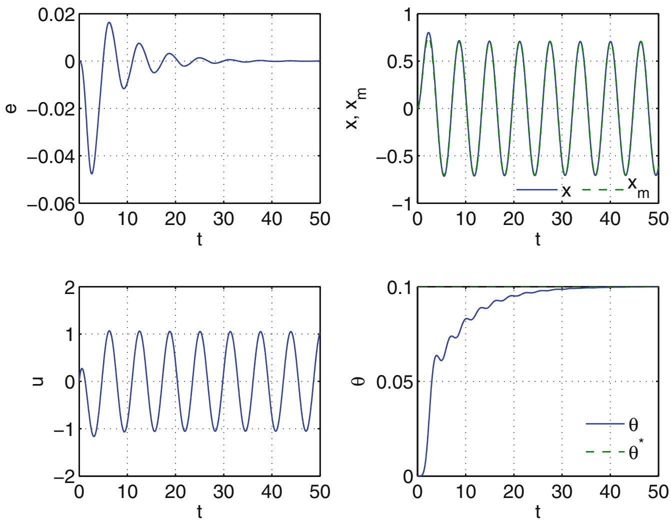

Fig. 5.3 Adaptive control system response

The results are shown in Fig. 5.3. It can be seen that $e(t) \to 0$ and $x(t) \to x_m(t)$ as $t \to \infty$. The estimate $\theta(t)$ also converges to the correct value of the uncertain parameter θ^*, although the convergence is quite gradual. The convergence rate can be increased by increasing the adaptation rate γ, but a large value of γ can lead to an increase in the sensitivity of the control system to noise and unmodeled dynamics that can lead to instability. In other words, a larger value of γ results in a better tracking performance but at the same time degrades the system robustness. In a practical design, the adaptation rate must be chosen carefully to maintain a sufficient robustness while achieving a desired level of performance.

5.3 Indirect MRAC for First-Order SISO Systems

Consider the system in Sect. 5.2.1 with a and b unknown, but sign of b is known. From the model matching conditions, if a and b can be estimated, then the gain k_x and k_r can be obtained. Therefore, the objective of indirect adaptive control is to estimate system parameters which are then used to update the control gains. Hence, indirect adaptive control is essentially a parameter identification method.

Let

$$k_x(t) = \frac{a_m - \hat{a}(t)}{\hat{b}(t)} \tag{5.40}$$

$$k_r(t) = \frac{b_m}{\hat{b}(t)} \tag{5.41}$$

Let $\tilde{a}(t) = \hat{a}(t) - a$ and $\tilde{b}(t) = \hat{b}(t) - b$ be the estimation errors. Now, the plant model is expressed as

$$\dot{x} = ax + \left(\hat{b} - \tilde{b}\right)\left[u + \Theta^{*^\top}\Phi(x)\right] \tag{5.42}$$

Then, substituting Eqs. (5.16), (5.40), and (5.41) into Eq. (5.42) yields

$$\begin{aligned}\dot{x} &= ax + \hat{b}\left[\frac{a_m - \hat{a}}{\hat{b}}x + \frac{b_m}{\hat{b}}r - \Theta^\top\Phi(x) + \Theta^{*^\top}\Phi(x)\right]\\ &\quad - \tilde{b}\left[\frac{a_m - \hat{a}}{\hat{b}}x + \frac{b_m}{\hat{b}}r - \Theta^\top\Phi(x) + \Theta^{*^\top}\Phi(x)\right]\\ &= (a_m - \tilde{a})x + b_m r - b\tilde{\Theta}^\top\Phi(x) - \tilde{b}\left(\frac{a_m - \hat{a}}{\hat{b}}x + \frac{b_m}{\hat{b}}r\right)\end{aligned} \tag{5.43}$$

Let

$$\bar{u} = k_x(t)\,x + k_r(t)\,r \tag{5.44}$$

Then, the tracking error equation is established as

$$\dot{e} = \dot{x}_m - \dot{x} = a_m e + \tilde{a}x + \tilde{b}\bar{u} + b\tilde{\Theta}^\top\Phi(x) \tag{5.45}$$

The Lyapunov's direct method is now used to find the adaptive laws as follows:

Proof Choose a Lyapunov candidate function

$$V\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) = e^2 + \frac{\tilde{a}^2}{\gamma_a} + \frac{\tilde{b}^2}{\gamma_b} + |b|\,\tilde{\Theta}^\top\Gamma^{-1}\tilde{\Theta} > 0 \tag{5.46}$$

where $\gamma_a > 0$ and $\gamma_b > 0$ are the adaptation rates for $\hat{a}(t)$ and $\hat{b}(t)$, respectively.

Then, $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ is evaluated as

$$\begin{aligned}\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) &= 2e\dot{e} + \frac{2\tilde{a}\dot{\tilde{a}}}{\gamma_a} + \frac{2\tilde{b}\dot{\tilde{b}}}{\gamma_b} + 2|b|\,\tilde{\Theta}^\top\Gamma^{-1}\dot{\tilde{\Theta}}\\ &= 2a_m e^2 + 2\tilde{a}\left(xe + \frac{\dot{\tilde{a}}}{\gamma_a}\right) + 2\tilde{b}\left(\bar{u}e + \frac{\dot{\tilde{b}}}{\gamma_b}\right)\\ &\quad + 2|b|\,\tilde{\Theta}^\top\left[\Phi(x)\,e\,\mathrm{sgn}b + \Gamma^{-1}\dot{\tilde{\Theta}}\right]\end{aligned} \tag{5.47}$$

Since a and b are constant, then $\dot{\tilde{a}} = \dot{\hat{a}}$ and $\dot{\tilde{b}} = \dot{\hat{b}}$. Thus, the adaptive laws are obtained as

$$\dot{\hat{a}} = -\gamma_a x e \tag{5.48}$$

$$\dot{\hat{b}} = -\gamma_b \bar{u} e \tag{5.49}$$

$$\dot{\Theta} = -\Gamma \Phi(x) \, e \operatorname{sgn} b \tag{5.50}$$

Then,

$$\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) = 2a_m e^2 \le 0 \tag{5.51}$$

Since $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) \le 0$, then $e(t)$, $\hat{a}(t)$, $\hat{b}(t)$, and $\Theta(t)$ are bounded. Then,

$$\lim_{t\to\infty} V\left(e, \tilde{k}_x, \tilde{k}_r, \tilde{\Theta}\right) = V\left(e_0, \tilde{a}_0, \tilde{b}_0, \tilde{\Theta}_0\right) + 2a_m \|e\|_2^2 \tag{5.52}$$

where e_0 and $\tilde{\Theta}_0$ are defined previously, $\tilde{a}(0) = a_0$, and $\tilde{b}(0) = b_0$.

So, $V\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|_2$ exists, therefore $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $\|\dot{e}\| \in \mathcal{L}_\infty$.

$\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ can be shown to be uniformly continuous by examining its derivative to see whether it is bounded. $\ddot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ is computed as

$$\ddot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) = 4a_m e\dot{e} = 4a_m e\left[a_m e + \tilde{a}x + \tilde{b}\bar{u} + b\tilde{\Theta}^\top \Phi(x)\right] \tag{5.53}$$

Since $e(t)$, $\hat{a}(t)$, $\hat{b}(t)$, and $\Theta(t)$ are bounded by the virtue that $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) \le 0$, $x(t)$ is bounded because $e(t)$ and $x_m(t)$ are bounded, $\bar{u}(t)$ is bounded because $x(t)$ is bounded and $r(t)$ is a bounded reference command signal, and $\Phi(x)$ is bounded because $x(t)$ is bounded; therefore, $\ddot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ is bounded. Thus, $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) \to 0$, hence $e(t) \to 0$ as $t \to \infty$. The tracking error is asymptotically stable.

■

It should be noted that the possibility of $\hat{b}(t) = 0$ does exist, and in such a case, the control gains will "blow up." Thus, indirect MRAC may not be as robust as direct MRAC. To prevent this from occurring, the adaptive law for $\hat{b}(t)$ must be modified so that the adaptation can take place in a closed subset of $\mathbb{R}$ that does not include $\hat{b}(t) = 0$. One technique for modification is the projection method which assumes a priori knowledge of b [2].

Suppose a lower bound on b is known, i.e., $0 < b_0 \le |b|$, then the adaptive law can be modified by the projection method as

$$\dot{\hat{b}} = \begin{cases} -\gamma_b \bar{u} e & \text{if } \left|\hat{b}\right| > b_0 \text{ or if } \left|\hat{b}\right| = b_0 \text{ and } \frac{d\left|\hat{b}\right|}{dt} \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.54}$$

The projection method is essentially a constrained optimization which will be discussed in detail in Chap. 9. In the simplest term, the projection method allows the adaptation to take place in such a manner that an adaptive parameter will not violate its a priori bound. A simple explanation of the modified adaptive law is as follows:

Suppose one knows that b is bounded from below by b_0, i.e., $|b| \geq b_0$, then as long as $\left|\hat{b}\right| > b_0$, the unmodified adaptive law can be used normally. Now, suppose $\left|\hat{b}\right| = b_0$, there are two cases to consider: $\frac{d\left|\hat{b}\right|}{dt} < 0$ and $\frac{d\left|\hat{b}\right|}{dt} \geq 0$.

1. If $\frac{d\left|\hat{b}\right|}{dt} < 0$, then $\left|\hat{b}\right|$ is decreasing and $\left|\hat{b}\right| < b_0$ at some time $t + \Delta t$, which would violate the constraint $\left|\hat{b}\right| \geq b_0$. Therefore, to satisfy the constraint on b, $\frac{d\left|\hat{b}\right|}{dt} = 0$.
2. On the other hand, if $\frac{d\left|\hat{b}\right|}{dt} \geq 0$, then $\left|\hat{b}\right|$ is non-decreasing and $\left|\hat{b}\right| \geq b_0$, so that the unmodified adaptive law can be used normally.

The modified adaptive law thus guarantees that $\left|\hat{b}\right|$ will always be greater than or equal to b_0.

Proof Because of the modification, $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ will no longer be the same and is now dependent on the conditions on $\left|\hat{b}\right|$ and $\frac{d\left|\hat{b}\right|}{dt}$. Thus,

$$\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) = 2a_m e^2 + 2\tilde{b}\left(\bar{u}e + \frac{\dot{\tilde{b}}}{\gamma_b}\right)$$

$$= \begin{cases} 2a_m e^2 \leq 0 & \text{if } \left|\hat{b}\right| \geq b_0 \text{ or if } \left|\hat{b}\right| = b_0 \text{ and } \frac{d\left|\hat{b}\right|}{dt} \geq 0 \\ a_m e^2 + 2\tilde{b}\bar{u}e & \text{if } \left|\hat{b}\right| = b_0 \text{ and } \frac{d\left|\hat{b}\right|}{dt} < 0 \end{cases} \tag{5.55}$$

Consider the second case when $\left|\hat{b}\right| = b_0$ and $\frac{d\left|\hat{b}\right|}{dt} < 0$ for which the sign definiteness of $\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right)$ is still undefined. The condition $\frac{d\left|\hat{b}\right|}{dt} < 0$ gives

$$\frac{d\left|\hat{b}\right|}{dt} = \dot{\hat{b}}\,\mathrm{sgn} b = -\gamma_b \bar{u}e\,\mathrm{sgn} b < 0 \Rightarrow \bar{u}e\,\mathrm{sgn} b > 0 \tag{5.56}$$

Since $\left|\hat{b}\right| = b_0$, then

$$2\tilde{b}\bar{u}e = 2\left(\hat{b} - b\right)\bar{u}e = 2\left[\left|\hat{b}\right|\mathrm{sgn} b - |b|\,\mathrm{sgn} b\right]\bar{u}e = 2\left(b_0 - |b|\right)\bar{u}e\,\mathrm{sgn} b \tag{5.57}$$

Since $|b| \geq b_0$, which implies $|b| = b_0 + \delta > 0$ where $\delta \geq 0$, then $\bar{u} e \operatorname{sgn} b > 0$ implies

$$2\tilde{b}\bar{u}e = 2\left(b_0 - b_0 - \delta\right)\bar{u}e\operatorname{sgn} b = -2\delta\bar{u}e\operatorname{sgn} b \leq 0 \tag{5.58}$$

Therefore,

$$\dot{V}\left(e, \tilde{a}, \tilde{b}, \tilde{\Theta}\right) = 2a_m e^2 + 2\tilde{b}\bar{u}e = 2a_m e^2 - 2\delta\bar{u}e\operatorname{sgn} b \leq 2a_m e^2 \leq 0 \tag{5.59}$$

Using the usual Barbalat's lemma, one can conclude that $e(t) \to 0$ as $t \to \infty$.

5.4 Direct MRAC for Second-Order SISO Systems

Consider a second-order nonlinear SISO system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = b\left[u + f(y, \dot{y})\right] \tag{5.60}$$

where ζ and ω_n are unknown and $f(y, \dot{y}) = \Theta^{*\top}\Phi(y, \dot{y})$ is defined in a manner similar to Eq. (5.10).

Let $x_1(t) = y(t)$, $x_2 = \dot{y}(t)$, and $x(t) = \left[x_1(t)\ x_2(t)\right]^\top \in \mathbb{R}^2$. The state-space form of the system is

$$\dot{x} = Ax + B\left[u + \Theta^*\Phi(x)\right] \tag{5.61}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix} \tag{5.62}$$

Let $x_{m_1}(t) = y_m(t)$, $x_{m_2} = \dot{y}_m(t)$, and $x_m(t) = \left[x_{m_1}(t)\ x_{m_2}(t)\right]^\top \in \mathbb{R}^2$. A reference model is given by

$$\dot{x}_m = A_m x_m + B_m r \tag{5.63}$$

where $r(t) \in \mathbb{R}$ is a bounded command signal, $A_m \in \mathbb{R}^2 \times \mathbb{R}^2$ is Hurwitz and known, and $B_m \in \mathbb{R}^2$ is also known.

5.4.1 Case I: A and B Unknown but Sign of b known

Firstly, the ideal controller is defined as

$$u^* = K_x^* x + k_r^* r - \Theta^{*\top}\Phi(x) \tag{5.64}$$

where $K_x^* \in \mathbb{R}^2$ and $k_r^* \in \mathbb{R}$ are constant but unknown ideal gains.

Comparing the ideal closed-loop plant to the reference model, the model matching conditions are

$$A + BK_x^* = A_m \tag{5.65}$$

$$Bk_r^* = B_m \tag{5.66}$$

Note that in general, one cannot always assume that K_x^* and k_r^* exist because A, A_m, B, and B_m may have different structures that do not allow the solutions for K_x^* and k_r^* to be determined. In most cases, if A and B are known, then K_x^* and k_r^* can be designed by any standard non-adaptive control techniques to stabilize the closed-loop system and enable it to follow a command. Then, A_m and B_m can be computed from A, B, K_x^*, and k_r^*.

Example 5.10 A second-order SISO system and a reference model are specified as

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; A_m = \begin{bmatrix} 0 & 1 \\ -16 & -2 \end{bmatrix}, \; B_m = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Then, utilizing the pseudo-inverse, K_x^* and k_r^* can be solved as

$$K_x^* = \left(B^\top B\right)^{-1} B^\top \left(A_m - A\right) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ -16 & -2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} -15 & -1 \end{bmatrix}$$

$$k_r^* = \left(B^\top B\right)^{-1} B^\top B_m = 2$$

Now, suppose

$$A_m = \begin{bmatrix} 1 & 1 \\ -16 & -2 \end{bmatrix}$$

Then, the solution of K_x^* is the same (verify!), but the model matching condition is not satisfied since

$$A + BK = \begin{bmatrix} 0 & 1 \\ -16 & -2 \end{bmatrix} \neq A_m$$

■

Thus, it is important to state an explicit assumption that there exist constant but unknown K_x^* and k_r^* such that the model matching conditions are satisfied. For a second-order SISO system, the model matching conditions are satisfied if A_m and B_m have the same structures as those of A and B, respectively.

A full-state feedback adaptive controller is designed as

$$u = K_x(t)\, x + k_r(t)\, r - \Theta^\top \Phi(x) \tag{5.67}$$

where $K_x(t) \in \mathbb{R}^2$ and $k_r(t) \in \mathbb{R}$.

Let $\tilde{K}_x(t) = K_x(t) - K_x^*$, $\tilde{k}_r(t) = k_r(t) - k_r^*$, and $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ be the estimation errors, then the closed-loop plant becomes

$$\dot{x} = \left(\underbrace{A + BK_x^*}_{A_m} + B\tilde{K}_x\right)x + \left(\underbrace{Bk_r^*}_{B_m} + B\tilde{k}_r\right)r - B\tilde{\Theta}^\top \Phi(x) \tag{5.68}$$

The closed-loop tracking error equation is now obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x x - B\tilde{k}_r r + B\tilde{\Theta}^\top \Phi(x) \tag{5.69}$$

where $e(t) = x_m(t) - x(t) \in \mathbb{R}^2$.

Proof To find the adaptive laws, choose a Lyapunov candidate function

$$V\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = e^\top P e + |b|\left(\tilde{K}_x \Gamma_x^{-1}\tilde{K}_x^\top + \frac{\tilde{k}_r^2}{\gamma_r} + \tilde{\Theta}^\top \Gamma_\Theta^{-1}\tilde{\Theta}\right) > 0 \tag{5.70}$$

where $\Gamma_x = \Gamma_x^\top > 0 \in \mathbb{R}^2 \times \mathbb{R}^2$ is a positive-definite adaptation rate matrix for $K_x(t)$ and $P = P^\top > 0 \in \mathbb{R}^2 \times \mathbb{R}^2$ that solves the following Lyapunov equation:

$$PA_m + A_m^\top P = -Q \tag{5.71}$$

where $Q = Q^\top > 0 \in \mathbb{R}^2 \times \mathbb{R}^2$.

Then, $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is evaluated as

$$\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = \dot{e}^\top P e + e^\top P \dot{e} + |b|\left(2\tilde{K}_x \Gamma_x^{-1}\dot{\tilde{K}}_x^\top + \frac{2\tilde{k}_r \dot{\tilde{k}}_r}{\gamma_r} + 2\tilde{\Theta}^\top \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right) \tag{5.72}$$

Substituting the tracking error equation yields

$$\begin{aligned}\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) &= e^\top\left(PA_m + A_m^\top P\right)e + 2e^\top PB\left[-\tilde{K}_x x - \tilde{k}_r r + \tilde{\Theta}^\top \Phi(x)\right] \\ &\quad + |b|\left(2\tilde{K}_x \Gamma_x^{-1}\dot{\tilde{K}}_x^\top + \frac{2\tilde{k}_r \dot{\tilde{k}}_r}{\gamma_r} + 2\tilde{\Theta}^\top \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right)\end{aligned} \tag{5.73}$$

Let p_{ij}, $i = 1, 2$, $j = 1, 2$, be the elements of P and notice that

$$2e^\top PB = 2e^\top \bar{P} b \in \mathbb{R} \tag{5.74}$$

where $\bar{P} = \left[\, p_{12} \;\; p_{22} \,\right]^\top$.

Then, $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ can be expressed as

$$\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = -e^\top Qe + 2\,|b|\,\mathrm{sgn}\,(b)\left[-\tilde{K}_x x - \tilde{k}_r r + \tilde{\Theta}^\top \Phi\,(x)\right] e^\top \bar{P} \\ + |b|\left(2\tilde{K}_x \Gamma_x^{-1}\dot{\tilde{K}}_x^\top + \frac{2\tilde{k}_r \dot{\tilde{k}}_r}{\gamma_r} + 2\tilde{\Theta}^\top \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right) \tag{5.75}$$

or

$$\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = -e^\top Qe + 2\,|b|\,\tilde{K}_x\left(-xe^\top \bar{P}\mathrm{sgn}b + \Gamma_x^{-1}\dot{\tilde{K}}_x^\top\right) \\ + 2\,|b|\,\tilde{k}_r\left(-re^\top \bar{P}\mathrm{sgn}b + \frac{\dot{\tilde{k}}_r}{\gamma_r}\right) \\ + 2\,|b|\,\tilde{\Theta}^\top\left[\Phi\,(x)\,e^\top \bar{P}\mathrm{sgn}b + \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right] \tag{5.76}$$

Thus, the following adaptive laws are obtained:

$$\dot{K}_x^\top = \Gamma_x x e^\top \bar{P}\mathrm{sgn}b \tag{5.77}$$

$$\dot{k}_r = \gamma_r r e^\top \bar{P}\mathrm{sgn}b \tag{5.78}$$

$$\dot{\Theta} = -\Gamma_\Theta \Phi\,(x)\,e^\top \bar{P}\mathrm{sgn}b \tag{5.79}$$

It follows that

$$\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = -e^\top Qe \le -\lambda_{min}\,(Q)\,\|e\|_2^2 \le 0 \tag{5.80}$$

Since $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) \le 0$, therefore $e\,(t)$, $K_x\,(t)$, $k_r\,(t)$, and $\Theta\,(t)$ are bounded. Then,

$$\lim_{t\to\infty} V\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = V\left(e_0, \tilde{K}_{x_0}, \tilde{k}_{r_0}, \tilde{\Theta}_0\right) - \lambda_{min}\,(Q)\,\|e\|_2^2 \tag{5.81}$$

So, $V\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|_2$ exists, therefore $e\,(t) \in \mathscr{L}_2 \cap \mathscr{L}_\infty$, but $\|\dot{e}\| \in \mathscr{L}_\infty$.

$\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ can be shown to be uniformly continuous by examining its derivative to see whether it is bounded, where

$$\ddot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) = -\dot{e}^\top Qe - e^\top Q\dot{e} = -e^\top\left(QA + A^\top Q\right)e \\ - 2e^\top Q\left[A_m e - B\tilde{K}_x x - B\tilde{k}_r r + B\tilde{\Theta}^\top \Phi\,(x)\right] \tag{5.82}$$

Since $e(t)$, $K_x(t)$, $k_r(t)$, and $\Theta(t)$ are bounded by the virtue of $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) \leq 0$, $x(t)$ is bounded because $e(t)$ and $x_m(t)$ are bounded, $r(t)$ is a bounded reference command signal, and $\Phi(x)$ is bounded because $x(t)$ is bounded, therefore $\ddot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is bounded. Thus, $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{K}_x, \tilde{k}_r, \tilde{\Theta}\right) \to 0$, hence $e(t) \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable.

Example 5.11 Design an adaptive controller for a second-order system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = b\left[u + \Theta^{*\top}\Phi(y)\right]$$

where $\zeta > 0$, $\omega_n > 0$, $b > 0$, and $\Theta^{*\top} = \left[\theta_1^* \ \theta_2^*\right]$ are unknown, and

$$\Phi(y) = \begin{bmatrix} 1 \\ y^2 \end{bmatrix}$$

The reference model is given by

$$\ddot{y}_m + 2\zeta_m\omega_m\dot{y}_m + \omega_m^2 y_m = b_m r$$

where $\zeta_m = 0.5$, $\omega_m = 2$, $b_m = 4$, and $r(t) = \sin 2t$. For simulation purposes, the unknown parameters may be assumed to be $\zeta = -0.5$, $\omega_n = 1$, $b = 1$, and $\Theta^{*\top} = \left[0.5 \ -0.1\right]$.

Note that the open-loop plant is unstable with the eigenvalues $\lambda(A) = \frac{1\pm\sqrt{3}}{2}$ on the right half plane, and the ideal control gains K_x^* and k_r^* exist and are equal to

$$K_x^* = \left(B^\top B\right)^{-1} B^\top (A_m - A) = \left[-3 \ -3\right]$$

$$k_r^* = \frac{b_m}{b} = 4$$

Let $Q = I$, then the solution of the Lyapunov equation (5.71) is

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} \Rightarrow \bar{P} = \begin{bmatrix} \frac{1}{8} \\ \frac{5}{16} \end{bmatrix}$$

Let $\Gamma_x = \operatorname{diag}\left(\gamma_{x_1}, \gamma_{x_2}\right)$ and $\Gamma_\Theta = \operatorname{diag}\left(\gamma_{\theta_1}, \gamma_{\theta_2}\right)$. Then, the adaptive laws are

$$\dot{K}_x^\top = \begin{bmatrix} \dot{k}_{x_1} \\ \dot{k}_{x_2} \end{bmatrix} = \Gamma_x x e^\top \bar{P} \underbrace{\operatorname{sgn}(b)}_{1} = \begin{bmatrix} \gamma_{x_1} & 0 \\ 0 & \gamma_{x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \left[e_1 \ e_2\right] \begin{bmatrix} \frac{1}{8} \\ \frac{5}{16} \end{bmatrix}$$
$$= \left(\frac{1}{8}e_1 + \frac{5}{16}e_2\right) \begin{bmatrix} \gamma_{x_1} x_1 \\ \gamma_{x_2} x_2 \end{bmatrix}$$

$$\dot{k}_r = \gamma_r r e^\top \bar{P} \mathrm{sgn} b = \left(\frac{1}{8} e_1 + \frac{5}{16} e_2 \right) \gamma_r r$$

$$\dot{\Theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = -\Gamma_\Theta \Phi(x) e^\top \bar{P} \mathrm{sgn} b = -\begin{bmatrix} \gamma_{\theta_1} & 0 \\ 0 & \gamma_{\theta_2} \end{bmatrix} \begin{bmatrix} 1 \\ x_1^2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \frac{1}{8} \\ \frac{5}{16} \end{bmatrix}$$
$$= -\left(\frac{1}{8} e_1 + \frac{5}{16} e_2 \right) \begin{bmatrix} \gamma_{\theta_1} \\ \gamma_{\theta_2} x_1^2 \end{bmatrix}$$

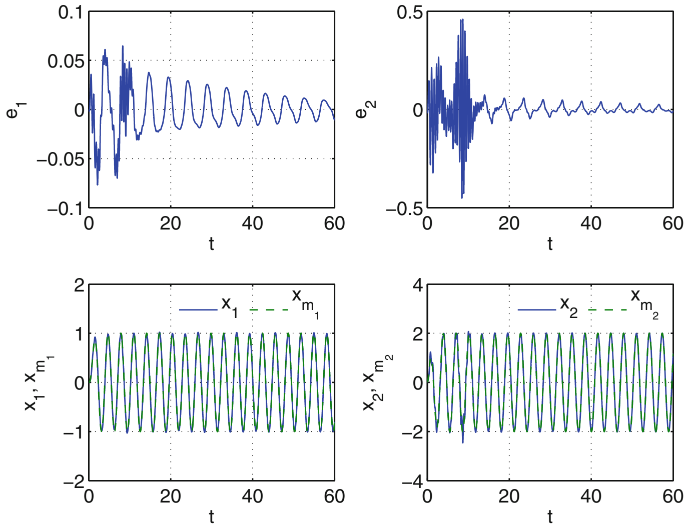

Fig. 5.4 Adaptive control system response

For simulations, all adaptation rates are chosen to be 100 and all initial conditions are set to zero. The results are shown in Figs. 5.4 and 5.5.

It is noted that the plant follows the reference model very well only after a short time, but the adaptive gains $K_x(t)$ and $k_r(t)$ and the adaptive parameter $\Theta(t)$ are converging much more slowly. When they converge, some of them do not converge to their true values, such as $k_{x_2}(t)$ and $k_r(t)$. This is one of the properties of MRAC whereby there is no assurance on the convergence of adaptive parameters to their true values.

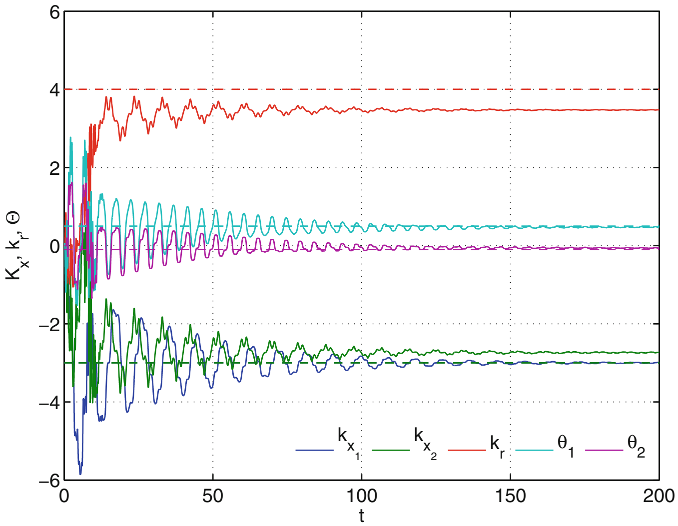

Fig. 5.5 Adaptive gains and adaptive parameters

5.4.2 *Case II: A and B Known*

If A and B are known, then it is assumed that there exist K_x and k_r that satisfy the model matching conditions

$$A + BK_x = A_m \tag{5.83}$$

$$Bk_r = B_m \tag{5.84}$$

For a second-order system, if A_m and B_m have the same structures as those of A and B, respectively, then K_x and k_r can be determined by using the pseudo-inverse method.

Let the adaptive controller be

$$u = K_x x + k_r r - \Theta^\top \Phi(x) \tag{5.85}$$

Then, the closed-loop plant is

$$\dot{x} = (A + BK_x)\,x + Bk_r r - B\tilde{\Theta}^\top \Phi(x) \tag{5.86}$$

and the tracking error equation is

$$\dot{e} = A_m e + B\tilde{\Theta}^\top \Phi(x) \tag{5.87}$$

Proof Choose a Lyapunov candidate function

$$V\left(e, \tilde{\Theta}\right) = e^\top P e + \tilde{\Theta}^\top \Gamma^{-1} \tilde{\Theta}^\top \tag{5.88}$$

Then, $\dot{V}\left(e, \tilde{\Theta}\right)$ is evaluated as

$$\dot{V}\left(e, \tilde{\Theta}\right) = -e^\top Q e + 2 e^\top P B \tilde{\Theta}^\top \Phi\left(x\right) + 2 \tilde{\Theta}^\top \Gamma^{-1} \dot{\tilde{\Theta}} \tag{5.89}$$

Since $e^\top P B \in \mathbb{R}$ is a scalar value, then

$$\begin{aligned}\dot{V}\left(e, \tilde{\Theta}\right) &= -e^\top Q e + 2 \tilde{\Theta}^\top \Phi\left(x\right) e^\top P B + 2 \tilde{\Theta}^\top \Gamma^{-1} \dot{\tilde{\Theta}} \\ &= -e^\top Q e + 2 \tilde{\Theta}^\top \left[\Phi\left(x\right) e^\top P B + \Gamma^{-1} \dot{\tilde{\Theta}}\right]\end{aligned} \tag{5.90}$$

Thus, the following adaptive law is obtained:

$$\dot{\Theta} = -\Gamma \Phi\left(x\right) e^\top P B \tag{5.91}$$

Then,

$$\dot{V}\left(e, \tilde{\Theta}\right) = -e^\top Q e \leq -\lambda_{min}\left(Q\right) \|e\|^2 \tag{5.92}$$

Therefore, $e\left(t\right)$ and $\Theta\left(t\right)$ are bounded. Using the same argument with the Barbalat's lemma as in the previous sections, one can conclude that $\dot{V}\left(e, \tilde{\Theta}\right)$ is uniformly continuous so $\dot{V}\left(e, \tilde{\Theta}\right) \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable with $e\left(t\right) \to 0$ as $t \to \infty$.

5.5 Indirect MRAC for Second-Order SISO Systems

Indirect MRAC for second-order systems is similar to that for first-order systems. Consider the second-order system in Sect. 5.4.1 with A and B unknown, but sign of b is known. Assuming that there exist K_x and k_r that satisfy the model matching conditions, and furthermore that A_m and B_m have the same structures as those of A and B, respectively, then A and B can be estimated. Let

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \quad A_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_m \end{bmatrix} \tag{5.93}$$

The model matching conditions are

$$\hat{A}(t) + \hat{B}(t) K_x(t) = A_m \tag{5.94}$$

$$\hat{B}(t) k_r(t) = B_m \tag{5.95}$$

from which $K_x(t)$ and $k_r(t)$ are determined by

$$\begin{aligned} K_x &= \left(\hat{B}^\top \hat{B}\right)^{-1} \hat{B}^\top \left(A_m - \hat{A}\right) = \frac{1}{\hat{b}^2} \left[0 \; \hat{b}\right] \begin{bmatrix} 0 & 0 \\ -\omega_m^2 + \hat{\omega}_n^2 & -2\zeta_m \omega_m + 2\hat{\zeta}\hat{\omega}_n \end{bmatrix} \\ &= \frac{1}{\hat{b}} \left[-\omega_m^2 + \hat{\omega}_n^2 \;\; -2\zeta_m \omega_m + 2\hat{\zeta}\hat{\omega}_n\right] \end{aligned} \tag{5.96}$$

$$k_r = \left(\hat{B}^\top \hat{B}\right)^{-1} \hat{B}^\top B_m = \frac{1}{\hat{b}^2} \left[0 \; \hat{b}\right] \begin{bmatrix} 0 \\ b_m \end{bmatrix} = \frac{b_m}{\hat{b}} \tag{5.97}$$

where $\hat{A}(t)$, $\hat{B}(t)$, $\hat{\omega}_n(t)$, and $\hat{\zeta}(t)$ are estimates of A, B, ω_n, and ζ, respectively.

Let $\tilde{A}(t) = \hat{A}(t) - A$ and $\tilde{B}(t) = \hat{B}(t) - B$ be the estimation errors. Now, the plant model is expressed as

$$\dot{x} = \left(\hat{A} - \tilde{A}\right) x + \left(\hat{B} - \tilde{B}\right) \left[u + \Theta^{*^\top} \Phi(x)\right] \tag{5.98}$$

Then, substituting Eqs. (5.67), (5.96), and (5.97) into Eq. (5.98) yields

$$\begin{aligned} \dot{x} &= \left(\hat{A} - \tilde{A}\right) x + \hat{B} \left[K_x x + k_r r - \Theta^\top \Phi(x) + \Theta^{*^\top} \Phi(x)\right] \\ &\quad - \tilde{B} \left[K_x x + k_r r - \Theta^\top \Phi(x) + \Theta^{*^\top} \Phi(x)\right] \\ &= \left(\underbrace{\hat{A} + \hat{B} K_x}_{A_m} - \tilde{A}\right) x + \underbrace{\hat{B} k_r}_{B_m} r - B \tilde{\Theta}^\top \Phi(x) - \tilde{B} \left(K_x x + k_r r\right) \end{aligned} \tag{5.99}$$

Let

$$\bar{u} = K_x(t) x + k_r(t) r \tag{5.100}$$

Then, the tracking error equation is established as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e + \tilde{A} x + \tilde{B} \bar{u} + B \tilde{\Theta}^\top \Phi(x) \tag{5.101}$$

Proof Proceed as usual by choosing a Lyapunov candidate function

$$V\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = e^\top P e + \operatorname{trace}\left(\tilde{A} \Gamma_A^{-1} \tilde{A}^\top\right) + \frac{\tilde{B}^\top \tilde{B}}{\gamma_b} + |b| \, \tilde{\Theta}^\top \Gamma_\Theta^{-1} \tilde{\Theta} \tag{5.102}$$

where $\Gamma_A = \Gamma_A^\top > 0 \in \mathbb{R}^2 \times \mathbb{R}^2$ is a positive-definite adaptation rate matrix for $\hat{A}(t)$.

Note that the matrix trace operator is used in the Lyapunov function to map a matrix product into a scalar quantity.

$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right)$ is evaluated as

$$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = -e^\top Q e + 2e^\top P\left[\tilde{A}x + \tilde{B}\bar{u} + B\tilde{\Theta}^\top \Phi(x)\right] + \operatorname{trace}\left(2\tilde{A}\Gamma_A^{-1}\dot{\tilde{A}}^\top\right) + \frac{2\tilde{B}^\top \dot{\tilde{B}}}{\gamma_b} + 2|b|\,\tilde{\Theta}^\top \Gamma_\Theta^{-1}\dot{\tilde{\Theta}} \tag{5.103}$$

Now, consider the trace operator of a product of two vectors $C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}^\top \in \mathbb{R}^n$ and $D = \begin{bmatrix} d_1 & d_2 & \dots & d_n \end{bmatrix}^\top \in \mathbb{R}^n$. Note that $C^\top D = D^\top C \in \mathbb{R}$ and $CD^\top \in \mathbb{R}^n \times \mathbb{R}^n$. Then, one of the identities of a trace operator is as follows:

$$\operatorname{trace}\left(CD^\top\right) = C^\top D = D^\top C \tag{5.104}$$

This can be shown by evaluating both sides of the identity as

$$C^\top D = D^\top C = \sum_{i=1}^{n} c_i d_i \tag{5.105}$$

$$CD^\top = \left\{c_i d_j\right\},\ i, j = 1, 2, \dots, n \tag{5.106}$$

The trace operator is the sum of all the diagonal elements. Therefore,

$$\operatorname{trace}\left(CD^\top\right) = \sum_{i=1}^{j} c_i d_i = C^\top D = D^\top C \tag{5.107}$$

Now, utilizing this trace identity, one can express

$$2\left(e^\top P\right)\left(\tilde{A}x\right) = \operatorname{trace}\left(2\tilde{A}xe^\top P\right) \tag{5.108}$$

Also note that

$$2\left(e^\top P\right)\left(\tilde{B}\right) = 2\tilde{B}^\top P e \tag{5.109}$$

and

$$2\left(e^\top P B\right)\left[\tilde{\Theta}^\top \Phi(x)\right] = 2\tilde{\Theta}^\top \Phi(x)\, e^\top P B = 2\tilde{\Theta}^\top \Phi(x)\, e^\top \bar{P}\,|b|\operatorname{sgn}(b) \tag{5.110}$$

since the terms $e^\top P B$ and $e^\top P \tilde{B}$ are scalar quantities (verify!), where $\bar{P}$ is defined previously. Therefore,

$$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = -e^\top Qe + \text{trace}\left[2\tilde{A}\left(xe^\top P + \Gamma_A^{-1}\dot{\hat{A}}^\top\right)\right] \quad +2\tilde{B}^\top\left(Pe\bar{u} + \frac{\dot{\hat{B}}}{\gamma_b}\right)$$
$$+ 2\left|b\right|\tilde{\Theta}^\top\left[\Phi\left(x\right)e^\top\bar{P}\text{sgn}b + \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right] \tag{5.111}$$

The following adaptive laws are then obtained:

$$\dot{\hat{A}}^\top = -\Gamma_A xe^\top P \tag{5.112}$$

$$\dot{\hat{B}} = -\gamma_b Pe\bar{u} \tag{5.113}$$

$$\dot{\Theta} = -\Gamma_\Theta \Phi\left(x\right)e^\top \bar{P}\text{sgn}b \tag{5.114}$$

It follows that $e\left(t\right)$, $\tilde{A}\left(t\right)$, $\tilde{B}\left(t\right)$, and $\tilde{\Theta}\left(t\right)$ are bounded since

$$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = -e^\top Qe \le -\lambda_{min}\left(Q\right)\|e\|^2 \tag{5.115}$$

$V\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$ since

$$V\left(t \to \infty\right) = V\left(t_0\right) - \int_{t_0}^{\infty}\lambda_{min}\left(Q\right)\|e\|^2 dt < \infty \tag{5.116}$$

It can be shown that $\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right)$ is uniformly continuous because $\ddot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right)$ is bounded. Then, applying the Barbalat's lemma, one can conclude that the tracking error is asymptotically stable with $e\left(t\right) \to 0$ as $t \to \infty$.

■

Let

$$\hat{\bar{A}} = \begin{bmatrix} 0 & 1 \end{bmatrix}\hat{A} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -\hat{\omega}_n^2 & -2\hat{\zeta}\hat{\omega}_n \end{bmatrix} = \begin{bmatrix} -\hat{\omega}_n^2 & -2\hat{\zeta}\hat{\omega}_n \end{bmatrix} \tag{5.117}$$

and since

$$\hat{b} = \begin{bmatrix} 0 & 1 \end{bmatrix}\hat{B} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ \hat{b} \end{bmatrix} \tag{5.118}$$

then the adaptive laws can be expressed in terms of the estimates of the unknown quantities ω_n and ζ as

$$\dot{\hat{\bar{A}}}^\top = -\Gamma_A xe^\top P\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\Gamma_A xe^\top \bar{P} \tag{5.119}$$

$$\dot{\hat{b}} = -\gamma_b \begin{bmatrix} 0 & 1 \end{bmatrix} Pe\bar{u} = -\gamma_b \bar{P}^\top e\bar{u} = -\gamma_b \bar{u} e^\top \bar{P} \tag{5.120}$$

Let

$$\Gamma_A = \begin{bmatrix} \gamma_\omega & 0 \\ 0 & \gamma_\zeta \end{bmatrix} > 0 \tag{5.121}$$

Then,

$$\frac{d}{dt}\left(-\hat{\omega}_n^2\right) = -\gamma_\omega x_1 e^\top \bar{P} \tag{5.122}$$

or

$$\dot{\hat{\omega}}_n = \frac{\gamma_\omega x_1 e^\top \bar{P}}{2\hat{\omega}_n} \tag{5.123}$$

and

$$\frac{d}{dt}\left(-2\hat{\zeta}\hat{\omega}_n\right) = -2\hat{\omega}_n \dot{\hat{\zeta}} - 2\hat{\zeta}\dot{\hat{\omega}}_n = -\gamma_\zeta x_2 e^\top \bar{P} \tag{5.124}$$

or

$$\dot{\hat{\zeta}} = \frac{\left(\gamma_\zeta x_2 \hat{\omega}_n - \gamma_\omega x_1 \hat{\zeta}\right) e^\top \bar{P}}{2\hat{\omega}_n^2} \tag{5.125}$$

To prevent the possibility of $\hat{\omega}_n(t) = 0$ or $\hat{b}(t) = 0$ that will cause the adaptive laws to blow up, both the adaptive laws for estimating $\hat{\omega}_n(t)$ and $\hat{b}(t)$ need to be modified by the projection method according to

$$\dot{\hat{\omega}}_n = \begin{cases} \frac{\gamma_\omega x_1 e^\top \bar{P}}{2\hat{\omega}_n} & \text{if } \hat{\omega}_n > \omega_0 > 0 \text{ or if } \hat{\omega}_n = \omega_0 \text{ and } \dot{\hat{\omega}}_n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.126}$$

$$\dot{\hat{b}} = \begin{cases} -\gamma_b \bar{u} e^\top \bar{P} & \text{if } \left|\hat{b}\right| > b_0 \text{ or if } \left|\hat{b}\right| = b_0 \text{ and } \frac{d\left|\hat{b}\right|}{dt} \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.127}$$

In the modified adaptive law for $\hat{\omega}_n(t)$, it is assumed that $\hat{\omega}_n(t)$ is always a positive quantity for a physically realizable system.

5.6 Direct MRAC for MIMO Systems

Consider a MIMO system with a matched uncertainty

$$\dot{x} = Ax + B\Lambda\left[u + f(x)\right] \tag{5.128}$$

where $x(t) \in \mathbb{R}^n$ is a state vector, $u(t) \in \mathbb{R}^m$ is a control vector, $A \in \mathbb{R}^n \times \mathbb{R}^n$ is a constant, known or unknown matrix, $B \in \mathbb{R}^n \times \mathbb{R}^m$ is a known matrix, $\Lambda = \Lambda^\top = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m \times \mathbb{R}^m$ is a control input uncertainty and a diagonal matrix, and $f(x) \in \mathbb{R}^m$ is a matched uncertainty that can be linearly parametrized as

$$f(x) = \Theta^{*\top} \Phi(x) \tag{5.129}$$

where $\Theta^* \in \mathbb{R}^l \times \mathbb{R}^m$ is a constant, unknown matrix, and $\Phi(x) \in \mathbb{R}^l$ is a vector of known and bounded basis functions.

Furthermore, it is assumed that the pair $(A, B\Lambda)$ is controllable. Recall that the controllability condition ensures that the control input $u(t)$ has a sufficient access to the state space to stabilize all unstable modes of a plant. The controllability condition can be checked by the rank condition of the controllability matrix C, where

$$C = \left[B\Lambda \mid AB\Lambda \mid A^2 B\Lambda \mid \ldots \mid A^{n-1} B\Lambda \right] \tag{5.130}$$

The pair $(A, B\Lambda)$ is controllable if $\operatorname{rank}(C) = n$.

The reference model is specified by

$$\dot{x}_m = A_m x_m + B_m r \tag{5.131}$$

where $x_m(t) \in \mathbb{R}^n$ is a reference state vector, $A_m \in \mathbb{R}^n \times \mathbb{R}^n$ is known and Hurwitz, $B_m \in \mathbb{R}^n \times \mathbb{R}^q$ is known, and $r(t) \in \mathbb{R}^q$ is a piecewise continuous and bounded command vector.

The objective is to design a full-state adaptive controller to allow $x(t)$ to follow $x_m(t)$.

5.6.1 Case I: A and Λ Unknown, but B and Sign of Λ Known

Firstly, it must be assumed that there exist ideal control gains K_x^* and K_r^* such that the following model matching conditions are satisfied:

$$A + B\Lambda K_x^* = A_m \tag{5.132}$$

$$B\Lambda K_r^* = B_m \tag{5.133}$$

If A_m and B_m have the same structures as those of A and $B\Lambda$, respectively, or if $B\Lambda$ is a square and invertible matrix, then there exist K_x^* and K_r^* that satisfy the model matching conditions.

Example 5.12 A MIMO system and a reference model are specified as

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ B\Lambda = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ A_m = \begin{bmatrix} 0 & 1 \\ -16 & -2 \end{bmatrix},\ B_m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Then,

$$K_x^* = (B\Lambda)^{-1}(A_m - A) = \begin{bmatrix} 14 & 1 \\ -15 & -1 \end{bmatrix}$$

$$K_r^* = (B\Lambda)^{-1} B_m = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

■

Define an adaptive controller as

$$u = K_x(t)\,x + K_r(t)\,r - \Theta^\top \Phi(x) \tag{5.134}$$

where $K_x(t) \in \mathbb{R}^m \times \mathbb{R}^n$, $K_r(t) \in \mathbb{R}^m \times \mathbb{R}^q$, and $\Theta(t) \in \mathbb{R}^l \times \mathbb{R}^m$ are estimates of K_x^*, K_r^*, and Θ^*, respectively.

Let $\tilde{K}_x(t) = K_x(t) - K_x^*$, $\tilde{K}_r(t) = K_r(t) - K_r^*$, and $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ be the estimation errors. Then, the closed-loop plant model is expressed as

$$\dot{x} = \left(\underbrace{A + B\Lambda K_x^*}_{A_m} + B\Lambda\tilde{K}_x\right)x + \left(\underbrace{B\Lambda K_r^*}_{B_m} + B\Lambda\tilde{K}_r\right)r - B\Lambda\tilde{\Theta}^\top\Phi(x) \tag{5.135}$$

The closed-loop tracking error equation can now be formulated as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\Lambda\tilde{K}_x x - B\Lambda\tilde{K}_r r + B\Lambda\tilde{\Theta}^\top\Phi(x) \tag{5.136}$$

Proof To derive the adaptive laws, choose the following Lyapunov candidate function:

$$\begin{aligned} V\left(e, \tilde{K}_x, \tilde{K}_r, \tilde{\Theta}\right) &= e^\top P e + \operatorname{trace}\left(|\Lambda|\,\tilde{K}_x \Gamma_x^{-1}\tilde{K}_x^\top\right) + \operatorname{trace}\left(|\Lambda|\,\tilde{K}_r\Gamma_r^{-1}\tilde{K}_r^\top\right) \\ &\quad + \operatorname{trace}\left(|\Lambda|\,\tilde{\Theta}^\top\Gamma_\Theta^{-1}\tilde{\Theta}\right) \end{aligned} \tag{5.137}$$

$\dot{V}\left(e, \tilde{K}_x, \tilde{K}_r, \tilde{\Theta}\right)$ is evaluated as

$$\begin{aligned} \dot{V}\left(e, \tilde{K}_x, \tilde{K}_r, \tilde{\Theta}\right) &= -e^\top Q e + 2e^\top P\left[-B\Lambda\tilde{K}_x x - B\Lambda\tilde{K}_r r + B\Lambda\tilde{\Theta}^\top\Phi(x)\right] \\ &\quad + 2\operatorname{trace}\left(|\Lambda|\,\tilde{K}_x\Gamma_x^{-1}\dot{\tilde{K}}_x^\top\right) \\ &\quad + 2\operatorname{trace}\left(|\Lambda|\,\tilde{K}_r\Gamma_r^{-1}\dot{\tilde{K}}_r^\top\right) + 2\operatorname{trace}\left(|\Lambda|\,\tilde{\Theta}^\top\Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right) \end{aligned} \tag{5.138}$$

Utilizing the trace property $\operatorname{trace}\left(CD^\top\right) = D^\top C$ and $\Lambda = \operatorname{sgn}\Lambda\,|\Lambda|$ where $\operatorname{sgn}\Lambda = \operatorname{diag}(\operatorname{sgn}\lambda_1, \operatorname{sgn}\lambda_2, \ldots, \operatorname{sgn}\lambda_m)$, then notice that

$$e^\top PB\Lambda\tilde{K}_x x = e^\top PB\operatorname{sgn}\Lambda\,|\Lambda|\,\tilde{K}_x x = \operatorname{trace}\left(|\Lambda|\,\tilde{K}_x x e^\top PB\operatorname{sgn}\Lambda\right) \tag{5.139}$$

$$e^\top PB\Lambda\tilde{K}_r r = \operatorname{trace}\left(|\Lambda|\,\tilde{K}_r r e^\top PB\operatorname{sgn}\Lambda\right) \tag{5.140}$$

$$e^\top PB\Lambda\tilde{\Theta}^\top\Phi(x) = \operatorname{trace}\left(|\Lambda|\,\tilde{\Theta}^\top\Phi(x)\,e^\top PB\operatorname{sgn}\Lambda\right) \tag{5.141}$$

Then,

$$\begin{aligned}\dot{V}\left(e,\tilde{K}_x,\tilde{K}_r,\tilde{\Theta}\right) &= -e^\top Qe + 2\operatorname{trace}\left(|\Lambda|\,\tilde{K}_x\left[-xe^\top PB\operatorname{sgn}\Lambda + \Gamma_x^{-1}\dot{\tilde{K}}_x^\top\right]\right)\\ &\quad + 2\operatorname{trace}\left(|\Lambda|\,\tilde{K}_r\left[-re^\top PB\operatorname{sgn}\Lambda + \Gamma_r^{-1}\dot{\tilde{K}}_r^\top\right]\right)\\ &\quad + 2\operatorname{trace}\left(|\Lambda|\,\tilde{\Theta}^\top\left[\Phi(x)\,e^\top PB\operatorname{sgn}\Lambda + \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right]\right)\end{aligned} \tag{5.142}$$

Thus, the adaptive laws are obtained as

$$\dot{K}_x^\top = \Gamma_x x e^\top PB\operatorname{sgn}\Lambda \tag{5.143}$$

$$\dot{K}_r^\top = \Gamma_r r e^\top PB\operatorname{sgn}\Lambda \tag{5.144}$$

$$\dot{\Theta} = -\Gamma_\Theta\Phi(x)\,e^\top PB\operatorname{sgn}\Lambda \tag{5.145}$$

It follows that $e(t)$, $\tilde{K}_x(t)$, $\tilde{K}_r(t)$, and $\tilde{\Theta}(t)$ are bounded since

$$\dot{V}\left(e,\tilde{K}_x,\tilde{K}_r,\tilde{\Theta}\right) = -e^\top Qe \le -\lambda_{min}(Q)\,\|e\|^2 \le 0 \tag{5.146}$$

Using the usual argument with the Barbalat's lemma, the tracking error is asymptotically stable with $e(t) \to 0$ as $t \to \infty$.

Example 5.13 Let $x(t) = \left[x_1(t)\ x_2(t)\right]^\top$, $u(t) = \left[u_1(t)\ u_2(t)\right]^\top$, $\Phi(x) = \left[x_1^2\ x_2^2\right]^\top$, A is unknown, Λ is unknown but $\Lambda > 0$ so $s\mathrm{gn}\Lambda = I$, and B is known and given by

$$B = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

A second-order reference model is specified by

$$\ddot{x}_{1m} + 2\zeta_m\omega_m\dot{x}_{1m} + \omega_m^2 x_{1m} = b_m r$$

where $\zeta_m = 0.5$, $\omega_m = 2$, $b_m = 4$, and $r(t) = \sin 2t$. For simulation purposes, the true A, Λ, and Θ^* matrices are

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \ \Lambda = \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix} \ \Theta^* = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$$

Since $B\Lambda$ is non-singular and invertible, K_x^* and K_r^* exist and are equal to

$$K_x^* = (B\Lambda)^{-1}(A_m - A) = \begin{bmatrix} \frac{4}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right) = \begin{bmatrix} \frac{5}{2} & \frac{5}{4} \\ -\frac{15}{4} & -\frac{5}{4} \end{bmatrix}$$

$$K_r^* = (B\Lambda)^{-1} B_m = \begin{bmatrix} \frac{4}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

Let $Q = I$, then the solution of the Lyapunov equation is

$$P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

Let $\Gamma_x = \operatorname{diag}\left(\gamma_{x_1}, \gamma_{x_2}\right)$, $\Gamma_r = \gamma_r$, and $\Gamma_\Theta = \operatorname{diag}\left(\gamma_{\theta_1}, \gamma_{\theta_2}\right)$. Then, the adaptive laws are

$$\begin{aligned} \dot{K}_x^\top &= \Gamma_x x e^\top P B \underbrace{\operatorname{sgn}\Lambda}_{I} = \begin{bmatrix} \gamma_{x_1} & 0 \\ 0 & \gamma_{x_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{13}{8} \\ \frac{1}{8} & \frac{7}{16} \end{bmatrix} \\ &= \begin{bmatrix} \gamma_{x_1} x_1 \left(\frac{3}{2} e_1 + \frac{1}{8} e_2\right) & \gamma_{x_1} x_1 \left(\frac{13}{8} e_1 + \frac{7}{16} e_2\right) \\ \gamma_{x_2} x_2 \left(\frac{3}{2} e_1 + \frac{1}{8} e_2\right) & \gamma_{x_2} x_2 \left(\frac{13}{8} e_1 + \frac{7}{16} e_2\right) \end{bmatrix} \end{aligned}$$

$$\dot{K}_r^\top = \Gamma_r r e^\top P B \operatorname{sgn}\Lambda = \gamma_r r \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{13}{8} \\ \frac{1}{8} & \frac{7}{16} \end{bmatrix} = \begin{bmatrix} \gamma_r r \left(\frac{3}{2} e_1 + \frac{1}{8} e_2\right) & \gamma_r r \left(\frac{13}{8} e_1 + \frac{7}{16} e_2\right) \end{bmatrix}$$

$$\begin{aligned} \dot{\Theta} &= -\Gamma_\Theta \Phi(x) e^\top P B \operatorname{sgn}\Lambda = -\begin{bmatrix} \gamma_{\theta_1} & 0 \\ 0 & \gamma_{\theta_2} \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{13}{8} \\ \frac{1}{8} & \frac{7}{16} \end{bmatrix} \\ &= -\begin{bmatrix} \gamma_{\theta_1} x_1^2 \left(\frac{3}{2} e_1 + \frac{1}{8} e_2\right) & \gamma_{\theta_1} x_1^2 \left(\frac{13}{8} e_1 + \frac{7}{16} e_2\right) \\ \gamma_{\theta_2} x_2^2 \left(\frac{3}{2} e_1 + \frac{1}{8} e_2\right) & \gamma_{\theta_2} x_2^2 \left(\frac{13}{8} e_1 + \frac{7}{16} e_2\right) \end{bmatrix} \end{aligned}$$

For simulation, all adaptation rates are chosen to be 10 and all initial conditions are set to zero. The results are shown in Figs. 5.6, 5.7, 5.8, and 5.9.

It can be seen that the tracking error tends to zero so that $x(t)$ follows $x_m(t)$. Also note that $K_r(t)$ and some elements of $K_x(t)$ do not converge to their ideal values.

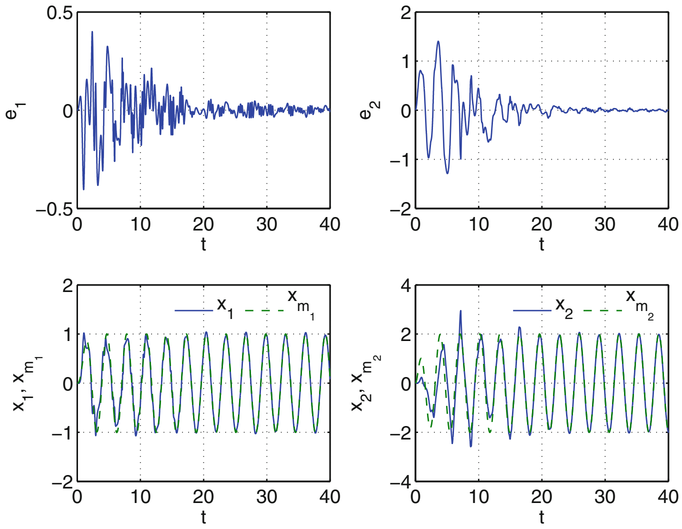

Fig. 5.6 Adaptive control system response

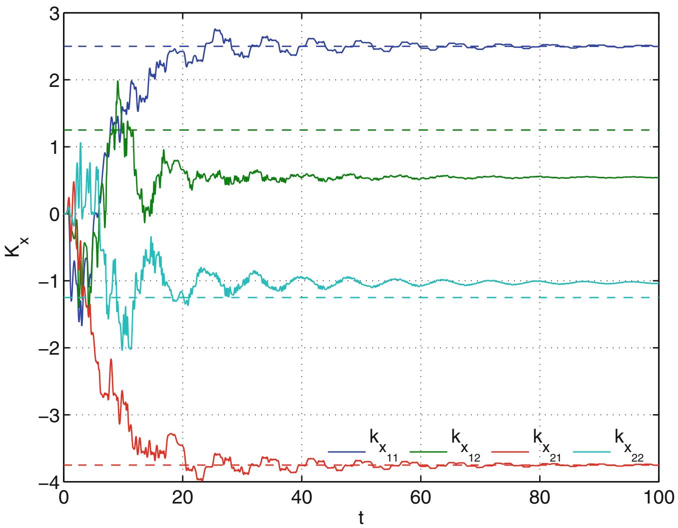

Fig. 5.7 Adaptive feedback gain K_x

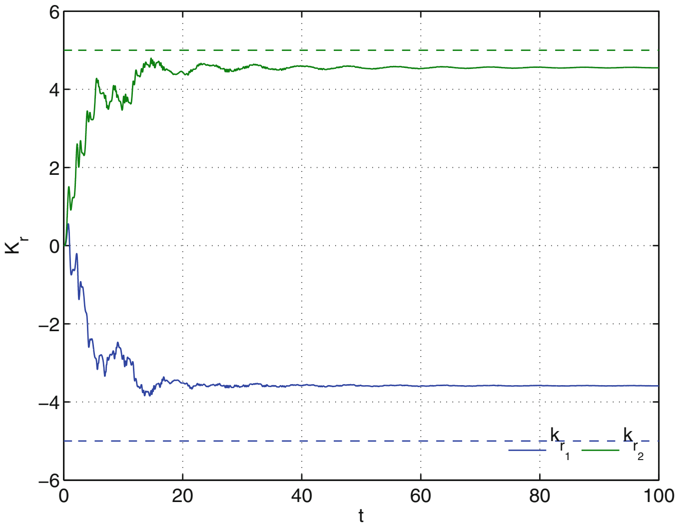

Fig. 5.8 Adaptive command gain K_r

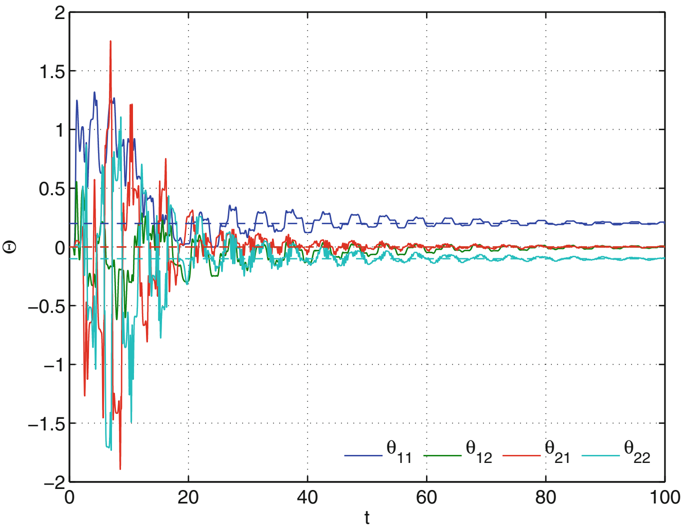

Fig. 5.9 Adaptive parameter Θ

5.6.2 Case II: A, B, $\Lambda = I$ Known

The plant model is given by

$$\dot{x} = Ax + B\left[u + \Theta^{*\top}\Phi\left(x\right)\right] \tag{5.147}$$

where both A and B are known.

Assuming that there exist K_x and K_r that satisfy the model matching conditions

$$A + BK_x = A_m \tag{5.148}$$

$$BK_r = B_m \tag{5.149}$$

then an adaptive controller is designed as

$$u = K_x x + K_r r - \Theta^\top\left(t\right)\Phi\left(x\right) \tag{5.150}$$

Let $\tilde{\Theta}\left(t\right) = \Theta\left(t\right) - \Theta^*$ be the estimation error, then the closed-loop plant model is

$$\dot{x} = \left(\underbrace{A + BK_x}_{A_m}\right)x + \underbrace{BK_r}_{B_m}r - B\tilde{\Theta}^\top\Phi\left(x\right) \tag{5.151}$$

The closed-loop tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e + B\tilde{\Theta}^\top\Phi\left(x\right) \tag{5.152}$$

Proof Choose a Lyapunov candidate function

$$V\left(e, \tilde{\Theta}\right) = e^\top Pe + \text{trace}\left(\tilde{\Theta}^\top\Gamma^{-1}\tilde{\Theta}\right) \tag{5.153}$$

Then,

$$\begin{aligned}\dot{V}\left(e, \tilde{\Theta}\right) &= -e^\top Qe + 2e^\top PB\tilde{\Theta}^\top\Phi\left(x\right) + 2\text{trace}\left(\tilde{\Theta}^\top\Gamma\dot{\tilde{\Theta}}\right)\\ &= -e^\top Qe + 2\text{trace}\left(\tilde{\Theta}^\top\left[\Phi\left(x\right)e^\top PB + \Gamma\dot{\tilde{\Theta}}\right]\right)\end{aligned} \tag{5.154}$$

The adaptive law is

$$\dot{\Theta} = -\Gamma\Phi\left(x\right)e^\top PB \tag{5.155}$$

Using the Barbalat's lemma, the tracking error can be shown to be asymptotically stable with $e\left(t\right) \to 0$ as $t \to \infty$.

5.7 Indirect MRAC for MIMO Systems

For the system in Sect. 5.6.1 with A and Λ unknown but B and sign of Λ known, assuming that $B\Lambda \in \mathbb{R}^n \times \mathbb{R}^m$ is invertible and $n \le m$, then there exist K_x^* and K_r^* that satisfy the model matching conditions such that

$$K_x^* = (B\Lambda)^{-1}(A_m - A) \tag{5.156}$$

$$K_r^* = (B\Lambda)^{-1} B_m \tag{5.157}$$

If $\hat{A}(t)$ and $\hat{\Lambda}(t)$ are estimates of A and Λ, then the estimates of K_x^* and K_r^* are given by

$$K_x(t) = \left[B\hat{\Lambda}(t)\right]^{-1}\left[A_m - \hat{A}(t)\right] \tag{5.158}$$

$$K_r(t) = \left[B\hat{\Lambda}(t)\right]^{-1} B_m \tag{5.159}$$

Note that if $n < m$, then $\left[B\hat{\Lambda}(t)\right]^{-1}$ is defined by the right pseudo-inverse $\hat{\Lambda}^\top(t)B^\top\left[B\hat{\Lambda}(t)\hat{\Lambda}^\top(t)B^\top\right]^{-1}$.

Let the adaptive controller be

$$u = K_x(t)x + K_r(t)r - \Theta^\top(t)\Phi(x) \tag{5.160}$$

Let $\tilde{A}(t) = \hat{A}(t) - A$ and $\tilde{\Lambda}(t) = \hat{\Lambda}(t) - \Lambda$ be the estimation errors. Then, the closed-loop plant model is expressed as

$$\begin{aligned}\dot{x} &= Ax + B\left(\hat{\Lambda} - \tilde{\Lambda}\right)\left[K_x x + K_r r - \tilde{\Theta}^\top \Phi(x)\right] \\ &= \left(A + A_m - \hat{A}\right)x + B_m r - B\tilde{\Lambda}(K_x x + K_r r) - B\Lambda\tilde{\Theta}^\top\Phi(x)\end{aligned} \tag{5.161}$$

Let

$$\bar{u} = K_x x + K_r r \tag{5.162}$$

Then, the tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e + \tilde{A}x + B\tilde{\Lambda}\bar{u} + B\Lambda\tilde{\Theta}^\top\Phi(x) \tag{5.163}$$

Proof Choose a Lyapunov candidate function

$$V\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = e^\top P e + \text{trace}\left(\tilde{A}\Gamma_A^{-1}\tilde{A}^\top\right) + \text{trace}\left(\tilde{\Lambda}\Gamma_\Lambda^{-1}\tilde{\Lambda}^\top\right) + \text{trace}\left(|\Lambda|\,\tilde{\Theta}^\top\Gamma_\Theta^{-1}\tilde{\Theta}\right) \tag{5.164}$$

Then,

$$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = -e^\top Qe + 2e^\top P\left[\tilde{A}x + B\tilde{\Lambda}\bar{u} + B\Lambda\tilde{\Theta}^\top \Phi(x)\right] \\ + 2\text{trace}\left(\tilde{A}\Gamma_A^{-1}\dot{\tilde{A}}^\top\right) + 2\text{trace}\left(\tilde{\Lambda}\Gamma_\Lambda^{-1}\dot{\tilde{\Lambda}}^\top\right) + 2\text{trace}\left(|\Lambda|\tilde{\Theta}^\top\Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right) \quad (5.165)$$

Utilizing the following relationships:

$$e^\top P\tilde{A}x = \text{trace}\left(\tilde{A}xe^\top P\right) \quad (5.166)$$

$$e^\top PB\tilde{\Lambda}\bar{u} = \text{trace}\left(\tilde{\Lambda}\bar{u}e^\top PB\right) \quad (5.167)$$

$$e^\top PB\Lambda\tilde{\Theta}^\top\Phi(x) = e^\top PB\text{sgn}(\Lambda)|\Lambda|\tilde{\Theta}^\top\Phi(x) = \text{trace}\left(|\Lambda|\tilde{\Theta}^\top\Phi(x)e^\top PB\text{sgn}(\Lambda)\right) \quad (5.168)$$

we obtain

$$\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) = -e^\top Qe + 2\text{trace}\left(\tilde{A}\left[xe^\top P + \Gamma_A^{-1}\dot{\tilde{A}}^\top\right]\right) \\ + 2\text{trace}\left(\tilde{\Lambda}\left[\bar{u}e^\top PB + \Gamma_\Lambda^{-1}\dot{\tilde{\Lambda}}^\top\right]\right) \\ + 2\text{trace}\left(|\Lambda|\tilde{\Theta}^\top\left[\Phi(x)e^\top PB\text{sgn}(\Lambda) + \Gamma_\Theta^{-1}\dot{\tilde{\Theta}}\right]\right) \quad (5.169)$$

from which the adaptive laws are obtained as

$$\dot{\hat{A}}^\top = -\Gamma_A xe^\top P \quad (5.170)$$

$$\dot{\hat{\Lambda}}^\top = -\Gamma_\Lambda \bar{u}e^\top PB \quad (5.171)$$

$$\dot{\Theta} = -\Gamma_\Theta\Phi(x)e^\top PB\text{sgn}\Lambda \quad (5.172)$$

Since $\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) \le -\lambda_{min}(Q)\|e\|^2 \le 0$ and $\ddot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right) \in \mathcal{L}_\infty$, then $\dot{V}\left(e, \tilde{A}, \tilde{B}, \tilde{\Theta}\right)$ is uniformly continuous. In addition, $V(t \to \infty) \le V(t_0)$. Therefore, according to the Barbalat's lemma, the tracking error is asymptotically stable with $e(t) \to 0$ as $t \to \infty$.

■

Because $\hat{\Lambda}(t)$ is involved in a matrix inversion operation, $\hat{\Lambda}(t)$ cannot be singular assuming B is non-singular. Therefore, the adaptive law for $\hat{\Lambda}(t)$ needs to be modified by the projection method if a priori knowledge of the bounds on the elements of Λ is available. Suppose the diagonal elements are bounded by $\lambda_{i_0} \le \left|\hat{\lambda}_{ii}\right| \le 1$, and the non-diagonal elements are bounded to be close to zero such that $\left|\hat{\lambda}_{ij}\right| \le \epsilon, i \ne j$. Then, the modified adaptive law for the diagonal elements can be expressed as

$$\dot{\hat{\lambda}}_{ii} = \begin{cases} -\left(\Gamma_\Lambda \bar{u} e^\top P B\right)_{ii} & \text{if } 1 \geq \left|\hat{\lambda}_{ii}\right| \geq \lambda_{i_0}, \text{ or if } \left|\hat{\lambda}_{ii}\right| = \lambda_{i_0} \text{ and } \frac{d\left|\hat{\lambda}_{ii}\right|}{dt} \geq 0, \text{ or if } \left|\hat{\lambda}_{ii}\right| = 1 \text{ and } \frac{d\left|\hat{\lambda}_{ii}\right|}{dt} \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.173}$$

and the modified adaptive law for the non-diagonal elements can be expressed as

$$\dot{\hat{\lambda}}_{ij} = \begin{cases} -\left(\Gamma_\Lambda \bar{u} e^\top P B\right)_{ji} & \text{if } \left|\hat{\lambda}_{ij}\right| \leq \epsilon, \text{ or if } \left|\hat{\lambda}_{ij}\right| = \epsilon \text{ and } \frac{d\left|\hat{\lambda}_{ij}\right|}{dt} \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.174}$$

5.8 Summary

In situations when the system uncertainty may become significant beyond a level of desired tolerance that can adversely affect the performance of a controller, adaptive control can play an important role in reducing the effects of the system uncertainty on the controller performance. Situations that may warrant the use of adaptive control could include unintended consequences of off-nominal modes of operation such as system failures or highly uncertain operating conditions, and complex system behaviors that can result in an increase in the complexity and hence cost of the modeling efforts.

There are generally two classes of adaptive control: (1) direct adaptive control and (2) indirect adaptive control. Adaptive control architectures that combine both types of adaptive control are also frequently used and are referred to as composite, combined, or hybrid direct–indirect adaptive control. Adaptive control can deal with either linear or nonlinear plants with various types of uncertainty which can be structured uncertainty, unstructured uncertainty, or unmodeled dynamics. Matched uncertainty is a type of structured uncertainty that can be matched by the control input for a class of MIMO linear affine-in-control systems. Adaptive control systems can be designed to provide cancellation of a matched uncertainty. When an uncertainty cannot be matched, it is called an unmatched uncertainty. Adaptive control systems can be designed to accommodate unmatched uncertainty, but cannot cancel the unmatched uncertainty in general. Control input uncertainty is a type of uncertainty that exists in the control input matrix for a class of MIMO linear affine-in-control systems. Control input uncertainty can be in the amplitude or in the sign or both. When the control input uncertainty is in the amplitude, a control saturation can occur and may worsen the performance of a controller. When the input uncertainty is in the sign, a control reversal can occur and potentially can cause instability. Control input uncertainty in general presents more challenges to adaptive control designers.

A reference model is used to specify a desired response of an adaptive control system to a command input. It is essentially a command shaping filter to achieve a desired command following. Since adaptive control is formulated as a command following or tracking control, the adaptation is operated on the tracking error between the reference model and the system output. A reference model must be designed properly for an adaptive control system to be able to follow.

Various direct and indirect model-reference adaptive control techniques for first-order and second-order SISO systems and MIMO systems are presented. MRAC can be shown to achieve asymptotic tracking, but it does not guarantee that adaptive parameters converge to their true values. The Lyapunov stability theory shows that adaptive parameter estimation errors are only bounded but not asymptotic.

5.9 Exercises

1. Consider a first-order nonlinear SISO system with a matched uncertainty

$$\dot{x} = ax + b\left[u + \theta^* \phi(x)\right]$$

where a is unknown, but b is known, θ^* is unknown, and $\phi(x) = x^2$.
A reference model is specified by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m < 0$ and b_m are known, and $r(t)$ is a bounded command signal.

a. Design and implement in Simulink a direct adaptive controller that enables the plant output $x(t)$ to track the reference model signal $x_m(t)$, given $b = 2$, $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$. For adaptation rates, use $\gamma_x = 1$ and $\gamma = 1$. For simulation purposes, assume $a = 1$ and $\theta^* = 0.2$ for the unknown parameters. Plot $e(t)$, $x(t)$, $x_m(t)$, $u(t)$, and $\theta(t)$ for $t \in [0, 50]$.
b. Show by the Lyapunov stability analysis that the tracking error is asymptotically stable, i.e., $e(t) \to 0$ as $t \to \infty$.
c. Repeat part (a) for $r(t) = 1(t)$ where $1(t)$ is the unit step function. Plot the same sets of data as in part (a). Comment on the convergence of $k_x(t)$ and $\theta(t)$ to the ideal values k_x^* and θ^*.

2. Consider the following first-order plant

$$\dot{x} = ax + b\left[u + \theta^* \phi(x)\right]$$

where a , $b > 0$, θ^* is unknown, and $\phi(x) = x^2$. Design an indirect adaptive controller in Simulink by estimating a, b, and θ^* so that the plant follows a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$. For simulation purposes, use $a = 1$, $b = 1$, $\theta^* = 0.1$, $x(0) = x_m(0) = 1$, $\hat{a}(0) = 0$, $\hat{b}(0) = 1.5$, $\gamma_a = \gamma_b = \gamma_\theta = 1$. Also assume that a lower bound of b is $b_0 = 0.5$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{\theta}(t)$ for $t \in [0, 50]$

3. Derive direct MRAC laws for a second-order SISO system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = b\left[u + \Theta^{*\top}\Phi(y)\right]$$

where ζ and ω_n are unknown, but b is known. Show by applying the Barbalat's lemma that the tracking error is asymptotically stable.
Design a direct adaptive controller for a second-order system using the following information: $b = 1$, $\zeta_m = 0.5$, $\omega_m = 2$, $b_m = 4$, $r(t) = \sin 2t$, and

$$\Phi(y) = \begin{bmatrix} 1 \\ y^2 \end{bmatrix}$$

For simulation purposes, the unknown parameters may be assumed to be $\zeta = -0.5$, $\omega_n = 1$, and $\Theta^{*\top} = \begin{bmatrix} 0.5 & -0.1 \end{bmatrix}$, and all initial conditions are assumed to be zero. Use $\Gamma_x = \Gamma_\Theta = 100I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $K_x(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

4. For Exercise 3, suppose b is unknown, but $b > 0$ is known. Design an indirect adaptive controller in Simulink. For simulation purposes, all initial conditions are assumed to be zero, except for $\hat{\omega}_n(0) = 0.8$ and $\hat{b}(0) = 0.6$. For simplicity, use the unmodified adaptive laws for $\hat{\omega}_n(t)$ and $\hat{b}(t)$. Use $\gamma_\omega = \gamma_\zeta = \gamma_b = 10$ and $\Gamma_\Theta = 10I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $\hat{\omega}_n(t)$, $\hat{\zeta}(t)$, $\hat{b}(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

5. Thus far, we have considered adaptive control with a matched uncertainty as a function of x. In physical systems, an external disturbance is generally a function of t. Adaptive control can be used for disturbance rejection if the disturbance structure is known. Suppose the matched uncertainty is a function of t, then all the adaptive laws can still be used by just replacing $\Phi(x)$ by $\Phi(t)$, assuming $\Phi(t)$ is known and bounded.
Consider the following first-order plant:

$$\dot{x} = ax + b\left[u + \theta^*\phi(t)\right]$$

where a , b, and θ^* are unknown, but $b > 0$ is known, and $\phi(t) = \sin 2t - \cos 4t$. Design an indirect adaptive controller in Simulink by estimating a, b, and θ^* so that the plant follows a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$. For simulation purposes, use $a = 1$, $b = 1$, $\theta^* = 0.1$, $x(0) = x_m(0) = 1$, $\hat{a}(0) = 0$, $\hat{b}(0) = 1.5$, $\gamma_a = \gamma_b = \gamma_\theta = 1$. Also assume that a lower bound of b is $b_0 = 0.5$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{\theta}(t)$ for $t \in [0, 50]$.

6. Derive direct MRAC laws for a MIMO system

$$\dot{x} = Ax + B\left[u + \Theta^{*\top}\Phi(x)\right]$$

where A is unknown, but B is known. Show by applying the Barbalat's lemma that the tracking error is asymptotically stable.
Given $x(t) = \left[x_1(t)\ x_2(t)\right]^\top$, $u(t) = \left[u_1(t)\ u_2(t)\right]^\top$, $\Phi(x) = \left[x_1^2\ x_2^2\right]^\top$, and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

design a direct adaptive controller in Simulink for the MIMO system to follow a second-order SISO system specified by

$$\dot{x}_m = A_m x + B_m r$$

where $r(t) = \sin 2t$ and

$$A_m = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix},\ B_m = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

For simulation purposes, the unknown parameters may be assumed to be

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ \Theta^* = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$$

and all initial conditions are assumed to be zero. Use $\Gamma_x = \Gamma_\Theta = 10I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $K_x(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

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