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L_1–"Adaptive" Control Always Converges to a Linear PI Control and Does Not Perform Better than the PI

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Abstract: We show in the paper that the, so–called, "new architecture" of L_1–adaptive control is, indeed, different from classical model reference adaptive control. Alas, it is not new, since it exactly coincides with a full–state feedback, linear time–invariant proportional plus integral (PI) controller with a decaying additive disturbance. Moreover, it is shown that if the PI controller does not stabilize the plant the L_1–adaptive controller will not stabilize it either.

Keywords: adaptive control, PI control, nonlinear systems

1. INTRODUCTION

The basic premise upon which adaptive control is based is the existence of a parameterized controller that achieves the control objective. It is, moreover, assumed that these parameters are not known but that they can be estimated on–line from measurements of the plant signals. Towards this end, an identifier is added to generate the parameter estimates. Then, applying in an *ad–hoc* manner a certainty equivalence principle, these estimates are directly applied in the aforementioned control law.

Let us illustrate the discussion above with the simplest example of direct, adaptive, state–feedback stabilization of single–input, linear time–invariant (LTI) system of the form

$$\dot{x} = Ax + bu \tag{1}$$

where the state $x \in \mathbb{R}^n$ is assumed to be *measurable*, $u \in \mathbb{R}$ is the control signal, $A \in \mathbb{R}^{n\times n}$ is the system matrix and $b \in \mathbb{R}^n$ the input vector. It is assumed that there exists a vector $\theta \in \mathbb{R}^n$ such that

$$A + b\theta^\top =: A_m$$

is a Hurwitz matrix, but this vector is *unknown.* In this case, the ideal control law takes the form

$$u = \theta^\top x, \tag{2}$$

that, as mentioned above, is made adaptive adding an identifier that generates the estimated parameters $\hat{\theta} \in \mathbb{R}^n$. In this way, we obtain the adaptive control law

$$u = \hat{\theta}^\top x. \tag{3}$$

Defining the parameter error

$$\tilde{\theta} := \hat{\theta} - \theta, \tag{4}$$

the control law may be written as

$$u = \theta^\top x + \tilde{\theta}^\top x.$$

If the parameter estimates converge to the desired value θ the control signal converges to the ideal control law (2) and asymptotic stabilization is achieved—provided x remains bounded. [1]

A key observation is that the ideal control signal (2) *cannot be implemented* without knowledge of the unknown parameters. If this were not the case adaptation would be unnecessary and we simply would plug in the controller that results when $\tilde{\theta} = 0$!

In a (long) series of recent papers—see, *e.g.*, Hovakimyan et al. (2011) and the extensive list of references therein—it has been proposed to replace (3) by

$$\dot{u} = -k(u - \hat{\theta}^\top x), \tag{5}$$

where $k > 0$ is a design parameter. Combining (5) with a standard, state prediction–based estimator is called in Hovakimyan et al. (2011) L_1–adaptive control, which in the sequel we refer to as L_1–AC.

The purpose of this paper is to prove the following facts regarding L_1–AC.

- For any parameter estimation law, the control signal (5) exactly coincides with the output of the LTI, full–state feedback, perturbed, PI controller
$$\begin{aligned} \dot{v} &= -K_I^\top x + \mu k\tilde{\theta}^\top x \\ u &= v - K_P^\top x, \end{aligned} \tag{6}$$
whose gains $K_P, K_I \in \mathbb{R}^n$ are *independent* of the parameters θ and $\mu = 1$.
- If the parameters of controller (5) are updated with a standard state predictor–based estimator the term $\tilde{\theta}^\top x$ *always* converges to zero. Hence, the L_1–AC converges to a controller that can be obtained *without* knowledge of the unknown parameters.
- If the PI controller obtained setting $\mu = 0$ in (6) *does not stabilize* the plant (1) then the L_1–AC does not stabilize it either.

[1] Actually, to achieve stabilization it is enough that $\hat{\theta}$ converges to the set $\{f \in \mathbb{R}^n \mid A + bf^\top \text{ is Hurwitz}\}$. This is the fundamental self–tuning property of direct adaptive control.

2. MAIN RESULT

We analyze in this paper the L_1–AC proposed in Hovakimyan et al. (2011) to address the basic problem of stabilization of single–input, LTI systems discussed in the previous section. In L_1–AC, besides the (overly restrictive) assumption of measurable state, it is assumed that the input vector b is *known*. Without loss of generality, the system can be represented in canonical form

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_1 & -a_2 & -a_3 & \dots & -a_n \end{bmatrix}$$

where $a_i \in \mathbb{R},\ i \in \bar{n} := \{1, \dots, n\}$ are unknown coefficients, and $b = e_n$, the n–th vector of the Euclidean basis. The system can also be expressed in the form

$$\dot{x} = A_m x - b(\theta^\top x - u) \tag{7}$$

where $A_m \in \mathbb{R}^{n\times n}$, a Hurwitz matrix representing the desired behavior for the closed–loop system, has the form

$$A_m = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_1^m & -a_2^m & -a_3^m & \dots & -a_n^m \end{bmatrix}$$

where $a_i^m \in \mathbb{R}_+,\ i \in \bar{n}$ are designer chosen coefficients and $\theta \in \mathbb{R}^n$ is a vector of *unknown* parameters, given by

$$\theta = \mathrm{col}(a_1 - a_1^m, a_2 - a_2^m, \dots, a_n - a_n^m), \tag{8}$$

where $\mathrm{col}(\cdot)$ denotes column vector. In L_1–AC the control law is computed via (5). The parameters are updated using the classical state predictor–based estimator

$$\begin{aligned} \dot{\hat{x}} &= A_m \hat{x} - b(\hat{\theta}^\top x - u) \\ \dot{\hat{\theta}} &= \gamma x(\hat{x} - x)^\top P b \end{aligned} \tag{9}$$

where $\gamma > 0$ is the adaptation gain and $P > 0$ is a Lyapunov matrix for A_m, that is,

$$PA_m + A_m^\top P < 0.$$

Proposition 1. Consider the plant (7).

P1 Independently of the parameter estimation, the signal u generated by the L_1–AC control law (5) *exactly coincides with the output of the perturbed, full-state feedback, LTI, PI controller* (6) *with $\mu = 1$ and*

$$\begin{aligned} K_I &= k\ \mathit{col}(a_1^m, a_2^m, a_3^m, \dots, a_n^m) \\ K_P &= k\ e_n. \end{aligned} \tag{10}$$

P2 There exists $k_c > 0$ such that the implementable[2] *PI controller* (6), (10) *with $\mu = 0$, ensures global asymptotic stability (GAS) of the closed–loop system for all $k > k_c$, all unknown parameter vectors θ and all Hurwitz matrices A_m.*

P3 If the L_1–AC controller (5), (9) *ensures boundedness of trajectories then the perturbation term verifies*

$$\lim_{t\to\infty} |\tilde{\theta}^\top(t) x(t)| = 0. \tag{11}$$

Consequently, the (bounded state) L_1–AC always *converges to the PI controller.*

P4 If the PI (6) *with $\mu = 0$* does not *ensure stability of the closed–loop system then the L_1–AC* (5), (9) *does not ensure this property either.*

Proof. To establish P1 we use the definition of the parameter error to write the the control signal (5) as

$$\dot{u} = -k(u + \theta^\top x) + k\tilde{\theta}^\top x. \tag{12}$$

Now, pre–multiplying (7) by $e_n^\top$, and rearranging terms, we get

$$u - \theta^\top x = e_n^\top(\dot{x} - A_m x),$$

that, upon replacement in (12), yields

$$\dot{u} = -k e_n^\top(\dot{x} - A_m x) + k\tilde{\theta}^\top x.$$

The proof is completed defining the signal

$$v = u + k x_n.$$

To establish P2 we simply analyze the resulting closed–loop LTI system

$$\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \mathbf{Q} \begin{bmatrix} x \\ u \end{bmatrix},$$

where

$$\mathbf{Q} = \begin{bmatrix} A & b \\ k\theta^\top & -k \end{bmatrix}.$$

Now, the characteristic polynomial of $\mathbf{Q}$ is computed as follows

$$\begin{aligned} \det(sI_n - \mathbf{Q}) &= \det \begin{bmatrix} sI_n - A & -b \\ -k\theta^\top & s + k \end{bmatrix} \\ &= \det(sI_n - A)\left[(s+k) - k\theta^\top(sI_n - A)^{-1} b\right] \\ &= \det(sI_n - A)\left[(s+k) - k\frac{N(s)}{\det(sI_n - A)}\right], \end{aligned}$$

where

$$N(s) = s^n + \theta_n s^{n-1} + \dots + \theta_1.$$

From the definition of θ in (8) we have that

$$\det(sI_n - A) + N(s) = \det(sI_n - A_m).$$

Hence, grouping terms we obtain

$$\det(sI_n - \mathbf{Q}) = s\det(sI_n - A) + k\det(sI_n - A_m). \tag{13}$$

The proof is completed applying a root–locus arguments to the expression above and recalling that A_m is a Hurwitz matrix.

To prove P3 we first write the dynamics of the system in closed–loop with the L_1–AC (9), (12),

$$\begin{aligned} \dot{\tilde{x}} &= A_m \tilde{x} - b\tilde{\theta}^\top x \\ \dot{\tilde{\theta}} &= \gamma x \tilde{x}^\top P b \\ \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} &= \mathbf{Q}_0 \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ k\tilde{\theta}^\top x \end{bmatrix}, \end{aligned} \tag{14}$$

where $\tilde{x} = \hat{x} - x$ is the prediction error. Consider the function

$$V(\tilde{x}, \tilde{\theta}) = \frac{1}{2}\tilde{x}^\top P \tilde{x} + \frac{1}{2\gamma}|\tilde{\theta}|^2,$$

whose derivative along the trajectories of (14) is

$$\dot{V} = -\frac{1}{2}\tilde{x}^\top Q \tilde{x}.$$

Since it has been assumed that all trajectories are bounded we can invoke LaSalle's invariance principle to conclude that all trajectories converge to the largest invariant set

[2] By "implementable" we mean here that the controller is independent of the unknown plant parameters.

contained in $\{\tilde{x} = 0\}$. The proof is completed analyzing the first equation of (14).

The proof of P4 is established proving the converse implication, *i.e.*, that the trajectories of the L_1–AC are bounded implies stability of the PI. In point P3 we proved that if the trajectories of (14) are bounded (11) holds true. Now, the system in the third equation of (14) is an LTI system whose input, *i.e.*, $\tilde{\theta}^\top x$ converges to zero and whose output $\mathrm{col}(x, u)$ is bounded, for all initial conditions $\mathrm{col}(x(0), u(0))$, consequently the matrix $\mathbf{Q}$ is stable.[3]

3. CONCLUDING REMARKS

The present paper extends the results of Ortega and Panteley (2014a), where we treat only scalar systems. It is similar in spirit to the proof of Heusden and Dumont (2012) that output feedback L_1–AC is, actually, nonadaptive. P1 in Proposition 1 underscores that the stabilization mechanism of L_1–AC has nothing to do with the parameter adaptation, but it's an elementary linear systems principle.

The qualifier "implementable" in P2 is essential to appreciate the significance of the statement. Of course, all adaptive controllers can be implemented as an LTI system perturbed by the parameter error but the resulting LTI system depends on *unknown plant* parameters. Due to the inclusion of the input filter, this is not the case in L_1–AC. Moreover, since it is shown in P3 that the term $\tilde{\theta}^\top x$ *always* converges to zero, the closed–loop system asymptotically coincides with a system that could have been obtained *without* adaptation rendering irrelevant—and even harmful—the use of adaptation. Indeed, it is hard to expect that adaptation, whose effect appears only in the additive term $\tilde{\theta}^\top x$, can improve the performance of the PI.

Proposition 1 has be generalized in Ortega and Panteley (2014b) in several directions. We have assumed for simplicity the case of regulation to zero and taken the input filter used in L_1–AC as $D(s) = \frac{k}{s+k}$. As shown in that paper the proposition extends *verbatim* to the case of nonconstant reference and general (stable, strictly proper) LTI filters $D(s)$. Also, we have assumed that the pair (A_m, b) is in canonical form to simplify the proof of P2 in Proposition 1. A similar result is obtained in Ortega and Panteley (2014b) for general (A_m, b), in which case we pre–multiply (7) by the Moore–Penrose pseudo-inverse of b, that is,

$$b^\dagger = (b^\top b)^{-1} b^\top,$$

instead of $e_n^\top$.

The paper complements the recent report Ioannou et al. (2013) where the claims of robustness and performance improvement of L_1–AC are scrutinized via theoretical analysis and a series of numerical examples. The interested reader is also referred to Boskovic and Mehra (2013); Ioannou et al. (2013) where the issues of numerical instability due to high–gain adaptation and bang–bang behavior of the control due to parameter projection L_1–AC, are discussed. The inability of L_1–AC to track non–constant references is widely acknowledged, see Ortega and Panteley (2014a) for a particular example. A freezing property of high–gain estimators, that puts a question mark on the interest of using it, is proven in Barabanov et al. (2005), see also Ortega and Panteley (2014a).

[3] From (13) it follows that $\mathbf{Q}$ may not have an eigenvalue at zero, but it may have eigenvalues in the $j\omega$ axis. The authors thank Denis Efimov for this insightful remark.

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