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Practice calculus! ✍️

Calculus cheatsheet 🚑 10 min

Calculus is the mathematical study of continuous change. It has two major branches, differential calculus and integral calculus; the former concerns instantaneous rates of change, and the slopes of curves, while integral calculus concerns accumulation of quantities, and areas under or between curves.

Definitions and notations

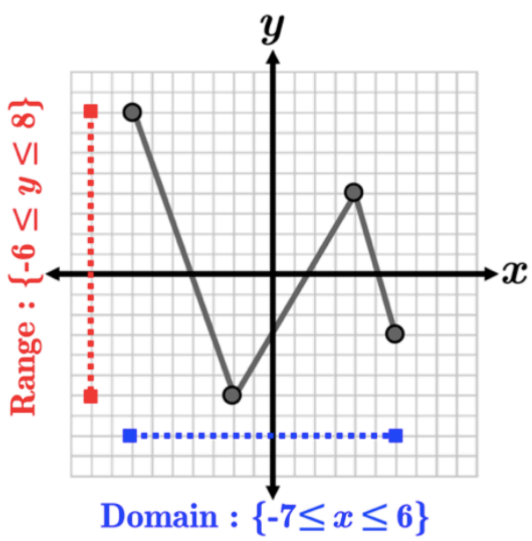
Function

A **function** $f(x)$ is a relation that consists of a set of ordered pairs $(x, y = f(x))$ in which each value of x is connected to a unique value of y, based on the rule of the function f. For each value of x, there is one and only one corresponding value of y.

Domain and range

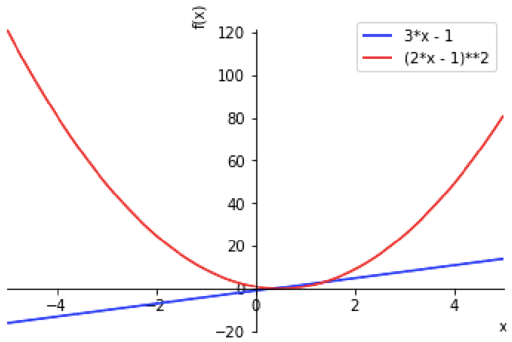
The domain of a function $y = f(x)$ is the set into which all of the input x of the function is constrained to fall.

The range of a function target set $y = f(x)$ is the set into which all of the output y of f is constrained to fall.



Function composition

In mathematics, **function composition** is an operation that takes two functions f and g and produces a function h such that $h(x) = g(f(x))$. In this operation, the function g is applied to the result of applying the function f to x.

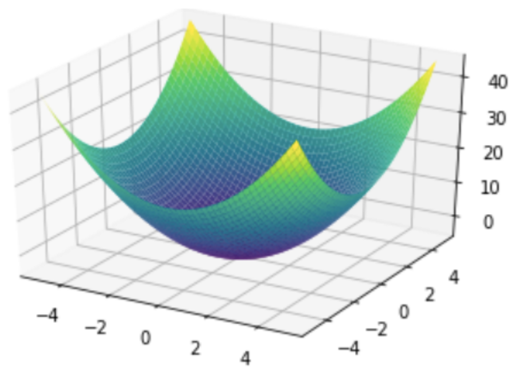


Notation

$$g(f(x)) = g \circ f(x)$$

Multivariate function

A **function of several variables** or **multivariate function** is a function with more than one argument. For example, $y = f(x, y, z)$ is a function of three variables x, y and z.

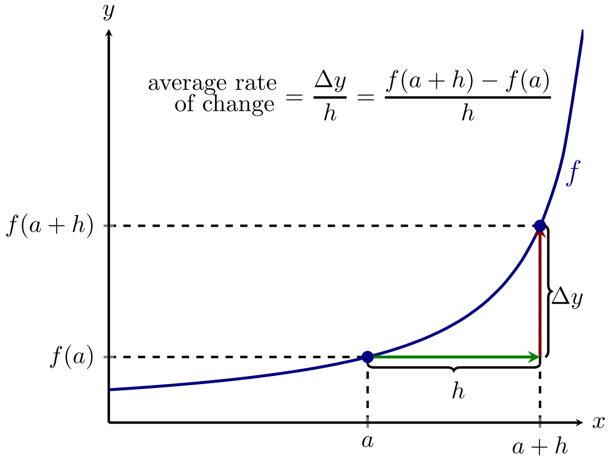


Derivative

The **derivative of a function** $y = f(x)$ of a variable x is a measure of the rate at which the value y of the function changes with respect to the change of the variable x. It is called the derivative of f with respect to x.

If $y = f(x)$ then the derivative is defined to be :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



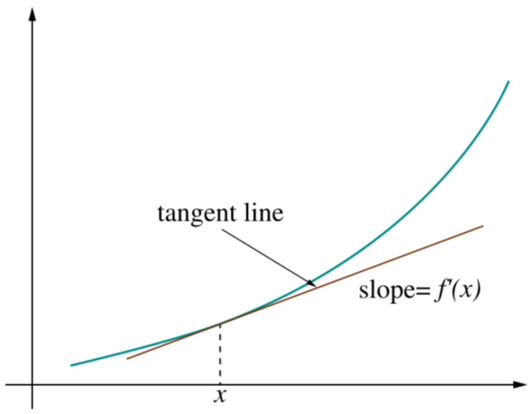
Notation

If $y = f(x)$ then all of the following are equivalent notations for the derivative :

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$

Interpretation

$m = f'(a)$ is the slope of the tangent line to the function $y = f(x)$ evaluated at $x = a$.



Partial derivative

In mathematics, a **partial derivative** of a **multivariate function** f is its derivative with respect to one of those variables, with the others held constant.

Notation

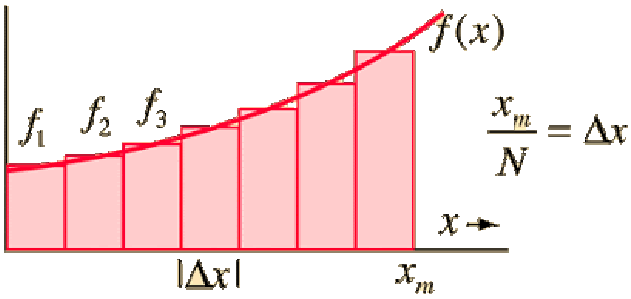
If $y = f(x, y, z)$ is a function of several variables x, y and z then the partial derivative of f with respect to x is noted :

$$\frac{\partial f}{\partial x}$$

Integral

Suppose $f(x)$ is continuous on an interval $[a, b]$. Divide $[a, b]$ into n subintervals of width Δx and for each subinterval i, choose x_i^*, a representative value of x in the subinterval i (for example, the average value of x_i). Then the (definite) **integral** of $f(x)$ with respect to x on the interval $[a, b]$ is defined as follows :

$$\int_a^b f(x)dx = \lim_{n \to +\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$



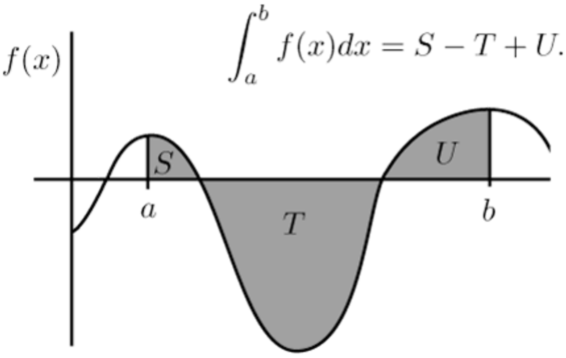
Notation

If $f(x)$ is continuous on $\mathbb{R}$ (the set of real numbers) then its integral with respect to x over $\mathbb{R}$ is noted :

$$\int f(x)dx = \int_{-\infty}^{+\infty} f(x)dx$$

Interpretation

The integral can be interpreted formally as the signed area of the region in the plane that is bounded by the graph of a given function f between two points a and b in the real line.



Multiple integral

A **multiple integral** is a definite integral of a function of several real variables, for instance, $f(x, y, z)$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ (the real-number plane) are called double integrals, and integrals of a function of three variables over a region in $\mathbb{R}^3$ (real-number 3D space) are called triple integrals.

Notation

The double integral of $f(x, y)$ with respect to variables x and y, over $x \in [x_1, x_2]$ and $y \in [y_1, y_2]$ is written :

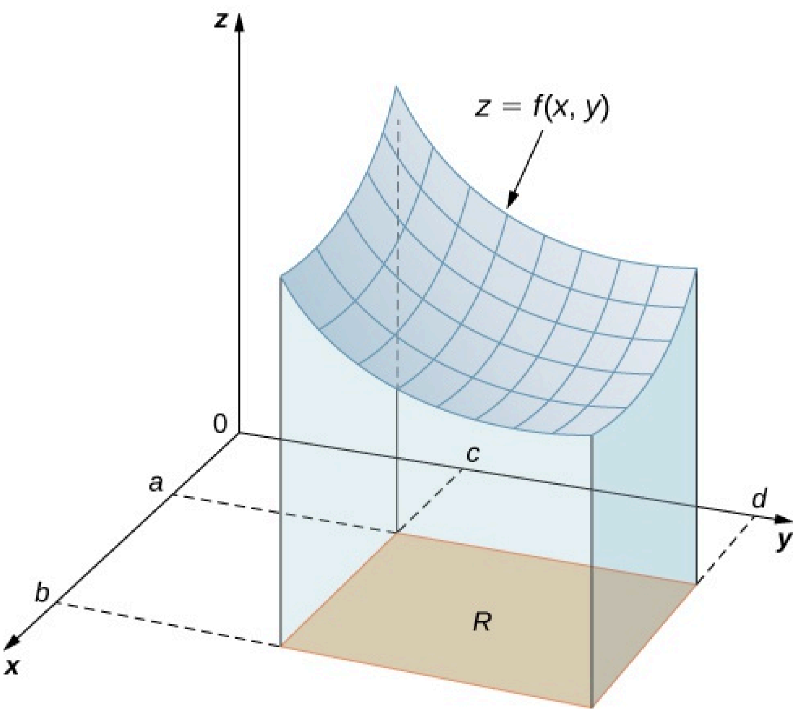
$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} f(x, y) dx dy$$

The triple integral of $f(x, y, z)$ with respect to variables x, y and z, over $x \in [x_1, x_2]$, $y \in [y_1, y_2]$ and $z \in [z_1, z_2]$ is written :

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z) dx dy dz$$

Interpretation

The double integral $\int_a^b \int_c^d f(x, y) dx dy$ can be interpreted as the volume under the 2D-curve of the function $z = f(x, y)$:



Common functions

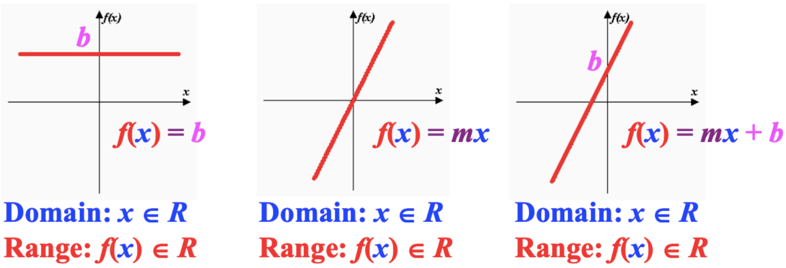
Linear functions

A linear function can be written as follows :

$$f(x) = mx + b$$

where :

- m is the slope of the line
- b is the y-intercept



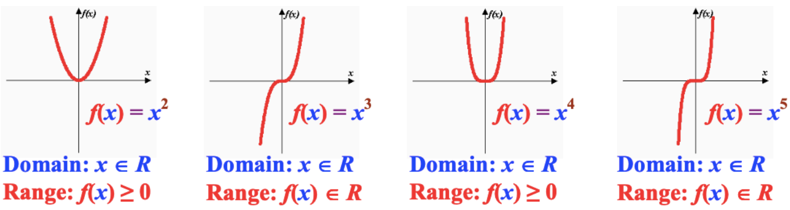
Power and root functions

Power function

A power function is written as follows :

$$f(x) = x^n$$

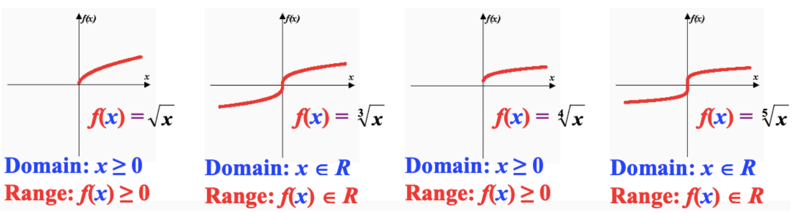
where $n \neq 1$ and $n \in \mathbb{N}$



Root function

A root function is a function defined for positive integer $n \neq 1$ and $n \in \mathbb{N}$:

$$f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$$



Identities

- $x^0 = 1$
- $x^{an} = (x^n)^a$
- $x^{n+m} = x^n x^m$
- $x^{n-m} = \frac{x^n}{x^m}$
- $x^{-n} = \frac{1}{x^n}$
- $(xy)^n = x^n y^n$
- $\sqrt[n]{x^n} = (\sqrt[n]{x})^n = x$
- $\sqrt[n]{ax} = \sqrt[n]{a}\sqrt[n]{x}$ (for a, x both positive real numbers)

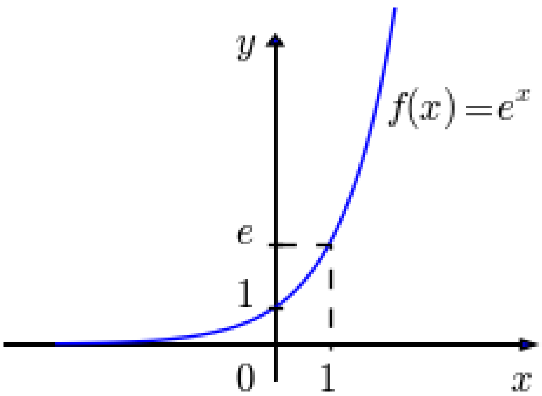
Exponential and logarithm

Exponential

The exponential function is the function that is equal to its derivative for each value of $x \in \mathbb{R}$:

$$f(x) = \exp(x) = e^x$$
$$f'(x) = \frac{df}{dx} = e^x$$

where $e \simeq 2.71828$ is Euler's constant.



Remark : The identities listed above for the power functions can also be applied to the case of the exponential function, by replacing x by e (for example : $e^{n+m} = e^n e^m$)

Natural logarithm

The natural logarithm (usually written $\ln$) is the inverse function to exponentiation. That means the logarithm of a given number x is the exponent to which e must be raised to produce that number x:

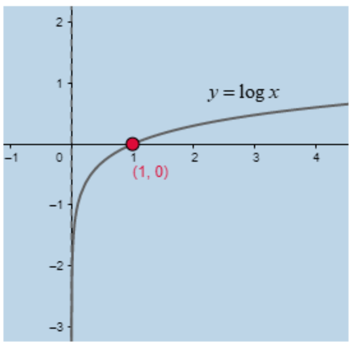
$$e^y = x$$
$$\Updownarrow$$
$$\ln(x) = y$$

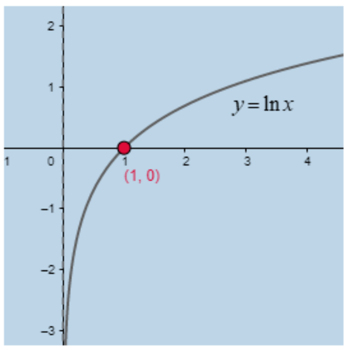
Common logarithm

The common logarithm (usually written $\log$) is the logarithm with base 10. It behaves like the natural logarithm, by replacing Euler's constant e by 10 :

$$10^y = x$$
$$\Updownarrow$$
$$\log(x) = y$$

Remark : the domain of $\ln(x)$ and $\log(x)$ is $\mathbb{R}^{+*}$ (real positive non-zero numbers)





Identities

Fundamental properties :

- $\ln(e^x) = x$
- $e^{\ln(x)} = x$ if x > 0
- $\log(10^x) = x$
- $10^{\log(x)} = x$ if x > 0
- $\log(10) = 1$
- $\ln(e) = 1$
- $\ln(1) = \log(1) = 0$

The following identities are true for both natural and common logarithms (you can replace $\ln$ by $\log$) :

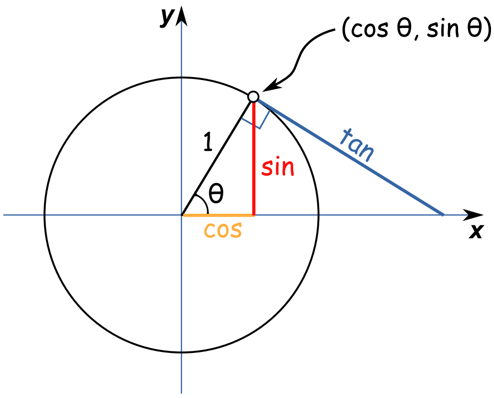
- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$

- $\ln(x^d) = d\ln(x)$
- $c\ln(x) + d\ln(y) = \ln(x^c y^d)$

Trigonometric functions

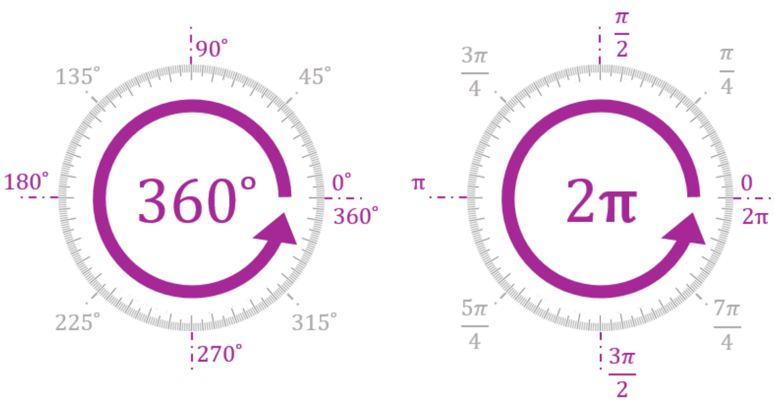
Definitions

The trigonometric functions $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$, can be defined thanks to the unit circle :



Angle measure

The measure of an angle can be expressed either in degrees or in radians : π rad = 180°



Common values

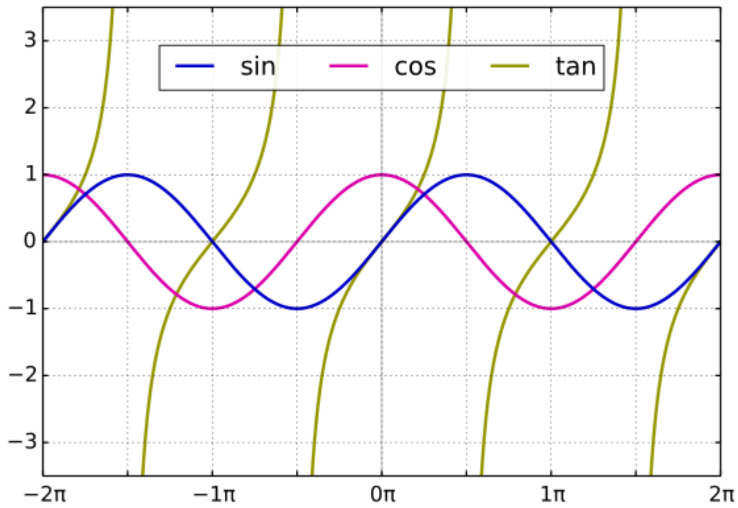
The table below shows some very common values of the trigonometric functions for different angles :

θ (°)	0	45	90	180	360
θ (rad)	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	2π
$\sin(\theta)$	0	$\frac{1}{\sqrt{2}}$	1	0	0
$\cos(\theta)$	1	$\frac{1}{\sqrt{2}}$	0	-1	1

θ (°)	0	45	90	180	360
$\tan(\theta)$	0	1	not defined	0	0

Graphical representation

The functions $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ are represented below, as a function of the angle measure expressed in radians :



Derivatives

Basic properties

If $f(x)$ and g(x) are differentiable functions (i.e. the derivative exists), c and n are any real numbers, then :

- $(cf(x))' = cf'(x)$
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(fg)' = f'g + fg'$ **Product rule :** $\mathbf{(uv)' = u'v + uv'}$
- $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$ **Quotient rule :** $\mathbf{(u/v)' = (u'v - uv')/v^2}$
- $\frac{d}{dx}(c) = 0$
- $\frac{d}{dx}(f(g(x))) = f'(g)g'(x)$ **Chain rule :** $\mathbf{\frac{d}{dx}u(v(x)) = \frac{du}{dv}\frac{dv}{dx}}$

Remark : all these properties are also true for partial derivatives 😉

Common derivatives

- $\frac{d}{dx}(x) = 1$
- $\frac{d}{dx}(\sin(x)) = \cos(x)$
- $\frac{d}{dx}(\cos(x)) = -\sin(x)$
- $\frac{d}{dx}(x^n) = nx^{n-1}$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
- $\frac{d}{dx}(\log(x)) = \frac{1}{x\ln(10)}$ (for $x > 0$)

Application : first derivative test

The first-derivative test examines a function's monotonic properties (where the function is increasing or decreasing), focusing on a particular point in its domain.

Monotonicity and sign of the derivative

Let $f(x)$ be a continuous function and differentiable at point x. Then :

- $f'(x) \geq 0 \iff f(x)$ is **increasing** at x
- $f'(x) \leq 0 \iff f(x)$ is **decreasing** at x

Let's use the properties above to find the local extrema (local maximum or minimum) of the function f :

If the function "switches" from increasing to decreasing at the point, then the function will achieve a highest value at that point. Similarly, if the function "switches" from decreasing to increasing at the point, then it will achieve a least value at that point. If the function fails to "switch" and remains increasing or remains decreasing, then no highest or least value is achieved.

Precise statement

- If $f(x)$ is increasing ($f'(x) > 0$) for all x in some interval $[a, x_0]$ and $f(x)$ is decreasing ($f'(x) < 0$) for all x in some interval $[x_0, b]$, then $f(x)$ has a local maximum at x_0.
- If $f(x)$ is decreasing ($f'(x) < 0$) for all x in some interval $[a, x0]$ and $f(x)$ is increasing ($f'(x) > 0$) for all x in

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some interval $[x0, b]$, then $f(x)$ has a local minimum at x_0.

Local maximum/minimum vs. global maximum/minimum

The function f is said to have a **local maximum** at the point x_0, if there exists some $\epsilon > 0$ such that $f(x_0) \geq f(x)$ for all x in the domain of f within distance ϵ of x_0. Similarly, the function has a **local minimum** at x_0, if $f(x_0) \leq f(x)$ for all x in the domain of f within distance ϵ of x_0.

The function f defined on a domain X has a **global maximum** at x_0, if $f(x_0) \geq f(x)$ for all x in its domain X. Similarly, the function has a **global minimum** at x_0, if $f(x_0) \leq f(x)$ for all x in its domain X.

From the definitions above, one can notice that a **local maximum/minimum** is not necessarily a **global maximum/minimum** ! (This has important consequences in machine learning 😉)