SQG BOUNDARY, May 6, 2019

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Contents

We're gonna consider the equation

$$\partial_t \theta + u \cdot \nabla\theta + \Lambda\theta = 0, u = \nabla^{\perp}\Lambda^{-1}\theta. \tag{1}$$

Here the operator

$$\Lambda := \sqrt{-\Delta_D}$$

where Δ_D is the Laplacian with Dirichlet boundary condition.

We're going to linearize the equation by fixing u independent of θ. What property do we want u to have? For some constant κ, we'll want

$$u = \sum_{j\in\mathbb{Z}} u_j,$$

$$\|u_j\|_\infty \le \kappa,$$

$$\left\|\Lambda^{-1/4}u_j\right\|_\infty \le \kappa 2^{-j/4},$$

$$\|\nabla u_j\|_\infty \le \kappa 2^j.$$

The convergence of that sum is in, say, weak L^2.

Recall the notation

$$[f]_\alpha := \sup_{x,y\in\Omega, x\ne y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

1. Lemmas

Lemma 1.1. *If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then*

$$\int \Lambda^s f \Lambda^s g\, dx \le 0.$$

This proves, in particular, that $-\int \theta_+\Lambda\theta_-$ is a positive term (hence dissipational and extraneous) and that $\int \Lambda^{1/2}(\theta-\psi)\Lambda^{1/2}(\theta-\psi)$ breaks down (bilinearly) into the doubly positive, the doubly negative, and the cross term, all of which are positive and hence each of which is bounded.

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Proof. Use the characterization from Caffarelli-Stinga. There exist non-negative functions $K(x,y)$ and $B(x)$, depending on the parameter s, such that

$$\int \Lambda^s f \Lambda^s g\, dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\, dxdy + \int f(x)g(x)B(x)\, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \le 0.$$

Since K is non-negative, the result follows. □

Lemma 1.2. *For all functions f in H^1_D,*

$$\int |\nabla f|^2 = \int |\Lambda f|^2 .$$

Moreover, if $f \in H^1_D$ then $\mathrm{tr}(f) = 0$.

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω. Because Ω has Lipschitz boundary, and because $\eta_i \nabla \eta_j$ is smooth on Ω and countinuous and bounded on $\overline{\Omega}$ vanishing on the boundary, therefore

$$\int_\Omega \mathrm{div}(\eta_i \nabla \eta_j) = \int_{\partial\Omega} \eta_i \nabla \eta_j.$$

But $\eta_i \nabla \eta_j$ vanishes on the boundary, so the right hand side vanishes. Moreover, $\mathrm{div}(\eta_i\nabla\eta_j) = \nabla\eta_i \cdot \nabla\eta_j + \eta_i \Delta \eta_j$. Therefore

$$\int \nabla\eta_i \cdot \nabla\eta_j = -\int \eta_i \Delta\eta_j = \lambda_k \int \eta_i\eta_j.$$

Of course, the inner product of two eigenfunctions is 0 unless they are the same eigenfunction, in which case it is 1.

Consider a function $f = \sum f_k \eta_k$ which is an element of H^1_D, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned}
\int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla \eta_i\right)\left(\sum_j f_j \nabla\eta_j\right) \\
&= \int \sum_{i,j} (f_i f_j) \nabla\eta_i \cdot \nabla\eta_j \\
&= \sum_{i,j} (f_i f_j) \int \nabla\eta_i \cdot \nabla\eta_j.
\end{aligned}$$

Since this double-sum vanishes except on the diagonal, we see from [citation] that in fact

$$\|\nabla f\|_{L^2(\Omega)} = \|\Lambda f\|_{L^2(\Omega)} .$$

To see that $\mathrm{tr}(f)$ vanishes, note that $f = \sum_{k=0}^\infty f_k \eta_k$ and that each finite partial sum for this series satisfies the Dirichlet boundary condition. Since tr is a bounded operator on H^1, we need only show that this series is Cauchy in H^1, in which case its H^1 limit will exist and be equal to its L^2 limit which will be equal to f.

For each k,

$$\|f_k\eta_k\|_{H^1} \le C_{\text{Poincare}} f_k \|\nabla \eta_k\|_2 = C f_k \sqrt{\lambda_k}.$$

This sequence is ℓ^2 summable, since $f \in H^1_D$ by assumption. Therefore f, being an H^1 limit of functions with vanishing trace, also has vanishing trace. □

Lemma 1.3. *For any function f, and any $0 < s < 1$,*

$$\int |\Lambda^s f|^2 \simeq \int \left|(-\Delta)^{s/2} \bar{f}\right|^2 .$$

Here $\bar{f}$ is the extension of f to $\mathbb{R}^2$ and $(-\Delta)^s$ is defined in the fourier sense.

Proof. Let g be any Schwarz function in $L^2(\mathbb{R}^2)$, and let f be a function in H_D^{s+1}. Let $E : H^1(\Omega) \to H^1(\mathbb{R}^2)$ be a bounded extension operator, where H^1 denotes the classical Sobolev space defined using the gradient. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f.$$

When $\Re(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|\Lambda^s f\|_2$. Hence

$$\Phi(z) \le \|g\|_2 \|f\|_{H_D^s} .$$

When $\Re(z) = 1$, then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$ and

$$\left\|(-\Delta)^{1/2} E \Lambda^{s-z} f\right\|_{L^2(\mathbb{R}^2)} = \|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} \le \|E\| \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} .$$

It remains to ask whether $\Lambda^{s-z} f$ is in H_D^1 so that we can apply lemma [citation]. However, this is true based on our assumption $f \in H_D^{1+s}$, since the various powers of Λ all commute and form a semigroup. Ergo

$$\|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|\Lambda \Lambda^{s-z} f\|_2 \le \|\Lambda^s f\|_2$$

and we can bound

$$\Phi(z) \le \|E\| \|g\|_2 \|f\|_{H_D^s} .$$

Now we will bound the derivative of $\Phi(z)$. Specifically, compute the derivative in z of the integrand, for $0 < \Re(z) < 1$, and hope that it is integrable. To this end, we rewrite the integrand of Φ as

$$\mathcal{F}^{-1} \left(|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1} \left(\ln(|\xi|)|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1} \left(|\xi|^z \hat{g}\right) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k.$$

Since $0 < \Re(z) < 1$, $\ln(|\xi|)|\xi|$ is bounded as a multiplier operator from Schwarz functions to L^2. Moreover, $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C \lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Since $f \in H_D^{1+s}$ this sum converges in L^2, in fact in H_D^1. This makes our differentiated integrand a sum of two H^1 functions with compact support multiplied by two Schwarz functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \Re(z) < 1$.

This is sufficient now to apply the Hadamard three-lines lemma to our function Φ.

It follows that for any Schwarz function $g \in L^2(\mathbb{R}^n)$ and H_D^{s+1} function f,

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} g E f = \Phi(s) \le \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{H_D^s} .$$

Since Schwarz functions are dense in $L^2(\mathbb{R}^2)$, this means by density that

$$\int \left|(-\Delta)^{s/2} E f\right|^2 \le \int |\Lambda^s f|^2$$

or in other words it means that E is a bounded operator from H_D^s to H^s, at least on the subset $H_D^{s+1} \cap H_D^s$. It remains to extend this bound to the whole space by density.

We know from [citation] Caffarelli and Stinga that $\mathcal{D}(\Omega)$ is dense in H^s_D for all $0 \le s < 1$. In fact, this takes a bit of interpretation, so I ought to illucidate that this is because $H^s_D = H^s_0$ (the latter in the Slobodekij sense) for most s and at $s = 1/2$ we get the Lions-Magenes spaces which still has $\mathcal{D}(\Omega)$ dense.

Surely, right(?), test functions are all inside of H^{1+s}_D. I should meditate on this, but it must be true. □

2. Littlewood-Paley Theory

Logan: The japanese paper's Lemma 3.6, used extensively here, only applies in the case $j \ge 0$. Obviously I need it and use it for $j > j_0$. This is equivalent, I can see from the proof, but maybe mention the issue somewhere so it doesn't seem like I didn't notice.

In this section we will prove that u breaks up into pieces with various norms under control.

We know that $\theta \in L^\infty$. Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. That is, for example, $\phi(2^j x)\phi(2^i x) = 0$ unless $|i - j| \le 1$ and $\sum \phi(2^j) = 1$. We have some projections

$$P_j f := \sum_k \phi(2^j \lambda_k^{1/2}) f_k \eta_k.$$

Recall that $P_j = 0$ for j sufficiently small, because $-\Delta_D$ has a smallest eigenvalue.

For each $j \in \mathbb{Z}$, I'll define

$$u_j := \nabla^\perp \Lambda^{-1} P_j \theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

Firstly, we know by [citation] Fornare, Metafune and Priola that if Ω is $C^{2,\alpha}$ then

$$\left\| \nabla e^{-t\Delta_D} \right\|_{L^\infty \to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le 1.$$

According to [citation] Iwabuchi, Matsuyama, and Tanaguchi's paper Bilinear Estimates, Lemma 3.6, this is enough to show that

$$\|u_j\|_\infty \le C \|\theta\|_\infty .$$

We'll need a lemma now,

Lemma 2.1. *For any function f,*

$$\|P_i \nabla P_j f\|_\infty \le C \min(2^j, 2^i) \|f\|_\infty .$$

Proof. Let g be an L^1 function. Then

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \le C 2^j \|g\|_1 \|f\|_\infty$$

from [citation] IMT-Bilinear, Lemma 3.6 and Proposition 3.3 (which is also IMT Boundedness of Spectral Multiplies for Schrodinger Operators on Open Sets, Theorem 1.1).

Further integrating by parts,

$$\int g P_i \nabla P_j f = -\int (\nabla P_i g) P_j f \le C 2^i \|g\|_1 \|f\|_\infty .$$

This follows from the same theorems as above.

The result follows. □

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4} u_j = \sum_{i \in \mathbb{Z}} P_i \Lambda^{-1/4} u_j .$$

Define $\bar{P}_j$ a projection which is 1 on the support of P_j (functional calculus-wise). Then $\bar{P}_j P_j = P_j$, and since both types of projections are spectral operators, they both commute with Λ^s. We therefore rewrite

$$\left(P_i \Lambda^{-1/4} u_j\right)^{\perp} = \left(\Lambda^{-1/4} \bar{P}_i\right) P_i \nabla P_j \left(\Lambda^{-1} \bar{P}_j\right) \theta.$$

We apply sequentially three bounded operators on L^∞. The outer two operators have bounded norm by [citation] IMT-Bilinear Proposition 3.3, and the inner operator has bounded norm by the above lemma, (and of course the perp operator is an isometry,) so

$$\left\| P_i \Lambda^{-1/4} u_j \right\|_\infty \le C 2^{-i/4} \min(2^j, 2^i) 2^{-j} \left\|\theta\right\|_\infty .$$

Summing these bounds on the projections of $\Lambda^{-1/4} u_j$, and noting that

$$\sum_{i \in \mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i \le j} 2^{i3/4} + \sum_{i > j} 2^{-i/4} \le C 2^{-j/4},$$

we obtain

$$\left\| \Lambda^{-1/4} u_j \right\|_\infty \le C 2^{-j/4} \left\|\theta\right\|_\infty .$$

Lastly, we'll show that ∇u_j is in L^∞. Equivalently, we'll show that $\Lambda^{-1} P_j \theta$ is $C^{1,1}$. This is essentially Schauder theory. We will obtain our $C^{1,1}$ bound by interpolating between a $C^{0,1}$ bound and a $C^{2,\alpha}$ bound. We could also obtain a $C^{1,\alpha}$ bound directly using the main theorem of [citation] Caffarelli-Stinga, but those estimates are not well-articulated in the specific context of our problem (namely, it's hard to make good use of the fact that f near the boundary). So instead, we use interpolation.

The $C^{0,1}$ bound is already known, it's the estimate

$$\left\| \nabla \Lambda^{-1} P_j \theta \right\|_\infty \le C \left\|\theta\right\|_\infty .$$

The $C^{2,\alpha}$ bound is classical Schauder theory. For convenience, define

$$F := \Lambda^{-1} P_j \theta$$

and recall that F is a finite linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Moreover, its Laplacian is

$$f := \Delta F = \Lambda P_j \theta$$

which is also smooth, vanishes at the boundary, and has various bounds. Specifically, we want to apply Theorem 6.6 from [citation] Gilbarg and Trudinger, page 98 in my library copy. It says that since Ω is $C^{2,\alpha}$ and $F \in C^{2,\alpha}(\bar{\Omega})$, and since $f \in C^\alpha(\bar{\Omega})$, and since the boundary conditions are homogeneous (hence smooth), then

$$\sup_{x,y \in \Omega} \frac{\left|D^2 F(x) - D^2 F(y)\right|}{|x-y|^\alpha} \le C \left\|F\right\|_\infty + C \left\|f\right\|_\infty + C \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

A lemma with two interpolations:

Lemma 2.2. *If $f \in L^\infty(\Omega) \cap C^{0,1}(\Omega)$ then for some universal constant C,*

$$[f]_\alpha \le C \left\|f\right\|_\infty^{1-\alpha} \left\|\nabla f\right\|_\infty^\alpha .$$

If $f \in C^{0,1}(\Omega) \cap C^{2,\alpha}(\Omega)$ where Ω satisfies the cone condition, then for some constants C and ℓ depending on Ω,

$$\left\|D^2 f\right\|_\infty \le C \delta^{-1} \left\|\nabla f\right\|_\infty + \delta^\alpha \left[D^2 f\right]_\alpha$$

for all $\delta < \ell$.

Proof. The first claim is incredibly straigtforward. We include it for completeness.

$$\begin{aligned}\sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{|x-y|^\alpha} &= \sup|f(x)-f(y)|^{1-\alpha}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha\\ &\le (2\|f\|_\infty)^{1-\alpha}\left(\sup\frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha\\ &\le C\|f\|_\infty^{1-\alpha}\|\nabla f\|_\infty^\alpha.\end{aligned}$$

The second claim is more complicated. We'll prove the equivalent claim that for f smooth,

$$\|\nabla f\|_\infty \le C\delta^{-1}\|f\|_{L^\infty(\bar\Omega)} + \delta^\alpha[\nabla f]_{\alpha;\bar\Omega}.$$

Since Ω satisfies the cone condition, we know that there exist positive constants ℓ and a such that, at each point $x\in\bar\Omega$, there exist two unit vectors e_1 and e_2 such that $e_1\cdot e_2\le a$ and $x+\tau e_i\in\Omega$ for $i=1,2$, $0<\tau\le\ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

The idea of the proof is that the average of ∇f along an interval is bounded since f is bounded, and the same average is close to the value of ∇f at a point because ∇f is continuous, hence the value of ∇f at any point must be bounded. By varying the length δ of the aforementioned interval, we actually get a parameterized family of bounds.

If we consider the directional derivative $\partial_i f$ of f along the direction e_i, then observe that for any $0<\delta\le\ell$,

$$\int_0^\delta \partial_i f(x+\tau e_i)\,d\tau = f(x+\delta e_i) - f(x).$$

This quantity on the right is bounded by the L^∞ norm of f.

On the other hand, since ∇f and hence $\partial_i f$ are continous functions, for any $\tau\in(0,\ell]$

$$|\partial_i f(x) - \partial_i f(x+\tau e_i)| \le [\nabla f]_\alpha\,\tau^\alpha.$$

From this bound, we obtain that

$$\int_0^\delta \partial_i f(x+\tau e_i)\,d\tau \le \int_0^\delta \partial_i f(x) + [\nabla f]_\alpha\,\tau^\alpha\,d\tau = \delta\partial_i f(x) + [\nabla f]_\alpha\frac{\delta^{1+\alpha}}{1+\alpha}$$

and similarly from below, so

$$\delta\partial_i f(x) - [\nabla f]_\alpha\frac{\delta^{1+\alpha}}{1+\alpha} \le \int_0^\delta \partial_i f(x+\tau e_i)\,d\tau \le \delta\partial_i f(x) + [\nabla f]_\alpha\frac{\delta^{1+\alpha}}{1+\alpha}.$$

What we have shown is that the integral of $\partial_i f$ over an interval of length δ is small, and also it differs not very much from $\delta\partial_i f(x)$. By rearranging, we find that $\partial_i f(x)$ must therefore be small:

$$|\partial_i f(x)| \le \frac{2}{\delta}\|f\|_\infty + \frac{\delta^\alpha}{1+\alpha}[\nabla f]_\alpha.$$

This is true independent of x and of $i=1,2$. Since $e_1\cdot e_2\le a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta\in(0,\ell]$,

$$\|\nabla f\|_\infty \le \frac{C}{1-a^2}\left(\delta^{-1}\|f\|_\infty + \delta^\alpha[\nabla f]_\alpha\right).$$

□

Let's bound the terms of the Gilbarg-Trudinger inequality. By [citation] IMT-Bilinear Proposition 3.3

$$\|f\|_\infty = \|\Lambda P_j\theta\|_\infty \le C2^j\|\theta\|_\infty$$

while by [citation] IMT-Bilinear Lemma 3.6

$$\|\nabla f\|_\infty = \|\nabla\Lambda P_j\theta\|_\infty \le C2^{2j}\|\theta\|_\infty.$$

Therefore we can interpolate

$$[f]_\alpha \leq C 2^{j(1+\alpha)} \|\theta\|_\infty .$$

And of course, by [citation] IMT-Bilinear Proposition 3.3

$$\|F\|_\infty = \left\|\Lambda^{-1} P_j \theta\right\|_\infty \leq C 2^{-j} \|\theta\|_\infty .$$

Combining these estimates with [citation: the Gilbarg-Trudinger thingy] we find

$$\left[D^2 F\right]_\alpha \leq C\left(2^{-j} + 2^j + 2^{j(1+\alpha)}\right) \|\theta\|_\infty .$$

At long last, the big estimate,

$$\left\|D^2 F\right\|_\infty \leq C\left(\delta^{-1} \|\nabla F\|_\infty + \delta^\alpha \left[D^2 F\right]_\alpha\right).$$

Since Ω is bounded, there exists a j_0 such that $P_j = 0$ if $j < j_0$. Therefore we assume withouth loss of generality that $j \geq j_0$. Thus $2^{-j} \leq 2^{j(1+\alpha)} 2^{-j(2+\alpha)} \leq 2^{j(1+\alpha)} 2^{-j_0(2+\alpha)}$ and similarly $2^j \leq 2^{j(1+\alpha)} 2^{-j\alpha} \leq 2^{j(1+\alpha)} 2^{-j_0\alpha}$. We can therefore say that for all $\delta \leq \ell$,

$$\left[D^2 F\right]_\infty \leq C\left(\delta^{-1} C + \delta^\alpha 2^{j(1+\alpha)}\right) \|\theta\|_\infty .$$

Set $\delta = 2^{-j} 2^{j_0} \ell \leq \ell$. Then

$$\left[D^2 F\right]_\infty \leq C\left(C 2^j + 2^{-j\alpha} 2^{j(1+\alpha)}\right) \|\theta\| = C(\Omega) 2^j \|\theta\| .$$

But $D^2 F = \nabla u_j$ so we are done.

3. Control on u

We've shown that our drift term u is a sum of u_j for $j \in \mathbb{Z}$, $j \geq j_0$. (Equivalently, u is a sum of u_j for $j \in \mathbb{Z}$ and $u_j = 0$ for $j < j_0$.) Each u_j is an L^∞ function, and their sum converges in L^2 to u. Each u_j satisfies a collection of bounds which are exponential in j.

In this section, we will first show that these terms sum to two functions u_l and u_h with appropriate bounds. Then we will show that these bounds remain true as we zoom in space and time.

We begin by stating what we mean by "appropriate" bounds on the u_j.

Definition 1 (Calibrated sequence)**.** We call a sequence u_j **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\begin{aligned}
\|u_j\|_\infty &\leq \kappa, \\
\|\nabla u_j\|_\infty &\leq 2^j 2^{-N} \kappa, \\
[u_j]_{3/4} &\leq 2^{j\frac{3}{4}} 2^{-N\frac{3}{4}} \kappa, \\
\left\|\Lambda^{-1/4} u_j\right\|_\infty &\leq 2^{-j/4} 2^{N/4} \kappa.
\end{aligned}$$

The most important property of a calibrated sequence is that its sum decomposes into two functions, which we call the high-pass term and the low-pass term.

Proposition 3.1. *Let*

$$u = \sum_{j_0}^\infty u_j$$

with the sum converging the the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center.

Then there exist some universal constants C_i such that

$$u = u_l + u_h$$

with

$$\|u_l\|_{Lip} \leq C_1 \kappa$$

and

$$[u_l]_{3/4} \leq C_2 \kappa$$

and

$$\left\|\Lambda^{-1/4} u_h\right\|_\infty \leq C_3 \kappa.$$

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j>N} u_j$$

and

$$u_l = \sum_{j\leq N} u_j.$$

Since $u_j \in L^\infty$ in particular they are L^2 functions which sum in L^2. Remember that only finitely many negative j have $u_j \neq 0$. The sequence u_j is thus singly infinite and in particular is a Cauchy sequence, so u_h also converges in L^2. Since $\Lambda^{-1/4}$ is a continuous linear operator, it passes to the partial sums and so

$$\Lambda^{-1/4} u_h = \lim_{L^2} \sum_{j>N} \Lambda^{-1/4} u_j.$$

In particular, the sum converges in the sense of distributions, i.e. in $\mathcal{D}(\Omega)'$. Since test functions are dense in $L^1(\Omega)$, and the partial sums are uniformly bounded in the dual of $L^1(\Omega)$ (namely $L^\infty(\Omega)$), therefore the limit $\Lambda^{-1/4} u_h$ is also bounded in the dual of $L^1(\Omega)$.

$$\left\|\Lambda^{-1/4} u_h\right\|_\infty \leq \sum_{j>N} \left\|\Lambda^{-1/4} u_j\right\|_\infty \leq \kappa \frac{2^{-1/4}}{1 - 2^{-1/4}}.$$

As for u_l, we have that $\sum_{j\leq N} u_j$ is a finite sum of Lipschitz and Hölder continuous functions. We can simply bound

$$\|\nabla u_l\|_\infty \leq \sum_{j\leq N} \|\nabla u_j\|_\infty \leq \kappa \frac{1}{1 - 2^{-1}}$$

and

$$[u_l]_{3/4} \leq \sum_{j\leq N} [u_j]_{3/4} \leq \kappa \frac{1}{1 - 2^{-3/4}}.$$

□

We showed in section 2 that u is a sum of a calibrated sequence, and now we have shown that the sum of a calibrated sequence is actually a finite sum of functions that are bounded in certain function spaces. Any bound we place on u directly will blow up as we zoom in, but a calibrated sequence remains calibrated (with increasing center). In the next lemma, we show that, thanks to this notion of calibration, our PDE is scale-invariant.

Lemma 3.2 (Scaling). *Suppose that θ and u solve the PDE*

$$[\partial_t + u \cdot \nabla + \Lambda]\, \theta = 0,$$

$$\operatorname{div} u = 0,$$

where the velocity u satisfies

$$u = \sum_{j=j_0}^{\infty} u_j$$

with that sum converging in $L^2(\Omega)$ and $(u_j)_j$ calibrated with constant κ and center N. Suppose that the domain of definition is $(-T, 0) \times \Omega$.

Let $\varepsilon > 0$ be a small constant.

Then

$$\bar{\theta}(t,x) := \theta(\varepsilon t, \varepsilon x)$$

and

$$\bar{u}(t,x) := \sum_{j=j_0}^{\infty} u_j(\varepsilon t, \varepsilon x)$$

satisfies the same PDE for $(t,x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$.

Moreover, $(u_j)_j$ is calibrated with the same constant κ and center $N - \ln_2(\varepsilon)$.

Proof. We calculate

$$\partial_t \bar{\theta}(p) = \varepsilon \partial_t \theta(\bar{p})$$

and

$$\nabla \bar{\theta}(p) = \varepsilon \nabla \theta(\bar{p})$$

and

$$\Lambda \bar{\theta}(p) = \varepsilon \Lambda \theta(\bar{p}).$$

& cetera...

It remains to show that $(u_j(eps\,\cdot, \varepsilon\,\cdot))_j$ is still calibrated. Define

$$\bar{u}_j(t,x) := u_j(\varepsilon t, \varepsilon x).$$

Then

$$\|\bar{u}_j\|_\infty = \|u_j\|_\infty \le \kappa$$

and

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \le 2^{\ln_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N - \ln_2(\varepsilon))} \kappa.$$

The entire thing is so straightforward I literally can't bring myself to type out the rest.

□

4. De Giorgi Estimates

First let us derive an energy inequality.

Lemma 4.1 (Caccioppoli Estimate). *Let $\theta \in L^2(0,T; H_D^{1/2}(\Omega))$ and $u \in L^\infty(0,T; L^2(\Omega))$ solve*

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0,$$
$$\operatorname{div} u = 0$$

in the sense of distributions. Let $\psi \in L^\infty(0,T; H_D^{1/2}(\Omega))$ be non-negative, Lipschitz-in-space, and Hölder continuous-in-space with exponent $\gamma < 1/2$. Then the decomposition

$$\theta = \theta_+ + \psi - \theta_-$$

satisfies the inequality

$$\frac{d}{dt} \int \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \le C \left(\int \chi_{\{\theta_+ > 0\}} + \int \theta_+ (\partial_t \psi + u \cdot \nabla \psi) \right)$$

with the constant C depending on $\|\nabla \psi\|_\infty$ and $\sup_t [\psi(t,\cdot)]_\gamma$.

Proof. We decompose $\theta - \psi = \theta_+ - \theta_-$ with both θ_+ and θ_- non-negative and having disjoint support. Since at a.e. time t, θ and ψ are both in $H_D^{1/2}$, we can write

$$\infty > \int \left| \Lambda^{1/2} (\theta_+ - \theta_-) \right|^2 = \int \left| \Lambda^{1/2} \theta_+ \right|^2 + \int \left| \Lambda^{1/2} \theta_- \right|^2 - \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_-.$$

By a lemma proved above, the final term (by disjoint support) is non-positive, hence all three terms are finite and in particular θ_+ and θ_- are in $H_D^{1/2}$ at a.e. time.

We can therefore multiply our equation [cite] by θ_+ and integrate in space to obtain

$$0 = \int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right] (\theta_+ + \psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right] \theta_+ &= (1/2)\frac{d}{dt}\int \theta_+^2 + (1/2)\int \operatorname{div} u\, \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \\ &= (1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The ψ term produces important error terms:

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right] \psi &= \int \theta_+ \partial_t \psi + \int \theta_+ u \cdot \nabla\psi + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi \\ &= \int \theta_+ (\partial_t\psi + u\cdot\nabla\psi) + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma 1.1:

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right] \theta_- &= (1/2)\int \theta_+\partial_t\theta_- + \int \theta_+ u\cdot\nabla\theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \\ &= \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le 0.\end{aligned}$$

Put together, we arrive at

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 + \iint \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi - \langle\theta_+,\theta_-\rangle_{1/2} \le \left|\int \theta_+(\partial_t\psi + u\cdot\nabla\psi)\cdot\nabla\psi\right|.$$

At this point we break down the $\Lambda^{1/2}\theta_+\Lambda^{1/2}\psi$ term using the formula from [citation] Caffarelli-Stinga.

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi = \iint [\theta_+(x) - \theta_+(y)][\psi(x)-\psi(y)]K(x,y) + \int \theta_+\psi B.$$

Since $B \ge 0$ (see Caff-Stinga [citation]) and ψ is non-negative by assumption, the B term is non-negative and so

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi \ge \iint [\theta_+(x) - \theta_+(y)][\psi(x)-\psi(y)]K(x,y).$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\iint [\theta_+(x)-\theta_+(y)][\psi(x)-\psi(y)]K(x,y) \le 2\iint \chi_{\{\theta_+>0\}}(x)\left|[\theta_+(x)-\theta_+(y)][\psi_t(x)-\psi_t(y)]\right| K(x,y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\left|\iint [\theta_+(x)-\theta_+(y)][\psi(x)-\psi(y)]K(x,y)\right| \le \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2 + \frac{1}{\varepsilon}\iint \chi_{\{\theta_+>0\}}(x)[\psi(x)-\psi(y)]^2K(x,y).$$

It remains to bound the quantity $[\psi(x)-\psi(y)]^2K(x,y)$. By Caffarelli-Stinga theorem 2.4 [citation], there is a universal constant C such that

$$K(x,y) \le \frac{C}{|x-y|^3}.$$

The cutoff ψ is Lipschitz, and Hölder continuous with exponent $\gamma < 1/2$ by assumption. Therefore

$$[\psi(x)-\psi(y)]^2K(x,y) \le |x-y|^{-1} \wedge |x-y|^{2\gamma-3}.$$

Since $3-2\gamma > 2$, this quantity is integrable. Thus

$$\int \chi_{\{\theta_+>0\}}(x)\int [\psi(x)-\psi(y)]^2K(x,y)\,dxdy \le C(\|\psi\|_{\mathrm{Lip}}, [\psi]_\gamma)\int \chi_{\{\theta_+>0\}}\,dx.$$

Combining [citation, like 4 different things are combined] we arrive at

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+,\theta_-\rangle_{1/2} \lesssim \int \theta_+(\partial_t\psi + u\cdot\nabla\psi) + \int \chi_{\{\theta_+>0\}}.$$

□

We've completed the essential version of the Caccioppoli estimate. However, much more can be said about the drift-term u. In particular, we can design a cutoff ψ in order to minimize the expression $\partial_t\psi + u\cdot\nabla\psi$.

Begin with a lemma, a product rule for Λ.

Lemma 4.2. *For two functions $f \geq 0$ with compact support and $g \in L^\infty \cap Lip$, their product is bounded*

$$\left\|\Lambda^{1/4} fg\right\|_{L^1(\Omega)} \leq C\left(\|g\|_\infty + \|\nabla g\|_\infty\right)\left(\|f\|_1 + |\operatorname{supp}(f)|^{1/2}\left(\|f\|_{L^2} + \|f\|_{H_D^{1/2}}\right)\right).$$

The constant C depends on Ω, but is independent of scaling.

Logan: you don't actually prove L^1 because you don't know the quantity is a function. As proven, it could contain a dirac or a Banach limit.

Proof. Consider some L^∞ test function h. To determine the L^1 norm of $\Lambda^{1/4}fg$ we consider the integral

$$\int h\Lambda^{1/4}(fg) = \iint [h(x)-h(y)][f(x)g(x)-f(y)g(y)]K_{1/4}\,dxdy + \int hfgB_{1/4}\,dx.$$

This breaks up, according to the formula of Caffarelli-Stinga [citation], into a K-term and a B-term. We consider first the K term.

Decompose this into the far part $|x-y|>1$ whose integral we call I and the far part $|x-y|\leq 1$ whose integral we call II. For the far part,

$$\begin{aligned} I &\leq 2\|h\|_\infty \iint_{|x-y|>1} |f(x)g(x)-f(y)g(y)|\,K_{1/4}\,dxdy \\ &\leq 4\|h\|_\infty \int fg\int_{|x-y|>1}\frac{C_{1/4}dy}{|x-y|^{2.25}}\,dx \\ &\leq C\|h\|_\infty\|g\|_\infty\|f\|_1. \end{aligned}$$

For the near part, recall first that by the upper- and lower-bounds on K in Caffarelli-Stinga [citation], we know that

$$K_{1/4}(x,y) \leq C(\Omega)|x-y|^{3/4}K_1(x,y).$$

In principle the constant relating these quantities may depend on Ω, though since both sides of the inequality scale the same way, the constant must be the same for scalings of Ω.

Since the integrand of II vanishes unless at least one of x or y is in the support of θ_+, we can say

$$II \leq 2\iint_{|x-y|\leq 1}\chi_{\{f\neq 0\}}(x)[h(x)-h(y)][f(x)g(x)-f(y)g(y)].$$

Therefore, applying Hölder's inequality,

$$\begin{aligned} II &\leq C\left(\iint_{\leq 1}\chi_{\{f>0\}}(x)[h(x)-h(y)]^2|x-y|^{3/4}K_{1/4}\right)^{1/2}\left(\iint_{\leq 1}[f(x)g(x)-f(y)g(y)]^2K_1\right)^{1/2} \\ &\leq \|h\|_\infty\left(\int_{|y|\leq 1}|y|^{-1.5}\,dy|\operatorname{supp}(f)|\right)^{1/2}\left(\iint [f(x)g(x)-f(y)g(y)]^2K_1\,dxdy\right)^{1/2} \\ &= C\|h\|_\infty|\operatorname{supp}(f)|^{1/2}\left(\iint [f(x)g(x)-f(y)g(y)]^2K_1\,dxdy\right)^{1/2}. \end{aligned}$$

For this final term,

$$\iint [f(x)g(x) - f(y)g(y)]^2 K_1 \le \iint f(x)^2 [g(x) - g(y)]^2 K_1 + \iint g(y)^2 [f(x) - f(y)]^2 K_1$$

$$\le C \int f(x)^2 \int \frac{\|g\|_{\mathrm{Lip}}^2 |x-y|^2 \wedge 2\,\|g\|_\infty^2}{|x-y|^3}\, dxdy + \|g\|_\infty^2 \iint [f(x) - f(y)]^2 K_1 \, dxdy$$

$$\le C(\|g\|_{\mathrm{Lip}}^2 + \|g\|_\infty^2)\, \|f\|_2^2 + \|g\|_\infty^2 \, \|f\|_{H_D^{1/2}}^2 .$$

Therefore

$$K - \text{term} \lesssim \|h\|_\infty \left(\|g\|_\infty + \|g\|_{\mathrm{Lip}}\right) \left(\|f\|_1 + |\operatorname{supp}(f)|^{1/2} \|f\|_2 + |\operatorname{supp}(f)|^{1/2} \left\|\Lambda^{1/2} f\right\|_2\right).$$

For the B part, on the other hand,

$$\int hfgB_{1/4} \le \|h\|_\infty \|g\|_\infty \int fB_{1/4}$$

and, since $[f(x) - f(y)][\chi_{\{f\neq 0\}}(x) - \chi_{\{f\neq 0\}}(y)] \ge 0$,

$$\int fB_{1/4} \le \int \chi_{\{f>0\}} \Lambda^{1/4} f \le |\operatorname{supp}(f)|^{1/2} \left(\int \left|\Lambda^{1/4} f\right|^2\right)^{1/2}.$$

Now that we have an L^2 norm of $\Lambda^{1/4} f$, since the H_D^s norm is simply an ℓ^2 norm of the eigndecomposition and $\lambda_k^{1/4} \le 1 + \lambda_k^{1/2}$, we can easily bound this $H_D^{1/4}$ norm by interpolation and find

$$B - \text{term} \lesssim \|h\|_\infty \|g\|_\infty |\operatorname{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{H_D^{1/2}}\right).$$

From [cite] and [cite] we find that for any $\varepsilon > 0$ arbitrarily small and some constant C depending on ε, $\|g\|_\infty$, $\|g\|_{\mathrm{Lip}}$, and on the shape (but not the size) of Ω,

$$\int h\Lambda^{1/4}(fg)\, dx \le C \|h\|_\infty \left(\|g\|_\infty + \|g\|_{\mathrm{Lip}}\right) \left(\|f\|_1 + |\operatorname{supp}(f)|^{1/2} \|f\|_2 + |\operatorname{supp}(f)|^{1/2} \left\|\Lambda^{1/2} f\right\|_2\right).$$

□

Lemma 4.3 (Energy inequality)**.** *Let $\theta \in L^2(0,T; H_D^{1/2}(\Omega))$ and $u \in L^\infty(0,T; L^2(\Omega))$ solve*

$$\partial_t \theta + u \cdot \nabla\theta + \Lambda\theta = 0,$$
$$\operatorname{div} u = 0$$

in the sense of distributions. Let

$$u = u_l + u_h$$

where $\Lambda^{-1/4} u_h \in L^\infty(0,T; L^\infty(\Omega))$ and $u_l \in L^\infty(0,T; Lip(\Omega)) \cap L^\infty(0,T; \dot{C}^{3/4}(\Omega))$. Suppose that $\Gamma, \gamma \in Lip(0,T)$ satisfy $\|\dot\gamma\|_\infty \le C_\gamma$ and

$$\dot\Gamma(t) + \dot\gamma(t) = u_l(t, \Gamma(t) + \gamma(t)).$$

Then there exists a $\phi \in C^2(\Omega)$ such that

$$\theta = \theta_+ + \phi(\cdot - \Gamma) - \theta_-$$

satisfies the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \le C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+ + \int \theta_+^2\right)$$

with the constant C depending on C_γ and T, on $\left\|\Lambda^{-1/4} u_h\right\|_\infty$, $[u_l]_{3/4}$, and $\|u_l\|_{Lip}$, and on $\left\|D^2\phi\right\|_\infty$, $\|\nabla\phi\|_\infty$, and $\sup \left\||x|^{3/4}\nabla\phi(x)\right\|_\infty$.

Proof. We'll apply the Caccioppoli estimate with

$$\psi(t,x) := \phi(x-\Gamma(t)),$$
$$\phi \in C^2(\mathbb{R}^2) \cap \dot{C}^{1/4}(\mathbb{R}^2).$$

Now

$$\partial_t \psi + u \cdot \nabla \psi = (u - \dot{\Gamma}) \cdot \nabla \phi(x - \Gamma(t)).$$

We arrive at

$$\frac{d}{dt} \int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \le C \left(\int \chi_{\{\theta_+ > 0\}} + \int \theta_+ (u - \dot{\Gamma}(t)) \cdot \nabla \phi(x - \Gamma(t)) \right).$$

Consider the high pass term $\int \theta_+ u_h \cdot \nabla \phi$. Integrating by parts (or, since Λ is self adjoint) and then applying the Lemma 4.2

$$\int \Lambda^{-1/4} u_h \Lambda^{1/4}(\theta_+ \nabla \phi) \le C \left\| \Lambda^{-1/4} u_h \right\|_\infty \left(\|\nabla \phi\|_\infty + \left\|D^2\phi\right\|_\infty \right) \left(\|\theta_+\|_1 + |\operatorname{supp}(\theta_+)|^{1/2} \left(\|\theta_+\|_{L^2} + \|\theta_+\|_{H_D^{1/2}} \right) \right).$$

From Hölder's inequality with Peter-Paul, we obtain

$$\int u_h \theta_+ \nabla \phi(x - \gamma(t))\, dx \le C(\phi, \varepsilon) \left\|\Lambda^{-1/4} u_h\right\|_\infty \left(\int \chi_{\{\theta_+ > 0\}} + \int \theta_+ + \int \theta_+^2 \right) + \varepsilon \int \left|\Lambda^{1/2}\theta_+\right|^2.$$

Time for the low pass term.

Recall that

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \Gamma + \gamma)$$

and the low pass term is

$$\int (u_l - \dot{\Gamma}) \theta_+ \nabla \phi(x - \Gamma).$$

Calculate

$$u_l(t,x) - \dot{\Gamma}(t) = u_l(t,x) - u_l(t, \Gamma + \gamma) - \dot{\gamma} \le |x - \Gamma - \gamma|^{3/4}.$$

It's easy to bound

$$\dot{\gamma} \nabla \phi(x - \Gamma) \le C_\gamma \|\nabla \phi\|_\infty.$$

We can bound

$$u_l(t,x) - u_l(t, \Gamma + \gamma) = (x) - (\Gamma) + (\Gamma) - (\Gamma + \gamma) \le [u_l]_{3/4} |x - \Gamma|^{3/4} + \|u_l\|_{\mathrm{Lip}} |t| C_\gamma.$$

All in all, for $t \in [-T, 0]$,

$$\left|(u_l - \dot{\Gamma}) \nabla \phi(x - \Gamma)\right| \le C(\phi, C_\gamma, [u_l]_{3/4}, \|u_l\|_{\mathrm{Lip}}, T).$$

Thus

$$\int (u_l - \dot{\Gamma}) \theta_+ \nabla \phi(x - \Gamma).$$

From this the result follows. □

At last we can prove the De Giorgi lemmas.

Lemma 4.4 (First De Giorgi Lemma)**.** *Suppose that θ and $u = u_l + u_h$ solve [cite] on $[-T, 0] \times \Omega$ for some open $C^{2,\alpha}$ set $\Omega \subseteq \mathbb{R}^2$.*

Suppose that for some $\Gamma : [-T, 0] \to \mathbb{R}^2$,

$$\theta(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+ \qquad \forall x \notin B_2(\Gamma(t)).$$

Suppose also that

$$u_l(t, \Gamma(t) + \gamma(t)) = \dot{\Gamma}(t) + \dot{\gamma}(t)$$

for some γ with Lipschitz norm less than C_γ.

Then there exist constants $\delta_0 > 0$ and $\varepsilon > 0$ such that

$$\int_{-2}^0 \int_{B_2(\Gamma(t))} \max(\theta, 0)^2 \, dx dt \le \delta_0$$

implies that

$$\theta(t,x) \le 1 \qquad \forall (t,x) \in [-1,0] \times B_1(\Gamma(t)).$$

Proof. Let ϕ be such that $\phi = 0$ on B_1 and $\phi(x) \ge 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 2$ while ϕ is Lipschitz and C^2 and its gradient decays like $|x|^{-3/4}$.

Consider the sequence of functions

$$\theta_k := (\theta(t,x) - \phi(x - \Gamma(t)) - 1 + 2^{-k})_+$$

and define

$$\mathcal{E}_k := \int_{-1-2^{-k}}^0 \int_\Omega \theta_k^2 \, dxdt.$$

Notice that

$$\mathcal{E}_0 = \int_{-2}^0 \int_\Omega (\theta - \phi(x - \Gamma))_+ \, dxdt \le \delta_0.$$

Moreover, as $k \to \infty$ we have

$$\mathcal{E}_k \to \int_{-1}^0 \int_\Omega (\theta - \phi(x - \Gamma) - 1)_+$$

so in particular, if we can show $\mathcal{E}_k \to 0$ then $\theta \le 1$ for $t \in [-1,0]$ and $x \in B_1(\Gamma)$.

That's enough setup, let's argue that $\mathcal{E}_k \to 0$. Notice that when $\theta_{k+1} > 0$, then in particular $\theta_k \ge 2^{-k}$ [or something similar]. Thus for any finite p, there exists a constant C so

$$\chi_{\{\theta_{k+1} > 0\}} \le C^k \theta_k^p.$$

In particular,

$$\mathcal{E}_{k+1} \le C^k \int_{-1-2^{-k}}^0 \int \theta_k^3.$$

Applying the energy inequality θ, ϕ, and Γ we obtain

$$\sup_{-1-2^{-k-1} < t < 0} \int \theta_{k+1}^2 + \int_{-1-2^{-k-1}}^0 \int \left|\Lambda^{1/2} \theta_{k+1}\right|^2 \le C^k \int_{-1-2^{-k}}^0 \theta_k^2 = \mathcal{E}_k.$$

However, by Sobolev embedding and the fact that $H_D^{1/2}$ controls classical $H^{1/2}$ controls L^4,

$$\|\theta_{k+1}\|_{L^3([-1-2^{-k-1},0] \times \Omega)} \le C^k \mathcal{E}_k^{1/2}.$$

Therefore

$$\mathcal{E}_{k+1} \le C^k \mathcal{E}_k^{3/2}.$$

It follows by a well known result [citation] that for $\mathcal{E}_0$ sufficiently small (say less than δ_0), $\mathcal{E}_k \to 0$ as $k \to \infty$ which we already established is sufficient to obtain our result. □

This is coming along quite nicely. We can move on to DG2, the isoperimetric inequality.

Lemma 4.5 (Second De Giorgi Lemma). *Let θ and $u = u_l + u_h$ be solutions to [cite] satisfying the desired bounds. Let Γ and γ be paths with the desired properties, in particular*

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \gamma + \Gamma).$$

Suppose that for $t \in [-5,0]$ and any $x \in \Omega$,

$$\theta(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

There exists a small constant μ such that the three conditions

$$|\{\theta \ge 1\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0,$$

$$|\{0 < \theta < 1\} \cap [-4,0] \times B_4(\Gamma)| \le \mu,$$

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge \frac{4|B_4|}{2}$$

cannot simultaneously be met.

In particular, our bound on θ implies $\theta \le 3$ on $[-5,0]\times B_3(\Gamma)$.

Proof. Suppose that θ_n is a sequence of counterexamples for $\mu = 1/n$. We wish to show that these counterexamples have a limit which violates basic principles.

Let ϕ be a function which vanishes on B_2 but has all the growth and smoothness properties. In particular assume that ϕ exceeds $2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 3$. Apply the energy inequality with $\phi(x-\Gamma)$, and find that

$$\sup_{[-2,0]} \int \theta_+^2 + \int_{-2}^0 \int \left|\Lambda^{1/2}\theta_+\right|^2 \le C \int_0^3 \int \left(\chi_{\{\theta_+>0\}} + \theta_+ + \theta_+^2\right).$$

This proves in particular that $\theta_+ \in L^2([-2,0]; H_D^{1/2}(\Omega))$ is uniformly bounded. Because, by equivalence of H^s and H_D^s, this space is compact in $L^2(\Omega)$, this is the first ingredient in applying the Aubin-Lions lemma.

Since θ_n solves the desired equation,

$$\partial_t\theta_+ + \chi_{\{+\}}\partial_t\phi(\cdot-\Gamma) + u\cdot\nabla\theta_+ + \chi_{\{+\}}u\cdot\nabla\phi(\cdot-\Gamma) + \Lambda\theta_+ + \chi_{\{+\}}\Lambda\phi(\cdot-\Gamma) - \chi_{\{+\}}\Lambda\theta_- = 0.$$

In fact we can simplify

$$-\chi_{\{+\}}\dot\Gamma\cdot\nabla\phi + u\cdot\nabla\theta_+ + \chi_{\{+\}}u\cdot\nabla\phi + = \chi_{\{+\}}(u-\dot\Gamma)\nabla\phi + u\nabla\theta_+.$$

Therefore

$$-\partial_t\theta_+ = \chi_{\{+\}}(u-\dot\Gamma)\nabla\phi + u\nabla\theta_+ + \chi_{\{+\}}\Lambda\theta_+ + \chi_{\{+\}}\Lambda\phi - \chi_{\{+\}}\Lambda\theta_-.$$

This doesn't look great. Notice though that θ_+^2 is also bounded in $L^2(H_D^{1/2})$ since

$$\int \left|\Lambda^{1/2}\theta_+^2\right|^2 = \text{no}$$

Anyways, assume that we've found that θ_{n+} is compact on $[-4,0]\times\Omega$, which I will actually prove later when I'm smarter. Then θ_+ has an L^2 limit f which is supported on $B_3(\Gamma)$ and which has the established mass bounds. Moreover, since $\frac{d}{dt}\int\theta_+^2$ is uniformly bounded, so too is $\frac{d}{dt}\int f^2$ for almost every time. Note that we mean $\frac{d}{dt}$ to be bounded in the sence that $\int f^2$ doesn't grow much over small intervals.

Since $H_D^{1/2}$ is a Hilbert space, it is reflexive, and hence for any subsequence of θ_n there exist a further subsequence such that for every $g \in H_D^{1/2}$, $\langle g, \theta_+\rangle \to \langle g, \tilde\theta\rangle$. However, since $g \in L^2$ in particular, that subsequential weak limit must be f. Therefore f must be a weak limit of the θ_n and in particular it must be bounded in $L^2(H_D^{1/2})$.

It follows, since indicator functions have infinite $H^{1/2}$ norm and hence infinite $H_D^{1/2}$ norm, that at a.e. time slice f is either entirely above or entirely below the interval $(0,1)$.

In other words, at any time slice either $\theta_n \le 0$ and $\theta_+ = 0$ or else $\theta_n \ge 1$ and $\int\theta_+^2 \ge |B_2|$. Moreover, since $\frac{d}{dt}\int\theta_+^2$ is bounded, and at least half of the interval $[-4,0]$ is spent below 0, surely the back half (namely $[-2,0]$) is spent entirely below 0. But this contradicts our assumption about δ_0, so we conclude that there must exist a minimal μ for which solutions exist.

The proof concludes. Still gotta prove compactness in time though. □

For compactness in time, it might be useful to know that $\left\|f^2\right\|_s \le \|f\|_\infty \|f\|_s$.

$$\left\|\Lambda^{1/2}(f^2)\right\|_2^2 = \iint [f(x)^2 - f(y)^2]^2 K + \int f^4 B$$
$$\le \iint f(x)^2[f(x)-f(y)]^2K + \iint f(y)^2[f(x)-f(y)]^2K + \|f\|_\infty^2 \int f^2 B$$
$$\le \|f\|_\infty^2 \int |\Lambda^{1/2} f|^2.$$

5. Harnack Inequality

We put together Propositions 4.4 and 4.5 to produce a Harnack inequality.

Proposition 5.1 (Harnack Inequality). *Let θ and $u = u_l + u_h$ be solutions to [cite] the PDE. Let $\Lambda^{-1/4}u_h \in L^\infty$ while $u_l \in Lip \cap L^{3/4}$. Moreover, let γ and Γ be such that*

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \Gamma + \gamma)$$

and $\|\dot{\gamma}\|_\infty \le C_\gamma$.

There exists a number k_0 and a small constant $\lambda > 0$ such that if for all $t \in [-5, 0]$, $x \in \Omega$,

$$\theta(t,x) \le 2 + 2^{-k_0}\left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+$$

and

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge \frac{4|B_4|}{2}$$

then for all $t \in [-1, 0]$, $x \in B_1(\Gamma)$ we have

$$\theta(t,x) \le 2 - \lambda.$$

Proof. Let μ and δ_0 as in Proposition 4.5, and take k_0 large enough that $(k_0 - 1)\mu > 4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x) := 2 + 2^k(\theta(t,x) - 2).$$

That is, $\theta_0 = \theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies [cite, boundedness], each θ_k for $k \le k_0$ and $(t,x) \in [-5,0] \times \Omega$ satisfies

$$\theta_k(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 4.5.

Note also that

$$|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma)|$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed δ_0 for all $k < k_0$ as well.

Applying Proposition 4.5 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0] \times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity [cite] increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity [cite] is bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption [cite] must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 4.4. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall (t,x) \in [-1,0] \times B_1(\Gamma).$$

For the original function θ, this means that for $(t,x) \in [-1,0] \times B_1(\Gamma)$

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

□

That's the absolute gain. Now let us consider how this gain can be shifted to our new reference frame. But first, a quice technical lemma:

Lemma 5.2. *There exist a constant $\bar{\lambda} > 0$ and $\alpha > 1$ such that, for any $0 < \varepsilon \le 1/2$ and any $z \ge 1$*

$$\left(|\varepsilon^{-1}(z-1)+3|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+ \ge \bar{\lambda}.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon = 1/2$. Consider

$$\left(|2z+1|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+.$$

When $\alpha = 1$, this quantity is clearly non-negative and in fact strictly positive when $z \ge 1$. On any compact interval $[0, N]$, the quantity with $\alpha = 1$ is bounded below, and the quantity $\left(|z|^{1/4} - 2^{1/4}\right)_+$ is bounded above, so if $\alpha - 1$ is less than the ratio of those bounds then the total quantity will be bounded below.

However, the range of acceptable α depends on N, and it is possible that no single α is acceptable for the whole of $z \in [1, \infty)$.

For $z > 2$, the expression reduces to

$$(2z+1)^{1/4} - \alpha z^{1/4} - (\alpha-1)2^{1/4} = z^{1/4}\left((2+1/z)^{1/4} - \alpha\right) - (\alpha-1)2^{1/4}.$$

This quantity is increasing as α decreases, and for any $\alpha < 2^{1/4}$ it tends to ∞ as z increases.

This is sufficient to show that for some $\alpha > 1$, there exists a lower bound $\bar{\lambda}$ on the quantity [cite], and thus the lemma holds. ☐

We are ready to prove the shifted version of the Harnack Inequality.

Lemma 5.3 (Harnack Inequality, with shift)**.** *Let θ and $u = u_l + u_h$ be as desired. Let Γ and γ be paths such that*

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \Gamma + \gamma)$$

and $\|\gamma\|_{Lip} \le C_\gamma$. If $0 < \varepsilon < 1/5$ is such that

$$5C_\gamma \le \varepsilon^{-1} - 3$$

then the following holds:

Let k_0 be as in Lemma 5.1 and assume that for $(t, x) \in [-5, 0] \times \Omega$

$$\theta(t, x) \le 2 + 2^{-k_0}\left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

Then there exist a $\lambda > 0$ small enough that for $(t, x) \in [-5, 0] \times \varepsilon^{-1}\Omega$

$$\frac{2}{2-\lambda}\theta(\varepsilon t, \varepsilon x) \le 2 + 2^{-k_0}\left(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|^{1/4} - 2^{1/4}\right)_+.$$

If we only wish to show that by zooming horizontally by a large amount and zooming vertically by a small amount we stay under the barrier, this is obvious and merely requires being written down. Even the shift itself is clearly not a problem when considered in the un-zoomed coordinates. Since the velocity of γ is bounded by C_γ, the shift γ is arbitrarily small over very small time periods. The difficulty comes from the fact that k_0 itself depends on C_γ, and as we will see in the next section C_γ depends on ε, so ε cannot depend on k_0. In time, the co-dependence of ε and C_γ is easy to untangle (so long as εC_γ is less than some universal constant, the proof will go through). In space, it is less clear that ε will not depend on k_0, and of course we need to zoom in time and space by the same factor so both issues are interconnected.

Proof. Take λ such that

$$\lambda \le 2^{-k_0}, \qquad \frac{2}{2-\lambda} \le 1 + 2^{-k_0}\bar{\lambda}/2, \qquad \frac{2}{2-\lambda} \le \alpha.$$

for $\bar{\lambda}$ and α from Lemma 5.2.

Denote

$$\bar{\theta}(t,x) := \frac{2}{2-\lambda}\theta(\varepsilon t, \varepsilon x)$$

and

$$\phi(x) = \left(|x|^{1/4} - 2^{1/4}\right)_+ .$$

We already proved in Lemma 5.1 that $\theta \le 2 - 2^{-k_0}$ on $[-1,0] \times B_1(\Gamma)$. For $\bar{\theta}$, this means that $\bar{\theta}(t,x) \le 2$ when $(t,x) \in [-1/\varepsilon, 0] \times B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$.

Similarly, the bound [cite] on θ becomes the equivalent bound on $\bar{\theta}$, for all $(t,x) \in [-5/\varepsilon, 0] \times \varepsilon^{-1}\Omega$

$$\bar{\theta}(t,x) \le \frac{2}{2-\lambda}\left[2 + 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right].$$

Let $t \in [-5,0]$ and $x \in \varepsilon^{-1}\Omega$, and define $y = x - \varepsilon^{-1}\Gamma(\varepsilon t)$. If $|y| \le \varepsilon^{-1}$ then

$$\bar{\theta}(t,x) \le 2 \le 2 + 2^{-k_0}\phi(x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)).$$

If $|y| \ge \varepsilon^{-1}$ then from Lemma 5.2,

$$\bar{\lambda} + \alpha\phi(\varepsilon|y|) \le \phi(|y| - \varepsilon^{-1} + 3).$$

Thus we rewrite the bound [cite] as

$$\begin{aligned}\bar{\theta}(t,x) &\le \frac{2}{2-\lambda}\left[2 + 2^{-k_0}\phi(\varepsilon|y|)\right] \\ &\le 2(1 + 2^{-k_0}\bar{\lambda}/2) + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ &= 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right] \\ &\le 2 + 2^{-k_0}\phi(|y| - \varepsilon^{-1} + 3).\end{aligned}$$

For $t \in [-5,0]$,

$$|y| - 5C_\gamma \le |y - \varepsilon^{-1}\gamma(\varepsilon t)|.$$

Thus, since by assumption $5C_\gamma \le \varepsilon^{-1} - 3$,

$$|y| - \varepsilon^{-1} + 3 \le |y - \varepsilon^{-1}\gamma(\varepsilon t)|.$$

Therefore, for $|y| \ge \varepsilon^{-1}$,

$$\bar{\theta}(t,x) \le 2 + 2^{-k_0}\phi(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|).$$

This concludes the proof. □

6. Hölder Continuity

Denote

$$\sum_k = \sum_{j > -k\ln(\varepsilon)}$$

while

$$\sum^k = \sum_{j \le -k\ln(\varepsilon)} .$$

Define

$$
\begin{aligned}
\theta_k(t,x) &:= (1-\lambda)^{-k}\theta(\varepsilon^k t, \varepsilon^t x),\\
u_l^k(t,x) &:= \sum^k u_j(\varepsilon^k t, \varepsilon^k x),\\
u_h^k(t,x) &:= \sum_k u_j(\varepsilon^k t, \varepsilon^k x),\\
\Gamma_0(t) &:= 0\\
\dot\gamma_k(t) &:= u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot\Gamma_k(t)\\
\gamma_k(0) &:= 0\\
\Gamma_k(t) &:= \varepsilon^{-1}\gamma_{k-1}(\varepsilon t) + \varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t) + \cdots + \varepsilon^{-k}\gamma_0(\varepsilon^k t).
\end{aligned}
$$

Use [citation] some lemma from Bahouri-Chemin-Danchin that's a generalization of Picard-Lindelof.

Recall the bounds, which will be rephrased to match the proofs above later (the above will be rephrased, that is),

$$
\left\|\Lambda^{-1/4} u_h^k\right\|_\infty \le \sum_k \varepsilon^{-k/4} 2^{-j/4}\kappa \le \varepsilon^{-k/4} C 2^{k\ln_2(\varepsilon)/4}\kappa = C\kappa
$$

for some universal constant C (it's literally a decimal, it's that universal). Similarly, for some different universal constant,

$$
\left\|\nabla u_h^k\right\|_\infty \le \sum^k \varepsilon^k 2^j \kappa \le \varepsilon^k C 2^{-k\ln_2(\varepsilon)}\kappa = C\kappa.
$$

For each norm of interest, there's a constant corresponding to that norm such that, for u_l and u_h, it's less than $C\kappa$. Note that these bounds do not depend on k.

I want to claim that

$$
\dot\Gamma_k(t) = \sum^{k-1}(\varepsilon^k t, \varepsilon^k \Gamma_k(t)).
$$

This makes sense, because $\Gamma_{k+1}(t) = \varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)$.

Well, $\dot\gamma_0(t) = u_l^0(t, \gamma_0(t))$. Moreover,

$$
u_l^k(t, \Gamma_{k+1}(t)) = u_l^k(\varepsilon^{-1}\varepsilon t, \varepsilon^{-1}\left[\gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t)\right])
$$

or

$$
\begin{aligned}
\sum^k u_j(\varepsilon^{k+1} t, \varepsilon^{k+1}\Gamma_{k+1}(t)) &= \sum^k u_j(\varepsilon^k \varepsilon t, \varepsilon^k\left[\gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t)\right])\\
&= u_l^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t))\\
&= \dot\gamma_k(\varepsilon t) + \dot\Gamma_k(\varepsilon t)\\
&= \partial_t\left[\varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)\right]\\
&= \dot\Gamma_{k+1}(t).
\end{aligned}
$$

In other words,

$$
\dot\Gamma_k(t) = \sum^{k-1} u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t)) \qquad k \ge 5.
$$

I should think more about what exactly happens at the edge case of $j = 0, 1$.

With this in hand, we can bound the size of γ_k. Namely,

$$\begin{aligned}\dot\gamma_k(t) &= u_l^k(t,\Gamma_k(t)+\gamma_k(t)) - \dot\Gamma_k(t)\\ &= \sum^k u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t)+\varepsilon^k\gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t))\\ &= \sum^{k-1}\left[u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t)+\varepsilon^k\gamma_k(t)) - u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t,\varepsilon^k\ldots).\end{aligned}$$

The sum $\sum^{k-1} u_j(\varepsilon^k\cdot,\varepsilon^k\cdot) = u_l^{k-1}(\varepsilon\,\cdot,\varepsilon\,\cdot)$ is Lipschitz in space, with Lipschitz constant less than $\varepsilon C\kappa$. Moreover, each u_j has $\|u_j\|_\infty \le \kappa$. Thus both terms of $\dot\gamma_k(t)$ are bounded

$$|\dot\gamma_k(t)| \le \varepsilon C\kappa|\gamma_k(t)| - \kappa\ln_2(\varepsilon).$$

This, by Gronwall's inequality, tells us that for $t\in[-T,0]$,

$$|\gamma_k(t)| \le \frac{-\ln_2(\varepsilon)}{\varepsilon C}\left(e^{\varepsilon C\kappa T}-1\right)$$

and hence

$$|\dot\gamma_k(t)| \le -\kappa\ln_2(\varepsilon)e^{\varepsilon C\kappa T} = C_\gamma = C_\gamma(T,\varepsilon,\kappa).$$

Note in particular that C_γ cannot be made small by altering any quantity that we have fine control over. Since we may have trouble making the same argument for γ_0, we can if necessary take the max of C_γ as defined above and $C\kappa j_0$. The key is that the γ_k are uniformly Lipschitz. Trivially

$$|\dot\Gamma_k(t)| \le kC_\gamma.$$

Lastly, let us state for the record that

$$\partial_t\left[\gamma_k(t)+\Gamma_k(t)\right] = u_l^k(t,\gamma_k(t)+\Gamma_k(t)).$$

Let θ be a function defined on $[-T,0]\times\Omega$ which satisfies the PDE.

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