HOLDER REGULARITY UP TO THE BOUNDARY FOR CRITICAL SQG ON BOUNDED DOMAINS

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Abstract. We consider the dissipative SQG equation in bounded domains first introduced in [Constantin Ignatova]. We show global Holder regularity up to the boundary of the solution. The method is based on the De Giorgi techniques as in [Caffarelli Vasseur]. The boundary introduces several difficulties, as the lack of translation invariance of the Laplacian operator, or the fact that the gradient does not commute with the Dirichlet Laplacian.

Contents

The surface quasigeostrophic equation (SQG) is a special case of the quasi-geostrophic equation (QG) when the atmosphere is at rest. The (QG) model is used extensively in meteorology and oceanography [citations]. These models are described in [?]. The SQG model was popularized by Constantin, Majda and Tabak in [citation], due to its similarities with the Euler and Navier-Stokes equation. They proposed it as a toy model for the study of 3D Fluid equations (see also Held, Garner, Pierrehumbert, and Swanson [citation]).

We consider in this paper critical SQG on a bounded domain. We will focus on the the following model, which was introduced by Constantin and Ignatova in [CI17] and [CI16]. Consider Ω a connected bounded domain in $\mathbb{R}^2$ with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and the Laplacian with homogeneous Dirichlet boundary conditions $(-\Delta_D)$. If $(\eta_k)_{k\in\mathbb{N}}$ is a family of eigenfunctions of $-\Delta_D$ with corresponding eigenvalues λ_k listed in increasing order, define

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

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The critical SQG problem on Ω with initial data $\theta_0 \in L^2(\Omega)$ is

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0 & (0,T)\times\Omega, \\ u = \nabla^\perp\Lambda^{-1}\theta & [0,T]\times\Omega, \\ \theta = \theta_0 & \{0\}\times\Omega. \end{cases} \tag{1}$$

In the model, the dissipation $\Lambda = \Delta_D^{1/2}$ is due to the Ekman pumping, while the nonlinear velocity u comes from the geostrophic and hydrostatic balance (see [**?**]).

The main result of this paper is the following.

Theorem 0.1. *Let θ be a smooth solution to* (1) *with initial data $\theta_0 \in L^2(\Omega)$ on a bounded open set $\Omega \subseteq \mathbb{R}^2$ with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, and on a time interval $[0,T]$.*

Then for any $t \in (0,T)$, θ is Hölder continuous uniformly on $(t,T)\times\bar{\Omega}$.

More precisely, there exists $\alpha \in (0,1)$ depending only on Ω and $\|\theta_0\|_{L^2(\Omega)}$, and some constant C depending only on Ω and t, such that

$$\|\theta\|_{C^\alpha((t,T)\times\Omega)} \le C\,\|\theta_0\|_{L^2(\Omega)}\,.$$

This model was first thoroughly studied in the cases without boundaries (either $\mathbb{R}^2$ or the torus $\mathbb{T}^2$). Global weak solutions were first constructed in Resnick [Resnick]. Global regularity was first showed with small initial values [Constantin Cordoba, Wu], or extra C^α regularity on the velocity [Constantin Wu, Dong Pavlovic]. in [citation] Kiselev, Nazarov and Volberg showed the propagation of C^∞ regularity. The global C^∞ regularity for any L^2 initial values was first proved in [CV10] (see also [Kiselev-Nazarov] Constantin Vicol [**?**]).

In the presence of boundaries, there are several distinct ways to define SQG. This can be attributed to alternative definitions of the half-Laplacian. Kriventsov [citation, Dennis Kriventsov] considered a two-phase problem which satisfied SQG only in a region, and was able to prove Hölder regularity in the time-independent case. This problem, intended to model air currents over a region containing both land and water, contains a half-Laplacian and a Riesz transform defined, not spectrally, but in terms of extension. In [citation, Matt and Vasseur], the authors derived from the Euler-Coriolis-Boussinesq model, the full 3D inviscid quasigeostrophic system in an impermeable cylinder (see also [MV2] for the construction of small time smooth solutions to the model). They obtain the natural boundary conditions in this case. It is noteworthy to mention that these boundary conditions lead a distinct set of boundary conditions for SQG than the one introduced in [CI17] [CI16] and described above. However, due to the complexity of the model described in [MV], we focus in this paper only on the homogenous case.

Existence of weak solutions for (1) is proven in [CI17]. The interior regularity of solutions is proven in [CI17] (together with propagation of L^∞ bounds). The method of proof for interior regularity uses the nonlinear maximum principles, introduced in [Constantin Vicol]. However, the bounds obtained in that paper blow up near the boundary and do not provide global regularity up to the boundary. In [CI16] Remark 1, questions about global regularity are suggested as open problems. Both the $C^\alpha(\bar{\Omega})$ regularity, and the bootstrapping to the $C^\infty(\bar{\Omega})$ regularity are mentioned as open problems. Our result answer the first question, by showing that solutions θ to (1) are globally Hölder continuous. Bootstrapping to $\mathbb{C}^\infty$ involves different techniques, and will be studied in a forthcoming paper [**?**].

Our proof is based on the De Giorgi method [DG57]. The method was applied to the SQG problem first in [CV10]. The method is powerful to show C^α regularity to nonlocal equations (see Caffarelli-Chan-Vasseur). It has been applied in a variety of situations for non-local problems, as the 3D Quasigeostrophic problem [**?**], the time-fractional case (Allen Caffarelli Vasseur), the kinetic setting (Silvestre- Imbert, Stokols -Vasseur), or even more exotic situations as the Hamilton-Jacobi equations (see Chan Vasseur, Stokols-Vasseur).

The first broad idea consists in decoupling the velocity u from θ to work on a linear equation, and prove alternating regularity results for θ and u independently. We first show that θ is in L^∞ regardless of the value of u (Section 2). The idea is to prove a decrease-in-oscillation lemma for fractional diffusion equations with drift (Section 5). By applying this oscillation lemma, then scaling our equation, and then applying the oscillation lemma again iteratively, we can show that θ is Hölder continuous (Section 6).

However, for this, we need to obtain scaling invariant controls on the drift $u = \nabla\Lambda^{-1}\theta$ (Section 3). Though L^∞ is the obvious choice, since solutions to SQG are easily shown to be L^∞ bounded under minimal assumptions, the Riesz transform is not bounded from L^∞ to L^∞. The usual solution is to consider BMO (as in [CV10] and [?]), but in the case of bounded domains the Riesz transform is not known to be bounded in this space either. The idea is to use extensions of the littlewood-Paley theory to bounded domains.

The adaptation of Fourier analysis and Littlewood-Paley theory to Schrodinger operators has is a well-studied subject [citations]. As an application of this theory, [IMT18] and [citation, Dong et al] have considered operators defined on open subsets of $\mathbb{R}^n$, which includes as a special case the operator $-\Delta_D$ (a Schrodinger operator with zero potential). In particular, in [IMT17], Iwabuchi, Matsuyama, and Taniguchi derives many important results, including the Bernstein inequalities, for Besov spaces adapted to the operator $-\Delta_D$ on bounded open subsets of $\mathbb{R}^n$ with smooth boundary. This theory turns out to greatly improve our understanding of the Riesz transform $\nabla\Lambda^{-1}$ on bounded domains.

Using the results of [IMT17], we will be able to show that the Riesz transform of an L^∞ function whose Fourier decomposition $f = \sum f_k \eta_k$ is supported on high frequencies $k > N$ will be bounded in the weak sobolev space $W^{-1/4,\infty}$, and the Riesz transform of an L^∞ function whose Fourier decomposition is supported on low frequencies $k < N$ will have bounded Lipschitz constant. The cutoff N for dividing high frequencies from low frequencies must depend however on the size of the domain Ω. In the case of $\mathbb{R}^2$, where ∇ and Λ^{-1} commute, this is equivalent to the observation that the Riesz transform is bounded from L^∞ to the Besov space $B^0_{\infty,\infty}$. In the case of bounded domains, the argument must be more subtle. We must decompose θ into its Littlewood-Paley projections, individually bound the Riesz transform of each projection in multiple spaces, and then recombine these infinitely-many functions into a low-frequency collection and a high-frequency collection depending on the scale of oscillation we are trying to detect.

We make this notion precise with the following definition.

Definition 1 (Calibrated sequence)**.** Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set and $0 < T \in \mathbb{R}$. We call a function $u \in L^2([0,T]\times\Omega)$ **calibrated** if it can be decomposed as the sum of a calibrated sequence

$$u = \sum_{j\in\mathbb{Z}} u_j$$

with each $u_j \in L^2([0,T]\times\Omega)$ and the infinite sum converging in the sense of L^2.

We call a sequence $(u_j)_{j\in\mathbb{Z}}$ **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\|u_j\|_{L^\infty([0,T]\times\Omega)} \le \kappa,$$
$$\|\nabla u_j\|_{L^\infty([0,T]\times\Omega)} \le 2^j 2^{-N}\kappa,$$
$$\left\|\Lambda^{-1/4} u_j\right\|_{L^\infty([0,T]\times\Omega)} \le 2^{-j/4}2^{N/4}\kappa.$$

In Section 3 we will show that u is calibrated and in Section 6 we will show that it remains calibrated at all scales (specifically, with fixed constant κ but not always with the same center N). Therefore we will consider the linear equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, & [-T, 0] \times \Omega \\ \operatorname{div} u = 0 & [-T, 0] \times \Omega. \end{cases} \tag{2}$$

In Section 2 we show that solutions to (2), with minimally regular velocity and L^2 initial data, become L^∞ instantly, and in Sections 4 and 5 we will show that solutions to (2) with calibrated velocity have decreasing oscillation between scales.

Previous applications of the De Giorgi method to equations related to (2) generally make extensive use of either an extension representation (c.f. [CV10]) or a singular-integral representation (c.f. [**?**]). In this paper, we use the kernel representation of the Dirichlet fractional Laplacian derived by Caffarelli and Stinga [CS16]. It is an generalization of [citation Stinga Torrea] and [CS16] the extension representation of [citation Caff Silvestre]. This theory is pivotal in translating the existing non-local De Giorgi techniques to the problem at hand(see Section 1).

Throughout the paper, we will use the following notations. If $f = \sum_k f_k \eta_k$ then

$$\begin{aligned} \|f\|_{\mathcal{H}^s} &:= \left(\sum_k \lambda_k^s f_k^2 \right)^{1/2} \\ &= \int |\Lambda^s f|^2 . \end{aligned}$$

We suppress the dependence on Ω, though in fact the λ_k and Λ are defined in terms of the domain Ω. The relevant domain will be clear from context. This is in fact a norm, not a seminorm, since $\|f\|_{L^2(\Omega)} \le \lambda_0^{s/2} \|f\|_{\mathcal{H}^s}$.

Recall the notation

$$\begin{aligned} [f]_\alpha &:= \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, & \alpha \in (0,1], \\ \|f\|_{C^\alpha} &:= \|f\|_\infty + [f]_\alpha, & \alpha \in (0,1], \\ \|f\|_{C^{k,\alpha}} &:= \sum_{n=0}^{k} \|D^n f\|_\infty + \left[D^k f \right]_\alpha, & \alpha \in (0,1], k \in \mathbb{N}. \end{aligned}$$

We will use the notation $(x)_+ := \max(0,x)$. When the parentheses are ommited, the subscript $+$ is merely a label. Throughout this paper, if an integral sign is written $\int$ without a specified domain, the domain is implied to be Ω, with Ω defined in context. For any vector $v = (v_1, v_2)$, by $v^\perp$ we mean $(-v_2, v_1)$, and by $\nabla^\perp$ we mean $(-\partial_y, \partial_x)$.

The Paper is organized as follows. Section 1 is dedicated to the properties of Λ.....

1. Properties of Λ

We begin by recounting the result of [CS16] which gives us a singular integral representation of the $\mathcal{H}^s$ norm.

Proposition 1.1 (Caffarelli-Stinga Representation)**.** *Let $s \in (0,1)$ and $f, g \in \mathcal{H}^s$ on a bounded $C^{2,\alpha}$ domain $\Omega \subseteq \mathbb{R}^2$. Then*

$$\int_\Omega \Lambda^s f \Lambda^s g \, dx = \iint_{\Omega^2} [f(x) - f(y)][g(x) - g(y)] K_{2s}(x,y) \, dx dy + \int_\Omega f(x) g(x) B_{2s}(x) \, dx$$

for kernels K_{2s} and B_{2s} which depend on the parameter s and the domain Ω.

Moreover, these kernels are bounded

$$0 \le K_{2s}(x,y) \le \frac{C(\Omega,s)}{|x-y|^{2+2s}}$$

for all $x \ne y \in \Omega$ *and*

$$0 \le B_{2s}(x)$$

for all $x \in \Omega$.

Finally, for any $s,t \in (0,2)$ *there exists a constant* $c = c(s,t,\Omega)$ *such that for all* $x \ne y \in \Omega$

$$K_t(x,y) \le |x-y|^{s-t} K_s(x,y). \tag{3}$$

Proof. See [CS16] Theorems 2.3 and 2.4.

Theorem 2.4 in [CS16] does not explicitly state the result (3). However, it does state that for each kernel K_s there exists a constant c_s dependent on s and Ω such that

$$\frac{1}{c_s}|x-y|^{2+s}K_s(x,y) \le \min\left(1, \frac{\eta_0(x)\eta_0(y)}{|x-y|^2}\right) \le c_s|x-y|^{2+s}K_s(x,y).$$

Since the middle term does not depend on s, we can say that

$$|x-y|^{2+t}K_t(x,y) \le c_t c_s |x-y|^{2+s}K_s(x,y)$$

from which (3) follows. □

From the explicit formulae given in [CS16], we see that K_{2s} is roughly equal to the standard kernel for the $\mathbb{R}^2$ fractional Laplacian $(-\Delta)^s$ when both x and y are in the interior of Ω or when x and y are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel B_{2s} is well-behaved in the interior but has a singularity at the boundary $\partial\Omega$. This justifies our thinking of the K_{2s} term as the interior term and B_{2s} as a boundary term.

When comparing the computations in this paper to corresponding computations on $\mathbb{R}^2$, it is convenient to say that the interior term behaves roughly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

Many useful results can be derived from Caffarelli-Stinga representation formula, including the following:

Lemma 1.2. *Let* $\Omega \subseteq \mathbb{R}^2$ *be a bounded open set with* $C^{2,\beta}$ *boundary.*

(a) Let $s \in (0,1)$. *If* f *and* g *are non-negative functions with disjoint support (i.e.* $f(x)g(x) = 0$ *for all* x*), then*

$$\int \Lambda^s f \Lambda^s g \, dx \le 0.$$

(b) Let $s \in (0,1)$. *If* $g \in C^{0,1}(\Omega)$ *then*

$$\|fg\|_{\mathcal{H}^s} \le 2\|g\|_\infty \|f\|_{\mathcal{H}^s} + 2\|f\|_2 \sup_y \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} \, dx.$$

(c) Let $s \in (0,1)$. *If* $g \in C^{0,1}(\Omega)$ *then for some constant* $C = C(s)$

$$\|fg\|_{\mathcal{H}^s} \le C\|g\|_{C^{0,1}(\Omega)} \left(\|f\|_2 + \|f\|_{\mathcal{H}^s}\right).$$

(d) Let $s \in (0,1/2)$. *Let* g *an* $L^\infty(\Omega)$ *function and* $f \in \mathcal{H}^{2s}$ *be non-negative with compact support. Let there be a constant* C_Ω *such that*

$$K_s(x,y) \le C_\Omega |x-y|^{3s} K_{4s}(x,y). \tag{4}$$

Then

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \le C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}\right).$$

(e) Let g an $L^\infty(\Omega)$ function and $f \in \mathcal{H}^{1/2}$ be non-negative with compact support. Let there be a constant C_Ω such that

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{3/4} K_1(x,y).$$

Then

$$\int g\Lambda^{1/4} f \le C \|g\|_\infty |\operatorname{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{1/2}}\right).$$

Proof. We prove these corollaries one at a time.

Proof of (a)**:** From Proposition 1.1

$$\int \Lambda^s f \Lambda^s g\, dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\, dxdy + \int f(x)g(x)B(x)\, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \le 0.$$

Since K is non-negative, the result follows.

Proof of (b)**:** From Proposition 1.1

$$\int |\Lambda^s(fg)|^2 = \iint \left(g(x)[f(x)-f(y)] + f(y)[g(x)-g(x)]\right)^2 K + \int f^2 g^2 B$$
$$\le 2\|g\|_\infty^2 \|f\|_{\mathcal{H}^s}^2 + 2\int f(y)^2 \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}}.$$

Proof of (c)**:** This follows immediately from (b).

Proof of (d)**:** From Proposition 1.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_< + I_\ge + II$$

where

$$I_< := \iint_{|x-y|<1} [g(x)-g(y)][f(x)-f(y)]K_s,$$
$$I_\ge := \iint_{|x-y|\ge 1} [g(x)-g(y)][f(x)-f(y)]K_s,$$
$$II := \int fgB_s.$$

First we estimate $I_<$. From (4) and from the symmetry of the integrand and the fact that $[f(x)-f(y)]$ vanishes unless at least one of $f(x)$ or $f(y)$ is non-zero,

$$|I_<| \le 2\iint_{|x-y|<1} \chi_{\{f>0\}}(x)\, |g(x)-g(y)| \cdot |f(x)-f(y)| \cdot |x-y|^{3s} K_{4s}.$$

We can break this up by Holder's inequality

$$|I_<| \le 2\left(\iint_{|x-y|<1} \chi_{\{f>0\}}(x)[g(x)-g(y)]^2 |x-y|^{6s} K_{4s}\right)^{1/2} \left(\iint [f(x)-f(y)]^2 K_{4s}\right)^{1/2}.$$

The kernel $|x-y|^{6s} K_{4s} \chi_{\{|x-y|<1\}}$ is integrable in y for x fixed. Therefore

$$|I_<| \le 2\left(2\|g\|_\infty^2 \int C\chi_{\{f>0\}}(x)\, dx\right)^{1/2} \left(\|f\|_{\mathcal{H}^{2s}}^2\right)^{1/2}. \tag{5}$$

For the term $I_\ge$, by the symmetry of the integrand we have

$$|I_\ge| \le 2\|g\|_\infty 2\int |f(x)| \int_{|x-y|\ge 1} K_s(x,y) dy\, dx.$$

Since $K_s \chi_{\{|x-y|\ge 1\}}$ is integrable in y for x fixed,

$$|I_\ge| \le C\|g\|_\infty \|f\|_1. \tag{6}$$

For the boundary term II,

$$|II| \le \|g\|_\infty \int \chi_{\{f>0\}} f B_s.$$

Since $f \ge 0$, $[f(x) - f(y)][\chi_{\{f>0\}}(x) - \chi_{\{f>0\}}(y)] \ge 0$. Therefore

$$\int \chi_{\{f>0\}} f B_s \le \int \Lambda^{s/2} \chi_{\{f>0\}} \Lambda^{s/2} f = \int \chi_{\{f>0\}} \Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \le \|g\|_\infty |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s}.$$

This combined with (5) and (6) gives us

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \le C \|g\|_\infty \left(|\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^{2s}} + \|f\|_1 + |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s} \right).$$

The lemma follows since $\|f\|_1 \le |\mathrm{supp}(f)|^{1/2} \|f\|_2$ and since $\|f\|_{\mathcal{H}^s} \le \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$.

Proof of (e)**:** This is an immediate application of part (d).

□

Let us consider the relationship between the norm $\mathcal{H}^s$ and the classical H^s norm.

It is known (see [CI16] and [CS16]) that for $s \in (0,1)$ the spaces $\mathcal{H}^s$ are equivalent to certain subsets of $H^s(\Omega)$ spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in $\mathcal{H}^s$ and that elements of $\mathcal{H}^s$ have trace zero for $s \in [\frac{1}{2}, 1]$.

The most important fact for us is that the fractional Sobolev norms defined in terms of extension (for which we have access to a variety of theorems regarding compactness and Sobolev embeddings) are dominated by our $\mathcal{H}^s$ norm with a constant that is independent of Ω.

Consider the extension-by-zero operator $E : L^2(\Omega) \to L^2(\mathbb{R}^2)$

$$Ef(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Proposition 1.3. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For any $s \in [\frac{1}{2}, 1]$ and function $f \in \mathcal{H}^s$,*

$$\int_{\mathbb{R}^2} \left| (-\Delta)^{s/2} Ef \right|^2 \le \int_\Omega |\Lambda^s f|^2.$$

Here $(-\Delta)^s$ is defined in the fourier sense.

We will prove this proposition by interpolating between $s = 0$ and $s = 1$. Before we can do this, we must prove the same in the $s = 1$ case.

Lemma 1.4. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For all functions f in $\mathcal{H}^1$,*

$$\int_\Omega |\nabla f|^2 = \int_\Omega |\Lambda f|^2.$$

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω and vanish at the boundary, so we can apply the divergence theorem and find

$$\int \nabla \eta_i \cdot \nabla \eta_j = -\int \eta_i \Delta \eta_j = \lambda_j \int \eta_i \eta_j = \lambda_j \delta_{i=j}.$$

Consider a function $f = \sum f_k \eta_k$ which is an element of $\mathcal{H}^1$, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned}
\int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla \eta_i\right)\left(\sum_j f_j \nabla \eta_j\right) \\
&= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\
&= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j \\
&= \sum_j \lambda_j f_j^2.
\end{aligned}$$

From this the result follows. □

We come now to the proof of Proposition 1.3.

Proof. Let g be any Schwartz function in $L^2(\mathbb{R}^2)$, and let f be a function in $\mathcal{H}^s$. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2}\, g E \Lambda^{s-z} f, \qquad z \in \mathbb{C}, \mathrm{Re}(z) \in [0,1].$$

Recall (see e.g. [citation Jerison, Kenig]) that when $t \in \mathbb{R}$, $(-\Delta)^{it}$ is a unitary transformation on $L^2(\mathbb{R}^2)$, and Λ^{it} is a unitary transformation on $L^2(\Omega)$.

When $\mathrm{Re}(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|f\|_{\mathcal{H}^s}$. Hence

$$\Phi(z) \le \|g\|_2 \, \|f\|_{\mathcal{H}^s}.$$

When $\mathrm{Re}(z) = 1$, integrate by parts to obtain

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{(z-1)/2} g \, (-\Delta)^{1/2} E \Lambda^{s-z} f.$$

Then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$, while $\|\Lambda^{s-z} f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^s}$. As an $\mathcal{H}^1$ function, $\Lambda^{s-z} f$ has trace zero so

$$\|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} = \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|f\|_{\mathcal{H}^s}.$$

Of course $\left\|(-\Delta)^{1/2} \cdot\right\|_{L^2(\mathbb{R}^2)} = \|\nabla \cdot\|_{L^2(\mathbb{R}^2)}$ in general so

$$\Phi(z) \le \|g\|_2 \, \|f\|_{\mathcal{H}^s}.$$

In order to apply the Hadamard three-lines lemma, we must show that Φ is differentiable in the interior of its domain.

Rewrite the integrand of Φ as

$$\mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}\left(\ln(|\xi|)|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k. \tag{7}$$

Fix some $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in (0,1)$. Since g is a Schwartz function, $\ln(|\xi|)|\xi|^z \hat{g}$ is in L^2. Moreover, for any $\varepsilon > 0$ we have $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C \lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Take $\varepsilon < \mathrm{Re}(z)$ and, since $f \in \mathcal{H}^s$, this sum will converge in L^2.

The differentiated integrand (7) is therefore a sum of two products of L^2 functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \mathrm{Re}(z) < 1$.

By the Hadamard three-lines lemma, for any $z \in (0,1)$ we have $\Phi(z) \leq \|g\|_2 \|f\|_{\mathcal{H}^s}$. Evaluating $\Phi(s)$, we see

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} gEf \leq \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

This inequality holds for any Schwartz function $g \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{H}^s$,

Since Schwartz functions are dense in $L^2(\mathbb{R}^2)$, the proof is complete. □

2. L^∞ bounds for θ

First let us derive an energy inequality.

Lemma 2.1 (Caccioppoli Estimate). *Let $\theta \in L^\infty(0,T;L^4(\Omega)\cap H^1(\Omega))$ and $u \in L^\infty(0,T;L^4(\Omega))$ solve (2) in the sense of distributions. Let $\Psi : [-T,0]\times\Omega \to \mathbb{R}$ be non-negative, Lipschitz-in-space, and Hölder continuous-in-space with exponent $\gamma < 1/2$. Then the decomposition*

$$\theta = \theta_+ + \Psi - \theta_-$$

satisfies the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \leq C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\right)$$

with the constant C depending on $\|\nabla\Psi\|_\infty$ and $\sup_t [\Psi(t,\cdot)]_\gamma$.

Proof. We multiply (2) by θ_+ and integrate in space to obtain

$$0 = \int \theta_+ [\partial_t + u\cdot\nabla + \Lambda](\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+ [\partial_t + u\cdot\nabla + \Lambda]\theta_+ &= (1/2)\frac{d}{dt}\int \theta_+^2 + (1/2)\int \operatorname{div} u\, \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \\ &= (1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The Ψ term produces important error terms:

$$\begin{aligned}\int \theta_+ [\partial_t + u\cdot\nabla + \Lambda]\Psi &= \int \theta_+\partial_t\Psi + \int \theta_+ u\cdot\nabla\Psi + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \\ &= \int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi) + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma a:

$$\begin{aligned}\int \theta_+ [\partial_t + u\cdot\nabla + \Lambda]\theta_- &= (1/2)\int \theta_+\partial_t\theta_- + \int \theta_+ u\cdot\nabla\theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \\ &= \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \leq 0.\end{aligned}$$

Put together, we arrive at

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \leq \left|\int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\cdot\nabla\Psi\right|.$$

At this point we break down the $\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi$ term using the formula from Proposition 1.1.

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi = \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y) + \int \theta_+\Psi B.$$

Since $B \geq 0$ and Ψ is non-negative by assumption, the B term is non-negative and so

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \geq \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y).$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\iint [\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y) \leq 2\iint \chi_{\{\theta_+>0\}}(x)\,|\theta_+(x)-\theta_+(y)|\cdot|\Psi_t(x)-\Psi_t(y)|\,K(x,y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\left|\iint [\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y)\right| \leq \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2+\frac{1}{\varepsilon}\iint \chi_{\{\theta_+>0\}}(x)[\Psi(x)-\Psi(y)]^2K(x,y).$$

It remains to bound the quantity $[\Psi(x)-\Psi(y)]^2K(x,y)$. By Proposition 1.1, there is a universal constant C such that

$$K(x,y)\leq\frac{C}{|x-y|^3}.$$

The cutoff Ψ is Lipschitz, and Hölder continuous with exponent $\gamma<1/2$ by assumption. Therefore

$$[\Psi(x)-\Psi(y)]^2K(x,y)\leq|x-y|^{-1}\wedge|x-y|^{2\gamma-3}.$$

Since $3-2\gamma>2$, this quantity is integrable. Thus

$$\int\chi_{\{\theta_+>0\}}(x)\int[\Psi(x)-\Psi(y)]^2K(x,y)\,dxdy\leq C(\|\Psi\|_{\mathrm{Lip}},[\Psi]_\gamma)\int\chi_{\{\theta_+>0\}}\,dx.$$

Combining [citation, like 4 different things are combined] we arrive at

$$\frac{d}{dt}\int\theta_+^2+\int\left|\Lambda^{1/2}\theta_+\right|^2-\langle\theta_+,\theta_-\rangle_{1/2}\lesssim\int\theta_+(\partial_t\Psi+u\cdot\nabla\Psi)+\int\chi_{\{\theta_+>0\}}.$$

□

This is sufficient to prove that a solution to (2) with L^2 initial data has bounded L^∞-norm after any small time.

Proposition 2.2 (L^2 to L^∞)**.** *If θ and u solve* (2) *on $[0,T]\times\Omega$ and $\theta_0\in L^2$, then for any time $S\in(0,T)$ there exists a constant $C=C(S)$ such that*

$$\|\theta\|_{L^\infty([S,T]\times\Omega)}\leq C\,\|\theta_0\|_{L^2(\Omega)}\,.$$

Proof. It is trivial to show that the $L^2(\Omega)$ norm of any smooth solution θ of (2) does not increase in time. Simply multiply the function by θ and integrate.

Moreover, using Lemma 2.1 with $\Psi(t,x)=\|\theta(T,\cdot)\|_{L^\infty(\Omega)}$ tells us that the $L^\infty(\Omega)$ norm of a solution, once finite, is non-increasing in time.

To show that the $L^\infty(\Omega)$ norm of a solution with $L^2(\Omega)$ initial data is bounded after a small time, consider the sequence of functions

$$\theta_k:=(\theta(t,x)-1+2^{-k})_+$$

and define

$$\mathcal{E}_k:=\int_{-1-2^{-k}}^0\int_\Omega\theta_k^2\,dxdt.$$

When $\theta_{k+1}>0$, then in particular $\theta_k\geq2^{-k-1}$. Thus for any finite p, there exists a constant C so

$$\chi_{\{\theta_{k+1}>0\}}\leq C^k\theta_k^p.$$

In particular,

$$\mathcal{E}_{k+1}\leq C^k\int_{-1-2^{-k}}^0\int\theta_k^3.$$

Applying the energy inequality θ, ϕ, and Γ we obtain

$$\sup_{-1-2^{-k-1}<t<0}\int\theta_{k+1}^2+\int_{-1-2^{-k-1}}^0\int\left|\Lambda^{1/2}\theta_{k+1}\right|^2\leq C^k\int_{-1-2^{-k}}^0\theta_k^2=\mathcal{E}_k.$$

However, by Sobolev embedding and the fact that $\mathcal{H}^{1/2}$ controls classical $H^{1/2}$ controls L^4,

$$\|\theta_{k+1}\|_{L^3([-1-2^{-k-1},0]\times\Omega)} \leq C^k \mathcal{E}_k^{1/2}.$$

Therefore

$$\mathcal{E}_{k+1} \leq C^k \mathcal{E}_k^{3/2}.$$

It follows by a well known result [citation] that for $\mathcal{E}_0$ sufficiently small (say less than $\bar{C}$), $\mathcal{E}_k \to 0$ as $k \to \infty$.

Notice that, since the $L^2(\Omega)$ norm of θ does not increase in time,

$$\mathcal{E}_0 = \int_{-2}^{0} \int_{\Omega} (\theta)_+ \, dxdt \leq 2 \int \theta_0^2 \, dx.$$

Moreover, as $k \to \infty$ we have

$$\mathcal{E}_k \to \int_{-1}^{0} \int_{\Omega} (\theta - 1)_+ \, dxdt$$

Thus, if $\|\theta_0\|_{L^2(\Omega)} \leq \sqrt{\bar{C}/2}$ then $\theta \leq 1$ on $[-1, 0]$.

Since (2) is linear and scales in time and space as in Lemma 6.1 (and since the constant $\bar{C}$ does not depend on Ω), we can take a solution θ with arbitrary initial L^2 norm and apply this result to a scaled version.

The result follows. □

3. Littlewood-Paley Theory

In this section we will prove that, because θ is uniformly bounded in L^∞, the velocity $u = \nabla^\perp \Lambda^{-1}\theta$ is calibrated (see Definition 1). The proof will utilize a Littlewood-Paley theory adapted to a bounded set Ω.

Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. Specifically, ϕ is non-negative, supported on $[1/2, 2]$, and has the property that

$$\sum_{j\in\mathbb{Z}} \phi(2^j \xi) = 1 \qquad \forall \xi \neq 0.$$

This allows us to define the Littlewood-Paley projections. For any $f = \sum f_k \eta_k$ in $L^2(\Omega)$

$$P_j f := \sum_k \phi(2^j \lambda_k^{1/2}) f_k \eta_k.$$

Note that P_j depends strongly on the domain Ω.

Recall that $-\Delta_D$ has some smallest eigenvalue λ_0 (depending on Ω) so if we define $j_0 = \log_2(\lambda_0) - 1$ then $P_j = 0$ for all $j < j_0$.

Our goal in this section is to prove the following proposition:

Proposition 3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $\theta \in L^\infty(\Omega)$. Then there exists a sequence of divergence-free functions $(u_j)_{j\in\mathbb{Z}}$ calibrated for some constant $\kappa = \kappa(\Omega, \|\theta\|_\infty)$ with center 0 (see Definition 1) such that*

$$\nabla^\perp \Lambda^{-1}\theta = \sum_{j\in\mathbb{Z}} u_j$$

with the infinite sum converging in the sense of L^2.

Moreover there exists some $j_0 \in \mathbb{Z}$ such that $u_j = 0$ for all $j < j_0$.

Before we can prove this, we state a few important lemmas.

The Bernstein Inequalities adapted for a bounded domain are proved in [IMT17]. We restate their result here:

Lemma 3.2 (Bernstein Inequalities)**.** *Let $1 \le p \le \infty$, and $\Omega \subset \mathbb{R}^2$ a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $(P_j)_{j\in\mathbb{Z}}$ be a Littlewood-Paley decomposition as defined above.*

There exists a constant C depending on p and Ω such that the following hold for any $f \in L^p(\Omega)$:

For any $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$\|\Lambda^\alpha P_j f\|_{L^p(\Omega)} \le C 2^{\alpha j} \|f\|_{L^p(\Omega)}.$$

For any $\alpha \in \mathbb{R}$ and $j \ge j_0$

$$\|\nabla \Lambda^\alpha P_j f\|_{L^p(\Omega)} \le C 2^{(1+\alpha)j} \|f\|_{L^p(\Omega)}.$$

Proof. The first claim is Lemma 3.5 in [IMT17]. It is also an immediate corollary of [IMT18] Theorem 1.1.

The second claim is similar to Lemma 3.6 in [IMT17]. A hypothesis of Lemma 3.6 is that

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le 1$$

(a property of Ω). The result of Lemma 3.6 only covers the case $j > 0$.

In [FMP04] it is proved that that if Ω is $C^{2,\alpha}$ then

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le T$$

which, by taking some T depending on j_0, is enough to prove the desired result for $j \ge j_0$ by a trivial modification of the proof in [IMT17]. □

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. In the case of $\mathbb{R}^2$, the Littlewood-Paley projections commute with the gradient so $P_i \nabla P_j = 0$ unless $|i-j| \le 1$. On a bounded domain, this is not the case. One can say that the gradient does not maintain localization in frequency-space. However, the following lemma formalizes the observation that $P_i \nabla P_j \approx 0$ when $i << j$.

Lemma 3.3. *There exists a constant C depending on Ω such that or any function $f \in L^\infty(\Omega)$,*

$$\|P_i \nabla P_j f\|_\infty \le C \min(2^j, 2^i) \|f\|_\infty.$$

Proof. Let g be an L^1 function. Then since P_i is self-adjoint

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \le C 2^j \|g\|_1 \|f\|_\infty$$

by Lemma 3.2.

Further integrating by parts,

$$\int g P_i \nabla P_j f = -\int (\nabla P_i g) P_j f \le C 2^i \|g\|_1 \|f\|_\infty.$$

This also follows from Lemma 3.2.

The result follows. □

This final lemma allows us to interpolate using Hölder seminorms. The results are not presumed to be novel, but since their proofs were difficult to find in the literature we include them below.

Lemma 3.4. *Let $\alpha \in (0,1)$. There exists a constant $C = C(\alpha)$ such that, for any set Ω and any $f \in C^{0,1}(\Omega)$,*

$$[f]_\alpha \le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha.$$

Let $\alpha \in (0,1)$ and Ω a set that satisfies the cone condition. There exist constants $C = C(\alpha, \Omega)$ and $\ell = \ell(\Omega)$ such that, for any $f \in C^{2,\alpha}(\Omega)$

$$\left\|D^2 f\right\|_\infty \le C\left(\delta^{-1} \|\nabla f\|_\infty + \delta^\alpha \left[D^2 f\right]_\alpha\right)$$

for all $\delta < \ell$.

We are now ready to prove Proposition 3.1.

Proof. For each $j \in \mathbb{Z}$, we define u_j to be the $\frac{\pi}{2}$-rotation of the Riesz transform of the j^{th} Littlewood-Paley projection of θ:

$$u_j := \nabla^\perp \Lambda^{-1} P_j \theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

We must bound u_j, $\Lambda^{-1/4} u_j$, and ∇u_j all in $L^\infty(\Omega)$.

By straightforward application of Lemma 3.2,

$$\|u_j\|_\infty \le C \|\theta\|_\infty . \tag{8}$$

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4} u_j = \sum_{i \in \mathbb{Z}} P_i \Lambda^{-1/4} u_j .$$

Define $\bar{P}_k := P_{k-1} + P_k + P_{k+1}$. Then $\bar{P}_k P_k = P_k$, and since the projections P_k are spectral operators, they commute with Λ^s. We therefore rewrite

$$\left(P_i \Lambda^{-1/4} u_j \right)^\perp = \left(\Lambda^{-1/4} \bar{P}_i \right) P_i \nabla P_j \left(\Lambda^{-1} \bar{P}_j \right) \theta .$$

On the right hand side we have three bounded linear operators applied sequentially to L^∞. The first operator has norm $C 2^{-j}(2^1 + 2^0 + 2^{-1})$ by Lemma 3.2. The second operator has norm $C \min(2^j, 2^i)$ by Lemma 3.3. The third operator has norm $C 2^{-i/4}(2^{1/4} + 2^0 + 2^{-1/4})$ by Lemma 3.2. (Of course, the perp operator is an isometry.) Therefore

$$\left\| P_i \Lambda^{-1/4} u_j \right\|_\infty \le C 2^{-i/4} \min(2^j, 2^i) 2^{-j} \|\theta\|_\infty .$$

Summing these bounds on the projections of $\Lambda^{-1/4} u_j$, and noting that

$$\sum_{i \in \mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i \le j} 2^{i3/4} + \sum_{i > j} 2^{-i/4} \le C 2^{-j/4},$$

we obtain

$$\left\| \Lambda^{-1/4} u_j \right\|_\infty \le C 2^{-j/4} \|\theta\|_\infty . \tag{9}$$

Lastly, we must show that ∇u_j is in L^∞. Equivalently, we will show that $\Lambda^{-1} P_j \theta$ is $C^{1,1}$. The method of proof is Schauder theory.

For convenience, define

$$F := \Lambda^{-1} P_j \theta .$$

Notice that F is a linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Therefore

$$-\Delta F = \Lambda^2 F = \Lambda P_j \theta .$$

We apply the standard Schauder estimate from [GT01] Theorem 6.6 to bound some $C^{2,\alpha}$ semi-norm of F by the L^∞ norm of F and the C^α norm of its Laplacian. By assumption there exists $\beta \in (0,1)$ such that Ω is $C^{2,\beta}$, and for this β we have by the Schauder estimate

$$\left[D^2 F \right]_\beta \le C \left\| \Lambda^{-1} P_j \theta \right\|_\infty + C \|\Lambda P_j \theta\|_\infty + C [\Lambda P_j \theta]_\beta . \tag{10}$$

By Lemma 3.2,

$$\begin{aligned} \left\| \Lambda^{-1} P_j \theta \right\|_\infty &\le C 2^{-j} \|\theta\|_\infty , \\ \|\Lambda P_j \theta\|_\infty &\le C 2^{j} \|\theta\|_\infty , \\ \|\nabla \Lambda P_j \theta\|_\infty &\le C 2^{2j} \|\theta\|_\infty . \end{aligned}$$

By Lemma 3.4 we can interpolate these last two bounds to obtain

$$[\Lambda P_j\theta]_\beta \le C2^{j(1+\beta)}\,\|\theta\|_\infty .$$

Plugging these estimates into (10) yields

$$\left[D^2F\right]_\beta \le C\left(2^{-j}+2^j+2^{j(1+\beta)}\right)\|\theta\|_\infty .$$

Recall that without loss of generality we can assume $j \ge j_0$. Therefore up to a constant depending on j_0, the term $2^{j(1+\beta)}$ bounds 2^j and 2^{-j} so we can write

$$\left[D^2F\right]_\alpha \le C2^{j(1+\beta)}\,\|\theta\|_\infty .$$

Using this estimate and the fact that $\|\nabla F\|_\infty = \left\|\nabla\Lambda^{-1}P_j\theta\right\|_\infty \le C\,\|\theta\|_\infty$ (see (8)), we can use Lemma 3.4 to interpolate. For some constant ℓ depending on Ω, for any $\delta \le \ell$ we have

$$\begin{aligned}\left\|D^2F\right\|_\infty &\le C\left(\delta^{-1}\,\|\nabla F\|_\infty + \delta^\beta\left[D^2F\right]_\beta\right)\\ &\le C\left(\delta^{-1}+\delta^\beta 2^{j(1+\beta)}\right)\|\theta\|_\infty .\end{aligned}$$

Set $\delta = 2^{-j}(2^{j_0}\ell) \le \ell$. Then

$$\left[D^2F\right]_\infty \le C\left(C2^j + 2^{-j\beta}2^{j(1+\beta)}\right)\|\theta\|_\infty = C(\Omega)2^j\,\|\theta\|_\infty .$$

Since $D^2F = \nabla u_j$, this estimate together with (8) and (9) complete the proof. □

Now that we know that u is calibrated, we end the section with a final lemma to show why calibrated sequences are useful.

4. De Giorgi Estimates

Our goal in this section is to prove De Giorgi's first and second lemmas for solutions to (2) with u uniformly calibrated.

The first step is to show that a calibrated velocity can be rewritten in a more practical format:

Lemma 4.1. *Let*

$$u = \sum_{j_0}^{\infty} u_j$$

with the sum converging in the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center, and that $\mathrm{div}(u_j) = 0$*.*

Then

$$u = u_l + u_h$$

with

$$\begin{aligned}\|\nabla u_l\|_{L^\infty([-T,0]\times\Omega)} &\le 2\kappa,\\ \left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-T,0]\times\Omega)} &\le 6\kappa.\end{aligned}$$

and $\mathrm{div}(u_l) = \mathrm{div}(u_h) = 0$*.*

We call u_l the low-pass term, and u_h the high-pass term.

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j>N} u_j$$

and bound

$$\left\|\Lambda^{-1/4}u_h\right\|_\infty \le \sum_{j>N}\left\|\Lambda^{-1/4}u_j\right\|_\infty \le \kappa\frac{2^{-1/4}}{1-2^{-1/4}}.$$

We define

$$u_l = \sum_{j=j_0}^{N} u_j.$$

and bound

$$\|\nabla u_l\|_\infty \le \sum_{j\le N} \|\nabla u_j\|_\infty \le \kappa \frac{1}{1-2^{-1}}.$$

□

In order to prove the De Giorgi lemmas, we must derive an energy inequality for the function $(\theta-\Psi)_+$ where $\Psi(t,x)$ is not constant, but rather grows sublinearly in $|x|$. However, applying Lemma 2.1 to such a function, we see that its derivatives are only small if the quantity $\partial_t\Psi + u\cdot\nabla\Psi$ is bounded.

To that end, we shall consider a family of functions θ, u_l, and $u_h : [-T,0]\times\Omega \to \mathbb{R}$, and paths Γ and $\gamma : [-5,0] \to \mathbb{R}^2$ which satisfy

$$\begin{cases} \partial_t\theta + (u_l+u_h)\cdot\nabla\theta + \Lambda\theta = 0 & \text{on } [-T,0]\times\Omega, \\ \operatorname{div}(u_l) = \operatorname{div}(u_h) = 0 & \text{on } [-T,0]\times\Omega, \\ \dot\Gamma(t) + \dot\gamma(t) = u_l(t,\gamma(t)+\Gamma(t)) & \text{on } [-T,0], \\ \gamma(0) = \Gamma(0) = 0. \end{cases} \tag{11}$$

Here it is implicitly assumed that $\gamma(t)+\Gamma(t)\in\Omega$. Generally speaking u_l and γ will be Lipschitz functions while u_h is merely in a weak space $W^{-1/4,\infty}$ and Γ will trace out points in Ω where θ is well behaved by assumption. See Section 6 for the construction of these functions Γ and γ.

Now we prove an energy inequality for solutions to (11).

Lemma 4.2 (Energy inequality). *Let κ, C_Ω, C_g, T, and R be positive constants, and let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a function with bounded first and second derivatives. Then there exists a constant $C > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 1.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_l, u_h, Γ and γ solve (11) on $[-T,0]\times\Omega$, and satisfy $\|\Lambda^{-1/4}u_h\|_{L^\infty([-T,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_l\|_{L^\infty([-T,0]\times\Omega)} \le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-T,0])} \le C_g$.

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ have bounded derivative and bounded second derivative, and consider the functions

$$\theta_+ := (\theta - \psi(\cdot-\Gamma))_+, \qquad \theta_- := (\psi(\cdot-\Gamma)-\theta)_+.$$

If θ_+ is supported on $x\in\Omega\cap B_R(\Gamma(t))$ for some $R>0$, then θ_+ and θ_- satisfy the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+,\theta_-\rangle_{1/2} \le C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+ + \int \theta_+^2\right).$$

Proof. Define

$$\Psi(t,x) := \psi(x-\Gamma(t))$$

so that

$$\partial_t\Psi + (u_l+u_h)\cdot\nabla\Psi = (u_l - \dot\Gamma + u_h)\cdot\nabla\psi(x-\Gamma(t)).$$

Applying Lemma 2.1 to θ and Ψ we arrive at

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+,\theta_-\rangle_{1/2} \le C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+(u_l - \dot\Gamma(t) + u_h)\cdot\nabla\psi(x-\Gamma(t))\right).$$

Consider first the high-pass term $\int \theta_+ u_h \cdot \nabla\psi$. By inserting $\Lambda^{1/4}\Lambda^{-1/4}$ and then integrating by parts, we can apply Lemma 1.2 parts (e) and (c) to obtain

$$\int \Lambda^{-1/4} u_h \Lambda^{1/4}(\theta_+ \nabla\psi) \le C \left\|\Lambda^{-1/4} u_h\right\|_\infty \left(\|\nabla\psi\|_\infty + \left\|D^2\psi\right\|_\infty\right) |\operatorname{supp}(\theta_+)|^{1/2} \left(\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}\right).$$

From Hölder's inequality with Peter-Paul, we obtain

$$\int u_h \theta_+ \nabla\psi(x-\gamma(t))\, dx \le C(\psi,\varepsilon)\kappa \left(\int \chi_{\{\theta_+>0\}} + \int \theta_+^2\right) + \varepsilon \int \left|\Lambda^{1/2}\theta_+\right|^2.$$

Consider now the low-pass term. Recall that

$$\dot\Gamma + \dot\gamma = u_l(t, \Gamma+\gamma)$$

so

$$u_l(t,x) - \dot\Gamma(t) = u_l(t,x) - u_l(t,\Gamma+\gamma) + \dot\gamma.$$

By assumption, $|\dot\gamma| \le C_g$ and so for $t \in [-T,0]$ we have $|\gamma(t)| \le TC_g$.

Since u_l is has bounded derivative,

$$\begin{aligned} |u_l(t,x) - u_l(t,\Gamma+\gamma)| &\le |u_l(t,x) - u_l(t,\Gamma)| + |u_l(t,\Gamma) - u_l(t,\Gamma+\gamma)| \\ &\le 2\kappa|x-\Gamma| + 2\kappa T C_g. \end{aligned}$$

Plugging these bounds into [cite] we obtain

$$\left|u_l(t,x) - \dot\Gamma(t)\right| \le (1+2\kappa T)C_g + 2\kappa|x-\Gamma|.$$

Now we can bound the low pass term

$$\int (u_l - \dot\Gamma)\theta_+ \nabla\psi(x-\Gamma) \le (1+2\kappa T)C_g \|\nabla\psi\|_\infty \int \theta_+\, dx + \|\nabla\psi\|_\infty 2\kappa \int |x-\Gamma|\theta_+\, dx.$$

By assumption, $|x-\Gamma|\theta_+ \le R\theta_+$, so the result follows. □

This energy inequality is sufficient to apply the method of De Giorgi.

Lemma 4.3 (First De Giorgi Lemma). *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $\delta_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 1.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_l, u_h, Γ and γ solve (11) on $[-2,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-2,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_l\|_{L^\infty([-2,0]\times\Omega)} \le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-2,0])} \le C_g$.

If

$$\theta(t,x) \le 2 + \left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \qquad \forall t \in [-2,0], x \in \Omega \setminus B_2(\Gamma(t))$$

and

$$\int_{-2}^0 \int_{\Omega\cap B_2(\Gamma(t))} (\theta)_+^2\, dxdt \le \delta_0$$

then

$$\theta(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma(t)).$$

Proof. Let ψ be such that $\psi = 0$ on B_1 and $\psi(x) \ge 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 2$ while ψ is Lipschitz and C^2 and its gradient decays like $|x|^{-3/4}$.

Consider the sequence of functions

$$\theta_k := (\theta(t,x) - \psi(x-\Gamma(t)) - 1 + 2^{-k})_+$$

and define

$$\mathcal{E}_k := \int_{-1-2^{-k}}^0 \int_\Omega \theta_k^2\, dxdt.$$

Notice that

$$\mathcal{E}_0 = \int_{-2}^{0} \int_{\Omega} (\theta - \psi(x - \Gamma))_+^2 \, dxdt \leq \delta_0.$$

Moreover, as $k \to \infty$ we have

$$\mathcal{E}_k \to \int_{-1}^{0} \int_{\Omega} (\theta - \psi(x - \Gamma) - 1)_+^2 \, dxdt$$

so in particular, if we can show $\mathcal{E}_k \to 0$ then $\theta \leq 1$ for $t \in [-1, 0]$ and $x \in B_1(\Gamma)$.

That's enough setup, let's argue that $\mathcal{E}_k \to 0$. Notice that when $\theta_{k+1} > 0$, then in particular $\theta_k \geq 2^{-k}$ [or something similar]. Thus for any finite p, there exists a constant C so

$$\chi_{\{\theta_{k+1} > 0\}} \leq C^k \theta_k^p.$$

In particular,

$$\mathcal{E}_{k+1} \leq C^k \int_{-1-2^{-k}}^{0} \int \theta_k^3.$$

Applying the energy inequality θ, ψ, and Γ we obtain

$$\sup_{-1-2^{-k-1} < t < 0} \int \theta_{k+1}^2 + \int_{-1-2^{-k-1}}^{0} \int \left| \Lambda^{1/2} \theta_{k+1} \right|^2 \leq C^k \int_{-1-2^{-k}}^{0} \int \theta_k^2 = \mathcal{E}_k.$$

However, by Sobolev embedding and the fact that $\mathcal{H}^{1/2}$ controls classical $H^{1/2}$ controls L^4, we know from Reisz-Thorin that the left side of the energy inequality controls the L^3 norm of θ_{k+1} so

$$\|\theta_{k+1}\|_{L^3([-1-2^{-k-1}, 0] \times \Omega)} \leq C^k \mathcal{E}_k^{1/2}.$$

Therefore

$$\mathcal{E}_{k+1} \leq C^k \mathcal{E}_k^{3/2}.$$

It follows by a well known result [citation] that for $\mathcal{E}_0$ sufficiently small (say less than δ_0), $\mathcal{E}_k \to 0$ as $k \to \infty$ which we already established is sufficient to obtain our result. $\square$

This is coming along quite nicely. We can move on to DG2, the isoperimetric inequality.

Lemma 4.4 (Second De Giorgi Lemma). *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $\mu > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 1.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x, y) \leq C_\Omega |x - y|^{1/2} K_1.$$

Let θ, u_l, u_h, Γ and γ solve (11) on $[-2, 0] \times \Omega$, and satisfy $\|\Lambda^{-1/4} u_h\|_{L^\infty([-2,0]\times\Omega)} \leq 6\kappa$, $\|\nabla u_l\|_{L^\infty([-2,0]\times\Omega)} \leq 2\kappa$, and $\|\dot{\gamma}\|_{L^\infty([-2,0])} \leq C_g$.

Suppose that for $t \in [-5, 0]$ and any $x \in \Omega$,

$$\theta(t, x) \leq 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+.$$

Then the three conditions

$$|\{\theta \geq 1\} \cap [-2, 0] \times B_2(\Gamma)| \geq \delta_0 / 4, \tag{12}$$

$$|\{0 < \theta < 1\} \cap [-4, 0] \times B_4(\Gamma)| \leq \mu,$$

$$|\{\theta \leq 0\} \cap [-4, 0] \times B_4(\Gamma)| \geq 2|B_4| \tag{13}$$

cannot simultaneously be met.

Here δ_0 is the constant from Lemma 4.3, which of course depends on κ, C_g, and C_Ω.

Proof. Suppose that the theorem is false. Then there must exist, for each $n \in \mathbb{N}$, a bounded $C^{2,\alpha}$ set Ω_n and function $\theta_n : [-5,0] \times \Omega_n \to \mathbb{R}$, functions $u_l^n, u_h^n : [-5,0] \times \Omega_n \to \mathbb{R}^2$, and paths $\Gamma_n, \gamma_n : [-5,0] \to \mathbb{R}^2$ which solve (11) and satisfy all of the the assumptions of our lemma (with the same constants κ, C_g, and C_Ω), except that

$$|\{0 < \theta_n < 1\} \cap [-4,0] \times B_4(\Gamma_n)| \le 1/n. \tag{14}$$

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function which vanishes on B_2 such that $\psi(x) = 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 3$.

Fix n and define

$$\theta_+ := (\theta_n - \psi(x - \Gamma_n))_+ .$$

Then θ_+ is supported on $\Omega \cap B_3(\Gamma_n)$ and is less than $2 + 3^{1/4} - 2^{1/4} \le 3$ everywhere.

Our goal is to bound the derivatives of θ_+^3 so that we can apply a compactness argument to the sequence θ_n. (For the curious reader, we will point out the steps below in which it is important to consider θ_+^3 instead of θ_+.)

Apply the energy inequality Lemma 4.2 to θ and $\psi(x-\Gamma_n)$, and find that for some C independent of n

$$\frac{d}{dt}\int \theta_+^2 \le C \tag{15}$$

and moreover that

$$\sup_{[-4,0]}\int \theta_+^2 + \int_{-4}^0\int \left|\Lambda^{1/2}\theta_+\right|^2 + \int_{-4}^0\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C. \tag{16}$$

This proves in particular that $\theta_+ \in L^2(-4,0;\mathcal{H}^{1/2}(\Omega))$ is uniformly bounded.

What's more, $\|\theta_+^3\|_{L^2(-4,0;\mathcal{H}^{1/2}(\Omega_n))}$ is uniformly bounded because

$$\begin{aligned}\left\|\Lambda^{1/2}(\theta_+^3)\right\|_2^2 &= \iint [\theta_+(x)^3 - \theta_+(y)^3]^2 K + \int \theta_+^6 B\\ &\le 2\iint \theta_+(x)^4[\theta_+(x)-\theta_+(y)]^2K + 2\iint \theta_+(y)^4[\theta_+(x)-\theta_+(y)]^2K + \|\theta_+\|_\infty^4\int\theta_+^2 B\\ &\le C\,\|\theta_+\|_\infty^4\,\|\theta_+\|_{\mathcal{H}^{1/2}}^2 .\end{aligned}$$

By Lemma 1.3, if $E\theta_+^3$ is the zero-extension of θ_+^3 from Ω_n to $\mathbb{R}^2$, then

$$\left\|E\theta_+^3\right\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \le C \tag{17}$$

where C does not depend on n.

Since θ_n solves the equation

$$\partial_t\theta_n + (u_h + u_l)\cdot\nabla\theta_n + \Lambda\theta_n = 0,$$

multiply this equation by $\varphi\theta_+^2$, where φ is any function in $C^2(\mathbb{R}^2)$ restricted to Ω_n, and integrate to obtain

$$\begin{aligned}\frac{1}{3}\int\varphi\partial_t\theta_+^3 + \frac{1}{3}\int\varphi\dot{\Gamma}_n\cdot\nabla\theta_+^3 = \frac{-1}{3}\int\varphi(u_l^n - \dot{\Gamma}_n + u_h^n)\cdot\nabla\theta_+^3 - \int\varphi\theta_+^2(u_l^n - \dot{\Gamma}_n + u_h^n)\cdot\nabla\psi\\ - \int\varphi\theta_+^2\Lambda\theta_+ - \int\varphi\theta_+^2\Lambda\psi + \int\varphi\theta_+^2\Lambda\theta_-.\end{aligned}$$

Further rearranging, this becomes

$$\begin{aligned}\int\varphi\partial_t\theta_+^3 + \int\varphi\dot{\Gamma}_n\cdot\nabla\theta_+^3 = \int(u_l^n - \dot{\Gamma}_n)\cdot(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi) + \int\Lambda^{-1/4}u_h^n\Lambda^{1/4}\left(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi\right)\\ - 3\int\varphi\theta_+^2\Lambda\theta_+ - 3\int\varphi\theta_+^2\Lambda\psi + 3\int\varphi\theta_+^2\Lambda\theta_-.\end{aligned}$$

We will bound the five terms on the right hand side one at a time.

Each instance of C in the following bounds is independent of n.

- Consider the low-pass term. As in the proof of Lemma 4.2, we have $|u_l^n(t,x) - \dot{\Gamma}_n(t)| \leq (1+8\kappa)C_g + 6\kappa$ for $t \in [-4,0]$ and $x \in \operatorname{supp}(\theta_+) \subseteq \Omega_n \cap B_3(\Gamma_n(t))$. Thus for $t \in [-4,0]$ we have for C independent of n and of φ

$$\int (u_l^n - \dot{\Gamma}_n) \cdot (\theta_+^3 \nabla \varphi - 3\varphi\theta_+^2 \nabla\psi) \leq C\left(\|\nabla\varphi(t,\cdot)\|_{L^\infty(\Omega)} + \|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\right).$$

- Consider the high-pass term. By Lemma 1.2 parts e and c,

$$\int \Lambda^{-1/4} u_h^n \Lambda^{1/4}\left(\theta_+^3 \nabla\varphi - 3\varphi\theta_+^2\nabla\psi\right) \leq C\kappa|\operatorname{supp}(\theta_+)|^{1/2}\left(\left\|\theta_+^3\nabla\varphi\right\|_{L^2} + \left\|\varphi\theta_+^2\nabla\psi\right\|_{L^2} + \left\|\theta_+^3\nabla\varphi\right\|_{\mathcal{H}^{1/2}} + \left\|\varphi\theta_+^2\nabla\psi\right\|_{\mathcal{H}^{1/2}}\right).$$

$$\leq C\left(\|\varphi(t,\cdot)\|_{C^1(\Omega)} + \|\varphi(t,\cdot)\|_{C^2(\Omega)}\|\theta(t,\cdot)\|_{\mathcal{H}^{1/2}}\right).$$

- Consider the $\Lambda\theta_+$ term. Decomposing this term using Proposition 1.1 we have first an interior term $\iint [\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\theta_+(x) - \theta_+(y)]K$ which decomposes as

$$\iint \varphi(x)(\theta_+(x) + \theta_+(y))[\theta_+(x) - \theta_+(y)]^2 K + \iint \theta_+(y)^2[\varphi(x) - \varphi(y)][\theta_+(x) - \theta_+(y)]K.$$

The first part is bounded by the L^∞ norms of φ and θ_+ and the square of the $\mathcal{H}^{1/2}$ norm of θ_+, while the second part is bounded

$$\iint \theta_+(y)^2[\varphi(x) - \varphi(y)][\theta_+(x) - \theta_+(y)]K \leq \|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}\sqrt{\int \theta_+(y)^2 \int \frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^3}dx\,dy}$$

which is bounded by the C^1 norm of φ and the $\mathcal{H}^{1/2}$ norm of θ_+.

The boundary term $\int \varphi\theta_+^3 B$ is bounded by the L^∞ norms of φ and θ_+, and by $\int \theta_+^2 B$ which is less than $\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}$. Taken together we have

$$\int \varphi\theta_+^2\Lambda\theta_+ \leq C\left(\|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2 + \|\varphi(t,\cdot)\|_{C^1(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}\right).$$

- Consider the $\Lambda\theta_-$ term. For any non-negative function f we know by Lemma 1.2 part (a) that $\int f\theta_+\Lambda\theta_- \leq 0$. It follows that $-\theta_+\Lambda\theta_-$ is a pointwise non-negative distribution. Moreover, the integral over $[-4,0]\times\Omega$ of $-\theta_+\Lambda\theta_-$ is bounded by (16). Thus $\theta_+\Lambda\theta_-$ is a measure with bounded total-variation norm. In fact, because φ is a continuous function,

$$\int_{-4}^0 \int \varphi\theta_+^2\Lambda\theta \leq \|\theta_+\|_\infty \|\theta_+\Lambda\theta_-\|_{\mathcal{M}}\|\varphi\|_{C^0} \leq C\|\varphi\|_{L^\infty([-4,0]\times\Omega)}.$$

- Consider the $\Lambda\psi$ term. Decomposing this term using Proposition 1.1 we have first an interior term $\iint [\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\psi(x) - \psi(y)]K$ which decomposes as

$$\int \theta_+(y)^2 \int [\varphi(x) - \varphi(y)][\psi(x) - \psi(y)]K + \iint \varphi(x)[\theta_+(x)^2 - \theta_+(y)^2][\psi(x) - \psi(y)]K.$$

The first part is bounded by the C^1 norms of φ and ψ and the L^2 norm of θ_+, while the second part is bounded

$$\iint \varphi(x)[\theta_+(x)^2 - \theta_+(y)^2][\psi(x) - \psi(y)]K \leq \left\|\theta_+^2(t,\cdot)\right\|_{\mathcal{H}^{1/2}}\sqrt{\int \varphi(x)^2 \int \frac{[\psi(x)-\psi(y)]^2}{|x-y|^3}dy\,dx}$$

which is bounded, because ψ is smooth and globally 1/4-Hölder continuous, by L^2 norm of φ and the $\mathcal{H}^{1/2}$ norm of θ_+.

The boundary term $\int \varphi\theta_+^2\psi B$ is bounded by the L^∞ norms of φ and $\psi\chi_{\{\theta_+>0\}}$ and by $\int \theta_+^2 B$ which is less than $\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2$. Taken together we have

$$\int \varphi\theta_+^2\Lambda\psi \leq C\left(\|\varphi(t,\cdot)\|_{C^1(\Omega)} + \|\varphi(t,\cdot)\|_{L^2(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}} + \|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2\right).$$

Remark. We are attempting to bound $\partial_t \theta_+^3$. If we had attempted to bound $\partial_t \theta_+^p$ instead, the final three terms above would have been problematic for $p = 1$ and the very final term would have been problematic for $p = 2$.

Combining all of these bounds, and using the fact that $\theta_+ \in L^2(-4, 0; \mathcal{H}^{1/2})$ uniformly, we conclude that there exists a constant C independent of n such that, for any $\varphi \in L^\infty(-4, 0; C^2(\mathbb{R}^2)) \cap L^\infty(-4, 0; L^2(\mathbb{R}^2))$,

$$\int_{-4}^{0} \int_{\Omega_n} \left(\partial_t \theta_+^3 + \dot{\Gamma}_n \cdot \nabla \theta_+^3\right) \varphi \, dx dt \le C \left\|\varphi\right\|_{L^\infty(-4,0;C^2(\mathbb{R}^2))} + C \left\|\varphi\right\|_{L^\infty(-4,0;L^2(\mathbb{R}^2))} . \tag{18}$$

Over time, the support of θ_+^3 moves around in Ω_n following the path Γ_n. In order to take a meaningful limit in n, we must shift these functions so that their supports remain in a compact set. To that end, define a new function on $[-4, 0] \times \mathbb{R}^2$ by

$$v_n(t, x) := \begin{cases} \theta_+(t, x + \Gamma_n(t))^3, & x + \Gamma_n(t) \in \Omega_n, \\ 0, & x + \Gamma_n(t) \notin \Omega_n. \end{cases}$$

In other words,

$$v_n(t, x) = (\theta_n(t, x + \Gamma_n(t)) - \psi(x))_+^3 \tag{19}$$

when the right hand side is defined.

Let $X \subseteq C^2(\mathbb{R}^2)$ be the Banach space of C^2 functions with norm $\|\cdot\|_X = \|\cdot\|_{C^2(\mathbb{R}^2)} + \|\cdot\|_{L^2(\mathbb{R}^2)}$ finite. Note that

$$\partial_t v_n(t, x) = \partial_t \theta_+^3(t, x + \Gamma_n) + \dot{\Gamma}_n \cdot \nabla \theta_+^3(t, x + \Gamma_n).$$

We know from (17) that

$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \le C$$

and from (18) that

$$\|\partial_t v_n\|_{L^1(-4,0;X^*)} \le C.$$

According to the Aubin-Lions Lemma, the set $\{v_n\}_n$ is therefore compactly embedded in $L^2([-4, 0] \times \mathbb{R}^2)$. Up to a subsequence, there is a function $v \in L^2([-4, 0] \times \mathbb{R}^2)$ such that

$$v_n \to v.$$

By elementary properties of L^2 convergence, we know that $v \in L^\infty$, $\mathrm{supp}(v) \subseteq [-4, 0] \times B_3(0)$, and $v \in L^2(H^{1/2})$.

By (15)

$$\frac{d}{dt} \int_{\mathbb{R}^2} v_n^{2/3} \, dx = \frac{d}{dt} \int_{\Omega_n} \theta_+^2 \, dx \le C \tag{20}$$

so the same must be true of v.

By (12), (14), and (13) applied to v_n (recalling the relation (19)), we conclude that

$$\begin{cases} |\{v \ge 1\} \cap [-2, 0] \times B_2(0)| & \ge \delta_0/4, \\ \left|\{0 < v < [1 - \psi]^3\} \cap [-4, 0] \times B_4(0)\right| & \le 0, \\ |\{v \le 0\} \cap [-4, 0] \times B_4(0)| & \ge 2|B_4| \end{cases} \tag{21}$$

For any $(t, x) \in [-4, 0] \times B_4(0)$, either $v(t, x) \ge [1 - \psi(x)]^3$ or else $v(t, x) = 0$. In fact, since $\|v(t, \cdot)\|_{H^{1/2}} < \infty$ for almost every t and $H^{1/2}$ does not contain functions with jump discontinuities, the function v is either identically 0 or else $\ge [1 - \psi(x)]^3$ at each t.

Thus $\int v(t, x)^{2/3} \, dx$ is either 0 or else $\ge \int [1 - \psi(x)]^3 \, dx > 0$ at each t. By (20) and (21), v must be identically zero for all $t > -2$ but also must be non-zero for some $t > -2$, which is a contradiction. Our assumption that the sequence θ_n exists must have been false. The proposition must be true.

$\square$

5. A Decrease in Oscillation

We put together Propositions 4.3 and 4.4 to produce an Oscillation lemma. It allows us to improve a bound on the supremum of a function based on information about its positive support.

Proposition 5.1 (Oscillation Lemma). *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $k_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 1.1 *hold on Ω with kernels that satisfy*

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_l, u_h, Γ and γ solve (11) *on $[-2,0]\times\Omega$, and satisfy $\|\Lambda^{-1/4} u_h\|_{L^\infty([-2,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_l\|_{L^\infty([-2,0]\times\Omega)} \le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-2,0])} \le C_g$.*

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$\theta(t,x) \le 2 + 2^{-k_0}\left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+,$$

and that

$$|\{\theta \le 0\} \cap [-4,0]\times B_4(\Gamma)| \ge \frac{4|B_4|}{2}.$$

Then for all $t \in [-1,0]$, $x \in \Omega \cap B_1(\Gamma)$ we have

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

Proof. Let μ and δ_0 as in Proposition 4.4, and take k_0 large enough that $(k_0-1)\mu > 4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x) := 2 + 2^k(\theta(t,x) - 2).$$

That is, $\theta_0 = \theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies [cite, boundedness], each θ_k for $k \le k_0$ and $(t,x) \in [-5,0]\times\Omega$ satisfies

$$\theta_k(t,x) \le 2 + \left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 4.4.

Note also that

$$|\{\theta_k \le 0\} \cap [-4,0]\times B_4(\Gamma)|$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0]\times B_2(\Gamma)| \ge \delta_0/4$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed $\delta_0/4$ for all $k < k_0$ as well.

Applying Proposition 4.4 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0]\times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity [cite] increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity [cite] is bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption [cite] must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 4.3. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

For the original function θ, this means that

$$\theta(t,x) \le 2 - 2^{-k_0} \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

□

That's the absolute gain. Now let us consider how this gain can be shifted to our new reference frame. But first, a quick technical lemma:

Lemma 5.2. *There exist constants $\bar{\lambda} > 0$ and $\alpha > 1$ such that, for any $0 < \varepsilon \le 1/2$ and any $z \ge 1$*

$$\left(|\varepsilon^{-1}(z-1)+3|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+ \ge \bar{\lambda}.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon = 1/2$. Consider

$$\left(|2z+1|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+.$$

When $\alpha = 1$, this quantity is clearly non-negative and in fact strictly positive when $z \ge 1$. On any compact interval $[0, N]$, the quantity with $\alpha = 1$ is bounded below, and the quantity $\left(|z|^{1/4} - 2^{1/4}\right)_+$ is bounded above, so if $\alpha - 1$ is less than the ratio of those bounds then the total quantity will be bounded below.

However, the range of acceptable α depends on N, and it is possible that no single α is acceptable for the whole of $z \in [1, \infty)$.

For $z > 2$, the expression reduces to

$$(2z+1)^{1/4} - \alpha z^{1/4} - (\alpha-1)2^{1/4} = z^{1/4}\left((2+1/z)^{1/4} - \alpha\right) - (\alpha-1)2^{1/4}.$$

This quantity is increasing as α decreases, and for any $\alpha < 2^{1/4}$ it tends to ∞ as z increases.

This is sufficient to show that for some $\alpha > 1$, there exists a lower bound $\bar{\lambda}$ on the quantity [cite], and thus the lemma holds. □

We are ready to prove the shifted version of the Harnack Inequality.

Lemma 5.3 (Oscillation Lemma, with shift)**.** *Let κ, C_Ω, and C_g, be positive constants, and let k_0 be as in Lemma 5.1. Then there exists a constant $\lambda > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 1.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_l, u_h, Γ and γ solve (11) on $[-5,0]\times\Omega$, and satisfy $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_l\|_{L^\infty([-5,0]\times\Omega)} \le 2\kappa$, and $\|\dot{\gamma}\|_{L^\infty([-5,0])} \le C_g$.

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$|\theta(t,x)| \le 2 + 2^{-k_0}\left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \tag{22}$$

and that

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge 2|B_4|.$$

Then for any $\varepsilon \in (0, 1/5)$ such that

$$5C_g \le \varepsilon^{-1} - 3 \tag{23}$$

we have

$$\left|\frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]\right| \le 2 + 2^{-k_0}\left(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|^{1/4} - 2^{1/4}\right)_+.$$

for all $t \in [-5,0]$ and x such that $\varepsilon x \in \Omega$.

If we only wish to show that by zooming horizontally by a large amount and zooming and translating vertically by a small amount we stay under the barrier, this is obvious and merely requires being written down. Even the shift itself is clearly not a problem when considered in the un-zoomed coordinates. Since the velocity of γ is bounded by C_g, the shift γ is arbitrarily small over very small time periods. The important thing to pay attention for is the dependence of ε and C_g and k_0 on eachother.

As we will see in Section 6 when we apply this lemma, the constant C_g depends on ε and k_0 depends on C_g. In the following proof, the constant ε will need to be small relative to C_g. The assumption (23) in this lemma turns out to be satisfiable, and now we must prove that it is sufficient.

Proof. Take λ such that

$$2\lambda \le 2^{-k_0}, \qquad (2+\lambda)\left(\frac{2}{2-\lambda}\right) \le 2 + 2^{-k_0}\bar{\lambda}, \qquad \frac{2}{2-\lambda} \le \alpha. \tag{24}$$

for $\bar{\lambda}$ and α from Lemma 5.2.

Denote

$$\bar{\theta}(t,x) := \frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]$$

defined for $t \in [-5/\varepsilon, 0]$ and

$$x \in \Omega_\varepsilon := \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$$

and denote

$$\phi(x) := \left(|x|^{1/4} - 2^{1/4}\right)_+.$$

We already proved in Lemma 5.1 that $\theta \le 2 - 2^{-k_0}$ for $t \in [-1,0]$ and $x \in \Omega \cap B_1(\Gamma)$. On this same set, $\theta \ge -2$ by assumption. For $\bar{\theta}$, this means that when $t \in [-1/\varepsilon, 0]$ and $x \in \Omega \cap B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$,

$$\begin{cases} \bar{\theta}(t,x) & \le \frac{2}{2-\lambda}\left[2 - 2^{-k_0} + \lambda\right] \le \frac{2}{2-\lambda}\left[2-\lambda\right] = 2. \\ \bar{\theta}(t,x) & \ge \frac{2}{2-\lambda}\left[-2+\lambda\right] = -2. \end{cases} \tag{25}$$

Similarly, the bound (22) on θ becomes the equivalent bounds on $\bar{\theta}$, for all $(t,x) \in [-5/\varepsilon, 0] \times \Omega_\varepsilon$

$$\bar{\theta}(t,x) \le \frac{2}{2-\lambda}\left[2 + \lambda + 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right] \tag{26}$$

and

$$\bar{\theta}(t,x) \ge \frac{2}{2-\lambda}\left[-2 + \lambda - 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right]. \tag{27}$$

It remains to show that these bounds (25), (26), and (27) on $\bar{\theta}$ imply the bound stipulated by the proposition.

Let $t \in [-5, 0]$ and $x \in \Omega_\varepsilon$, and define

$$y := x - \varepsilon^{-1}\Gamma(\varepsilon t).$$

From (26) and the assumptions (24), we can bound

$$\begin{aligned} \bar{\theta}(t,x) &\le \frac{2}{2-\lambda}\left[2 + \lambda + 2^{-k_0}\phi(\varepsilon|y|)\right] \\ &\le 2 + 2^{-k_0}\bar{\lambda} + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ &= 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]. \end{aligned}$$

From (27) and the assumptions (24), we can bound

$$\begin{aligned} -\bar{\theta}(t,x) &\le \frac{2}{2-\lambda}\left[2 - \lambda + 2^{-k_0}\phi(\varepsilon|y|)\right] \\ &\le 2 + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ &\le 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]. \end{aligned}$$

Therefore

$$\left|\bar{\theta}(t,x)\right| \le 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]. \tag{28}$$

If $|y| \le \varepsilon^{-1}$ then from (25) we have

$$\left|\bar{\theta}(t,x)\right| \le 2 \le 2 + 2^{-k_0}\phi(x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t))$$

and the proof would be complete. Therefore assume without loss of generality that $|y| \geq \varepsilon^{-1}$. In this case we can apply Lemma 5.2 so

$$2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right] \leq 2 + 2^{-k_0}\phi(|y| - \varepsilon^{-1} + 3).$$

For $t \in [-5, 0]$, we have by assumption (23)

$$|y| - \varepsilon^{-1} + 3 \leq |y| - 5C_g \leq |y - \varepsilon^{-1}\gamma(\varepsilon t)|.$$

The estimate (28) becomes

$$\left|\bar{\theta}(t, x)\right| \leq 2 + 2^{-k_0}\phi(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|).$$

This concludes the proof. □

6. Hölder Continuity

In this section we shall prove the main theorem, Theorem 0.1. We begin with a final lemma to describe the scaling properties of (2).

Lemma 6.1 (Scaling). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\alpha}$ boundary. Suppose that $\theta : [-T, 0] \times \Omega \to \mathbb{R}$ and $u : [-T, 0] \times \Omega \to \mathbb{R}^2$ solve (2) and u satisfies*

$$u = \sum_{j=j_0}^{\infty} u_j$$

with that sum converging in $L^2(\Omega)$ and $(u_j)_j$ calibrated with constant κ and center N. Suppose that on Ω the functions $K_{1/4}$ and K_1 (defined in Proposition 1.1) satisfy the relation

$$K_{1/4}(x, y) \leq C_\Omega |x - y|^{3/4} K_1(x, y) \qquad \forall x \neq y \in \Omega. \tag{29}$$

Let $\varepsilon > 0$ be a small constant.

Then

$$\bar{\theta}(t, x) := \theta(\varepsilon t, \varepsilon x)$$

and

$$\bar{u}(t, x) := \sum_{j=j_0}^{\infty} u_j(\varepsilon t, \varepsilon x)$$

satisfies the same PDE on $[-T/\varepsilon, 0] \times \Omega_\varepsilon$ where $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$.

Moreover, $(u_j)_j$ is calibrated with the same constant κ but with center $N - \ln_2(\varepsilon)$, and the estimate

$$\bar{K}_{1/4}(x, y) \leq C_\Omega |x - y|^{3/4} \bar{K}_1(x, y) \qquad \forall x \neq y \in \Omega_\varepsilon$$

holds.

Proof. Denote by $\bar{\Lambda}$ the square root of the Laplacian with Dirichlet boundary conditions on Ω_ε. One can calculate (see e.g. [CS16] Section 2.4) that for $(t, x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$

$$\Lambda\theta(\varepsilon t, \varepsilon x) = \varepsilon\bar{\Lambda}\bar{\theta}(t, x).$$

Similarly, in the Caffarelli-Stinga representation from Proposition 1.1 the operator $\bar{\Lambda}^s$ will have kernel

$$\bar{K}_s(x, y) = \varepsilon^s K_s(\varepsilon x, \varepsilon y).$$

From these facts it is clear that the scaled functions satisfy (2) and (29).

Define

$$\bar{u}_j(t, x) := u_j(\varepsilon t, \varepsilon x).$$

To show that $(\bar{u}_j)_{j\in\mathbb{Z}}$ is calibrated, we must translate the various bounds on u_j to corresponding bounds on $\bar{u}_j$. Each of the calculations are similar, so we show only one:

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \leq 2^{\ln_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N - \ln_2(\varepsilon))} \kappa.$$

□

Proof of Theorem 0.1. We'll show that if θ with $\|\theta\|_{L^\infty([-5,0]\times\Omega)} \le 2$ solves (1) on $[-5,0]\times\Omega$ then θ is Hölder continuous at the point $(0,0)$ (with possibly $0 \in \bar{\Omega}$). Up to translation and scaling, this will be sufficient to show continuity at all points in the domain, with a constant depending on Ω and on the time we wait.

From Section 3, we know that

$$\mathbb{R}^\perp\theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence $(u_j)_j$ calibrated with some constant $\kappa = \kappa(\Omega)$ and center 0.

Choose a constant $0 < \varepsilon < 1/5$ such that

$$5\max\left(-\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}, (1-j_0)\kappa\right) \le \varepsilon^{-1} - 3, \tag{30}$$

For notational convenience, denote

$$\sum_k = \sum_{j>-k\ln(\varepsilon)}, \qquad \sum^k = \sum_{j\le -k\ln(\varepsilon)}.$$

For integers $k \ge 0$ consider the domains

$$\Omega_k := \{x \in \mathbb{R}^2 : \varepsilon^k x \in \Omega\}$$

and define the following functions on $[-5,0]\times\Omega_k$:

$$\begin{aligned} u_l^k(t,x) &:= \sum^k u_j(\varepsilon^k t, \varepsilon^k x), \\ u_h^k(t,x) &:= \sum_k u_j(\varepsilon^k t, \varepsilon^k x). \end{aligned}$$

For $t \in [-5,0]$ and $k \ge 0$ define $\Gamma_k, \gamma_k : [-5,0] \to \mathbb{R}^2$ by the following ODEs:

$$\begin{aligned} &\Gamma_0(t) := 0, \\ &\gamma_k(0) := 0, \\ &\dot{\gamma}_k(t) := u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t) \\ &\Gamma_k(t) := \varepsilon^{-1}\gamma_{k-1}(\varepsilon t) + \varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t) + \cdots + \varepsilon^{-k}\gamma_0(\varepsilon^k t), \qquad k \ge 1. \end{aligned}$$

Use [citation] some lemma from Bahouri-Chemin-Danchin that's a generalization of Picard-Lindelof to prove that these γ exist. Each u_l^k is a Lipschitz-in-space vector field, and each $\Gamma_k + \gamma_k$ is a flow along that vector field which ends up at the origin at $t = 0$. In particular, since u_l^k is tangential to the boundary of Ω_k and has unique flows, the flow $\Gamma_k + \gamma_k$ cannot exit the region Ω_k and so our expressions remain well-defined.

By Lemmas 6.1 and 4.1, we know the sequence $(u_j(\varepsilon^k\cdot, \varepsilon^k\cdot))_j$ is calibrated and hence that independently of k

$$\left\|\Lambda^{-1/4}u_h^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le C\kappa$$

etc. Particularly

$$\left\|\nabla u_l^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le 2\kappa.$$

By construction, for $k \geq 0$ we have $\Gamma_{k+1}(t) = \varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)$. Therefore

$$\begin{aligned}\dot{\Gamma}_{k+1}(t) &= \partial_t \left[\varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)\right] \\ &= \dot{\gamma}_k(\varepsilon t) + \dot{\Gamma}_k(\varepsilon t) \\ &= u_l^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t)) \\ &= u_l^k(\varepsilon t, \varepsilon\Gamma_{k+1}(t)).\end{aligned}$$

With this in hand, we can bound the size of γ_k. Namely, for $k \geq 1$,

$$\begin{aligned}\dot{\gamma}_k(t) &= u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t) \\ &= u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - u_l^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t)) \\ &= \sum^{k} u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t)) \\ &= \sum^{k-1}\left[u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k \ldots) \\ &= \left[u_l^{k-1}\big(\varepsilon t, \varepsilon\Gamma_k(t) + \varepsilon\gamma_k(t)\big) - u_l^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k \ldots).\end{aligned}$$

The sum $\sum^{k-1} u_j(\varepsilon^k\cdot, \varepsilon^k\cdot) = u_l^{k-1}(\varepsilon\,\cdot, \varepsilon\,\cdot)$ is Lipschitz in space, with Lipschitz constant less than $2\varepsilon\kappa$. Moreover, each u_j has $\|u_j\|_\infty \leq \kappa$. Thus both terms of $\dot{\gamma}_k(t)$ are bounded

$$|\dot{\gamma}_k(t)| \leq 2\varepsilon\kappa|\gamma_k(t)| - \kappa\ln_2(\varepsilon).$$

This, by Gronwall's inequality, tells us that for $t \in [-5, 0]$,

$$|\gamma_k(t)| \leq \frac{-\ln_2(\varepsilon)}{2\varepsilon}\left(e^{10\varepsilon\kappa} - 1\right)$$

and hence

$$|\dot{\gamma}_k(t)| \leq -\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}.$$

To account for γ_0, define

$$C_g = \max\left(-\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}, j_0\kappa\right)$$

so that for all $k \geq 0$ and $t \in [-5, 0]$

$$|\dot{\gamma}_k(t)| \leq C_g.$$

Let us now produce a sequence of solutions θ_k. Define

$$\theta_0(t, x) := \theta(t, x)$$

and for each $k \geq 0$, if $|\{\theta_k \leq 0\} \cap [-5, 0] \times B_4(\Gamma_k(t))| \geq 2|B_4|$ then set

$$\theta_{k+1}(t, x) := \frac{2}{2-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) + \lambda\right].$$

Otherwise, set

$$\theta_{k+1}(t, x) := \frac{1}{1-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) - \lambda\right].$$

From Lemma 6.1, we know that θ_k and the calibrated sequence $(u_j(\varepsilon^k\cdot, \varepsilon^k\cdot))_j$ solve (2).

We will now show that

$$|\theta_k| \leq 2 + 2^{-k_0}\left(|x - \Gamma_k(t)|^{1/4} - 2^{1/4}\right)_+ \tag{31}$$

holds for all $k \geq 0$.

Since $|\theta_0| \leq 2$ by assumption, we know in particular that (31) holds at $k = 0$.

This is sufficient for us to apply Lemma 5.3 to each θ_k (or to $-\theta_k$ as appropriate) in order. We conclude that (31) holds for all $k \geq 0$.

Each θ_k is between -2 and 2 on $[-5,0]\times B_2(\Gamma_k)$. But recall that each Γ_k is Lipschitz with constant kC_g. Thus $|\Gamma_k(t)|\le 1$ for $t\in[-(kC_g)^{-1},0]$. On that time interval,

$$|\theta_k(t,x)|\le 2 \qquad \forall x\in B_1(0).$$

We conclude that

$$\left|\sup_{[-\varepsilon^k(kC_g)^{-1},0]\times B_{\varepsilon^k}(0)}\theta(t,x)-\inf_{[-\varepsilon^k(kC_g)^{-1},0]\times B_{\varepsilon^k}(0)}\theta(t,x)\right|\le 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In particular, for some positive constant C such that

$$\varepsilon^{Ck}\le (kC_g)^{-1} \qquad \forall k\ge 0,$$

we can say that

$$|t|^2+|x|^2\le \varepsilon^{(1+C)k}$$

implies that $(t,x)\in[-\varepsilon^k(kC_g)^{-1},0]\times B_{\varepsilon^k}(0)$ which in turn implies that

$$|\theta(t,x)-\theta(0,0)|\le 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In other words,

$$\begin{aligned}|\theta(t,x)-\theta(0,0)| &\le 4\left(\frac{2}{2-\lambda}\right)^{-\frac{1}{1+C}\log_\varepsilon(|t|^2-|x|^2)+1}\\ &= 4\left(\frac{2}{2-\lambda}\right)\exp\left[\ln\left(\frac{2}{2-\lambda}\right)\frac{\ln(|t|^2+|x|^2)}{-(1+C)\ln(\varepsilon)}\right]\\ &= \frac{8}{2-\lambda}(|t|^2+|x|^2)^{-\frac{\ln(2)-\ln(2-\lambda)}{(1+C)\ln(\varepsilon)}}.\end{aligned}$$

□

Appendix A. Proof of Hölder Interpolation

In this appendix we prove the lemma 3.4.

Proof of 3.4. The first claim is straigtforward.

$$\begin{aligned}\sup_{x,y\in\Omega}\frac{|f(x)-f(y)|}{|x-y|^\alpha} &= \sup|f(x)-f(y)|^{1-\alpha}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha\\ &\le (2\|f\|_\infty)^{1-\alpha}\left(\sup\frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha\\ &\le C\|f\|_\infty^{1-\alpha}\|\nabla f\|_\infty^\alpha.\end{aligned}$$

The second claim is more complicated. We will prove the stronger claim that for $f\in C^{1,\alpha}$

$$\|\nabla f\|_\infty\le C\delta^{-1}\|f\|_{L^\infty(\bar\Omega)}+\delta^\alpha[\nabla f]_{\alpha;\bar\Omega}.$$

The idea of the proof is to average ∇f along an interval of length δ with endpoint x. The magnitude of the average will be small, since $f\in L^\infty$, and the average will differ not very much from $\nabla f(x)$ since $\nabla f\in C^{1,\alpha}$.

Since Ω satisfies the cone condition, there exist positive constants ℓ and $a<1$ such that, at each point $x\in\bar\Omega$, there exist two unit vectors e_1 and e_2 such that $|e_1\cdot e_2|\le a$ and $x+\tau e_i\in\Omega$ for $i=1,2$ and $0<\tau\le\ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

Consider the directional derivative $\partial_i f$ of f along the direction e_i, and observe that for any $0 < \delta \le \ell$,

$$\left| \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \right| = |f(x + \delta e_i) - f(x)| \le 2 \|f\|_\infty . \tag{32}$$

On the other hand, $\partial_i f$ is continous so, for any $\tau \in (0, \ell]$,

$$|\partial_i f(x) - \partial_i f(x + \tau e_i)| \le [\nabla f]_\alpha \tau^\alpha .$$

From this, we obtain that

$$\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \le \int_0^\delta \left(\partial_i f(x) + [\nabla f]_\alpha \tau^\alpha \right) d\tau = \delta \partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below. Thus

$$\left| \delta \partial_i f(x) - \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \right| \le [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha} .$$

Combining this bound with (32), we obtain

$$|\partial_i f(x)| \le \frac{2}{\delta} \|f\|_\infty + \frac{\delta^\alpha}{1+\alpha} [\nabla f]_\alpha .$$

This bound is independent of x and of $i = 1, 2$. Since $e_1 \cdot e_2 \le a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta \in (0, \ell]$,

$$\|\nabla f\|_\infty \le \frac{C}{1 - a^2} \left(\delta^{-1} \|f\|_\infty + \delta^\alpha [\nabla f]_\alpha \right) .$$

□

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