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We're gonna consider the equation

$$(1) \quad \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, u = \nabla^\perp \Lambda^{-1} \theta.$$

Here the operator

$$\Lambda := \sqrt{-\Delta_D}$$

where  $\Delta_D$  is the Laplacian with Dirichlet boundary condition.

We're going to linearize the equation by fixing  $u$  independent of  $\theta$ . What property do we want  $u$  to have? For some constant  $\kappa$ , we'll want

$$\begin{aligned} u &= \sum_{j \in \mathbb{Z}} u_j, \\ \|\Lambda^{-1/4} u_j\|_\infty &\leq \kappa 2^{-j/4}, \\ \|\nabla u_j\|_\infty &\leq \kappa 2^j. \end{aligned}$$

The convergence of that sum is in, say, weak  $L^2$ .

## 1. LEMMAS

**Lemma 1.1.** *If  $f$  and  $g$  are non-negative functions with disjoint support (i.e.  $f(x)g(x) = 0$  for all  $x$ ), then*

$$\int \Lambda^s f \Lambda^s g \, dx \leq 0.$$

This proves, in particular, that  $-\int \theta_+ \Lambda \theta_-$  is a positive term (hence dissipational and extraneous) and that  $\int \Lambda^{1/2}(\theta - \psi) \Lambda^{1/2}(\theta - \psi)$  breaks down (bilinearly) into the doubly positive, the doubly negative, and the cross term, all of which are positive and hence each of which is positive.

*Proof.* Use the characterization from Caffarelli-Stinga. There exist non-negative functions  $K(x, y)$  and  $B(x)$ , depending on the parameter  $s$ , such that

$$\int \Lambda^s f \Lambda^s g \, dx = \iint [f(x) - f(y)][g(x) - g(y)] K(x, y) \, dx dy + \int f(x) g(x) B(x) \, dx.$$

Since  $f$  and  $g$  are non-negative and disjoint, the  $B$  term vanishes. Moreover, the product inside the  $K$  term becomes

$$[f(x) - f(y)][g(x) - g(y)] = -f(x)g(y) - f(y)g(x) \leq 0.$$

Since  $K$  is non-negative, the result follows. □

**Lemma 1.2.** *For all functions  $f$  in  $H_D^1$ ,*

$$\int |\nabla f|^2 = \int |\Lambda f|^2.$$

*Moreover, if  $f \in H_D^1$  then  $\text{tr}(f) = 0$ .*

*Proof.* Let  $\eta_i$  and  $\eta_j$  be two eigenfunctions of the Dirichlet Laplacian on  $\Omega$ . Note that these functions are smooth in the interior of  $\Omega$ . Because  $\Omega$  has Lipschitz boundary, and because  $\eta_i \nabla \eta_j$  is smooth on  $\Omega$  and continuous and bounded on  $\bar{\Omega}$  vanishing on the boundary, therefore

$$\int_{\Omega} \operatorname{div}(\eta_i \nabla \eta_j) = \int_{\partial\Omega} \eta_i \nabla \eta_j.$$

But  $\eta_i \nabla \eta_j$  vanishes on the boundary, so the right hand side vanishes. Moreover,  $\operatorname{div}(\eta_i \nabla \eta_j) = \nabla \eta_i \cdot \nabla \eta_j + \eta_i \Delta \eta_j$ . Therefore

$$\int \nabla \eta_i \cdot \nabla \eta_j = - \int \eta_i \Delta \eta_j = \lambda_k \int \eta_i \eta_j.$$

Of course, the inner product of two eigenfunctions is 0 unless they are the same eigenfunction, in which case it is 1.

Consider a function  $f = \sum f_k \eta_k$  which is an element of  $H_D^1$ , by which we mean  $\sum \lambda_k f_k^2 < \infty$ . Since  $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$ , the following sums all converge in  $L^2(\Omega)$  and hence the calculation is justified:

$$\begin{aligned} \int |\nabla f|^2 &= \int \left( \sum_i f_i \nabla \eta_i \right) \left( \sum_j f_j \nabla \eta_j \right) \\ &= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j. \end{aligned}$$

Since this double-sum vanishes except on the diagonal, we see from [citation] that in fact

$$\|\nabla f\|_{L^2(\Omega)} = \|\Lambda f\|_{L^2(\Omega)}.$$

To see that  $\operatorname{tr}(f)$  vanishes, note that  $f = \sum_{k=0}^{\infty} f_k \eta_k$  and that each finite partial sum for this series satisfies the Dirichlet boundary condition. Since  $\operatorname{tr}$  is a bounded operator on  $H^1$ , we need only show that this series is Cauchy in  $H^1$ , in which case its  $H^1$  limit will exist and be equal to its  $L^2$  limit which will be equal to  $f$ .

For each  $k$ ,

$$\|f_k \eta_k\|_{H^1} \leq C_{\text{Poincare}} f_k \|\nabla \eta_k\|_2 = C f_k \sqrt{\lambda_k}.$$

This sequence is  $\ell^2$  summable, since  $f \in H_D^1$  by assumption. Therefore  $f$ , being an  $H^1$  limit of functions with vanishing trace, also has vanishing trace.  $\square$

**Lemma 1.3.** *For any function  $f$ , and any  $0 < s < 1$ ,*

$$\int |\Lambda^s f|^2 \simeq \int \left| (-\Delta)^{s/2} \bar{f} \right|^2.$$

Here  $\bar{f}$  is the extension of  $f$  to  $\mathbb{R}^2$  and  $(-\Delta)^s$  is defined in the fourier sense.

*Proof.* Let  $g$  be any Schwarz function in  $L^2(\mathbb{R}^2)$ , and let  $f$  be a function in  $H_D^{s+1}$ . Let  $E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^2)$  be a bounded extension operator, where  $H^1$  denotes the classical Sobolev space defined using the gradient. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f.$$

When  $\Re(z) = 0$ , then  $\|(-\Delta)^{z/2} g\|_2 = \|g\|_2$  and  $\|\Lambda^{s-z} f\|_2 = \|\Lambda^s f\|_2$ . Hence

$$\Phi(z) \leq \|g\|_2 \|f\|_{H_D^s}.$$

When  $\Re(z) = 1$ , then  $\|(-\Delta)^{(z-1)/2} g\|_2 = \|g\|_2$  and

$$\|(-\Delta)^{1/2} E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} = \|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} \leq \|E\| \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)}.$$

It remains to ask whether  $\Lambda^{s-z} f$  is in  $H_D^1$  so that we can apply lemma [citation]. However, this is true based on our assumption  $f \in H_D^{1+s}$ , since the various powers of  $\Lambda$  all commute and form a semigroup. Ergo

$$\|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|\Lambda \Lambda^{s-z} f\|_2 \leq \|\Lambda^s f\|_2$$

and we can bound

$$\Phi(z) \leq \|E\| \|g\|_2 \|f\|_{H_D^s}.$$

Now we will bound the derivative of  $\Phi(z)$ . Specifically, compute the derivative in  $z$  of the integrand, for  $0 < \Re(z) < 1$ , and hope that it is integrable. To this end, we rewrite the integrand of  $\Phi$  as

$$\mathcal{F}^{-1}(|\xi|^z \hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative  $\frac{d}{dz}$  commutes with linear operators like  $\mathcal{F}^{-1}$  and  $E$ , so the derivative is

$$\mathcal{F}^{-1}(\ln(|\xi|)|\xi|^z \hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}(|\xi|^z \hat{g}) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k.$$

Since  $0 < \Re(z) < 1$ ,  $\ln(|\xi|)|\xi|$  is bounded as a multiplier operator from Schwarz functions to  $L^2$ . Moreover,  $\ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} \leq C \lambda_k^{\frac{s-z+\varepsilon}{2}}$  for some  $C$  independent of  $k$  but dependent on  $z, \varepsilon$ . Since  $f \in H_D^{1+s}$  this sum converges in  $L^2$ , in fact in  $H_D^1$ . This makes our differentiated integrand a sum of two  $H^1$  functions with compact support multiplied by two Schwarz functions. In particular it is integrable, which means we can interchange the integral sign and the derivative  $\frac{d}{dz}$  and prove that  $\Phi'(z)$  is finite for all  $0 < \Re(z) < 1$ .

This is sufficient now to apply the Hadamard three-lines lemma to our function  $\Phi$ .

It follows that for any Schwarz function  $g \in L^2(\mathbb{R}^n)$  and  $H_D^{s+1}$  function  $f$ ,

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} g E f = \Phi(s) \leq \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{H_D^s}.$$

Since Schwarz functions are dense in  $L^2(\mathbb{R}^2)$ , this means by density that

$$\int \left| (-\Delta)^{s/2} E f \right|^2 \leq \int |\Lambda^s f|^2$$

or in other words it means that  $E$  is a bounded operator from  $H_D^s$  to  $H^s$ , at least on the subset  $H_D^{s+1} \cap H_D^s$ . It remains to extend this bound to the whole space by density.

We know from [citation] Caffarelli and Stinga that  $\mathcal{D}(\Omega)$  is dense in  $H_D^s$  for all  $0 \leq s < 1$ . In fact, this takes a bit of interpretation, so I ought to illucidate that this is because  $H_D^s = H_0^s$  (the latter in the Slobodekij sense) for most  $s$  and at  $s = 1/2$  we get the Lions-Magenes spaces which still has  $\mathcal{D}(\Omega)$  dense.

Surely, right(?), test functions are all inside of  $H_D^{1+s}$ . I should meditate on this, but it must be true.  $\square$

## 2. DE GIORGI ESTIMATES

First let us derive an energy inequality.

We know a priori that  $\theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_D^{1/2}(\Omega))$ . Let  $\psi : \Omega \rightarrow \mathbb{R}^+$  be a non-negative function in  $H_D^{1/2}$  non-uniformly, and define  $\theta = \theta_+ + \psi - \theta_-$ . Since  $\theta - \psi$  is in  $H_D^{1/2}$ , by the lemma above, both  $\theta_+$  and  $\theta_-$  are in that space as well. In particular, our weak solution can eat  $\theta_+$ .

We end up with

$$0 = \int \theta_+ \left[ \frac{d}{dt} + u \cdot \nabla + \Lambda \right] (\theta_+ + \psi - \theta_-)$$

which decomposes into three terms, corresponding to  $\theta_+$ ,  $\psi$ , and  $\theta_-$ . We analyze them one at a time.

Firstly,

$$\begin{aligned} \int \theta_+ \left[ \frac{d}{dt} + u \cdot \nabla + \Lambda \right] \theta_+ &= (1/2) \frac{d}{dt} \int \theta_+^2 + (1/2) \int \operatorname{div} u \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \\ &= (1/2) \frac{d}{dt} \int \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2. \end{aligned}$$

The  $\psi$  term produces important error terms:

$$\int \theta_+ \left[ \frac{d}{dt} + u \cdot \nabla + \Lambda \right] \psi = \frac{d}{dt} \int \theta_+ \psi + \int \theta_+ u \cdot \nabla \psi + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \psi.$$

Since  $\theta_+$  and  $\theta_-$  have disjoint support, the  $\theta_-$  term is nonnegative by lemma [citation]:

$$\begin{aligned} \int \theta_+ \left[ \frac{d}{dt} + u \cdot \nabla + \Lambda \right] \theta_- &= (1/2) \int \theta_+ \partial_t \theta_- + \int \theta_+ u \cdot \theta_- + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- \\ &= \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- \leq 0. \end{aligned}$$

Put together, we arrive at

$$(1/2) \frac{d}{dt} \int \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \leq \left| \iint \Lambda^{1/2} \theta_+ \Lambda^{1/2} \psi \right| + \left| \int \theta_+ u \cdot \nabla \psi \right|.$$

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$$(1/2) \frac{d}{dt} \int \theta_+^2 + \int u \cdot \nabla \frac{\theta_+^2}{2} + \int \theta_+ u \cdot \nabla \psi - \int \theta_+ u \cdot \nabla \theta_- + \int \theta_+ \Lambda \theta = 0.$$

We break up the  $\theta_+ \Lambda \theta$  term into

$$\begin{aligned} \int \theta_+ \Lambda \theta &= \int \left| \Lambda^{1/2} \theta_+ \right|^2 + \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)] K(x, y) + \int \theta_+ \psi B - \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- \\ &= \int \left| \Lambda^{1/2} \theta_+ \right|^2 + \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)] K(x, y) + \int \theta_+ \psi B - \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_-. \end{aligned}$$

The  $\theta_-$  term is non-negative by lemma [citation], and the  $B$  term is non-negative since  $B \geq 0$ , so we have the inequality

$$(1/2) \frac{d}{dt} \int \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \leq \left| \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)] K(x, y) \right| + \left| \int \theta_+ u \cdot \nabla \psi \right|.$$

This integral is symmetric in  $x$  and  $y$ , and the integrand is only nonzero if one of  $\theta_+(x)$  and  $\theta_+(y)$  is nonzero. Hence

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)] K(x, y) \leq 2 \chi_{\{\theta_+ > 0\}}(x) |[\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]| K(x, y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)] K(x, y) \leq \varepsilon \int \left| \Lambda^{1/2} \theta_+ \right|^2 + \frac{1}{\varepsilon} \iint \chi_{\{\theta_+ > 0\}}(x) [\psi(x) - \psi(y)]^2 K(x, y).$$

It remains to bound the quantity  $[\psi(x) - \psi(y)]^2 K(x, y)$ . By Caffarelli-Stinga theorem 2.4 [citation], there is a universal constant  $C$  such that

$$K(x, y) \leq \frac{C}{|x - y|^3}.$$

The cutoff  $\psi$  is Lipschitz, and it grows at a rate  $|x|^\gamma$ . Therefore

$$[\psi(x) - \psi(y)]^2 K(x, y) \leq |x - y|^{-1} \wedge |x - y|^{2\gamma-3}.$$

If  $3 - 2\gamma > 2$  then this quantity is integrable. Thus

$$\frac{d}{dt} \int \theta_+^2 + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \lesssim \int \theta_+ u \cdot \nabla \psi + \int \chi_{\{\theta_+ > 0\}}.$$

For the drift term, let's say that  $u$  is broken down into  $u_l$  and  $u_h$  (standing for high-pass and low-pass) and that they have the desired properties.

For the high-pass term, for each  $i = 1, 2$ ,

$$\int (u_l)_i \theta_+ \partial_i \psi = \int \Lambda^{-1/4} u_l \Lambda^{1/4} (\theta_+ \nabla \psi) \leq \left( \int \left| \Lambda^{1/4} \theta_+ \nabla \psi \right|^2 \right)^{1/2} \leq$$

We're looking at  $\int g \Lambda^{1/4} \theta_+$  where  $\Lambda^{1/4} g = u_h$  and  $g \in L^\infty$ . This breaks down as

$$\iint [g - g^*][\theta_+ - \theta_+^*] K_{1/4} + \int g \theta_+ B_{1/4}.$$

For a given parameter  $\lambda$ , break up into the region where  $|x - y|$  is bigger and smaller than  $\lambda$ . Considering the bigger part,

$$\iint_{\geq \lambda} [g - g^*][\theta_+ - \theta_+^*] K_{1/4} \leq 2(2 \|g\|_\infty) \iint_{\geq \lambda} |\theta_+ - \theta_+^*| \frac{dx dx^*}{|x - y|^{2+1/4}} \leq 8\lambda^{-2.25} \|g\|_\infty \|\theta_+\|_1.$$

The  $B$  part may be a real problem. The  $g$  means it doesn't have a sign, so we actually have to bound it. Consider this.

$$\begin{aligned} \int g \theta_k B_{1/4} &\leq \|g\|_\infty \int \theta_{k-1} \theta_k B_{1/4} \\ &\leq \|g\|_\infty \int \theta_{k-1}^2 B_{1/4} \\ &\leq \|g\|_\infty \int \left| \Lambda^{1/8} \theta_{k-1} \right|^2. \end{aligned}$$

Lastly, we have the near part.

$$\begin{aligned} \iint_{< \lambda} [g - g^*][\theta_+ - \theta_+^*] K_{1/4} &\leq 2 \iint_{< \lambda} \chi_+ [g - g^*]^2 |x - y|^{3/4} K_{1/4} + \iint_{< \lambda} [\theta_+ - \theta_+^*]^2 |x - y|^{-3/4} K_{1/4} \\ &\leq 4 \|g\|_\infty^2 \int \chi_+ \int_{< \lambda} |x - y|^{-1.5} + \iint_{< \lambda} [\theta_+ - \theta_+^*]^2 K_1 \\ &\leq \|g\|_\infty^2 \lambda^{1/2} |\chi_+| + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \end{aligned}$$

Taken together,

$$\int \Lambda^{-1/4} \theta_+ u_h \cdot \nabla \psi \leq \frac{1}{2} \int \left| \Lambda^{1/2} \theta_+ \nabla \psi \right| + \int \chi_+ + \int \theta_+ + \int \theta_+ \nabla \psi B_{1/4}.$$

### 3. CONTROL ON $u$

Let's assume that our drift term is a sum of  $u_j$  for  $j \in \mathbb{Z}$  and a constant which is tbd. Assume that each  $u_j$  is an  $L^\infty$  function, and that their sum converges to  $u$  in  $L^2(\Omega)$ , and that each  $u_j$  specifies the bounds as stated. First we show that they sum up to  $u_h$  and  $u_l$  in the ways desired. Then we show that the properties required are maintained as we zoom. Then at last we argue that, before any zooming,  $u$  really does have this property.

Firstly, assume that

$$\begin{aligned} u &= \lim_{L^2} \sum_{-N}^N u_j, \\ \left\| \Lambda^{-1/4} u_j \right\|_\infty &\leq \kappa 2^{-j/4}, \\ \left\| \nabla u_j \right\|_\infty &\leq \kappa 2^j. \end{aligned}$$

We then define

$$u_h = \sum_{j=0}^{\infty} u_j$$

and

$$u_l = \sum_{j=-1}^{-\infty} u_j.$$

Since  $u_j \in L^\infty$  in particular they are  $L^2$  functions which sum in  $L^2$ . Remember that only finitely many negative  $j$  have  $u_j \neq 0$ . The sequence  $u_j$  is thus singly infinite and in particular is a Cauchy sequence, so  $u_h$  also converges in  $L^2$ . Since  $\Lambda^{-1/4}$  is a continuous linear operator, it passes to the partial sums and so

$$\Lambda^{-1/4}u = \lim_{L^2} \sum_{j=0}^{\infty} \Lambda^{-1/4}u_j.$$

In particular, the sum converges in the sense of distributions, i.e. in  $\mathcal{D}(\Omega)'$ . Since test functions are dense in  $L^1(\Omega)$ , and the partial sums are uniformly bounded in the dual of  $L^1(\Omega)$  (namely  $L^\infty(\Omega)$ ), therefore the limit  $\Lambda^{-1/4}u_h$  is also bounded in the dual of  $L^1(\Omega)$ .

$$\|\Lambda^{-1/4}u_h\|_\infty \leq \sum_{j=0}^{\infty} \|\Lambda^{-1/4}u_j\|_\infty \leq \kappa \frac{1}{1 - 2^{-1/4}}.$$

As for  $u_\ell$ , we have that  $\sum_{j<0} u_j$  converges in  $L^2$ , and hence also in the sense of distributions  $\mathcal{D}(\Omega)'$ . Since  $\nabla$  is continuous on distributions, also  $\sum_{j<0} \nabla u_j$  converges to  $\nabla u_\ell$ . But each partial sum is uniformly bounded in the dual of  $L^1(\Omega)$ , meaning that the limit  $\nabla u_\ell$  is also bounded in the dual,  $L^\infty(\Omega)$ .

$$\|\nabla u_\ell\|_\infty \leq \sum_{j<0} \|\nabla u_j\|_\infty \leq \kappa \frac{1/2}{1 - 2^{-1}} = \kappa.$$

**Lemma 3.1.** *Scaling Suppose that  $\theta$  solves the PDE*

$$[\partial_t + u \cdot \nabla + \Lambda] \theta = 0$$

*where the velocity  $u$  satisfies*

$$u = \gamma(t) + \sum_{j=-\infty}^{\infty} u_j$$

*with that sum converging in  $L^2(\Omega)$ . Suppose that the domain of definition is  $(-T, 0) \times \Omega$ , and  $0 \in \partial\Omega$ . Suppose that*

$$\begin{aligned} \gamma(t) &= \sum_{j<0} u_j(0), \\ \|\Lambda^{-1/4}u_j\|_\infty &\leq \kappa 2^{-j/4}, \\ \|\nabla u_j\|_\infty &\leq \kappa 2^j. \end{aligned}$$

*Then for any small constant  $\varepsilon$ , there exists some  $\bar{\gamma} : [-T/\varepsilon, 0] \rightarrow \mathbb{R}^2$  such that*

$$\bar{\theta}(t, x) = \theta(\varepsilon t, \varepsilon x + \bar{\gamma}(\varepsilon t))$$

*satisfies all of the same constraints for  $(t, x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$ .*

*Proof.* Denote  $p = (t, x)$  and  $\bar{p} = (\varepsilon t, \varepsilon x + \bar{\gamma}(\varepsilon t))$ .

We calculate

$$\partial_t \bar{\theta}(p) = \varepsilon \partial_t \theta(\bar{p}) + \varepsilon \dot{\bar{\gamma}}(\bar{p}) \cdot \nabla \theta(\bar{p})$$

and

$$\nabla \bar{\theta}(p) = \varepsilon \nabla \theta(\bar{p})$$

and

$$\Lambda \bar{\theta}(p) = \varepsilon \Lambda \theta(\bar{p}).$$

Thus, if we define

$$\bar{u}(p) = u(\bar{p}) - \dot{\bar{\gamma}}(\bar{p})$$

then it will be the case that, for  $p \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$ ,

$$[\partial_t + \bar{u} \cdot \nabla + \Lambda] \bar{\theta}(p) = \varepsilon [\partial_t + u \cdot \nabla + \Lambda] \theta(\bar{p}).$$

It remains to demonstrate the decomposition of  $\bar{u}$  and that  $\bar{\gamma}$  still makes  $\bar{u}_\ell(0) = 0$ , let alone that 0 is still on the boundary of  $\Omega_\varepsilon$ .  $\square$

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