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Contents

We're gonna consider the equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0 \qquad (0,T) \times \Omega, \tag{1}$$

$$u = \nabla^\perp \Lambda^{-1} \theta \qquad [0,T] \times \Omega, \tag{2}$$

$$\theta = \theta_0 \qquad \{0\} \times \Omega \tag{3}$$

on an open domain $\Omega \subseteq \mathbb{R}^2$ and a time interval $[0,T]$, with given initial data θ_0.

Here the operator

$$\Lambda := \sqrt{-\Delta_D}$$

is the square root of $-\Delta_D$, the Laplacian on Ω with Dirichlet boundary condition. More specifically, if $(\eta_k)_{k\in\mathbb{Z}}$ is a family of eigenfunctions of $-\Delta_D$ with corresponding eigenvalues λ_k, then

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

Our main result will be to show that θ is Hölder continuous.

Theorem 0.1. *Let $\theta_0 \in L^2(\Omega)$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set and $T > 0$ a time. Then there exist functions $\theta, u \in L^\infty(0,T;L^2(\Omega))$ which solve (1). Moreover, for any $t \in (0,T)$, θ is Hölder continuous uniformly on $(t,T) \times \Omega$.*

In fact, for some $\alpha \in (0,1)$ depending only on Ω and some constant C depending only on Ω, T, and t

$$\|\theta\|_{C^\alpha((t,T)\times\Omega)} \leq C \|\theta_0\|_{L^2(\Omega)}.$$

The existence of a weak solution (meaning solution in the sense of distributions) $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathcal{H}^{1/2}(\Omega))$ is proven in [citation, Constantin and Ignatova].

The technique to prove this is, like in [citation, Caff & Vasseur], to linearize the equation by forgetting the dependence of u on θ, and then prove a Harnack inequality for fractional diffusion equations with "bounded" drift. Then we zoom in on the solution and apply the Harnack inequality again. By iterating this process, we can show that θ is Hölder continuous.

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The difficulty is in finding a bound on u which remains bounded no matter how much we zoom in. Ideally this would simply be L^∞ which is of course scaling invariant. The problem is that the Riesz operator $\nabla\Lambda^{-1}$ is not bounded from L^∞ to L^∞. In [citation Caff & Vasseur], Caffareli and Vasseur utilize the fact that the Riesz operator is bounded $L^\infty \to \mathrm{BMO}$. The space of functions with Bounded Mean Oscillation is scaling invariant, and one can show the Harnack inequality with BMO drift using De Giorgi's method with a shifting reference frame.

In the case of bounded domains, it is not known that the Riesz operator is bounded $L^\infty \to \mathrm{BMO}$. Another well known scaling invariant function space is the Besov space $B^0_{\infty,\infty}$, and this is closer to what we want.

One complication is that, on bounded domains, we have no access to the Fourier transform. However, an analogue involving the spectral decomposition of the Dirichlet Laplacian (where classical Littlewood-Paley theory involves the spectral decomposition of the Laplacian) has been developed by e.g. [citation, IMT] and [citation, Bui-Duong-Yan]. This theory is an outgrowth of the theory of Schrodinger Operators from mathematical physics. We will continue to refer to this theory using the terminology of Littlewood-Paley, but it is a significant generalization.

What's more, instead of considering the Littlewood-Paley projections of the Riesz transform of θ, we will actually seek to control the Riesz transforms of the Littlewood-Paley projections of θ. Because the Dirichlet Laplacian is not translation invariant, the gradient is not a spectral operator and the Riesz transform does not commute with the Littlewood-Paley projection operators. For this reason we cannot utilize the theory of Besov spaces, but the bounds on u that we do utilize are computationally similar.

Finally, because the gradient does not commute with the Dirichlet Laplacian, saying that u is bounded in this way which is analogous to $B^0_{\infty,\infty}$ is not equivalent to saying that $\Lambda^{-1/4}u$ is bounded in a way analogous to the space $B^{1/4}_{\infty,\infty}$ or that ∇u is bounded in a way analogous to $B^{-1}_{\infty,\infty}$. Therefore we must bound u in each sense indendently, though in the classical case all of these bounds would be identical.

We make this notion precise with the following definition.

Definition 1 (Calibrated sequence)**.** We call a sequence u_j **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\begin{aligned}
\|u_j\|_\infty &\le \kappa,\\
\|\nabla u_j\|_\infty &\le 2^j 2^{-N}\kappa,\\
[u_j]_{3/4} &\le 2^{j\frac{3}{4}} 2^{-N\frac{3}{4}}\kappa,\\
\left\|\Lambda^{-1/4}u_j\right\|_\infty &\le 2^{-j/4}2^{N/4}\kappa.
\end{aligned}$$

We call a function **calibrated** if it is the sum of a calibrated sequence, with the infinite sum converging in the sense of L^2.

In sections 2 and 3 we will show that u is calibrated and that it remains calibrated at all scales. Thereafter we will consider the linear equation

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0,\\ \operatorname{div} u = 0\end{cases} \tag{4}$$

where u is assumed to be calibrated. In sections 4 and 5 we will show a Harnack inequality for solutions to (4).

Recall the notation

$$[f]_\alpha := \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Throughout, we will use the notation $(x)_+ := \max(0,x)$. When the parentheses are ommited, the subscript + is merely a label.

We will denote

$$\|f\|_{\mathcal{H}^s} = \int |\Lambda^s f|^2 .$$

We suppress the dependence on Ω, though in fact Ω is defined in terms of Ω in a way that will be clear from context. This norm is in fact a norm, not a seminorm, for $s \geq 1/2$, because any $f \in \mathcal{H}^s$ vanishes at the boundary so $\|f\|_{L^2(\Omega)} \leq C\,\|f\|_{\mathcal{H}^s}$ but the constant depends on Ω.

If $f = \sum_k f_k \eta_k$ then

$$\|f\|_{\mathcal{H}^s} = \left(\sum_k \lambda_k^s f_k^2\right)^{1/2} .$$

1. Properties of Λ

We begin by recounting the result of [citation] Caffarelli and Stinga which gives us a singular integral representation of the $\mathcal{H}^s$ norm.

Proposition 1.1. *Let $f, g \in \mathcal{H}^s$ on a bounded $C^{2,\alpha}$ domain $\Omega \subseteq \mathbb{R}^2$. Then*

$$\int_\Omega \Lambda^s f \Lambda^s g\,dx = \iint_{\Omega^2} [f(x)-f(y)][g(x)-g(y)]K_{2s}(x,y)\,dxdy + \int_\Omega f(x)g(x)B_{2s}(x)\,dx$$

for kernels K_{2s} and B_{2s} which depend on the parameter s and the domain Ω.

Moreover, these kernels are bounded

$$0 \leq K_{2s}(x,y) \leq \frac{C(\Omega,s)}{|x-y|^{2+2s}}$$

for all $x \neq y \in \Omega$ and

$$0 \leq B_{2s}(x)$$

for all $x \in \Omega$.

Finally, there exists a function $\tilde{K}(x,y)$ depending on Ω but independent of s, and a constant c_s depending on s and Ω, such that for all $x \neq y \in \Omega$

$$0 \leq \tilde{K}(x,y)\frac{c_s}{|x-y|^{2+2s}} \leq K_{2s}(x,y).$$

Proof. See [citation] Theorems 2.3 and 2.4. □

From the explicit formulae given in [cite], we see that K_{2s} is roughly equal to the standard kernel for the $\mathbb{R}^2$ fractional Laplacian when both x and y are in the interior of Ω or when x and y are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel B_{2s} is well-behaved in the interior but has a singularity at the boundary $\partial\Omega$. This justifies our thinking of the K_{2s} term as the interior term and B_{2s} as a boundary term.

When comparing the computations in this paper to corresponding computations on $\mathbb{R}^2$, it is convenient to say that the interior term behaves roughly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

We will now prove a collection of lemmas which follow from the Caffarelli-Stinga representation.

Lemma 1.2. *We present 4 elementary corallaries of the representation of the Caffarelli-Stinga representation.*

(a) If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then

$$\int \Lambda^s f \Lambda^s g\,dx \leq 0.$$

(b) Let $s \in (0,1)$. If $g \in L^\infty \cap Lip(\Omega)$ then

$$\|fg\|_{\mathcal{H}^s} \le \|g\|_\infty \|f\|_{\mathcal{H}^s} + \|f\|_2 \sup_y \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}}\,dx.$$

(c) Let $s \in (0,1)$. If $g \in L^\infty \cap Lip(\Omega)$ then

$$\|fg\|_{\mathcal{H}^s} \le \|g\|_\infty \|f\|_{\mathcal{H}^s} + \|f\|_2 \left(\|g\|_\infty + \|g\|_{Lip}\right).$$

(d) Let g an L^∞ function and $f \in H^{2s}$ be non-negative with compact support. Let there be a constant C_Ω such that

$$K_s(x,y) \le C|x-y|^{3s} K_{4s}(x,y). \tag{5}$$

Then

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \le C \|g\|_\infty |\operatorname{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}\right).$$

Proof. We prove these corollaries one at a time.

Proof of (a)**:** From Proposition **??**

$$\int \Lambda^s f \Lambda^s g\,dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\,dxdy + \int f(x)g(x)B(x)\,dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \le 0.$$

Since K is non-negative, the result follows.

Proof of (b)**:**

$$\begin{aligned}\int |\Lambda^s(fg)|^2 &= \iint \left(g(x)[f(x)-f(y)] + f(y)[g(x)-g(x)]\right)^2 K + \int f^2 g^2 B \\ &\le \|g\|_\infty^2 \|f\|_{\mathcal{H}^s}^2 + \int f(y)^2 \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}}.\end{aligned}$$

Proof of (c)**:** This follows immediately from (b).

Proof of (d)**:** From Proposition 1.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_< + I_\ge + II$$

where

$$\begin{aligned} I_< &:= \iint_{|x-y|<1} [g(x)-g(y)][f(x)-f(y)]K_s, \\ I_\ge &:= \iint_{|x-y|\ge 1} [g(x)-g(y)][f(x)-f(y)]K_s, \\ II &:= \int fgB_s. \end{aligned}$$

First we estimate $I_<$. From (5) and the fact that $[f(x)-f(y)]$ vanishes unless at least one of $f(x)$ or $f(y)$ is non-zero,

$$|I_<| \le 2 \iint_{|x-y|<1} \chi_{\{f>0\}}(x)\,|g(x)-g(y)| \cdot |f(x)-f(y)| \cdot |x-y|^{3s} K_{4s}.$$

We can break this up by Holder's inequality

$$|I_<| \le 2 \left(\iint_{|x-y|<1} \chi_{\{f>0\}}(x)[g(x)-g(y)]^2 |x-y|^{6s} K_{4s}\right)^{1/2} \left(\iint [f(x)-f(y)]^2 K_{4s}\right)^{1/2}.$$

The kernel $|x-y|^{6s} K_{4s}\chi_{\{|x-y|<1\}}$ is integrable in y for x fixed. Therefore

$$|I_<| \le 2\left(2\,\|g\|_\infty^2 \int C\chi_{\{f>0\}}(x)\,dx\right)^{1/2}\left(\|f\|^2_{\mathcal{H}^{2s}}\right)^{1/2}. \tag{6}$$

To estimate $I_\ge$,

$$|I_\ge| \le 2\,\|g\|_\infty\, 2\int |f(x)| \int_{|x-y|\ge 1} K_s(x,y)dy\,dx.$$

Since $K_s\chi_{\{|x-y|\ge1\}}$ is integrable in y for x fixed,

$$|I_\ge| \le C\,\|g\|_\infty\,\|f\|_1\,. \tag{7}$$

For the boundary term II,

$$|II| \le \|g\|_\infty \int \chi_{\{f>0\}} f B_s.$$

Since $f \ge 0$, $[f(x)-f(y)][\chi_{\{f>0\}}(x)-\chi_{\{f>0\}}(y)] \ge 0$. Therefore

$$\int \chi_{\{f>0\}} f B_s \le \int \Lambda^{s/2}\chi_{\{f>0\}}\Lambda^{s/2} f = \int \chi_{\{f>0\}}\Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \le \|g\|_\infty\,|\operatorname{supp}(f)|^{1/2}\,\|f\|_{\mathcal{H}^s}\,.$$

This combined with (6) and (7) gives us

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \le C\,\|g\|_\infty\left(\|f\|_1 + |\operatorname{supp}(f)|^{1/2}\,\|f\|_{\mathcal{H}^s} + \|f\|_{\mathcal{H}^{2s}}\right).$$

The lemma follows since $\|f\|_1 \le |\operatorname{supp}(f)|^{1/2}\,\|f\|_2$ and since $\|f\|_{\mathcal{H}^s} \le \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$.

□

Let us consider the relationship between the norm $\mathcal{H}^s$ and the classical H^s norm.

It is known (see [citation] and [citation]) that for $s \in (0,1)$ the spaces $\mathcal{H}^s$ are equivalent to certain $H^s(\Omega)$ spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in $\mathcal{H}^s$ and that elements of $\mathcal{H}^s$ have trace zero for $s \in [1/2, 1]$.

The most important fact for us is that the fractional Sobolev norms defined in terms of extension (for which we have access to a variety of theorems regarding compactness and Sobolev embeddings) are dominated by our $\mathcal{H}^s$ norm with a constant that is independent of Ω.

Lemma 1.3. *For any function f, and any $1/2 \le s < 1$,*

$$\int_\Omega |\Lambda^s f|^2 \ge \int_{\mathbb{R}^2} \left|(-\Delta)^{s/2}\,\bar f\right|^2.$$

Here $\bar f$ is the extension-by-zero of f to $\mathbb{R}^2$ and $(-\Delta)^s$ is defined in the fourier sense.

We will prove this by interpolating between $s = 0$ and $s = 1$. Before we can do this, we must prove the

Lemma 1.4. *For all functions f in $\mathcal{H}^1$,*

$$\int_\Omega |\nabla f|^2 = \int_\Omega |\Lambda f|^2\,.$$

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω. Because Ω has Lipschitz boundary, and because $\eta_i\nabla\eta_j$ is smooth on Ω and countinuous and bounded on $\overline{\Omega}$ vanishing on the boundary, therefore

$$\int_\Omega \operatorname{div}(\eta_i\nabla\eta_j) = \int_{\partial\Omega} \eta_i\nabla\eta_j.$$

But $\eta_i \nabla \eta_j$ vanishes on the boundary, so the right hand side vanishes. Moreover, $\operatorname{div}(\eta_i \nabla \eta_j) = \nabla \eta_i \cdot \nabla \eta_j + \eta_i \Delta \eta_j$. Therefore

$$\int \nabla \eta_i \cdot \nabla \eta_j = -\int \eta_i \Delta \eta_j = \lambda_j \int \eta_i \eta_j = \lambda_j \delta_{i=j}.$$

Of course, the inner product of two eigenfunctions is 0 unless they are the same eigenfunction, in which case it is 1.

Consider a function $f = \sum f_k \eta_k$ which is an element of $\mathcal{H}^1$, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned} \int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla \eta_i\right)\left(\sum_j f_j \nabla \eta_j\right) \\ &= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j. \end{aligned}$$

Since this double-sum vanishes except on the diagonal, we see from [citation] that in fact

$$\|\nabla f\|_{L^2(\Omega)} = \|\Lambda f\|_{L^2(\Omega)}.$$

□

We come now to the proof of Proposition 1.3.

Proof. Let g be any Schwarz function in $L^2(\mathbb{R}^2)$, and let f be a function in $\mathcal{H}^s$. Let $E : \mathcal{H}^1(\Omega) \to H^1(\mathbb{R}^2)$ be a the extension-by-zero operator, where H^1 denotes the classical Sobolev space defined using the gradient. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f, \qquad z \in \mathbb{C}, \Re(z) \in [0,1].$$

When $\Re(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|\Lambda^s f\|_2$ since Λ^{it} is a unitary operator on L^2 for any $t \in \mathbb{R}$. Hence

$$\Phi(z) \leq \|g\|_2 \|f\|_{\mathcal{H}^s}.$$

When $\Re(z) = 1$, then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$ and

$$\left\|(-\Delta)^{1/2} E \Lambda^{s-z} f\right\|_{L^2(\mathbb{R}^2)} = \|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} \leq \|E\| \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)}.$$

Since $\Lambda^s f \in L^2(\Omega)$, $\Lambda^{s-z} f \in \mathcal{H}^1$ so we can apply Lemma 1.4. Ergo

$$\|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|\Lambda \Lambda^{s-z} f\|_2 \leq \|\Lambda^s f\|_2$$

and we can bound

$$\Phi(z) \leq \|E\| \|g\|_2 \|f\|_{\mathcal{H}^s}.$$

Now we will bound the derivative of $\Phi(z)$. Specifically, compute the derivative in z of the integrand, for $0 < \Re(z) < 1$, and hope that it is integrable. To this end, we rewrite the integrand of Φ as

$$\mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}\left(\ln(|\xi|)|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k.$$

Since $0 < \Re(z) < 1$, $\ln(|\xi|)|\xi|^z$ is bounded as a multiplier operator from Schwarz functions to L^2. Moreover, $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \leq C\lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. For z fixed, we take $\varepsilon < \Re(z)$ and use $f \in \mathcal{H}^s$ to see that this sum converges in L^2. This makes our differentiated integrand a sum of two products of L^2 functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \Re(z) < 1$.

This is sufficient now to apply the Hadamard three-lines lemma to our function Φ.

It follows that for any Schwarz function $g \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{H}^s$,

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} gEf = \Phi(s) \leq \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

Since Schwarz functions are dense in $L^2(\mathbb{R}^2)$, this means by density that

$$\int \left|(-\Delta)^{s/2} Ef\right|^2 \leq \int |\Lambda^s f|^2$$

or in other words it means that E is a bounded operator from $\mathcal{H}^s$ to H^s with norm 1. □

2. Littlewood-Paley Theory

Logan: The japanese paper's Lemma 3.6, used extensively here, only applies in the case $j \geq 0$. Obviously I need it and use it for $j > j_0$. This is equivalent, I can see from the proof, but maybe mention the issue somewhere so it doesn't seem like I didn't notice.

In this section we will prove that u breaks up into pieces with various norms under control.

Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. Specifically, we mean that ϕ is non-negative, supported on $[1/2, 2]$, and has the property that

$$\sum_{j\in\mathbb{Z}} \phi(2^j\xi) = 1 \qquad \forall \xi \neq 0.$$

This allows us to define the Littlewood-Paley projections. For any $f = \sum f_k\eta_k$

$$P_j f := \sum_k \phi(2^j\lambda_k^{1/2}) f_k\eta_k.$$

Recall that $-\Delta_D$ has some smallest eigenvalue λ_0 (depending on Ω) so if we define $j_0 = \log_2(\lambda_0) - 1$ then $P_j = 0$ for all $j < j_0$.

Our goal in this section is to prove the following proposition:

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\gamma}$ boundary for some $\gamma \in (0,1)$. Let $\theta \in L^\infty(\Omega)$. Then there exists a calibrated sequence of functions $(u_j)_{j\in\mathbb{Z}}$ for some constant $\kappa = \kappa(\Omega)$ with center 0 (see Definition 1) such that*

$$\nabla^\perp\Lambda^{-1}\theta = \sum_{j\in\mathbb{Z}} u_j$$

with the infinite sum converging in the sense of L^2.

Moreover there exists some $j_0 \in \mathbb{Z}$ such that $u_j = 0$ for all $j < j_0$.

Before we can prove this, we state a few important lemmas.

First we restate a known result from the literature.

Lemma 2.2. *There exists a constant C depending on Ω and on ϕ such that the following hold:*

For any $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}$

$$\|\Lambda^\alpha P_j f\|_{L^\infty(\Omega)} \leq C2^{\alpha j} \|f\|_{L^\infty(\Omega)}.$$

For any $\alpha \in \mathbb{R}$ and $j \geq j_0$

$$\|\nabla\Lambda^\alpha P_j f\|_{L^\infty(\Omega)} \leq C2^{(1+\alpha)j} \|f\|_{L^\infty(\Omega)}.$$

Proof. The first claim is [**?**] Lemma 3.5, and it is also an immediate corollary of [**?**] Theorem 1.1.

The second claim follows from [**?**] Lemma 3.6. This lemma requires that

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le 1$$

and only states the result for $j > 0$.

In [**?**] it is proved that that if Ω is $C^{2,\alpha}$ then

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le T$$

which, by a trivial modification of the proof in [**?**], is enough to prove the result stated here. □

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. Classically, if a function is supported in fourier space in a single dyadic ring, then after taking the gradient it remains supported in that same dyadic ring. In the case of our adapted Littlewood-Paley theory, the gradient operator may take a function supported on a small range of freqencies and "smear it out" over all of freqency space, particularly over the high frequencies. We prove below that if a function is supported on a small range of frequencies, its gradient will at least decay quickly on low frequencies.

Lemma 2.3. *For any function f,*

$$\|P_i \nabla P_j f\|_\infty \le C \min(2^j, 2^i) \|f\|_\infty .$$

Proof. Let g be an L^1 function. Then

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \le C 2^j \|g\|_1 \|f\|_\infty$$

by Lemma 2.2.

Further integrating by parts,

$$\int g P_i \nabla P_j f = -\int (\nabla P_i g) P_j f \le C 2^i \|g\|_1 \|f\|_\infty .$$

This also follows from Lemma 2.2.

The result follows. □

This final lemma allows us to interpolate using Hölder seminorms. The results are not presumed to be novel, but since their proofs were difficult to find in the literature we include them below.

Lemma 2.4. *If $f \in L^\infty(\Omega) \cap C^{0,1}(\Omega)$ then for some universal constant C,*

$$[f]_\alpha \le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha .$$

If $f \in C^{0,1}(\Omega) \cap C^{2,\alpha}(\Omega)$ where Ω satisfies the cone condition, then for some constants C and ℓ depending on Ω,

$$\left\|D^2 f\right\|_\infty \le C\delta^{-1} \|\nabla f\|_\infty + \delta^\alpha \left[D^2 f\right]_\alpha$$

for all $\delta < \ell$.

Proof. The first claim is incredibly straigtforward. We include it for completeness.

$$\begin{aligned} \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} &= \sup |f(x) - f(y)|^{1-\alpha} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^\alpha \\ &\le (2\|f\|_\infty)^{1-\alpha} \left(\sup \frac{|f(x) - f(y)|}{|x-y|} \right)^\alpha \\ &\le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha . \end{aligned}$$

The second claim is more complicated. We'll prove the sufficient claim that for f smooth,

$$\|\nabla f\|_\infty \leq C\delta^{-1}\|f\|_{L^\infty(\bar{\Omega})} + \delta^\alpha [\nabla f]_{\alpha;\bar{\Omega}}.$$

Since Ω satisfies the cone condition, we know that there exist positive constants ℓ and $a < 1$ such that, at each point $x \in \bar{\Omega}$, there exist two unit vectors e_1 and e_2 such that $|e_1 \cdot e_2| \leq a$ and $x + \tau e_i \in \Omega$ for $i = 1, 2$, $0 < \tau \leq \ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

The idea of the proof is that the average of ∇f along an interval is bounded since f is bounded, and the same average is close to the value of ∇f at a point because ∇f is continuous, hence the value of ∇f at any point must be bounded. By varying the length δ of the aforementioned interval, we actually get a parameterized family of bounds.

If we consider the directional derivative $\partial_i f$ of f along the direction e_i, then observe that for any $0 < \delta \leq \ell$,

$$\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau = f(x + \delta e_i) - f(x).$$

This quantity on the right is bounded by the L^∞ norm of f.

On the other hand, since ∇f and hence $\partial_i f$ are continous functions, for any $\tau \in (0, \ell]$

$$|\partial_i f(x) - \partial_i f(x + \tau e_i)| \leq [\nabla f]_\alpha \tau^\alpha.$$

From this bound, we obtain that

$$\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \leq \int_0^\delta \partial_i f(x) + [\nabla f]_\alpha \tau^\alpha\, d\tau = \delta\partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below, so

$$\left|\delta\partial_i f(x) - \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau\right| \leq [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}.$$

What we have shown is that the integral of $\partial_i f$ over an interval of length δ is small, and also it differs not very much from $\delta\partial_i f(x)$. By rearranging, we find that $\partial_i f(x)$ must therefore be small:

$$|\partial_i f(x)| \leq \frac{2}{\delta}\|f\|_\infty + \frac{\delta^\alpha}{1+\alpha}[\nabla f]_\alpha.$$

This is true independent of x and of $i = 1, 2$. Since $e_1 \cdot e_2 \leq a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta \in (0, \ell]$,

$$\|\nabla f\|_\infty \leq \frac{C}{1-a^2}\left(\delta^{-1}\|f\|_\infty + \delta^\alpha [\nabla f]_\alpha\right).$$

□

We are now ready to prove Proposition 2.1.

Proof. For each $j \in \mathbb{Z}$, we define u_j to be the rotation of the Riesz transform of the j^{th} Littlewood-Paley projection of θ:

$$u_j := \nabla^\perp \Lambda^{-1} P_j \theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

By straightforward application of Lemma 2.2 we know that

$$\|u_j\|_\infty \leq C\|\theta\|_\infty.$$

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4} u_j = \sum_{i\in\mathbb{Z}} P_i \Lambda^{-1/4} u_j.$$

Define $\bar{P}_k := P_{k-1}+P_k+P_{k+1}$. Then $\bar{P}_k P_k = P_k$, and since the projections P_k are spectral operators, they commute with Λ^s. We therefore rewrite

$$\left(P_i \Lambda^{-1/4} u_j\right)^{\perp} = \left(\Lambda^{-1/4} \bar{P}_i\right) P_i \nabla P_j \left(\Lambda^{-1} \bar{P}_j\right) \theta.$$

We apply sequentially three bounded operators on L^∞. The first operator has norm $C2^{-j}(2^1 + 2^0 + 2^{-1})$ by Lemma 2.2. The second operator has norm $C\min(2^j, 2^i)$ by Lemma 2.3. The third operator has norm $C2^{-i/4}(2^{1/4} + 2^0 + 2^{-1/4})$ by Lemma 2.2. (Of course, the perp operator is an isometry.) Therefore

$$\left\| P_i \Lambda^{-1/4} u_j \right\|_\infty \le C2^{-i/4} \min(2^j, 2^i) 2^{-j} \left\|\theta\right\|_\infty .$$

Summing these bounds on the projections of $\Lambda^{-1/4} u_j$, and noting that

$$\sum_{i\in\mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i\le j} 2^{i3/4} + \sum_{i>j} 2^{-i/4} \le C2^{-j/4},$$

we obtain

$$\left\| \Lambda^{-1/4} u_j \right\|_\infty \le C2^{-j/4} \left\|\theta\right\|_\infty .$$

Lastly, we'll show that ∇u_j is in L^∞. Equivalently, we'll show that $\Lambda^{-1} P_j \theta$ is $C^{1,1}$. The method of proof is standard Schauder theory.

For convenience, define

$$F := \Lambda^{-1} P_j \theta$$

and recall that F is a finite linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Moreover, its Laplacian is

$$f := \Delta F = \Lambda P_j \theta$$

which is also smooth and vanishes at the boundary.

We apply the standard Schauder estimate from [**?**] Theorem 6.6, which says that since, for some α, Ω is $C^{2,\alpha}$ and $F \in C^{2,\alpha}(\bar{\Omega})$, and since $f \in C^{\alpha}(\bar{\Omega})$, and since the boundary conditions are homogeneous (hence smooth), then

$$\left[D^2 F\right]_\alpha \le C \left\|F\right\|_\infty + C \left\|f\right\|_\infty + C \left[f\right]_\alpha .$$

By Lemma 2.2,

$$\left\|F\right\|_\infty = \left\|\Lambda^{-1} P_j \theta\right\|_\infty \le C2^{-j} \left\|\theta\right\|_\infty$$

and

$$\left\|f\right\|_\infty = \left\|\Lambda P_j \theta\right\|_\infty \le C2^{j} \left\|\theta\right\|_\infty$$

and

$$\left\|\nabla f\right\|_\infty = \left\|\nabla \Lambda P_j \theta\right\|_\infty \le C2^{2j} \left\|\theta\right\|_\infty .$$

Therefore we can interpolate by Lemma 2.4 to obtain

$$\left[f\right]_\alpha \le C2^{j(1+\alpha)} \left\|\theta\right\|_\infty .$$

Plugging these estimates into [cite] yields

$$\left[D^2 F\right]_\alpha \le C \left(2^{-j} + 2^j + 2^{j(1+\alpha)}\right) \left\|\theta\right\|_\infty .$$

Recall that without loss of generality we can assume $j \ge j_0$. Therefore up to a constant depending on j_0, the term $2^{j(1+\alpha)}$ bounds 2^j and 2^{-j} so we can write

$$\left[D^2 F\right]_\alpha \le C2^{j(1+\alpha)} \left\|\theta\right\|_\infty .$$

Using this estimate and the fact that $\|\nabla F\|_\infty = \left\|\nabla\Lambda^{-1}P_j\theta\right\|_\infty \le C\,\|\theta\|_\infty$ we can use Lemma 2.4 to interpolate. For some constant ℓ depending on Ω, for any $\delta \le \ell$ we have

$$\begin{aligned}\left\|D^2F\right\|_\infty &\le C\left(\delta^{-1}\|\nabla F\|_\infty + \delta^\alpha\left[D^2F\right]_\alpha\right)\\ &\le C\left(\delta^{-1}C + \delta^\alpha C2^{j(1+\alpha)}\right)\|\theta\|_\infty.\end{aligned}$$

Set $\delta = 2^{-j}2^{j_0}\ell \le \ell$. Then

$$\left[D^2F\right]_\infty \le C\left(C2^j + 2^{-j\alpha}2^{j(1+\alpha)}\right)\|\theta\|_\infty = C(\Omega)2^j\,\|\theta\|_\infty.$$

Since $D^2F = \nabla u_j$, this estimate together with [cite], [cite], and [cite] complete the proof. □

3. Properties of Calibrated Sequences

We've shown in the previous section that our drift term u is a sum of a calibrated sequence $(u_j)_{j\in\mathbb{Z}}$.

In this section, we will first show that these terms sum to two functions u_l and u_h with appropriate bounds. Then we will show that these bounds remain true as we zoom in space and time.

The most important property of a calibrated sequence is that its sum decomposes into two functions, which we call the high-pass term and the low-pass term.

Proposition 3.1. *Let*

$$u = \sum_{j_0}^{\infty} u_j$$

with the sum converging in the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center.

Then there exist some universal constants C_i such that

$$u = u_l + u_h$$

with

$$\|\nabla u_l\|_{L^\infty(-T,0)\times\Omega} \le C_1\kappa$$

and

$$[u_l]_{3/4} \le 2\kappa$$

and

$$\left\|\Lambda^{-1/4}u_h\right\|_\infty \le C_3\kappa.$$

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j>N} u_j$$

and

$$u_l = \sum_{j\le N} u_j.$$

Since $u_j \in L^\infty$ in particular they are L^2 functions which sum in L^2. Remember that only finitely many negative j have $u_j \ne 0$. The sequence u_j is thus singly infinite and in particular is a Cauchy sequence, so u_h also converges in L^2. Since $\Lambda^{-1/4}$ is a continuous linear operator, it passes to the partial sums and so

$$\Lambda^{-1/4}u_h = \lim_{L^2}\sum_{j>N}\Lambda^{-1/4}u_j.$$

In particular, the sum converges in the sense of distributions, i.e. in $\mathcal{D}(\Omega)'$. Since test functions are dense in $L^1(\Omega)$, and the partial sums are uniformly bounded in the dual of $L^1(\Omega)$ (namely $L^\infty(\Omega)$), therefore the limit $\Lambda^{-1/4}u_h$ is also bounded in the dual of $L^1(\Omega)$.

$$\left\|\Lambda^{-1/4}u_h\right\|_\infty \leq \sum_{j>N}\left\|\Lambda^{-1/4}u_j\right\|_\infty \leq \kappa\frac{2^{-1/4}}{1-2^{-1/4}}.$$

As for u_l, we have that $\sum_{j\leq N} u_j$ is a finite sum of Lipschitz and Hölder continuous functions. We can simply bound

$$\|\nabla u_l\|_\infty \leq \sum_{j\leq N}\|\nabla u_j\|_\infty \leq \kappa\frac{1}{1-2^{-1}}$$

and

$$[u_l]_{3/4} \leq \sum_{j\leq N}[u_j]_{3/4} \leq \kappa\frac{1}{1-2^{-3/4}}.$$

□

We showed in section 2 that u is a sum of a calibrated sequence, and now we have shown that the sum of a calibrated sequence is actually a finite sum of functions that are bounded in certain function spaces. Any bound we place on u directly will blow up as we zoom in, but a calibrated sequence remains calibrated (with increasing center). In the next lemma, we show that, thanks to this notion of calibration, our PDE is scale-invariant.

Lemma 3.2 (Scaling). *Suppose that θ and u solve the PDE*

$$[\partial_t + u\cdot\nabla + \Lambda]\,\theta = 0,$$

$$\operatorname{div} u = 0,$$

where the velocity u satisfies

$$u = \sum_{j=j_0}^{\infty} u_j$$

with that sum converging in $L^2(\Omega)$ and $(u_j)_j$ calibrated with constant κ and center N. Suppose that the domain of definition is $(-T,0)\times\Omega$.

Let $\varepsilon>0$ be a small constant.

Then

$$\bar\theta(t,x) := \theta(\varepsilon t,\varepsilon x)$$

and

$$\bar u(t,x) := \sum_{j=j_0}^{\infty} u_j(\varepsilon t,\varepsilon x)$$

satisfies the same PDE for $(t,x)\in[-T/\varepsilon,0]\times\Omega_\varepsilon$.

Moreover, $(u_j)_j$ is calibrated with the same constant κ and center $N-\ln_2(\varepsilon)$.

Proof. We calculate

$$\partial_t\bar\theta(p) = \varepsilon\partial_t\theta(\bar p)$$

and

$$\nabla\bar\theta(p) = \varepsilon\nabla\theta(\bar p)$$

and

$$\Lambda\bar\theta(p) = \varepsilon\Lambda\theta(\bar p).$$

& cetera...

It remains to show that $(u_j(eps\,\cdot,\varepsilon\,\cdot))_j$ is still calibrated. Define

$$\bar u_j(t,x) := u_j(\varepsilon t,\varepsilon x).$$

Then

$$\|\bar{u}_j\|_\infty = \|u_j\|_\infty \le \kappa$$

and

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \le 2^{\ln_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N - \ln_2(\varepsilon))} \kappa.$$

The entire thing is so straightforward I literally can't bring myself to type out the rest.

□

4. De Giorgi Estimates

First let us derive an energy inequality.

Lemma 4.1 (Caccioppoli Estimate). *Let $\theta \in L^2(0,T;\mathcal{H}^{1/2}(\Omega))$ and $u \in L^\infty(0,T;L^2(\Omega))$ solve (4) in the sense of distributions. Let $\psi : [-T,0] \times \Omega \to \mathbb{R}$ be non-negative, Lipschitz-in-space, and Hölder continuous-in-space with exponent $\gamma < 1/2$. Then the decomposition*

$$\theta = \theta_+ + \psi - \theta_-$$

satisfies the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \le C\left(\int \chi_{\{\theta_+ > 0\}} + \int \theta_+(\partial_t \psi + u \cdot \nabla \psi)\right)$$

with the constant C depending on $\|\nabla \psi\|_\infty$ and $\sup_t [\psi(t,\cdot)]_\gamma$.

Proof. From Lemma **??**, we know that θ_+ is in $\mathcal{H}^{1/2}(\Omega)$ for a.e. $t \in [0,T]$. We can therefore multiply our equation [cite] by θ_+ and integrate in space to obtain

$$0 = \int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right](\theta_+ + \psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right]\theta_+ &= (1/2)\frac{d}{dt}\int \theta_+^2 + (1/2)\int \operatorname{div} u\, \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \\ &= (1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The ψ term produces important error terms:

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right]\psi &= \int \theta_+ \partial_t \psi + \int \theta_+ u \cdot \nabla \psi + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi \\ &= \int \theta_+(\partial_t \psi + u \cdot \nabla \psi) + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma a:

$$\begin{aligned}\int \theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right]\theta_- &= (1/2)\int \theta_+ \partial_t \theta_- + \int \theta_+ u \cdot \nabla \theta_- + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- \\ &= \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- \le 0.\end{aligned}$$

Put together, we arrive at

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi \le \left|\int \theta_+(\partial_t \psi + u \cdot \nabla \psi) \cdot \nabla \psi\right|.$$

At this point we break down the $\Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi$ term using the formula from [citation] Caffarelli-Stinga.

$$\int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi = \iint \left[\theta_+(x) - \theta_+(y)\right]\left[\psi(x) - \psi(y)\right] K(x,y) + \int \theta_+ \psi B.$$

Since $B \geq 0$ (see Caff-Stinga [citation]) and ψ is non-negative by assumption, the B term is non-negative and so

$$\int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi \geq \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y).$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) \leq 2 \iint \chi_{\{\theta_+>0\}}(x) \left|\theta_+(x) - \theta_+(y)\right| \cdot \left|\psi_t(x) - \psi_t(y)\right| K(x,y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\left|\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y)\right| \leq \varepsilon \int \left|\Lambda^{1/2}\theta_+\right|^2 + \frac{1}{\varepsilon} \iint \chi_{\{\theta_+>0\}}(x)[\psi(x) - \psi(y)]^2 K(x,y).$$

It remains to bound the quantity $[\psi(x) - \psi(y)]^2 K(x,y)$. By Caffarelli-Stinga theorem 2.4 [citation], there is a universal constant C such that

$$K(x,y) \leq \frac{C}{|x-y|^3}.$$

The cutoff ψ is Lipschitz, and Hölder continuous with exponent $\gamma < 1/2$ by assumption. Therefore

$$[\psi(x) - \psi(y)]^2 K(x,y) \leq |x-y|^{-1} \wedge |x-y|^{2\gamma - 3}.$$

Since $3 - 2\gamma > 2$, this quantity is integrable. Thus

$$\int \chi_{\{\theta_+>0\}}(x) \int [\psi(x) - \psi(y)]^2 K(x,y)\, dxdy \leq C(\|\psi\|_{\text{Lip}}, [\psi]_\gamma) \int \chi_{\{\theta_+>0\}}\, dx.$$

Combining [citation, like 4 different things are combined] we arrive at

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle \theta_+, \theta_- \rangle_{1/2} \lesssim \int \theta_+(\partial_t \psi + u \cdot \nabla \psi) + \int \chi_{\{\theta_+>0\}}.$$

□

This is sufficient to prove that a solution to [cite] the PDE with L^2 initial data is L^∞ in finite time.

Proposition 4.2 (L^2 to L^∞)**.** *If θ and u solve [cite] on $[0,T] \times \Omega$ and $\theta_0 \in L^2$, then for any time $S \in (0,T)$ there exists a constant $C = C(S)$ such that*

$$\|\theta\|_{L^\infty([S,T]\times\Omega)} \leq C \|\theta_0\|_{L^2(\Omega)}.$$

Proof. It is trivial to show that the $L^2(\Omega)$ norm of any solution θ to (4) does not increase in time. Simply multiply the function by θ and integrate.

Moreover, using Lemma 4.1 with $\psi(t,x) = \|\theta(T,\cdot)\|_{L^\infty(\Omega)}$ tells us that the $L^\infty(\Omega)$ norm of a solution, once finite, is non-increasing in time.

To show that the $L^\infty(\Omega)$ norm of a solution with $L^2(\Omega)$ initial data becomes finite in finite time, consider the sequence of functions

$$\theta_k := (\theta(t,x) - 1 + 2^{-k})_+$$

and define

$$\mathcal{E}_k := \int_{-1-2^{-k}}^{0} \int_\Omega \theta_k^2 \, dxdt.$$

When $\theta_{k+1} > 0$, then in particular $\theta_k \geq 2^{-k}$ [or something similar]. Thus for any finite p, there exists a constant C so

$$\chi_{\{\theta_{k+1}>0\}} \leq C^k \theta_k^p.$$

In particular,

$$\mathcal{E}_{k+1} \leq C^k \int_{-1-2^{-k}}^{0} \int \theta_k^3.$$

Applying the energy inequality θ, ϕ, and Γ we obtain

$$\sup_{-1-2^{-k-1}<t<0} \int \theta_{k+1}^2 + \int_{-1-2^{-k-1}}^0 \int \left|\Lambda^{1/2}\theta_{k+1}\right|^2 \le C^k \int_{-1-2^{-k}}^0 \theta_k^2 = \mathcal{E}_k.$$

However, by Sobolev embedding and the fact that $\mathcal{H}^{1/2}$ controls classical $H^{1/2}$ controls L^4,

$$\|\theta_{k+1}\|_{L^3([-1-2^{-k-1},0]\times\Omega)} \le C^k \mathcal{E}_k^{1/2}.$$

Therefore

$$\mathcal{E}_{k+1} \le C^k \mathcal{E}_k^{3/2}.$$

It follows by a well known result [citation] that for $\mathcal{E}_0$ sufficiently small (say less than $\bar{C}$), $\mathcal{E}_k \to 0$ as $k \to \infty$.

Notice that, since the $L^2(\Omega)$ norm of θ does not increase in time,

$$\mathcal{E}_0 = \int_{-2}^0 \int_\Omega (\theta)_+ \, dxdt \le 2 \int \theta_0^2 \, dx.$$

Moreover, as $k \to \infty$ we have

$$\mathcal{E}_k \to \int_{-1}^0 \int_\Omega (\theta - 1)_+ \, dxdt$$

Thus, if $\|\theta_0\|_{L^2(\Omega)} \le \sqrt{\bar{C}/2}$ then $\theta \le 1$ on $[-1,0]$.

Since (4) is linear and scales in time and space as in Lemma 3.2 (and since the constant $\bar{C}$ does not depend on Ω), we can take a solution θ with arbitrary initial L^2 norm and apply this result to a scaled version.

The result follows. □

We've completed the essential version of the Caccioppoli estimate. However, much more can be said about the drift-term u. In particular, we can design a cutoff ψ in order to minimize the expression $\partial_t \psi + u \cdot \nabla \psi$.

From here on, we will consider sets $\Omega \subseteq \mathbb{R}^2$ and collections of functions θ, u_l, and $u_h : [-T,0]\times\Omega \to \mathbb{R}$, and paths Γ and $\gamma : [-5,0] \to \mathbb{R}^2$ which satisfy

$$\begin{cases} \partial_t\theta + (u_l + u_h)\cdot\nabla\theta + \Lambda\theta = 0 & \text{on } [-5,0]\times\Omega, \\ \operatorname{div}(u_l) = \operatorname{div}(u_h) = 0 & \text{on } [-5,0]\times\Omega, \\ \dot{\Gamma}(t) + \dot{\gamma}(t) = u_l(t, \gamma(t) + \Gamma(t)) & \text{on } [-5,0], \\ \gamma(0) = \Gamma(0) = 0. \end{cases} \tag{8}$$

Here it is implicitly assumed that $\gamma(t) + \Gamma(t) \in \Omega$. We assume also that Ω is bounded with $C^{2,\alpha}$ boundary for some $\alpha \in (0,1)$, and that for some constant C_Ω we have

$$K_{1/2}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

For some constant κ we will assume

$$\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-5,0]\times\Omega}, \|\nabla u_l\|_{L^\infty([-5,0]\times\Omega)}, \left\|[u_l]_{3/4;\Omega}\right\|_{L^\infty([-5,0])} \le 2\kappa$$

and for some constant C_g we will assume

$$\|\dot{\gamma}\|_{L^\infty([-5,0])} \le C_g.$$

Note that we make no assumptions about the size of Γ. Generally we will assume something about the size of θ when $|x - \Gamma|$ is small, though the form of that assumption will vary from lemma to lemma.

Lemma 4.3 (Energy inequality)**.** *Let θ, u_l, u_h, Γ and γ solve* (8)*.*

$$\dot{\Gamma}(t) + \dot{\gamma}(t) = u_l(t, \Gamma(t) + \gamma(t)).$$

Then for any $\phi \in C^2(\Omega)$ such that $|x|^{3/4}\nabla\phi(x) \in L^\infty$, the functions

$$\theta_+ := (\theta - \phi(\cdot - \Gamma))_+, \qquad \theta_- := (\phi(\cdot - \Gamma) - \theta)_+$$

satisfy the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+, \theta_-\rangle_{1/2} \le C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+ + \int \theta_+^2\right)$$

with the constant C depending on C_g and T, on $\left\|\Lambda^{-1/4}u_h\right\|_\infty$, $[u_l]_{3/4}$, and $\|u_l\|_{Lip}$, and on $\left\|D^2\phi\right\|_\infty$, $\|\nabla\phi\|_\infty$, and $\sup\left\||x|^{3/4}\nabla\phi(x)\right\|_\infty$.

Proof. We'll apply the Caccioppoli estimate with

$$\psi(t,x) := \phi(x - \Gamma(t)),$$
$$\phi \in C^2(\mathbb{R}^2) \cap \dot{C}^{1/4}(\mathbb{R}^2).$$

Now

$$\partial_t\psi + u\cdot\nabla\psi = (u - \dot{\Gamma})\cdot\nabla\phi(x - \Gamma(t)).$$

We arrive at

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+, \theta_-\rangle_{1/2} \le C\left(\int \chi_{\{\theta_+>0\}} + \int \theta_+(u - \dot{\Gamma}(t))\cdot\nabla\phi(x - \Gamma(t))\right).$$

Consider the high pass term $\int \theta_+ u_h \cdot \nabla\phi$. By inserting $\Lambda^{1/4}\Lambda^{-1/4}$ and then integrating by parts, we can apply Lemma **??** and obtain

$$\int \Lambda^{-1/4}u_h\Lambda^{1/4}(\theta_+\nabla\phi) \le C\left\|\Lambda^{-1/4}u_h\right\|_\infty\left(\|\nabla\phi\|_\infty + \left\|D^2\phi\right\|_\infty\right)\left(\|\theta_+\|_1 + |\operatorname{supp}(\theta_+)|^{1/2}\left(\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}\right)\right).$$

From Hölder's inequality with Peter-Paul, we obtain

$$\int u_h\theta_+\nabla\phi(x - \gamma(t))\,dx \le C(\phi,\varepsilon)\left\|\Lambda^{-1/4}u_h\right\|_\infty\left(\int \chi_{\{\theta_+>0\}} + \int\theta_+ + \int\theta_+^2\right) + \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2.$$

Time for the low pass term.

Recall that

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \Gamma + \gamma)$$

so

$$u_l(t,x) - \dot{\Gamma}(t) = u_l(t,x) - u_l(t, \Gamma + \gamma) + \dot{\gamma}.$$

By assumption, $|\dot{\gamma}| \le C_g$ and so for $t \in [-T, 0]$ we have $|\gamma(t)| \le TC_g$.

Since u_l is Lipschitz and Hölder continuous,

$$\begin{aligned}|u_l(t,x) - u_l(t,\Gamma+\gamma)| &\le |u_l(t,x) - u_l(t,\Gamma)| + |u_l(t,\Gamma) - u_l(t,\Gamma+\gamma)|\\ &\le [u_l]_{3/4}|x - \Gamma|^{3/4} + \|u_l\|_{\mathrm{Lip}}TC_g.\end{aligned}$$

Plugging these bounds int [cite] we obtain

$$\left|u_l(t,x) - \dot{\Gamma}(t)\right| \le (1 + \|\nabla u_l\|_\infty T)C_g + [u_l]_{3/4}|x - \Gamma|^{3/4}.$$

Now we can bound the low pass term

$$\begin{aligned}\int (u_l - \dot{\Gamma})\theta_+\nabla\phi(x-\Gamma) &\le (1 + \|\nabla u_l\|_\infty T)C_g\int|\nabla\phi(x-\Gamma)|\theta_+\,dx + [u_l]_{3/4}\int|x-\Gamma|^{3/4}\theta_+|\nabla\phi(x-\Gamma)|\,dx\\ &\le (1 + \|\nabla u_l\|_\infty T)C_g\|\nabla\phi\|_\infty\int\theta_+\,dx + [u_l]_{3/4}\left\||x|^{3/4}\nabla\phi(x)\right\|_\infty\int\theta_+\,dx.\end{aligned}$$

From this the result follows. ☐

At last we can prove the De Giorgi lemmas.

Lemma 4.4 (First De Giorgi Lemma)**.** *There exists a constan δ_0 such that the following holds: For any functions θ, u_l, u_h, Γ, and γ which solve (8), if*

$$\theta(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \qquad \forall x \in \Omega \setminus B_2(\Gamma(t))$$

and

$$\int_{-2}^{0} \int_{B_2(\Gamma(t))} \max(\theta,0)^2 \, dxdt \le \delta_0$$

then

$$\theta(t,x) \le 1 \qquad \forall (t,x) \in [-1,0] \times B_1(\Gamma(t)).$$

Proof. Let ϕ be such that $\phi = 0$ on B_1 and $\phi(x) \ge 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 2$ while ϕ is Lipschitz and C^2 and its gradient decays like $|x|^{-3/4}$.

Consider the sequence of functions

$$\theta_k := (\theta(t,x) - \phi(x - \Gamma(t)) - 1 + 2^{-k})_+$$

and define

$$\mathcal{E}_k := \int_{-1-2^{-k}}^{0} \int_{\Omega} \theta_k^2 \, dxdt.$$

Notice that

$$\mathcal{E}_0 = \int_{-2}^{0} \int_{\Omega} (\theta - \phi(x - \Gamma))_+^2 \, dxdt \le \delta_0.$$

Moreover, as $k \to \infty$ we have

$$\mathcal{E}_k \to \int_{-1}^{0} \int_{\Omega} (\theta - \phi(x - \Gamma) - 1)_+^2 \, dxdt$$

so in particular, if we can show $\mathcal{E}_k \to 0$ then $\theta \le 1$ for $t \in [-1,0]$ and $x \in B_1(\Gamma)$.

That's enough setup, let's argue that $\mathcal{E}_k \to 0$. Notice that when $\theta_{k+1} > 0$, then in particular $\theta_k \ge 2^{-k}$ [or something similar]. Thus for any finite p, there exists a constant C so

$$\chi_{\{\theta_{k+1} > 0\}} \le C^k \theta_k^p.$$

In particular,

$$\mathcal{E}_{k+1} \le C^k \int_{-1-2^{-k}}^{0} \int \theta_k^3.$$

Applying the energy inequality θ, ϕ, and Γ we obtain

$$\sup_{-1-2^{-k-1} < t < 0} \int \theta_{k+1}^2 + \int_{-1-2^{-k-1}}^{0} \int \left|\Lambda^{1/2} \theta_{k+1}\right|^2 \le C^k \int_{-1-2^{-k}}^{0} \int \theta_k^2 = \mathcal{E}_k.$$

However, by Sobolev embedding and the fact that $\mathcal{H}^{1/2}$ controls classical $H^{1/2}$ controls L^4, we know from Reisz-Thorin that the left side of the energy inequality controls the L^3 norm of θ_{k+1} so

$$\|\theta_{k+1}\|_{L^3([-1-2^{-k-1},0] \times \Omega)} \le C^k \mathcal{E}_k^{1/2}.$$

Therefore

$$\mathcal{E}_{k+1} \le C^k \mathcal{E}_k^{3/2}.$$

It follows by a well known result [citation] that for $\mathcal{E}_0$ sufficiently small (say less than δ_0), $\mathcal{E}_k \to 0$ as $k \to \infty$ which we already established is sufficient to obtain our result. □

This is coming along quite nicely. We can move on to DG2, the isoperimetric inequality.

Lemma 4.5 (Second De Giorgi Lemma)**.** *Let θ and $u = u_l + u_h$ be solutions to [cite] satisfying the desired bounds. Let Γ and γ be paths with the desired properties, in particular*

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \gamma + \Gamma).$$

Suppose that for $t \in [-5, 0]$ and any $x \in \Omega$,

$$\theta(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

There exists a small constant $\mu > 0$ such that the three conditions

$$|\{\theta \ge 1\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0,$$

$$|\{0 < \theta < 1\} \cap [-4,0] \times B_4(\Gamma)| \le \mu,$$

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge \frac{4|B_4|}{2}$$

cannot simultaneously be met.

Proof. Suppose that the theorem is false, i.e. that there exist functions $\theta_n : [-5,0] \times \Omega_n \to \mathbb{R}$ and $u_l^n, u_h^n : [-5,0] \times \Omega_n \to \mathbb{R}^2$ which satisfy the desired bounds on Ω_n some scaling of Ω, namely that

$$\dot{\Gamma}_n + \dot{\gamma}_n = u_l^n(t, \Gamma_n + \gamma_n)$$

for paths $\Gamma_n : [-5,0] \to \Omega_n$ and $\gamma_n : [-5,0] \to \Omega_n$ satisfying $|\dot{\gamma}| \le C_g$, but for which

$$|\{0 < \theta_n < 1\} \cap [-4,0] \times B_4(\Gamma_n)| \le 1/n.$$

Let ϕ be a function which vanishes on B_2 but has all the growth and smoothness properties. In particular assume that ϕ exceeds $2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 3$.

Fix n and define

$$\theta_+ := (\theta_n - \phi(x - \Gamma_n))_+.$$

Then θ_+ is supported on $B_3(\Gamma_n)$ and is less than $2 + 3^{1/4} - 2^{1/4} \le 3$ everywhere.

Apply the energy inequality Lemma 4.3 with $\phi(x - \Gamma_n)$, and find that

$$\sup_{[-4,0]} \int \theta_+^2 + \int_{-4}^0 \int \left|\Lambda^{1/2}\theta_+\right|^2 \le C \int_0^4 \int \left(\chi_{\{\theta_+ > 0\}} + \theta_+ + \theta_+^2\right).$$

This proves in particular that $\theta_+ \in L^2([-2,0]; \mathcal{H}^{1/2}(\Omega))$ is uniformly bounded.

What's more, $\left\|\theta_+^3\right\|_{\mathcal{H}^{1/2}(\Omega_n)}$ is uniformly bounded because

$$\begin{aligned}
\left\|\Lambda^{1/2}(\theta_+^3)\right\|_2^2 &= \iint [\theta_+(x)^3 - \theta_+(y)^3]^2 K + \int \theta_+^6 B \\
&\le 2 \iint \theta_+(x)^4 [\theta_+(x) - \theta_+(y)]^2 K + 2 \iint \theta_+(y)^4 [\theta_+(x) - \theta_+(y)]^2 K + \|\theta_+\|_\infty^4 \int \theta_+^2 B \\
&\le C \|\theta_+\|_\infty^4 \|\theta_+\|_{\mathcal{H}^{1/2}}^2 \le C.
\end{aligned}$$

By Lemma 1.3, if $E\theta_+^3$ is the zero-extension of θ_+^3 from Ω_n to $\mathbb{R}^2$, then

$$\left\|E\theta_+^3\right\|_{L^2(-5,0;H^{1/2}(\mathbb{R}^2))} \le C$$

where C does not depend on n.

Since θ_n solves the equation [cite], multiply the equation by $\varphi\theta_+^2$, where φ is any function in $C^2(\mathbb{R}^2)$ restricted to Ω_n, and integrate to obtain

$$\frac{1}{3}\int \varphi \partial_t \theta_+^3 + \frac{1}{3}\int \varphi \dot{\Gamma}_n \cdot \nabla\theta_+^3 = \frac{-1}{3}\int \varphi(u^n - \dot{\Gamma}_n)\cdot\nabla\theta_+^3 - \int \varphi\theta_+^2 \Lambda\theta_+ - \int \varphi\theta_+^2 (u^n - \dot{\Gamma}_n)\cdot\nabla\phi - \int \varphi\theta_+^2 \Lambda\phi + \int \varphi\theta_+^2 \Lambda\theta_-.$$

We will bound the terms on the right hand side one at a time.

Each instance of C in the following bounds is independent of n.

Recall $|u_l^n - \dot\Gamma_n| \le C\kappa + 4C_g$ on $[-4,0]\times B_3(\Gamma_n)$ which is precisely the support of θ_+. Integrating by parts,

$$\int \varphi(u_l^n - \dot\Gamma_n)\nabla\theta_+^3 = \int \nabla\varphi(u_l^n - \dot\Gamma_n)\theta_+^3 \le C\,\|\nabla\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))}.$$

Similarly,

$$\int \varphi\theta_+^2(u_l^n - \dot\Gamma_n)\cdot\nabla\phi \le C\,\|\nabla\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))}.$$

By Lemmas d and b

$$\int \varphi u_h^n\nabla\theta_+^3 = \int \Lambda^{-1/4}u_h^n\Lambda^{1/4}(\theta_+^3\nabla\varphi) \le C(\|\nabla\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))} + \left\|D^2\varphi\right\|_{L^\infty(-4,0;L^\infty(\Omega_n))}).$$

By the same lemmas,

$$\begin{aligned}\int \varphi\theta_+^2u_h^n\nabla\phi &\le C\left(\|\varphi\nabla\phi\|_{L^1(L^\infty)} + \|\nabla(\varphi\nabla\phi)\|_{L^1(L^\infty)}\right)\\ &\le C\left(\|\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))} + \|\nabla\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))}\right).\end{aligned}$$

This completes both of the drift terms. The Λ terms remain. These are bounded by relentless use of the Caffarelli-Stinga representation formula.

For the θ_+ term

$$\begin{aligned}\int \varphi\theta_+^2\Lambda\theta_+ &= \iint [\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\theta_+(x)-\theta_+(y)]K + \int \varphi\theta_+^3B\\ &= \iint \varphi(x)(\theta_+(x)+\theta_+(y))[\theta_+(x)-\theta_+(y)]^2K + \iint \theta_+(y)^2[\varphi(x)-\varphi(y)][\theta_+(x)-\theta_+(y)]K + \iint \varphi\theta_+^3B\\ &\le C\,\|\varphi\|_{L^1(-4,0;L^\infty(\Omega_n))} + C\,\|\nabla\varphi\|_{L^1(-4,0;L^\infty(\Omega_n))}\end{aligned}$$

For any non-negative function f we know by Lemma a that

$$\int f\theta_+^2\Lambda\theta_- \le 0.$$

It follows that $-\theta_+^2\Lambda\theta_-$ is a pointwise non-negative distribution. If we can bound its integral, then we will have bounded it as an element of $L^\infty(\mathcal{M}(\Omega_n))$. From [cite], its integral is simply

$$-\int \theta_+^2\Lambda\theta_- \le -\|\theta_+\|_\infty \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C.$$

Since φ is continuous,

$$\int \varphi\theta_+^2\Lambda\theta_- \le \|\varphi\|_\infty \left\|\theta_+^2\Lambda\theta_-\right\|_{\mathcal{M}(\Omega_n)} \le C\,\|\varphi\|_{L^1(-4,0;L^\infty(\Omega_n))}.$$

Lastly,

$$\begin{aligned}\int \varphi\theta_+^2\Lambda\phi &= \iint [\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\phi(x)-\phi(y)]K + \int \varphi\theta_+^2\phi B\\ &= \iint \varphi(x)[\theta_+(x)^2-\theta_+(y)^2][\phi(x)-\phi(y)]K + \iint \theta_+(y)^2[\varphi(x)-\varphi(y)][\phi(x)-\phi(y)]K + \int \varphi\theta_+^2\phi B\\ &\le C\left(\iint \varphi(x)^2[\phi(x)-\phi(y)]^2K\right)^{1/2} + (\|\varphi\|_\infty + \|\nabla\varphi\|_\infty)\int \theta_+(y)^2\,dy + \int \varphi\theta_+^2\phi B\\ &\le C\,\|\varphi\|_{L^2} + C\,\|\varphi\|_\infty + C\,\|\nabla\varphi\|_\infty + \left\|\phi\chi_{\{\theta_+>0\}}\right\|_\infty \|\theta_+\|_{\mathcal{H}^{1/2}}^2\,\|\varphi\|_\infty\\ &= C\,\|\phi\|_{L^\infty(-4,0;L^2(\Omega_n))} + C\,\|\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))} + C\,\|\nabla\varphi\|_{L^\infty(-4,0;L^\infty(\Omega_n))}.\end{aligned}$$

Taken all together, we conclude that there exists a constant C indepnendent of n such that, for any $\varphi \in L^\infty(-4,0;C^2(\mathbb{R}^2)) \cap L^\infty(-4,0;L^2(\mathbb{R}^2))$,

$$\int_{-4}^0\int_{\Omega_n} \left(\partial_t\theta_+^3 + \dot\Gamma_n\cdot\nabla\theta_+^3\right)\varphi\,dxdt \le C\,\|\varphi\|_{L^\infty(-4,0;C^2(\mathbb{R}^2))} + C\,\|\varphi\|_{L^\infty(-4,0;L^2(\mathbb{R}^2))}.$$

Over time, the support of θ_+^3 moves around in Ω_n following the path Γ_n. If we try to take a limit, that limit will vanish except at points where infinitely many Γ pass nearby, which is generally unhelpful. Instead, we extend each θ_+^3 to a function on $\mathbb{R}^2$ and then shift them around to remain supported near the origin. To that end, define a new function on $[-4,0]\times\mathbb{R}^2$ by

$$v_n(t,x) := \begin{cases} \theta_+^3(t, x+\Gamma_n(t)), & x+\Gamma_n(t)\in\Omega_n, \\ 0, & x+\Gamma_n(t)\notin\Omega_n. \end{cases}$$

Let $X\subseteq C^2(\mathbb{R}^2)$ be the Banach space of C^2 functions with norm $\|\cdot\|_X = \|\cdot\|_{C^2(\mathbb{R}^2)} + \|\cdot\|_{L^2(\mathbb{R}^2)}$ finite. Note that

$$\partial_t v_n(t,x) = \partial_t\theta_+^3(t,x+\Gamma_n) + \dot{\Gamma}_n\cdot\nabla\theta_+^3(t,x+\Gamma_n).$$

We know that

$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2)}\leq C$$

and

$$\|\partial_t v_n\|_{L^1(-4,0;X^*)}\leq C.$$

Moreover,

$$\frac{d}{dt}\int_{\mathbb{R}^2} v_n^{2/3}\,dx = \frac{d}{dt}\int_{\Omega_n}\theta_+^3\,dx\leq C.$$

Finally, from [cite],

$$|\{v_n\geq 1\}\cap[-2,0]\times B_2(0)|\geq\delta_0,$$
$$\left|\{0<v_n<[1-\phi(x)]^3\}\cap[-4,0]\times B_4(0\right|\leq 1/n,$$
$$|\{v_n\leq 0\}\cap[-4,0]\times B_4(0)|\geq\frac{4|B_4|}{2}.$$

By [cite], [cite], and the Aubin-Lions lemma, the set $\{v_n\}_n$ is compactly embedded in $L^2(-4,0;L^2(\mathbb{R}^2))$. Up to a subsequence, there is a function $v\in L^2(-4,0;L^2(\mathbb{R}^2))$ such that

$$v_n\to v.$$

By elementary properties of L^2 convergence, we know that $v\in L^\infty$, $\mathrm{supp}(v)\subseteq[-4,0]\times B_3(0)$, $v\in L^2(H^{1/2})$ and

$$\frac{d}{dt}\int v(t,\cdot)^{2/3}\leq C. \tag{9}$$

Also, the properties [cite] hold still in the limit.

For any $(t,x)\in[-4,0]\times B_4(0)$, either $v(t,x)\geq[1-\phi(x)]^3$ or else $v(t,x)=0$. In fact, since $\|v(t,\cdot)\|_{H^{1/2}}<\infty$ for almost every t and $H^{1/2}$ does not cantain functions with jump discontinuities, the function v is either identically 0 or else $\geq[1-\phi(x)]^3$ at each t.

Thus $\int v(t,x)^{2/3}\,dx$ is either 0 or else $\geq\int[1-\phi(x)]^3\,dx>0$ at each t. By (9) and [cite, mass bounds], v must be identically zero for all $t>-2$. This contradicts [cite], so our assumption that the sequence θ_n exists must have been false. The proposition must be true.

□

5. The Oscillation Lemmas

We put together Propositions 4.4 and 4.5 to produce an Oscillation lemma. It allows us to improve a bound on the supremum of a function based on information about its positive support.

Proposition 5.1 (Oscillation Lemma)**.** *Let θ and $u=u_l+u_h$ be solutions to [cite] the PDE. Let $\Lambda^{-1/4}u_h\in L^\infty$ while $u_l\in Lip\cap L^{3/4}$. Moreover, let γ and Γ be such that*

$$\dot{\Gamma}+\dot{\gamma}=u_l(t,\Gamma+\gamma)$$

and $\|\dot{\gamma}\|_\infty\leq C_g$.

There exists a number k_0 such that if for all $t\in[-5,0]$, $x\in\Omega$,

$$\theta(t,x)\leq 2+2^{-k_0}\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+$$

and

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge \frac{4|B_4|}{2}$$

then for all $t \in [-1,0]$*,* $x \in B_1(\Gamma)$ *we have*

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

Proof. Let μ and δ_0 as in Proposition 4.5, and take k_0 large enough that $(k_0 - 1)\mu > 4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x) := 2 + 2^k(\theta(t,x) - 2).$$

That is, $\theta_0 = \theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies [cite, boundedness], each θ_k for $k \le k_0$ and $(t,x) \in [-5,0] \times \Omega$ satisfies

$$\theta_k(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 4.5.

Note also that

$$|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma)|$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed δ_0 for all $k < k_0$ as well.

Applying Proposition 4.5 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0] \times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity [cite] increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity [cite] is bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption [cite] must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 4.4. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall (t,x) \in [-1,0] \times B_1(\Gamma).$$

For the original function θ, this means that for $(t,x) \in [-1,0] \times B_1(\Gamma)$

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

□

That's the absolute gain. Now let us consider how this gain can be shifted to our new reference frame. But first, a quick technical lemma:

Lemma 5.2. *There exist a constant* $\bar{\lambda} > 0$ *and* $\alpha > 1$ *such that, for any* $0 < \varepsilon \le 1/2$ *and any* $z \ge 1$

$$\left(|\varepsilon^{-1}(z-1) + 3|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+ \ge \bar{\lambda}.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon = 1/2$. Consider

$$\left(|2z+1|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+.$$

When $\alpha = 1$, this quantity is clearly non-negative and in fact strictly positive when $z \ge 1$. On any compact interval $[0,N]$, the quantity with $\alpha = 1$ is bounded below, and the quantity $\left(|z|^{1/4} - 2^{1/4}\right)_+$ is bounded above, so if $\alpha - 1$ is less than the ratio of those bounds then the total quantity will be bounded below.

However, the range of acceptable α depends on N, and it is possible that no single α is acceptable for the whole of $z \in [1,\infty)$.

For $z > 2$, the expression reduces to

$$(2z+1)^{1/4} - \alpha z^{1/4} - (\alpha - 1)2^{1/4} = z^{1/4}\left((2+1/z)^{1/4} - \alpha\right) - (\alpha-1)2^{1/4}.$$

This quantity is increasing as α decreases, and for any $\alpha < 2^{1/4}$ it tends to ∞ as z increases.

This is sufficient to show that for some $\alpha > 1$, there exists a lower bound $\bar{\lambda}$ on the quantity [cite], and thus the lemma holds. □

We are ready to prove the shifted version of the Harnack Inequality.

Lemma 5.3 (Oscillation Lemma, with shift). *Let θ and $u = u_l + u_h$ be as desired. Let Γ and γ be paths such that*

$$\dot{\Gamma} + \dot{\gamma} = u_l(t, \Gamma + \gamma)$$

and $\|\gamma\|_{Lip} \le C_g$. If $0 < \varepsilon < 1/5$ is such that

$$5C_g \le \varepsilon^{-1} - 3 \tag{10}$$

then the following holds:

*Let k_0 be as in Lemma **??** and assume that for $(t,x) \in [-5,0] \times \Omega$*

$$|\theta(t,x)| \le 2 + 2^{-k_0}\left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+$$

and

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge \frac{4|B_4|}{2}.$$

Then there exist a $\lambda > 0$ small enough that for $(t,x) \in [-5,0] \times \varepsilon^{-1}\Omega$

$$\left|\frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]\right| \le 2 + 2^{-k_0}\left(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|^{1/4} - 2^{1/4}\right)_+.$$

If we only wish to show that by zooming horizontally by a large amount and zooming and translating vertically by a small amount we stay under the barrier, this is obvious and merely requires being written down. Even the shift itself is clearly not a problem when considered in the un-zoomed coordinates. Since the velocity of γ is bounded by C_g, the shift γ is arbitrarily small over very small time periods. The important thing to pay attention for is the dependence of ε and C_g and k_0 on eachother.

As we will see in Section 6 when we apply this lemma, the constant C_g depends on ε and k_0 depends on C_g. In the following proof, the constant ε will need to be small relative to C_g. The assumption (10) in this lemma turns out to be satisfiable, and now we must prove that it is sufficient.

Proof. Take λ such that

$$2\lambda \le 2^{-k_0}, \qquad (2+\lambda)\left(\frac{2}{2-\lambda}\right) \le 2 + 2^{-k_0}\bar{\lambda}, \qquad \frac{2}{2-\lambda} \le \alpha. \tag{11}$$

for $\bar{\lambda}$ and α from Lemma 5.2.

Denote

$$\bar{\theta}(t,x) := \frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]$$

and

$$\phi(x) = \left(|x|^{1/4} - 2^{1/4}\right)_+.$$

We already proved in Lemma 5.1 that $\theta \le 2 - 2^{-k_0}$ on $[-1,0] \times B_1(\Gamma)$. For $\bar{\theta}$, this means that when $(t,x) \in [-1/\varepsilon, 0] \times B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$,

$$\bar{\theta}(t,x) \le \frac{2}{2-\lambda}\left[2 - 2^{-k_0} + \lambda\right] \le \frac{2}{2-\lambda}\left[2 - \lambda\right] = 2$$

and

$$\bar{\theta}(t,x) \geq \frac{2}{2-\lambda}\left[-2+\lambda\right] = -2.$$

Similarly, the bound [cite] on θ becomes the equivalent bounds on $\bar{\theta}$, for all $(t,x) \in [-5/\varepsilon, 0] \times \varepsilon^{-1}\Omega$

$$\bar{\theta}(t,x) \leq \frac{2}{2-\lambda}\left[2+\lambda+2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right]$$

and

$$\bar{\theta}(t,x) \geq \frac{2}{2-\lambda}\left[\lambda-2-2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right].$$

It remains to show that these various bounds on $\bar{\theta}$ imply the bound stipulated by the proposition. Let $t \in [-5,0]$ and $x \in \varepsilon^{-1}\Omega$, and define $y = x - \varepsilon^{-1}\Gamma(\varepsilon t)$.
If $|y| \leq \varepsilon^{-1}$ then

$$\left|\bar{\theta}(t,x)\right| \leq 2 \leq 2 + 2^{-k_0}\phi(x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)).$$

If $|y| \geq \varepsilon^{-1}$ then from Lemma 5.2,

$$\bar{\lambda} + \alpha\phi(\varepsilon|y|) \leq \phi(|y| - \varepsilon^{-1} + 3).$$

From the assumptions (11), we can rewrite the bound [cite] as

$$\begin{aligned}\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2+\lambda+2^{-k_0}\phi(\varepsilon|y|)\right]\\ &\leq 2 + 2^{-k_0}\bar{\lambda} + 2^{-k_0}\alpha\phi(\varepsilon|y|)\\ &= 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]\\ &\leq 2 + 2^{-k_0}\phi(|y| - \varepsilon^{-1} + 3).\end{aligned}$$

For $t \in [-5,0]$,

$$|y| - 5C_g \leq |y - \varepsilon^{-1}\gamma(\varepsilon t)|.$$

Thus, with assumption (10),

$$|y| - \varepsilon^{-1} + 3 \leq |y - \varepsilon^{-1}\gamma(\varepsilon t)|.$$

Therefore, for $|y| \geq \varepsilon^{-1}$,

$$\bar{\theta}(t,x) \leq 2 + 2^{-k_0}\phi(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|).$$

On the other hand,

$$\begin{aligned}-\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2-\lambda+2^{-k_0}\phi(\varepsilon|y|)\right]\\ &\leq 2 + 2^{-k_0}\alpha\phi(\varepsilon|y|)\\ &\leq 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]\\ &\leq 2 + 2^{-k_0}\phi(|y| - \varepsilon^{-1} + 3)\\ &\leq 2 + 2^{-k_0}\phi(|y - \varepsilon^{-1}\gamma(\varepsilon t)|).\end{aligned}$$

This concludes the proof. $\square$

6. Hölder Continuity

In this section we prove the main theorem.

Proof. We'll show that if θ with $\|\theta\|_{L^\infty([-5,0]\times\Omega)} \leq 2$ solves (1) on $[-5,0]\times\Omega$ then θ is Hölder continuous at the point $(0,0)$ (with possibly $0\in\bar{\Omega}$). Up to translation and scaling, this will be sufficient to show continuity at all points in the domain, with a constant depending on Ω and on the time we wait.

From Section 2, we know that

$$\mathbb{R}^\perp\theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence $(u_j)_j$ calibrated with some constant $\kappa = \kappa(\Omega)$ and center 0.

Choose a constant $0<\varepsilon<1/5$ such that

$$5\max\left(-\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}, (1-j_0)\kappa\right) \leq \varepsilon^{-1}-3, \tag{12}$$

For notational convenience, denote

$$\sum_k = \sum_{j>-k\ln(\varepsilon)}, \qquad \sum^k = \sum_{j\leq -k\ln(\varepsilon)}.$$

For integers $k\geq 0$ consider the domains

$$\Omega_k := \{x\in\mathbb{R}^2 : \varepsilon^k x\in\Omega\}$$

and define the following functions on $[-5,0]\times\Omega_k$:

$$\begin{aligned} u_l^k(t,x) &:= \sum^k u_j(\varepsilon^k t, \varepsilon^k x),\\ u_h^k(t,x) &:= \sum_k u_j(\varepsilon^k t, \varepsilon^k x). \end{aligned}$$

For $t\in[-5,0]$ and $k\geq 0$ define $\Gamma_k,\gamma_k : [-5,0]\to\mathbb{R}^2$ by the following ODEs:

$$\begin{aligned} \Gamma_0(t) &:= 0,\\ \gamma_k(0) &:= 0,\\ \dot{\gamma}_k(t) &:= u_l^k(t, \Gamma_k(t)+\gamma_k(t)) - \dot{\Gamma}_k(t)\\ \Gamma_k(t) &:= \varepsilon^{-1}\gamma_{k-1}(\varepsilon t) + \varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t) + \cdots + \varepsilon^{-k}\gamma_0(\varepsilon^k t), \qquad k\geq 1. \end{aligned}$$

Use [citation] some lemma from Bahouri-Chemin-Danchin that's a generalization of Picard-Lindelof to prove that these γ exist. Each u_l^k is a Lipschitz-in-space vector field, and each $\Gamma_k+\gamma_k$ is a flow along that vector field which ends up at the origin at $t=0$. In particular, since u_l^k is tangential to the boundary of Ω_k and has unique flows, the flow $\Gamma_k+\gamma_k$ cannot exit the region Ω_k and so our expressions remain well-defined.

By Lemmas 3.2 and 3.1, we know the sequence $(u_j(\varepsilon^k\cdot,\varepsilon^k\cdot))_j$ is calibrated and hence that independently of k

$$\left\|\Lambda^{-1/4}u_h^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \leq C\kappa$$

etc. Particularly

$$\left\|\nabla u_l^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \leq 2\kappa.$$

By construction, for $k \geq 0$ we have $\Gamma_{k+1}(t) = \varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)$. Therefore

$$\begin{aligned}\dot\Gamma_{k+1}(t) &= \partial_t\left[\varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)\right]\\ &= \dot\gamma_k(\varepsilon t) + \dot\Gamma_k(\varepsilon t)\\ &= u_l^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t))\\ &= u_l^k(\varepsilon t, \varepsilon\Gamma_{k+1}(t)).\end{aligned}$$

With this in hand, we can bound the size of γ_k. Namely, for $k \geq 1$,

$$\begin{aligned}\dot\gamma_k(t) &= u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot\Gamma_k(t)\\ &= u_l^k(t, \Gamma_k(t) + \gamma_k(t)) - u_l^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t))\\ &= \sum^k u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t))\\ &= \sum^{k-1}\left[u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k\ldots)\\ &= \left[u_l^{k-1}\big(\varepsilon t, \varepsilon\Gamma_k(t) + \varepsilon\gamma_k(t)\big) - u_l^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k\ldots).\end{aligned}$$

The sum $\sum^{k-1} u_j(\varepsilon^k\cdot, \varepsilon^k\cdot) = u_l^{k-1}(\varepsilon\,\cdot, \varepsilon\,\cdot)$ is Lipschitz in space, with Lipschitz constant less than $2\varepsilon\kappa$. Moreover, each u_j has $\|u_j\|_\infty \leq \kappa$. Thus both terms of $\dot\gamma_k(t)$ are bounded

$$|\dot\gamma_k(t)| \leq 2\varepsilon\kappa|\gamma_k(t)| - \kappa\ln_2(\varepsilon).$$

This, by Gronwall's inequality, tells us that for $t \in [-5, 0]$,

$$|\gamma_k(t)| \leq \frac{-\ln_2(\varepsilon)}{2\varepsilon}\left(e^{10\varepsilon\kappa} - 1\right)$$

and hence

$$|\dot\gamma_k(t)| \leq -\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}.$$

To account for γ_0, define

$$C_g = \max\left(-\kappa\ln_2(\varepsilon)e^{10\varepsilon\kappa}, j_0\kappa\right)$$

so that for all $k \geq 0$ and $t \in [-5, 0]$

$$|\dot\gamma_k(t)| \leq C_g.$$

Let us now produce a sequence of solutions θ_k. Define

$$\theta_0(t, x) := \theta(t, x)$$

and for each $k \geq 0$, if $|\{\theta_k \leq 0\} \cap [-5, 0] \times B_4(\Gamma_k(t))| \geq 2|B_4|$ then set

$$\theta_{k+1}(t, x) := \frac{2}{2-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) + \lambda\right].$$

Otherwise, set

$$\theta_{k+1}(t, x) := \frac{1}{1-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) - \lambda\right].$$

From Lemma 3.2, we know that θ_k and the calibrated sequence $(u_j(\varepsilon^k\cdot, \varepsilon^k\cdot))_j$ solve (4).

We will now show that

$$|\theta_k| \leq 2 + 2^{-k_0}\left(|x - \Gamma_k(t)|^{1/4} - 2^{1/4}\right)_+ \tag{13}$$

holds for all $k \geq 0$.

Since $|\theta_0| \leq 2$ by assumption, we know in particular that (13) holds at $k = 0$.

This is sufficient for us to apply Lemma **??** to each θ_k (or to $-\theta_k$ as appropriate) in order. We conclude that (13) holds for all $k \geq 0$.

Each θ_k is between -2 and 2 on $[-5,0] \times B_2(\Gamma_k)$. But recall that each Γ_k is Lipschitz with constant kC_g. Thus $|\Gamma_k(t)| \le 1$ for $t \in [-(kC_g)^{-1}, 0]$. On that time interval,

$$|\theta_k(t,x)| \le 2 \qquad \forall x \in B_1(0).$$

We conclude that

$$\left| \sup_{[-\varepsilon^k (kC_g)^{-1},0] \times B_{\varepsilon^k}(0)} \theta(t,x) - \inf_{[-\varepsilon^k (kC_g)^{-1},0] \times B_{\varepsilon^k}(0)} \theta(t,x) \right| \le 4 \left(\frac{2}{2-\lambda} \right)^{-k}.$$

In particular, for some positive constant C such that

$$\varepsilon^{Ck} \le (kC_g)^{-1} \qquad \forall k \ge 0,$$

we can say that

$$|t|^2 + |x|^2 \le \varepsilon^{(1+C)k}$$

implies that $(t,x) \in [-\varepsilon^k (kC_g)^{-1}, 0] \times B_{\varepsilon^k}(0)$ which in turn implies that

$$|\theta(t,x) - \theta(0,0)| \le 4 \left(\frac{2}{2-\lambda} \right)^{-k}.$$

In other words,

$$\begin{aligned} |\theta(t,x) - \theta(0,0)| &\le 4 \left(\frac{2}{2-\lambda} \right)^{-\frac{1}{1+C} \log_\varepsilon (|t|^2 - |x|^2) + 1} \\ &= 4 \left(\frac{2}{2-\lambda} \right) \exp \left[\ln \left(\frac{2}{2-\lambda} \right) \frac{\ln(|t|^2 + |x|^2)}{-(1+C)\ln(\varepsilon)} \right] \\ &= \frac{8}{2-\lambda} (|t|^2 + |x|^2)^{-\frac{\ln(2) - \ln(2-\lambda)}{(1+C)\ln(\varepsilon)}}. \end{aligned}$$

□

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