

# HOLDER REGULARITY UP TO THE BOUNDARY FOR CRITICAL SQG ON BOUNDED DOMAINS

LOGAN F. STOKOLS AND ALEXIS F. VASSEUR

ABSTRACT. We consider the dissipative SQG equation in bounded domains, first introduced by Constantin and Ignatova in 2016. We show global Holder regularity up to the boundary of the solution, with a method based on the De Giorgi techniques. The boundary introduces several difficulties. In particular, the Dirichlet Laplacian is not translation invariant near the boundary, which leads to complications involving the Riesz transform.

## CONTENTS

1. Preliminaries	1
2. Properties of the Fractional Dirichlet Laplacian	6
3. Existence of suitable solutions	11
4. Littlewood-Paley Theory	17
5. De Giorgi Estimates	20
6. A Decrease in Oscillation	26
7. Hölder Continuity	29
Appendix A. Technical Lemmas	33
References	36

## 1. PRELIMINARIES

The surface quasigeostrophic equation (SQG) is a special case of the quasi-geostrophic system (QG) with uniform potential vorticity. The QG model is used extensively in meteorology and oceanography (e.g. Charney [Cha71]). These models are described in Pedlosky [Ped92]. The SQG model was popularized by Constantin, Majda and Tabak in [CMT94], due to its similarities with the Euler and Navier-Stokes equation. They proposed it as a toy model for the study of 3D Fluid equations (see also Held, Garner, Pierrehumbert, and Swanson [HPGS95]).

We consider in this paper critical SQG on a bounded domain. We will focus on the following model, which was introduced by Constantin and Ignatova in [CI17] and [CI16]. Consider  $\Omega$  a connected bounded domain in  $\mathbb{R}^2$  with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and the Laplacian with homogeneous Dirichlet boundary conditions  $-\Delta_D$ . If  $(\eta_k)_{k \in \mathbb{N}}$  is the sequence of  $L^2$ -normalized

---

*Date:* December 20, 2019.

*2010 Mathematics Subject Classification.* 35Q35, 35Q86.

*Key words and phrases.* SQG, Bounded domain, Besov spaces on bounded domains, Hölder regularity, De Giorgi method.

**Acknowledgment.** This work was partially supported by the NSF Grant DMS 1614918 and the NSF RTG Grant DMS 1840314. We would also like to thank the referee for suggesting several improvements to the paper, especially the construction of global suitable solutions.

eigenfunctions of  $-\Delta_D$  with corresponding eigenvalues  $\lambda_k$  listed in non-decreasing order, define

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

The critical SQG problem on  $\Omega$  with initial data  $\theta_0 \in L^2(\Omega)$  is

$$(1) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0 & (0, T) \times \Omega, \\ u = \nabla^\perp \Lambda^{-1} \theta & [0, T] \times \Omega, \\ \theta = \theta_0 & \{0\} \times \Omega. \end{cases}$$

In the model, the dissipation  $\Lambda = (-\Delta_D)^{1/2}$  is due to the Ekman pumping, while the nonlinear velocity  $u$  comes from the geostrophic and hydrostatic balance (see [Ped92]).

The main result of this paper is the following:

**Theorem 1.1.** *There exists a universal constant  $C_1 > 0$  such that the following holds:*

*For any  $\Omega \subseteq \mathbb{R}^2$  open and bounded with  $C^{2,\beta}$  boundary,  $\beta \in (0, 1)$ , there exists for any  $S > 0$  a constant  $C_S > 0$  (depending also on  $\Omega$ ), and for any  $k > 0$  a constant  $\alpha_{k,S} \in (0, 1)$  (depending also on  $\Omega$ ) so the following holds:*

*For any  $\theta_0 \in L^2(\Omega)$  there exists a global-in-time weak solution  $\theta \in L^\infty([0, \infty); L^2(\Omega)) \cap L^2([0, \infty); \mathcal{H}^{1/2})$  to (1) verifying  $\theta(t, x) = 0$  on  $(0, \infty) \times \partial\Omega$  and  $\lim_{t \rightarrow 0} \theta(t, \cdot) = \theta_0$  in the  $L^2$ -weak sense. For  $k \geq \|\theta_0\|_{L^2(\Omega)}$  and for every  $S > 0$*

$$\theta \in C^{\alpha_{k,S}}([S, \infty) \times \bar{\Omega})$$

*where  $\bar{\Omega}$  denotes the closure of  $\Omega$ .*

*Moreover,*

$$\|\theta\|_{L^\infty([S, \infty) \times \bar{\Omega})} \leq \frac{C_1}{S} \|\theta_0\|_{L^2(\Omega)}$$

*and*

$$\|\theta\|_{C^{\alpha_{k,S}}([S, \infty) \times \bar{\Omega})} \leq C_S \|\theta_0\|_{L^2(\Omega)}.$$

This model was first thoroughly studied in the cases without boundaries (either  $\mathbb{R}^2$  or the torus  $\mathbb{T}^2$ ). Global weak solutions were first constructed in Resnick [Res95]. Global regularity was first shown with small initial values by Constantin, Cordoba, and Wu [CCW01], or extra  $C^\alpha$  regularity on the velocity in Constantin and Wu [CW08] and Dong and Pavlović [DP09]. In [KNV07], Kiselev, Nazarov and Volberg showed the propagation of  $C^\infty$  regularity. The global  $C^\infty$  regularity for any  $L^2$  initial values was first proved in [CV10] (see also Kiselev and Nazarov [KN09] and Constantin and Vicol [CV12]).

In the presence of boundaries, there are several distinct ways to define SQG. This can be attributed to alternative generalizations of the fractional Laplacian. Kriventsov [Kri15] considered a two-phase problem which satisfies critical SQG only in part of the domain, and was able to prove Hölder regularity in the time-independent case. This problem, intended to model air currents over a region containing both land and water, contains a half-Laplacian and a Riesz transform defined, not spectrally, but in terms of extension. In [NV18b], the authors consider the Euler-Coriolis-Boussinesq model and derive the full 3D inviscid quasigeostrophic system in an impermeable cylinder (see also [NV19] for the construction of small time smooth solutions to the model). They obtain natural boundary conditions for SQG distinct from the homogeneous conditions introduced in [CI17], [CI16] and described above. However, due to the complexity of the model described in [NV18b], we focus in this paper only on the homogenous case.

Existence of weak solutions for (1) is proven in [CI17], and local existence and uniqueness for strong solutions with sufficiently smooth initial data is proven by Constantin and Nguyen in [CN18b]

(see also Constantin and Nguyen [CN18a] and Constantin, Ignatova, and Nguyen [CIN18] for the inviscid case). The interior regularity of solutions is proven in [CI16] (together with propagation of  $L^\infty$  bounds). The method of proof for interior regularity uses nonlinear maximum principles, introduced by Constantin and Vicol [CV12]. However, the bounds obtained in [CI16] blow up near the boundary and do not provide global regularity. In [CI16] Remark 1, questions about global regularity are suggested as open problems. Both the  $C^\alpha(\Omega)$  regularity, and bootstrapping to  $C^\infty(\bar{\Omega})$  regularity, are identified as interesting problems. Our result answers the first question, by showing that solutions  $\theta$  to (1) are globally Hölder continuous. Bootstrapping to  $C^\infty$  involves different techniques, and will be studied in a forthcoming work [SV].

Our proof is based on the De Giorgi method pioneered by De Giorgi in [DG57]. The method was applied to the SQG problem first in [CV10]. The method is powerful for showing  $C^\alpha$  regularity of elliptic- and parabolic-type equations. It has been applied in a variety of situations for non-local problems, such as the fractional heat equation in [CCV11], the time-fractional case in [ACV16], the 3D Quasigeostrophic problem in [NV18a], or the kinetic setting by Imbert and Silvestre [IS16] or in [Sto18]. The method has also been applied in more exotic, non-elliptic situations such as Hamilton-Jacobi equations (see [CV17], [SV18]).

The De Giorgi method involves rescaling our equation by zooming in iteratively, and applying regularity results at each scale. Therefore it is important that certain results be proven independently of the domain  $\Omega$ . The particular dependence on  $\Omega$  will be made clear in each lemma of this paper. As a general overview, in the proof of Theorem 1.1 we will apply the results of Sections 3 and 4 only on a single fixed domain, while the results of Sections 5 and 6 must be applied at each level of zoom with a different rescaled domain each time.

The first broad idea of our proof consists in decoupling the velocity  $u$  from  $\theta$  to work on a linear equation, and prove alternating regularity results for  $\theta$  and  $u$  independently. We can show that  $\theta$  is in  $L^\infty$  without any assumption on  $u$  (see Section 3). Using that  $L^\infty$  bound, we will need to obtain scaling invariant controls on the drift  $u = \nabla \Lambda^{-1} \theta$ . By scaling invariant, we mean that the bound, once proven on  $\Omega$  fixed, will remain true of the scaled function  $u(\varepsilon \cdot, \varepsilon \cdot)$  for all  $\varepsilon$ . Unfortunately, although the Riesz transform is bounded from  $L^p$  to  $L^p$  for all  $p$  finite, it is not bounded for  $p = \infty$ . The usual technique, therefore, is to consider BMO (as in [CV10] and [NV18a]), but in the case of bounded domains the Riesz transform is not known to be bounded in this space either. Our solution is to use extensions of the Littlewood-Paley theory to bounded domains.

The adaptation of Fourier analysis and Littlewood-Paley theory to Schrodinger operators is a well-studied subject (e.g. Zheng [Zhe06], Benedetto and Zheng [BZ10]). As an application of this theory, Iwabuchi, Matsuyama, and Taniguchi [IMT19], [IMT18], and Bui, Duong, and Yang [BDY12] have considered operators defined on open subsets of  $\mathbb{R}^n$ , which includes as a special case the operator  $-\Delta_D$  (a Schrodinger operator with zero potential). In particular, in [IMT17], Iwabuchi, Matsuyama, and Taniguchi derive many important results, including the Bernstein inequalities, for Besov spaces adapted to the operator  $-\Delta_D$  on bounded open subsets of  $\mathbb{R}^n$  with smooth boundary. This theory turns out to greatly improve our understanding of the Riesz transform  $\nabla \Lambda^{-1}$  on bounded domains.

Using the results of [IMT17], we will be able to show that the Riesz transform of an  $L^\infty$  function whose Fourier decomposition  $f = \sum f_k \eta_k$  is supported on high frequencies  $k > N$  will be bounded in the weak sobolev space  $W^{-1/4, \infty}$ , and the Riesz transform of an  $L^\infty$  function whose Fourier decomposition is supported on low frequencies  $k < N$  will have bounded Lipschitz constant. The cutoff  $N$  for dividing high frequencies from low frequencies must depend however on the size of the domain  $\Omega$ . In the case of  $\mathbb{R}^2$ , where  $\nabla$  and  $\Lambda^{-1}$  commute, this is equivalent to the observation that the Riesz transform is bounded from  $L^\infty$  to the Besov space  $B_{\infty, \infty}^0$ . In the case of bounded domains, the argument must be more subtle. We must decompose  $\theta$  into its Littlewood-Paley

projections, individually bound the Riesz transform of each projection in multiple spaces, and then recombine these infinitely-many functions into a low-frequency collection and a high-frequency collection depending on the scale of oscillation we are trying to detect (see Section 4 and Lemma 5.1).

We make this notion precise with the following definition:

**Definition 1** (Calibrated sequence). Let  $\Omega \subseteq \mathbb{R}^2$  be any bounded open set and  $0 < T \in \mathbb{R}$ . We call a function  $u \in L^2([0, T] \times \Omega)$  **calibrated** if it can be decomposed as the sum of a calibrated sequence

$$u = \sum_{j \in \mathbb{Z}} u_j$$

with each  $u_j \in L^2([0, T] \times \Omega)$  and the infinite sum converging in the sense of  $L^2$ .

We call a sequence  $(u_j)_{j \in \mathbb{Z}}$  **calibrated** for a constant  $\kappa$  and a center  $N$  if each term of the sequence satisfies the following bounds.

$$\begin{aligned} \|u_j\|_{L^\infty([0, T] \times \Omega)} &\leq \kappa, \\ \|\nabla u_j\|_{L^\infty([0, T] \times \Omega)} &\leq 2^j 2^{-N} \kappa, \\ \|\Lambda^{-1/4} u_j\|_{L^\infty([0, T] \times \Omega)} &\leq 2^{-j/4} 2^{N/4} \kappa. \end{aligned}$$

In Section 7 we will show that a calibrated velocity remains calibrated at all scales (specifically, with fixed constant  $\kappa$  but a changing center  $N$ ). Therefore we can consider, for any domain  $\Omega$  and time  $T$ , the system of linear equations

$$(2) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, & [-T, 0] \times \Omega \\ \operatorname{div} u = 0 & [-T, 0] \times \Omega. \end{cases}$$

In Section 3 we show that solutions to (1) with  $L^2$  initial data exist and regularize instantly into  $L^\infty$ , and in Section 4 we show that the Riesz transform of  $L^\infty$  data is calibrated. Then in Sections 5 and 6 we will show that solutions to (2) with calibrated velocity have decreasing oscillation between scales. By iteratively applying this oscillation lemma and scaling our equation, we show in Section 7 that  $\theta$  is Hölder continuous.

The low-frequency component of a calibrated velocity  $u$  will be uniformly Lipschitz, which means it is only bounded up to a constant. This is similar to the case of BMO velocity functions in [CV10] and [NV18a], which by the John-Nirenberg inequality are also bounded up to a constant. As in these cases, we consider a moving reference frame, denoted  $\Gamma : [0, T] \rightarrow \mathbb{R}^2$ , in which our velocity is shifted by a constant, making the low-frequency component of  $u$  bounded. There are two differences between our implementation of this technique and the implementation in [CV10] and [NV18a]: firstly, we subtract off the value of the low-frequency part of  $u$  at a point, rather than subtracting off the average of  $u$  on a ball. Secondly, rather than applying the standard De Giorgi argument to  $\tilde{\theta}(t, x) := \theta(t, x + \Gamma(t))$ , we must reformulate the De Giorgi argument to “follow” the path  $\Gamma(t)$  explicitly. This is a purely notational difference, but it is necessary because otherwise  $\Omega$  would be time-dependent.

At each scale, there will be a natural Lagrangian path  $\Gamma_\ell$  corresponding to the low-frequency part of  $u$ . However, the low-frequency part of  $u$  changes non-trivially as we zoom, so  $\Gamma_\ell$  will be different at each scale. Throughout Sections 5 and 6, we will use  $\Gamma_\ell$  to denote the “current” Lagrangian path and  $\Gamma$  to denote the Lagrangian path at the previous scale. In the proof of Theorem 1.1 in Section 7, these are denoted  $\Gamma_k(t)$  and  $\varepsilon^{-1} \Gamma_{k-1}(\varepsilon t)$  respectively. In Lemmas 5.2, 5.3, 5.4 and 6.1, we will make assumptions about  $\theta$  which are centered on  $x \approx \Gamma(t)$  and obtain conclusions which are similarly centered on  $x \approx \Gamma(t)$ , conditioned on  $\gamma := \Gamma_\ell - \Gamma$  being small in Lipschitz norm. Finally in Lemma 6.2, we will show that, given bounds on  $\theta$  for  $x \approx \Gamma(t)$ , we can bound  $\theta$  for  $x \approx \Gamma_\ell(t)$  for  $t$  sufficiently small, again conditioned on  $\gamma := \Gamma_\ell - \Gamma$  being small in Lipschitz norm. Controlling  $\gamma$

amounts to controlling the change in  $\Gamma_\ell$  between consecutive scales, which is much easier to obtain than scale-independent bounds on  $\Gamma_\ell$ .

Previous applications of the De Giorgi method to non-local equations such as (2) generally make extensive use of either an extension representation (c.f. [CV10]) or a singular integral representation (c.f. [NV18a]). In this paper, we use the singular integral representation for the Dirichlet fractional Laplacian derived by Caffarelli and Stinga [CS16]. It is based on the results of Stinga and Torrea [ST10] which generalize the extension representation of Caffarelli and Silvestre [CS07]. This theory is pivotal in translating the existing non-local De Giorgi techniques to the problem at hand (see Section 2).

In order to apply De Giorgi's method to weak solutions of (2), we will need to assume a certain a priori estimate which holds, in particular, for  $L^2(H_0^1)$  weak solutions. However, such solutions are only known to exist for short time and for  $H^2$  initial data, as shown by Constantin and Nguyen in [CN18b]. We call weak solutions in  $L^2(\mathcal{H}^{1/2})$  which happen to verify this a priori estimate “suitable solutions,” by analogy to suitable solutions to Navier-Stokes as in [CKN82]. We give the formal definition of suitable solutions in Section 3, where we also construct global-in-time suitable solutions using the vanishing viscosity method. Compared to [CI17], our solutions verify a full family of localized energy inequalities which allow us to apply the De Giorgi method.

The Paper is organized as follows. Section 2 is dedicated to basic properties of the operator  $\Lambda$  and the corresponding Sobolev spaces  $\mathcal{H}^s$ . In Section 3 we construct weak solutions which verify the suitability conditions. In Section 4 we prove that the Riesz transform of the  $L^\infty$  function  $\theta$  is calibrated. Section 5 contains the De Giorgi Lemmas. Section 6 is dedicated to the local decrease in oscillation through an analog of the Harnack inequality. Finally in Section 7 we prove the main theorem, Theorem 1.1. In the Appendix A we prove a few technical lemmas which are needed in the main paper.

**Notation.** Throughout the paper, we will use the following notations. By  $\eta_k$  and  $\lambda_k$  we mean the eigenfunctions and eigenvalues of  $-\Delta_D$ , with  $\lambda_0 \leq \lambda_1 \leq \dots$  and  $\|\eta_k\|_2 = 1$  for all  $k$ . If  $f = \sum_k f_k \eta_k$  then

$$\begin{aligned} \|f\|_{\mathcal{H}^s} &:= \left( \sum_k \lambda_k^s f_k^2 \right)^{1/2} \\ &= \int |\Lambda^s f|^2. \end{aligned}$$

We suppress the dependence on  $\Omega$ , though in fact  $\Lambda$ ,  $\lambda_k$ , and  $\eta_k$  are defined in terms of the domain  $\Omega$ . The relevant domain will be clear from context. The norm on  $\mathcal{H}^s$  is in fact a norm, not a seminorm, since  $\|f\|_{L^2(\Omega)} \leq \lambda_0^{-s/2} \|f\|_{\mathcal{H}^s}$ .

For a set  $A$  and a function  $f : A \rightarrow \mathbb{R}$ , denote

$$\begin{aligned} [f]_{\alpha;A} &:= \sup_{x,y \in A, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}, & \alpha \in (0, 1], \\ \|f\|_{C^\alpha(A)} &:= \|f\|_{L^\infty(A)} + [f]_{\alpha;A}, & \alpha \in (0, 1], \\ \|f\|_{C^{k,\alpha}(A)} &:= \sum_{n=0}^k \|D^n f\|_{L^\infty(A)} + [D^k f]_{\alpha;A}, & \alpha \in (0, 1], k \in \mathbb{N}. \end{aligned}$$

When the domain  $A$  is omitted, the relevant spatial domain  $\Omega$  is implied.

We will use the notation  $(x)_+ := \max(0, x)$ . When the parentheses are omitted, the subscript  $+$  is merely a label.

Throughout this paper, if an integral sign is written  $\int$  without a specified domain, the domain is implied to be  $\Omega$ , with  $\Omega$  defined in context.

For any vector  $v = (v_1, v_2)$ , by  $v^\perp$  we mean  $(-v_2, v_1)$ . By  $\nabla^\perp$  we mean  $(-\partial_y, \partial_x)$ .

In the remainder of this paper, the differential operator  $D^2$  refers to the Hessian in space, excluding time derivatives. The function space  $C_c^\infty$  consists of smooth functions with compact support. By abuse of notation, if  $\Gamma : [a, b] \rightarrow \mathbb{R}^2$ , we write  $[a, b] \times B_R(\Gamma)$  to denote  $\{(t, x) \in [a, b] \times \mathbb{R}^2 : |x - \Gamma(t)| \leq R\}$ .

The symbol  $C$  represents a constant which may change value each time it is written.

## 2. PROPERTIES OF THE FRACTIONAL DIRICHLET LAPLACIAN

In this section we will investigate the basic properties of the operator  $\Lambda$  and the space  $\mathcal{H}^s$  on a general domain  $\Omega$ .

We begin by stating a result of [CS16] which gives us a singular integral representation of the  $\mathcal{H}^s$  norm.

**Proposition 2.1** (Caffarelli-Stinga Representation). *Let  $s \in (0, 1)$  and  $f, g \in \mathcal{H}^s$  on a bounded  $C^{2,\beta}$  domain  $\Omega \subseteq \mathbb{R}^2$ . Then*

$$(3) \quad \int_{\Omega} \Lambda^s f \Lambda^s g \, dx = \iint_{\Omega^2} [f(x) - f(y)][g(x) - g(y)] K_{2s}(x, y) \, dx dy + \int_{\Omega} f(x) g(x) B_{2s}(x) \, dx$$

for kernels  $K_{2s}$  and  $B_{2s}$  which depend on the parameter  $s$  and the domain  $\Omega$ .

There exists a constant  $C = C(s)$  independent of  $\Omega$  such that

$$0 \leq K_{2s}(x, y) \leq \frac{C(s)}{|x - y|^{2+2s}}$$

for all  $x \neq y \in \Omega$  and

$$0 \leq B_{2s}(x)$$

for all  $x \in \Omega$ .

Moreover, for any  $s, t \in (0, 2)$  there exists a constant  $c = c(s, t, \Omega)$  such that for all  $x \neq y \in \Omega$

$$(4) \quad K_t(x, y) \leq c |x - y|^{s-t} K_s(x, y).$$

*Proof.* See [CS16] Theorems 2.3 and 2.4.

Theorem 2.4 in [CS16] does not explicitly state the result (4). However, it does state that for each kernel  $K_s$  there exists a constant  $c_s$  dependent on  $s$  and  $\Omega$  such that

$$\frac{1}{c_s} |x - y|^{2+s} K_s(x, y) \leq \min \left( 1, \frac{\eta_0(x) \eta_0(y)}{|x - y|^2} \right) \leq c_s |x - y|^{2+s} K_s(x, y).$$

Since the middle term does not depend on  $s$ , we can say that

$$|x - y|^{2+t} K_t(x, y) \leq c_t c_s |x - y|^{2+s} K_s(x, y)$$

from which (4) follows.  $\square$

Though the result is proven in [CS16] only for  $f, g \in \mathcal{H}^s$ , the result applies much more generally by a standard continuity argument.

From the explicit formulae given in [CS16], we see that  $K_{2s}$  is approximately equal to the standard kernel for the  $\mathbb{R}^2$  fractional Laplacian  $(-\Delta)^s$  when both  $x$  and  $y$  are in the interior of  $\Omega$  or when  $x$  and  $y$  are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel  $B_{2s}$  is well-behaved in the interior but has a singularity at the boundary  $\partial\Omega$ . This justifies our thinking of the  $K_{2s}$  term as the interior term and  $B_{2s}$  as a boundary term.

When comparing the computations in this paper to corresponding computations on  $\mathbb{R}^2$ , one finds that the interior term behaves nearly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

Many useful results can be derived from Caffarelli-Stinga representation formula. We summarize them in the following lemma.

**Lemma 2.2.** *Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ .*

(a) *Let  $s \in (0, 1)$ . If  $f$  and  $g$  are non-negative functions with disjoint support (i.e.  $f(x)g(x) = 0$  for all  $x$ ), then*

$$\int \Lambda^s f \Lambda^s g dx \leq 0.$$

(b) *Let  $s \in (0, 1)$ . If  $g \in C^{0,1}(\Omega)$  then for some constant  $C = C(s)$  independent of  $\Omega$*

$$\|fg\|_{\mathcal{H}^s} \leq 2 \|g\|_{\infty} \|f\|_{\mathcal{H}^s} + C \|f\|_2 \sup_y \int \frac{|g(x) - g(y)|^2}{|x - y|^{2+2s}} dx.$$

(c) *Let  $s \in (0, 1)$ . If  $g \in C^{0,1}(\Omega)$  then for some constant  $C = C(s)$  independent of  $\Omega$*

$$\|fg\|_{\mathcal{H}^s} \leq C \|g\|_{C^{0,1}(\Omega)} (\|f\|_2 + \|f\|_{\mathcal{H}^s}).$$

(d) *Let  $s \in (0, 1/2)$ . Let  $g$  an  $L^\infty(\Omega)$  function and  $f \in \mathcal{H}^{2s}$  be non-negative with compact support. Let  $C_{dmn}$  be a constant such that*

$$(5) \quad K_s(x, y) \leq C_{dmn} |x - y|^{3s} K_{4s}(x, y).$$

*Then there exists a constant  $C$  depending only on  $s$  and  $C_{dmn}$  such that*

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C \|g\|_{\infty} |\text{supp}(f)|^{1/2} (\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}).$$

(e) *Let  $g$  an  $L^\infty(\Omega)$  function and  $f \in \mathcal{H}^{1/2}$  be non-negative with compact support. Let  $C_{dmn}$  be a constant such that*

$$K_{1/4}(x, y) \leq C_{dmn} |x - y|^{3/4} K_1(x, y).$$

*Then there exists a constant  $C$  depending only on  $C_{dmn}$  such that*

$$\int g \Lambda^{1/4} f \leq C \|g\|_{\infty} |\text{supp}(f)|^{1/2} (\|f\|_2 + \|f\|_{\mathcal{H}^{1/2}}).$$

*Proof.* We prove these corollaries one at a time.

**Proof of (a):** From Proposition 2.1

$$\int \Lambda^s f \Lambda^s g dx = \iint [f(x) - f(y)][g(x) - g(y)] K(x, y) dx dy + \int f(x) g(x) B(x) dx.$$

Since  $f$  and  $g$  are non-negative and disjoint, the  $B$  term vanishes. Moreover, the product inside the  $K$  term becomes

$$[f(x) - f(y)][g(x) - g(y)] = -f(x)g(y) - f(y)g(x) \leq 0.$$

Since  $K$  is non-negative, the result follows.

**Proof of (b):** From Proposition 2.1

$$\begin{aligned} \int |\Lambda^s(fg)|^2 &= \iint (g(x)[f(x) - f(y)] + f(y)[g(x) - g(y)])^2 K + \int f^2 g^2 B \\ &\leq 2 \|g\|_{\infty}^2 \|f\|_{\mathcal{H}^s}^2 + C(s) \int f(y)^2 \int \frac{|g(x) - g(y)|^2}{|x - y|^{2+2s}} dx dy. \end{aligned}$$

**Proof of (c):** This follows immediately from (b), since

$$|g(x) - g(y)| \leq (\|g\|_{\infty}) \wedge (\|\nabla g\|_{\infty} |x - y|)$$

and

$$\int \frac{1 \wedge |x - y|^2}{|x - y|^{2+2s}} dx$$

is bounded uniformly in  $y$ .

**Proof of (d):** From Proposition 2.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_{<} + I_{\geq} + II$$

where

$$\begin{aligned} I_{<} &:= \iint_{|x-y|<1} [g(x) - g(y)][f(x) - f(y)] K_s, \\ I_{\geq} &:= \iint_{|x-y|\geq 1} [g(x) - g(y)][f(x) - f(y)] K_s, \\ II &:= \int f g B_s. \end{aligned}$$

First we estimate  $I_{<}$ . From (5) and from the symmetry of the integrand and the fact that  $[f(x) - f(y)]$  vanishes unless at least one of  $f(x)$  or  $f(y)$  is non-zero,

$$|I_{<}| \leq 2 \iint_{|x-y|<1} \chi_{\{f>0\}}(x) |g(x) - g(y)| \cdot |f(x) - f(y)| \cdot |x - y|^{3s} K_{4s}.$$

We can break this up by Hölder's inequality

$$|I_{<}| \leq 2 \left( \iint_{|x-y|<1} \chi_{\{f>0\}}(x) [g(x) - g(y)]^2 |x - y|^{6s} K_{4s} \right)^{1/2} \left( \iint [f(x) - f(y)]^2 K_{4s} \right)^{1/2}.$$

The kernel  $|x - y|^{6s} K_{4s} \chi_{\{|x-y|<1\}}$  is integrable in  $y$  for  $x$  fixed. Therefore

$$(6) \quad |I_{<}| \leq 2 \left( (2 \|g\|_{\infty})^2 \int C \chi_{\{f>0\}}(x) dx \right)^{1/2} (\|f\|_{\mathcal{H}^{2s}}^2)^{1/2}.$$

For the term  $I_{\geq}$ , by the symmetry of the integrand we have

$$|I_{\geq}| \leq 2 \|g\|_{\infty} 2 \int |f(x)| \int_{|x-y|\geq 1} K_s(x, y) dy dx.$$

Since  $K_s \chi_{\{|x-y|\geq 1\}}$  is integrable in  $y$  for  $x$  fixed,

$$(7) \quad |I_{\geq}| \leq C \|g\|_{\infty} \|f\|_1.$$

For the boundary term  $II$ ,

$$|II| \leq \|g\|_{\infty} \int \chi_{\{f>0\}} f B_s.$$

Since  $f \geq 0$ ,  $[f(x) - f(y)][\chi_{\{f>0\}}(x) - \chi_{\{f>0\}}(y)] \geq 0$ . Therefore

$$\int \chi_{\{f>0\}} f B_s \leq \int \Lambda^{s/2} \chi_{\{f>0\}} \Lambda^{s/2} f = \int \chi_{\{f>0\}} \Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \leq \|g\|_{\infty} |\text{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s}.$$

This combined with (6) and (7) gives us

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C \|g\|_{\infty} \left( |\text{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^{2s}} + \|f\|_1 + |\text{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s} \right).$$

The lemma follows since  $\|f\|_1 \leq |\text{supp}(f)|^{1/2} \|f\|_2$  and since  $\|f\|_{\mathcal{H}^s} \leq \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$ .

**Proof of (e):** This is an immediate application of part (d).



□

Let us consider the relationship between the norm  $\mathcal{H}^s$  and the  $H^s$  norm on  $\mathbb{R}^2$ .

It is known (see [CI16] and [CS16]) that for  $s \in (0, 1)$  the spaces  $\mathcal{H}^s$  are equivalent to certain subsets of  $H^s(\Omega)$  spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in  $\mathcal{H}^s$  for  $s \in [0, 1]$  and that elements of  $\mathcal{H}^s$  have trace zero for  $s \in [\frac{1}{2}, 1]$ .

The most important fact for us is that the fractional Sobolev norms defined in terms of extension are dominated by our  $\mathcal{H}^s$  norm with a constant that is independent of  $\Omega$ .

We do not claim that this result is new, but we present a detailed proof because the result is crucial to the De Giorgi method. The De Giorgi lemmas require Sobolev embeddings and Rellich-Kondrachov embeddings which are independent of scale.

Define the extension-by-zero operator  $E : L^2(\Omega) \rightarrow L^2(\mathbb{R}^2)$

$$Ef(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

**Proposition 2.3.** *Let  $\Omega \subseteq \mathbb{R}^2$  be any bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ . For any  $s \in [0, 1]$  and function  $f \in \mathcal{H}^s$ ,*

$$\int_{\mathbb{R}^2} |(-\Delta)^{s/2} Ef|^2 \leq \int_{\Omega} |\Lambda^s f|^2.$$

Here  $(-\Delta)^s$  is defined in the fourier sense.

We will prove this proposition by interpolating between  $s = 0$  and  $s = 1$ . Before we can do this, we must prove the same in the  $s = 1$  case. This result is known (see e.g. Jerison and Kenig [JK95]) but we include the proof for completeness.

**Lemma 2.4.** *Let  $\Omega \subseteq \mathbb{R}^2$  be any bounded open set with Lipschitz boundary. For all functions  $f$  in  $\mathcal{H}^1$ ,*

$$\int_{\Omega} |\nabla f|^2 = \int_{\Omega} |\Lambda f|^2.$$

*Proof.* Let  $\eta_i$  and  $\eta_j$  be two eigenfunctions of the Dirichlet Laplacian on  $\Omega$ . Note that these functions are smooth in the interior of  $\Omega$  and vanish at the boundary, so we can apply the divergence theorem and find

$$\int \nabla \eta_i \cdot \nabla \eta_j = - \int \eta_i \Delta \eta_j = \lambda_j \int \eta_i \eta_j = \lambda_j \delta_{i=j}.$$

Consider a function  $f = \sum f_k \eta_k$  which is an element of  $\mathcal{H}^1$ , by which we mean  $\sum \lambda_k f_k^2 < \infty$ . Since  $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$ , the following sums all converge in  $L^2(\Omega)$  and hence the calculation is justified:

$$\begin{aligned} \int |\nabla f|^2 &= \int \left( \sum_i f_i \nabla \eta_i \right) \left( \sum_j f_j \nabla \eta_j \right) \\ &= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_j \lambda_j f_j^2. \end{aligned}$$

From this the result follows. □

We come now to the proof of Proposition 2.3. The proof is by complex interpolations using the Hadamard three-lines theorem.

*Proof.* Let  $g$  be any Schwartz function in  $L^2(\mathbb{R}^2)$ , and let  $f$  be a function in  $\mathcal{H}^s$ . Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f, \quad z \in \mathbb{C}, \operatorname{Re}(z) \in [0, 1].$$

Recall (see e.g. [JK95]) that when  $t \in \mathbb{R}$ ,  $(-\Delta)^{it}$  is a unitary transformation on  $L^2(\mathbb{R}^2)$ , and  $\Lambda^{it}$  is a unitary transformation on  $L^2(\Omega)$ .

When  $\operatorname{Re}(z) = 0$ , then  $\|(-\Delta)^{z/2} g\|_2 = \|g\|_2$  and  $\|\Lambda^{s-z} f\|_2 = \|f\|_{\mathcal{H}^s}$ . Hence

$$\Phi(z) \leq \|g\|_2 \|f\|_{\mathcal{H}^s}, \quad \operatorname{Re}(z) = 0.$$

When  $\operatorname{Re}(z) = 1$ , integrate by parts to obtain

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{(z-1)/2} g (-\Delta)^{1/2} E \Lambda^{s-z} f.$$

Then  $\|(-\Delta)^{(z-1)/2} g\|_2 = \|g\|_2$ , while  $\|\Lambda^{s-z} f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^s}$ . As an  $\mathcal{H}^1$  function,  $\Lambda^{s-z} f$  has trace zero so

$$\|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} = \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|f\|_{\mathcal{H}^s}.$$

Of course  $\|(-\Delta)^{1/2} \cdot\|_{L^2(\mathbb{R}^2)} = \|\nabla \cdot\|_{L^2(\mathbb{R}^2)}$  in general so

$$\Phi(z) \leq \|g\|_2 \|f\|_{\mathcal{H}^s}, \quad \operatorname{Re}(z) = 1.$$

In order to apply the Hadamard three-lines theorem, we must show that  $\Phi$  is differentiable in the interior of its domain.

Rewrite the integrand of  $\Phi$  as

$$\mathcal{F}^{-1}(|\xi|^z \hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative  $\frac{d}{dz}$  commutes with linear operators like  $\mathcal{F}^{-1}$  and  $E$ , so the derivative is

$$(8) \quad \mathcal{F}^{-1}(\ln(|\xi|)|\xi|^z \hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}(|\xi|^z \hat{g}) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k.$$

Fix some  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) \in (0, 1)$ . Since  $g$  is a Schwartz function,  $\ln(|\xi|)|\xi|^z \hat{g}$  is in  $L^2$ . Moreover, for any  $\varepsilon > 0$  we have  $\ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} \leq C \lambda_k^{\frac{s-z+\varepsilon}{2}}$  for some  $C$  independent of  $k$  but dependent on  $z, \varepsilon$ . Take  $\varepsilon < \operatorname{Re}(z)$  and, since  $f \in \mathcal{H}^s$ , this sum will converge in  $L^2$ .

The differentiated integrand (8) is therefore a sum of two products of  $L^2$  functions. In particular it is integrable, which means we can interchange the integral sign and the derivative  $\frac{d}{dz}$  and prove that  $\Phi'(z)$  is finite for all  $0 < \operatorname{Re}(z) < 1$ .

By the Hadamard three-lines theorem, for any  $z \in (0, 1)$  we have  $\Phi(z) \leq \|g\|_2 \|f\|_{\mathcal{H}^s}$ . Evaluating  $\Phi(s)$ , we see

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} g E f \leq \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

This inequality holds for any Schwartz function  $g \in L^2(\mathbb{R}^n)$  and any  $f \in \mathcal{H}^s$ .

Since Schwartz functions are dense in  $L^2(\mathbb{R}^2)$  and  $(-\Delta)^{s/2}$  is self-adjoint, the proof is complete.  $\square$

## 3. EXISTENCE OF SUITABLE SOLUTIONS

In this section, we define the needed notion of suitable solutions. This involves two families of localized energy inequalities. The first family (11) concerns the time evolution of  $\int_{\Omega}(\theta - \Psi)_+^2$  for generic cutoff functions  $\Psi$ . We need also to control the time derivative  $\partial_t(\theta - \Psi)_+^2$  in the sense of distributions for the second De Giorgi lemma (see Proposition 5.4, step 2). This control comes in the family of inequalities (12).

It is important that the universal constant  $C^*$  appearing in the suitability conditions (11) and (12) is independent of  $\Omega$ . The De Giorgi argument requires that we apply the same bound iteratively as we rescale the solution, so our bounds must be scale independent. For this reason, we will define the constant through Proposition 3.1 before stating the definition of suitable solutions. As with the Navier-Stokes equations, it is not obvious that weak solutions constructed directly from the Galerkin scheme are suitable. Therefore we will construct our weak solutions as vanishing viscosity limits of

$$(9) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = \varepsilon \Delta \theta & (0, \infty) \times \Omega, \\ u = \nabla^\perp \Lambda^{-1} \theta & [0, \infty) \times \Omega, \\ \theta = \theta_0 & \{0\} \times \Omega. \end{cases}$$

The construction of solutions to (9) will follow the Galerkin method (as in [CI17]).

We begin by defining the universal constant  $C^*$  and simultaneously showing that the inequalities (11) and (12) are valid for sufficiently smooth solutions to the linear equation

$$(10) \quad \begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = \varepsilon \Delta \theta, \\ \operatorname{div} u = 0, \end{cases}$$

uniformly with respect to  $\varepsilon \in [0, 1]$ . This smoothness requirement will be shown to be valid when  $\varepsilon > 0$ .

**Proposition 3.1** (Energy Inequalities). *There exists a universal constant  $C^*$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be bounded and open with  $C^{2,\beta}$  boundary,  $\beta \in (0, 1)$ , and let  $0 < T < \infty$  a time, and let  $\varepsilon \in [0, 1]$ . Let  $\theta, u$  be a solution to (10) on  $\Omega \times [0, T]$ , with  $\theta \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$  and  $u \in L^\infty(0, T; L^2(\Omega)) \cap L^4(0, T; L^4(\Omega))$ .*

*Then for any smooth non-negative function  $\Psi \in C^\infty([0, \infty) \times \mathbb{R}^2)$  satisfying  $\|\nabla \Psi\|_{L^\infty([0, \infty) \times \mathbb{R}^2)} \leq k$  and the Hölder seminorm  $\sup_{[0, \infty)} [\Psi(t, \cdot)]_{1/4; \mathbb{R}^2} \leq k$  for some constant  $k$ , any time  $S \in (0, T)$ , and any smooth non-negative  $\varphi \in C_c^\infty(S, T; C^\infty(\Omega))$ , the function  $\theta_+ := (\theta - \Psi)_+$  satisfies*

$$(11) \quad \frac{d}{dt} \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 \leq C^* \left( k^2 \int \chi_{\{\theta \geq \Psi\}} + \left| \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) \right| \right) \quad \forall t \in [0, T]$$

*in the sense of distributions and*

$$(12) \quad \begin{aligned} -\frac{1}{2} \int_S^T \int \theta_+^2 \partial_t \varphi &\leq \frac{1}{2} \int_S^T \int \theta_+^2 u \cdot \nabla \varphi - \int_S^T \int \varphi \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) + C^* \|\varphi\|_{C^0(S, T; C^2)} \left( \left(1 + \frac{1}{S}\right) \int_0^T \int \theta_+^2 \right. \\ &\quad \left. + k^2 \int_0^T \int \chi_{\{\theta \geq \Psi\}} + \int_0^T \left| \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) \right| \right). \end{aligned}$$

*Remark.* Note that since  $C^*$  is universal, Proposition 3.1 does not depend on the values of  $\|\theta\|_{L^\infty(L^2)}$ ,  $\|\theta\|_{L^2(H_0^1)}$ ,  $\|u\|_{L^\infty(L^2)}$ , or  $\|u\|_{L^4(L^4)}$ , but only on the fact that these quantities are finite. Therefore, using the natural scaling of (10), if  $(\theta, u)$  verify the assumptions of Proposition 3.1 on  $[0, T] \times \Omega$ , then so does  $(\lambda \theta(\mu \cdot, \mu \cdot), u(\mu \cdot, \mu \cdot))$  on  $[0, \mu^{-1}T] \times \mu^{-1}\Omega$ , for any  $\lambda \in \mathbb{R}$  and  $\mu > 0$  such that  $\mu^{-1}\varepsilon \in [0, 1]$ .

Therefore the proposition applies also to these scaled functions, with the same universal constant  $C^*$ .

*Proof of Proposition 3.1.* Since  $\theta_+ \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ , we can multiply (10) by  $\theta_+$  and integrate in space to obtain

$$0 = \int \theta_+ [\partial_t + u \cdot \nabla + \Lambda - \varepsilon \Delta] (\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to  $\theta_+$ ,  $\Psi$ , and  $\theta_-$ . We analyze them one at a time.

Firstly,

$$\begin{aligned} \int \theta_+ [\partial_t + u \cdot \nabla + \Lambda - \varepsilon \Delta] \theta_+ &= \left(\frac{1}{2}\right) \frac{d}{dt} \int \theta_+^2 + \left(\frac{1}{2}\right) \int \operatorname{div} u \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 + \varepsilon \int |\nabla \theta_+|^2 \\ &= \left(\frac{1}{2}\right) \frac{d}{dt} \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 + \varepsilon \int |\nabla \theta_+|^2. \end{aligned}$$

The  $\Psi$  term produces important error terms:

$$\begin{aligned} \int \theta_+ [\partial_t + u \cdot \nabla + \Lambda - \varepsilon \Delta] \Psi &= \int \theta_+ \partial_t \Psi + \int \theta_+ u \cdot \nabla \Psi + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi + \varepsilon \nabla \theta_+ \cdot \nabla \Psi \\ &= \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi + \varepsilon \nabla \theta_+ \cdot \nabla \Psi. \end{aligned}$$

Since  $\theta_+$  and  $\theta_-$  have disjoint support, the  $\theta_-$  term is nonnegative by Lemma 2.2 part (a):

$$\int \theta_+ [\partial_t + u \cdot \nabla + \Lambda] \theta_- = \left(\frac{1}{2}\right) \int \theta_+ \partial_t \theta_- + \int \theta_+ u \cdot \nabla \theta_- + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- + \varepsilon \int \nabla \theta_+ \cdot \nabla \theta_- \leq 0.$$

Put together, we arrive at

$$(13) \quad \left(\frac{1}{2}\right) \frac{d}{dt} \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 + \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi \leq - \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) - \varepsilon \left[ \int \nabla \theta_+ \cdot \nabla \Psi + \int |\nabla \theta_+|^2 \right].$$

The  $\varepsilon$  term is bounded, using the fact that  $\nabla \theta_+ = \chi_{\{\theta_+ > 0\}} \nabla \theta_+$  and  $\varepsilon \in [0, 1]$ , by

$$(14) \quad \begin{aligned} -\varepsilon \left[ \int |\nabla \theta_+|^2 + \int \nabla \theta_+ \cdot \nabla \Psi \right] &\leq \frac{-\varepsilon}{2} \int |\nabla \theta_+|^2 + \frac{\varepsilon}{2} \int |\nabla \Psi|^2 \chi_{\{\theta_+ > 0\}} \\ &\leq \frac{k^2}{2} \int \chi_{\{\theta_+ > 0\}}. \end{aligned}$$

At this point we break down the  $\Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi$  term using the formula from Proposition 2.1.

$$\int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi = \iint [\theta_+(x) - \theta_+(y)] [\Psi(x) - \Psi(y)] K(x, y) + \int \theta_+ \Psi B.$$

Since  $B \geq 0$  and  $\Psi$  is non-negative by assumption, the  $B$  term is non-negative and so

$$(15) \quad \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi \geq \iint [\theta_+(x) - \theta_+(y)] [\Psi(x) - \Psi(y)] K(x, y).$$

The remaining integral is symmetric in  $x$  and  $y$ , and the integrand is only nonzero if at least one of  $\theta_+(x)$  and  $\theta_+(y)$  is nonzero. Hence

$$\left| \iint [\theta_+(x) - \theta_+(y)] [\Psi(x) - \Psi(y)] K(x, y) \right| \leq 2 \iint \chi_{\{\theta_+ > 0\}}(x) |\theta_+(x) - \theta_+(y)| \cdot |\Psi(x) - \Psi(y)| K(x, y).$$

Now we can break up this integral using Young's inequality, and since  $\iint [\theta_+(x) - \theta_+(y)]^2 K \leq \|\theta_+\|_{\mathcal{H}^{1/2}}^2$  the inequality (15) becomes

$$(16) \quad \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \Psi + \frac{1}{2} \int |\Lambda^{1/2} \theta_+|^2 \geq -2 \iint \chi_{\{\theta_+ > 0\}}(x) [\Psi(x) - \Psi(y)]^2 K(x, y).$$

It remains to bound the quantity  $[\Psi(x) - \Psi(y)]^2 K(x, y)$ . By Proposition 2.1, there is a universal constant  $C$  such that

$$K(x, y) \leq \frac{C}{|x - y|^3}.$$

The cutoff  $\Psi$  is locally Lipschitz, and Hölder continuous with exponent  $1/4$ , by assumption. Therefore

$$[\Psi(x) - \Psi(y)]^2 K(x, y) \leq Ck^2 |x - y|^{-1} \wedge |x - y|^{-2.5}.$$

Since  $1 < 2 < 2.5$ , this quantity is integrable. Thus

$$\int \chi_{\{\theta_+ > 0\}}(x) \int [\Psi(x) - \Psi(y)]^2 K(x, y) dy dx \leq Ck^2 \int \chi_{\{\theta_+ > 0\}} dx.$$

Combining this with (13), (16), and (14) we obtain (11).

We begin now the proof of (12). Since  $\theta_+ \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ , by interpolation we can further conclude  $\theta_+, u \in L^4(0, T; L^4(\Omega))$ . Therefore we can multiply (10) by  $\varphi\theta_+$  and integrate in space to obtain

$$0 = \int \varphi\theta_+ [\partial_t + u \cdot \nabla + \Lambda] (\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to  $\theta_+$ ,  $\Psi$ , and  $\theta_-$ . After rearranging and integrating by parts, this becomes

$$(17) \quad \frac{1}{2} \int \varphi \partial_t \theta_+^2 = \frac{1}{2} \int \theta_+^2 u \cdot \nabla \varphi - \int \varphi \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) - \int \varphi \theta_+ \Lambda \theta_+ - \int \varphi \theta_+ \Lambda \Psi + \varepsilon \int \varphi \theta_+ \Delta (\theta_+ + \Psi).$$

The  $\varepsilon$  term decomposes as

$$(18) \quad \begin{aligned} \varepsilon \int \varphi \theta_+ \Delta (\theta_+ + \Psi) &= -\varepsilon \int \varphi \nabla \theta_+ \cdot \nabla (\theta_+ + \Psi) - \varepsilon \int \theta_+ \nabla \varphi \cdot \nabla (\theta_+ + \Psi) \\ &= -\varepsilon \int \varphi |\nabla \theta_+|^2 - \varepsilon \int \varphi \nabla \theta_+ \cdot \nabla \Psi + \frac{\varepsilon}{2} \int \theta_+^2 \Delta \varphi - \varepsilon \int \theta_+ \nabla \varphi \cdot \nabla \Psi \\ &\leq \frac{\varepsilon}{2} \int \varphi |\nabla \Psi|^2 \chi_{\{\theta_+ > 0\}} + \frac{\varepsilon}{2} \int \theta_+^2 \Delta \varphi + \frac{\varepsilon}{2} \int \theta_+^2 |\nabla \varphi| + \frac{\varepsilon}{2} \int \chi_{\{\theta_+ > 0\}} |\nabla \varphi| |\nabla \Psi|^2 \\ &\leq k^2 \|\varphi\|_{C^1} \int \chi_{\{\theta_+ > 0\}} + \|\varphi\|_{C^2} \int \theta_+^2. \end{aligned}$$

The  $\int \varphi \theta_+ \Lambda \theta_+$  term is bounded by Lemma 2.2 part (c)

$$(19) \quad - \int \varphi \theta_+ \Lambda \theta_+ \leq C \|\varphi\|_{C^1} \left( \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 \right)$$

and the  $\int \varphi \theta_+ \Lambda \Psi$  term is bounded, just as for the  $\int \theta_+ \Lambda \Psi$  term in the previous family of inequalities but with the addition of Lemma 2.2 part (c),

$$(20) \quad \begin{aligned} - \int \varphi \theta_+ \Lambda \Psi &\leq \iint [\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)] [\Psi(x) - \Psi(y)] K_1 \\ &\leq 2 \iint \chi_{\{\theta_+ > 0\}} |\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)| \cdot |\Psi(x) - \Psi(y)| K_1 \\ &= 2 \iint \left( \|\varphi\|_{C^1}^{-1/2} |\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)| \right) \left( \|\varphi\|_{C^1}^{1/2} \chi_{\{\theta_+ > 0\}} |\Psi(x) - \Psi(y)| \right) K_1 \\ &\leq \|\varphi\|_{C^1}^{-1} \|\varphi\theta_+\|_{\mathcal{H}^{1/2}}^2 + \|\varphi\|_{C^1} \iint \chi_{\{\theta_+ > 0\}} [\Psi(x) - \Psi(y)]^2 K_1 \\ &\leq C \|\varphi\|_{C^1} \left( \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 \right) + Ck^2 \|\varphi\|_{C^1} \left( \int_{\mathbb{R}^2} |y|^{-1} \wedge |y|^{-2.5} dy \right) \int \chi_{\{\theta_+ > 0\}}. \end{aligned}$$

From the inequality (11) already proven, we can obtain by a standard argument

$$(21) \quad \int_S^T \int |\Lambda^{1/2} \theta_+|^2 \leq \frac{1}{S} \int_0^T \int \theta_+^2 + k^2 \int_0^T \int \chi_{\{\theta_+ > 0\}} + \int_0^T \left| \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) \right|.$$

By combining (17) with (18), (19), and (20) we obtain

$$\frac{1}{2} \int \varphi \partial_t \theta_+^2 = \frac{1}{2} \int \theta_+^2 u \cdot \nabla \varphi - \int \varphi \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) + C \|\varphi\|_{C^2} \left( \int \theta_+^2 + k^2 \int \chi_{\{\theta_+ > 0\}} + \int |\Lambda^{1/2} \theta_+|^2 \right).$$

Integrating this inequality from  $S$  to  $T$  and applying (21), we obtain (12).  $\square$

We will construct global-in-time solutions to (2) (equivalently (10) with  $\varepsilon = 0$ ) for any initial value  $\theta_0 \in L^2$  which verify these energy inequalities (11) and (12) with the same universal constant  $C^*$  at all scales, but which may not be in  $L^2(H_0^1)$ .

**Definition 2.** A pair  $\theta, u$  is called a **suitable solution** to (2) on  $[0, \infty) \times \Omega$  if  $\Omega \subseteq \mathbb{R}^2$  open and bounded,  $\theta, u \in L^\infty(\mathbb{R}_+; L^2(\Omega))$ ,  $\theta \in L^2(\mathbb{R}_+; \mathcal{H}^{1/2}(\Omega))$ ,  $u \in L^3(\mathbb{R}_+; L^3(\Omega))$  and  $\theta, u$  is a suitable solution on  $[0, T] \times \Omega$  for all  $0 < T < \infty$ .

A pair  $\theta, u$  is called a **suitable solution** to (2) on a space time domain  $[0, T] \times \Omega$  if  $T < \infty$ ,  $\Omega \subseteq \mathbb{R}^2$  open and bounded,  $\theta, u \in L^\infty(0, T; L^2(\Omega))$ ,  $\theta \in L^2(0, T; \mathcal{H}^{1/2}(\Omega))$ ,  $u \in L^3(0, T; L^3(\Omega))$  and

(1)  $\theta, u$  solve (2) in the sense of distributions on  $[0, T] \times \Omega$ ,

(2)  $\theta, u$  satisfy (11) and (12) at all scales with the same universal constant  $C^*$  defined in Proposition 3.1. More specifically, the following holds:

Let  $\lambda \in \mathbb{R}$  and  $\mu \in (0, 1)$  be given and let  $\Psi \in C^\infty([0, \infty) \times \mathbb{R}^2)$  be any smooth non-negative function satisfying  $\|\nabla \Psi\|_{L^\infty([0, \infty) \times \mathbb{R}^2)} \leq k$  and  $\sup_{[0, \infty)} [\Psi(t, \cdot)]_{1/4; \mathbb{R}^2} \leq k$  for some constant  $k$ .

Define  $\tilde{\Omega} := \{x \in \mathbb{R}^2 : \mu x \in \Omega\}$ ,  $\tilde{T} := \mu^{-1}T$ ,  $\tilde{\theta}_+(t, x) := (\lambda \theta(\mu t, \mu x) - \Psi(t, x))_+$ , and  $\tilde{u}(t, x) := u(\mu t, \mu x)$ . Let  $S \in (0, \tilde{T})$  and let  $\varphi \in C_c^\infty(S, T; C^\infty(\tilde{\Omega}))$  be non-negative.

Then  $\tilde{\theta}_+$  and  $\tilde{u}$  and  $\varphi$  and  $\Psi$  satisfy (11) and (12) on  $\tilde{\Omega}$  with times 0,  $S$  and  $\tilde{T}$ , with the same universal constant  $C^*$ .

The rest of this section is dedicated to the proof of the following proposition:

**Proposition 3.2** (Existence of global suitable solutions). *There exists a universal constant  $C > 0$  such that the following holds:*

*Given an open, bounded domain  $\Omega \subseteq \mathbb{R}^2$  with  $C^{2, \beta}$  boundary,  $\beta \in (0, 1)$ , and initial data  $\theta_0 \in L^2(\Omega)$ , there exists a global-in-time weak solution  $\theta$  to (1) such that, for any  $0 < T < \infty$ ,  $\theta$  and  $u := \nabla^\perp \Lambda^{-1} \theta$  are a suitable solution to (2) on  $[0, T] \times \Omega$ .*

*Moreover,  $\theta \in L^\infty([0, \infty); L^2(\Omega)) \cap L^2([0, \infty); \mathcal{H}^{1/2}(\Omega))$ , and  $\theta(t, \cdot) \rightarrow \theta_0(\cdot)$  weakly in  $L^2(\Omega)$  as  $t \rightarrow 0$ , and for any  $S > 0$*

$$\|\theta\|_{L^\infty([S, \infty) \times \Omega)} \leq \frac{C}{S} \|\theta_0\|_{L^2(\Omega)}.$$

To construct global suitable solutions, we will use the vanishing viscosity method. First, we must prove existence of global weak solutions to (9).

**Lemma 3.3** (Existence for viscous equation). *There exists a universal constant  $C$  such that the following holds:*

*Given an open, bounded domain  $\Omega \subseteq \mathbb{R}^2$ , initial data  $\theta_0 \in L^2(\Omega)$  and a constant  $\varepsilon > 0$ , there exists a global-in-time weak solution  $\theta$  to (9).*

*In particular,  $\theta \in C^0([0, \infty); L^2(\Omega)) \cap L^2([0, \infty); H_0^1(\Omega))$ ,  $u \in C^0([0, \infty); L^2(\Omega)) \cap L^4([0, \infty) \times \Omega)$ , and  $\partial_t \in L^2([0, \infty); H^{-1}(\Omega))$ , and  $\theta(t, \cdot) \rightarrow \theta_0(\cdot)$  weakly in  $L^2(\Omega)$  as  $t \rightarrow 0$ , and for any  $S > 0$*

$$\|\theta\|_{L^\infty([S, \infty) \times \Omega)} \leq \frac{C}{S} \|\theta_0\|_{L^2(\Omega)}.$$

The proof of existence is by Galerkin's method, while the  $L^\infty$  bound uses a De Giorgi argument.

*Proof.* Recall that  $\eta_j$  are the eigenfunctions of  $-\Delta_D$ . Let  $N$  be an integer parameter, and  $W_N := \text{span}(\eta_0, \dots, \eta_N)$ , which consists only of smooth functions which vanish on  $\partial\Omega$ . We seek first a solution  $\theta_N \in W_N$  to the weak equation

$$(22) \quad \int \varphi \partial_t \theta_N + \int \varphi \nabla^\perp \Lambda^{-1} \theta_N \cdot \nabla \theta_N + \int \varphi \Lambda \theta_N + \varepsilon \int \nabla \theta_N \nabla \varphi = 0, \quad \forall t \in \mathbb{R}_{\geq 0}, \varphi \in W_N.$$

If we write

$$\theta_N(t, x) := \sum_{i=0}^N \alpha_{i,N}(t) \eta_i(x)$$

and choose  $\varphi = \eta_i$  as a test function, then  $\theta_N$  solves (22) if and only if, for all  $i \leq N$ ,

$$\alpha'_{i,N}(t) + \sum_{j=0}^N \sum_{k=0}^N \alpha_{j,N}(t) \alpha_{k,N}(t) B_{ijk} + \lambda_i^{1/2} \alpha_{i,N}(t) + \varepsilon \lambda_i \alpha_{i,N}(t) = 0$$

with

$$B_{ijk} = \lambda_j^{-1/2} \int \eta_i \nabla^\perp \eta_j \cdot \nabla \eta_k$$

a constant tensor.

By Peano's existence theorem for ODEs, solutions to this system exist on some interval  $[0, T]$  where  $T$  depends on  $\Omega$  and  $N$  and (since  $W_N$  is finite dimensional and all norms are equivalent) the  $L^2$  norm of the initial data.

Since  $\theta_N \in W_N$  we can take  $\theta_N$  as a test function and obtain, for any solution  $\theta_N$  to (22),

$$\frac{d}{dt} \int \theta_N^2 + \int \left| \Lambda^{1/2} \theta_N \right|^2 + \varepsilon \int |\nabla \theta_N|^2 = 0.$$

Therefore in particular  $\|\theta_N\|_{L^2(\Omega)}$  is non-increasing in time and we conclude that  $\theta_N$  exists for all time. Moreover,  $\theta_N$  is uniformly bounded in  $L^\infty(L^2(\Omega))$  and  $L^2(H_0^1(\Omega))$ .

To take a limit in  $N$ , we need uniform regularity in time. From (22) we can bound

$$\int_0^\infty \int \partial_t \theta_N \varphi \leq \|\theta_N\|_{L^4(L^4)}^2 \|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(L^2)} \|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(H_0^1)} \|\varphi\|_{L^2(H_0^1)}.$$

Note that  $\theta_N$  is uniformly bounded in  $L^4(L^4)$  by interpolation and  $L^2(L^2)$  by Poincaré's inequality. Therefore  $\iint \varphi \partial_t \theta_N \leq C \|\varphi\|_{L^2(H_0^1)}$  for all  $\varphi \in L^2(W_N)$  for a constant  $C$  independent of  $N$ . Since  $\partial_t \theta_N \in W_N$ , this is sufficient to show that  $\|\partial_t \theta_N\|_{L^2(H^{-1})}$  is uniformly bounded.

By Aubin-Lions, we conclude that  $\theta_N$  is a compact sequence in  $L^2([0, \infty) \times \Omega)$  and so it has an  $L^2$  limit  $\theta$ . This limit  $\theta$  is in  $L^\infty(L^2(\Omega))$  and  $L^2(H_0^1(\Omega))$  and  $\partial_t \theta \in L^2(H^{-1}(\Omega))$ .

We must prove that  $\theta$  is a weak solution to (9). Let  $\varphi \in C_c^\infty(W_M)$  for some  $M$ . For  $N \geq M$ ,

$$-\iint \theta_N \partial_t \varphi - \iint \theta_N \nabla^\perp \Lambda^{-1} \theta_N \cdot \nabla \varphi + \iint \theta_N \Lambda \varphi - \varepsilon \iint \theta_N \Delta \varphi = 0.$$

This expression is continuous for  $\theta_N \in L^2(L^2)$ , so by taking  $N \rightarrow \infty$  we obtain

$$-\int \theta \partial_t \varphi - \int \theta \nabla^\perp \Lambda^{-1} \theta \cdot \nabla \varphi + \int \theta \Lambda \varphi + \varepsilon \int \nabla \theta \cdot \nabla \varphi = 0$$

for any  $\varphi \in C_c^\infty(W_M)$  for any  $M \in \mathbb{N}$ . By density,  $\theta$  solves (9) in the sense of distributions.

Since  $\partial_t \theta_N$  is uniformly bounded in  $L^2(H^{-1})$ , we know  $\theta_N(t, \cdot) \rightarrow \theta_0$  weakly in  $L^2$  uniformly in  $N$  and so the same holds for  $\theta$ .

Lastly, for any constant  $a \geq 0$ , the function  $(\theta - a)_+$  satisfies

$$\begin{aligned} \frac{d}{dt} \int \frac{1}{2} (\theta - a)_+^2 + \int |\Lambda^{1/2} (\theta - a)_+|^2 &= \frac{-1}{2} \int u \cdot \nabla (\theta - a)_+^2 - \int a \Lambda (\theta - a)_+ - \varepsilon \int |\nabla (\theta - a)_+|^2 \\ &= - \int a (\theta - a)_+ B_1 - \varepsilon \int |\nabla (\theta - a)_+|^2 \\ &\leq 0. \end{aligned}$$

This inequality is scaling-invariant, so the same holds for  $\lambda \theta(\mu t, \mu x)$  for any  $\lambda, \mu > 0$ .

By the standard De Giorgi argument (see Lemma A.1 in the Appendix for details), there exists a universal constant  $\delta$  such that  $\int_0^2 \int (\lambda \theta(\mu t, \mu x)_+^2 dx dt \leq \delta$  implies  $\theta \leq \lambda^{-1}$  on  $[\mu 1, \mu 2]$ . In fact, by comparison with a constant super-solution,  $\theta \leq \lambda^{-1}$  on  $[\mu, \infty)$ . Taking  $\lambda = \sqrt{\frac{\delta}{2\mu^{-2} \|\theta_0\|_{L^2(\Omega)}^2}}$ , we find  $\theta(t, \cdot) \leq Ct^{-1} \|\theta_0\|_{L^2(\Omega)}$  for a universal constant  $C$ . Applying the same argument to  $-\theta$  gives the  $L^\infty$  bound.  $\square$

Now that we have global existence of solutions to (9) for  $\varepsilon > 0$ , we can prove Proposition 3.2 by taking a limit as  $\varepsilon \rightarrow 0$ .

*Proof of Proposition 3.2.* For any parameter  $\varepsilon > 0$ , define  $\theta_\varepsilon \in L^2(H_0^1)$  the weak solution to (9) constructed in Lemma 3.3. The  $\theta_\varepsilon$  are uniformly bounded in  $L^\infty(L^2)$  and  $L^2(\mathcal{H}^{1/2})$  by the standard energy argument, so by interpolation they are also uniformly bounded in  $L^4(L^{8/3})$ . Recall that  $\theta_\varepsilon$  are uniformly bounded in  $L^\infty(L^\infty)$  after any positive time.

For any smooth  $\varphi$ , we have

$$\int_0^\infty \int \partial_t \theta_\varepsilon \varphi \leq \|\theta_\varepsilon\|_{L^4(L^{8/3})}^2 \|\varphi\|_{L^2(W^{1,4})} + \|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})} \|\varphi\|_{L^2(\mathcal{H}^{1/2})} + \varepsilon \|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})} \|\varphi\|_{L^2(\mathcal{H}^{3/2})}.$$

Therefore  $\partial_t \theta_\varepsilon$  is uniformly bounded in  $L^2(\mathcal{H}^{-3/2})$ . By Aubin-Lions, the sequence  $\theta_\varepsilon$ , up to a subsequence, has a strong limit in  $L^2(L^2)$ . Call this limit  $\theta$ .

Since  $\partial_t \theta_\varepsilon$  is uniformly bounded and  $\theta(t, \cdot) \rightarrow \theta_0$  weakly in  $L^2$ , the same holds for  $\theta$ .

Define  $u_\varepsilon := \nabla^\perp \Lambda^{-1} \theta_\varepsilon$ , and by continuity of the Riesz transform we have  $u_\varepsilon \rightarrow u$  in  $L^2([0, \infty) \times \Omega)$  where  $u := \nabla^\perp \Lambda^{-1} \theta$ .

It remains only to prove that  $\theta$  and  $u$  are a suitable pair. Recall that  $\theta_\varepsilon$  and  $u_\varepsilon$  satisfy (11) and (12) by Proposition 3.1, so we need only show that these inequalities hold also in the limit. The details of this calculation are given below.

Let  $0 < T < \infty$  be a constant, and let  $\lambda, \mu, \Psi, S$  and  $\varphi$  be as in the definition of suitable solutions. Define

$$\begin{aligned} \tilde{\theta}_\varepsilon(t, x) &:= \lambda \theta_\varepsilon(\mu t, \mu x), & \tilde{\theta}(t, x) &= \lambda \theta(\mu t, \mu x), \\ \tilde{u}_\varepsilon(t, x) &:= u_\varepsilon(\mu t, \mu x), & \tilde{u}(t, x) &:= u(\mu t, \mu x), \\ \tilde{\theta}_{\varepsilon,+}(t, x) &:= (\tilde{\theta}_\varepsilon(t, x) - \Psi(t, x))_+, & \tilde{\theta}_+(t, x) &:= (\tilde{\theta}(t, x) - \Psi(t, x))_+, \end{aligned}$$

and let  $\tilde{\Omega} := \{x \in \mathbb{R}^2 : \mu x \in \Omega\}$ ,  $\tilde{T} := \mu^{-1} T$ ,  $\tilde{\varepsilon} := \mu^{-1} \varepsilon$ .

Note that  $\tilde{\theta}_\varepsilon$  and  $\tilde{u}_\varepsilon$  are weak solutions to (10) with viscosity  $\tilde{\varepsilon}$ . Therefore, if  $\varepsilon \leq \mu$  then  $\tilde{\theta}_{\varepsilon,+}$  and  $\tilde{u}_\varepsilon$  satisfy (11). The terms  $\int \tilde{\theta}_{\varepsilon,+} (\partial_t \Psi + \tilde{u}_\varepsilon \cdot \nabla \Psi)$  and  $\int \chi_{\{\tilde{\theta}_{\varepsilon,+} \geq 0\}}$  are continuous under  $L^2$  limits, and the quantities  $\frac{d}{dt} \int \tilde{\theta}_{\varepsilon,+}^2$  and  $\int |\Lambda^{1/2} \tilde{\theta}_{\varepsilon,+}|^2$  are lower-semicontinuous under  $L^2$  limits, so we conclude that  $\tilde{\theta}_+$  and  $\tilde{u}$  satisfy (11).

Similarly,  $\tilde{\theta}_{\varepsilon,+}$  and  $\tilde{u}_\varepsilon$  satisfy (12) if  $\varepsilon \leq \mu$ . On  $[S, \tilde{T}]$  we have a uniform  $L^\infty(L^\infty)$  bound for  $\tilde{\theta}_\varepsilon$ . Therefore  $\tilde{\theta}_{\varepsilon,+}$  converges in  $L^3(L^3)$ , and so  $\int_S^{\tilde{T}} \int \tilde{\theta}_{\varepsilon,+}^2 \tilde{u}_\varepsilon \cdot \nabla \varphi$  is conserved in the limit  $\varepsilon \rightarrow 0$ . The remaining terms in (12) are  $L^2(L^2)$  continuous, so  $\tilde{\theta}_+$  and  $\tilde{u}$  satisfy (12).  $\square$



## 4. LITTLEWOOD-PALEY THEORY

In this section we will prove that, because  $\theta$  is uniformly bounded in  $L^\infty$ , the velocity  $u = \nabla^\perp \Lambda^{-1} \theta$  is calibrated (see Definition 1). The proof will utilize a Littlewood-Paley theory adapted to a bounded set  $\Omega$ .

Because the Littlewood-Paley theory depends in an essential way on the domain  $\Omega$ , any results proven in this way will also be domain-dependent. Therefore, in the proof of Hölder continuity in Section 7, we will apply the following Proposition only to the unscaled function  $\theta$  on the unscaled domain  $\Omega$ . As we zoom in, the velocity will remain calibrated, so there will be no further need for this result.

**Proposition 4.1.** *Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded set with  $C^{2,\beta}$  boundary for some  $\beta \in (0,1)$ . Let  $\theta \in L^\infty(\Omega)$ . Then there exists an integer  $j_0 = j_0(\Omega)$  and a sequence of divergence-free functions  $(u_j)_{j \geq j_0}$  calibrated for some constant  $\kappa = \kappa(\Omega, \|\theta\|_\infty)$  with center 0 (see Definition 1) such that*

$$\nabla^\perp \Lambda^{-1} \theta = \sum_{j \geq j_0} u_j$$

with the infinite sum converging in the sense of  $L^2$ .

Before we can prove this, we define the Littlewood-Paley projections and prove some of their properties:

Let  $\phi$  be a Schwartz function on  $\mathbb{R}$  which is suited to Littlewood-Paley decomposition. Specifically,  $\phi$  is non-negative, supported on  $[1/2, 2]$ , and has the property that

$$\sum_{j \in \mathbb{Z}} \phi(2^j \xi) = 1 \quad \forall \xi \neq 0.$$

For any  $f = \sum f_k \eta_k$  in  $L^2(\Omega)$ , we define the Littlewood-Paley projections

$$P_j f := \sum_k \phi(2^j \lambda_k^{1/2}) f_k \eta_k.$$

Note that  $P_j$  depends strongly on the domain  $\Omega$ .

Recall that  $-\Delta_D$  has some smallest eigenvalue  $\lambda_0$  (depending on  $\Omega$ ) so if we define  $j_0 = \log_2(\lambda_0) - 1$  then  $P_j = 0$  for all  $j < j_0$ .

The Bernstein Inequalities adapted for a bounded domain are proved in [IMT17]. We restate their result here:

**Lemma 4.2** (Bernstein Inequalities). *Let  $1 \leq p \leq \infty$  and  $\Omega \subset \mathbb{R}^2$  a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0,1)$ , and let  $(P_j)_{j \in \mathbb{Z}}$  be the Littlewood-Paley decomposition defined above.*

*There exists a constant  $C$  depending on  $p$  and  $\Omega$  such that the following hold for any  $f \in L^p(\Omega)$ : For any  $\alpha \in \mathbb{R}$  and  $j \in \mathbb{Z}$ ,*

$$\|\Lambda^\alpha P_j f\|_{L^p(\Omega)} \leq C 2^{\alpha j} \|f\|_{L^p(\Omega)}.$$

*For any  $\alpha \in \mathbb{R}$  and  $j \geq j_0$*

$$\|\nabla \Lambda^\alpha P_j f\|_{L^p(\Omega)} \leq C 2^{(1+\alpha)j} \|f\|_{L^p(\Omega)}.$$

*Proof.* The first claim is Lemma 3.5 in [IMT17]. It is also an immediate corollary of [IMT18] Theorem 1.1.

The second claim is similar to Lemma 3.6 in [IMT17]. A hypothesis of Lemma 3.6 is that

$$\|\nabla e^{-t\Delta_D}\|_{L^\infty \rightarrow L^\infty} \leq \frac{C}{\sqrt{t}} \quad 0 < t \leq 1$$

(a property of  $\Omega$ ). The result of Lemma 3.6 only covers the case  $j > 0$ .

In [FMP04] it is proved that if  $\Omega$  is  $C^{2,\beta}$  then

$$\|\nabla e^{-t\Delta_D}\|_{L^\infty \rightarrow L^\infty} \leq \frac{C}{\sqrt{t}} \quad 0 < t \leq T$$

which, by taking some  $T$  depending on  $j_0$ , is enough to prove the desired result for  $j \geq j_0$  by a trivial modification of the proof in [IMT17].  $\square$

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. In the case of  $\mathbb{R}^2$ , the Littlewood-Paley projections commute with the gradient so  $P_i \nabla P_j = 0$  unless  $|i - j| \leq 1$ . On a bounded domain, this is not the case; the gradient does not maintain localization in frequency-space. However, the following lemma formalizes the observation that  $P_i \nabla P_j \approx 0$  when  $i \ll j$ .

**Lemma 4.3.** *Let  $1 \leq p \leq \infty$ . There exists a constant  $C$  depending on  $p$  and  $\Omega$  such that for any function  $f \in L^p(\Omega)$ ,*

$$\|P_i \nabla P_j f\|_p \leq C \min(2^j, 2^i) \|f\|_p.$$

*Proof.* Let  $q$  be the Hölder conjugate of  $p$  and  $g$  be an  $L^q$  function. Then since  $P_i$  is self-adjoint

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \leq C 2^j \|g\|_q \|f\|_p$$

by Lemma 4.2.

Further integrating by parts,

$$\int g P_i \nabla P_j f = - \int (\nabla P_i g) P_j f \leq C 2^i \|g\|_q \|f\|_p.$$

This also follows from Lemma 4.2.

The result follows.  $\square$

We are now ready to prove Proposition 4.1.

*Proof of 4.1.* For each integer  $j \geq j_0$ , we define  $u_j$  to be the  $\frac{\pi}{2}$ -rotation of the Riesz transform of the  $j^{\text{th}}$  Littlewood-Paley projection of  $\theta$ :

$$u_j := \nabla^\perp \Lambda^{-1} P_j \theta.$$

Qualitatively, we know that  $\theta \in L^2$  and hence  $u_j \in L^2$ . In fact,  $u = \sum u_j$  in the  $L^2$  sense.

We must bound  $u_j$ ,  $\Lambda^{-1/4} u_j$ , and  $\nabla u_j$  all in  $L^\infty(\Omega)$ .

By straightforward application of Lemma 4.2,

$$(23) \quad \|u_j\|_\infty \leq C \|\theta\|_\infty.$$

Since  $u_j \in L^2$ , we know that

$$\Lambda^{-1/4} u_j = \sum_{i \in \mathbb{Z}} P_i \Lambda^{-1/4} u_j.$$

Define  $\bar{P}_k := P_{k-1} + P_k + P_{k+1}$ . Then  $\bar{P}_k P_k = P_k$ , and since the projections  $P_k$  are spectral operators, they commute with  $\Lambda^s$  and each other. We therefore rewrite

$$\left( P_i \Lambda^{-1/4} u_j \right)^\perp = \left( \Lambda^{-1/4} \bar{P}_i \right) (P_i \nabla P_j) (\Lambda^{-1} \bar{P}_j) \theta.$$

On the right hand side we have three bounded linear operators applied sequentially to  $\theta \in L^\infty$ . The first operator has norm  $C 2^{-j} (2^1 + 2^0 + 2^{-1})$  by Lemma 4.2. The second operator has norm  $C \min(2^j, 2^i)$  by Lemma 4.3. The third operator has norm  $C 2^{-i/4} (2^{1/4} + 2^0 + 2^{-1/4})$  by Lemma 4.2. Therefore

$$\left\| P_i \Lambda^{-1/4} u_j \right\|_\infty \leq C 2^{-i/4} \min(2^j, 2^i) 2^{-j} \|\theta\|_\infty.$$

Summing these bounds on the projections of  $\Lambda^{-1/4}u_j$ , and noting that

$$\sum_{i \in \mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i \leq j} 2^{i3/4} + \sum_{i > j} 2^{-i/4} \leq C 2^{-j/4},$$

we obtain

$$(24) \quad \left\| \Lambda^{-1/4} u_j \right\|_{\infty} \leq C 2^{-j/4} \|\theta\|_{\infty}.$$

Lastly, we must show that  $\nabla u_j$  is in  $L^{\infty}$ . Equivalently, we will show that  $\Lambda^{-1}P_j\theta$  is  $C^{1,1}$ . The method of proof is Schauder theory.

For convenience, define

$$F := \Lambda^{-1}P_j\theta.$$

Notice that  $F$  is a linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Therefore

$$-\Delta F = \Lambda^2 F = \Lambda P_j \theta.$$

We apply the standard Schauder estimate from Gilbarg and Trudinger [GT01] Theorem 6.6 to bound some  $C^{2,\alpha}$  semi-norm of  $F$  by the  $L^{\infty}$  norm of  $F$  and the  $C^{\alpha}$  norm of its Laplacian. By assumption there exists  $\beta \in (0, 1)$  such that  $\Omega$  is  $C^{2,\beta}$ , and for this  $\beta$  we have by the Schauder estimate

$$(25) \quad [D^2 F]_{\beta} \leq C \left\| \Lambda^{-1} P_j \theta \right\|_{\infty} + C \left\| \Lambda P_j \theta \right\|_{\infty} + C [\Lambda P_j \theta]_{\beta}.$$

By Lemma 4.2,

$$\begin{aligned} \left\| \Lambda^{-1} P_j \theta \right\|_{\infty} &\leq C 2^{-j} \|\theta\|_{\infty}, \\ \left\| \Lambda P_j \theta \right\|_{\infty} &\leq C 2^j \|\theta\|_{\infty}, \\ \left\| \nabla \Lambda P_j \theta \right\|_{\infty} &\leq C 2^{2j} \|\theta\|_{\infty}. \end{aligned}$$

By Lemma A.2 (see Appendix A) we can interpolate these last two bounds to obtain

$$[\Lambda P_j \theta]_{\beta} \leq C 2^{j(1+\beta)} \|\theta\|_{\infty}.$$

Plugging these estimates into (25) yields

$$[D^2 F]_{\beta} \leq C \left( 2^{-j} + 2^j + 2^{j(1+\beta)} \right) \|\theta\|_{\infty}.$$

Recall that without loss of generality we can assume  $j \geq j_0$ . Therefore up to a constant depending on  $j_0$ , the term  $2^{j(1+\beta)}$  bounds  $2^j$  and  $2^{-j}$  so we can write

$$[D^2 F]_{\beta} \leq C 2^{j(1+\beta)} \|\theta\|_{\infty}.$$

Using this estimate and the fact that  $\|\nabla F\|_{\infty} = \|\nabla \Lambda^{-1} P_j \theta\|_{\infty} \leq C \|\theta\|_{\infty}$  (see (23)), we can interpolate to obtain an  $L^{\infty}$  bound on  $D^2 F$ . Lemma A.3 states that since  $F \in C^{2,\beta}$  and  $\Omega$  is sufficiently regular, there exist a constant  $\ell = \ell(\Omega)$  such that for any  $\delta \in [0, \ell]$  we have

$$\begin{aligned} \|D^2 F\|_{\infty} &\leq C \left( \delta^{-1} \|\nabla F\|_{\infty} + \delta^{\beta} [D^2 F]_{\beta} \right) \\ &\leq C \left( \delta^{-1} + \delta^{\beta} 2^{j(1+\beta)} \right) \|\theta\|_{\infty}. \end{aligned}$$

Set  $\delta = 2^{-j} (2^{j_0} \ell) \leq \ell$ . Then

$$\|D^2 F\|_{\infty} \leq C \left( 2^j + 2^{-j\beta} 2^{j(1+\beta)} \right) \|\theta\|_{\infty} = C(\Omega) 2^j \|\theta\|_{\infty}.$$

Since  $D^2 F = \nabla u_j$ , this estimate together with (23) and (24) complete the proof.  $\square$

## 5. DE GIORGI ESTIMATES

Our goal in this section is to prove De Giorgi's first and second lemmas for suitable solutions to (2) with  $u$  uniformly calibrated. The De Giorgi lemmas will eventually be applied iteratively to various rescalings of the solution  $\theta$ , so the following results must be independent of the size of the domain  $\Omega$ . Any properties we do assume for the domain, such as the regularity of the boundary, must be scaling invariant.

Rather than working directly with the calibrated sequence, we will decompose  $u$  into just two terms, a low-pass term and a high-pass term. The construction is described in the following lemma. Note that we make no assumption on the center of calibration, which means this result is independent of scale.

**Lemma 5.1.** *Let*

$$u = \sum_{j_0}^{\infty} u_j$$

*with the sum converging in the  $L^2$  sense. Assume that  $(u_j)_{j \in \mathbb{Z}}$  is a calibrated sequence with constant  $\kappa$  and some center, and that  $\operatorname{div}(u_j) = 0$  for all  $j$ .*

*Then*

$$u = u_\ell + u_h$$

*with*

$$\begin{aligned} \|\nabla u_\ell\|_{L^\infty([-T,0] \times \Omega)} &\leq 2\kappa, \\ \|\Lambda^{-1/4} u_h\|_{L^\infty([-T,0] \times \Omega)} &\leq 6\kappa. \end{aligned}$$

*and  $\operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0$ .*

We call  $u_\ell$  the low-pass term, and  $u_h$  the high-pass term.

*Proof.* Let  $N$  be the center to which  $(u_j)_{j \in \mathbb{Z}}$  is calibrated.

We define

$$u_h = \sum_{j=N+1}^{\infty} u_j$$

and bound

$$\|\Lambda^{-1/4} u_h\|_{\infty} \leq \sum_{j>N} \|\Lambda^{-1/4} u_j\|_{\infty} \leq \kappa \frac{2^{-1/4}}{1 - 2^{-1/4}}.$$

We define

$$u_\ell = \sum_{j=j_0}^N u_j$$

and bound

$$\|\nabla u_\ell\|_{\infty} \leq \sum_{j \leq N} \|\nabla u_j\|_{\infty} \leq \kappa \frac{1}{1 - 2^{-1}}.$$

□

In order to prove the De Giorgi lemmas, we must derive an energy inequality for the function  $(\theta - \Psi)_+$  where  $\Psi(t, x)$  grows sublinearly in  $|x|$ . Considering the suitability condition (11), we see that control can only be gained if the quantity  $\partial_t \Psi + u \cdot \nabla \Psi$  is bounded. This requires a barrier function which is moving in space along a Lagrangian path  $\Gamma_\ell$  of  $u_\ell$ .

To that end, we shall consider, for any domain  $\Omega$  and time  $T$ , functions  $\theta : [-T, 0] \times \Omega \rightarrow \mathbb{R}$ ,  $L^2$  functions  $u_\ell$  and  $u_h : [-T, 0] \times \Omega \rightarrow \mathbb{R}^2$ , and a Lipschitz path  $\Gamma_\ell : [-T, 0] \rightarrow \Omega$  which satisfy

$$(26) \quad \begin{cases} \theta, (u_\ell + u_h) \text{ suitable solution to (2)} & \text{on } [-T, 0] \times \Omega, \\ \operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0 & \text{on } [-T, 0] \times \Omega, \\ \dot{\Gamma}_\ell(t) = u_\ell(t, \Gamma_\ell(t)) & \text{on } [-T, 0]. \end{cases}$$

Because  $\Gamma_\ell$  depends on  $u_\ell$  which depends on  $N$ , the path  $\Gamma_\ell$  will change significantly between scales. In particular, though  $\Gamma_\ell \in \operatorname{Lip}([-T, 0]; \mathbb{R}^2)$ , we cannot assume any uniform bound on its Lipschitz constant. We can bound, however, the difference between  $\Gamma_\ell$  at consecutive scales. Therefore we must consider in the following lemmas an arbitrary Lipschitz path  $\Gamma$ , which was produced at a previous scale, and denote  $\gamma := \Gamma_\ell - \Gamma$  which will be uniformly bounded.

Now we prove an energy inequality for solutions to (26). Though this lemma is independent of the size of the domain, it depends on the geometry of the domain in a way encoded by the constant  $C_{dmn}$ . We will later show that this constraint on  $\Omega$  is scaling invariant.

**Lemma 5.2** (Energy inequality). *Let  $\kappa$ ,  $C_{dmn}$ ,  $C_{pth}$ ,  $T$ , and  $R$  be positive constants, and let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function such that  $\|\nabla \psi\|_\infty$ ,  $\|D^2 \psi\|_\infty$ , and  $\sup_t [\psi(t, \cdot)]_{1/4}$  are all finite. Then there exists a constant  $C > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $\Gamma : [-T, 0] \rightarrow \mathbb{R}^2$  be Lipschitz. Assume that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn} |x - y|^{3/4} K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Let  $\theta$ ,  $u_\ell$ ,  $u_h$ ,  $\Gamma_\ell$  solve (26) on  $[-T, 0] \times \Omega$  with  $\theta$  and  $u_\ell + u_h$  a suitable pair, and satisfy  $\|\Lambda^{-1/4} u_h\|_{L^\infty([-T, 0] \times \Omega)} \leq 6\kappa$ , and  $\|\nabla u_\ell\|_{L^\infty([-T, 0] \times \Omega)} \leq 2\kappa$ . Denote  $\gamma := \Gamma_\ell - \Gamma$  and assume  $\|\dot{\gamma}\|_{L^\infty([-T, 0])} \leq C_{pth}$  and  $\gamma(0) = 0$ .*

*Consider the functions*

$$\theta_+ := (\theta - \psi(\cdot - \Gamma))_+, \quad \theta_- := (\psi(\cdot - \Gamma) - \theta)_+.$$

*If  $\theta_+$  is supported on  $x \in \Omega \cap B_R(\Gamma(t))$  then  $\theta_+$  and  $\theta_-$  satisfy the inequality*

$$\frac{d}{dt} \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 - \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- \leq C \left( \int \chi_{\{\theta \geq \psi\}} + \int \theta_+ + \int \theta_+^2 \right).$$

*Proof.* Define

$$\Psi(t, x) := \psi(x - \Gamma(t))$$

so that

$$\partial_t \Psi + (u_\ell + u_h) \cdot \nabla \Psi = (u_\ell - \dot{\Gamma} + u_h) \cdot \nabla \psi(x - \Gamma(t)).$$

Applying (11) we arrive at

$$(27) \quad \frac{d}{dt} \int \theta_+^2 + \int |\Lambda^{1/2} \theta_+|^2 \leq C \left( \int \chi_{\{\theta \geq \psi\}} + \left| \int \theta_+ (u_\ell - \dot{\Gamma}(t) + u_h) \cdot \nabla \psi(x - \Gamma(t)) \right| \right).$$

Consider first the high-pass term  $\int \theta_+ u_h \cdot \nabla \psi$ . This term is equal to  $\int \Lambda^{1/4} (\theta_+ \nabla \Psi) \Lambda^{-1/4} u_h$ , as can be calculated by first decomposing  $\theta_+ \nabla \Psi$  and  $u_h$  as sums of eigenfunctions. The operations on these infinite sums are justified because  $\theta_+ \nabla \Psi$ ,  $\Lambda^{1/4} (\theta_+ \nabla \Psi)$ ,  $u_h$ , and  $\Lambda^{-1/4} u_h$  are all in  $L^2$ . Therefore we can apply Lemma 2.2 parts (e) and (c) to obtain

$$\int \Lambda^{-1/4} u_h \Lambda^{1/4} (\theta_+ \nabla \psi) \leq C \left\| \Lambda^{-1/4} u_h \right\|_\infty \left( \|\nabla \psi\|_\infty + \|D^2 \psi\|_\infty \right) |\operatorname{supp}(\theta_+)|^{1/2} (\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}).$$

We apply Young's inequality to find that for any constant  $\varepsilon > 0$  there exists  $C = C(\psi, \kappa, C_{dmn}, \varepsilon)$  such that

$$(28) \quad \int u_h \theta_+ \nabla \psi(x - \Gamma(t)) dx \leq C \left( |\text{supp}(\theta_+)| + \int \theta_+^2 \right) + \varepsilon \int |\Lambda^{1/2} \theta_+|^2.$$

Consider now the low-pass term. By (26)

$$(29) \quad u_\ell(t, x) - \dot{\Gamma}(t) = u_\ell(t, x) - u_\ell(t, \Gamma + \gamma) + \dot{\gamma}.$$

Since  $u_\ell$  is has derivative bounded by  $2\kappa$ ,

$$\begin{aligned} |u_\ell(t, x) - u_\ell(t, \Gamma + \gamma)| &\leq |u_\ell(t, x) - u_\ell(t, \Gamma)| + |u_\ell(t, \Gamma) - u_\ell(t, \Gamma + \gamma)| \\ &\leq 2\kappa|x - \Gamma| + 2\kappa|\gamma|. \end{aligned}$$

By assumption  $|\dot{\gamma}| \leq C_{pth}$  and  $\gamma(0) = 0$ , and so for  $t \in [-T, 0]$  we have  $|\gamma(t)| \leq TC_{pth}$ .

Plugging these bounds into (29) we obtain

$$|u_\ell(t, x) - \dot{\Gamma}(t)| \leq 2\kappa|x - \Gamma| + 2\kappa TC_{pth} + C_{pth}.$$

Now we can bound the low pass term

$$\int (u_\ell - \dot{\Gamma}) \theta_+ \nabla \psi(x - \Gamma) \leq (2\kappa T + 1) C_{pth} \|\nabla \psi\|_\infty \int \theta_+ dx + \|\nabla \psi\|_\infty 2\kappa \int |x - \Gamma| \theta_+ dx.$$

By assumption,  $|x - \Gamma| \theta_+ \leq R \theta_+$ , so from this, (28), and (27) the result follows.  $\square$

This energy inequality is sufficient to prove the De Giorgi Lemmas.

The first lemma is a local version of the  $L^2$  to  $L^\infty$  regularization, stating that solutions with small  $L^2$  norm in a region will have small  $L^\infty$  norm in a smaller region.

**Proposition 5.3** (First De Giorgi Lemma). *Let  $\kappa$ ,  $C_{dmn}$ , and  $C_{pth}$ , be positive constants. Then there exists a constant  $\delta_0 > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $\Gamma : [-2, 0] \rightarrow \mathbb{R}^2$  be Lipschitz. Assume that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn}|x - y|^{3/4} K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Let  $\theta$ ,  $u_\ell$ ,  $u_h$ , and  $\Gamma_\ell$  solve (26) on  $[-2, 0] \times \Omega$  with  $\theta$  and  $u_\ell + u_h$  a suitable pair, and satisfy  $\|\Lambda^{-1/4} u_h\|_{L^\infty([-2, 0] \times \Omega)} \leq 6\kappa$ , and  $\|\nabla u_\ell\|_{L^\infty([-2, 0] \times \Omega)} \leq 2\kappa$ . Denote  $\gamma := \Gamma_\ell - \Gamma$  and assume  $\|\dot{\gamma}\|_{L^\infty([-2, 0])} \leq C_{pth}$  and  $\gamma(0) = 0$ .*

*If*

$$\theta(t, x) \leq 2 + \left( |x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+ \quad \forall t \in [-2, 0], x \in \Omega \setminus B_2(\Gamma(t))$$

*and*

$$\int_{-2}^0 \int_{\Omega \cap B_2(\Gamma(t))} (\theta)_+^2 dx dt \leq \delta_0$$

*then*

$$\theta(t, x) \leq 1 \quad \forall t \in [-1, 0], x \in \Omega \cap B_1(\Gamma(t)).$$

*Proof.* Let  $\psi$  be such that  $\psi = 0$  for  $|x| \leq 1$  and  $\psi(x) = 2 + (|x|^{1/4} - 2^{1/4})_+$  for  $|x| > 2$ , and let  $\nabla \psi$  and  $D^2 \psi$  be bounded.

For any constant  $a > 0$ , we can apply Lemma 5.2 to the function

$$\theta_a := (\theta(t, x) - \psi(x - \Gamma(t)) - a)_+$$

and obtain

$$\frac{d}{dt} \int \theta_a^2 + \int |\Lambda^{1/2} \theta_a|^2 \leq C \left( \int \chi_{\{\theta \geq \psi + a\}} + \int \theta_a + \int \theta_a^2 \right).$$

Thus  $\theta - \psi(x - \Gamma)$  satisfies the assumptions of Lemma A.1. There exists a constant, which we call  $\delta_0$ , so that if

$$\int_{-2}^0 \int (\theta(t, x) - \psi(x - \Gamma(t)))_+ dx dt \leq \delta_0$$

then

$$\theta(t, x) \leq 1 + \psi(x - \Gamma(t)) \quad \forall t \in [-1, 0], x \in \Omega.$$

By construction of  $\psi$ , our result follows immediately.  $\square$

Next, we will prove De Giorgi's second lemma, a quantitative analog of the isoperimetric inequality.

**Proposition 5.4** (Second De Giorgi Lemma). *Let  $\kappa$ ,  $C_{dmn}$ ,  $C_{pth}$ , and  $\beta \in (0, 1)$  be positive constants. Then there exists a constant  $\mu > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $\Gamma : [-5, 0] \rightarrow \mathbb{R}^2$  be Lipschitz. Assume that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn}|x - y|^{3/4}K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Let  $\theta$ ,  $u_\ell$ ,  $u_h$ , and  $\Gamma_\ell$  solve (26) on  $[-5, 0] \times \Omega$  with  $\theta$  and  $u_\ell + u_h$  a suitable pair, and satisfy  $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5, 0] \times \Omega)} \leq 6\kappa$ , and  $\|\nabla u_\ell\|_{L^\infty([-5, 0] \times \Omega)} \leq 2\kappa$ . Denote  $\gamma := \Gamma_\ell - \Gamma$  and assume  $\|\dot{\gamma}\|_{L^\infty([-5, 0])} \leq C_{pth}$  and  $\gamma(0) = 0$ .*

*Suppose that for  $t \in [-5, 0]$  and any  $x \in \Omega$ ,*

$$\theta(t, x) \leq 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

*Then the three conditions*

$$(30) \quad |\{\theta \geq 1\} \cap [-2, 0] \times B_2(\Gamma)| \geq \delta_0/4,$$

$$|\{0 < \theta < 1\} \cap [-4, 0] \times B_4(\Gamma)| \leq \mu,$$

$$(31) \quad |\{\theta \leq 0\} \cap [-4, 0] \times B_4(\Gamma)| \geq 2|B_4|$$

*cannot simultaneously be met.*

Here  $\delta_0$  is the constant from Proposition 5.3, which of course depends on  $\kappa$ ,  $C_{pth}$ , and  $C_{dmn}$ .

*Proof.* Suppose that the proposition is false. Then there must exist, for each  $n \in \mathbb{N}$ , a bounded open set  $\Omega_n$  with  $C^{2,\beta_n}$  boundary for  $\beta_n \in (0, 1)$ , a Lipschitz path  $\Gamma_n : [-5, 0] \rightarrow \mathbb{R}^2$ , a function  $\theta_n : [-5, 0] \times \Omega_n \rightarrow \mathbb{R}$ , functions  $u_\ell^n, u_h^n : [-5, 0] \times \Omega_n \rightarrow \mathbb{R}^2$ , and paths  $\Gamma_\ell^n = \Gamma_n + \gamma_n : [-5, 0] \rightarrow \mathbb{R}^2$  which solve (26) and satisfy all of the assumptions of our proposition (with the same constants  $\kappa$ ,  $C_{pth}$ , and  $C_{dmn}$ ), except that

$$(32) \quad |\{0 < \theta_n < 1\} \cap [-4, 0] \times B_4(\Gamma_n)| \leq 1/n.$$

Let  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth function which vanishes on  $B_2$  such that  $\psi(x) = 2 + (|x|^{1/4} - 2^{1/4})_+$  for  $|x| > 3$ .

Fix  $n$  and define

$$\theta_+ := (\theta_n - \psi(x - \Gamma_n))_+.$$

Then  $\theta_+$  is supported on  $\Omega \cap B_3(\Gamma_n)$  and is less than  $2 + 3^{1/4} - 2^{1/4} \leq 3$  everywhere.

Our goal is to bound the derivatives of  $\theta_+^2$  so that we can apply a compactness argument to the sequence  $\theta_n$ . (It is the calculations in Step 2 below in which it becomes necessary to consider  $\theta_+^2$  instead of  $\theta_+$ .)

The remainder of the proof is divided in three steps. First we show that the sequence of  $\theta_+$  is compact in space, then we show that it is compact in time, and finally we show that the limiting function implies a contradiction.

**Step 1:** Compactness in space

Apply the energy inequality Lemma 5.2 to  $\theta$  and  $\psi(x - \Gamma_n)$ , and find that for some  $C$  independent of  $n$

$$(33) \quad \frac{d}{dt} \int \theta_+^2 \leq C.$$

Moreover, by integrating Lemma 5.2 in time from  $-5$  to  $s \in [-4, 0]$  and taking a supremum over  $s$ , we find

$$(34) \quad \sup_{[-4, 0]} \int \theta_+^2 + \int_{-4}^0 \int |\Lambda^{1/2} \theta_+|^2 + \int_{-4}^0 \int \Lambda^{1/2} \theta_+ \Lambda^{1/2} \theta_- \leq C.$$

This proves in particular that  $\theta_+ \in L^2(-4, 0; \mathcal{H}^{1/2}(\Omega))$  is uniformly bounded.

Furthermore,  $\|\theta_+^2\|_{L^2(-4, 0; \mathcal{H}^{1/2}(\Omega_n))}$  is uniformly bounded because

$$\begin{aligned} \|\Lambda^{1/2}(\theta_+^2)\|_2^2 &= \iint [\theta_+(x)^2 - \theta_+(y)^2]^2 K + \int \theta_+^4 B \\ &\leq 2 \iint \theta_+(x)^2 [\theta_+(x) - \theta_+(y)]^2 K + 2 \iint \theta_+(y)^2 [\theta_+(x) - \theta_+(y)]^2 K + \|\theta_+\|_\infty^2 \int \theta_+^2 B \\ &\leq C \|\theta_+\|_\infty^2 \|\theta_+\|_{\mathcal{H}^{1/2}}^2. \end{aligned}$$

By Proposition 2.3, for  $E$  the extension-by-zero operator from  $L^2(\Omega_n)$  to  $L^2(\mathbb{R}^2)$ ,

$$(35) \quad \|E\theta_+^2\|_{L^2(-4, 0; H^{1/2}(\mathbb{R}^2))} \leq C$$

where  $C$  does not depend on  $n$ .

**Step 2:** Compactness in time

Let  $\varphi \in C_0^\infty([-4, 0]; C^\infty(\Omega))$  a test function. Since each  $\theta_n$  and  $u_h^n + u_\ell^n$  is a suitable pair on  $[-5, 0] \times \Omega_n$  by assumption, we can apply the inequality (12) to find that, for some constant  $C$  independent of  $n$  and of  $\varphi$ , on  $[-4, 0] \times \Omega_n$

$$(36) \quad \begin{aligned} \iint \varphi \partial_t \theta_+^2 + \iint \varphi \dot{\Gamma}_n \cdot \nabla \theta_+^2 &\leq \iint \theta_+^2 (u_\ell^n - \dot{\Gamma}_n + u_h^n) \cdot \nabla \varphi - 2 \iint \varphi \theta_+ (u_\ell^n - \dot{\Gamma}_n + u_h^n) \cdot \nabla \psi \\ &\quad + C \|\varphi\|_{C^0(C^2)} \left( 1 + \int_{-5}^0 \left| \int \theta_+ (u_\ell^n - \dot{\Gamma}_n + u_h^n) \cdot \nabla \psi \right| \right). \end{aligned}$$

For the low pass terms, as in the proof of Lemma 5.2, we have  $|u_\ell^n(t, x) - \dot{\Gamma}_n(t)| \leq (1 + 8\kappa)C_{pth} + 6\kappa$  for  $t \in [-4, 0]$  and  $x \in \text{supp}(\theta_+) \subseteq B_3(\Gamma_n(t))$ . Thus for  $t \in [-4, 0]$  we have for  $C$  independent of  $n$  and  $\varphi$

$$(37) \quad \begin{aligned} \int (u_\ell^n - \dot{\Gamma}_n) \cdot (\theta_+^2 \nabla \varphi) &\leq C \|\nabla \varphi\|_{L^\infty(\Omega)}, \\ \int (u_\ell^n - \dot{\Gamma}_n) \cdot (\theta_+ \varphi \nabla \psi) &\leq C \|\varphi\|_{L^\infty(\Omega)}, \\ \int (u_\ell^n - \dot{\Gamma}_n) \cdot (\theta_+ \nabla \psi) &\leq C. \end{aligned}$$

For the high pass terms, we have  $u_h^n$  uniformly bounded in  $\dot{W}^{-1/4, \infty}$ . From step 1, we know  $\theta_+^2$  is uniformly bounded in  $L^2(-4, 0; \mathcal{H}^{1/2})$  so, by Lemma 2.2 parts e and c, there is a constant  $C$



independent of  $n$  and  $\varphi$  such that

$$(38) \quad \begin{aligned} \iint u_h^n \cdot (\theta_+^2 \nabla \varphi) &\leq C \left( \|\nabla \varphi\|_{C^0(-4,0;L^\infty(\Omega))} + \|\varphi\|_{C^0(-4,0;C^2(\Omega))} \right), \\ \iint u_h^n \cdot (\theta_+ \varphi \nabla \psi) &\leq C \left( \|\varphi\|_{C^0(-4,0;L^\infty(\Omega))} + \|\varphi\|_{C^0(-4,0;C^1(\Omega))} \right), \\ \int_{-5}^0 \left| \int u_h^n \cdot (\theta_+ \nabla \psi) \right| &\leq C. \end{aligned}$$

Plugging these six bounds into (36), for a constant  $C$  independent of  $n$  and  $\varphi$ , for any  $\varphi$  non-negative we have

$$(39) \quad \int_{-4}^0 \int_{\Omega_n} (\partial_t \theta_+^2 + \dot{\Gamma}_n \cdot \nabla \theta_+^2) \varphi \, dx dt \leq C \|\varphi\|_{C^0(-4,0;C^2(\Omega_n))}.$$

Note that

$$\int_{-4}^0 \int_{\Omega_n} (\partial_t \theta_+^2 + \dot{\Gamma}_n \cdot \nabla \theta_+^2) \, dx dt = \theta_+(0, \Gamma_n(0))^2 - \theta_+(-4, \Gamma_n(-4))^2$$

is uniformly bounded above and below. Therefore, by decomposing  $\varphi = (\varphi + \|\varphi\|_{C^0}) - \|\varphi\|_{C^0}$  into a non-negative smooth function plus a constant, we can see that (39) holds for general  $\varphi$ .

**Step 3:** Taking the limit

We wish to analyze the limiting behavior of  $\theta_+^2$  in the vicinity of  $\Gamma_n$ . First we shift these functions following  $\Gamma_n$  and define new functions on  $[-4, 0] \times \mathbb{R}^2$  by

$$v_n(t, x) := \begin{cases} \theta_+(t, x + \Gamma_n(t))^2, & x + \Gamma_n(t) \in \Omega_n, \\ 0, & x + \Gamma_n(t) \notin \Omega_n. \end{cases}$$

Each  $v_n$  is supported on  $|x| \leq 3$ , and

$$(40) \quad v_n(t, x) = (\theta_n(t, x + \Gamma_n(t)) - \psi(x))_+^2$$

whenever the right hand side is defined.

Note that

$$\partial_t v_n(t, x) = \partial_t \theta_+^2(t, x + \Gamma_n) + \dot{\Gamma}_n \cdot \nabla \theta_+^2(t, x + \Gamma_n).$$

For  $C$  independent of  $n$ , we know from (35) that

$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \leq C$$

and from (39) that

$$\|\partial_t v_n\|_{\mathcal{M}(-4,0;C^{-2}(\Omega))} \leq C$$

where  $\mathcal{M}$  is the space of Radon measures with total-variation norm and  $C^{-2}(\Omega)$  is the dual of  $C^2(\Omega)$ .

Therefore, by the Aubin-Lions Lemma, the set  $\{v_n\}_n$  is compactly embedded in  $L^2([-4, 0] \times \mathbb{R}^2)$ . Up to a subsequence, there is a function  $v \in L^2([-4, 0] \times \mathbb{R}^2)$  such that

$$v_n \xrightarrow{L^2} v.$$

We know that  $v \in L^\infty$ ,  $\text{supp}(v) \subseteq [-4, 0] \times B_3(0)$ , and  $v \in L^2(H^{1/2})$  because these properties hold uniformly on  $v_n$ .

By (33)

$$(41) \quad \frac{d}{dt} \int_{\mathbb{R}^2} v_n \, dx = \frac{d}{dt} \int_{\Omega_n} \theta_+^2 \, dx \leq C$$

so the same must be true of  $v$ , for  $\frac{d}{dt}$  interpreted in the sense of distributions.

By (30), (32), and (31) applied to  $v_n$  (recalling the relation (40)), we conclude that

$$(42) \quad \begin{cases} |\{v \geq 1\} \cap [-2, 0] \times B_2(0)| \geq \delta_0/4, \\ |\{0 < v < [1 - \psi]^2\} \cap [-4, 0] \times B_4(0)| \leq 0, \\ |\{v \leq 0\} \cap [-4, 0] \times B_4(0)| \geq 2|B_4|. \end{cases}$$

For any  $(t, x) \in [-4, 0] \times B_4(0)$ , either  $v(t, x) \geq [1 - \psi(x)]^2$  or else  $v(t, x) = 0$ . In fact, since  $\|v(t, \cdot)\|_{H^{1/2}} < \infty$  for almost every  $t$  and  $H^{1/2}$  does not contain functions with jump discontinuities, the function  $v$  is either identically 0 or else  $\geq [1 - \psi(x)]^2$  at each  $t$ .

Thus  $\int v(t, x) dx$  is either 0 or else  $\geq \int [1 - \psi(x)]^2 dx > 0$  at each  $t$ . By (41) and (42),  $v$  must be identically zero for all  $t > -2$  but also must be non-zero for some  $t > -2$ , which is a contradiction.

Our assumption that the sequence  $\theta_n$  exists must have been false, and the proposition must be true. □

## 6. A DECREASE IN OSCILLATION

We combine the two De Giorgi lemmas (Propositions 5.3 and 5.4) to produce an oscillation lemma. This result is similar to the weak Harnack inequality for harmonic functions. As in the previous section, all of the following results must be independent of the size of  $\Omega$ , and any assumptions made on  $\Omega$  must be scaling invariant.

**Lemma 6.1** (Oscillation Lemma). *Let  $\kappa$ ,  $C_{dmn}$ , and  $C_{pth}$ , be positive constants. Then there exists a constant  $k_0 > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $\Gamma : [-5, 0] \rightarrow \mathbb{R}^2$  be Lipschitz. Assume that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn}|x - y|^{3/4}K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Let  $\theta$ ,  $u_\ell$ ,  $u_h$ , and  $\Gamma_\ell$  solve (26) on  $[-5, 0] \times \Omega$ , and satisfy  $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5, 0] \times \Omega)} \leq 6\kappa$ , and  $\|\nabla u_\ell\|_{L^\infty([-5, 0] \times \Omega)} \leq 2\kappa$ . Denote  $\gamma := \Gamma_\ell - \Gamma$  and assume  $\|\dot{\gamma}\|_{L^\infty([-5, 0])} \leq C_{pth}$  and  $\gamma(0) = 0$ .*

*Suppose that for all  $t \in [-5, 0]$  and any  $x \in \Omega$*

$$(43) \quad \theta(t, x) \leq 2 + 2^{-k_0} \left( |x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+,$$

*and that*

$$|\{\theta \leq 0\} \cap [-4, 0] \times B_4(\Gamma)| \geq 2|B_4|.$$

*Then for all  $t \in [-1, 0]$ ,  $x \in \Omega \cap B_1(\Gamma)$  we have*

$$\theta(t, x) \leq 2 - 2^{-k_0}.$$

*Proof.* Let  $\mu$  and  $\delta_0$  as in Proposition 5.4, and take  $k_0$  large enough that  $(k_0 - 1)\mu > 4|B_4|$ .

Consider the sequence of functions,

$$\theta_k(t, x) := 2 + 2^k(\theta(t, x) - 2).$$

That is,  $\theta_0 = \theta$  and as  $k$  increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since  $\theta$  satisfies (43), each  $\theta_k$  for  $k \leq k_0$  and  $(t, x) \in [-5, 0] \times \Omega$  satisfies

$$\theta_k(t, x) \leq 2 + \left( |x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+.$$

This is precisely the assumption in Proposition 5.4.

Note also that

$$(44) \quad |\{\theta_k \leq 0\} \cap [-4, 0] \times B_4(\Gamma)|$$

is an increasing function of  $k$ , and hence is greater than  $2|B_4|$  for all  $k$ .

Assume, for means of contradiction, that

$$(45) \quad |\{1 \leq \theta_k\} \cap [-2, 0] \times B_2(\Gamma)| \geq \delta_0/4$$

for  $k = k_0 - 1$ . Since this quantity is decreasing in  $k$ , it must then exceed  $\delta_0/4$  for all  $k < k_0$  as well.

Applying Proposition 5.4 to each  $\theta_k$ , we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4, 0] \times B_4(\Gamma)| \geq \mu.$$

In particular, this means that the quantity (44) increases by atleast  $\mu$  every time  $k$  increases by 1. By choice of  $k_0$  and the fact that quantity (44) is trivially bounded by  $4|B_4|$ , we obtain a contradicton. Therefore, the assumption (45) must fail for  $k = k_0 - 1$ .

Therefore  $\theta_{k_0}$  must satisfy the assumptions of Proposition 5.3. In particular, we conclude that

$$\theta_{k_0}(t, x) \leq 1 \quad \forall t \in [-1, 0], x \in \Omega \cap B_1(\Gamma).$$

For the original function  $\theta$ , this means that

$$\theta(t, x) \leq 2 - 2^{-k_0} \quad \forall t \in [-1, 0], x \in \Omega \cap B_1(\Gamma).$$

□

By assuming that  $\theta$  is small near  $x = \Gamma(t)$ , we have shown that the oscillation of  $\theta$  is decreased in a smaller neighborhood of  $\Gamma(t)$ . However, our goal is to control the oscillation near  $x = \Gamma_\ell(t)$ . Therefore we will prove the following proposition:

**Proposition 6.2** (Oscillation Lemma with shift). *Let  $\kappa$ ,  $C_{dmn}$ , and  $C_{pth}$ , be positive constants, and let  $k_0$  be as in Lemma 6.1. Then there exists a constant  $\lambda > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $\Gamma : [-5, 0] \rightarrow \mathbb{R}^2$  be Lipschitz. Assume that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn}|x - y|^{3/4}K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Let  $\theta$ ,  $u_\ell$ ,  $u_h$ , and  $\Gamma_\ell$  solve (26) on  $[-5, 0] \times \Omega$ , and satisfy  $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5, 0] \times \Omega)} \leq 6\kappa$ , and  $\|\nabla u_\ell\|_{L^\infty([-5, 0] \times \Omega)} \leq 2\kappa$ . Denote  $\gamma := \Gamma_\ell - \Gamma$  and assume  $\|\dot{\gamma}\|_{L^\infty([-5, 0])} \leq C_{pth}$  and  $\gamma(0) = 0$ .*

*Suppose that for all  $t \in [-5, 0]$  and any  $x \in \Omega$*

$$(46) \quad |\theta(t, x)| \leq 2 + 2^{-k_0} \left( |x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+$$

*and that*

$$|\{\theta \leq 0\} \cap [-4, 0] \times B_4(\Gamma)| \geq 2|B_4|.$$

*Then for any  $\varepsilon \in (0, 1/5]$  such that*

$$(47) \quad 5C_{pth} \leq \varepsilon^{-1} - 3$$

*we have*

$$\left| \frac{2}{2 - \lambda} [\theta(\varepsilon t, \varepsilon x) + \lambda] \right| \leq 2 + 2^{-k_0} \left( |x - \varepsilon^{-1}\Gamma_\ell(\varepsilon t)|^{1/4} - 2^{1/4} \right)_+.$$

*for all  $t \in [-5, 0]$  and  $x$  such that  $\varepsilon x \in \Omega$ .*

The idea of the proof is to consider a small enough time interval that  $\Gamma(t)$  is very close to  $\Gamma_\ell(t)$ . This is possible because  $\Gamma_\ell - \gamma$  is uniformly Lipschitz by assumption.

If, in this proposition, we only wished to show the existence of some  $\varepsilon = \varepsilon(k_0, C_{pth})$  satisfying the proposition's conclusion, then a simpler non-constructive proof would suffice. However, in Section 7 we will apply this proposition with parameters  $k_0$  and  $C_{pth}$  depending on  $\varepsilon$ . To avoid circularity, we must prove the result for all  $\varepsilon$  satisfying (47).

*Proof.* Let  $\bar{\lambda} > 0$  and  $\alpha > 1$  be the universal constants defined in Lemma A.4. Take  $\lambda > 0$  such that

$$(48) \quad 2\lambda \leq 2^{-k_0}, \quad (2 + \lambda)\left(\frac{2}{2 - \lambda}\right) \leq 2 + 2^{-k_0}\bar{\lambda}, \quad \frac{2}{2 - \lambda} \leq \alpha.$$

Denote

$$\bar{\theta}(t, x) := \frac{2}{2 - \lambda} [\theta(\varepsilon t, \varepsilon x) + \lambda]$$

defined for  $t \in [-5/\varepsilon, 0]$  and

$$x \in \Omega_\varepsilon := \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$$

and denote

$$\phi(x) := \left(|x|^{1/4} - 2^{1/4}\right)_+.$$

We proved in Lemma 6.1 that  $\theta(t, x) \leq 2 - 2^{-k_0}$  for  $t \in [-1, 0]$  and  $x \in \Omega \cap B_1(\Gamma)$ . On this same set,  $\theta(t, x) \geq -2$  by assumption. By the definition of  $\bar{\theta}$  and by (48), for all  $t \in [-1/\varepsilon, 0]$  and  $x \in \Omega \cap B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$  we have therefore

$$(49) \quad \begin{cases} \bar{\theta}(t, x) & \leq \frac{2}{2 - \lambda} [2 - 2^{-k_0} + \lambda] \leq \frac{2}{2 - \lambda} [2 - \lambda] = 2. \\ \bar{\theta}(t, x) & \geq \frac{2}{2 - \lambda} [-2 + \lambda] = -2. \end{cases}$$

Similarly, the bound (46) on  $\theta$  becomes the equivalent bounds on  $\bar{\theta}$ , for all  $(t, x) \in [-5/\varepsilon, 0] \times \Omega_\varepsilon$

$$(50) \quad \bar{\theta}(t, x) \leq \frac{2}{2 - \lambda} [2 + 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|) + \lambda]$$

and

$$(51) \quad \bar{\theta}(t, x) \geq \frac{2}{2 - \lambda} [-2 - 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|) + \lambda].$$

Let  $t \in [-5, 0]$  and  $x \in \Omega_\varepsilon$ , and define

$$y := x - \varepsilon^{-1}\Gamma(\varepsilon t).$$

From (50) and the assumptions (48), we can bound

$$\begin{aligned} \bar{\theta}(t, x) & \leq \frac{2}{2 - \lambda} [2 + \lambda + 2^{-k_0}\phi(\varepsilon|y|)] \\ & \leq 2 + 2^{-k_0}\bar{\lambda} + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ & = 2 + 2^{-k_0} [\bar{\lambda} + \alpha\phi(\varepsilon|y|)]. \end{aligned}$$

From (51) and the assumptions (48), we can bound

$$\begin{aligned} -\bar{\theta}(t, x) & \leq \frac{2}{2 - \lambda} [2 - \lambda + 2^{-k_0}\phi(\varepsilon|y|)] \\ & \leq 2 + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ & \leq 2 + 2^{-k_0} [\bar{\lambda} + \alpha\phi(\varepsilon|y|)]. \end{aligned}$$

Therefore

$$(52) \quad |\bar{\theta}(t, x)| \leq 2 + 2^{-k_0} [\bar{\lambda} + \alpha\phi(\varepsilon|y|)].$$

If  $|y| \leq \varepsilon^{-1}$  then from (49) we have

$$|\bar{\theta}(t, x)| \leq 2 \leq 2 + 2^{-k_0} \phi(x - \varepsilon^{-1} \Gamma(\varepsilon t) - \varepsilon^{-1} \gamma(\varepsilon t))$$

which is our desired result. Therefore assume without loss of generality that  $|y| \geq \varepsilon^{-1}$ . In this case we can apply Lemma A.4 which states that, since  $\varepsilon < 1/2$  and  $\varepsilon|y| \geq 1$ , it is a property of  $\phi$ ,  $\alpha$ , and  $\bar{\lambda}$  that

$$2 + 2^{-k_0} [\bar{\lambda} + \alpha \phi(\varepsilon|y|)] \leq 2 + 2^{-k_0} [\phi(|y| - \varepsilon^{-1} + 3)].$$

For  $t \in [-5, 0]$ , we have by assumption (47)

$$|y| - \varepsilon^{-1} + 3 \leq |y| - 5C_{pth} \leq |y - \varepsilon^{-1} \gamma(\varepsilon t)|.$$

The estimate (52) becomes

$$|\bar{\theta}(t, x)| \leq 2 + 2^{-k_0} \phi(|x - \varepsilon^{-1} \Gamma(\varepsilon t) - \varepsilon^{-1} \gamma(\varepsilon t)|).$$

This concludes the proof.  $\square$

## 7. HÖLDER CONTINUITY

In this section we shall prove the main theorem, Theorem 1.1. We will demonstrate Hölder continuity by iteratively applying Proposition 6.2 and rescaling.

We begin with a lemma to describe the scaling properties of (2).

**Lemma 7.1** (Scaling). *Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2,\beta}$  boundary for some  $\beta \in (0, 1)$ , and let  $j_0 \in \mathbb{Z}$  and  $\varepsilon > 0$  be constants.*

*Suppose that  $\theta : [-T, 0] \times \Omega \rightarrow \mathbb{R}$  and  $u : [-T, 0] \times \Omega \rightarrow \mathbb{R}^2$  are a suitable solution to (2) and  $u$  is calibrated by a sequence  $(u_j)_{j \geq j_0}$  with constant  $\kappa$  and center  $N$ .*

*Suppose that on  $\Omega$  the functions  $K_{1/4}$  and  $K_1$  (defined in (3)) satisfy the relation*

$$K_{1/4}(x, y) \leq C_{dmn} |x - y|^{3/4} K_1(x, y) \quad \forall x \neq y \in \Omega.$$

*Then*

$$\bar{\theta}(t, x) := \theta(\varepsilon t, \varepsilon x)$$

*and*

$$\bar{u}(t, x) := \sum_{j=j_0}^{\infty} \bar{u}_j(t, x), \quad \bar{u}_j(t, x) := u_j(\varepsilon t, \varepsilon x)$$

*are also a suitable solution to (2) on  $[-T/\varepsilon, 0] \times \Omega_\varepsilon$  where  $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$ .*

*Moreover,  $(\bar{u}_j)_{j \geq j_0}$  is calibrated with the same constant  $\kappa$  but with center  $N - \log_2(\varepsilon)$ , and the relation*

$$(53) \quad \bar{K}_{1/4}(x, y) \leq C_{dmn} |x - y|^{3/4} \bar{K}_1(x, y) \quad \forall x \neq y \in \Omega_\varepsilon$$

*holds.*

*Proof.* Denote by  $\bar{\Lambda}$  the square root of the Laplacian with Dirichlet boundary conditions on  $\Omega_\varepsilon$ . One can calculate (see e.g. [CS16] Section 2.4) that for  $(t, x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$

$$\Lambda \theta(\varepsilon t, \varepsilon x) = \varepsilon \bar{\Lambda} \bar{\theta}(t, x).$$

Similarly, in the Caffarelli-Stinga representation from Proposition 2.1 the operator  $\bar{\Lambda}^s$  will have kernel

$$\bar{K}_s(x, y) = \varepsilon^{s-2} K_s(\varepsilon x, \varepsilon y).$$

From these facts it is clear that the scaled functions satisfy (2) and (53).

To show that  $(\bar{u}_j)_{j \in \mathbb{Z}}$  is calibrated, we must translate the three bounds on  $u_j$  to corresponding bounds on  $\bar{u}_j$ . Each of the calculations are similar, so we show only one:

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \leq 2^{\log_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N - \log_2(\varepsilon))} \kappa.$$

□

The next lemma demonstrates Hölder continuity of suitable solutions. The proof method is to construct a sequence of rescaled functions all of which, by induction, satisfy the assumptions of Proposition 6.2. We will assume that the velocity  $u$  is the Riesz transform of an  $L^\infty$  function  $\Theta$ , which will in practice typically be  $\theta$  itself, up to scaling and translation.

**Lemma 7.2** (Continuity of suitable solutions). *There exists a universal constant  $C$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be an open, bounded domain with  $C^{2,\beta}$  boundary,  $\beta \in (0, 1)$ . Let  $\Theta \in L^\infty([-5, 0] \times \Omega)$ . Then there exists a constant  $\alpha \in (0, 1)$  depending on  $\Omega$  and  $\|\Theta\|_{L^\infty}$  such that the following holds:*

*Let  $\theta : [-5, 0] \times \Omega \rightarrow \mathbb{R}$  and  $u : [-5, 0] \times \Omega \rightarrow \mathbb{R}^2$  be a suitable solution to (2). Assume that  $\|\theta\|_{L^\infty([-5, 0] \times \Omega)} \leq 2$  and that  $u = \nabla^\perp \Lambda^{-1} \Theta$ .*

*Then for any point  $P \in \bar{\Omega}$ ,  $\theta$  is Hölder continuous at  $(0, P)$  and*

$$\sup_{(t,x) \in [-5,0] \times \Omega} \frac{|\theta(t,x) - \theta(0,P)|}{(|t|^2 + |x - P|^2)^{\alpha/2}} \leq C.$$

*Proof.* By relabelling our coordinate system, we can assume without loss of generality that  $P = 0$  is the origin in  $\mathbb{R}^2$ .

From Proposition 4.1, we know that

$$u = \nabla^\perp \Lambda^{-1} \Theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence  $(u_j)_{j \geq j_0}$  of divergence-free functions calibrated with some constant  $\kappa = \kappa(\Omega, \|\Theta\|_{L^\infty})$  and center 0. Assume without loss of generality that  $j_0 < 0$ .

Choose a constant  $0 < \varepsilon < 1/5$  such that

$$(54) \quad 5 \max(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1 - j_0)\kappa) \leq \varepsilon^{-1} - 3.$$

For integers  $k \geq 0$  consider the domains

$$\Omega_k := \{x \in \mathbb{R}^2 : \varepsilon^k x \in \Omega\}.$$

If  $K_s^k$  are the kernels defined in Proposition 2.1 corresponding to the operators  $\Lambda^s$  on  $\Omega_k$ , then by Proposition 2.1 and Lemma 7.1 the relation

$$K_{1/4}^k(x, y) \leq C_{dmn} |x - y|^{3/4} K_1^k(x, y) \quad \forall x \neq y \in \Omega_k$$

holds for some constant  $C_{dmn}$  independent of  $k$ .

For notational convenience, denote

$$\sum_k = \sum_{j > -k \log_2(\varepsilon)}, \quad \sum^k = \sum_{j \leq -k \log_2(\varepsilon)}$$

and define the following functions on  $[-5, 0] \times \Omega_k$ :

$$\begin{aligned} u_\ell^k(t, x) &:= \sum_{j \leq -k \log_2(\varepsilon)} u_j(\varepsilon^k t, \varepsilon^k x), \\ u_h^k(t, x) &:= \sum_{j > -k \log_2(\varepsilon)} u_j(\varepsilon^k t, \varepsilon^k x). \end{aligned}$$

By Lemmas 7.1 we know the sequence  $(u_j(\varepsilon^k \cdot, \varepsilon^k \cdot))_j$  is calibrated with constant  $\kappa$  and center  $-k \log_2(\varepsilon)$ , and hence by 5.1 we know that, independently of  $k$ ,

$$\left\| \Lambda^{-1/4} u_h^k \right\|_{L^\infty([-5,0] \times \Omega_k)} \leq 6\kappa$$

and

$$\left\| \nabla u_\ell^k \right\|_{L^\infty([-5,0] \times \Omega_k)} \leq 2\kappa.$$

Each  $u_\ell^k$  is a finite sum of  $L^\infty$  functions, hence  $L^\infty$  itself, though not uniformly in  $k$ .

Define  $\Gamma_k, \gamma_k : [-5,0] \rightarrow \mathbb{R}^2$  by the following recursive formulae and ODEs:

$$\begin{aligned} \Gamma_0(t) &:= 0, & t \in [-5,0], \\ \gamma_k(0) &:= 0, & k \geq 0, \\ \dot{\gamma}_k(t) &:= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t), & k \geq 0, t \in [-5,0] \\ \Gamma_k(t) &:= \varepsilon^{-1} \gamma_{k-1}(\varepsilon t) + \varepsilon^{-2} \gamma_{k-2}(\varepsilon^2 t) + \dots + \varepsilon^{-k} \gamma_0(\varepsilon^k t), & k \geq 1, t \in [-5,0]. \end{aligned}$$

Since each  $u_\ell^k$  is  $L^\infty$  in space-time and Lipschitz in space, these  $\gamma_k$  exist by a version of the Cauchy-Lipschitz theorem. For example, Theorem 3.7 of Bahouri, Chemin, and Danchin [BCD11] proves existence and uniqueness in our case. In particular, since  $u_\ell^k$  is a vector field which is tangential to the boundary of  $\Omega_k$  and has unique flows, the Lagrangian path

$$\Gamma_\ell^k(t) := \Gamma_k(t) + \gamma(k)$$

for  $u_\ell^k$  must remain inside  $\bar{\Omega}_k$  for all time and so our expressions remain well-defined.

The quantity  $\gamma_k$  here corresponds to the frequency packets  $u_j$  with  $-(k-1)\log_2(\varepsilon) < j \leq -k\log_2(\varepsilon)$ . These frequencies are included in the definition of  $u_\ell^k$  but not the definition of  $u_\ell^{k-1}$  (they would instead be included in  $u_h^{k-1}$ ).

By construction, for  $k \geq 0$  we have  $\Gamma_{k+1}(t) = \varepsilon^{-1} \gamma_k(\varepsilon t) + \varepsilon^{-1} \Gamma_k(\varepsilon t)$ . Therefore

$$\begin{aligned} \dot{\Gamma}_{k+1}(t) &= \partial_t [\varepsilon^{-1} \gamma_k(\varepsilon t) + \varepsilon^{-1} \Gamma_k(\varepsilon t)] \\ &= \dot{\gamma}_k(\varepsilon t) + \dot{\Gamma}_k(\varepsilon t) \\ &= u_\ell^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t)) \\ &= u_\ell^k(\varepsilon t, \varepsilon \Gamma_{k+1}(t)). \end{aligned}$$

With this in hand, we can bound the size of  $\gamma_k$ . Namely, for  $k \geq 1$ ,

$$\begin{aligned} \dot{\gamma}_k(t) &= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t) \\ &= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - u_\ell^{k-1}(\varepsilon t, \varepsilon \Gamma_k(t)) \\ &= \sum_{j=k} u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t) + \varepsilon^k \gamma_k(t)) - \sum_{j=k-1} u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t)) \\ &= \sum_{j=k-1} [u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t) + \varepsilon^k \gamma_k(t)) - u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t))] + \sum_{j=k} u_j(\varepsilon^k t, \varepsilon^k \dots) \\ &= [u_\ell^{k-1}(\varepsilon t, \varepsilon \Gamma_k(t) + \varepsilon \gamma_k(t)) - u_\ell^{k-1}(\varepsilon t, \varepsilon \Gamma_k(t))] + \sum_{j=k-1} u_j(\varepsilon^k t, \varepsilon^k \dots). \end{aligned}$$

The function  $x \mapsto u_\ell^{k-1}(\varepsilon t, x)$  is Lipschitz, with Lipschitz constant less than  $2\kappa$ . Moreover, each  $u_j$  has  $\|u_j\|_\infty \leq \kappa$ . Thus from the above calculation we can bound

$$(55) \quad |\dot{\gamma}_k(t)| \leq 2\kappa\varepsilon|\gamma_k(t)| - \kappa \log_2(\varepsilon).$$

Applying Gronwall's inequality, we find that for  $t \in [-5, 0]$

$$|\gamma_k(t)| \leq \frac{-\log_2(\varepsilon)}{2\varepsilon} (e^{10\varepsilon\kappa} - 1).$$

Plugging this estimate back into (55),

$$|\dot{\gamma}_k(t)| \leq -\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa} \quad \forall k \geq 1.$$

Trivially  $|\dot{\gamma}_0| \leq (1 - j_0)\kappa$ , so if we define

$$C_{pth} = \max(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1 - j_0)\kappa)$$

then for all  $k \geq 0$  and  $t \in [-5, 0]$

$$|\dot{\gamma}_k(t)| \leq C_{pth}.$$

Moreover, the assumption (47) then follows from (54).

Define

$$\theta_0(t, x) := \theta(t, x)$$

and for each  $k \geq 0$ , if  $|\{\theta_k \leq 0\} \cap [-4, 0] \times B_4(\Gamma_k(t))| \geq 2|B_4|$  then set

$$\theta_{k+1}(t, x) := \frac{2}{2 - \lambda} [\theta_k(\varepsilon t, \varepsilon x) + \lambda].$$

Otherwise, set

$$\theta_{k+1}(t, x) := \frac{1}{1 - \lambda} [\theta_k(\varepsilon t, \varepsilon x) - \lambda].$$

From Lemma 7.1, we know that  $\theta_k$  and the calibrated function  $\sum_{j \geq j_0} u_j(\varepsilon^k \cdot, \varepsilon^k \cdot)$  solve (2). By construction,  $\theta_k$ ,  $u_\ell^k$ ,  $u_h^k$ , and  $\Gamma_\ell^k$  solve (26).

Since  $|\theta_0| \leq 2$  by assumption, we know in particular that

$$(56) \quad |\theta_k| \leq 2 + 2^{-k_0} (|x - \Gamma_k(t)|^{1/4} - 2^{1/4})_+$$

holds for  $k = 0$ .

If (56) holds for  $k$ , then at least one of  $\theta_k$  or  $-\theta_k$  (depending on whether  $|\{\theta_k \leq 0\} \cap [-4, 0] \times B_4(\Gamma_k(t))|$  is more or less than  $2|B_4|$ ) will satisfy the assumptions of Proposition 6.2. In either case, we conclude that  $\theta_{k+1}$  satisfies (56). By induction, this bound holds for all  $\theta_k$ .

Each  $\theta_k$  is between  $-2$  and  $2$  on  $[-5, 0] \times B_2(\Gamma_k)$ . But recall that each  $\Gamma_k$  is Lipschitz with constant  $kC_{pth}$ . Thus  $|\Gamma_k(t)| \leq 1$  for  $t \in [-(kC_{pth})^{-1}, 0]$ . On that time interval,

$$|\theta_k(t, x)| \leq 2 \quad \forall x \in B_1(0).$$

We conclude that

$$\left| \sup_{[-\varepsilon^k(kC_{pth})^{-1}, 0] \times B_{\varepsilon^k}(0)} \theta(t, x) - \inf_{[-\varepsilon^k(kC_{pth})^{-1}, 0] \times B_{\varepsilon^k}(0)} \theta(t, x) \right| \leq 4 \left( \frac{2}{2 - \lambda} \right)^{-k}.$$

In particular, for some positive constant  $C$  such that

$$\varepsilon^{Ck} \leq (kC_{pth})^{-1} \quad \forall k \geq 0,$$

we can say that

$$|t|^2 + |x|^2 \leq \varepsilon^{(1+C)k}$$

implies that  $(t, x) \in [-\varepsilon^k(kC_{pth})^{-1}, 0] \times B_{\varepsilon^k}(0)$  which in turn implies that

$$|\theta(t, x) - \theta(0, 0)| \leq 4 \left( \frac{2}{2 - \lambda} \right)^{-k}.$$



In other words,

$$\begin{aligned}
|\theta(t, x) - \theta(0, 0)| &\leq 4 \left( \frac{2}{2-\lambda} \right)^{-\frac{1}{1+C} \log_\varepsilon(|t|^2 - |x|^2) + 1} \\
&= 4 \left( \frac{2}{2-\lambda} \right) \exp \left[ \ln \left( \frac{2}{2-\lambda} \right) \frac{\ln(|t|^2 + |x|^2)}{-(1+C) \ln(\varepsilon)} \right] \\
&= \frac{8}{2-\lambda} (|t|^2 + |x|^2)^{-\frac{\ln(2) - \ln(2-\lambda)}{(1+C) \ln(\varepsilon)}} \\
&\leq 8 (|t|^2 + |x|^2)^{-\frac{\ln(2) - \ln(2-\lambda)}{(1+C) \ln(\varepsilon)}}
\end{aligned}$$

□

We are now able to prove the main result, Theorem 1.1.

*Proof of Theorem 1.1.* Recall that  $\Omega$ ,  $S$ ,  $k$ , and  $\theta_0$  are given.

In Proposition 3.2 we construct global-in-time solutions to (1). By construction, there is a universal constant  $C_1$  so  $\|\theta(t, \cdot)\|_{L^\infty(\Omega)} \leq C_1 t^{-1} \|\theta_0\|_{L^2(\Omega)}$ .

Consider a point  $(t_0, x_0)$  with  $t_0 > S$ . Consider arbitrary constants  $\lambda, \mu \in (0, 1]$  and note that

$$\tilde{\theta}(t, x) := \lambda \theta(t_0 + \mu t, \mu x), \quad \tilde{u}(t, x) := u(t_0 + \mu t, \mu x)$$

is a suitable solution to (2) on  $[-t_0/\mu, \infty) \times \tilde{\Omega}$  where  $\tilde{\Omega} := \{x \in \mathbb{R}^2 : \mu x \in \Omega\}$ .

If  $S + \mu(-5) = \frac{S}{2}$ , or equivalently if  $\mu = S/10$ , then then we have

$$\|\Lambda^{-1} \nabla^\perp \cdot \tilde{u}\|_{L^\infty([-5, 0] \times \Omega)} \leq 2C_1 \frac{k}{S}$$

and

$$\|\tilde{\theta}\|_{L^\infty([-5, 0] \times \tilde{\Omega})} \leq \lambda 2C_1 \frac{k}{S}.$$

If  $\lambda = S/(C_1 k)$  then we can apply Lemma 7.2 to  $\tilde{\theta}$ ,  $\tilde{u}$  and find that  $\tilde{\theta}$  is Hölder continuous at  $(t_0, x_0)$  with constant  $\alpha$  depending on  $k$  and  $S$ .

Since the rescaled function satisfies

$$\sup_{(t, x) \in [-5, 0] \times \tilde{\Omega}} \frac{|\tilde{\theta}(t, x) - \tilde{\theta}(t_0, x_0)|}{(|t - t_0|^2 + |x - x_0|^2)^{\alpha_{k, S/2}}} \leq C,$$

the original function satisfies

$$\sup_{(t, x) \in [S/2, t_0] \times \Omega} \frac{|\theta(t, x) - \theta(t_0, x_0)|}{(|t - t_0|^2 + |x - x_0|^2)^{\alpha_{k, S/2}}} \leq C \lambda^{-1} \mu^{-\alpha_{k, S/2}} \leq C (\lambda \mu)^{-1} = C \frac{10C_1}{S^2} k.$$

□

## APPENDIX A. TECHNICAL LEMMAS

In this appendix we state and prove a few technical lemmas.

**Lemma A.1** (De Giorgi Iteration Argument). *For any constant  $\bar{C} \geq 0$ , there exists a  $\delta > 0$  such that the following holds:*

*Let  $\Omega \subseteq \mathbb{R}^2$  be a bounded open set with  $C^{2, \beta}$  boundary for some  $\beta \in (0, 1)$ . Let  $f \in L^2([-2, 0] \times \Omega)$  be a function with the property that for any positive constant  $a$*

$$(57) \quad \frac{d}{dt} \int (f - a)_+^2 + \int |\Lambda^{1/2} (f - a)_+|^2 \leq \bar{C} \left( \int \chi_{\{f \geq a\}} + \int (f - a)_+ + \int (f - a)_+^2 \right).$$

Then

$$\int_{-2}^0 \int (f - 0)_+^2 dx dt \leq \delta$$

implies that

$$f(t, x) \leq 1 \quad \forall t \in [-1, 0], x \in \Omega.$$

*Proof.* Consider for  $k \in \mathbb{N}$  the constants  $t_k := -1 - 2^{-k}$  (so that  $t_0 = -2$  and  $t_\infty = -1$ ), and functions

$$f_k := (f - 1 + 2^{-k})_+$$

(so that  $f_0 = (f)_+$  and  $f_\infty = (f - 1)_+$ ).

Define

$$\mathcal{E}_k := \int_{t_k}^0 \int_\Omega f_k^2 dx dt.$$

When  $f_{k+1} > 0$ , then in particular  $f_k \geq 2^{-k-1}$ . Thus for any finite  $p$ , there exists a constant  $C$  so

$$(58) \quad \chi_{\{f_{k+1} > 0\}} \leq C^k f_k^p.$$

Let  $k \geq 0$  and define  $\eta : [-2, 0] \rightarrow \mathbb{R}$  a continuous function

$$\eta(t) := \begin{cases} 0 & t \leq t_k \\ 2^{k+1}(t - t_k) & t_k \leq t \leq t_{k+1} \\ 1 & t_{k+1} \leq t. \end{cases}$$

Let  $s \in (t_{k+1}, 0)$ . Multiplying the inequality (57) with cutoff  $a_k$  by  $\eta(t)$  and integrating in time from  $-2$  to  $s$ , then integrating by parts, we obtain

$$\int f_k(s, x)^2 dx - 2^{k+1} \int_{t_k}^{t_{k+1}} \int f_k(t, x)^2 dx dt + \int_{t_{k+1}}^s \int |\Lambda^{1/2} f_k|^2 dx dt \leq \bar{C} \left( \int_{t_k}^0 \int \chi_{\{f_k > 0\}} + f_k + f_k^2 dx dt \right)$$

By taking the supremum over all  $s \in (t_{k+1}, 0)$ , we obtain

$$(59) \quad \sup_{[t_{k+1}, 0]} \int f_k^2 dx + \int_{t_{k+1}}^0 \int |\Lambda^{1/2} f_k|^2 dx dt \leq C \left( 2^{k+1} \int_{t_k}^0 \int f_k^2 dx dt + \int_{t_k}^0 \int \chi_{\{f_k > 0\}} + f_k dx dt \right)$$

From Proposition 2.3 and Sobolev embedding,

$$\int_{t_{k+1}}^0 \left( \int f_k^4 dx \right)^{1/2} dt \leq C \int_{t_{k+1}}^0 \int |\Lambda^{1/2} f_k|^2 dx dt.$$

Therefore by the Riesz-Thorin interpolation theorem,

$$\int_{t_{k+1}}^0 \int f_k^3 dx dt \leq C \left( \sup_{[t_{k+1}, 0]} \int f_k^2 dx + \int_{t_{k+1}}^0 \int |\Lambda^{1/2} f_k|^2 dx dt \right)^{3/2}.$$

This estimate, along with (59) and (58), and the fact that  $t_{k-1} < t_k$  and  $f_{k-1} \geq f_k$ , tell us that

$$\int_{t_{k+1}}^0 \int f_k^3 dx dt \leq C^k \mathcal{E}_{k-1}^{3/2}.$$

Now we can estimate, using again (58) and the fact  $f_k \geq f_{k+1}$ ,

$$\mathcal{E}_{k+1} \leq C^k \int_{t_{k+1}}^0 \int f_k^3 dx dt \leq C^k \mathcal{E}_{k-1}^{3/2}.$$

This nonlinear recursive inequality  $\mathcal{E}_{k+1} \leq C^k \mathcal{E}_{k-1}^{3/2}$ , by a standard fact about nonlinear recursions (see [DG57] or [Vas16]), tells us that there exists a constant  $\delta$  depending only on  $C$  (which in turn depends only on the constant  $\bar{C}$  in (57))

$$\mathcal{E}_0 \leq \delta \text{ implies } \lim_{k \rightarrow \infty} \mathcal{E}_k = 0.$$

By assumption

$$\mathcal{E}_0 = \int_{-2}^0 \int (f)_+ \leq \delta.$$

Therefore  $\mathcal{E}_k \rightarrow 0$  and, by the dominated convergence theorem,

$$\int_{-1}^0 \int (f - 1)_+ dx dt = 0.$$

The result follows.  $\square$

**Lemma A.2.** *Let  $\alpha \in (0, 1)$ . There exists a constant  $C = C(\alpha)$  such that, for any set  $\Omega$  and any  $f \in C^{0,1}(\Omega)$ ,*

$$[f]_\alpha \leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha.$$

*Proof.* This simple lemma is a straightforward calculation:

$$\begin{aligned} \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\alpha} &= \sup |f(x) - f(y)|^{1-\alpha} \left( \frac{|f(x) - f(y)|}{|x - y|} \right)^\alpha \\ &\leq (2 \|f\|_\infty)^{1-\alpha} \left( \sup \frac{|f(x) - f(y)|}{|x - y|} \right)^\alpha \\ &\leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha. \end{aligned}$$

$\square$

**Lemma A.3.** *Let  $\alpha \in (0, 1)$  and  $\Omega$  a set that satisfies the cone condition. There exist constants  $C = C(\alpha, \Omega)$  and  $\ell = \ell(\Omega)$  such that, for any  $f \in C^{1,\alpha}(\Omega)$*

$$\|\nabla f\|_\infty \leq C (\delta^{-1} \|f\|_\infty + \delta^\alpha [\nabla f]_\alpha)$$

for all  $\delta < \ell$ .

The idea of the proof is to average  $\nabla f$  along an interval of length  $\delta$  with endpoint  $x$ . The magnitude of the average will be small, since  $f \in L^\infty$ , and the average will differ not very much from  $\nabla f(x)$  since  $\nabla f \in C^{1,\alpha}$ .

*Proof.* Since  $\Omega$  satisfies the cone condition, there exist positive constants  $\ell$  and  $a < 1$  such that, at each point  $x \in \bar{\Omega}$ , there exist two unit vectors  $e_1$  and  $e_2$  such that  $|e_1 \cdot e_2| \leq a$  and  $x + \tau e_i \in \Omega$  for  $i = 1, 2$  and  $0 < \tau \leq \ell$ . In other words,  $\Omega$  contains rays at each point that extend for length  $\ell$ , end at  $x$ , and are non-parallel with angle at least  $\cos^{-1}(a)$ .

Consider the directional derivative  $\partial_i f$  of  $f$  along the direction  $e_i$ , and observe that for any  $0 < \delta \leq \ell$ ,

$$(60) \quad \left| \int_0^\delta \partial_i f(x + \tau e_i) d\tau \right| = |f(x + \delta e_i) - f(x)| \leq 2 \|f\|_\infty.$$

On the other hand,  $\partial_i f$  is continuous so, for any  $\tau \in (0, \ell]$ ,

$$|\partial_i f(x) - \partial_i f(x + \tau e_i)| \leq [\nabla f]_\alpha \tau^\alpha.$$

From this, we obtain that

$$\int_0^\delta \partial_i f(x + \tau e_i) d\tau \leq \int_0^\delta (\partial_i f(x) + [\nabla f]_\alpha \tau^\alpha) d\tau = \delta \partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below. Thus

$$\left| \delta \partial_i f(x) - \int_0^\delta \partial_i f(x + \tau e_i) d\tau \right| \leq [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}.$$

Combining this bound with (60), we obtain

$$|\partial_i f(x)| \leq \frac{2}{\delta} \|f\|_\infty + \frac{\delta^\alpha}{1+\alpha} [\nabla f]_\alpha.$$

This bound is independent of  $x$  and of  $i = 1, 2$ . Since  $e_1 \cdot e_2 \leq a$  by assumption, by a little linear algebra we can bound  $\nabla f$  in terms of the  $\partial_i f$  and obtain that, for all  $\delta \in (0, \ell]$ ,

$$\|\nabla f\|_\infty \leq \frac{C}{1-a^2} (\delta^{-1} \|f\|_\infty + \delta^\alpha [\nabla f]_\alpha).$$

□

**Lemma A.4.** *There exist constants  $\bar{\lambda} > 0$  and  $\alpha > 1$  such that, for any  $0 < \varepsilon \leq 1/2$  and any  $z \geq 1$*

$$\left( |\varepsilon^{-1}(z-1) + 3|^{1/4} - 2^{1/4} \right)_+ - \alpha \left( |z|^{1/4} - 2^{1/4} \right)_+ \geq \bar{\lambda}.$$

*Proof.* For  $z$  fixed, this function is increasing as  $\varepsilon$  decreases, so it will suffice to show the lemma when  $\varepsilon = 1/2$ , that is to show

$$f_\alpha(z) := \left( |2z+1|^{1/4} - 2^{1/4} \right)_+ - \alpha \left( |z|^{1/4} - 2^{1/4} \right)_+ \geq \bar{\lambda}$$

for all  $z \geq 1$ . Note that  $f_\alpha(z) \geq f_\beta(z)$  if  $\alpha < \beta$ .

For  $z \geq 2$ ,

$$f_\alpha(z) = (2z+1)^{1/4} - 2^{1/4} - \alpha z^{1/4} + \alpha 2^{1/4} = z^{1/4} \left( (2+1/z)^{1/4} - \alpha \right) + (\alpha-1)2^{1/4}.$$

For any  $\alpha < 2^{1/4}$ , clearly  $f_\alpha(z)$  tends to  $\infty$  as  $z$  increases. Therefore there exist  $N$  and  $\alpha_0 > 1$  such that

$$f_\alpha(z) \geq 1 \quad \forall z \geq N, \alpha \leq \alpha_0.$$

We can decompose  $f_\alpha(z) = g_1(z) - (\alpha-1)g_2(z)$  where

$$\begin{aligned} g_1(z) &:= \left( |2z+1|^{1/4} - 2^{1/4} \right)_+ - \left( |z|^{1/4} - 2^{1/4} \right)_+, \\ g_2(z) &:= \left( |z|^{1/4} - 2^{1/4} \right)_+. \end{aligned}$$

Note that  $g_1, g_2$  are both continuous, and  $g_1(z)$  is strictly positive for  $z \geq 1$ . Therefore we can take  $\alpha \in (1, \alpha_0]$  small enough that

$$\alpha - 1 < \frac{\inf_{[1, N]} g_1}{\sup_{[1, N]} g_2}.$$

For this  $\alpha$ ,  $f_\alpha(z)$  is strictly positive on the compact interval  $[1, N]$ , and  $f_\alpha(z) \geq 1$  on  $[N, \infty)$ . Therefore  $f_\alpha(z)$  has a positive lower bound  $\bar{\lambda}$  for all  $z \geq 1$ . □

## REFERENCES

- [ACV16] M. Allen, L. Caffarelli, and A. Vasseur. A parabolic problem with a fractional time derivative. *Arch. Ration. Mech. Anal.*, 221(2):603–630, 2016.
- [BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.
- [BDY12] H.-Q. Bui, X.-T. Duong, and L. Yan. Calderón reproducing formulas and new Besov spaces associated with operators. *Adv. Math.*, 229(4):2449–2502, 2012.
- [BZ10] J. Benedetto and S. Zheng. Besov spaces for the Schrödinger operator with barrier potential. *Complex Anal. Oper. Theory*, 4(4):777–811, 2010.

- [CCV11] L. Caffarelli, C.-H. Chan, and A. Vasseur. Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.*, 24(3):849–869, 2011.
- [CCW01] P. Constantin, D. Cordoba, and J. Wu. On the critical dissipative quasi-geostrophic equation. *Indiana Univ. Math. J.*, 50(Special Issue):97–107, 2001. Dedicated to Professors Ciprian Foias and Roger Temam (Bloomington, IN, 2000).
- [Cha71] J. Charney. Geostrophic turbulence. *Journal of the Atmospheric Sciences*, 28(6):1087–1095, 1971.
- [CI16] P. Constantin and M. Ignatova. Critical SQG in bounded domains. *Ann. PDE*, 2(2):Art. 8, 42, 2016.
- [CI17] P. Constantin and M. Ignatova. Remarks on the fractional Laplacian with Dirichlet boundary conditions and applications. *Int. Math. Res. Not. IMRN*, (6):1653–1673, 2017.
- [CIN18] P. Constantin, M. Ignatova, and H.-Q. Nguyen. Inviscid limit for SQG in bounded domains. *SIAM J. Math. Anal.*, 50(6):6196–6207, 2018.
- [CKN82] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.
- [CMT94] P. Constantin, A. J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7(6):1495–1533, 1994.
- [CN18a] P. Constantin and H. Nguyen. Global weak solutions for SQG in bounded domains. *Comm. Pure Appl. Math.*, 71(11):2323–2333, 2018.
- [CN18b] P. Constantin and H.-Q. Nguyen. Local and global strong solutions for SQG in bounded domains. *Phys. D*, 376/377:195–203, 2018.
- [CS07] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [CS16] L. A. Caffarelli and P. R. Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(3):767–807, 2016.
- [CV10] L. A. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (2)*, 171(3):1903–1930, 2010.
- [CV12] P. Constantin and V. Vicol. Nonlinear maximum principles for dissipative linear nonlocal operators and applications. *Geom. Funct. Anal.*, 22(5):1289–1321, 2012.
- [CV17] C.-H. Chan and A. Vasseur. De Giorgi techniques applied to the Hölder regularity of solutions to Hamilton-Jacobi equations. In *From particle systems to partial differential equations*, volume 209 of *Springer Proc. Math. Stat.*, pages 117–137. Springer, Cham, 2017.
- [CW08] P. Constantin and J. Wu. Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(6):1103–1110, 2008.
- [DG57] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.
- [DP09] H. Dong and N. Pavlović. Regularity criteria for the dissipative quasi-geostrophic equations in Hölder spaces. *Comm. Math. Phys.*, 290(3):801–812, 2009.
- [FMP04] S. Fornaro, G. Metafune, and E. Priola. Gradient estimates for Dirichlet parabolic problems in unbounded domains. *J. Differential Equations*, 205(2):329–353, 2004.
- [GT01] D. Gilbarg and N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Classics in Mathematics. U.S. Government Printing Office, 2001.
- [HPGS95] I. Held, R. Pierrehumbert, S. Garner, and K. Swanson. Surface quasi-geostrophic dynamics. *Journal of Fluid Mechanics*, 282:120, 1995.
- [IMT17] T. Iwabuchi, T. Matsuyama, and K. Taniguchi. Bilinear estimates in Besov spaces generated by the Dirichlet Laplacian. *arXiv e-prints*, page arXiv:1705.08595, May 2017.
- [IMT18] T. Iwabuchi, T. Matsuyama, and K. Taniguchi. Boundedness of spectral multipliers for Schrödinger operators on open sets. *Rev. Mat. Iberoam.*, 34(3):1277–1322, 2018.
- [IMT19] T. Iwabuchi, T. Matsuyama, and K. Taniguchi. Besov spaces on open sets. *Bull. Sci. Math.*, 152:93–149, 2019.
- [IS16] C. Imbert and L. Silvestre. The weak Harnack inequality for the Boltzmann equation without cut-off. *arXiv e-prints*, page arXiv:1608.07571, August 2016.
- [JK95] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.
- [KN09] A. Kiselev and F. Nazarov. A variation on a theme of Caffarelli and Vasseur. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 370(Kraevye Zadachi Matematicheskoi Fiziki i Smezhnye Voprosy Teorii Funktsii. 40):58–72, 220, 2009.
- [KNV07] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2D dissipative quasi-geostrophic equation. *Invent. Math.*, 167(3):445–453, 2007.
- [Kri15] D. Kriventsov. Regularity for a local-nonlocal transmission problem. *Arch. Ration. Mech. Anal.*, 217(3):1103–1195, 2015.

- [NV18a] M. D. Novack and A. Vasseur. Global in time classical solutions to the 3D quasi-geostrophic system for large initial data. *Comm. Math. Phys.*, 358(1):237–267, 2018.
- [NV18b] M. D. Novack and A. Vasseur. The Inviscid 3D Quasi-Geostrophic System on Bounded Domains. *arXiv e-prints*, page arXiv:1806.00547, Jun 2018.
- [NV19] M. D. Novack and A. Vasseur. Classical Solutions for the 3D Quasi-Geostrophic System on a Bounded Domain. *arXiv e-prints*, page arXiv:1904.13050, Apr 2019.
- [Ped92] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer study edition. Springer New York, 1992.
- [Res95] S. Resnick. *Dynamical problems in non-linear advective partial differential equations*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)—The University of Chicago.
- [ST10] P.-R. Stinga and J.-L. Torrea. Extension problem and Harnack’s inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.
- [Sto18] L. F. Stokols. Holder continuity for a family of nonlocal hypoelliptic kinetic equations. *arXiv e-prints*, page arXiv:1812.09006, Dec 2018.
- [SV] L. F. Stokols and A. Vasseur. Smooth global solutions for SQG on bounded domains. *Work in Progress*.
- [SV18] L. F. Stokols and A. Vasseur. De Giorgi techniques applied to Hamilton-Jacobi equations with unbounded right-hand side. *Commun. Math. Sci.*, 16(6):1465–1487, 2018.
- [Vas16] A. Vasseur. The De Giorgi method for elliptic and parabolic equations and some applications. In *Lectures on the analysis of nonlinear partial differential equations. Part 4*, volume 4 of *Morningside Lect. Math.*, pages 195–222. Int. Press, Somerville, MA, 2016.
- [Zhe06] S. Zheng. Littlewood-Paley theorem for Schrödinger operators. *Anal. Theory Appl.*, 22(4):353–361, 2006.

(L. F. Stokols)

DEPARTMENT OF MATHEMATICS,  
THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712, USA  
*E-mail address:* `lstokols@math.utexas.edu`

(A. F. Vasseur)

DEPARTMENT OF MATHEMATICS,  
THE UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712, USA  
*E-mail address:* `vasseur@math.utexas.edu`