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We're gonna consider the equation

$$\partial_t\theta + u \cdot \nabla\theta + \Lambda\theta = 0, u = \nabla^\perp\Lambda^{-1}\theta. \tag{1}$$

Here the operator

$$\Lambda := \sqrt{-\Delta_D}$$

where Δ_D is the Laplacian with Dirichlet boundary condition.

We're going to linearize the equation by fixing u independent of θ. What property do we want u to have? For some constant κ, we'll want

$$u = \sum_{j\in\mathbb{Z}} u_j,$$
$$\left\|\Lambda^{-1/4}u_j\right\|_\infty \le \kappa 2^{-j/4},$$
$$\|\nabla u_j\|_\infty \le \kappa 2^j.$$

The convergence of that sum is in, say, weak L^2.

1. Lemmas

Lemma 1.1. *If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then*

$$\int \Lambda^s f \Lambda^s g\, dx \le 0.$$

This proves, in particular, that $-\int \theta_+\Lambda\theta_-$ is a positive term (hence dissipational and extraneous) and that $\int \Lambda^{1/2}(\theta-\psi)\Lambda^{1/2}(\theta-\psi)$ breaks down (bilinearly) into the doubly positive, the doubly negative, and the cross term, all of which are positive and hence each of which is positive.

Proof. Use the characterization from Caffarelli-Stinga. There exist non-negative functions $K(x,y)$ and $B(x)$, depending on the parameter s, such that

$$\int \Lambda^s f \Lambda^s g\, dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\, dxdy + \int f(x)g(x)B(x)\, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \le 0.$$

Since K is non-negative, the result follows. □

Lemma 1.2. *For any function f, and any $0 < s < 1$,*

$$\int |\Lambda^s f|^2 \simeq \int \left|(-\Delta)^{s/2}\bar{f}\right|^2.$$

Here $\bar{f}$ is the extension of f to $\mathbb{R}^2$ and $(-\Delta)^s$ is defined in the fourier sense.

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Proof. Let g be any L^2 function defined on all of $\mathbb{R}^2$, and let f be a function in H^s_D. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2}\, g\overline{\Lambda^{s-z} f}.$$

When $\Re(z) = 0$, then $\left\|(-\Delta)^{z/2}\, g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|\Lambda^s f\|_2$.

When $\Re(z) = 1$, then $\left\|(-\Delta)^{(z-1)/2}\, g\right\|_2 = \|g\|_2$ and

$$\left\|(-\Delta)^{1/2}\, \overline{\Lambda^{s-z} f}\right\|_2 = \left\|\nabla \overline{\Lambda^{s-z} f}\right\|_2 = \|\Lambda\Lambda^{s-z} f\|_2$$

$\square$

2. De Giorgi Estimates

First let us derive an energy inequality.

We know a priori that $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^{1/2}_D(\Omega))$. Let $\psi : \Omega \to \mathbb{R}^+$ be a non-negative function in $H^{1/2}_D$ non-uniformly, and define $\theta = \theta_+ + \psi - \theta_-$. Since $\theta - \psi$ is in $H^{1/2}_D$, by the lemma above, both θ_+ and θ_- are in that space as well. In particular, our weak solution can eat θ_+.

We end up with

$$0 = \int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] (\theta_+ + \psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \theta_+ &= (1/2)\frac{d}{dt}\int \theta_+^2 + (1/2)\int \operatorname{div} u\, \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \\ &= (1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The ψ term produces important error terms:

$$\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \psi = \frac{d}{dt}\int \theta_+\psi + \int \theta_+ u \cdot \nabla\psi + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi.$$

Since $\theta_+ and \theta_-$ have disjoint support, the θ_- term is nonnegative by lemma [citation]:

$$\begin{aligned}\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \theta_- &= (1/2)\int \theta_+\partial_t\theta_- + \int \theta_+ u \cdot \theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \\ &= \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le 0.\end{aligned}$$

Put together, we arrive at

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \le \left|\iint \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi\right| + \left|\int \theta_+ u \cdot \nabla\psi\right|.$$

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int u \cdot \nabla\frac{\theta_+^2}{2} + \int \theta_+ u \cdot \nabla\psi - \int \theta_+ u \cdot \nabla\theta_- + \int \theta_+\Lambda\theta = 0.$$

We break up the $\theta_+\Lambda\theta$ term into

$$\begin{aligned}\int \theta_+\Lambda\theta &= \int \left|\Lambda^{1/2}\theta_+\right| + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \\ &= \int \left|\Lambda^{1/2}\theta_+\right|^2 + \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) + \int \theta_+\psi B - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_-.\end{aligned}$$

The θ_- term is non-negative by lemma [citation], and the B term is non-negative since $B \geq 0$, so we have the inequality

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \leq \left|\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y)\right| + \left|\int \theta_+ u \cdot \nabla\psi\right|.$$

This integral is symmetric in x and y, and the integrand is only nonzero if one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) \leq 2\chi_{\{\theta_+>0\}}(x)\left|[\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]\right|K(x,y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) \leq \varepsilon\int \left|\Lambda^{1/2}\theta_+\right|^2 + \frac{1}{\varepsilon}\iint \chi_{\{\theta_+>0\}}(x)[\psi(x) - \psi(y)]^2K(x,y).$$

It remains to bound the quantity $[\psi(x) - \psi(y)]^2K(x,y)$. By Caffarelli-Stinga theorem 2.4 [citation], there is a universal constant C such that

$$K(x,y) \leq \frac{C}{|x-y|^3}.$$

The cutoff ψ is Lipschitz, and it grows at a rate $|x|^\gamma$. Therefore

$$[\psi(x) - \psi(y)]^2K(x,y) \leq |x-y|^{-1} \wedge |x-y|^{2\gamma-3}.$$

If $3 - 2\gamma > 2$ then this quantity is integrable. Thus

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \lesssim \int \theta_+ u \cdot \nabla\psi + \int \chi_{\{\theta_+>0\}}.$$

For the drift term, let's say that u is broken down into u_l and u_h (standing for high-pass and low-pass) and that they have the desired properties.

For the high-pass term, for each $i = 1, 2$,

$$\int (u_l)_i\theta_+\partial_i\psi = \int \Lambda^{-1/4}u_l\Lambda^{1/4}(\theta_+\nabla\psi) \leq \left(\int \left|\Lambda^{1/4}\theta_+\nabla\psi\right|^2\right)^{1/2} \leq$$

We're looking at $\int g\Lambda^{1/4}\theta_+$ where $\Lambda^{1/4}g = u_h$ and $g \in L^\infty$. This breaks down as

$$\iint [g - g^*][\theta_+ - \theta_+^*]K_{1/4} + \int g\theta_+B_{1/4}.$$

For a given parameter λ, break up into the region where $|x-y|$ is bigger and smaller than λ. Considering the bigger part,

$$\iint_{\geq\lambda} [g - g^*][\theta_+ - \theta_+^*]K_{1/4} \leq 2(2\|g\|_\infty)\iint_{\geq\lambda} |\theta_+ - \theta_+^*|\frac{dxdx^*}{|x-y|^{2+1/4}} \leq 8\lambda^{-2.25}\|g\|_\infty \|\theta_+\|_1.$$

The B part may be a real problem. The g means it doesn't have a sign, so we actually have to bound it. Consider this.

$$\begin{aligned}\int g\theta_kB_{1/4} &\leq \|g\|_\infty \int \theta_{k-1}\theta_kB_{1/4}\\ &\leq \|g\|_\infty \int \theta_{k-1}^2B_{1/4}\\ &\leq \|g\|_\infty \int \left|\Lambda^{1/8}\theta_{k-1}\right|^2.\end{aligned}$$

Lastly, we have the near part.

$$\begin{aligned}\iint_{<\lambda}[g-g^*][\theta_+-\theta_+^*]K_{1/4} &\leq 2\iint_{<\lambda}\chi_+[g-g^*]^2|x-y|^{3/4}K_{1/4}+\iint_{<\lambda}[\theta_+-\theta_+^*]^2|x-y|^{-3/4}K_{1/4}\\ &\leq 4\,\|g\|_\infty^2\int\chi_+\int_{<\lambda}|x-y|^{-1.5}+\iint_{<\lambda}[\theta_+-\theta_+^*]^2K_1\\ &\leq \|g\|_\infty^2\,\lambda^{1/2}\,|\chi_+|+\int\left|\Lambda^{1/2}\theta_+\right|^2\end{aligned}$$

Taken together,

$$\int\Lambda^{-1/4}\theta_+u_h\cdot\nabla\psi\leq\frac{1}{2}\int\left|\Lambda^{1/2}\theta_+\nabla\psi\right|+\int\chi_++\int\theta_++\int\theta_+\nabla\psi B_{1/4}.$$

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