HOLDER REGULARITY UP TO THE BOUNDARY FOR CRITICAL SQG ON BOUNDED DOMAINS

LOGAN F. STOKOLS AND ALEXIS F. VASSEUR

Abstract. We consider the dissipative SQG equation in bounded domains, first introduced by Constantin and Ignatova in 2016. We show global Holder regularity up to the boundary of the solution. The method is based on the De Giorgi techniques as in the 2010 work of Caffarelli and V. The boundary introduces several difficulties, as the Dirichlet Laplacian is not translation invariant near the boundary which makes the Riesz transform poorly behaved.

Contents

1. Preliminaries

The surface quasigeostrophic equation (SQG) is a special case of the quasi-geostrophic equation (QG) when the atmosphere is at rest. The (QG) model is used extensively in meteorology and oceanography [citations]. These models are described in [Ped92]. The SQG model was popularized by Constantin, Majda and Tabak in [?], due to its similarities with the Euler and Navier-Stokes equation. They proposed it as a toy model for the study of 3D Fluid equations (see also Held, Garner, Pierrehumbert, and Swanson [?]).

We consider in this paper critical SQG on a bounded domain. We will focus on the the following model, which was introduced by Constantin and Ignatova in [CI17] and [CI16]. Consider Ω a connected bounded domain in $\mathbb{R}^2$ with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and the Laplacian with homogeneous Dirichlet boundary conditions $(-\Delta_D)$. If $(\eta_k)_{k\in\mathbb{N}}$ is a family of eigenfunctions of $-\Delta_D$ with corresponding eigenvalues λ_k listed in increasing order, define

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

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The critical SQG problem on Ω with initial data $\theta_0 \in L^2(\Omega)$ is

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0 & (0,T)\times\Omega, \\ u = \nabla^\perp\Lambda^{-1}\theta & [0,T]\times\Omega, \\ \theta = \theta_0 & \{0\}\times\Omega. \end{cases} \tag{1}$$

In the model, the dissipation $\Lambda = \Delta_D^{1/2}$ is due to the Ekman pumping, while the nonlinear velocity u comes from the geostrophic and hydrostatic balance (see [Ped92]).

The main result of this paper is the following.

Theorem 1.1. *Let θ be a smooth solution to* (1) *with initial data $\theta_0 \in L^2(\Omega)$ on a bounded open set $\Omega \subseteq \mathbb{R}^2$ with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, and on a time interval $[0,T]$.*

Then for any $t \in (0,T)$, θ is bounded and Hölder continuous uniformly on $(t,T)\times\bar{\Omega}$.

More precisely, there exists a constant C depending only on Ω and t such that

$$\|\theta\|_{L^\infty([t,T]\times\bar{\Omega})} \leq C\,\|\theta_0\|_{L^2(\Omega)}$$

and there exists $\alpha \in (0,1)$ depending on Ω, t, and $\|\theta_0\|_{L^2(\Omega)}$ such that

$$\|\theta\|_{C^\alpha([t,T]\times\bar{\Omega})} \leq C\,\|\theta_0\|_{L^2(\Omega)}\,.$$

This model was first thoroughly studied in the cases without boundaries (either $\mathbb{R}^2$ or the torus $\mathbb{T}^2$). Global weak solutions were first constructed in Resnick [**?**]. Global regularity was first showed with small initial values [**?**], or extra C^α regularity on the velocity [**?**], [**?**]. In [**?**], Kiselev, Nazarov and Volberg showed the propagation of C^∞ regularity. The global C^∞ regularity for any L^2 initial values was first proved in [CV10] (see also [**?**] Constantin Vicol [CV12]).

In the presence of boundaries, there are several distinct ways to define SQG. This can be attributed to alternative definitions of the half-Laplacian. Kriventsov [**?**] considered a two-phase problem which satisfied SQG only in a region, and was able to prove Hölder regularity in the time-independent case. This problem, intended to model air currents over a region containing both land and water, contains a half-Laplacian and a Riesz transform defined, not spectrally, but in terms of extension. In [citation, Matt and Vasseur], the authors derived from the Euler-Coriolis-Boussinesq model, the full 3D inviscid quasigeostrophic system in an impermeable cylinder (see also [MV2] for the construction of small time smooth solutions to the model). They obtain the natural boundary conditions in this case. It is noteworthy to mention that these boundary conditions lead a distinct set of boundary conditions for SQG than the one introduced in [CI17] [CI16] and described above. However, due to the complexity of the model described in [MV], we focus in this paper only on the homogenous case.

Existence of weak solutions for (1) is proven in [CI17]. The interior regularity of solutions is proven in [CI17] (together with propagation of L^∞ bounds). The method of proof for interior regularity uses the nonlinear maximum principles, introduced in [CV12]. However, the bounds obtained in that paper blow up near the boundary and do not provide global regularity up to the boundary. In [CI16] Remark 1, questions about global regularity are suggested as open problems. Both the $C^\alpha(\bar{\Omega})$ regularity, and the bootstrapping to the $C^\infty(\bar{\Omega})$ regularity are mentioned as open problems. Our result answers the first question, by showing that solutions θ to (1) are globally Hölder continuous. Bootstrapping to $\mathbb{C}^\infty$ involves different techniques, and will be studied in a forthcoming paper [SV].

Our proof is based on the De Giorgi method [DG57]. The method was applied to the SQG problem first in [CV10]. The method is powerful to show C^α regularity to nonlocal equations (see Caffarelli-Chan-Vasseur). It has been applied in a variety of situations for non-local problems, as the 3D Quasigeostrophic problem [NV18], the time-fractional case (Allen Caffarelli Vasseur), the kinetic setting (Silvestre- Imbert, Stokols -Vasseur), or even more exotic situations as the Hamilton-Jacobi equations (see Chan Vasseur, Stokols-Vasseur).

The first broad idea consists in decoupling the velocity u from θ to work on a linear equation, and prove alternating regularity results for θ and u independently. We first show that θ is in L^∞ regardless of the value of u (Section 3). The idea is to prove a decrease-in-oscillation lemma for fractional diffusion equations with drift (Section 6). By applying this oscillation lemma, then scaling our equation, and then applying the oscillation lemma again iteratively, we can show that θ is Hölder continuous (Section 7).

However, for this, we need to obtain scaling invariant controls on the drift $u = \nabla\Lambda^{-1}\theta$ (Section 4). Though L^∞ is the obvious choice, since solutions to SQG are easily shown to be L^∞ bounded under minimal assumptions, the Riesz transform is not bounded from L^∞ to L^∞. The usual solution is to consider BMO (as in [CV10] and [NV18]), but in the case of bounded domains the Riesz transform is not known to be bounded in this space either. The idea is to use extensions of the littlewood-Paley theory to bounded domains.

The adaptation of Fourier analysis and Littlewood-Paley theory to Schrodinger operators has is a well-studied subject [citations]. As an application of this theory, [IMT18] and [citation, Dong et al] have considered operators defined on open subsets of $\mathbb{R}^n$, which includes as a special case the operator $-\Delta_D$ (a Schrodinger operator with zero potential). In particular, in [IMT17], Iwabuchi, Matsuyama, and Taniguchi derives many important results, including the Bernstein inequalities, for Besov spaces adapted to the operator $-\Delta_D$ on bounded open subsets of $\mathbb{R}^n$ with smooth boundary. This theory turns out to greatly improve our understanding of the Riesz transform $\nabla\Lambda^{-1}$ on bounded domains.

Using the results of [IMT17], we will be able to show that the Riesz transform of an L^∞ function whose Fourier decomposition $f = \sum f_k\eta_k$ is supported on high frequencies $k > N$ will be bounded in the weak sobolev space $W^{-1/4,\infty}$, and the Riesz transform of an L^∞ function whose Fourier decomposition is supported on low frequencies $k < N$ will have bounded Lipschitz constant. The cutoff N for dividing high frequencies from low frequencies must depend however on the size of the domain Ω. In the case of $\mathbb{R}^2$, where ∇ and Λ^{-1} commute, this is equivalent to the observation that the Riesz transform is bounded from L^∞ to the Besov space $B^0_{\infty,\infty}$. In the case of bounded domains, the argument must be more subtle. We must decompose θ into its Littlewood-Paley projections, individually bound the Riesz transform of each projection in multiple spaces, and then recombine these infinitely-many functions into a low-frequency collection and a high-frequency collection depending on the scale of oscillation we are trying to detect.

We make this notion precise with the following definition.

Definition 1 (Calibrated sequence)**.** Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set and $0 < T \in \mathbb{R}$. We call a function $u \in L^2([0,T]\times\Omega)$ **calibrated** if it can be decomposed as the sum of a calibrated sequence

$$u = \sum_{j\in\mathbb{Z}} u_j$$

with each $u_j \in L^2([0,T]\times\Omega)$ and the infinite sum converging in the sense of L^2.

We call a sequence $(u_j)_{j\in\mathbb{Z}}$ **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\begin{aligned}
\|u_j\|_{L^\infty([0,T]\times\Omega)} &\le \kappa,\\
\|\nabla u_j\|_{L^\infty([0,T]\times\Omega)} &\le 2^j 2^{-N}\kappa,\\
\left\|\Lambda^{-1/4}u_j\right\|_{L^\infty([0,T]\times\Omega)} &\le 2^{-j/4}2^{N/4}\kappa.
\end{aligned}$$

In Section 4 we will show that u is calibrated and in Section 7 we will show that it remains calibrated at all scales (specifically, with fixed constant κ but not always with the same center N). Therefore we will consider the linear equation

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, & [-T,0]\times\Omega\\ \operatorname{div} u = 0 & [-T,0]\times\Omega.\end{cases} \tag{2}$$

In Section 3 we show that solutions to (2), with minimally regular velocity and L^2 initial data, become L^∞ instantly, and in Sections 5 and 6 we will show that solutions to (2) with calibrated velocity have decreasing oscillation between scales.

Previous applications of the De Giorgi method to equations related to (2) generally make extensive use of either an extension representation (c.f. [CV10]) or a singular-integral representation (c.f. [NV18]). In this paper, we use the kernel representation of the Dirichlet fractional Laplacian derived by Caffarelli and Stinga [CS16]. It is an generalization of [citation Stinga Torrea] and [CS16] the extension representation of [citation Caff Silvestre]. This theory is pivotal in translating the existing non-local De Giorgi techniques to the problem at hand(see Section 2).

The Paper is organized as follows. Section 2 is dedicated to basic properties of the operator Λ and the corresponding Sobolev spaces $\mathcal{H}^s$. In Section 3 we prove $L^2 \to L^\infty$ regularization. In Section 4 we prove that the Riesz transform of the L^∞ function θ is callibrated. Section 5 contains the De Giorgi Lemmas. Section 6 is dedicated to the local decrease in oscillation through an analog of the Harnack inequality. Finally in Section 7 we prove the main theorem, Theorem 1.1. In the Appendix A we prove a few technical lemmas which are needed in the main paper.

Notation. Throughout the paper, we will use the following notations. If $f = \sum_k f_k\eta_k$ then

$$\begin{aligned}
\|f\|_{\mathcal{H}^s} &:= \left(\sum_k \lambda_k^s f_k^2\right)^{1/2}\\
&= \int |\Lambda^s f|^2 .
\end{aligned}$$

We suppress the dependence on Ω, though in fact the λ_k and Λ are defined in terms of the domain Ω. The relevant domain will be clear from context. This is in fact a norm, not a seminorm, since $\|f\|_{L^2(\Omega)} \le \lambda_0^{s/2}\|f\|_{\mathcal{H}^s}$.

For a set D and a function $f : A \to \mathbb{R}$, denote

$$\begin{aligned}
[f]_{\alpha;D} &:= \sup_{x,y\in D, x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}, & \alpha\in(0,1],\\
\|f\|_{C^\alpha(D)} &:= \|f\|_\infty + [f]_{\alpha;D}\,, & \alpha\in(0,1],\\
\|f\|_{C^{k,\alpha}(D)} &:= \sum_{n=0}^k \|D^n f\|_\infty + \left[D^k f\right]_{\alpha;D}\,, & \alpha\in(0,1], k\in\mathbb{N}.
\end{aligned}$$

When the domain D is ommited, the relevant spatial domain Ω is implied.

We will use the notation $(x)_+ := \max(0,x)$. When the parentheses are ommited, the subscript $+$ is merely a label. Throughout this paper, if an integral sign is written $\int$ without a specified domain, the domain is implied to be Ω, with Ω defined in context. For any vector $v = (v_1, v_2)$, by $v^\perp$ we mean $(-v_2, v_1)$, and by $\nabla^\perp$ we mean $(-\partial_y, \partial_x)$. The differential operators ∇ and D^2 do not include time-derivatives.

The operator D^2 is the Hessian, the matrix-valued operator of second-order spatial derivatives. Specifically, D^2, like ∇, never includes time derivatives.

The symbol C represents a constant which may change value each time it is written.

2. Properties of the Fractional Dirichlet Laplacian

We begin by stating a result of [CS16] which gives us a singular integral representation of the $\mathcal{H}^s$ norm.

Proposition 2.1 (Caffarelli-Stinga Representation). *Let $s \in (0,1)$ and $f, g \in \mathcal{H}^s$ on a bounded $C^{2,\beta}$ domain $\Omega \subseteq \mathbb{R}^2$. Then*

$$\int_\Omega \Lambda^s f \Lambda^s g \, dx = \iint_{\Omega^2} [f(x) - f(y)][g(x) - g(y)] K_{2s}(x,y) \, dxdy + \int_\Omega f(x) g(x) B_{2s}(x) \, dx$$

for kernels K_{2s} and B_{2s} which depend on the parameter s and the domain Ω.

These kernels are bounded

$$0 \le K_{2s}(x,y) \le \frac{C(s)}{|x-y|^{2+2s}}$$

for all $x \neq y \in \Omega$ and

$$0 \le B_{2s}(x)$$

for all $x \in \Omega$.

Moreover, for any $s, t \in (0,2)$ there exists a constant $c = c(s,t,\Omega)$ such that for all $x \neq y \in \Omega$

$$K_t(x,y) \le c|x-y|^{s-t} K_s(x,y). \tag{3}$$

Proof. See [CS16] Theorems 2.3 and 2.4.

Theorem 2.4 in [CS16] does not explicitly state the result (3). However, it does state that for each kernel K_s there exists a constant c_s dependent on s and Ω such that

$$\frac{1}{c_s}|x-y|^{2+s} K_s(x,y) \le \min\left(1, \frac{\eta_0(x)\eta_0(y)}{|x-y|^2}\right) \le c_s |x-y|^{2+s} K_s(x,y).$$

Since the middle term does not depend on s, we can say that

$$|x-y|^{2+t} K_t(x,y) \le c_t c_s |x-y|^{2+s} K_s(x,y)$$

from which (3) follows. □

From the explicit formulae given in [CS16], we see that K_{2s} is approximately equal to the standard kernel for the $\mathbb{R}^2$ fractional Laplacian $(-\Delta)^s$ when both x and y are in the interior of Ω or when x and y are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel B_{2s} is well-behaved in the interior but has a singularity at the boundary $\partial\Omega$. This justifies our thinking of the K_{2s} term as the interior term and B_{2s} as a boundary term.

When comparing the computations in this paper to corresponding computations on $\mathbb{R}^2$, one finds that the interior term behaves nearly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

Many useful results can be derived from Caffarelli-Stinga representation formula. We summarize them in the following lemma.

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$.*

(a) Let $s \in (0,1)$. If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then

$$\int \Lambda^s f \Lambda^s g \, dx \leq 0.$$

(b) Let $s \in (0,1)$. If $g \in C^{0,1}(\Omega)$ then for some constant $C = C(s)$

$$\|fg\|_{\mathcal{H}^s} \leq 2\|g\|_\infty \|f\|_{\mathcal{H}^s} + C\|f\|_2 \sup_y \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} \, dx.$$

(c) Let $s \in (0,1)$. If $g \in C^{0,1}(\Omega)$ then for some constant $C = C(s)$

$$\|fg\|_{\mathcal{H}^s} \leq C\|g\|_{C^{0,1}(\Omega)} \left(\|f\|_2 + \|f\|_{\mathcal{H}^s}\right).$$

(d) Let $s \in (0,1/2)$. Let g an $L^\infty(\Omega)$ function and $f \in \mathcal{H}^{2s}$ be non-negative with compact support. Let C_Ω be a constant such that

$$K_s(x,y) \leq C_\Omega |x-y|^{3s} K_{4s}(x,y). \tag{4}$$

Then

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}\right).$$

(e) Let g an $L^\infty(\Omega)$ function and $f \in \mathcal{H}^{1/2}$ be non-negative with compact support. Let C_Ω be a constant such that

$$K_{1/4}(x,y) \leq C_\Omega |x-y|^{3/4} K_1(x,y).$$

Then

$$\int g\Lambda^{1/4} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{1/2}}\right).$$

Proof. We prove these corollaries one at a time.

Proof of (a)**:** From Proposition 2.1

$$\int \Lambda^s f \Lambda^s g \, dx = \iint [f(x)-f(y)][g(x)-g(y)] K(x,y) \, dxdy + \int f(x)g(x)B(x) \, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \leq 0.$$

Since K is non-negative, the result follows.

Proof of (b)**:** From Proposition 2.1

$$\int |\Lambda^s(fg)|^2 = \iint \left(g(x)[f(x)-f(y)] + f(y)[g(x)-g(x)]\right)^2 K + \int f^2 g^2 B$$

$$\leq 2\|g\|_\infty^2 \|f\|_{\mathcal{H}^s}^2 + C(s) \int f(y)^2 \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} \, dxdy.$$

Proof of (c)**:** This follows immediately from (b), since

$$|g(x)-g(y)| \leq \left(\|g\|_\infty\right) \wedge \left(\|\nabla g\|_\infty |x-y|\right)$$

and

$$\int \frac{1 \wedge |x-y|^2}{|x-y|^{2+2s}} \, dx$$

is bounded uniformly in y.

Proof of (d)**:** From Proposition 2.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_< + I_\geq + II$$

where

$$I_< := \iint_{|x-y|<1} [g(x)-g(y)][f(x)-f(y)]K_s,$$
$$I_\geq := \iint_{|x-y|\geq 1} [g(x)-g(y)][f(x)-f(y)]K_s,$$
$$II := \int fgB_s.$$

First we estimate $I_<$. From (4) and from the symmetry of the integrand and the fact that $[f(x)-f(y)]$ vanishes unless at least one of $f(x)$ or $f(y)$ is non-zero,

$$|I_<| \leq 2\iint_{|x-y|<1} \chi_{\{f>0\}}(x)\,|g(x)-g(y)|\cdot|f(x)-f(y)|\cdot|x-y|^{3s}K_{4s}.$$

We can break this up by Holder's inequality

$$|I_<| \leq 2\left(\iint_{|x-y|<1} \chi_{\{f>0\}}(x)[g(x)-g(y)]^2|x-y|^{6s}K_{4s}\right)^{1/2}\left(\iint [f(x)-f(y)]^2K_{4s}\right)^{1/2}.$$

The kernel $|x-y|^{6s}K_{4s}\chi_{\{|x-y|<1\}}$ is integrable in y for x fixed. Therefore

$$|I_<| \leq 2\left((2\,\|g\|_\infty)^2\int C\chi_{\{f>0\}}(x)\,dx\right)^{1/2}\left(\|f\|^2_{\mathcal{H}^{2s}}\right)^{1/2}. \tag{5}$$

For the term $I_\geq$, by the symmetry of the integrand we have

$$|I_\geq| \leq 2\,\|g\|_\infty\, 2\int |f(x)|\int_{|x-y|\geq 1} K_s(x,y)dy\,dx.$$

Since $K_s\chi_{\{|x-y|\geq 1\}}$ is integrable in y for x fixed,

$$|I_\geq| \leq C\,\|g\|_\infty\,\|f\|_1\,. \tag{6}$$

For the boundary term II,

$$|II| \leq \|g\|_\infty \int \chi_{\{f>0\}}fB_s.$$

Since $f\geq 0$, $[f(x)-f(y)][\chi_{\{f>0\}}(x)-\chi_{\{f>0\}}(y)]\geq 0$. Therefore

$$\int \chi_{\{f>0\}}fB_s \leq \int \Lambda^{s/2}\chi_{\{f>0\}}\Lambda^{s/2}f = \int \chi_{\{f>0\}}\Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \leq \|g\|_\infty\,|\mathrm{supp}(f)|^{1/2}\,\|f\|_{\mathcal{H}^s}\,.$$

This combined with (5) and (6) gives us

$$\int \Lambda^{s/2}g\Lambda^{s/2}f \leq C\,\|g\|_\infty\left(|\mathrm{supp}(f)|^{1/2}\,\|f\|_{\mathcal{H}^{2s}} + \|f\|_1 + |\mathrm{supp}(f)|^{1/2}\,\|f\|_{\mathcal{H}^s}\right).$$

The lemma follows since $\|f\|_1 \leq |\mathrm{supp}(f)|^{1/2}\,\|f\|_2$ and since $\|f\|_{\mathcal{H}^s} \leq \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$.

Proof of (e)**:** This is an immediate application of part (d).

□

Let us consider the relationship between the norm $\mathcal{H}^s$ and the H^s norm on $\mathbb{R}^2$.

It is known (see [CI16] and [CS16]) that for $s\in(0,1)$ the spaces $\mathcal{H}^s$ are equivalent to certain subsets of $H^s(\Omega)$ spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in $\mathcal{H}^s$ and that elements of $\mathcal{H}^s$ have trace zero for $s\in[\frac{1}{2},1]$.

The most important fact for us is that the fractional Sobolev norms defined in terms of extension are dominated by our $\mathcal{H}^s$ norm with a constant that is independent of Ω.

We do not claim that this result is new, but we present a detailed proof because the result is crucial to the De Giorgi method. The De Giorgi lemmas require Sobolev embeddings and Rellich-Kondrachov embeddings which are independent of scale.

Define the extension-by-zero operator $E : L^2(\Omega) \to L^2(\mathbb{R}^2)$

$$Ef(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Proposition 2.3. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For any $s \in [0,1]$ and function $f \in \mathcal{H}^s$,*

$$\int_{\mathbb{R}^2} \left| (-\Delta)^{s/2} Ef \right|^2 \le \int_\Omega |\Lambda^s f|^2 .$$

Here $(-\Delta)^s$ is defined in the fourier sense.

We will prove this proposition by interpolating between $s = 0$ and $s = 1$. Before we can do this, we must prove the same in the $s = 1$ case. This result is known (see e.g. Jerison and Kenig [**?**]) but we include the proof for completeness.

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For all functions f in $\mathcal{H}^1$,*

$$\int_\Omega |\nabla f|^2 = \int_\Omega |\Lambda f|^2 .$$

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω and vanish at the boundary, so we can apply the divergence theorem and find

$$\int \nabla \eta_i \cdot \nabla \eta_j = -\int \eta_i \Delta \eta_j = \lambda_j \int \eta_i \eta_j = \lambda_j \delta_{i=j}.$$

Consider a function $f = \sum f_k \eta_k$ which is an element of $\mathcal{H}^1$, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned} \int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla \eta_i \right) \left(\sum_j f_j \nabla \eta_j \right) \\ &= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j \\ &= \sum_j \lambda_j f_j^2. \end{aligned}$$

From this the result follows. □

We come now to the proof of Proposition 2.3. The proof is by complex interpolations using the Hadamard three-lines theorem.

Proof. Let g be any Schwartz function in $L^2(\mathbb{R}^2)$, and let f be a function in $\mathcal{H}^s$. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f, \qquad z \in \mathbb{C}, \mathrm{Re}(z) \in [0,1].$$

Recall (see e.g. [**?**]) that when $t \in \mathbb{R}$, $(-\Delta)^{it}$ is a unitary transformation on $L^2(\mathbb{R}^2)$, and Λ^{it} is a unitary transformation on $L^2(\Omega)$.

When $\mathrm{Re}(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|f\|_{\mathcal{H}^s}$. Hence

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 0.$$

When $\mathrm{Re}(z) = 1$, integrate by parts to obtain

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{(z-1)/2} g \, (-\Delta)^{1/2} E \Lambda^{s-z} f.$$

Then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$, while $\|\Lambda^{s-z} f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^s}$. As an $\mathcal{H}^1$ function, $\Lambda^{s-z} f$ has trace zero so

$$\|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} = \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)} = \|f\|_{\mathcal{H}^s}.$$

Of course $\left\|(-\Delta)^{1/2} \cdot \right\|_{L^2(\mathbb{R}^2)} = \|\nabla \cdot\|_{L^2(\mathbb{R}^2)}$ in general so

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 1.$$

In order to apply the Hadamard three-lines theorem, we must show that Φ is differentiable in the interior of its domain.

Rewrite the integrand of Φ as

$$\mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}\left(\ln(|\xi|)|\xi|^z \hat{g}\right) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}\left(|\xi|^z \hat{g}\right) E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k. \tag{7}$$

Fix some $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in (0,1)$. Since g is a Schwartz function, $\ln(|\xi|)|\xi|^z \hat{g}$ is in L^2. Moreover, for any $\varepsilon > 0$ we have $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C \lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Take $\varepsilon < \mathrm{Re}(z)$ and, since $f \in \mathcal{H}^s$, this sum will converge in L^2.

The differentiated integrand (7) is therefore a sum of two products of L^2 functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \mathrm{Re}(z) < 1$.

By the Hadamard three-lines theorem, for any $z \in (0,1)$ we have $\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}$. Evaluating $\Phi(s)$, we see

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} g E f \le \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

This inequality holds for any Schwartz function $g \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{H}^s$.

Since Schwartz functions are dense in $L^2(\mathbb{R}^2)$ and $(-\Delta)^{s/2}$ is self-adoint, the proof is complete. □

3. L^∞ bounds for θ

Our goal in this section is to show that the L^∞ norm of solutions to (2) are instantaneously bounded by the L^2 norm of the initial data. Note that global L^∞ bounds for L^∞ initial data are proven in [CI16].

Proposition 3.1 (L^2 to L^∞)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and $T \in \mathbb{R}^+$ a positive time.*

Let $\theta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $u \in L^\infty(0,T;L^4(\Omega))$ solve (2) in the sense of distributions with $\theta_0 \in L^2(\Omega)$.

Then for any time $S \in (0,T)$ there exists a constant $C = C(S)$ such that

$$\|\theta\|_{L^\infty([S,T]\times\Omega)} \le C \|\theta_0\|_{L^2(\Omega)}.$$

In order to prove this proposition, we require an energy inequality for $(\theta - C)_+$ for arbitrary constants C. We will prove a much more general lemma, which has further applications in Section 5.

Lemma 3.2 (Caccioppoli Estimate). *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and $T > 0$ and $0 < \gamma < 1/2$ be constants. Let $\Psi : [-T,0] \times \Omega \to \mathbb{R}$ be non-negative and smooth. Then there exists a constant C depending only on $\|\nabla\Psi\|_\infty$ and $\sup_t [\Psi(t,\cdot)]_{\gamma;\Omega}$ such that the following holds:*

Let $\theta \in L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $u \in L^\infty(0,T;L^4(\Omega))$ solve (2) in the sense of distributions. Then the functions

$$\theta_+ := (\theta - \psi(\cdot - \Gamma))_+, \qquad \theta_- := (\psi(\cdot - \Gamma) - \theta)_+$$

satisfy the inequlality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C\left(\int \chi_{\{\theta_+>0\}} + \left|\int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\right|\right).$$

Proof. We multiply (2) by θ_+ and integrate in space to obtain

$$0 = \int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right](\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\theta_+ &= \left(\frac{1}{2}\right)\frac{d}{dt}\int \theta_+^2 + \left(\frac{1}{2}\right)\int \operatorname{div} u\,\theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \\ &= \left(\frac{1}{2}\right)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The Ψ term produces important error terms:

$$\begin{aligned}\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\Psi &= \int \theta_+\partial_t\Psi + \int \theta_+ u\cdot\nabla\Psi + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \\ &= \int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi) + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma 2.2 part (a):

$$\begin{aligned}\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\theta_- &= \left(\frac{1}{2}\right)\int \theta_+\partial_t\theta_- + \int \theta_+ u\cdot\nabla\theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \\ &= \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le 0.\end{aligned}$$

Put together, we arrive at

$$\left(\frac{1}{2}\right)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \le \left|\int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\right|. \tag{8}$$

At this point we break down the $\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi$ term using the formula from Proposition 2.1.

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi = \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y) + \int \theta_+\Psi B.$$

Since $B \ge 0$ and Ψ is non-negative by assumption, the B term is non-negative and so

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \ge \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y). \tag{9}$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\left|\iint [\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y)\right| \le 2\iint \chi_{\{\theta_+>0\}}(x)\,|\theta_+(x)-\theta_+(y)|\cdot|\Psi(x)-\Psi(y)|\,K(x,y).$$

Now we can break up this integral using Young's inequality, and since $\iint[\theta_+(x)-\theta_+(y)]^2K \le \|\theta_+\|^2_{\mathcal{H}^{1/2}}$ the inequality (9) becomes

$$\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi + \frac{1}{2}\int\left|\Lambda^{1/2}\theta_+\right|^2 \ge -2\iint \chi_{\{\theta_+>0\}}(x)[\Psi(x)-\Psi(y)]^2K(x,y). \tag{10}$$

It remains to bound the quantity $[\Psi(x)-\Psi(y)]^2K(x,y)$. By Proposition 2.1, there is a universal constant C such that

$$K(x,y) \le \frac{C}{|x-y|^3}.$$

The cutoff Ψ is locally Lipschitz, and Hölder continuous with exponent $\gamma < 1/2$, by assumption. Therefore

$$[\Psi(x)-\Psi(y)]^2K(x,y) \le C|x-y|^{-1}\wedge|x-y|^{2\gamma-3}.$$

Since $3-2\gamma > 2$, this quantity is integrable. Thus

$$\int \chi_{\{\theta_+>0\}}(x)\int[\Psi(x)-\Psi(y)]^2K(x,y)\,dydx \le C(\|\nabla\Psi\|_\infty,[\Psi]_\gamma)\int \chi_{\{\theta_+>0\}}\,dx.$$

Combining this with (8) and (10) we obtain

$$\frac{d}{dt}\int\theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2 - \int\Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C\left(\left|\int\theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\right| + \int\chi_{\{\theta_+>0\}}\right).$$

□

With this energy inequality, Proposition 3.1 follows by a standard De Giorgi argument. Since we will need to make a similar argument again in Section 5, we will make the bulk of the argument in a more general lemma.

Lemma 3.3 (De Giorgi Iteration Argument)**.** *For any constant $\bar{C} > 0$, there exists a $\delta > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $f \in L^2([-2,0]\times\Omega)$ be a function with the property that for any positive constant a

$$\frac{d}{dt}\int(f-a)_+^2 + \int\left|\Lambda^{1/2}(f-a)_+\right|^2 \le \bar{C}\left(\int\chi_{\{f>a\}} + \int(f-a)_+ + \int(f-a)_+^2\right). \tag{11}$$

Then

$$\int_{-2}^0\int(f-a)_+^2\,dxdt \le \delta$$

implies that

$$f(t,x) \le 1 \qquad \forall t\in[-1,0], x\in\Omega.$$

Proof. Consider for $k\in\mathbb{N}$ the constants $t_k := -1-2^{-k}$ (so that $t_0 = -2$ and $t_\infty = -1$), and functions

$$f_k := (f-1+2^{-k})_+$$

(so that $f_0 = (f)_+$ and $f_\infty = (f-1)_+$).

Define

$$\mathcal{E}_k := \int_{t_k}^0\int_\Omega f_k^2\,dxdt.$$

When $f_{k+1} > 0$, then in particular $f_k \ge 2^{-k-1}$. Thus for any finite p, there exists a constant C so

$$\chi_{\{f_{k+1}>0\}} \le C^k f_k^p. \tag{12}$$

Let $k \geq 0$ and define $\eta : [-2,0] \to \mathbb{R}$ a continuous function

$$\eta(t) := \begin{cases} 0 & t \leq t_k \\ 2^{k+1}(t - t_k) & t_k \leq t \leq t_{k+1} \\ 1 & t_{k+1} \leq t. \end{cases}$$

Let $s \in (t_{k+1}, 0)$. Multiplying the inequality (11) with cutoff a_k by $\eta(t)$ and integrating in time from -2 to s, then integrating by parts, we obtain

$$\int f_k(s,x)^2\, dx - 2^{k+1} \int_{t_k}^{t_{k+1}} \int f_k(t,x)^2\, dxdt + \int_{t_{k+1}}^{s} \int \left|\Lambda^{1/2} f_k\right|^2 dxdt \leq \bar{C}\left(\int_{t_k}^{0} \int \chi_{\{f_k>0\}} + f_k + f_k^2\, dxdt \right)$$

By taking the supremum over all $s \in (t_{k+1}, 0)$, we obtain

$$\sup_{[t_{k+1},0]} \int f_k^2\, dx + \int_{t_{k+1}}^{0} \int \left|\Lambda^{1/2} f_k\right|^2 dxdt \leq C\left(2^{k+1} \int_{t_k}^{0} \int f_k^2\, dxdt + \int_{t_k}^{0} \int \chi_{\{f_k>0\}} + f_k\, dxdt \right) \tag{13}$$

From Proposition 2.3 and Sobolev embedding,

$$\int_{t_{k+1}}^{0} \left(\int f_k^4\, dx \right)^{1/2} dt \leq C \int_{t_{k+1}}^{0} \int \left|\Lambda^{1/2} f_k\right|^2 dxdt.$$

Therefore by the Riesz-Thorin interpolation theorem,

$$\int_{t_{k+1}}^{0} \int f_k^3\, dxdt \leq C\left(\sup_{[t_{k+1},0]} \int f_k^2\, dx + \int_{t_{k+1}}^{0} \int \left|\Lambda^{1/2} f_k\right|^2 \right)^{3/2}.$$

This estimate, along with (13) and (12), and the fact that $t_{k-1} < t_k$ and $f_{k-1} \geq f_k$, tell us that

$$\int_{t_{k+1}}^{0} \int f_k^3\, dxdt \leq C^k \mathcal{E}_{k-1}^{3/2}.$$

Now we can estimate, using again (12) and the fact $f_k \geq f_{k+1}$,

$$\mathcal{E}_{k+1} \leq C^k \int_{t_{k+1}}^{0} \int f_k^3\, dxdt \leq C^k \mathcal{E}_{k-1}^{3/2}.$$

This nonlinear recursive inequality $\mathcal{E}_{k+1} \leq C^k \mathcal{E}_{k-1}^{3/2}$, by a standard fact about nonlinear recursions (see [DG57] or [Vas16]), tells us that there exists a constant δ depending only on C (which in turn depends only on the constant $\bar{C}$ in (11))

$$\mathcal{E}_0 \leq \delta \text{ implies } \lim_{k\to\infty} \mathcal{E}_k = 0.$$

By assumption

$$\mathcal{E}_0 = \int_{-2}^{0} \int (f)_+ \leq \delta.$$

Therefore $\mathcal{E}_k \to 0$ and, by the dominated convergence theorem,

$$\int_{-1}^{0} \int (f-1)_+\, dxdt = 0.$$

The result follows. □

Proposition 3.1 is an easy consequence of these lemmas.

Proof of 3.1. Our regularity assumptions on θ and u are enough to show, by the usual energy argument, that $\|\theta(t,\cdot)\|_{L^2(\Omega)}$ is decreasing in time. By the main result of [CI16], or by applying Lemma 3.2 with constant cutoff $\Psi(t,x) = \|\theta(T,\cdot)\|_{L^\infty(\Omega)}$, the L^∞ norm of $\theta(t,\cdot)$ is also decreasing in time once finite. Thus for T large, $\|\theta\|_{L^\infty([1,T]\times\Omega)} \leq \|\theta\|_{L^\infty([1,2]\times\Omega)}$.

In the case $S \geq 1$, $T \geq 2$, we can apply Lemma 3.2 to establish the family of energy inequalities assumed in Lemma 3.3 and then apply Lemma 3.3 to the function $\frac{\sqrt{\delta}}{\sqrt{2}\|\theta_0\|_2}\theta(t+2,x)$ to show that

$$\|\theta\|_{L^\infty([1,T]\times\Omega)} \leq \left(\frac{2}{\delta}\right)^{1/2} \|\theta_0\|_2 .$$

Assuming without loss of generality that $S \leq T/2$, we can apply the above argument to the function $\theta(t/S, x/S)$ to obtain the result for small values of S. □

4. Littlewood-Paley Theory

In this section we will prove that, because θ is uniformly bounded in L^∞, the velocity $u = \nabla^\perp\Lambda^{-1}\theta$ is calibrated (see Definition 1). The proof will utilize a Littlewood-Paley theory adapted to a bounded set Ω.

Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. Specifically, ϕ is non-negative, supported on $[1/2, 2]$, and has the property that

$$\sum_{j\in\mathbb{Z}} \phi(2^j\xi) = 1 \qquad \forall \xi \neq 0.$$

This allows us to define the Littlewood-Paley projections. For any $f = \sum f_k\eta_k$ in $L^2(\Omega)$

$$P_j f := \sum_k \phi(2^j\lambda_k^{1/2}) f_k\eta_k.$$

Note that P_j depends strongly on the domain Ω.

Recall that $-\Delta_D$ has some smallest eigenvalue λ_0 (depending on Ω) so if we define $j_0 = \log_2(\lambda_0) - 1$ then $P_j = 0$ for all $j < j_0$.

Our goal in this section is to prove the following proposition:

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $\theta \in L^\infty(\Omega)$. Then there exists an integer $j_0 = j_0(\Omega)$ and a sequence of divergence-free functions $(u_j)_{j\geq j_0}$ calibrated for some constant $\kappa = \kappa(\Omega, \|\theta\|_\infty)$ with center 0 (see Definition 1) such that*

$$\nabla^\perp\Lambda^{-1}\theta = \sum_{j\geq j_0} u_j$$

with the infinite sum converging in the sense of L^2.

Before we can prove this, we state a few important lemmas.

The Bernstein Inequalities adapted for a bounded domain are proved in [IMT17]. We restate their result here:

Lemma 4.2 (Bernstein Inequalities)**.** *Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^2$ a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $(P_j)_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition defined above.*

There exists a constant C depending on p and Ω such that the following hold for any $f \in L^p(\Omega)$: For any $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$\|\Lambda^\alpha P_j f\|_{L^p(\Omega)} \leq C 2^{\alpha j} \|f\|_{L^p(\Omega)} .$$

For any $\alpha \in \mathbb{R}$ and $j \geq j_0$

$$\|\nabla \Lambda^{\alpha} P_j f\|_{L^p(\Omega)} \leq C 2^{(1+\alpha)j} \|f\|_{L^p(\Omega)}.$$

Proof. The first claim is Lemma 3.5 in [IMT17]. It is also an immediate corollary of [IMT18] Theorem 1.1.

The second claim is similar to Lemma 3.6 in [IMT17]. A hypothesis of Lemma 3.6 is that

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \leq \frac{C}{\sqrt{t}} \qquad 0 < t \leq 1$$

(a property of Ω). The result of Lemma 3.6 only covers the case $j > 0$.

In [FMP04] it is proved that that if Ω is $C^{2,\beta}$ then

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \leq \frac{C}{\sqrt{t}} \qquad 0 < t \leq T$$

which, by taking some T depending on j_0, is enough to prove the desired result for $j \geq j_0$ by a trivial modification of the proof in [IMT17]. □

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. In the case of $\mathbb{R}^2$, the Littlewood-Paley projections commute with the gradient so $P_i \nabla P_j = 0$ unless $|i-j| \leq 1$. On a bounded domain, this is not the case; the gradient does not maintain localization in frequency-space. However, the following lemma formalizes the observation that $P_i \nabla P_j \approx 0$ when $i << j$.

Lemma 4.3. *Let $1 \leq p \leq \infty$. There exists a constant C depending on p and Ω such that or any function $f \in L^p(\Omega)$,*

$$\|P_i \nabla P_j f\|_p \leq C \min(2^j, 2^i) \|f\|_p.$$

Proof. Let q be the Hölder conjugate of p and g be an L^q function. Then since P_i is self-adjoint

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \leq C 2^j \|g\|_q \|f\|_p$$

by Lemma 4.2.

Further integrating by parts,

$$\int g P_i \nabla P_j f = -\int (\nabla P_i g) P_j f \leq C 2^i \|g\|_q \|f\|_p.$$

This also follows from Lemma 4.2.

The result follows. □

We are now ready to prove Proposition 4.1.

Proof of 4.1. For each integer $j \geq j_0$, we define u_j to be the $\frac{\pi}{2}$-rotation of the Riesz transform of the j^{th} Littlewood-Paley projection of θ:

$$u_j := \nabla^{\perp} \Lambda^{-1} P_j \theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

We must bound u_j, $\Lambda^{-1/4} u_j$, and ∇u_j all in $L^\infty(\Omega)$.

By straightforward application of Lemma 4.2,

$$\|u_j\|_\infty \leq C \|\theta\|_\infty. \tag{14}$$

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4} u_j = \sum_{i \in \mathbb{Z}} P_i \Lambda^{-1/4} u_j.$$

Define $\bar{P}_k := P_{k-1} + P_k + P_{k+1}$. Then $\bar{P}_k P_k = P_k$, and since the projections P_k are spectral operators, they commute with Λ^s and each other. We therefore rewrite

$$\left(P_i \Lambda^{-1/4} u_j\right)^{\perp} = \left(\Lambda^{-1/4} \bar{P}_i\right)(P_i \nabla P_j)\left(\Lambda^{-1} \bar{P}_j\right)\theta.$$

On the right hand side we have three bounded linear operators applied sequentially to $\theta \in L^\infty$. The first operator has norm $C2^{-j}(2^1 + 2^0 + 2^{-1})$ by Lemma 4.2. The second operator has norm $C \min(2^j, 2^i)$ by Lemma 4.3. The third operator has norm $C2^{-i/4}(2^{1/4} + 2^0 + 2^{-1/4})$ by Lemma 4.2. Therefore

$$\left\| P_i \Lambda^{-1/4} u_j \right\|_\infty \leq C 2^{-i/4} \min(2^j, 2^i) 2^{-j} \left\|\theta\right\|_\infty.$$

Summing these bounds on the projections of $\Lambda^{-1/4} u_j$, and noting that

$$\sum_{i \in \mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i \leq j} 2^{i3/4} + \sum_{i > j} 2^{-i/4} \leq C 2^{-j/4},$$

we obtain

$$\left\|\Lambda^{-1/4} u_j\right\|_\infty \leq C 2^{-j/4} \left\|\theta\right\|_\infty. \tag{15}$$

Lastly, we must show that ∇u_j is in L^∞. Equivalently, we will show that $\Lambda^{-1} P_j \theta$ is $C^{1,1}$. The method of proof is Schauder theory.

For convenience, define

$$F := \Lambda^{-1} P_j \theta.$$

Notice that F is a linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Therefore

$$-\Delta F = \Lambda^2 F = \Lambda P_j \theta.$$

We apply the standard Schauder estimate from Gilbarg and Trudinger [GT01] Theorem 6.6 to bound some $C^{2,\alpha}$ semi-norm of F by the L^∞ norm of F and the C^α norm of its Laplacian. By assumption there exists $\beta \in (0,1)$ such that Ω is $C^{2,\beta}$, and for this β we have by the Schauder estimate

$$\left[D^2 F\right]_\beta \leq C \left\|\Lambda^{-1} P_j \theta\right\|_\infty + C \left\|\Lambda P_j \theta\right\|_\infty + C \left[\Lambda P_j \theta\right]_\beta. \tag{16}$$

By Lemma 4.2,

$$\begin{aligned} \left\|\Lambda^{-1} P_j \theta\right\|_\infty &\leq C 2^{-j} \left\|\theta\right\|_\infty, \\ \left\|\Lambda P_j \theta\right\|_\infty &\leq C 2^{j} \left\|\theta\right\|_\infty, \\ \left\|\nabla \Lambda P_j \theta\right\|_\infty &\leq C 2^{2j} \left\|\theta\right\|_\infty. \end{aligned}$$

By Lemma A.1 (see Appendix A) we can interpolate these last two bounds to obtain

$$\left[\Lambda P_j \theta\right]_\beta \leq C 2^{j(1+\beta)} \left\|\theta\right\|_\infty.$$

Plugging these estimates into (16) yields

$$\left[D^2 F\right]_\beta \leq C\left(2^{-j} + 2^j + 2^{j(1+\beta)}\right) \left\|\theta\right\|_\infty.$$

Recall that without loss of generality we can assume $j \geq j_0$. Therefore up to a constant depending on j_0, the term $2^{j(1+\beta)}$ bounds 2^j and 2^{-j} so we can write

$$\left[D^2 F\right]_\beta \leq C 2^{j(1+\beta)} \left\|\theta\right\|_\infty.$$

Using this estimate and the fact that $\|\nabla F\|_\infty = \left\|\nabla \Lambda^{-1} P_j \theta\right\|_\infty \leq C\|\theta\|_\infty$ (see (14)), we can interpolate to obtain an L^∞ bound on $D^2 F$. Lemma A.2 states that since $F \in C^{2,\beta}$ and Ω is sufficiently regular, there exist a constant $\ell = \ell(\Omega)$ such that for any $\delta \in [0, \ell]$ we have

$$\begin{aligned}\left\|D^2 F\right\|_\infty &\leq C\left(\delta^{-1}\|\nabla F\|_\infty + \delta^\beta\left[D^2 F\right]_\beta\right)\\ &\leq C\left(\delta^{-1} + \delta^\beta 2^{j(1+\beta)}\right)\|\theta\|_\infty .\end{aligned}$$

Set $\delta = 2^{-j}(2^{j_0}\ell) \leq \ell$. Then

$$\left\|D^2 F\right\|_\infty \leq C\left(2^j + 2^{-j\beta}2^{j(1+\beta)}\right)\|\theta\|_\infty = C(\Omega)2^j\|\theta\|_\infty .$$

Since $D^2 F = \nabla u_j$, this estimate together with (14) and (15) complete the proof. □

5. De Giorgi Estimates

Our goal in this section is to prove De Giorgi's first and second lemmas for solutions to (2) with u uniformly calibrated.

Rather than working directly with the calibrated sequence, we will decompose u into just two terms, a low-pass term and a high-pass term. The construction is described in the following lemma:

Lemma 5.1. *Let*

$$u = \sum_{j_0}^{\infty} u_j$$

with the sum converging in the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center, and that $\operatorname{div}(u_j) = 0$ for all j.

Then

$$u = u_\ell + u_h$$

with

$$\begin{aligned}\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)} &\leq 2\kappa,\\ \left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-T,0]\times\Omega)} &\leq 6\kappa.\end{aligned}$$

and $\operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0$.

We call u_ℓ the low-pass term, and u_h the high-pass term.

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j>N} u_j$$

and bound

$$\left\|\Lambda^{-1/4}u_h\right\|_\infty \leq \sum_{j>N}\left\|\Lambda^{-1/4}u_j\right\|_\infty \leq \kappa\frac{2^{-1/4}}{1-2^{-1/4}}.$$

We define

$$u_\ell = \sum_{j=j_0}^{N} u_j$$

and bound

$$\|\nabla u_\ell\|_\infty \leq \sum_{j\leq N}\|\nabla u_j\|_\infty \leq \kappa\frac{1}{1-2^{-1}}.$$

□

In order to prove the De Giorgi lemmas, we must derive an energy inequality for the function $(\theta-\Psi)_+$ where $\Psi(t,x)$ grows sublinearly in $|x|$. However, applying Lemma 3.2 to such a function, we see that control can only be gained if the quantity $\partial_t\Psi + u\cdot\nabla\Psi$ is bounded.

To that end, we shall consider a family of functions $\theta : [-T,0]\times\Omega \to \mathbb{R}$, u_ℓ and $u_h : [-T,0]\times\Omega\to\mathbb{R}^2$, and paths Γ and $\gamma : [-T,0]\to\mathbb{R}^2$ which satisfy

$$\begin{cases} \partial_t\theta + (u_\ell + u_h)\cdot\nabla\theta + \Lambda\theta = 0 & \text{on } [-T,0]\times\Omega, \\ \operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0 & \text{on } [-T,0]\times\Omega, \\ \dot\Gamma(t) + \dot\gamma(t) = u_\ell(t,\gamma(t)+\Gamma(t)) & \text{on } [-T,0], \\ \gamma(0) = 0. \end{cases} \tag{17}$$

Here it is implicitly assumed that $\gamma(t)+\Gamma(t)\in\Omega$. Generally speaking u_ℓ and γ will be locally Lipschitz functions while u_h is merely in a weak space $W^{-1/4,\infty}$ and Γ will trace out points in Ω where θ is well behaved by assumption. See Section 7 for the construction of these functions Γ and γ.

Now we prove an energy inequality for solutions to (17).

Lemma 5.2 (Energy inequality). *Let κ, C_Ω, C_g, T, and R be positive constants, and let $\psi : \mathbb{R}^2\to\mathbb{R}$ be a function such that $\|\nabla\psi\|_\infty$, $\|D^2\psi\|_\infty$, and $\sup_t[\psi(t,\cdot)]_{1/4}$ are all finite. Then there exists a constant $C>0$ such that the following holds:*

Let $\Omega\subseteq\mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$. Assume that Lemma 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega|x-y|^{1/2}K_1(x,y).$$

Let θ, u_ℓ, u_h, Γ and γ solve (17) on $[-T,0]\times\Omega$, and satisfy $\|\Lambda^{-1/4}u_h\|_{L^\infty([-T,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)}\le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-T,0])}\le C_g$.

Consider the functions

$$\theta_+ := (\theta - \psi(\cdot-\Gamma))_+, \qquad \theta_- := (\psi(\cdot-\Gamma)-\theta)_+.$$

If θ_+ is supported on $x\in\Omega\cap B_R(\Gamma(t))$ then θ_+ and θ_- satisfy the inequality

$$\frac{d}{dt}\int\theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2 - \int\Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C\left(\int\chi_{\{\theta_+>0\}} + \int\theta_+ + \int\theta_+^2\right).$$

Proof. Define

$$\Psi(t,x) := \psi(x-\Gamma(t))$$

so that

$$\partial_t\Psi + (u_\ell+u_h)\cdot\nabla\Psi = (u_\ell - \dot\Gamma + u_h)\cdot\nabla\psi(x-\Gamma(t)).$$

Applying Lemma 3.2 to θ and Ψ we arrive at

$$\frac{d}{dt}\int\theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2 - \langle\theta_+,\theta_-\rangle_{1/2} \le C\left(\int\chi_{\{\theta_+>0\}} + \left|\int\theta_+(u_\ell - \dot\Gamma(t) + u_h)\cdot\nabla\psi(x-\Gamma(t))\right|\right). \tag{18}$$

Consider first the high-pass term $\int\theta_+u_h\cdot\nabla\psi$. By inserting $\Lambda^{1/4}\Lambda^{-1/4}$ and then integrating by parts, we can apply Lemma 2.2 parts (e) and (c) to obtain

$$\int\Lambda^{-1/4}u_h\Lambda^{1/4}(\theta_+\nabla\psi) \le C\left\|\Lambda^{-1/4}u_h\right\|_\infty\left(\|\nabla\psi\|_\infty + \left\|D^2\psi\right\|_\infty\right)|\operatorname{supp}(\theta_+)|^{1/2}\left(\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}\right).$$

We apply Young's inequality to find that for any constant $\varepsilon>0$ there exists $C = C(\psi,\kappa,C_\Omega,\varepsilon)$ such that

$$\int u_h\theta_+\nabla\psi(x-\gamma(t))\,dx \le C\left(|\operatorname{supp}(\theta_+)| + \int\theta_+^2\right) + \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2. \tag{19}$$

Consider now the low-pass term. By (17)

$$u_\ell(t,x) - \dot\Gamma(t) = u_\ell(t,x) - u_\ell(t,\Gamma+\gamma) + \dot\gamma. \tag{20}$$

Since u_ℓ is has derivative bounded by 2κ,

$$\begin{aligned}|u_\ell(t,x) - u_\ell(t,\Gamma+\gamma)| &\le |u_\ell(t,x) - u_\ell(t,\Gamma)| + |u_\ell(t,\Gamma) - u_\ell(t,\Gamma+\gamma)| \\ &\le 2\kappa|x-\Gamma| + 2\kappa|\gamma|.\end{aligned}$$

By assumption $|\dot\gamma| \le C_g$ and $\gamma(0) = 0$, and so for $t \in [-T, 0]$ we have $|\gamma(t)| \le TC_g$.

Plugging these bounds into (20) we obtain

$$\left|u_\ell(t,x) - \dot\Gamma(t)\right| \le 2\kappa|x-\Gamma| + 2\kappa T C_g + C_g.$$

Now we can bound the low pass term

$$\int (u_\ell - \dot\Gamma)\theta_+ \nabla\psi(x-\Gamma) \le (2\kappa T + 1) C_g \left\|\nabla\psi\right\|_\infty \int \theta_+\, dx + \left\|\nabla\psi\right\|_\infty 2\kappa \int |x-\Gamma|\theta_+\, dx.$$

By assumption, $|x-\Gamma|\theta_+ \le R\theta_+$, so from this, (19), and (18) the result follows. ☐

This energy inequality is sufficient to prove the De Giorgi Lemmas.

The first lemma is a local version of the L^2 to L^∞ regularization, stating that solutions with small L^2 norm in a region will have small L^∞ norm in a smaller region.

Proposition 5.3 (First De Giorgi Lemma). *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $\delta_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1(x,y).$$

Let θ, u_ℓ, u_h, Γ and γ solve (17) on $[-2,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-2,0]\times\Omega)} \le 6\kappa$, $\left\|\nabla u_\ell\right\|_{L^\infty([-2,0]\times\Omega)} \le 2\kappa$, and $\left\|\dot\gamma\right\|_{L^\infty([-2,0])} \le C_g$.

If

$$\theta(t,x) \le 2 + \left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \qquad \forall t \in [-2,0], x \in \Omega \smallsetminus B_2(\Gamma(t))$$

and

$$\int_{-2}^0 \int_{\Omega\cap B_2(\Gamma(t))} (\theta)_+^2 \, dxdt \le \delta_0$$

then

$$\theta(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma(t)).$$

Proof. Let ψ be such that $\psi = 0$ for $|x| \le 1$ and $\psi(x) = 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 2$, and let $\nabla\psi$ and $D^2\psi$ be bounded.

For any constant $a > 0$, we can apply Lemma 5.2 to the function

$$\theta_a := (\theta(t,x) - \psi(x-\Gamma(t)) - a)_+$$

and obtain

$$\frac{d}{dt}\int \theta_a^2 + \int \left|\Lambda^{1/2}\theta_a\right|^2 \le C\left(\int \chi_{\{\theta_a>0\}} + \int \theta_a + \int \theta_a^2\right).$$

Thus $\theta - \psi(x-\Gamma)$ satisfies the assumptions of Lemma 3.3. There exists a constant, which we call δ_0, so that if

$$\int_{-2}^0 \int (\theta(t,x) - \psi(x-\Gamma(t)))_+ \, dxdt \le \delta_0$$

then

$$\theta(t,x) \le 1 + \psi(x-\Gamma(t)) \qquad \forall t \in [-1,0], x \in \Omega.$$

By construction of ψ, our result follows immediately.

$\square$

Next, we will prove De Giorgi's second lemma, a quantitative analog of the isoperimetric inequality.

Proposition 5.4 (Second De Giorgi Lemma). *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $\mu > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1(x,y).$$

Let θ, u_ℓ, u_h, Γ and γ solve (17) on $[-5,0]\times\Omega$, and satisfy $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)} \le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-5,0])} \le C_g$.

Suppose that for $t \in [-5,0]$ and any $x \in \Omega$,

$$\theta(t,x) \le 2 + \left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

Then the three conditions

$$|\{\theta \ge 1\} \cap [-2,0]\times B_2(\Gamma)| \ge \delta_0/4, \tag{21}$$

$$|\{0<\theta<1\} \cap [-4,0]\times B_4(\Gamma)| \le \mu,$$

$$|\{\theta \le 0\} \cap [-4,0]\times B_4(\Gamma)| \ge 2|B_4| \tag{22}$$

cannot simultaneously be met.

Here δ_0 is the constant from Lemma 5.3, which of course depends on κ, C_g, and C_Ω.

Proof. Suppose that the theorem is false. Then there must exist, for each $n \in \mathbb{N}$, a bounded open $C^{2,\beta}$ set Ω_n and function $\theta_n : [-5,0]\times\Omega_n \to \mathbb{R}$, functions $u_\ell^n, u_h^n : [-5,0]\times\Omega_n \to \mathbb{R}^2$, and paths $\Gamma_n, \gamma_n : [-5,0] \to \mathbb{R}^2$ which solve (17) and satisfy all of the the assumptions of our lemma (with the same constants κ, C_g, and C_Ω), except that

$$|\{0<\theta_n<1\} \cap [-4,0]\times B_4(\Gamma_n)| \le 1/n. \tag{23}$$

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function which vanishes on B_2 such that $\psi(x) = 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 3$.

Fix n and define

$$\theta_+ := (\theta_n - \psi(x-\Gamma_n))_+ .$$

Then θ_+ is supported on $\Omega \cap B_3(\Gamma_n)$ and is less than $2 + 3^{1/4} - 2^{1/4} \le 3$ everywhere.

Our goal is to bound the derivatives of θ_+^3 so that we can apply a compactness argument to the sequence θ_n. (For the curious reader, we will point out the steps below in which it is important to consider θ_+^3 instead of θ_+.)

Apply the energy inequality Lemma 5.2 to θ and $\psi(x-\Gamma_n)$, and find that for some C independent of n

$$\frac{d}{dt}\int \theta_+^2 \le C \tag{24}$$

and moreover that

$$\sup_{[-4,0]} \int \theta_+^2 + \int_{-4}^0 \int \left|\Lambda^{1/2}\theta_+\right|^2 + \int_{-4}^0 \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- \le C. \tag{25}$$

This proves in particular that $\theta_+ \in L^2(-4,0;\mathcal{H}^{1/2}(\Omega))$ is uniformly bounded.

What's more, $\left\|\theta_+^3\right\|_{L^2(-4,0;\mathcal{H}^{1/2}(\Omega_n))}$ is uniformly bounded because

$$\begin{aligned}\left\|\Lambda^{1/2}(\theta_+^3)\right\|_2^2 &= \iint [\theta_+(x)^3 - \theta_+(y)^3]^2 K + \int \theta_+^6 B \\ &\le 2\iint \theta_+(x)^4[\theta_+(x)-\theta_+(y)]^2 K + 2\iint \theta_+(y)^4[\theta_+(x)-\theta_+(y)]^2 K + \|\theta_+\|_\infty^4 \int \theta_+^2 B \\ &\le C\,\|\theta_+\|_\infty^4\,\|\theta_+\|_{\mathcal{H}^{1/2}}^2 .\end{aligned}$$

By Lemma 2.3, if $E\theta_+^3$ is the zero-extension of θ_+^3 from Ω_n to $\mathbb{R}^2$, then

$$\left\|E\theta_+^3\right\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \le C \tag{26}$$

where C does not depend on n.

Since θ_n solves the equation

$$\partial_t\theta_n + (u_h + u_\ell)\cdot\nabla\theta_n + \Lambda\theta_n = 0,$$

multiply this equation by $\varphi\theta_+^2$, where φ is any function in $C^2(\mathbb{R}^2)$ restricted to Ω_n, and integrate to obtain

$$\begin{aligned}\frac{1}{3}\int \varphi\partial_t\theta_+^3 + \frac{1}{3}\int \varphi\dot\Gamma_n\cdot\nabla\theta_+^3 = \frac{-1}{3}\int \varphi(u_\ell^n - \dot\Gamma_n + u_h^n)\cdot\nabla\theta_+^3 - \int \varphi\theta_+^2(u_\ell^n - \dot\Gamma_n + u_h^n)\cdot\nabla\psi \\ - \int \varphi\theta_+^2\Lambda\theta_+ - \int\varphi\theta_+^2\Lambda\psi + \int\varphi\theta_+^2\Lambda\theta_-.\end{aligned}$$

Further rearranging, this becomes

$$\begin{aligned}\int \varphi\partial_t\theta_+^3 + \int \varphi\dot\Gamma_n\cdot\nabla\theta_+^3 = \int (u_\ell^n - \dot\Gamma_n)\cdot(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi) + \int \Lambda^{-1/4}u_h^n\Lambda^{1/4}\left(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi\right) \\ - 3\int\varphi\theta_+^2\Lambda\theta_+ - 3\int\varphi\theta_+^2\Lambda\psi + 3\int\varphi\theta_+^2\Lambda\theta_-.\end{aligned}$$

We will bound the five terms on the right hand side one at a time.

Each instance of C in the following bounds is independent of n.

- Consider the low-pass term. As in the proof of Lemma 5.2, we have $|u_\ell^n(t,x) - \dot\Gamma_n(t)| \le (1+8\kappa)C_g + 6\kappa$ for $t\in[-4,0]$ and $x\in\operatorname{supp}(\theta_+)\subseteq\Omega_n\cap B_3(\Gamma_n(t))$. Thus for $t\in[-4,0]$ we have for C independent of n and of φ

$$\int (u_\ell^n - \dot\Gamma_n)\cdot(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi) \le C\left(\|\nabla\varphi(t,\cdot)\|_{L^\infty(\Omega)} + \|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\right).$$

- Consider the high-pass term. By Lemma 2.2 parts e and c,

$$\begin{aligned}\int \Lambda^{-1/4}u_h^n\Lambda^{1/4}\left(\theta_+^3\nabla\varphi - 3\varphi\theta_+^2\nabla\psi\right) &\le C\kappa|\operatorname{supp}(\theta_+)|^{1/2}\left(\left\|\theta_+^3\nabla\varphi\right\|_{L^2} + \left\|\varphi\theta_+^2\nabla\psi\right\|_{L^2} + \left\|\theta_+^3\nabla\varphi\right\|_{\mathcal{H}^{1/2}} + \left\|\varphi\theta_+^2\nabla\psi\right\|_{\mathcal{H}^{1/2}}\right). \\ &\le C\left(\|\varphi(t,\cdot)\|_{C^1(\Omega)} + \|\varphi(t,\cdot)\|_{C^2(\Omega)}\,\|\theta(t,\cdot)\|_{\mathcal{H}^{1/2}}\right).\end{aligned}$$

- Consider the $\Lambda\theta_+$ term. Decomposing this term using Proposition 2.1 we have first an interior term $\iint[\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\theta_+(x)-\theta_+(y)]K$ which decomposes as

$$\iint \varphi(x)(\theta_+(x)+\theta_+(y))[\theta_+(x)-\theta_+(y)]^2K + \iint \theta_+(y)^2[\varphi(x)-\varphi(y)][\theta_+(x)-\theta_+(y)]K.$$

  The first part is bounded by the L^∞ norms of φ and θ_+ and the square of the $\mathcal{H}^{1/2}$ norm of θ_+, while the second part is bounded

$$\iint \theta_+(y)^2[\varphi(x)-\varphi(y)][\theta_+(x)-\theta_+(y)]K \le \|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}\sqrt{\int\theta_+(y)^2\int\frac{[\varphi(x)-\varphi(y)]^2}{|x-y|^3}dx\,dy}$$

  which is bounded by the C^1 norm of φ and the $\mathcal{H}^{1/2}$ norm of θ_+.

The boundary term $\int \varphi\theta_+^3 B$ is bounded by the L^∞ norms of φ and θ_+, and by $\int \theta_+^2 B$ which is less than $\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}$. Taken together we have

$$\int \varphi\theta_+^2\Lambda\theta_+ \le C\left(\|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2 + \|\varphi(t,\cdot)\|_{C^1(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}\right).$$

- Consider the $\Lambda\theta_-$ term. For any non-negative function f we know by Lemma 2.2 part (a) that $\int f\theta_+\Lambda\theta_- \le 0$. It follows that $-\theta_+\Lambda\theta_-$ is a pointwise non-negative distribution. Moreover, the integral over $[-4,0]\times\Omega$ of $-\theta_+\Lambda\theta_-$ is bounded by (25). Thus $\theta_+\Lambda\theta_-$ is a measure with bounded total-variation norm. In fact, because φ is a continuous function,

$$\int_{-4}^0\int \varphi\theta_+^2\Lambda\theta \le \|\theta_+\|_\infty\|\theta_+\Lambda\theta_-\|_{\mathcal{M}}\|\varphi\|_{C^0} \le C\|\varphi\|_{L^\infty([-4,0]\times\Omega)}.$$

- Consider the $\Lambda\psi$ term. Decomposing this term using Proposition 2.1 we have first an interior term $\iint[\varphi(x)\theta_+(x)^2 - \varphi(y)\theta_+(y)^2][\psi(x)-\psi(y)]K$ which decomposes as

$$\int \theta_+(y)^2\int[\varphi(x)-\varphi(y)][\psi(x)-\psi(y)]K + \iint \varphi(x)[\theta_+(x)^2-\theta_+(y)^2][\psi(x)-\psi(y)]K.$$

The first part is bounded by the C^1 norms of φ and ψ and the L^2 norm of θ_+, while the second part is bounded

$$\iint \varphi(x)[\theta_+(x)^2-\theta_+(y)^2][\psi(x)-\psi(y)]K \le \left\|\theta_+^2(t,\cdot)\right\|_{\mathcal{H}^{1/2}}\sqrt{\int\varphi(x)^2\int\frac{[\psi(x)-\psi(y)]^2}{|x-y|^3}dy\,dx}$$

which is bounded, because ψ is smooth and globally 1/4-Hölder continuous, by L^2 norm of φ and the $\mathcal{H}^{1/2}$ norm of θ_+.

The boundary term $\int\varphi\theta_+^2\psi B$ is bounded by the L^∞ norms of φ and $\psi\chi_{\{\theta_+>0\}}$ and by $\int\theta_+^2 B$ which is less than $\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2$. Taken together we have

$$\int\varphi\theta_+^2\Lambda\psi \le C\left(\|\varphi(t,\cdot)\|_{C^1(\Omega)} + \|\varphi(t,\cdot)\|_{L^2(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}} + \|\varphi(t,\cdot)\|_{L^\infty(\Omega)}\|\theta_+(t,\cdot)\|_{\mathcal{H}^{1/2}}^2\right).$$

Remark. We are attempting to bound $\partial_t\theta_+^3$. If we had attempted to bound $\partial_t\theta_+^p$ instead, the final three terms above would have been problematic for $p=1$ and the very final term would have been problematic for $p=2$.

Combining all of these bounds, and using the fact that $\theta_+\in L^2(-4,0;\mathcal{H}^{1/2})$ uniformly, we conclude that there exists a constant C independent of n such that, for any $\varphi\in L^\infty(-4,0;C^2(\mathbb{R}^2))\cap L^\infty(-4,0;L^2(\mathbb{R}^2))$,

$$\int_{-4}^0\int_{\Omega_n}\left(\partial_t\theta_+^3 + \dot\Gamma_n\cdot\nabla\theta_+^3\right)\varphi\,dxdt \le C\|\varphi\|_{L^\infty(-4,0;C^2(\mathbb{R}^2))} + C\|\varphi\|_{L^\infty(-4,0;L^2(\mathbb{R}^2))}. \tag{27}$$

Over time, the support of θ_+^3 moves around in Ω_n following the path Γ_n. In order to take a meaningful limit in n, we must shift these functions so that their supports remain in a compact set. To that end, define a new function on $[-4,0]\times\mathbb{R}^2$ by

$$v_n(t,x) := \begin{cases}\theta_+(t,x+\Gamma_n(t))^3, & x+\Gamma_n(t)\in\Omega_n,\\ 0, & x+\Gamma_n(t)\notin\Omega_n.\end{cases}$$

In other words,

$$v_n(t,x) = (\theta_n(t,x+\Gamma_n(t)) - \psi(x))_+^3 \tag{28}$$

when the right hand side is defined.

Let $X\subseteq C^2(\mathbb{R}^2)$ be the Banach space of C^2 functions with norm $\|\cdot\|_X = \|\cdot\|_{C^2(\mathbb{R}^2)} + \|\cdot\|_{L^2(\mathbb{R}^2)}$ finite. Note that

$$\partial_t v_n(t,x) = \partial_t\theta_+^3(t,x+\Gamma_n) + \dot\Gamma_n\cdot\nabla\theta_+^3(t,x+\Gamma_n).$$

We know from (26) that

$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2)} \le C$$

and from (27) that

$$\|\partial_t v_n\|_{L^1(-4,0;X^*)} \le C.$$

According to the Aubin-Lions Lemma, the set $\{v_n\}_n$ is therefore compactly embedded in $L^2([-4,0]\times\mathbb{R}^2)$. Up to a subsequence, there is a function $v \in L^2([-4,0]\times\mathbb{R}^2)$ such that

$$v_n \to v.$$

By elementary properties of L^2 convergence, we know that $v \in L^\infty$, $\mathrm{supp}(v) \subseteq [-4,0]\times B_3(0)$, and $v \in L^2(H^{1/2})$.

By (24)

$$\frac{d}{dt}\int_{\mathbb{R}^2} v_n^{2/3}\,dx = \frac{d}{dt}\int_{\Omega_n} \theta_+^2\,dx \le C \tag{29}$$

so the same must be true of v.

By (21), (23), and (22) applied to v_n (recalling the relation (28)), we conclude that

$$\begin{cases} |\{v \ge 1\} \cap [-2,0]\times B_2(0)| & \ge \delta_0/4, \\ \left|\{0 < v < [1-\psi]^3\} \cap [-4,0]\times B_4(0)\right| & \le 0, \\ |\{v \le 0\} \cap [-4,0]\times B_4(0)| & \ge 2|B_4| \end{cases} \tag{30}$$

For any $(t,x) \in [-4,0]\times B_4(0)$, either $v(t,x) \ge [1-\psi(x)]^3$ or else $v(t,x) = 0$. In fact, since $\|v(t,\cdot)\|_{H^{1/2}} < \infty$ for almost every t and $H^{1/2}$ does not contain functions with jump discontinuities, the function v is either identically 0 or else $\ge [1-\psi(x)]^3$ at each t.

Thus $\int v(t,x)^{2/3}\,dx$ is either 0 or else $\ge \int [1-\psi(x)]^3\,dx > 0$ at each t. By (29) and (30), v must be identically zero for all $t > -2$ but also must be non-zero for some $t > -2$, which is a contradiction. Our assumption that the sequence θ_n exists must have been false. The proposition must be true.

□

6. A Decrease in Oscillation

We combine the two De Giorgi lemmas (Propositions 5.3 and 5.4) to produce an Oscillation lemma. This result is similar to the Harnack theory for harmonic functions.

Proposition 6.1 (Oscillation Lemma)**.** *Let κ, C_Ω, and C_g, be positive constants. Then there exists a constant $k_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Assume that Lemma 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_ℓ, u_h, Γ and γ solve (17) on $[-2,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-2,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_\ell\|_{L^\infty([-2,0]\times\Omega)} \le 2\kappa$, and $\|\dot\gamma\|_{L^\infty([-2,0])} \le C_g$.

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$\theta(t,x) \le 2 + 2^{-k_0}\left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+, \tag{31}$$

and that

$$|\{\theta \le 0\} \cap [-4,0]\times B_4(\Gamma)| \ge \frac{4|B_4|}{2}.$$

Then for all $t \in [-1,0]$, $x \in \Omega \cap B_1(\Gamma)$ *we have*

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

Proof. Let μ and δ_0 as in Proposition 5.4, and take k_0 large enough that $(k_0-1)\mu > 4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x) := 2 + 2^k(\theta(t,x) - 2).$$

That is, $\theta_0 = \theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies (31), each θ_k for $k \le k_0$ and $(t,x) \in [-5,0] \times \Omega$ satisfies

$$\theta_k(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 5.4.

Note also that

$$|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma)| \tag{32}$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0/4 \tag{33}$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed $\delta_0/4$ for all $k < k_0$ as well.

Applying Proposition 5.4 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0] \times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity (32) increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity (32) is trivially bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption (33) must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 5.3. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

For the original function θ, this means that

$$\theta(t,x) \le 2 - 2^{-k_0} \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

□

By assuming that θ is small near $x = \Gamma(t)$, we have shown that the oscillation of θ is decreased in a smaller neighborhood of $\Gamma(t)$. However, our goal is to control the oscillation near $x = \Gamma(t) + \gamma(t)$. Therefore we will prove the following proposition:

Lemma 6.2 (Oscillation Lemma with shift)**.** *Let* κ, C_Ω, *and* C_g, *be positive constants, and let* k_0 *be as in Lemma* 6.1. *Then there exists a constant* $\lambda > 0$ *such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ *be a bounded open set with* $C^{2,\beta}$ *boundary for some* $\beta \in (0,1)$. *Assume that Lemma* 2.1 *hold on* Ω *with kernels that satisfy*

$$K_{1/4}(x,y) \le C_\Omega |x-y|^{1/2} K_1.$$

Let θ, u_ℓ, u_h, Γ *and* γ *solve* (17) *on* $[-5,0] \times \Omega$, *and satisfy* $\left\|\Lambda^{-1/4} u_h\right\|_{L^\infty([-5,0]\times\Omega)} \le 6\kappa$, $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)} \le 2\kappa$, *and* $\|\dot\gamma\|_{L^\infty([-5,0])} \le C_g$.

Suppose that for all $t \in [-5,0]$ *and any* $x \in \Omega$

$$|\theta(t,x)| \le 2 + 2^{-k_0}\left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \tag{34}$$

and that

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge 2|B_4|.$$

Then for any $\varepsilon \in (0,1/3)$ *such that*

$$5C_g \le \varepsilon^{-1} - 3 \tag{35}$$

we have

$$\left|\frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]\right| \le 2 + 2^{-k_0}\left(|x - \varepsilon^{-1}\Gamma(\varepsilon t) - \varepsilon^{-1}\gamma(\varepsilon t)|^{1/4} - 2^{1/4}\right)_+ .$$

for all $t \in [-5,0]$ *and* x *such that* $\varepsilon x \in \Omega$.

The idea of the proof is to consider a small enough time interval that $\Gamma(t) + \gamma(t)$ is very close to $\Gamma(t)$. This is possible because γ is Lipschitz by assumption.

If, in this lemma, we only wished to show the existence of some $\varepsilon = \varepsilon(k_0, C_g)$ satisfying the lemma's conclusion, then a simpler non-constructive proof would suffice. However, in Section 7 we will apply this lemma with parameters k_0 and C_g depending on ε. To avoid circularity, we must prove the result for all ε satisfying (35).

Proof. Take λ such that

$$2\lambda \le 2^{-k_0}, \qquad (2+\lambda)\left(\frac{2}{2-\lambda}\right) \le 2 + 2^{-k_0}\bar{\lambda}, \qquad \frac{2}{2-\lambda} \le \alpha. \tag{36}$$

for $\bar{\lambda} > 0$ and $\alpha > 1$ from Lemma A.3.

Denote

$$\bar{\theta}(t,x) := \frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x) + \lambda\right]$$

defined for $t \in [-5/\varepsilon, 0]$ and

$$x \in \Omega_\varepsilon := \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$$

and denote

$$\phi(x) := \left(|x|^{1/4} - 2^{1/4}\right)_+ .$$

We already proved in Lemma 6.1 that $\theta \le 2 - 2^{-k_0}$ for $t \in [-1,0]$ and $x \in \Omega \cap B_1(\Gamma)$. On this same set, $\theta \ge -2$ by assumption. For $\bar{\theta}$, this means that when $t \in [-1/\varepsilon, 0]$ and $x \in \Omega \cap B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$,

$$\begin{cases} \bar{\theta}(t,x) & \le \frac{2}{2-\lambda}\left[2 - 2^{-k_0} + \lambda\right] \le \frac{2}{2-\lambda}\left[2-\lambda\right] = 2. \\ \bar{\theta}(t,x) & \ge \frac{2}{2-\lambda}\left[-2+\lambda\right] = -2. \end{cases} \tag{37}$$

Similarly, the bound (34) on θ becomes the equivalent bounds on $\bar{\theta}$, for all $(t,x) \in [-5/\varepsilon, 0] \times \Omega_\varepsilon$

$$\bar{\theta}(t,x) \le \frac{2}{2-\lambda}\left[2 + \lambda + 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right] \tag{38}$$

and

$$\bar{\theta}(t,x) \ge \frac{2}{2-\lambda}\left[-2 + \lambda - 2^{-k_0}\phi(|\varepsilon x - \Gamma(\varepsilon t)|)\right]. \tag{39}$$

It remains to show that these bounds (37), (38), and (39) on $\bar{\theta}$ imply the bound stipulated by the proposition.

Let $t \in [-5,0]$ and $x \in \Omega_\varepsilon$, and define

$$y := x - \varepsilon^{-1}\Gamma(\varepsilon t).$$

From (38) and the assumptions (36), we can bound

$$\begin{aligned} \bar{\theta}(t,x) &\le \frac{2}{2-\lambda}\left[2 + \lambda + 2^{-k_0}\phi(\varepsilon|y|)\right] \\ &\le 2 + 2^{-k_0}\bar{\lambda} + 2^{-k_0}\alpha\phi(\varepsilon|y|) \\ &= 2 + 2^{-k_0}\left[\bar{\lambda} + \alpha\phi(\varepsilon|y|)\right]. \end{aligned}$$

From (39) and the assumptions (36), we can bound

$$
\begin{aligned}
-\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2-\lambda+2^{-k_0}\phi(\varepsilon|y|)\right] \\
&\leq 2+2^{-k_0}\alpha\phi(\varepsilon|y|) \\
&\leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right].
\end{aligned}
$$

Therefore

$$
\left|\bar{\theta}(t,x)\right| \leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right]. \tag{40}
$$

If $|y| \leq \varepsilon^{-1}$ then from (37) we have

$$
\left|\bar{\theta}(t,x)\right| \leq 2 \leq 2+2^{-k_0}\phi(x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t))
$$

and the proof would be complete. Therefore assume without loss of generality that $|y| \geq \varepsilon^{-1}$. In this case we can apply Lemma A.3 which states that, since $\varepsilon < 1/2$ and $\varepsilon|y| \geq 1$, it is a property of ϕ, α, and $\bar{\lambda}$ that

$$
2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right] \leq 2+2^{-k_0}\left[\phi(|y|-\varepsilon^{-1}+3)\right].
$$

For $t \in [-5,0]$, we have by assumption (35)

$$
|y|-\varepsilon^{-1}+3 \leq |y|-5C_g \leq |y-\varepsilon^{-1}\gamma(\varepsilon t)|.
$$

The estimate (40) becomes

$$
\left|\bar{\theta}(t,x)\right| \leq 2+2^{-k_0}\phi(|x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t)|).
$$

This concludes the proof. □

7. Hölder Continuity

In this section we shall prove the main theorem, Theorem 1.1. We begin with a final lemma to describe the scaling properties of (2).

Lemma 7.1 (Scaling)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\alpha}$ boundary. Suppose that $\theta : [-T,0]\times\Omega \to \mathbb{R}$ and $u : [-T,0]\times\Omega\to\mathbb{R}^2$ solve* (2) *and u satisfies*

$$
u = \sum_{j=j_0}^{\infty} u_j
$$

with that sum converging in $L^2(\Omega)$ and $(u_j)_j$ calibrated with constant κ and center N. Suppose that on Ω the functions $K_{1/4}$ and K_1 (defined in Proposition 2.1*) satisfy the relation*

$$
K_{1/4}(x,y) \leq C_\Omega|x-y|^{3/4}K_1(x,y) \qquad \forall x \neq y \in \Omega. \tag{41}
$$

Let $\varepsilon > 0$ be a small constant.

Then

$$
\bar{\theta}(t,x) := \theta(\varepsilon t, \varepsilon x)
$$

and

$$
\bar{u}(t,x) := \sum_{j=j_0}^{\infty} u_j(\varepsilon t, \varepsilon x)
$$

satisfies the same PDE on $[-T/\varepsilon, 0]\times\Omega_\varepsilon$ where $\Omega_\varepsilon = \{x\in\mathbb{R}^2 : \varepsilon x \in \Omega\}$.

Moreover, $(u_j)_j$ is calibrated with the same constant κ but with center $N-\log_2(\varepsilon)$, and the estimate

$$
\bar{K}_{1/4}(x,y) \leq C_\Omega|x-y|^{3/4}\bar{K}_1(x,y) \qquad \forall x \neq y \in \Omega_\varepsilon
$$

holds.

Proof. Denote by $\bar{\Lambda}$ the square root of the Laplacian with Dirichlet boundary conditions on Ω_ε. One can calculate (see e.g. [CS16] Section 2.4) that for $(t,x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$

$$\Lambda\theta(\varepsilon t, \varepsilon x) = \varepsilon \bar{\Lambda}\bar{\theta}(t,x).$$

Similarly, in the Caffarelli-Stinga representation from Proposition 2.1 the operator $\bar{\Lambda}^s$ will have kernel

$$\bar{K}_s(x,y) = \varepsilon^s K_s(\varepsilon x, \varepsilon y).$$

From these facts it is clear that the scaled functions satisfy (2) and (41).

Define

$$\bar{u}_j(t,x) := u_j(\varepsilon t, \varepsilon x).$$

To show that $(\bar{u}_j)_{j\in\mathbb{Z}}$ is calibrated, we must translate the various bounds on u_j to corresponding bounds on $\bar{u}_j$. Each of the calculations are similar, so we show only one:

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \le 2^{\log_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N-\log_2(\varepsilon))}\kappa.$$

□

Proof of Theorem 1.1. By Proposition 3.1, the L^∞ norm of θ, after a short time, will be bounded by $\|\theta_0\|_{L^2}$. By translating and scaling, it will be sufficient to assume that θ solving (1) satisfies

$$\|\theta\|_{L^\infty([-5,0]\times\Omega)} \le 2$$

and prove that θ is Hölder continuous at the origin $(0,0) \in [-5,0] \times \bar{\Omega}$, meaning

$$\frac{|\theta(t,x) - \theta(0,0)|}{(|t|^2 + |x|^2)^{\alpha/2}} \le C$$

for all $(t,x) \in [-5,0] \times \Omega$ and some constants α and C depending on Ω.

From Proposition 4.1, we know that

$$u = \nabla^\perp \Lambda^{-1}\theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence $(u_j)_{j\ge j_0}$ of divergence-free functions calibrated with some constant $\kappa = \kappa(\Omega)$ and center 0.

Choose a constant $0 < \varepsilon < 1/5$ such that

$$5\max\left(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1-j_0)\kappa\right) \le \varepsilon^{-1} - 3, \tag{42}$$

For notational convenience, denote

$$\sum_k = \sum_{j > -k\log_2(\varepsilon)}, \qquad \sum^k = \sum_{j \le -k\log_2(\varepsilon)}.$$

For integers $k \ge 0$ consider the domains

$$\Omega_k := \{x \in \mathbb{R}^2 : \varepsilon^k x \in \Omega\}$$

and define the following functions on $[-5,0] \times \Omega_k$:

$$u_\ell^k(t,x) := \sum^k u_j(\varepsilon^k t, \varepsilon^k x),$$
$$u_h^k(t,x) := \sum_k u_j(\varepsilon^k t, \varepsilon^k x).$$

By Lemmas 7.1 and 5.1, we know the sequence $(u_j(\varepsilon^k\cdot, \varepsilon^k\cdot))_j$ is calibrated and hence that independently of k

$$\left\|\Lambda^{-1/4} u_h^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le 6\kappa$$

and

$$\left\|\nabla u_\ell^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le 2\kappa.$$

Each u_ℓ^k is a finite sum of L^∞ functions, hence L^∞ itself, though not uniformly in k.

For $t \in [-5,0]$ and $k \ge 0$ define $\Gamma_k, \gamma_k : [-5,0] \to \mathbb{R}^2$ by the following ODEs:

$$\begin{aligned}
&\Gamma_0(t) := 0,\\
&\gamma_k(0) := 0,\\
&\dot\gamma_k(t) := u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot\Gamma_k(t)\\
&\Gamma_k(t) := \varepsilon^{-1}\gamma_{k-1}(\varepsilon t) + \varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t) + \cdots + \varepsilon^{-k}\gamma_0(\varepsilon^k t), \qquad k \ge 1.
\end{aligned}$$

Since each u_ℓ^k is L^∞ in space-time and Lipschitz in space, these γ_k exist by a variant of the Cauchy-Lipschitz theorem. For example, Theorem 3.7 of [?] proves existence and uniqueness in our case. Moreover, since u_ℓ^k is a vector field which is tangential to the boundary of Ω_k and it has unique flows, the path $\Gamma_k + \gamma_k$ which follows this vector field must remain inside Ω_k for all time and so our expressions remain well-defined.

By construction, for $k \ge 0$ we have $\Gamma_{k+1}(t) = \varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)$. Therefore

$$\begin{aligned}
\dot\Gamma_{k+1}(t) &= \partial_t\left[\varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)\right]\\
&= \dot\gamma_k(\varepsilon t) + \dot\Gamma_k(\varepsilon t)\\
&= u_\ell^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t))\\
&= u_\ell^k(\varepsilon t, \varepsilon\Gamma_{k+1}(t)).
\end{aligned}$$

With this in hand, we can bound the size of γ_k. Namely, for $k \ge 1$,

$$\begin{aligned}
\dot\gamma_k(t) &= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot\Gamma_k(t)\\
&= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - u_\ell^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t))\\
&= \sum^k u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t))\\
&= \sum^{k-1}\left[u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t) + \varepsilon^k\gamma_k(t)) - u_j(\varepsilon^k t, \varepsilon^k\Gamma_k(t))\right] + \sum_{k-1}^k u_j(\varepsilon^k t, \varepsilon^k \ldots)\\
&= \left[u_\ell^{k-1}\big(\varepsilon t, \varepsilon\Gamma_k(t) + \varepsilon\gamma_k(t)\big) - u_\ell^{k-1}(\varepsilon t, \varepsilon\Gamma_k(t))\right] + \sum_{k-1}^k u_j(\varepsilon^k t, \varepsilon^k \ldots).
\end{aligned}$$

The function $x \mapsto u_\ell^{k-1}(\varepsilon t, \varepsilon x)$ is Lipschitz, with Lipschitz constant less than $2\varepsilon\kappa$. Moreover, each u_j has $\|u_j\|_\infty \le \kappa$. Thus from the above calculation we can bound

$$|\dot\gamma_k(t)| \le 2\varepsilon\kappa|\gamma_k(t)| - \kappa\log_2(\varepsilon). \tag{43}$$

Applying Gronwall's inequality, we find that for $t \in [-5,0]$

$$|\gamma_k(t)| \le \frac{-\log_2(\varepsilon)}{2\varepsilon}\left(e^{10\varepsilon\kappa} - 1\right).$$

Plugging this estimate back into (43),

$$|\dot\gamma_k(t)| \le -\kappa\log_2(\varepsilon)e^{10\varepsilon\kappa} \qquad \forall k \ge 1.$$

Trivially $|\dot\gamma_0| \le (j_0 - 1)\kappa$, so if we define

$$C_g = \max\left(-\kappa\log_2(\varepsilon)e^{10\varepsilon\kappa}, (j_0-1)\kappa\right)$$

then for all $k \ge 0$ and $t \in [-5,0]$

$$|\dot\gamma_k(t)| \le C_g.$$

Let us now produce a sequence of solutions θ_k. Define

$$\theta_0(t,x) := \theta(t,x)$$

and for each $k \geq 0$, if $|\{\theta_k \leq 0\} \cap [-5,0] \times B_4(\Gamma_k(t))| \geq 2|B_4|$ then set

$$\theta_{k+1}(t,x) := \frac{2}{2-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) + \lambda\right].$$

Otherwise, set

$$\theta_{k+1}(t,x) := \frac{1}{1-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) - \lambda\right].$$

From Lemma 7.1, we know that θ_k and the calibrated sequence $(u_j(\varepsilon^k \cdot, \varepsilon^k \cdot))_j$ solve (2).

We will now show that

$$|\theta_k| \leq 2 + 2^{-k_0}\left(|x - \Gamma_k(t)|^{1/4} - 2^{1/4}\right)_+ \tag{44}$$

holds for all $k \geq 0$.

Since $|\theta_0| \leq 2$ by assumption, we know in particular that (44) holds at $k = 0$.

This is sufficient for us to apply Lemma 6.2 to each θ_k (or to $-\theta_k$ as appropriate) in order. We conclude that (44) holds for all $k \geq 0$.

Each θ_k is between -2 and 2 on $[-5,0] \times B_2(\Gamma_k)$. But recall that each Γ_k is Lipschitz with constant kC_g. Thus $|\Gamma_k(t)| \leq 1$ for $t \in [-(kC_g)^{-1}, 0]$. On that time interval,

$$|\theta_k(t,x)| \leq 2 \qquad \forall x \in B_1(0).$$

We conclude that

$$\left| \sup_{[-\varepsilon^k (kC_g)^{-1},0]\times B_{\varepsilon^k}(0)} \theta(t,x) - \inf_{[-\varepsilon^k (kC_g)^{-1},0]\times B_{\varepsilon^k}(0)} \theta(t,x) \right| \leq 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In particular, for some positive constant C such that

$$\varepsilon^{Ck} \leq (kC_g)^{-1} \qquad \forall k \geq 0,$$

we can say that

$$|t|^2 + |x|^2 \leq \varepsilon^{(1+C)k}$$

implies that $(t,x) \in [-\varepsilon^k (kC_g)^{-1}, 0] \times B_{\varepsilon^k}(0)$ which in turn implies that

$$|\theta(t,x) - \theta(0,0)| \leq 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In other words,

$$\begin{aligned}
|\theta(t,x) - \theta(0,0)| &\leq 4\left(\frac{2}{2-\lambda}\right)^{-\frac{1}{1+C}\log_\varepsilon(|t|^2 - |x|^2)+1} \\
&= 4\left(\frac{2}{2-\lambda}\right)\exp\left[\ln\left(\frac{2}{2-\lambda}\right)\frac{\ln(|t|^2+|x|^2)}{-(1+C)\ln(\varepsilon)}\right] \\
&= \frac{8}{2-\lambda}(|t|^2+|x|^2)^{-\frac{\ln(2)-\ln(2-\lambda)}{(1+C)\ln(\varepsilon)}}.
\end{aligned}$$

□

Appendix A. Technical Lemmas

In this appendix we state and prove a few technical lemmas.

Lemma A.1. *Let $\alpha \in (0,1)$. There exists a constant $C = C(\alpha)$ such that, for any set Ω and any $f \in C^{0,1}(\Omega)$,*

$$[f]_\alpha \le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha .$$

Proof. This simple lemma is a straightforward calculation:

$$\begin{aligned}\sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} &= \sup |f(x)-f(y)|^{1-\alpha} \left(\frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha \\ &\le (2\|f\|_\infty)^{1-\alpha} \left(\sup \frac{|f(x)-f(y)|}{|x-y|}\right)^\alpha \\ &\le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha .\end{aligned}$$

□

Lemma A.2. *Let $\alpha \in (0,1)$ and Ω a set that satisfies the cone condition. There exist constants $C = C(\alpha,\Omega)$ and $\ell = \ell(\Omega)$ such that, for any $f \in C^{2,\alpha}(\Omega)$*

$$\left\|D^2 f\right\|_\infty \le C\left(\delta^{-1}\|\nabla f\|_\infty + \delta^\alpha \left[D^2 f\right]_\alpha\right)$$

for all $\delta < \ell$.

Proof. We will prove the stronger claim that for $f \in C^{1,\alpha}$

$$\|\nabla f\|_\infty \le C\delta^{-1}\|f\|_{L^\infty(\bar\Omega)} + \delta^\alpha [\nabla f]_{\alpha;\bar\Omega} .$$

The idea of the proof is to average ∇f along an interval of length δ with endpoint x. The magnitude of the average will be small, since $f \in L^\infty$, and the average will differ not very much from $\nabla f(x)$ since $\nabla f \in C^{1,\alpha}$.

Since Ω satisfies the cone condition, there exist positive constants ℓ and $a < 1$ such that, at each point $x \in \bar\Omega$, there exist two unit vectors e_1 and e_2 such that $|e_1 \cdot e_2| \le a$ and $x + \tau e_i \in \Omega$ for $i = 1,2$ and $0 < \tau \le \ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

Consider the directional derivative $\partial_i f$ of f along the direction e_i, and observe that for any $0 < \delta \le \ell$,

$$\left|\int_0^\delta \partial_i f(x+\tau e_i)\,d\tau\right| = |f(x+\delta e_i) - f(x)| \le 2\|f\|_\infty . \tag{45}$$

On the other hand, $\partial_i f$ is continous so, for any $\tau \in (0,\ell]$,

$$|\partial_i f(x) - \partial_i f(x+\tau e_i)| \le [\nabla f]_\alpha \tau^\alpha .$$

From this, we obtain that

$$\int_0^\delta \partial_i f(x+\tau e_i)\,d\tau \le \int_0^\delta \left(\partial_i f(x) + [\nabla f]_\alpha \tau^\alpha\right) d\tau = \delta\partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below. Thus

$$\left|\delta\partial_i f(x) - \int_0^\delta \partial_i f(x+\tau e_i)\,d\tau\right| \le [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}.$$

Combining this bound with (45), we obtain

$$|\partial_i f(x)| \le \frac{2}{\delta}\|f\|_\infty + \frac{\delta^\alpha}{1+\alpha}[\nabla f]_\alpha .$$

This bound is independent of x and of $i = 1, 2$. Since $e_1 \cdot e_2 \le a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta \in (0, \ell]$,

$$\|\nabla f\|_\infty \le \frac{C}{1 - a^2} \left(\delta^{-1} \|f\|_\infty + \delta^\alpha [\nabla f]_\alpha \right).$$

□

Lemma A.3. *There exist constants $\bar{\lambda} > 0$ and $\alpha > 1$ such that, for any $0 < \varepsilon \le 1/2$ and any $z \ge 1$*

$$\left(|\varepsilon^{-1}(z-1) + 3|^{1/4} - 2^{1/4} \right)_+ - \alpha \left(|z|^{1/4} - 2^{1/4} \right)_+ \ge \bar{\lambda}.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon = 1/2$, that is to show

$$f_\alpha(z) := \left(|2z+1|^{1/4} - 2^{1/4} \right)_+ - \alpha \left(|z|^{1/4} - 2^{1/4} \right)_+ \ge \bar{\lambda}$$

for all $z \ge 1$. Note that $f_\alpha(z) \ge f_\beta(z)$ if $\alpha < \beta$.

For $z \ge 2$,

$$f_\alpha(z) = (2z+1)^{1/4} - 2^{1/4} - \alpha z^{1/4} + \alpha 2^{1/4} = z^{1/4} \left((2 + 1/z)^{1/4} - \alpha \right) + (\alpha - 1) 2^{1/4}.$$

For any $\alpha < 2^{1/4}$, clearly $f_\alpha(z)$ tends to ∞ as z increases. Therefore there exist N and $\alpha_0 > 1$ such that

$$f_\alpha(z) \ge 1 \qquad \forall z \ge N, \alpha \le \alpha_0.$$

We can decompose $f_\alpha(z) = g_1(z) - (\alpha - 1) g_2(z)$ where

$$g_1(z) := \left(|2z+1|^{1/4} - 2^{1/4} \right)_+ - \left(|z|^{1/4} - 2^{1/4} \right)_+,$$
$$g_2(z) := \left(|z|^{1/4} - 2^{1/4} \right)_+.$$

Note that g_1, g_2 are both continuous, and $g_1(z)$ is strictly positive for $z \ge 1$. Therefore we can take $\alpha \in (1, \alpha_0]$ small enough that

$$\alpha - 1 < \frac{\inf_{[1,N]} g_1}{\sup_{[1,N]} g_2}.$$

For this α, $f_\alpha(z)$ is strictly positive on the compact interval $[1, N]$, and $f_\alpha(z) \ge 1$ on $[N, \infty)$. Therefore $f_\alpha(z)$ has a positive lower bound $\bar{\lambda}$ for all $z \ge 1$. □

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(L. F. Stokols)
Department of Mathematics,
The University of Texas at Austin, Austin, TX 78712, USA
E-mail address: lstokols@math.utexas.edu

(A. F. Vasseur)
Department of Mathematics,
The University of Texas at Austin, Austin, TX 78712, USA
E-mail address: vasseur@math.utexas.edu