SQG BOUNDARY, DRAFT 1

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We're gonna consider the equation

$$\partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, u = \nabla^\perp \Lambda^{-1} \theta. \tag{1}$$

Here the operator

$$\Lambda := \sqrt{-\Delta_D}$$

where Δ_D is the Laplacian with Dirichlet boundary condition.

We're going to linearize the equation by fixing u independent of θ. What property do we want u to have? For some constant κ, we'll want

$$u = \sum_{j \in \mathbb{Z}} u_j,$$
$$\left\| \Lambda^{-1/4} u_j \right\|_\infty \leq \kappa 2^{-j/4},$$
$$\left\| \nabla u_j \right\|_\infty \leq \kappa 2^j.$$

The convergence of that sum is in, say, weak L^2.

1. Lemmas

Lemma 1.1. *If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then*

$$\int \Lambda^s f \Lambda^s g \, dx \leq 0.$$

This proves, in particular, that $-\int \theta_+ \Lambda \theta_-$ is a positive term (hence dissipational and extraneous) and that $\int \Lambda^{1/2}(\theta - \psi)\Lambda^{1/2}(\theta - \psi)$ breaks down (bilinearly) into the doubly positive, the doubly negative, and the cross term, all of which are positive and hence each of which is positive.

Proof. Use the characterization from Caffarelli-Stinga. There exist non-negative functions $K(x, y)$ and $B(x)$, depending on the parameter s, such that

$$\int \Lambda^s f \Lambda^s g \, dx = \iint [f(x) - f(y)][g(x) - g(y)] K(x, y) \, dx dy + \int f(x) g(x) B(x) \, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x) - f(y)][g(x) - g(y)] = -f(x)g(y) - f(y)g(x) \leq 0.$$

Since K is non-negative, the result follows. □

Lemma 1.2. *For all functions f in H^1_D,*

$$\int |\nabla f|^2 = \int |\Lambda f|^2.$$

Moreover, if $f \in H^1_D$ then $\operatorname{tr}(f) = 0$.

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Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω. Because Ω has Lipschitz boundary, and because $\eta_i \nabla \eta_j$ is smooth on Ω and countinuous and bounded on $\overline{\Omega}$ vanishing on the boundary, therefore

$$\int_\Omega \operatorname{div}(\eta_i \nabla \eta_j) = \int_{\partial\Omega} \eta_i \nabla \eta_j.$$

But $\eta_i \nabla \eta_j$ vanishes on the boundary, so the right hand side vanishes. Moreover, $\operatorname{div}(\eta_i \nabla \eta_j) = \nabla \eta_i \cdot \nabla \eta_j + \eta_i \Delta \eta_j$. Therefore

$$\int \nabla \eta_i \cdot \nabla \eta_j = -\int \eta_i \Delta \eta_j = \lambda_k \int \eta_i \eta_j.$$

Of course, the inner product of two eigenfunctions is 0 unless they are the same eigenfunction, in which case it is 1.

Consider a function $f = \sum f_k \eta_k$ which is an element of H^1_D, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned}
\int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla \eta_i \right) \left(\sum_j f_j \nabla \eta_j \right) \\
&= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j \\
&= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j.
\end{aligned}$$

Since this double-sum vanishes except on the diagonal, we see from [citation] that in fact

$$\|\nabla f\|_{L^2(\Omega)} = \|\Lambda f\|_{L^2(\Omega)}.$$

To see that $\operatorname{tr}(f)$ vanishes, note that $f = \sum_{k=0}^\infty f_k \eta_k$ and that each finite partial sum for this series satisfies the Dirichlet boundary condition. Since tr is a bounded operator on H^1, we need only show that this series is Cauchy in H^1, in which case its H^1 limit will exist and be equal to its L^2 limit which will be equal to f.

For each k,

$$\|f_k \eta_k\|_{H^1} \le C_{\text{Poincare}} f_k \|\nabla \eta_k\|_2 = C f_k \sqrt{\lambda_k}.$$

This sequence is ℓ^2 summable, since $f \in H^1_D$ by assumption. Therefore f, being an H^1 limit of functions with vanishing trace, also has vanishing trace. □

Lemma 1.3. *For any function f, and any $0 < s < 1$,*

$$\int |\Lambda^s f|^2 \simeq \int \left| (-\Delta)^{s/2} \bar{f} \right|^2.$$

Here $\bar{f}$ is the extension of f to $\mathbb{R}^2$ and $(-\Delta)^s$ is defined in the fourier sense.

Proof. Let g be any Schwarz function in $L^2(\mathbb{R}^2)$, and let f be a function in H^{s+1}_D. Let $E : H^1(\Omega) \to H^1(\mathbb{R}^2)$ be a bounded extension operator, where H^1 denotes the classical Sobolev space defined using the gradient. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f.$$

When $\Re(z) = 0$, then $\left\| (-\Delta)^{z/2} g \right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|\Lambda^s f\|_2$. Hence

$$\Phi(z) \le \|g\|_2 \|f\|_{H^s_D}.$$

When $\Re(z) = 1$, then $\left\| (-\Delta)^{(z-1)/2} g \right\|_2 = \|g\|_2$ and

$$\left\| (-\Delta)^{1/2} E \Lambda^{s-z} f \right\|_{L^2(\mathbb{R}^2)} = \|\nabla E \Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} \le \|E\| \|\nabla \Lambda^{s-z} f\|_{L^2(\Omega)}.$$

It remains to ask whether $\Lambda^{s-z}f$ is in H^1_D so that we can apply lemma [citation]. However, this is true based on our assumption $f \in H^{1+s}_D$, since the various powers of Λ all commute and form a semigroup. Ergo

$$\|\nabla\Lambda^{s-z}f\|_{L^2(\Omega)} = \|\Lambda\Lambda^{s-z}f\|_2 \le \|\Lambda^s f\|_2$$

and we can bound

$$\Phi(z) \le \|E\|\,\|g\|_2\,\|f\|_{H^s_D}.$$

Now we will bound the derivative of $\Phi(z)$. Specifically, compute the derivative in z of the integrand, for $0 < \Re(z) < 1$, and hope that it is integrable. To this end, we rewrite the integrand of Φ as

$$\mathcal{F}^{-1}\left(|\xi|^z\hat{g}\right) E\sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}\left(\ln(|\xi|)|\xi|^z\hat{g}\right) E\sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}\left(|\xi|^z\hat{g}\right) E\sum_k \frac{-1}{2}\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} f_k.$$

Since $0 < \Re(z) < 1$, $\ln(|\xi|)|\xi|$ is bounded as a multiplier operator from Schwarz functions to L^2. Moreover, $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C\lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Since $f \in H^{1+s}_D$ this sum converges in L^2, in fact in H^1_D. This makes our differentiated integrand a sum of two H^1 functions with compact support multiplied by two Schwarz functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \Re(z) < 1$.

This is sufficient now to apply the Hadamard three-lines lemma to our function Φ.

It follows that for any Schwarz function $g \in L^2(\mathbb{R}^n)$ and H^{s+1}_D function f,

$$\int_{\mathbb{R}^2}(-\Delta)^{s/2} gEf = \Phi(s) \le \|g\|_{L^2(\mathbb{R}^2)}\|f\|_{H^s_D}.$$

Since Schwarz functions are dense in $L^2(\mathbb{R}^2)$, this means by density that

$$\int\left|(-\Delta)^{s/2}Ef\right|^2 \le \int|\Lambda^s f|^2$$

or in other words it means that E is a bounded operator from H^s_D to H^s, at least on the subset $H^{s+1}_D \cap H^s_D$. It remains to extend this bound to the whole space by density.

We know from [citation] Caffarelli and Stinga that $\mathcal{D}(\Omega)$ is dense in H^s_D for all $0 \le s < 1$. In fact, this takes a bit of interpretation, so I ought to illucidate that this is because $H^s_D = H^s_0$ (the latter in the Slobodekij sense) for most s and at $s = 1/2$ we get the Lions-Magenes spaces which still has $\mathcal{D}(\Omega)$ dense.

Surely, right(?), test functions are all inside of H^{1+s}_D. I should meditate on this, but it must be true. □

2. De Giorgi Estimates

First let us derive an energy inequality.

We know a priori that $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^{1/2}_D(\Omega))$. Let $\psi:\Omega \to \mathbb{R}^+$ be a non-negative function in $H^{1/2}_D$ non-uniformly, and define $\theta = \theta_+ + \psi - \theta_-$. Since $\theta - \psi$ is in $H^{1/2}_D$, by the lemma above, both θ_+ and θ_- are in that space as well. In particular, our weak solution can eat θ_+.

We end up with

$$0 = \int \theta_+\left[\frac{d}{dt} + u\cdot\nabla + \Lambda\right](\theta_+ + \psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \theta_+ = (1/2)\frac{d}{dt}\int \theta_+^2 + (1/2)\int \operatorname{div} u\, \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2$$
$$= (1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2.$$

The ψ term produces important error terms:

$$\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \psi = \frac{d}{dt}\int \theta_+\psi + \int \theta_+ u \cdot \nabla\psi + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi.$$

Since $\theta_+ and \theta_-$ have disjoint support, the θ_- term is nonnegative by lemma [citation]:

$$\int \theta_+ \left[\frac{d}{dt} + u \cdot \nabla + \Lambda\right] \theta_- = (1/2)\int \theta_+ \partial_t \theta_- + \int \theta_+ u \cdot \theta_- + \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_-$$
$$= \int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- \le 0.$$

Put together, we arrive at

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \le \left|\iint \Lambda^{1/2}\theta_+ \Lambda^{1/2}\psi\right| + \left|\int \theta_+ u \cdot \nabla\psi\right|.$$

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int u \cdot \nabla \frac{\theta_+^2}{2} + \int \theta_+ u \cdot \nabla\psi - \int \theta_+ u \cdot \nabla\theta_- + \int \theta_+ \Lambda\theta = 0.$$

We break up the $\theta_+\Lambda\theta$ term into

$$\int \theta_+\Lambda\theta = \int \left|\Lambda^{1/2}\theta_+\right| + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\psi - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_-$$
$$= \int \left|\Lambda^{1/2}\theta_+\right|^2 + \iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) + \int \theta_+\psi B - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_-.$$

The θ_- term is non-negative by lemma [citation], and the B term is non-negative since $B \ge 0$, so we have the inequality

$$(1/2)\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \le \left|\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y)\right| + \left|\int \theta_+ u \cdot \nabla\psi\right|.$$

This integral is symmetric in x and y, and the integrand is only nonzero if one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) \le 2\chi_{\{\theta_+>0\}}(x)\left|[\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]\right| K(x,y).$$

Now we can break up this integral using the Peter-Paul variant of Hölder's inequality.

$$\iint [\theta_+(x) - \theta_+(y)][\psi(x) - \psi(y)]K(x,y) \le \varepsilon \int \left|\Lambda^{1/2}\theta_+\right|^2 + \frac{1}{\varepsilon}\iint \chi_{\{\theta_+>0\}}(x)[\psi(x) - \psi(y)]^2 K(x,y).$$

It remains to bound the quantity $[\psi(x) - \psi(y)]^2 K(x,y)$. By Caffarelli-Stinga theorem 2.4 [citation], there is a universal constant C such that

$$K(x,y) \le \frac{C}{|x-y|^3}.$$

The cutoff ψ is Lipschitz, and it grows at a rate $|x|^\gamma$. Therefore

$$[\psi(x) - \psi(y)]^2 K(x,y) \le |x-y|^{-1} \wedge |x-y|^{2\gamma-3}.$$

If $3 - 2\gamma > 2$ then this quantity is integrable. Thus

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \lesssim \int \theta_+ u \cdot \nabla\psi + \int \chi_{\{\theta_+>0\}}.$$

For the drift term, let's say that u is broken down into u_l and u_h (standing for high-pass and low-pass) and that they have the desired properties.

For the high-pass term, for each $i = 1, 2$,

$$\int (u_l)_i \theta_+ \partial_i \psi = \int \Lambda^{-1/4} u_l \Lambda^{1/4} (\theta_+ \nabla \psi) \leq \left(\int \left| \Lambda^{1/4} \theta_+ \nabla \psi \right|^2 \right)^{1/2} \leq$$

We're looking at $\int g \Lambda^{1/4} \theta_+$ where $\Lambda^{1/4} g = u_h$ and $g \in L^\infty$. This breaks down as

$$\iint [g - g^*][\theta_+ - \theta_+^*] K_{1/4} + \int g \theta_+ B_{1/4}.$$

For a given parameter λ, break up into the region where $|x - y|$ is bigger and smaller than λ. Considering the bigger part,

$$\iint_{\geq \lambda} [g - g^*][\theta_+ - \theta_+^*] K_{1/4} \leq 2(2 \|g\|_\infty) \iint_{\geq \lambda} |\theta_+ - \theta_+^*| \frac{dx dx^*}{|x - y|^{2+1/4}} \leq 8 \lambda^{-2.25} \|g\|_\infty \|\theta_+\|_1 .$$

The B part may be a real problem. The g means it doesn't have a sign, so we actually have to bound it. Consider this.

$$\begin{aligned} \int g \theta_k B_{1/4} &\leq \|g\|_\infty \int \theta_{k-1} \theta_k B_{1/4} \\ &\leq \|g\|_\infty \int \theta_{k-1}^2 B_{1/4} \\ &\leq \|g\|_\infty \int \left| \Lambda^{1/8} \theta_{k-1} \right|^2 . \end{aligned}$$

Lastly, we have the near part.

$$\begin{aligned} \iint_{<\lambda} [g - g^*][\theta_+ - \theta_+^*] K_{1/4} &\leq 2 \iint_{<\lambda} \chi_+ [g - g^*]^2 |x - y|^{3/4} K_{1/4} + \iint_{<\lambda} [\theta_+ - \theta_+^*]^2 |x - y|^{-3/4} K_{1/4} \\ &\leq 4 \|g\|_\infty^2 \int \chi_+ \int_{<\lambda} |x - y|^{-1.5} + \iint_{<\lambda} [\theta_+ - \theta_+^*]^2 K_1 \\ &\leq \|g\|_\infty^2 \lambda^{1/2} |\chi_+| + \int \left| \Lambda^{1/2} \theta_+ \right|^2 \end{aligned}$$

Taken together,

$$\int \Lambda^{-1/4} \theta_+ u_h \cdot \nabla \psi \leq \frac{1}{2} \int \left| \Lambda^{1/2} \theta_+ \nabla \psi \right| + \int \chi_+ + \int \theta_+ + \int \theta_+ \nabla \psi B_{1/4}.$$

3. Control on u

Let's assume that our drift term is a sum of u_j for $j \in \mathbb{Z}$ and a constant which is tbd. Assume that each u_j is an L^∞ function, and that their sum converges to u in $L^2(\Omega)$, and that each u_j specfies the bounds as stated. First we show that they sum up to u_h and u_l in the ways desired. Then we show that the properties required are maintained as we zoom. Then at last we argue that, before any zooming, u really does have this property.

Firstly, assume that

$$u = \lim_{L^2} \sum_{-N}^{N} u_j,$$

$$\left\| \Lambda^{-1/4} u_j \right\|_\infty \leq \kappa 2^{-j/4},$$

$$\|\nabla u_j\|_\infty \leq \kappa 2^j.$$

We then define

$$u_h = \sum_{j=0}^{\infty} u_j$$

and

$$u_\ell = \sum_{-\infty}^{j=-1} u_j.$$

Since $u_j \in L^\infty$ in particular they are L^2 functions which sum in L^2. Remember that only finitely many negative j have $u_j \neq 0$. The sequence u_j is thus singly infinite and in particular is a Cauchy sequence, so u_h also converges in L^2. Since $\Lambda^{-1/4}$ is a continuous linear operator, it passes to the partial sums and so

$$\Lambda^{-1/4}u = \lim_{L^2} \sum_{j=0}^{\infty} \Lambda^{-1/4} u_j.$$

In particular, the sum converges in the sense of distributions, i.e. in $\mathcal{D}(\Omega)'$. Since test functions are dense in $L^1(\Omega)$, and the partial sums are uniformly bounded in the dual of $L^1(\Omega)$ (namely $L^\infty(\Omega)$), therefore the limit $\Lambda^{-1/4}u_h$ is also bounded in the dual of $L^1(\Omega)$.

$$\left\|\Lambda^{1/4}u\right\|_\infty \leq \sum_{j=0}^{\infty} \left\|\Lambda^{-1/4}u_j\right\|_\infty \leq \kappa \frac{1}{1-2^{-1/4}}.$$

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