HOLDER REGULARITY UP TO THE BOUNDARY FOR CRITICAL SQG ON BOUNDED DOMAINS

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ABSTRACT. We consider the dissipative SQG equation in bounded domains, first introduced by Constantin and Ignatova in 2016. We show global Holder regularity up to the boundary of the solution, with a method based on the De Giorgi techniques. The boundary introduces several difficulties. In particular, the Dirichlet Laplacian is not translation invariant near the boundary, which leads to complications involving the Riesz transform.

CONTENTS

1. PRELIMINARIES

The surface quasigeostrophic equation (SQG) is a special case of the quasi-geostrophic system (QG) with uniform potential vorticity. The QG model is used extensively in meteorology and oceanography (e.g. Charney [Cha71]). These models are described in Pedlosky [Ped92]. The SQG model was popularized by Constantin, Majda and Tabak in [CMT94], due to its similarities with the Euler and Navier-Stokes equation. They proposed it as a toy model for the study of 3D Fluid equations (see also Held, Garner, Pierrehumbert, and Swanson [HPGS95]).

We consider in this paper critical SQG on a bounded domain. We will focus on the the following model, which was introduced by Constantin and Ignatova in [CI17] and [CI16]. Consider Ω a connected bounded domain in $\mathbb{R}^2$ with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and the Laplacian with homogeneous Dirichlet boundary conditions $-\Delta_D$. If $(\eta_k)_{k\in\mathbb{N}}$ is the sequence of L^2-normalized

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eigenfunctions of $-\Delta_D$ with corresponding eigenvalues λ_k listed in non-decreasing order, define

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

The critical SQG problem on Ω with initial data $\theta_0 \in L^2(\Omega)$ is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0 & (0,T) \times \Omega, \\ u = \nabla^\perp \Lambda^{-1} \theta & [0,T] \times \Omega, \\ \theta = \theta_0 & \{0\} \times \Omega. \end{cases} \tag{1}$$

In the model, the dissipation $\Lambda = (-\Delta_D)^{1/2}$ is due to the Ekman pumping, while the nonlinear velocity u comes from the geostrophic and hydrostatic balance (see [Ped92]).

The main result of this paper is the following:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, and let $\bar{\Omega}$ denote the closure of Ω.*

For any $\theta_0 \in L^2(\Omega)$, there exists a global-in-time weak solution $\theta \in C^\alpha((0,\infty) \times \bar{\Omega})$ to (1) which verifies $\theta(t,x) = 0$ on $(0,\infty) \times \partial\Omega$ and verifies $\lim_{t \to 0} \theta(t,\cdot) = \theta_0$ in the L^2-weak sense.
Moreover, there exists a universal constant C_1 such that, for any time $t > 0$,

$$\|\theta\|_{L^\infty([t,\infty)\times\bar{\Omega})} \leq \frac{C_1}{t} \|\theta_0\|_{L^2(\Omega)}$$

and there exists a constant $\alpha \in (0,1)$ depending only on t, Ω, and $\|\theta_0\|_{L^2(\Omega)}$ and a constant C_2 depending on t and Ω such that

$$\|\theta\|_{C^\alpha([t,\infty)\times\bar{\Omega})} \leq C \|\theta_0\|_{L^2(\Omega)}.$$

This model was first thoroughly studied in the cases without boundaries (either $\mathbb{R}^2$ or the torus $\mathbb{T}^2$). Global weak solutions were first constructed in Resnick [Res95]. Global regularity was first shown with small initial values by Constantin, Cordoba, and Wu [CCW01], or extra C^α regularity on the velocity in Constantin and Wu [CW08] and Dong and Pavlović [DP09]. In [KNV07], Kiselev, Nazarov and Volberg showed the propagation of C^∞ regularity. The global C^∞ regularity for any L^2 initial values was first proved in [CV10] (see also Kiselev and Nazarov [KN09] and Constantin and Vicol [CV12]).

In the presence of boundaries, there are several distinct ways to define SQG. This can be attributed to alternative generalizations of the fractional Laplacian. Kriventsov [Kri15] considered a two-phase problem which satisfies critical SQG only in part of the domain, and was able to prove Hölder regularity in the time-independent case. This problem, intended to model air currents over a region containing both land and water, contains a half-Laplacian and a Riesz transform defined, not spectrally, but in terms of extension. In [NV18b], the authors consider the Euler-Coriolis-Boussinesq model and derive the full 3D inviscid quasigeostrophic system in an impermeable cylinder (see also [NV19] for the construction of small time smooth solutions to the model). They obtain natural boundary conditions for SQG distinct from the homogeneous conditions introduced in [CI17], [CI16] and described above. However, due to the complexity of the model described in [NV18b], we focus in this paper only on the homogenous case.

Existence of weak solutions for (1) is proven in [CI17], and local existence and uniqueness for strong solutions with sufficiently smooth initial data is proven by Constantin and Nguyen in [CN18b] (see also Constantin and Nguyen [CN18a] and Constantin, Ignatova, and Nguyen [CIN18] for the inviscid case). The interior regularity of solutions is proven in [CI16] (together with propagation of L^∞ bounds). The method of proof for interior regularity uses nonlinear maximum principles, introduced by Constantin and Vicol [CV12]. However, the bounds obtained in [CI16] blow up near the boundary and do not provide global regularity. In [CI16] Remark 1, questions about global regularity are suggested as open problems. Both the $C^\alpha(\bar{\Omega})$ regularity, and bootstrapping to $C^\infty(\bar{\Omega})$ regularity, are indentified as interesting problems. Our result answers the first question, by showing that solutions θ to (1) are globally Hölder continuous. Bootstrapping to C^∞ involves different techniques, and will be studied in a forthcoming work [SV].

Our proof is based on the De Giorgi method pioneered by De Giorgi in [DG57]. The method was applied to the SQG problem first in [CV10]. The method is powerful for showing C^α regularity of elliptic- and parabolic-type equations. It has been applied in a variety of situations for non-local problems, such as the fractional heat equation in [CCV11], the time-fractional case in [ACV16], the 3D Quasigeostrophic problem in [NV18a], or the kinetic setting by Imbert and Silvestre [IS16] or in [Sto18]. The method has also been applied in more exotic, non-elliptic situations such as Hamilton-Jacobi equations (see [CV17], [SV18]).

The De Giorgi method involves rescaling our equation by zooming in iteratively, and applying regularity results at each scale. Therefore it is important that certain results be proven independently of the domain Ω. The particular dependence on Ω will be made clear in each lemma of this paper. As a general overview, in the proof of Theorem 1.1 we will apply the results of Sections 3 and 4 only on a single fixed domain, while the results of Sections 5 and 6 must be applied at each level of zoom with a different rescaled domain each time.

The first broad idea of our proof consists in decoupling the velocity u from θ to work on a linear equation, and prove alternating regularity results for θ and u independently. We can show that θ is in L^∞ without any assumption on u (see Section 3). Using that L^∞ bound, we will need to obtain scaling invariant controls on the drift $u = \nabla\Lambda^{-1}\theta$. By scaling invariant, we mean that the bound, once proven on Ω fixed, will remain true of the scaled function $u(\varepsilon\,\cdot, \varepsilon\,\cdot)$ for all ε. Unfortunately, although the Riesz transform is bounded from L^p to L^p for all p finite, it is not bounded for $p = \infty$. The usual technique, therefore, is to consider BMO (as in [CV10] and [NV18a]), but in the case of bounded domains the Riesz transform is not known to be bounded in this space either. Our solution is to use extensions of the Littlewood-Paley theory to bounded domains.

The adaptation of Fourier analysis and Littlewood-Paley theory to Schrodinger operators is a well-studied subject (e.g. Zheng [Zhe06], Benedetto and Zheng [BZ10]). As an application of this theory, Iwabuchi, Matsuyama, and Taniguchi [IMT19], [IMT18], and Bui, Duong, and Yang [BDY12] have considered operators defined on open subsets of $\mathbb{R}^n$, which includes as a special case the operator $-\Delta_D$ (a Schrodinger operator with zero potential). In particular, in [IMT17], Iwabuchi, Matsuyama, and Taniguchi derive many important results, including the Bernstein inequalities, for Besov spaces adapted to the operator $-\Delta_D$ on bounded open subsets of $\mathbb{R}^n$ with smooth boundary. This theory turns out to greatly improve our understanding of the Riesz transform $\nabla\Lambda^{-1}$ on bounded domains.

Using the results of [IMT17], we will be able to show that the Riesz transform of an L^∞ function whose Fourier decomposition $f = \sum f_k \eta_k$ is supported on high frequencies $k > N$ will be bounded in the weak sobolev space $W^{-1/4,\infty}$, and the Riesz transform of an L^∞ function whose Fourier decomposition is supported on low frequencies $k < N$ will have bounded Lipschitz constant. The cutoff N for dividing high frequencies from low frequencies must depend however on the size of the domain Ω. In the case of $\mathbb{R}^2$, where ∇ and Λ^{-1} commute, this is equivalent to the observation that the Riesz transform is bounded from L^∞ to the Besov space $B^0_{\infty,\infty}$. In the case of bounded domains, the argument must be more subtle. We must decompose θ into its Littlewood-Paley projections, individually bound the Riesz transform of each projection in multiple spaces, and then recombine these infinitely-many functions into a low-frequency collection and a high-frequency collection depending on the scale of oscillation we are trying to detect (see Section 4 and Lemma 5.1).

We make this notion precise with the following definition:

Definition 1 (Calibrated sequence)**.** Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set and $0 < T \in \mathbb{R}$. We call a function $u \in L^2([0,T]\times\Omega)$ **calibrated** if it can be decomposed as the sum of a calibrated sequence

$$u = \sum_{j\in\mathbb{Z}} u_j$$

with each $u_j \in L^2([0,T]\times\Omega)$ and the infinite sum converging in the sense of L^2.

We call a sequence $(u_j)_{j\in\mathbb{Z}}$ **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\begin{aligned}\|u_j\|_{L^\infty([0,T]\times\Omega)} &\le \kappa,\\ \|\nabla u_j\|_{L^\infty([0,T]\times\Omega)} &\le 2^j 2^{-N}\kappa,\\ \left\|\Lambda^{-1/4}u_j\right\|_{L^\infty([0,T]\times\Omega)} &\le 2^{-j/4}2^{N/4}\kappa.\end{aligned}$$

In Section 7 we will show that a calibrated velocity remains calibrated at all scales (specifically, with fixed constant κ but a changing center N). Therefore we can consider, for any domain Ω and time T, the system of linear equations

$$\begin{cases}\partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, & [-T,0]\times\Omega\\ \operatorname{div} u = 0 & [-T,0]\times\Omega.\end{cases} \tag{2}$$

In Section 3 we show that solutions to (1) with L^2 initial data exist and regularize instantly into L^∞, and in Section 4 we show that the Riesz transform of L^∞ data is calibrated. Then in Sections 5 and 6 we will show that solutions to (2) with calibrated velocity have decreasing oscillation between scales. By iteratively applying this oscillation lemma and scaling our equation, we show in Section 7 that θ is Hölder continuous.

The low-freqency component of a calibrated velocity u will be uniformly Lipschitz, which means it is only bounded up to a constant. This is similar to the case of BMO velocity functions in [CV10] and [NV18a], which by the John-Nirenberg inequality are also bounded up to a constant. As in these cases, we consider a moving reference frame, denoted $\Gamma : [0,T] \to \mathbb{R}^2$, in which our velocity is shifted by a constant, making the low-frequency component of u bounded. There are two differences between our implementation of this technique and the implementation in [CV10] and [NV18a]: firstly, we subtract off the value of the low-frequency part of u at a point, rather than subtracting off the average of u on a ball. Secondly, rather than applying the standard De Giorgi argument to $\tilde\theta(t,x) := \theta(t, x+\Gamma(t))$, we must reformulate the De Giorgi argument to "follow" the path $\Gamma(t)$ explicitly. This is a purely notational difference, but it is necessary because otherwise Ω would be time-dependent.

At each scale, there will be a natural Lagrangian path Γ_ℓ corresponding to the low-frequency part of u. However, the low-frequency part of u changes non-trivially as we zoom, so Γ_ℓ will be different at each scale. Throughout Sections 5 and 6, we will use Γ_ℓ to denote the "current" Lagrangian path and Γ to denote the Lagrangian path at the previous scale. In the proof of Theorem 1.1 in Section 7, these are denoted $\Gamma_k(t)$ and $\varepsilon^{-1}\Gamma_{k-1}(\varepsilon t)$ respectively. In Lemmas 5.2, 5.3, 5.4 and 6.1, we will make assumptions about θ which are centered on $x \approx \Gamma(t)$ and obtain conclusions which are similarly centered on $x \approx \Gamma(t)$, conditioned on $\Gamma_\ell - \Gamma$ being small in Lipschitz norm. Finally in Lemma 6.2, we will show that, given bounds on θ for $x \approx \Gamma(t)$, we can bound θ for $x \approx \Gamma_\ell(t)$ for t sufficiently small, again conditioned on $\Gamma_\ell - \Gamma$ being small in Lipschitz norm.

Previous applications of the De Giorgi method to non-local equations such as (2) generally make extensive use of either an extension representation (c.f. [CV10]) or a singular integral representation (c.f. [NV18a]). In this paper, we use the singular integral representation for the Dirichlet fractional Laplacian derived by Caffarelli and Stinga [CS16]. It is based on the results of Stinga and Torrea [ST10] which generalize the extension representation of Caffarelli and Silvestre [CS07]. This theory is pivotal in translating the existing non-local De Giorgi techniques to the problem at hand (see Section 2).

In order to apply De Giorgi's method to weak solutions of (2), we will need to assume a certain a priori estimate which holds, in particular, for $L^2(H^1_0)$ weak solutions. However, such solutions are only known to exist for short time and for H^2 initial data, as shown by Constantin and Nguyen in [CN18b]. We call weak solutions in $L^2(\mathcal{H}^{1/2})$ which happen to verify this a priori estimate "suitable solutions," by analogy to suitable solutions to Navier-Stokes as in [cite]. We give the formal definition of suitable solutions in Section 3, where we also construct global-in-time suitable solutions using the vanishing viscosity method. These solutions are not a priori more regular than those constructed in [CI17], but because they verify an energy inequality we are able to apply the De Giorgi method.

The Paper is organized as follows. Section 2 is dedicated to basic properties of the operator Λ and the corresponding Sobolev spaces $\mathcal{H}^s$. In Section 3 we construct weak solutions which verify the suitability condition. In Section 4 we prove that the Riesz transform of the L^∞ function θ is callibrated. Section 5 contains the De Giorgi Lemmas. Section 6 is dedicated to the local decrease in oscillation through an analog of the Harnack inequality. Finally in Section 7 we prove the main theorem, Theorem 1.1. In the Appendix A we prove a few technical lemmas which are needed in the main paper.

Notation. Throughout the paper, we will use the following notations. By η_k and λ_k we mean the eigenfunctions and eigenvalues of $-\Delta_D$, with $\lambda_0 \le \lambda_1 \le \dots$ and $\|\eta_k\|_2 = 1$ for all k. If $f = \sum_k f_k \eta_k$ then

$$\begin{aligned}\|f\|_{\mathcal{H}^s} &:= \left(\sum_k \lambda_k^s f_k^2\right)^{1/2} \\ &= \int |\Lambda^s f|^2 .\end{aligned}$$

We suppress the dependence on Ω, though in fact Λ, λ_k, and η_k are defined in terms of the domain Ω. The relevant domain will be clear from context. The norm on $\mathcal{H}^s$ is in fact a norm, not a seminorm, since $\|f\|_{L^2(\Omega)} \le \lambda_0^{-s/2} \|f\|_{\mathcal{H}^s}$.

For a set A and a function $f : A \to \mathbb{R}$, denote

$$\begin{aligned}[f]_{\alpha;A} &:= \sup_{x,y\in A, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, && \alpha \in (0,1], \\ \|f\|_{C^\alpha(A)} &:= \|f\|_{L^\infty(A)} + [f]_{\alpha;A}, && \alpha \in (0,1], \\ \|f\|_{C^{k,\alpha}(A)} &:= \sum_{n=0}^{k} \|D^n f\|_{L^\infty(A)} + \left[D^k f\right]_{\alpha;A}, && \alpha \in (0,1], k \in \mathbb{N}.\end{aligned}$$

When the domain A is ommited, the relevant spatial domain Ω is implied.

We will use the notation $(x)_+ := \max(0,x)$. When the parentheses are ommited, the subscript $+$ is merely a label.

Throughout this paper, if an integral sign is written $\int$ without a specified domain, the domain is implied to be Ω, with Ω defined in context.

For any vector $v = (v_1, v_2)$, by $v^\perp$ we mean $(-v_2, v_1)$. By $\nabla^\perp$ we mean $(-\partial_y, \partial_x)$.

In the remainder of this paper, the differential operator D^2 refers to the Hessian in space, excluding time derivatives. The function space C_c^∞ consists of smooth functions with compact support.

The symbol C represents a constant which may change value each time it is written.

2. Properties of the Fractional Dirichlet Laplacian

In this section we will investigate the basic properties of the operator Λ and the space $\mathcal{H}^s$ on a general domain Ω.

We begin by stating a result of [CS16] which gives us a singular integral representation of the $\mathcal{H}^s$ norm.

Proposition 2.1 (Caffarelli-Stinga Representation). *Let $s \in (0,1)$ and $f, g \in \mathcal{H}^s$ on a bounded $C^{2,\beta}$ domain $\Omega \subseteq \mathbb{R}^2$. Then*

$$\int_\Omega \Lambda^s f \Lambda^s g \, dx = \iint_{\Omega^2} [f(x) - f(y)][g(x) - g(y)] K_{2s}(x,y) \, dx dy + \int_\Omega f(x) g(x) B_{2s}(x) \, dx$$

for kernels K_{2s} and B_{2s} which depend on the parameter s and the domain Ω.

There exists a constant $C = C(s)$ independent of Ω such that

$$0 \leq K_{2s}(x,y) \leq \frac{C(s)}{|x-y|^{2+2s}}$$

for all $x \neq y \in \Omega$ and

$$0 \leq B_{2s}(x)$$

for all $x \in \Omega$.

Moreover, for any $s, t \in (0,2)$ there exists a constant $c = c(s,t,\Omega)$ such that for all $x \neq y \in \Omega$

$$K_t(x,y) \leq c|x-y|^{s-t} K_s(x,y). \tag{3}$$

Proof. See [CS16] Theorems 2.3 and 2.4.

Theorem 2.4 in [CS16] does not explicitly state the result (3). However, it does state that for each kernel K_s there exists a constant c_s dependent on s and Ω such that

$$\frac{1}{c_s}|x-y|^{2+s} K_s(x,y) \leq \min\left(1, \frac{\eta_0(x)\eta_0(y)}{|x-y|^2}\right) \leq c_s |x-y|^{2+s} K_s(x,y).$$

Since the middle term does not depend on s, we can say that

$$|x-y|^{2+t} K_t(x,y) \leq c_t c_s |x-y|^{2+s} K_s(x,y)$$

from which (3) follows. □

Though the result is proven in [CS16] only for $f, g \in \mathcal{H}^s$, the result applies much more generality by a standard continuity argument.

From the explicit formulae given in [CS16], we see that K_{2s} is approximately equal to the standard kernel for the $\mathbb{R}^2$ fractional Laplacian $(-\Delta)^s$ when both x and y are in the interior of Ω or when x and y are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel B_{2s} is well-behaved in the interior but has a singularity at the boundary $\partial\Omega$. This justifies our thinking of the K_{2s} term as the interior term and B_{2s} as a boundary term.

When comparing the computations in this paper to corresponding computations on $\mathbb{R}^2$, one finds that the interior term behaves nearly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

Many useful results can be derived from Caffarelli-Stinga representation formula. We summarize them in the following lemma.

Lemma 2.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$.*

(a) Let $s \in (0,1)$. If f and g are non-negative functions with disjoint support (i.e. $f(x)g(x) = 0$ for all x), then

$$\int \Lambda^s f \Lambda^s g \, dx \leq 0.$$

(b) Let $s \in (0,1)$. If $g \in C^{0,1}(\Omega)$ then for some constant $C = C(s)$ independent of Ω

$$\|fg\|_{\mathcal{H}^s} \leq 2\|g\|_\infty \|f\|_{\mathcal{H}^s} + C\|f\|_2 \sup_y \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} \, dx.$$

(c) Let $s \in (0,1)$. If $g \in C^{0,1}(\Omega)$ then for some constant $C = C(s)$ independent of Ω

$$\|fg\|_{\mathcal{H}^s} \leq C\|g\|_{C^{0,1}(\Omega)} \left(\|f\|_2 + \|f\|_{\mathcal{H}^s}\right).$$

(d) Let $s \in (0,1/2)$. Let g an $L^\infty(\Omega)$ function and $f \in \mathcal{H}^{2s}$ be non-negative with compact support. Let C_{dmn} be a constant such that

$$K_s(x,y) \leq C_{dmn}|x-y|^{3s} K_{4s}(x,y). \tag{4}$$

Then there exists a constant C depending only on s and C_{dmn} such that

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}\right).$$

(e) Let g an $L^\infty(\Omega)$ function and $f \in \mathcal{H}^{1/2}$ be non-negative with compact support. Let C_{dmn} be a constant such that

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4} K_1(x,y).$$

Then there exists a constant C depending only on C_{dmn} such that

$$\int g \Lambda^{1/4} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} \left(\|f\|_2 + \|f\|_{\mathcal{H}^{1/2}}\right).$$

Proof. We prove these corollaries one at a time.

Proof of (a)**:** From Proposition 2.1

$$\int \Lambda^s f \Lambda^s g \, dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\, dxdy + \int f(x)g(x)B(x)\, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \leq 0.$$

Since K is non-negative, the result follows.

Proof of (b)**:** From Proposition 2.1

$$\begin{aligned}\int |\Lambda^s(fg)|^2 &= \iint \left(g(x)[f(x)-f(y)] + f(y)[g(x)-g(x)]\right)^2 K + \int f^2 g^2 B \\ &\leq 2\|g\|_\infty^2 \|f\|_{\mathcal{H}^s}^2 + C(s)\int f(y)^2 \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} \, dxdy.\end{aligned}$$

Proof of (c)**:** This follows immediately from (b), since

$$|g(x)-g(y)| \leq (\|g\|_\infty) \wedge (\|\nabla g\|_\infty |x-y|)$$

and

$$\int \frac{1 \wedge |x-y|^2}{|x-y|^{2+2s}} \, dx$$

is bounded uniformly in y.

Proof of (d)**:** From Proposition 2.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_{<} + I_{\geq} + II$$

where

$$I_{<} := \iint_{|x-y|<1} [g(x) - g(y)][f(x) - f(y)] K_s,$$
$$I_{\geq} := \iint_{|x-y|\geq 1} [g(x) - g(y)][f(x) - f(y)] K_s,$$
$$II := \int f g B_s.$$

First we estimate $I_{<}$. From (4) and from the symmetry of the integrand and the fact that $[f(x) - f(y)]$ vanishes unless at least one of $f(x)$ or $f(y)$ is non-zero,

$$|I_{<}| \leq 2 \iint_{|x-y|<1} \chi_{\{f>0\}}(x) \, |g(x) - g(y)| \cdot |f(x) - f(y)| \cdot |x-y|^{3s} K_{4s}.$$

We can break this up by Hölder's inequality

$$|I_{<}| \leq 2 \left(\iint_{|x-y|<1} \chi_{\{f>0\}}(x) [g(x) - g(y)]^2 |x-y|^{6s} K_{4s} \right)^{1/2} \left(\iint [f(x) - f(y)]^2 K_{4s} \right)^{1/2}.$$

The kernel $|x-y|^{6s} K_{4s} \chi_{\{|x-y|<1\}}$ is integrable in y for x fixed. Therefore

$$|I_{<}| \leq 2 \left((2 \|g\|_\infty)^2 \int C \chi_{\{f>0\}}(x) \, dx \right)^{1/2} \left(\|f\|^2_{\mathcal{H}^{2s}} \right)^{1/2}. \tag{5}$$

For the term $I_{\geq}$, by the symmetry of the integrand we have

$$|I_{\geq}| \leq 2 \|g\|_\infty 2 \int |f(x)| \int_{|x-y|\geq 1} K_s(x,y) dy \, dx.$$

Since $K_s \chi_{\{|x-y|\geq 1\}}$ is integrable in y for x fixed,

$$|I_{\geq}| \leq C \|g\|_\infty \|f\|_1 . \tag{6}$$

For the boundary term II,

$$|II| \leq \|g\|_\infty \int \chi_{\{f>0\}} f B_s.$$

Since $f \geq 0$, $[f(x) - f(y)][\chi_{\{f>0\}}(x) - \chi_{\{f>0\}}(y)] \geq 0$. Therefore

$$\int \chi_{\{f>0\}} f B_s \leq \int \Lambda^{s/2} \chi_{\{f>0\}} \Lambda^{s/2} f = \int \chi_{\{f>0\}} \Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \leq \|g\|_\infty |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s} .$$

This combined with (5) and (6) gives us

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C \|g\|_\infty \left(|\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^{2s}} + \|f\|_1 + |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s} \right).$$

The lemma follows since $\|f\|_1 \leq |\mathrm{supp}(f)|^{1/2} \|f\|_2$ and since $\|f\|_{\mathcal{H}^s} \leq \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$.

Proof of (e)**:** This is an immediate application of part (d).

□

Let us consider the relationship between the norm $\mathcal{H}^s$ and the H^s norm on $\mathbb{R}^2$.

It is known (see [CI16] and [CS16]) that for $s \in (0,1)$ the spaces $\mathcal{H}^s$ are equivalent to certain subsets of $H^s(\Omega)$ spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in $\mathcal{H}^s$ for $s \in [0,1]$ and that elements of $\mathcal{H}^s$ have trace zero for $s \in [\frac{1}{2}, 1]$.

The most important fact for us is that the fractional Sobolev norms defined in terms of extension are dominated by our $\mathcal{H}^s$ norm with a constant that is independent of Ω.

We do not claim that this result is new, but we present a detailed proof because the result is crucial to the De Giorgi method. The De Giorgi lemmas require Sobolev embeddings and Rellich-Kondrachov embeddings which are independent of scale.

Define the extension-by-zero operator $E : L^2(\Omega) \to L^2(\mathbb{R}^2)$

$$Ef(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Proposition 2.3. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. For any $s \in [0,1]$ and function $f \in \mathcal{H}^s$,*

$$\int_{\mathbb{R}^2} \left|(-\Delta)^{s/2} Ef\right|^2 \leq \int_\Omega |\Lambda^s f|^2 .$$

Here $(-\Delta)^s$ is defined in the fourier sense.

We will prove this proposition by interpolating between $s = 0$ and $s = 1$. Before we can do this, we must prove the same in the $s = 1$ case. This result is known (see e.g. Jerison and Kenig [JK95]) but we include the proof for completeness.

Lemma 2.4. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For all functions f in $\mathcal{H}^1$,*

$$\int_\Omega |\nabla f|^2 = \int_\Omega |\Lambda f|^2 .$$

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω and vanish at the boundary, so we can apply the divergence theorem and find

$$\int \nabla\eta_i \cdot \nabla\eta_j = -\int \eta_i \Delta\eta_j = \lambda_j \int \eta_i\eta_j = \lambda_j \delta_{i=j}.$$

Consider a function $f = \sum f_k \eta_k$ which is an element of $\mathcal{H}^1$, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla\eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\begin{aligned} \int |\nabla f|^2 &= \int \left(\sum_i f_i \nabla\eta_i\right)\left(\sum_j f_j \nabla\eta_j\right) \\ &= \int \sum_{i,j} (f_i f_j) \nabla\eta_i \cdot \nabla\eta_j \\ &= \sum_{i,j} (f_i f_j) \int \nabla\eta_i \cdot \nabla\eta_j \\ &= \sum_j \lambda_j f_j^2. \end{aligned}$$

From this the result follows. □

We come now to the proof of Proposition 2.3. The proof is by complex interpolations using the Hadamard three-lines theorem.

Proof. Let g be any Schwartz function in $L^2(\mathbb{R}^2)$, and let f be a function in $\mathcal{H}^s$. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} gE\Lambda^{s-z} f, \qquad z \in \mathbb{C}, \mathrm{Re}(z) \in [0,1].$$

Recall (see e.g. [JK95]) that when $t \in \mathbb{R}$, $(-\Delta)^{it}$ is a unitary transformation on $L^2(\mathbb{R}^2)$, and Λ^{it} is a unitary transformation on $L^2(\Omega)$.

When $\mathrm{Re}(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|f\|_{\mathcal{H}^s}$. Hence

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 0.$$

When $\mathrm{Re}(z) = 1$, integrate by parts to obtain

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{(z-1)/2} g (-\Delta)^{1/2} E\Lambda^{s-z} f.$$

Then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$, while $\|\Lambda^{s-z} f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^s}$. As an $\mathcal{H}^1$ function, $\Lambda^{s-z} f$ has trace zero so

$$\|\nabla E\Lambda^{s-z} f\|_{L^2(\mathbb{R}^2)} = \|\nabla\Lambda^{s-z} f\|_{L^2(\Omega)} = \|f\|_{\mathcal{H}^s}.$$

Of course $\left\|(-\Delta)^{1/2} \cdot\right\|_{L^2(\mathbb{R}^2)} = \|\nabla\cdot\|_{L^2(\mathbb{R}^2)}$ in general so

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 1.$$

In order to apply the Hadamard three-lines theorem, we must show that Φ is differentiable in the interior of its domain.

Rewrite the integrand of Φ as

$$\mathcal{F}^{-1}(|\xi|^z\hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}(\ln(|\xi|)|\xi|^z\hat{g}) E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}(|\xi|^z\hat{g}) E \sum_k \frac{-1}{2}\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} f_k. \tag{7}$$

Fix some $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in (0,1)$. Since g is a Schwartz function, $\ln(|\xi|)|\xi|^z\hat{g}$ is in L^2. Moreover, for any $\varepsilon > 0$ we have $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C\lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Take $\varepsilon < \mathrm{Re}(z)$ and, since $f \in \mathcal{H}^s$, this sum will converge in L^2.

The differentiated integrand (7) is therefore a sum of two products of L^2 functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \mathrm{Re}(z) < 1$.

By the Hadamard three-lines theorem, for any $z \in (0,1)$ we have $\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}$. Evaluating $\Phi(s)$, we see

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} gEf \le \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

This inequality holds for any Schwartz function $g \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{H}^s$.

Since Schwartz functions are dense in $L^2(\mathbb{R}^2)$ and $(-\Delta)^{s/2}$ is self-adoint, the proof is complete. $\square$

3. Existence of suitable solutions

In this section, we will define and then construct suitable solutions to (1).

We begin by calculating the energy inequalities that any sufficiently smooth weak solution to (2) will satisfy.

Proposition 3.1 (Energy Inequalities)**.** *There exists a universal constant C^* such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be bounded and open with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, and let $0 < S < T$ be times. Let θ, u be a solution to (2) on $\Omega \times [0,T]$, with $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$.

Then for any smooth non-negative function $\Psi \in C^\infty([0,\infty)\times\mathbb{R}^2)$ satisfying $\|\nabla\Psi\|_{L^\infty([0,\infty)\times\mathbb{R}^2)} \le k$ and the Hölder seminorm $\sup_{[0,\infty)}[\Psi(t,\cdot)]_{1/4;\mathbb{R}^2} \le k$ for some constant k and any smooth non-negative $\varphi \in C_c^\infty(S,T;C^\infty(\Omega))$, the function $\theta_+ := (\theta - \Psi)_+$ satisfies

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \le C\left(k^2\int \chi_{\{\theta_+\ge 0\}} + \left|\int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi)\right|\right) \tag{8}$$

for all $t \in [S,T]$, and

$$\begin{aligned}\frac{-1}{2}\int_S^T\!\!\int \theta_+^2\partial_t\varphi \le -\int_S^T\!\!\int \varphi\theta_+\left(\partial_t\Psi + u\cdot\nabla\Psi\right) + \frac{1}{2}\int_S^T\!\!\int \theta_+^2 u\cdot\nabla\varphi \\ - \int_S^T\!\!\iint_{\Omega\times\Omega}[\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)][\Psi(x)-\Psi(y)]K_1(x,y)\,dxdy \\ - \frac{1}{2}\int_S^T\!\!\iint_{\Omega\times\Omega}[\theta_+(x)^2 - \theta_+(y)^2][\varphi(x)-\varphi(y)]K_1(x,y)\,dxdy.\end{aligned} \tag{9}$$

Proof of Proposition 3.1. Since $\theta_+ \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, we can multiply (2) by θ_+ and integrate in space to obtain

$$0 = \int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right](\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\theta_+ &= \left(\frac{1}{2}\right)\frac{d}{dt}\int \theta_+^2 + \left(\frac{1}{2}\right)\int \operatorname{div} u\,\theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2 \\ &= \left(\frac{1}{2}\right)\frac{d}{dt}\int \theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2.\end{aligned}$$

The Ψ term produces important error terms:

$$\begin{aligned}\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\Psi &= \int \theta_+\partial_t\Psi + \int\theta_+ u\cdot\nabla\Psi + \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \\ &= \int \theta_+(\partial_t\Psi + u\cdot\nabla\Psi) + \int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma 2.2 part (a):

$$\int \theta_+\left[\partial_t + u\cdot\nabla + \Lambda\right]\theta_- = \left(\frac{1}{2}\right)\int\theta_+\partial_t\theta_- + \int\theta_+ u\cdot\nabla\theta_- + \int\Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le 0.$$

Put together, we arrive at

$$\left(\frac{1}{2}\right)\frac{d}{dt}\int\theta_+^2 + \int\left|\Lambda^{1/2}\theta_+\right|^2 + \int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi \le -\int\theta_+(\partial_t\Psi + u\cdot\nabla\Psi). \tag{10}$$

At this point we break down the $\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi$ term using the formula from Proposition 2.1.

$$\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi = \iint[\theta_+(x) - \theta_+(y)][\Psi(x)-\Psi(y)]K(x,y) + \int\theta_+\Psi B.$$

Since $B \geq 0$ and Ψ is non-negative by assumption, the B term is non-negative and so

$$\int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\Psi \geq \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y). \tag{11}$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\left| \iint [\theta_+(x) - \theta_+(y)][\Psi(x) - \Psi(y)]K(x,y) \right| \leq 2 \iint \chi_{\{\theta_+ > 0\}}(x) \, |\theta_+(x) - \theta_+(y)| \cdot |\Psi(x) - \Psi(y)| \, K(x,y).$$

Now we can break up this integral using Young's inequality, and since $\iint [\theta_+(x) - \theta_+(y)]^2 K \leq \|\theta_+\|^2_{\mathcal{H}^{1/2}}$ the inequality (11) becomes

$$\int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\Psi + \frac{1}{2}\int \left|\Lambda^{1/2}\theta_+\right|^2 \geq -2 \iint \chi_{\{\theta_+ > 0\}}(x)[\Psi(x) - \Psi(y)]^2 K(x,y). \tag{12}$$

It remains to bound the quantity $[\Psi(x) - \Psi(y)]^2 K(x,y)$. By Proposition 2.1, there is a universal constant C such that

$$K(x,y) \leq \frac{C}{|x-y|^3}.$$

The cutoff Ψ is locally Lipschitz, and Hölder continuous with exponent $1/4$, by assumption. Therefore

$$[\Psi(x) - \Psi(y)]^2 K(x,y) \leq Ck^2 |x-y|^{-1} \wedge |x-y|^{-2.5}.$$

Since $1 < 2 < 2.5$, this quantity is integrable. Thus

$$\int \chi_{\{\theta_+ > 0\}}(x) \int [\Psi(x) - \Psi(y)]^2 K(x,y) \, dy dx \leq Ck^2 \int \chi_{\{\theta_+ > 0\}} \, dx.$$

Combining this with (10) and (12) we obtain (8).

We begin now the proof of (9). Since $\theta_+ \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$, by interpolation we can further conclude $\theta_+, u \in L^4(0,T; L^4(\Omega))$. Therefore we can multiply (2) by $\varphi\theta_+$ and integrate in space to obtain

$$0 = \int \varphi\theta_+ \left[\partial_t + u \cdot \nabla + \Lambda\right](\theta_+ + \Psi - \theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. After rearranging and integrating by parts, this becomes

$$\frac{1}{2}\int \varphi \partial_t \theta_+^2 - \int \varphi\theta_+ \Lambda\theta_- = \frac{1}{2}\int \theta_+^2 u \cdot \nabla\varphi - \int \varphi\theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) - \int \varphi\theta_+ \Lambda\theta_+ - \int \varphi\theta_+ \Lambda\Psi. \tag{13}$$

Note that the $\Lambda\theta_-$ term on the left-hand side of this equality is non-negative by Lemma 2.2 part (a).

For the $\Lambda\theta_+$ term, by Cordoba's inequality (c.f. [CI17]),

$$\begin{aligned}
-\int \varphi\theta_+ \Lambda\theta_+ &\leq \frac{-1}{2}\int \varphi \Lambda\theta_+^2 \\
&= \frac{-1}{2}\int \theta_+^2 \Lambda\varphi \\
&\leq \frac{-1}{2}\iint_{\Omega\times\Omega} [\theta_+(x)^2 - \theta_+(y)^2][\varphi(x) - \varphi(y)]K_1(x,y) \, dxdy.
\end{aligned}$$

From the Caffarelli-Stinga Representation of Proposition 2.1,

$$-\int \varphi\theta_+ \Lambda\Psi \leq -\iint_{\Omega\times\Omega} [\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)][\Psi(x) - \Psi(y)]K_1(x,y) \, dxdy.$$

With these two bounds, the stated result follows from (13).

□

We will construct solutions which verify these energy inequalities proven above but which are not a priori in $L^2(H^1_0)$.

Definition 2. A pair θ, u is called a **suitable solution** on a space time domain $[S,T]\times\Omega$ if $S<T$ are finite, $\Omega\subseteq\mathbb{R}^2$ open and bounded, $\theta, u\in L^\infty(0,T;L^2(\Omega))$, $\theta\in L^2(0,T;\mathcal{H}^{1/2}(\Omega))$, $u\in L^4(0,T;L^4(\Omega))$ and

(1) θ, u solve (2) in the sense of distributions with L^2 norm decreasing in time,

(2) θ, u satisfy (8) and (9) at all scales with the same universal constant C^* defined in Proposition 3.1. More specifically, the following holds:
Let $\lambda>0$ and $\mu\in(0,1)$ be given and let $\Psi\in C^\infty([0,\infty)\times\mathbb{R}^2)$ be any smooth non-negative function satisfying $\|\nabla\Psi\|_{L^\infty([0,\infty)\times\mathbb{R}^2)}\le k$ and $\sup_{[0,\infty)}[\Psi(t,\cdot)]_{1/4;\mathbb{R}^2}\le k$ for some constant k.
Define $\tilde\Omega := \{x\in\mathbb{R}^2 : \mu x\in\Omega\}$, $\tilde\theta_+(t,x) := (\lambda\theta(\mu t,\mu x)-\Psi(t,x))_+$, and $\tilde u(t,x) := u(\mu t,\mu x)$. Let $\varphi\in C_c^\infty(S,T;C^\infty(\tilde\Omega))$ be non-negative.
Then $\tilde\theta_+$ and $\tilde u$ and φ and Ψ satisfy (8) and (9) on $[\mu^{-1}S,\mu^{-1}T]\times\tilde\Omega$ with the universal constant C^*.

Since the assumptions of Proposition 3.1 are preserved by the scaling $\theta\mapsto\lambda\theta(\mu\cdot,\mu\cdot)$, $u\mapsto u(\mu\cdot,\mu\cdot)$, and since the Riesz transform is bounded $L^p\to L^p$ for any $p\in(1,\infty)$, we see immediately that any solution $\theta\in L^\infty(L^2)\cap L^2(H^1_0)$ to (1) forms a suitable pair with the corresponding u. However, global solutions are only known to exist with $L^2(\mathcal{H}^{1/2})$ regularity (c.f. [CI17]).

To construct global suitable solutions, we will use the vanishing viscosity method. First, we must prove existence of global weak solutions to critical SQG with an epsilon of viscosity.

Consider the equation

$$\begin{cases}\partial_t\theta+u\cdot\nabla\theta+\Lambda\theta=\varepsilon\Delta\theta & (0,\infty)\times\Omega,\\ u=\nabla^\perp\Lambda^{-1}\theta & [0,\infty)\times\Omega,\\ \theta=\theta_0 & \{0\}\times\Omega.\end{cases}\tag{14}$$

Lemma 3.2 (Existence for viscous equation)**.** *Given an open, bounded domain $\Omega\subseteq\mathbb{R}^2$, initial data $\theta_0\in L^2(\Omega)$ and a constant $\varepsilon>0$, there exists a global-in-time weak solution θ to* (14)*.*

In particular, $\theta\in C^0([0,\infty);L^2(\Omega))\cap L^2([0,\infty);H^1_0(\Omega))$ and $\partial_t\in L^2([0,\infty);H^{-1}(\Omega))$, and $\theta(t,\cdot)\to\theta_0(\cdot)$ weakly in $L^2(\Omega)$ as $t\to 0$.

Moreover, there exists a universal constant C such that, for any $t>0$,

$$\|\theta\|_{L^\infty([t,\infty)\times\Omega)}\le\frac{C}{t}\|\theta_0\|_{L^2(\Omega)}.$$

The proof is by Galerkin's method.

Proof. Recall that η_j are the eigenfunctions of $-\Delta_D$. Let N be an integer parameter, and $W_N :=$ span$(\eta_0,\dots,\eta_N)$, which consists only of smooth functions which vanish on $\partial\Omega$. We seek first a solution $\theta_N\in W_N$ to the weak equation

$$\int\varphi\partial_t\theta_N+\int\varphi\nabla^\perp\Lambda^{-1}\theta_N\cdot\nabla\theta_N+\int\varphi\Lambda\theta_N+\varepsilon\int\nabla\theta_N\nabla\varphi=0,\qquad\forall t\in\mathbb{R}_{\ge0},\varphi\in W_N.\tag{15}$$

If we write

$$\theta_N(t,x):=\sum_{i=0}^N\alpha_{i,N}(t)\eta_i(x)$$

and choose $\varphi=\eta_i$ as a test function, then θ_N solves (15) if and only if, for all $i\le N$,

$$\alpha'_{i,N}(t)+\sum_{j=0}^N\sum_{k=0}^N\alpha_{j,N}(t)\alpha_{k,N}(t)B_{ijk}+\lambda_i^{1/2}\alpha_{i,N}(t)+\varepsilon\lambda_i\alpha_{i,N}(t)=0$$

with

$$B_{ijk} = \lambda_j^{-1/2} \int \eta_i \nabla^\perp \eta_j \cdot \nabla \eta_k$$

a constant tensor.

By Peano's existence theorem for ODEs, solutions to this system exist on some interval $[0,T]$ where T depends on Ω and N and the L^2 norm of the initial data.

Since $\theta_N \in W_N$ we can take θ_N as a test function and obtain, for any solution θ_N to (15),

$$\frac{d}{dt}\int \theta_N^2 + \int \left|\Lambda^{1/2}\theta_N\right|^2 + \varepsilon \int |\nabla \theta_N|^2 = 0.$$

Therefore in particular $\|\theta_N\|_{L^2(\Omega)}$ is non-increasing in time and we conclude that θ_N exists for all time. Moreover, θ_N is uniformly bounded in $L^\infty(L^2(\Omega))$ and $L^2(H_0^1(\Omega))$.

To take a limit in N, we need uniform regularity in time. From (15) we can bound

$$\int_0^\infty \int \partial_t \theta_N \varphi \le \|\theta_N\|_{L^4(L^4)} \|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(L^2)} \|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(H_0^1)} \|\varphi\|_{L^2(H_0^1)}.$$

Note that θ_N is uniformly bounded in $L^4(L^4)$ by interpolation and $L^2(L^2)$ by Poincaré's inequality. Therefore $\iint \varphi \partial_t \theta_N \le C \|\varphi\|_{L^2(H_0^1)}$ for all $\varphi \in L^2(W_N)$ for a constant C independent of N. Since $\partial_t\theta_N \in W_N$, this is sufficient to show that $\|\partial_t \theta_N\|_{L^2(H^{-1})}$ is uniformly bounded.

By Aubin-Lions, we conclude that θ_N is a compact sequence in $L^2([0,\infty)\times\Omega)$ and so it has an L^2 limit θ. This limit θ is in $L^\infty(L^2(\Omega))$ and $L^2(H_0^1(\Omega))$ and $\partial_t\theta \in L^2(H^{-1}(\Omega))$.

We must prove that θ is a weak solution to (14). Let $\varphi \in C_c^\infty(W_M)$ for some M. For $N \ge M$,

$$-\iint \theta_N \partial_t \varphi - \iint \theta_N \nabla^\perp \Lambda^{-1}\theta_N \cdot \nabla\varphi + \iint \theta_N \Lambda \varphi - \varepsilon \iint \theta_N \Delta\varphi = 0.$$

This expression is continuous for $\theta_N \in L^2(L^2)$, so by taking $N \to \infty$ we obtain

$$-\int \theta \partial_t \varphi - \int \theta \nabla^\perp \Lambda^{-1}\theta \cdot \nabla\varphi + \int \theta \Lambda\varphi + \varepsilon \int \nabla\theta \cdot \nabla\varphi = 0$$

for any $\varphi \in C_c^\infty(W_M)$ for any $M \in \mathbb{N}$. By density, θ solves (14) in the sense of distributions.

Since $\partial_t\theta_N$ is uniformly bounded in $L^2(H^{-1})$, we know $\theta_N(t,\cdot) \to \theta_0$ weakly in L^2 uniformly in N and so the same holds for θ.

Lastly, for any constant $a \ge 0$, the function $(\theta - a)_+$ satisfies

$$\frac{d}{dt}\int \frac{1}{2}(\theta-a)_+ + \int \left|\Lambda^{1/2}(\theta-a)_+\right|^2 = \frac{-1}{2}\int u\cdot\nabla(\theta-a)_+^2 - \int a\Lambda(\theta-a)_+ + \varepsilon\int |\nabla(\theta-a)_+|^2.$$

$$= -\int a(\theta-a)_+ B_1 \le 0.$$

This inequality is scaling-invariant, so the same holds for $\lambda\theta(\mu t, \mu(\varepsilon)$ for any $\lambda, \mu > 0$.

By the standard De Giorgi argument (see Lemma A.1 in the Appendix for details), there exists a universal constant δ such that $\int_0^2 \int (\lambda\theta(\mu t, \mu x)_+^2\, dxdt \le \delta$ implies $\theta \le \lambda^{-1}$ on $[\mu 1, \mu 2]$. In fact, by comparison with a constant solution, $\theta \le \lambda^{-1}$ on $[\mu,\infty)$. Taking $\lambda = \sqrt{\frac{\delta}{2\mu^{-2}\|\theta_0\|_{L^2(\Omega)}^2}}$, we find $\theta(t,\cdot) \le Ct^{-1}\|\theta_0\|_{L^2(\Omega)}$ for a universal constant C.

guarantees that $\theta \in L^\infty([t,\infty]\times\Omega)$ for all $t > 0$, with norm depending on t and $\|\theta_0\|_{L^2(\Omega)}$ and Ω but independent of ε.

In the case $S \ge 2$, we can apply (8) to establish the family of energy inequalities assumed in Lemma A.1 and then apply Lemma A.1 to the function $\frac{\sqrt{\delta}}{\sqrt{2}\|\theta_0\|_2}\theta(t+2,x)$ to show that

$$\|\theta\|_{L^\infty([2,T]\times\Omega)} \le \left(\frac{2}{\delta}\right)^{1/2} \|\theta_0\|_2.$$

For small S, we can apply the above argument to the function $\theta\left((S/2)t,(S/2)x\right)$. □

Now that we have global existence of solutions to the perturbed equation, we can take a limit as $\varepsilon\to 0$ and obtain a suitable solution to the original equation.

Proposition 3.3 (Existence of suitable solutions)**.** *Given an open, bounded domain $\Omega\subseteq\mathbb{R}^2$ with $C^{2,\beta}$ boundary, $\beta\in(0,1)$, and initial data $\theta_0\in L^2(\Omega)$, there exists a global-in-time weak solution θ to* (1) *such that, for any $0<S<T<\infty$, θ and $u:=\nabla^\perp\Lambda^{-1}\theta$ are a suitable pair on $[S,T]\times\Omega$.*

Moreover, $\theta\in L^\infty([0,\infty);L^2(\Omega))\cap L^2([0,\infty);\mathcal{H}^{1/2}(\Omega))$, and $\theta(t,\cdot)\to\theta_0(\cdot)$ weakly in $L^2(\Omega)$ as $t\to 0$.

Moreover, there exists a universal constant C such that, for any $t>0$,

$$\|\theta\|_{L^\infty([t,\infty)\times\Omega)}\le\frac{C}{t}\|\theta_0\|_{L^2(\Omega)}.$$

The proof is by the method of vanishing viscosity.

Proof. For any parameter $\varepsilon>0$, define $\theta_\varepsilon\in L^2(H^1_0)$ the weak solution to (14) constructed in Lemma 3.2. The θ_ε are uniformly bounded in $L^\infty(L^2)$ and $L^2(\mathcal{H}^{1/2})$ by the standard energy argument, so by interpolation they are also uniformly bounded in $L^4(L^{8/3})$. Recall that θ_ε are uniformly bounded in $L^\infty(L^\infty)$ after any positive time.

For any smooth φ, we have

$$\int_0^\infty\int\partial_t\theta_\varepsilon\varphi\le\|\theta_\varepsilon\|^2_{L^4(L^{8/3})}\|\varphi\|_{L^2(W^{1,4})}+\|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})}\|\varphi\|_{L^2(\mathcal{H}^{1/2})}+\varepsilon\|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})}\|\varphi\|_{L^2(\mathcal{H}^{3/2})}.$$

Therefore $\partial_t\theta_\varepsilon$ is uniformly bounded in $L^2(\mathcal{H}^{-3/2})$. By Aubin-Lions, the sequence θ_ε has a strong limit in $L^2(L^2)$. Call this limit θ.

Since $\partial_t\theta_\varepsilon$ is uniformly bounded and $\theta(t,\cdot)\to\theta_0$ weakly in L^2, the same holds for θ.

Define $u_\varepsilon:=\nabla^\perp\Lambda^{-1}\theta_\varepsilon$, and by continuity of the Riesz transform we have $u_\varepsilon\to u$ in $L^2([0,\infty)\times\Omega)$ where $u:=\nabla^\perp\Lambda^{-1}\theta$.

It remains only to prove that θ and u are a suitable pair. This will be true because, for θ_ε and u_ε, we can make the same calculation as in the proof of Proposition 3.1. The added viscosity term produces error terms which vanish as $\varepsilon\to 0$, and since the suitability conditions (8) and (9) are maintained by L^2 limits, the limiting functions θ and u will satisfy the suitability conditions with no error terms. The details of this calculation are given below.

Let $0<S<T<\infty$ be constant times, and let Ψ, φ, λ, and μ be as in the definition of suitable solutions. Define

$$\begin{aligned}\tilde\Omega&:=\{x\in\mathbb{R}^2:\mu x\in\Omega\},\\ \tilde T&:=\mu^{-1}T,\\ \tilde S&:=\mu^{-1}S,\\ \tilde\theta_\varepsilon(t,x)&:=\lambda\theta_\varepsilon(\mu t,\mu x),\\ \tilde u_\varepsilon(t,x)&:=u_\varepsilon(\mu t,\mu x),\\ \tilde\theta_{\varepsilon,+}(t,x)&:=\left(\tilde\theta_\varepsilon(t,x)-\Psi(t,x)\right)_+.\end{aligned}$$

Note that $\tilde\theta_\varepsilon$ and $\tilde u_\varepsilon$ are weak solutions to

$$\begin{cases}\partial_t\tilde\theta_\varepsilon+\tilde u_\varepsilon\cdot\nabla\tilde\theta_\varepsilon+\Lambda\tilde\theta_\varepsilon=\varepsilon\Delta\tilde\theta_\varepsilon,\\ \operatorname{div}\tilde u_\varepsilon=0.\end{cases}\tag{16}$$

Because $\tilde{\theta}_{\varepsilon,+} \in L^2(H^1_0)$, we can perform the exact same calculation as in the proof of Proposition 3.1 to arrive at (8) but with an additional term corresponding to $\varepsilon\Delta\theta_\varepsilon$: (all integrals are over $\tilde{\Omega}$)

$$\frac{d}{dt}\int \tilde{\theta}^2_{\varepsilon,+} + \int \left|\Lambda^{1/2}\tilde{\theta}_{\varepsilon,+}\right|^2 \le C\left(-\int \tilde{\theta}_{\varepsilon,+}(\partial_t\Psi + \tilde{u}_\varepsilon\cdot\nabla\Psi) + k^2\int \chi_{\{\tilde{\theta}_{\varepsilon,+}\ge 0\}} + \varepsilon\int \tilde{\theta}_{\varepsilon,+}\Delta\tilde{\theta}_\varepsilon\right).$$

Recall that $\tilde{\theta}_{\varepsilon,+}$ and $\tilde{\theta}_\varepsilon - \tilde{\theta}_{\varepsilon,+} - \Psi$ have disjoint support. The added $\Delta\tilde{\theta}_\varepsilon$ term therefore decomposes as

$$\begin{aligned}\varepsilon\int \tilde{\theta}_{\varepsilon,+}\Delta\tilde{\theta}_\varepsilon &= -\varepsilon\int \left|\nabla\tilde{\theta}_{\varepsilon,+}\right|^2 - \varepsilon\int \nabla\tilde{\theta}_{\varepsilon,+}\cdot\nabla\Psi\\ &\le -\frac{\varepsilon}{2}\int \left|\nabla\tilde{\theta}_{\varepsilon,+}\right|^2 + \frac{\varepsilon}{2}\int |\nabla\Psi|^2\\ &\le \varepsilon\frac{k^2}{2}\mu^{-2}|\Omega|.\end{aligned}$$

This term vanishes in the limit as $\varepsilon \to 0$. The terms $\int \tilde{\theta}_{\varepsilon,+}(\partial_t\Psi + \tilde{u}_\varepsilon\cdot\nabla\Psi)$ and $\int \chi_{\{\tilde{\theta}_{\varepsilon,+}\ge 0\}}$ are continuous under L^2 limits, and the quantities $\frac{d}{dt}\int \tilde{\theta}^2_{\varepsilon,+}$ and $\int \left|\Lambda^{1/2}\tilde{\theta}_{\varepsilon,+}\right|^2$ are lower-semicontinuous under L^2 limits, so we conclude that θ satisfies (8).

Similarly for the second suitability condition, we have as in the proof of Proposition 3.1 (all $\iint$ are $\int_{\tilde{S}}^{\tilde{T}}\int_{\tilde{\Omega}}$)

(17)
$$\begin{aligned}\frac{-1}{2}\iint \tilde{\theta}^2_{\varepsilon,+}\partial_t\varphi = \frac{1}{2}\iint \tilde{\theta}^2_{\varepsilon,+}\tilde{u}_\varepsilon\cdot\nabla\varphi - \iint \varphi\tilde{\theta}_{\varepsilon,+}\left(\partial_t\Psi + \tilde{u}_\varepsilon\cdot\nabla\Psi\right)\\ - \int_{\tilde{S}}^{\tilde{T}}\iint_{\tilde{\Omega}\times\tilde{\Omega}}[\varphi(x)\tilde{\theta}_{\varepsilon,+}(x) - \varphi(y)\tilde{\theta}_{\varepsilon,+}(y)][\Psi(x)-\Psi(y)]K_1\\ - \frac{1}{2}\int_{\tilde{S}}^{\tilde{T}}\iint_{\tilde{\Omega}\times\tilde{\Omega}}[\tilde{\theta}_{\varepsilon,+}(x)^2 - \tilde{\theta}_{\varepsilon,+}(y)^2][\varphi(x)-\varphi(y)]K_1 + \varepsilon\iint \tilde{\theta}_{\varepsilon,+}\Delta\tilde{\theta}_\varepsilon.\end{aligned}$$

The added $\Delta\tilde{\theta}_\varepsilon$ term decomposes as

$$\begin{aligned}\varepsilon\iint \tilde{\theta}_{\varepsilon,+}\Delta\tilde{\theta}_\varepsilon &= -\varepsilon\iint \varphi\nabla\tilde{\theta}_{\varepsilon,+}\cdot\nabla(\tilde{\theta}_{\varepsilon,+}+\Psi) - \varepsilon\iint \tilde{\theta}_{\varepsilon,+}\nabla\varphi\cdot\nabla(\tilde{\theta}_{\varepsilon,+}+\Psi)\\ &= -\varepsilon\iint \varphi\left|\nabla\tilde{\theta}_{\varepsilon,+}\right|^2 - \varepsilon\iint \varphi\nabla\tilde{\theta}_{\varepsilon,+}\cdot\nabla\Psi + \frac{\varepsilon}{2}\iint \tilde{\theta}^2_{\varepsilon,+}\Delta\varphi - \varepsilon\iint \tilde{\theta}_{\varepsilon,+}\nabla\varphi\cdot\nabla\Psi\\ &\le -\frac{\varepsilon}{2}\iint \varphi\left|\nabla\tilde{\theta}_{\varepsilon,+}\right|^2 + \frac{\varepsilon}{2}\iint \varphi|\nabla\Psi|^2 + \frac{\varepsilon}{2}\iint \tilde{\theta}^2_{\varepsilon,+}\Delta\varphi - \varepsilon\iint \tilde{\theta}_{\varepsilon,+}\nabla\varphi\cdot\nabla\Psi\\ &\le \varepsilon\frac{k^2}{2}\mu^{-1}T\left\|\varphi\right\|_{C^0(L^1)} + \varepsilon\frac{1}{2}\left\|\Delta\varphi\right\|_{L^\infty(L^\infty)}\left\|\tilde{\theta}_{\varepsilon,+}\right\|^2_{L^2(L^2)} + \varepsilon k\mu^{-1}T\left\|\varphi\right\|_{C^0(H^1)}\left\|\tilde{\theta}_{\varepsilon,+}\right\|_{L^\infty(L^2)}.\end{aligned}$$

For T, μ, λ, and φ fixed, this term vanishes in the limit as $\varepsilon \to 0$.

On $[\tilde{S},\tilde{T}]$ we have a uniform $L^\infty(L^\infty)$ bound for $\tilde{\theta}_\varepsilon$. Therefore $\tilde{\theta}_{\varepsilon,+}$ converges in $L^4(L^4)$, and so $\iint \tilde{\theta}^2_{\varepsilon,+}\tilde{u}_\varepsilon\cdot\nabla\varphi$ is conserved in the limit $\varepsilon \to 0$.

Since Ψ is smooth by assumption, $\iint \varphi\tilde{\theta}_{\varepsilon,+}\left(\partial_t\Psi + \tilde{u}_\varepsilon\cdot\nabla\Psi\right)$ is $L^2(L^2)$ continuous and is conserved in the limit $\varepsilon \to 0$.

To show continuity of the $\iint[\varphi\tilde{\theta}_{\varepsilon,+} - \varphi\tilde{\theta}_{\varepsilon,+}][\Psi - \Psi]K_1$ term, let f be an arbitrary $\mathcal{H}^{1/4}$ function. Then

$$\iint [\varphi(x)f(x) - \varphi(y)f(y)][\Psi(x)-\Psi(y)]K_1(x,y)\,dxdy$$

$$\le \left(\iint [\varphi(x)f(x) - \varphi(y)f(y)]^2 K_{1/2}(x,y) \right)^{1/2} \left(\iint [\Psi(x) - \Psi(y)]^2 \frac{K_1(x,y)^2}{K_{1/2}(x,y)} \right)^{1/2}$$
$$\le \|\varphi f\|_{\mathcal{H}^{1/4}} \left(\iint [\Psi(x) - \Psi(y)]^2 \frac{K_1(x,y)^2}{K_{1/2}(x,y)} \right)^{1/2}$$
$$\le C \|f\|_{\mathcal{H}^{1/4}} \|\varphi\|_{C^1} \|\nabla\Psi\|_{L^\infty} \iint |x-y|^2 |x-y|^{-1/2} \frac{1}{|x-y|^3}.$$

In this last line we used Proposition 2.1 and Lemma 2.2 part (c). It follows that the $\iint [\varphi\tilde{\theta}_{\varepsilon,+} - \varphi\tilde{\theta}_{\varepsilon,+}][\Psi - \Psi]K_1$ term of (17) is continuous under $L^1(\mathcal{H}^{1/4})$ limits of $\tilde{\theta}_{\varepsilon,+}$. Since $\tilde{\theta}_\varepsilon$ are convergent in $L^2(L^2)$ and bounded in $L^2(\mathcal{H}^{1/2})$, this term is conserved in the limit $\varepsilon \to 0$.

To show continuity of the $\iint [\tilde{\theta}^2_{\varepsilon,+} - \tilde{\theta}^2_{\varepsilon,+}][\varphi - \varphi]K_1$ term, let f be an arbitrary $\mathcal{H}^{1/4}$ function. Then

$$\iint [f(x)^2 - f(y)^2][\varphi(x) - \varphi(y)]K_1 \le \left(\iint [f(x)^2 - f(y)^2]^2 K_{1/2} \right)^{1/2} \left(\iint [\varphi(x) - \varphi(y)]^2 \frac{K_1^2}{K_{1/2}} \right)^{1/2}$$
$$\le \left(4 \iint f(x)^2 [f(x) - f(y)]^2 K_{1/2} \right)^{1/2} \|\nabla\varphi\|_{L^\infty} \left(\iint \frac{1}{|x-y|^{1.5}} \right)^{1/2}$$
$$\le C \|f\|_{L^\infty} \|f\|_{\mathcal{H}^{1/4}}.$$

Since all of the terms of (17) behave well in the limit as $\varepsilon \to 0$, it follows that θ satisfies (9). □

4. Littlewood-Paley Theory

In this section we will prove that, because θ is uniformly bounded in L^∞, the velocity $u = \nabla^\perp \Lambda^{-1}\theta$ is calibrated (see Definition 1). The proof will utilize a Littlewood-Paley theory adapted to a bounded set Ω.

Because the Littlewood-Paley theory depends in an essential way on the domain Ω, any results proven in this way will also be domain-dependent. Therefore, in the proof of Hölder continuity in Section 7, we will apply the following Proposition only to the unscaled function θ on the unscaled domain Ω. As we zoom in, the velocity will remain calibrated, so there will be no further need for this result.

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $\theta \in L^\infty(\Omega)$. Then there exists an integer $j_0 = j_0(\Omega)$ and a sequence of divergence-free functions $(u_j)_{j \ge j_0}$ calibrated for some constant $\kappa = \kappa(\Omega, \|\theta\|_\infty)$ with center 0 (see Definition 1) such that*

$$\nabla^\perp \Lambda^{-1}\theta = \sum_{j \ge j_0} u_j$$

with the infinite sum converging in the sense of L^2.

Before we can prove this, we define the Littlewood-Paley projections and prove some of their properties:

Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. Specifically, ϕ is non-negative, supported on $[1/2, 2]$, and has the property that

$$\sum_{j \in \mathbb{Z}} \phi(2^j \xi) = 1 \qquad \forall \xi \ne 0.$$

For any $f = \sum f_k \eta_k$ in $L^2(\Omega)$, we define the Littlewood-Paley projections

$$P_j f := \sum_k \phi(2^j \lambda_k^{1/2}) f_k \eta_k.$$

Note that P_j depends strongly on the domain Ω.

Recall that $-\Delta_D$ has some smallest eigenvalue λ_0 (depending on Ω) so if we define $j_0 = \log_2(\lambda_0)-1$ then $P_j = 0$ for all $j < j_0$.

The Bernstein Inequalities adapted for a bounded domain are proved in [IMT17]. We restate their result here:

Lemma 4.2 (Bernstein Inequalities)**.** *Let $1 \le p \le \infty$ and $\Omega \subset \mathbb{R}^2$ a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $(P_j)_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition defined above.*

There exists a constant C depending on p and Ω such that the following hold for any $f \in L^p(\Omega)$:

For any $\alpha \in \mathbb{R}$ and $j \in \mathbb{Z}$,

$$\|\Lambda^\alpha P_j f\|_{L^p(\Omega)} \le C2^{\alpha j} \|f\|_{L^p(\Omega)}.$$

For any $\alpha \in \mathbb{R}$ and $j \ge j_0$

$$\|\nabla \Lambda^\alpha P_j f\|_{L^p(\Omega)} \le C2^{(1+\alpha)j} \|f\|_{L^p(\Omega)}.$$

Proof. The first claim is Lemma 3.5 in [IMT17]. It is also an immediate corollary of [IMT18] Theorem 1.1.

The second claim is similar to Lemma 3.6 in [IMT17]. A hypothesis of Lemma 3.6 is that

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty\to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le 1$$

(a property of Ω). The result of Lemma 3.6 only covers the case $j > 0$.

In [FMP04] it is proved that that if Ω is $C^{2,\beta}$ then

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty\to L^\infty} \le \frac{C}{\sqrt{t}} \qquad 0 < t \le T$$

which, by taking some T depending on j_0, is enough to prove the desired result for $j \ge j_0$ by a trivial modification of the proof in [IMT17]. □

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. In the case of $\mathbb{R}^2$, the Littlewood-Paley projections commute with the gradient so $P_i \nabla P_j = 0$ unless $|i-j| \le 1$. On a bounded domain, this is not the case; the gradient does not maintain localization in frequency-space. However, the following lemma formalizes the observation that $P_i \nabla P_j \approx 0$ when $i << j$.

Lemma 4.3. *Let $1 \le p \le \infty$. There exists a constant C depending on p and Ω such that or any function $f \in L^p(\Omega)$,*

$$\|P_i \nabla P_j f\|_p \le C \min(2^j, 2^i) \|f\|_p.$$

Proof. Let q be the Hölder conjugate of p and g be an L^q function. Then since P_i is self-adjoint

$$\int g P_i \nabla P_j f = \int (P_i g) \nabla P_j f \le C2^j \|g\|_q \|f\|_p$$

by Lemma 4.2.

Further integrating by parts,

$$\int g P_i \nabla P_j f = -\int (\nabla P_i g) P_j f \le C2^i \|g\|_q \|f\|_p.$$

This also follows from Lemma 4.2.

The result follows. □

We are now ready to prove Proposition 4.1.

Proof of 4.1. For each integer $j \geq j_0$, we define u_j to be the $\frac{\pi}{2}$-rotation of the Riesz transform of the j^{th} Littlewood-Paley projection of θ:

$$u_j := \nabla^{\perp}\Lambda^{-1}P_j\theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

We must bound u_j, $\Lambda^{-1/4}u_j$, and ∇u_j all in $L^\infty(\Omega)$.

By straightforward application of Lemma 4.2,

$$\|u_j\|_\infty \leq C\,\|\theta\|_\infty . \tag{18}$$

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4}u_j = \sum_{i\in\mathbb{Z}} P_i\Lambda^{-1/4}u_j.$$

Define $\bar{P}_k := P_{k-1} + P_k + P_{k+1}$. Then $\bar{P}_kP_k = P_k$, and since the projections P_k are spectral operators, they commute with Λ^s and each other. We therefore rewrite

$$\left(P_i\Lambda^{-1/4}u_j\right)^{\perp} = \left(\Lambda^{-1/4}\bar{P}_i\right)\left(P_i\nabla P_j\right)\left(\Lambda^{-1}\bar{P}_j\right)\theta.$$

On the right hand side we have three bounded linear operators applied sequentially to $\theta \in L^\infty$. The first operator has norm $C2^{-j}(2^1 + 2^0 + 2^{-1})$ by Lemma 4.2. The second operator has norm $C\min(2^j, 2^i)$ by Lemma 4.3. The third operator has norm $C2^{-i/4}(2^{1/4} + 2^0 + 2^{-1/4})$ by Lemma 4.2. Therefore

$$\left\|P_i\Lambda^{-1/4}u_j\right\|_\infty \leq C2^{-i/4}\min(2^j, 2^i)2^{-j}\,\|\theta\|_\infty .$$

Summing these bounds on the projections of $\Lambda^{-1/4}u_j$, and noting that

$$\sum_{i\in\mathbb{Z}} 2^{-j}2^{-i/4}\min(2^j, 2^i) = 2^{-j}\sum_{i\leq j}2^{i3/4} + \sum_{i>j}2^{-i/4} \leq C2^{-j/4},$$

we obtain

$$\left\|\Lambda^{-1/4}u_j\right\|_\infty \leq C2^{-j/4}\,\|\theta\|_\infty . \tag{19}$$

Lastly, we must show that ∇u_j is in L^∞. Equivalently, we will show that $\Lambda^{-1}P_j\theta$ is $C^{1,1}$. The method of proof is Schauder theory.

For convenience, define

$$F := \Lambda^{-1}P_j\theta.$$

Notice that F is a linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Therefore

$$-\Delta F = \Lambda^2 F = \Lambda P_j\theta.$$

We apply the standard Schauder estimate from Gilbarg and Trudinger [GT01] Theorem 6.6 to bound some $C^{2,\alpha}$ semi-norm of F by the L^∞ norm of F and the C^α norm of its Laplacian. By assumption there exists $\beta \in (0,1)$ such that Ω is $C^{2,\beta}$, and for this β we have by the Schauder estimate

$$\left[D^2F\right]_\beta \leq C\left\|\Lambda^{-1}P_j\theta\right\|_\infty + C\,\|\Lambda P_j\theta\|_\infty + C\,[\Lambda P_j\theta]_\beta . \tag{20}$$

By Lemma 4.2,

$$\begin{aligned}
\left\|\Lambda^{-1}P_j\theta\right\|_\infty &\leq C2^{-j}\,\|\theta\|_\infty ,\\
\|\Lambda P_j\theta\|_\infty &\leq C2^{j}\,\|\theta\|_\infty ,\\
\|\nabla\Lambda P_j\theta\|_\infty &\leq C2^{2j}\,\|\theta\|_\infty .
\end{aligned}$$

By Lemma A.2 (see Appendix A) we can interpolate these last two bounds to obtain

$$[\Lambda P_j\theta]_\beta \le C2^{j(1+\beta)} \|\theta\|_\infty .$$

Plugging these estimates into (20) yields

$$\left[D^2F\right]_\beta \le C\left(2^{-j} + 2^j + 2^{j(1+\beta)}\right)\|\theta\|_\infty .$$

Recall that without loss of generality we can assume $j \ge j_0$. Therefore up to a constant depending on j_0, the term $2^{j(1+\beta)}$ bounds 2^j and 2^{-j} so we can write

$$\left[D^2F\right]_\beta \le C2^{j(1+\beta)} \|\theta\|_\infty .$$

Using this estimate and the fact that $\|\nabla F\|_\infty = \left\|\nabla\Lambda^{-1}P_j\theta\right\|_\infty \le C\|\theta\|_\infty$ (see (18)), we can interpolate to obtain an L^∞ bound on D^2F. Lemma A.3 states that since $F \in C^{2,\beta}$ and Ω is sufficiently regular, there exist a constant $\ell = \ell(\Omega)$ such that for any $\delta \in [0,\ell]$ we have

$$\begin{aligned}\left\|D^2F\right\|_\infty &\le C\left(\delta^{-1}\|\nabla F\|_\infty + \delta^\beta\left[D^2F\right]_\beta\right)\\ &\le C\left(\delta^{-1} + \delta^\beta 2^{j(1+\beta)}\right)\|\theta\|_\infty .\end{aligned}$$

Set $\delta = 2^{-j}(2^{j_0}\ell) \le \ell$. Then

$$\left\|D^2F\right\|_\infty \le C\left(2^j + 2^{-j\beta}2^{j(1+\beta)}\right)\|\theta\|_\infty = C(\Omega)2^j\|\theta\|_\infty .$$

Since $D^2F = \nabla u_j$, this estimate together with (18) and (19) complete the proof. □

5. De Giorgi Estimates

Our goal in this section is to prove De Giorgi's first and second lemmas for suitable solutions to (2) with u uniformly calibrated. The De Giorgi lemmas will eventually be applied iteratively to various rescalings of the solution θ, so the following results must be independent of the size of the domain Ω. Any properties we do assume for the domain, such as the regularity of the boundary, must be scaling invariant.

Rather than working directly with the calibrated sequence, we will decompose u into just two terms, a low-pass term and a high-pass term. The construction is described in the following lemma. Note that we make no assumption on the center of calibration, which means this result is indendent of scale.

Lemma 5.1. *Let*

$$u = \sum_{j_0}^\infty u_j$$

with the sum converging in the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center, and that $\operatorname{div}(u_j) = 0$ *for all j.*

Then

$$u = u_\ell + u_h$$

with

$$\begin{aligned}\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)} &\le 2\kappa,\\ \left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-T,0]\times\Omega)} &\le 6\kappa.\end{aligned}$$

and $\operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0$.

We call u_ℓ the low-pass term, and u_h the high-pass term.

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j=N+1}^{\infty} u_j$$

and bound

$$\left\|\Lambda^{-1/4}u_h\right\|_\infty \le \sum_{j>N}\left\|\Lambda^{-1/4}u_j\right\|_\infty \le \kappa\frac{2^{-1/4}}{1-2^{-1/4}}.$$

We define

$$u_\ell = \sum_{j=j_0}^{N} u_j$$

and bound

$$\|\nabla u_\ell\|_\infty \le \sum_{j\le N}\|\nabla u_j\|_\infty \le \kappa\frac{1}{1-2^{-1}}.$$

□

In order to prove the De Giorgi lemmas, we must derive an energy inequality for the function $(\theta-\Psi)_+$ where $\Psi(t,x)$ grows sublinearly in $|x|$. Considering the suitability condition (8), we see that control can only be gained if the quantity $\partial_t\Psi + u\cdot\nabla\Psi$ is bounded. This requires a barrier function which is moving in space along a Lagrangian path Γ_ℓ of u_ℓ.

To that end, we shall consider, for any domain Ω and time T, functions $\theta:[-T,0]\times\Omega\to\mathbb{R}$, u_ℓ and $u_h:[-T,0]\times\Omega\to\mathbb{R}^2$, and a Lipschitz path $\Gamma_\ell:[-T,0]\to\Omega$ which satisfy

$$\begin{cases}\theta,(u_\ell+u_h)\text{ suitable solution to (2)} & \text{on } [-T,0]\times\Omega,\\ \operatorname{div}(u_\ell)=\operatorname{div}(u_h)=0 & \text{on } [-T,0]\times\Omega,\\ \dot\Gamma_\ell(t)=u_\ell(t,\Gamma_\ell(t)) & \text{on } [-T,0].\end{cases}\tag{21}$$

Because Γ_ℓ depends on u_ℓ which depends on N, the path Γ_ℓ will change significantly between scales. In particular, though $\Gamma_\ell\in\mathrm{Lip}([-T,0];\mathbb{R}^2)$, we cannot assume any uniform bound on it Lipschitz constant. We can bound, however, the difference between Γ_ℓ at consecutive scales. Therefore we must consider in the following lemmas an arbitrary Lipschitz path Γ, which was produced at a previous scale, and denote $\gamma := \Gamma_\ell-\Gamma$ which will be uniformly bounded.

Now we prove an energy inequality for solutions to (21). Though this lemma is independent of the size of the domain, it depends on the geometry of the domain in a way encoded by the constant C_{dmn}. We will later show that this constraint on Ω is scaling invariant.

Lemma 5.2 (Energy inequality)**.** *Let κ, C_{dmn}, C_{pth}, T, and R be positive constants, and let $\psi:\mathbb{R}^2\to\mathbb{R}$ be a function such that $\|\nabla\psi\|_\infty$, $\left\|D^2\psi\right\|_\infty$, and $\sup_t[\psi(t,\cdot)]_{1/4}$ are all finite. Then there exists a constant $C>0$ such that the following holds:*

Let $\Omega\subseteq\mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$, and let $\Gamma:[-T,0]\to\mathbb{R}^2$ be Lipschitz. Assume that Proposition 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y)\le C_{dmn}|x-y|^{3/4}K_1(x,y).$$

Let θ, u_ℓ, u_h, Γ_ℓ solve (21) on $[-T,0]\times\Omega$ with θ and $u_\ell+u_h$ a suitable pair, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-T,0]\times\Omega)}\le 6\kappa$, and $\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)}\le 2\kappa$. Denote $\gamma := \Gamma_\ell-\gamma$ and assume $\|\dot\gamma\|_{L^\infty([-T,0])}\le C_{pth}$ and $\gamma(0)=0$.

Consider the functions

$$\theta_+ := (\theta-\psi(\cdot-\Gamma))_+,\qquad \theta_- := (\psi(\cdot-\Gamma)-\theta)_+.$$

If θ_+ is supported on $x \in \Omega \cap B_R(\Gamma(t))$ then θ_+ and θ_- satisfy the inequality

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C\left(\int \chi_{\{\theta_+\ge 0\}} + \int \theta_+ + \int \theta_+^2\right).$$

Proof. Define

$$\Psi(t,x) := \psi(x - \Gamma(t))$$

so that

$$\partial_t \Psi + (u_\ell + u_h)\cdot\nabla\Psi = (u_\ell - \dot\Gamma + u_h)\cdot\nabla\psi(x-\Gamma(t)).$$

Applying (8) to θ and Ψ we arrive at

(22)

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C\left(\int \chi_{\{\theta_+\ge 0\}} + \left|\int \theta_+(u_\ell - \dot\Gamma(t) + u_h)\cdot\nabla\psi(x-\Gamma(t))\right|\right).$$

Consider first the high-pass term $\int \theta_+ u_h \cdot \nabla\psi$. By inserting $\Lambda^{1/4}\Lambda^{-1/4}$ and then integrating by parts, we can apply Lemma 2.2 parts (e) and (c) to obtain

$$\int \Lambda^{-1/4}u_h\Lambda^{1/4}(\theta_+\nabla\psi) \le C\left\|\Lambda^{-1/4}u_h\right\|_\infty\left(\|\nabla\psi\|_\infty + \left\|D^2\psi\right\|_\infty\right)|\operatorname{supp}(\theta_+)|^{1/2}\left(\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}\right).$$

We apply Young's inequality to find that for any constant $\varepsilon > 0$ there exists $C = C(\psi,\kappa,C_{dmn},\varepsilon)$ such that

$$\int u_h\theta_+\nabla\psi(x-\Gamma(t))\,dx \le C\left(|\operatorname{supp}(\theta_+)| + \int \theta_+^2\right) + \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2. \tag{23}$$

Consider now the low-pass term. By (21)

$$u_\ell(t,x) - \dot\Gamma(t) = u_\ell(t,x) - u_\ell(t,\Gamma+\gamma) + \dot\gamma. \tag{24}$$

Since u_ℓ is has derivative bounded by 2κ,

$$\begin{aligned}|u_\ell(t,x) - u_\ell(t,\Gamma+\gamma)| &\le |u_\ell(t,x) - u_\ell(t,\Gamma)| + |u_\ell(t,\Gamma) - u_\ell(t,\Gamma+\gamma)|\\ &\le 2\kappa|x-\Gamma| + 2\kappa|\gamma|.\end{aligned}$$

By assumption $|\dot\gamma| \le C_{pth}$ and $\gamma(0) = 0$, and so for $t \in [-T,0]$ we have $|\gamma(t)| \le TC_{pth}$.

Plugging these bounds into (24) we obtain

$$\left|u_\ell(t,x) - \dot\Gamma(t)\right| \le 2\kappa|x-\Gamma| + 2\kappa T C_{pth} + C_{pth}.$$

Now we can bound the low pass term

$$\int (u_\ell - \dot\Gamma)\theta_+\nabla\psi(x-\Gamma) \le (2\kappa T + 1)C_{pth}\|\nabla\psi\|_\infty\int\theta_+\,dx + \|\nabla\psi\|_\infty 2\kappa\int|x-\Gamma|\theta_+\,dx.$$

By assumption, $|x-\Gamma|\theta_+ \le R\theta_+$, so from this, (23), and (22) the result follows. □

This energy inequality is sufficient to prove the De Giorgi Lemmas.

The first lemma is a local version of the L^2 to L^∞ regularization, stating that solutions with small L^2 norm in a region will have small L^∞ norm in a smaller region.

Proposition 5.3 (First De Giorgi Lemma)**.** *Let κ, C_{dmn}, and C_{pth}, be positive constants. Then there exists a constant $\delta_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma : [-2,0] \to \mathbb{R}^2$ be Lipschitz. Assume that Proposition 2.1 holds on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_{dmn}|x-y|^{3/4}K_1(x,y).$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (21) *on* $[-2,0]\times\Omega$ *with θ and $u_\ell+u_h$ a suitable pair, and satisfy* $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-2,0]\times\Omega)}\le 6\kappa$, *and* $\|\nabla u_\ell\|_{L^\infty([-2,0]\times\Omega)}\le 2\kappa$. *Denote* $\gamma:=\Gamma_\ell-\Gamma$ *and assume* $\|\dot\gamma\|_{L^\infty([-2,0])}\le C_{pth}$ *and* $\gamma(0)=0$.

If

$$\theta(t,x)\le 2+\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+\qquad \forall t\in[-2,0],x\in\Omega\smallsetminus B_2(\Gamma(t))$$

and

$$\int_{-2}^{0}\int_{\Omega\cap B_2(\Gamma(t))}(\theta)_+^2\,dxdt\le\delta_0$$

then

$$\theta(t,x)\le 1\qquad\forall t\in[-1,0],x\in\Omega\cap B_1(\Gamma(t)).$$

Proof. Let ψ be such that $\psi=0$ for $|x|\le 1$ and $\psi(x)=2+\left(|x|^{1/4}-2^{1/4}\right)_+$ for $|x|>2$, and let $\nabla\psi$ and $D^2\psi$ be bounded.

For any constant $a>0$, we can apply Lemma 5.2 to the function

$$\theta_a:=(\theta(t,x)-\psi(x-\Gamma(t))-a)_+$$

and obtain

$$\frac{d}{dt}\int\theta_a^2+\int\left|\Lambda^{1/2}\theta_a\right|^2\le C\left(\int\chi_{\{\theta_a\ge0\}}+\int\theta_a+\int\theta_a^2\right).$$

Thus $\theta-\psi(x-\Gamma)$ satisfies the assumptions of Lemma A.1. There exists a constant, which we call δ_0, so that if

$$\int_{-2}^{0}\int(\theta(t,x)-\psi(x-\Gamma(t)))_+\,dxdt\le\delta_0$$

then

$$\theta(t,x)\le 1+\psi(x-\Gamma(t))\qquad\forall t\in[-1,0],x\in\Omega.$$

By construction of ψ, our result follows immediately.

□

Next, we will prove De Giorgi's second lemma, a quantitative analog of the isoperimetric inequality.

Proposition 5.4 (Second De Giorgi Lemma)**.** *Let κ, C_{dmn}, C_{pth}, and $\beta\in(0,1)$ be positive constants. Then there exists a constant $\mu>0$ such that the following holds:*

Let $\Omega\subseteq\mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$, and let $\Gamma:[-5,0]\to\mathbb{R}^2$ be Lipschitz. Assume that Proposition 2.1 *holds on Ω with kernels that satisfy*

$$K_{1/4}(x,y)\le C_{dmn}|x-y|^{3/4}K_1(x,y).$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (21) *on* $[-5,0]\times\Omega$ *with θ and $u_\ell+u_h$ a suitable pair, and satisfy* $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-5,0]\times\Omega)}\le 6\kappa$, *and* $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)}\le 2\kappa$. *Denote* $\gamma:=\Gamma_\ell-\Gamma$ *and assume* $\|\dot\gamma\|_{L^\infty([-5,0])}\le C_{pth}$ *and* $\gamma(0)=0$.

Suppose that for $t\in[-5,0]$ and any $x\in\Omega$,

$$\theta(t,x)\le 2+\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+.$$

Then the three conditions

$$|\{\theta\ge1\}\cap[-2,0]\times B_2(\Gamma)|\ge\delta_0/4,\tag{25}$$

$$|\{0<\theta<1\}\cap[-4,0]\times B_4(\Gamma)|\le\mu,$$

$$|\{\theta\le0\}\cap[-4,0]\times B_4(\Gamma)|\ge2|B_4|\tag{26}$$

cannot simultaneously be met.

Here δ_0 is the constant from Proposition 5.3, which of course depends on κ, C_{pth}, and C_{dmn}.

Proof. Suppose that the proposition is false. Then there must exist, for each $n \in \mathbb{N}$, a bounded open set Ω_n with C^{2,β_n} boundary for $\beta_n \in (0,1)$, a Lipschitz path $\Gamma_n : [-5,0] \to \mathbb{R}^2$, a function $\theta_n : [-5,0] \times \Omega_n \to \mathbb{R}$, functions $u_\ell^n, u_h^n : [-5,0] \times \Omega_n \to \mathbb{R}^2$, and paths $\Gamma_\ell^n = \Gamma_n + \gamma_n : [-5,0] \to \mathbb{R}^2$ which solve (21) and satisfy all of the the assumptions of our proposition (with the same constants κ, C_{pth}, and C_{dmn}), except that

$$|\{0 < \theta_n < 1\} \cap [-4,0] \times B_4(\Gamma_n)| \le 1/n. \tag{27}$$

Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function which vanishes on B_2 such that $\psi(x) = 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 3$.

Fix n and define

$$\theta_+ := \left(\theta_n - \psi(x - \Gamma_n)\right)_+ .$$

Then θ_+ is supported on $\Omega \cap B_3(\Gamma_n)$ and is less than $2 + 3^{1/4} - 2^{1/4} \le 3$ everywhere.

Our goal is to bound the derivatives of θ_+^2 so that we can apply a compactness argument to the sequence θ_n. (For the curious reader, it is the calculations in Step 2 below in which it becomes necessary to consider θ_+^2 instead of θ_+.)

The remainder of the proof is divided in three steps. First we show that the sequence of θ_+ is compact in space, then we show that it is compact in time, and finally we show that the limiting function implies a contradiction.

Step 1: Compactness in space

Apply the energy inequality Lemma 5.2 to θ and $\psi(x-\Gamma_n)$, and find that for some C independent of n

$$\frac{d}{dt}\int \theta_+^2 \le C. \tag{28}$$

Moreover, by integrating Lemma 5.2 in time from -5 to $s \in [-4,0]$ and taking a supremum over s, we find

$$\sup_{[-4,0]}\int \theta_+^2 + \int_{-4}^0\int \left|\Lambda^{1/2}\theta_+\right|^2 + \int_{-4}^0\int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \le C. \tag{29}$$

This proves in particular that $\theta_+ \in L^2(-4,0;\mathcal{H}^{1/2}(\Omega))$ is uniformly bounded.

Furthermore, $\left\|\theta_+^2\right\|_{L^2(-4,0;\mathcal{H}^{1/2}(\Omega_n))}$ is uniformly bounded because

$$\begin{aligned}\left\|\Lambda^{1/2}(\theta_+^2)\right\|_2^2 &= \iint [\theta_+(x)^2 - \theta_+(y)^2]^2 K + \int \theta_+^4 B \\ &\le 2\iint \theta_+(x)^2[\theta_+(x) - \theta_+(y)]^2 K + 2\iint \theta_+(y)^2[\theta_+(x) - \theta_+(y)]^2 K + \|\theta_+\|_\infty^2 \int \theta_+^2 B \\ &\le C\,\|\theta_+\|_\infty^2\,\|\theta_+\|_{\mathcal{H}^{1/2}}^2 .\end{aligned}$$

By Proposition 2.3, for E the extension-by-zero operator from $L^2(\Omega_n)$ to $L^2(\mathbb{R}^2)$,

$$\left\|E\theta_+^2\right\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \le C \tag{30}$$

where C does not depend on n.

Step 2: Compactness in time

Let $\varphi \in C_0^\infty([-4,0];C^\infty(\Omega))$ a test function. Since each θ_n and $u_h^n + u_\ell^n$ is a suitable pair by assumption, we can apply the inequality (9) to find that, for some constant C independent of n

and of φ, on $[-4,0]\times\Omega_n$
(31)
$$\iint \varphi\partial_t\theta_+^2 + \iint \varphi\dot{\Gamma}_n\cdot\nabla\theta_+^2 \le \iint \theta_+^2\left(u_\ell^n - \dot{\Gamma}_n + u_h^n\right)\cdot\nabla\varphi - 2\iint \varphi\theta_+\left(u_\ell^n - \dot{\Gamma}_n + u_h^n\right)\cdot\nabla\psi$$
$$- \iint \theta_+^2\Lambda\varphi - 2\int\iint [\varphi(x)\theta_+(x) - \varphi(y)\theta_+(y)][\psi(x-\Gamma_n) - \psi(y-\Gamma_n)]K_1.$$

For the low pass terms, as in the proof of Lemma 5.2, we have $\left|u_\ell^n(t,x) - \dot{\Gamma}_n(t)\right| \le (1+8\kappa)C_{pth} + 6\kappa$ for $t\in[-4,0]$ and $x\in\operatorname{supp}(\theta_+)\subseteq B_3(\Gamma_n(t))$. Thus for $t\in[-4,0]$ we have for C independent of n and φ

$$\begin{aligned}\int\left(u_\ell^n - \dot{\Gamma}_n\right)\cdot\left(\theta_+^2\nabla\varphi\right) &\le C\left\|\nabla\varphi\right\|_{L^\infty(\Omega)},\\ \int\left(u_\ell^n - \dot{\Gamma}_n\right)\cdot\left(\theta_+\varphi\nabla\psi\right) &\le C\left\|\varphi\right\|_{L^\infty(\Omega)}.\end{aligned} \tag{32}$$

For the high pass terms, we have u_h^n uniformly bounded in $\dot{W}^{-1/4,\infty}$. From step 1, we know θ_+^2 is uniformly bounded in $L^2(-4,0;\mathcal{H}^{1/2})$ so, by Lemma 2.2 parts e and c, there is a constant C independent of n and φ such that

$$\begin{aligned}\iint u_h^n\cdot(\theta_+^2\nabla\varphi) &\le C\left(\|\nabla\varphi\|_{C^0(-4,0;L^\infty(\Omega))} + \|\varphi\|_{C^0(-4,0;C^2(\Omega))}\right),\\ \iint u_h^n\cdot(\theta_+\varphi\nabla\psi) &\le C\left(\|\varphi\|_{C^0(-4,0;L^\infty(\Omega))} + \|\varphi\|_{C^0(-4,0;C^1(\Omega))}\right).\end{aligned} \tag{33}$$

Since $\|\Lambda\varphi\|_{L^2} = \|\nabla\varphi\|_{L^2}$, we have

$$-\iint \theta_+^2\Lambda\varphi \le \|\varphi\| \tag{34}$$

Plugging these six bounds into (31), we have for a constant C independent of n and φ

$$\int_{-4}^0\int_{\Omega_n}\left(\partial_t\theta_+^2 + \dot{\Gamma}_n\cdot\nabla\theta_+^2\right)\varphi\,dxdt \le C\|\varphi\|_{C^0(0,T;C^2(\Omega))}. \tag{35}$$

Step 3: Taking the limit

We wish to analyze the limiting behavior of θ_+^2 in the vicinity of Γ_n. First we shift these functions to remove the dependence on Γ_n, and define new functions on $[-4,0]\times\mathbb{R}^2$ by

$$v_n(t,x) := \begin{cases}\theta_+(t,x+\Gamma_n(t))^2, & x+\Gamma_n(t)\in\Omega_n,\\ 0, & x+\Gamma_n(t)\notin\Omega_n.\end{cases}$$

Each v_n is supported on $|x|\le 3$, and

$$v_n(t,x) = (\theta_n(t,x+\Gamma_n(t)) - \psi(x))_+^2 \tag{36}$$

whenever the right hand side is defined.

Note that
$$\partial_t v_n(t,x) = \partial_t\theta_+^2(t,x+\Gamma_n) + \dot{\Gamma}_n\cdot\nabla\theta_+^2(t,x+\Gamma_n).$$

For C independent of n, we know from (30) that
$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))} \le C$$
and from (35) that
$$\|\partial_t v_n\|_{\mathcal{M}(-4,0;C^{-2}(\Omega))} \le C$$
where $\mathcal{M}$ is the space of Radon measures with total-variation norm and $C^{-2}(\Omega)$ is the dual of $C^2(\Omega)$.

Therefore, by the Aubin-Lions Lemma, the set $\{v_n\}_n$ is compactly embedded in $L^2([-4,0]\times\mathbb{R}^2)$. Up to a subsequence, there is a function $v \in L^2([-4,0]\times\mathbb{R}^2)$ such that

$$v_n \xrightarrow{L^2} v.$$

By elementary properties of L^2 convergence, we know that $v \in L^\infty$, $\mathrm{supp}(v) \subseteq [-4,0]\times B_3(0)$, and $v \in L^2(H^{1/2})$.

By (28)

$$\frac{d}{dt}\int_{\mathbb{R}^2} v_n\,dx = \frac{d}{dt}\int_{\Omega_n} \theta_+^2\,dx \le C \tag{37}$$

so the same must be true of v, for $\frac{d}{dt}$ interpreted in the sense of distributions.

By (25), (27), and (26) applied to v_n (recalling the relation (36)), we conclude that

$$\begin{cases} |\{v \ge 1\} \cap [-2,0]\times B_2(0)| \ge \delta_0/4, \\ \left|\{0 < v < [1-\psi]^2\} \cap [-4,0]\times B_4(0)\right| \le 0, \\ |\{v \le 0\} \cap [-4,0]\times B_4(0)| \ge 2|B_4| \end{cases} \tag{38}$$

For any $(t,x) \in [-4,0]\times B_4(0)$, either $v(t,x) \ge [1-\psi(x)]^2$ or else $v(t,x) = 0$. In fact, since $\|v(t,\cdot)\|_{H^{1/2}} < \infty$ for almost every t and $H^{1/2}$ does not contain functions with jump discontinuities, the function v is either identically 0 or else $\ge [1-\psi(x)]^2$ at each t.

Thus $\int v(t,x)\,dx$ is either 0 or else $\ge \int [1-\psi(x)]^2\,dx > 0$ at each t. By (37) and (38), v must be identically zero for all $t > -2$ but also must be non-zero for some $t > -2$, which is a contradiction.

Our assumption that the sequence θ_n exists must have been false, and the proposition must be true.

□

6. A Decrease in Oscillation

We combine the two De Giorgi lemmas (Propositions 5.3 and 5.4) to produce an oscillation lemma. This result is similar to the weak Harnack inequality for harmonic functions. As in the previous section, all of the following results must be independent of the size of Ω, and any assumptions made on Ω must be scaling invariant.

Lemma 6.1 (Oscillation Lemma)**.** *Let κ, C_{dmn}, and C_{pth}, be positive constants. Then there exists a constant $k_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma_\ell : [-5,0] \to \mathbb{R}^2$ be Lipschitz. Assume that Proposition 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_{dmn}|x-y|^{3/4}K_1.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (21) on $[-5,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-5,0]\times\Omega)} \le 6\kappa$, and $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)} \le 2\kappa$. Denote $\gamma := \Gamma_\ell - \Gamma$ and assume $\|\dot\gamma\|_{L^\infty([-5,0])} \le C_{pth}$ and $\gamma(0) = 0$.

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$\theta(t,x) \le 2 + 2^{-k_0}\left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+, \tag{39}$$

and that

$$|\{\theta \le 0\} \cap [-4,0]\times B_4(\Gamma)| \ge 2|B_4|.$$

Then for all $t \in [-1,0]$, $x \in \Omega \cap B_1(\Gamma)$ we have

$$\theta(t,x) \le 2 - 2^{-k_0}.$$

Proof. Let μ and δ_0 as in Proposition 5.4, and take k_0 large enough that $(k_0 - 1)\mu > 4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x) := 2 + 2^k(\theta(t,x) - 2).$$

That is, $\theta_0 = \theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies (39), each θ_k for $k \le k_0$ and $(t,x) \in [-5,0] \times \Omega$ satisfies

$$\theta_k(t,x) \le 2 + \left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 5.4.

Note also that

$$|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma)| \tag{40}$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0/4 \tag{41}$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed $\delta_0/4$ for all $k < k_0$ as well.

Applying Proposition 5.4 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0] \times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity (40) increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity (40) is trivially bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption (41) must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 5.3. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

For the original function θ, this means that

$$\theta(t,x) \le 2 - 2^{-k_0} \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

□

By assuming that θ is small near $x = \Gamma(t)$, we have shown that the oscillation of θ is decreased in a smaller neighborhood of $\Gamma(t)$. However, our goal is to control the oscillation near $x = \Gamma_\ell(t))$. Therefore we will prove the following proposition:

Proposition 6.2 (Oscillation Lemma with shift). *Let κ, C_{dmn}, and C_{pth}, be positive constants, and let k_0 be as in Lemma 6.1. Then there exists a constant $\lambda > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma_\ell : [-5,0] \to \mathbb{R}^2$ be Lipschitz. Assume that Proposition 2.1 hold on Ω with kernels that satisfy

$$K_{1/4}(x,y) \le C_{dmn}|x-y|^{3/4}K_1.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (21) on $[-5,0] \times \Omega$, and satisfy $\|\Lambda^{-1/4}u_h\|_{L^\infty([-5,0]\times\Omega)} \le 6\kappa$, and $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)} \le 2\kappa$. Denote $\gamma := \Gamma_\ell - \Gamma$ and assume $\|\dot\gamma\|_{L^\infty([-5,0])} \le C_{pth}$ and $\gamma(0) = 0$.

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$|\theta(t,x)| \le 2 + 2^{-k_0}\left(|x - \Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \tag{42}$$

and that

$$|\{\theta \le 0\} \cap [-4,0] \times B_4(\Gamma)| \ge 2|B_4|.$$

Then for any $\varepsilon \in (0,1/5]$ such that

$$5C_{pth} \le \varepsilon^{-1} - 3 \tag{43}$$

we have

$$\left|\frac{2}{2-\lambda}\left[\theta(\varepsilon t,\varepsilon x)+\lambda\right]\right| \le 2+2^{-k_0}\left(|x-\varepsilon^{-1}\Gamma_\ell(\varepsilon t)|^{1/4}-2^{1/4}\right)_+ .$$

for all $t\in[-5,0]$ *and* x *such that* $\varepsilon x\in\Omega$.

The idea of the proof is to consider a small enough time interval that $\Gamma(t)$ is very close to $\Gamma_\ell(t)$. This is possible because $\Gamma_\ell-\gamma$ is uniformly Lipschitz by assumption.

If, in this proposition, we only wished to show the existence of some $\varepsilon=\varepsilon(k_0,C_{pth})$ satisfying the proposition's conclusion, then a simpler non-constructive proof would suffice. However, in Section 7 we will apply this proposition with parameters k_0 and C_{pth} depending on ε. To avoid circularity, we must prove the result for all ε satisfying (43).

Proof. Let $\bar\lambda>0$ and $\alpha>1$ be the universal constants defined in Lemma A.4. Take $\lambda>0$ such that

$$2\lambda\le 2^{-k_0},\qquad (2+\lambda)\left(\frac{2}{2-\lambda}\right)\le 2+2^{-k_0}\bar\lambda,\qquad \frac{2}{2-\lambda}\le\alpha. \tag{44}$$

Denote

$$\bar\theta(t,x):=\frac{2}{2-\lambda}\left[\theta(\varepsilon t,\varepsilon x)+\lambda\right]$$

defined for $t\in[-5/\varepsilon,0]$ and

$$x\in\Omega_\varepsilon:=\{x\in\mathbb{R}^2:\varepsilon x\in\Omega\}$$

and denote

$$\phi(x):=\left(|x|^{1/4}-2^{1/4}\right)_+ .$$

We proved in Lemma 6.1 that $\theta(t,x)\le 2-2^{-k_0}$ for $t\in[-1,0]$ and $x\in\Omega\cap B_1(\Gamma)$. On this same set, $\theta(t,x)\ge-2$ by assumption. By the definition of $\bar\theta$ and by (44), for all $t\in[-1/\varepsilon,0]$ and $x\in\Omega\cap B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$ we have therefore

$$\begin{cases}\bar\theta(t,x) & \le \frac{2}{2-\lambda}\left[2-2^{-k_0}+\lambda\right]\le\frac{2}{2-\lambda}\left[2-\lambda\right]=2.\\ \bar\theta(t,x) & \ge\frac{2}{2-\lambda}\left[-2+\lambda\right]=-2.\end{cases} \tag{45}$$

Similarly, the bound (42) on θ becomes the equivalent bounds on $\bar\theta$, for all $(t,x)\in[-5/\varepsilon,0]\times\Omega_\varepsilon$

$$\bar\theta(t,x)\le\frac{2}{2-\lambda}\left[2+2^{-k_0}\phi(|\varepsilon x-\Gamma(\varepsilon t)|)+\lambda\right] \tag{46}$$

and

$$\bar\theta(t,x)\ge\frac{2}{2-\lambda}\left[-2-2^{-k_0}\phi(|\varepsilon x-\Gamma(\varepsilon t)|)+\lambda\right]. \tag{47}$$

Let $t\in[-5,0]$ and $x\in\Omega_\varepsilon$, and define

$$y:=x-\varepsilon^{-1}\Gamma(\varepsilon t).$$

From (46) and the assumptions (44), we can bound

$$\begin{aligned}\bar\theta(t,x)&\le\frac{2}{2-\lambda}\left[2+\lambda+2^{-k_0}\phi(\varepsilon|y|)\right]\\&\le 2+2^{-k_0}\bar\lambda+2^{-k_0}\alpha\phi(\varepsilon|y|)\\&=2+2^{-k_0}\left[\bar\lambda+\alpha\phi(\varepsilon|y|)\right].\end{aligned}$$

From (47) and the assumptions (44), we can bound

$$\begin{aligned}
-\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2-\lambda+2^{-k_0}\phi(\varepsilon|y|)\right] \\
&\leq 2+2^{-k_0}\alpha\phi(\varepsilon|y|) \\
&\leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right].
\end{aligned}$$

Therefore

$$\left|\bar{\theta}(t,x)\right| \leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right]. \tag{48}$$

If $|y| \leq \varepsilon^{-1}$ then from (45) we have

$$\left|\bar{\theta}(t,x)\right| \leq 2 \leq 2+2^{-k_0}\phi(x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t))$$

which is our desired result. Therefore assume without loss of generality that $|y| \geq \varepsilon^{-1}$. In this case we can apply Lemma A.4 which states that, since $\varepsilon < 1/2$ and $\varepsilon|y| \geq 1$, it is a property of ϕ, α, and $\bar{\lambda}$ that

$$2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right] \leq 2+2^{-k_0}\left[\phi(|y|-\varepsilon^{-1}+3)\right].$$

For $t \in [-5,0]$, we have by assumption (43)

$$|y|-\varepsilon^{-1}+3 \leq |y|-5C_{pth} \leq |y-\varepsilon^{-1}\gamma(\varepsilon t)|.$$

The estimate (48) becomes

$$\left|\bar{\theta}(t,x)\right| \leq 2+2^{-k_0}\phi(|x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t)|).$$

This concludes the proof. □

7. Hölder Continuity

In this section we shall prove the main theorem, Theorem 1.1. We begin with a final lemma to describe the scaling properties of (2).

Lemma 7.1 (Scaling)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $j_0 \in \mathbb{Z}$ and $\varepsilon > 0$ be constants.*

Suppose that $\theta : [-T,0]\times\Omega \to \mathbb{R}$ and $u : [-T,0]\times\Omega \to \mathbb{R}^2$ solve (2) and u is calibrated by a sequence $(u_j)_{j\geq j_0}$ with constant κ and center N.

Suppose that on Ω the functions $K_{1/4}$ and K_1 (defined in Proposition 2.1) satisfy the relation

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4}K_1(x,y) \qquad \forall x \neq y \in \Omega.$$

Then

$$\bar{\theta}(t,x) := \theta(\varepsilon t, \varepsilon x)$$

and

$$\bar{u}(t,x) := \sum_{j=j_0}^{\infty} \bar{u}_j(t,x), \qquad \bar{u}_j(t,x) := u_j(\varepsilon t, \varepsilon x)$$

also solve (2) on $[-T/\varepsilon, 0]\times\Omega_\varepsilon$ where $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$.

Moreover, $(\bar{u}_j)_{j\geq j_0}$ is calibrated with the same constant κ but with center $N-\log_2(\varepsilon)$, and the relation

$$\bar{K}_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4}\bar{K}_1(x,y) \qquad \forall x \neq y \in \Omega_\varepsilon \tag{49}$$

holds.

Proof. Denote by $\bar{\Lambda}$ the square root of the Laplacian with Dirichlet boundary conditions on Ω_ε. One can calculate (see e.g. [CS16] Section 2.4) that for $(t,x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$

$$\Lambda\theta(\varepsilon t, \varepsilon x) = \varepsilon\bar{\Lambda}\bar{\theta}(t,x).$$

Similarly, in the Caffarelli-Stinga representation from Proposition 2.1 the operator $\bar{\Lambda}^s$ will have kernel

$$\bar{K}_s(x,y) = \varepsilon^{s-2} K_s(\varepsilon x, \varepsilon y).$$

From these facts it is clear that the scaled functions satisfy (2) and (49).

To show that $(\bar{u}_j)_{j\in\mathbb{Z}}$ is calibrated, we must translate the three bounds on u_j to corresponding bounds on $\bar{u}_j$. Each of the calculations are similar, so we show only one:

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon \|\nabla u_j\|_\infty \le 2^{\log_2(\varepsilon)} 2^j 2^{-N} \kappa = 2^j 2^{-(N-\log_2(\varepsilon))}\kappa.$$

□

Proof of Theorem 1.1. By Proposition **??**, the L^∞ norm of θ, after a short time, will be bounded by $\|\theta_0\|_{L^2}$. By translating and scaling, it will be sufficient to assume that θ solving (1) satisfies

$$\|\theta\|_{L^\infty([-5,0]\times\Omega)} \le 2$$

and prove that θ is Hölder continuous at the origin $(0,0) \in [-5,0] \times \bar{\Omega}$, meaning

$$\frac{|\theta(t,x) - \theta(0,0)|}{(|t|^2 + |x|^2)^{\alpha/2}} \le C$$

for all $(t,x) \in [-5,0] \times \Omega$ and some constants α and C depending on Ω.

From Proposition 4.1, we know that

$$u = \nabla^\perp \Lambda^{-1}\theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence $(u_j)_{j\ge j_0}$ of divergence-free functions calibrated with some constant $\kappa = \kappa(\Omega)$ and center 0. Assume without loss of generality that $j_0 < 0$.

Choose a constant $0 < \varepsilon < 1/5$ such that

$$5\max\left(-\kappa\log_2(\varepsilon)e^{10\varepsilon\kappa}, (1-j_0)\kappa\right) \le \varepsilon^{-1} - 3. \tag{50}$$

For integers $k \ge 0$ consider the domains

$$\Omega_k := \{x \in \mathbb{R}^2 : \varepsilon^k x \in \Omega\}.$$

If K_s^k are the kernels defined in Proposition 2.1 corresponding to the operators Λ^s on Ω_k, then by Proposition 2.1 and Lemma 7.1 the relation

$$K_{1/4}^k(x,y) \le C_{dmn}|x-y|^{3/4} K_1^k(x,y) \qquad \forall x \ne y \in \Omega_k$$

holds for some constant C_{dmn} independent of k.

For notational convenience, denote

$$\sum_k = \sum_{j>-k\log_2(\varepsilon)}, \qquad \sum^k = \sum_{j\le -k\log_2(\varepsilon)}$$

and define the following functions on $[-5,0]\times\Omega_k$:

$$u_\ell^k(t,x) := \sum^k u_j(\varepsilon^k t, \varepsilon^k x),$$
$$u_h^k(t,x) := \sum_k u_j(\varepsilon^k t, \varepsilon^k x).$$

By Lemmas 7.1 we know the sequence $(u_j(\varepsilon^k\cdot,\varepsilon^k\cdot))_j$ is calibrated with constant κ and center $-k\log_2(\varepsilon)$, and hence by 5.1 we know that, independently of k,

$$\left\|\Lambda^{-1/4}u_h^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le 6\kappa$$

and

$$\left\|\nabla u_\ell^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \le 2\kappa.$$

Each u_ℓ^k is a finite sum of L^∞ functions, hence L^∞ itself, though not uniformly in k.

Define $\Gamma_k,\gamma_k:[-5,0]\to\mathbb{R}^2$ by the following recursive formulae and ODEs:

$$\begin{aligned}
\Gamma_0(t) &:= 0, && t\in[-5,0],\\
\gamma_k(0) &:= 0, && k\ge 0,\\
\dot\gamma_k(t) &:= u_\ell^k(t,\Gamma_k(t)+\gamma_k(t)) - \dot\Gamma_k(t), && k\ge 0, t\in[-5,0)\\
\Gamma_k(t) &:= \varepsilon^{-1}\gamma_{k-1}(\varepsilon t)+\varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t)+\cdots+\varepsilon^{-k}\gamma_0(\varepsilon^k t), && k\ge 1, t\in[-5,0].
\end{aligned}$$

The quantity γ_k here corresponds to the part of the drift coming from frequency packets u_j which are part of the definition of u_ℓ^k but are not contained in u_ℓ^{k-1} (they would instead be included in u_h^{k-1}).

Since each u_ℓ^k is L^∞ in space-time and Lipschitz in space, these γ_k exist by a version of the Cauchy-Lipschitz theorem. For example, Theorem 3.7 of Bahouri, Chemin, and Danchin [BCD11] proves existence and uniqueness in our case. In particular, since u_ℓ^k is a vector field which is tangential to the boundary of Ω_k and has unique flows, the path $\Gamma_k+\gamma_k$ which follows this vector field must remain inside $\bar\Omega_k$ for all time and so our expressions remain well-defined.

By construction, for $k\ge 0$ we have $\Gamma_{k+1}(t)=\varepsilon^{-1}\gamma_k(\varepsilon t)+\varepsilon^{-1}\Gamma_k(\varepsilon t)$. Therefore

$$\begin{aligned}
\dot\Gamma_{k+1}(t) &= \partial_t\left[\varepsilon^{-1}\gamma_k(\varepsilon t)+\varepsilon^{-1}\Gamma_k(\varepsilon t)\right]\\
&= \dot\gamma_k(\varepsilon t)+\dot\Gamma_k(\varepsilon t)\\
&= u_\ell^k(\varepsilon t,\gamma_k(\varepsilon t)+\Gamma_k(\varepsilon t))\\
&= u_\ell^k(\varepsilon t,\varepsilon\Gamma_{k+1}(t)).
\end{aligned}$$

With this in hand, we can bound the size of γ_k. Namely, for $k\ge 1$,

$$\begin{aligned}
\dot\gamma_k(t) &= u_\ell^k(t,\Gamma_k(t)+\gamma_k(t)) - \dot\Gamma_k(t)\\
&= u_\ell^k(t,\Gamma_k(t)+\gamma_k(t)) - u_\ell^{k-1}(\varepsilon t,\varepsilon\Gamma_k(t))\\
&= \sum^k u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t)+\varepsilon^k\gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t))\\
&= \sum^{k-1}\left[u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t)+\varepsilon^k\gamma_k(t)) - u_j(\varepsilon^k t,\varepsilon^k\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t,\varepsilon^k\ldots)\\
&= \left[u_\ell^{k-1}\big(\varepsilon t,\varepsilon\Gamma_k(t)+\varepsilon\gamma_k(t)\big) - u_\ell^{k-1}(\varepsilon t,\varepsilon\Gamma_k(t))\right] + \sum_{k-1}^{k} u_j(\varepsilon^k t,\varepsilon^k\ldots).
\end{aligned}$$

The function $x\mapsto u_\ell^{k-1}(\varepsilon t,x)$ is Lipschitz, with Lipschitz constant less than 2κ. Moreover, each u_j has $\|u_j\|_\infty\le\kappa$. Thus from the above calculation we can bound

$$|\dot\gamma_k(t)| \le 2\kappa\varepsilon|\gamma_k(t)| - \kappa\log_2(\varepsilon). \tag{51}$$

Applying Gronwall's inequality, we find that for $t\in[-5,0]$

$$|\gamma_k(t)| \le \frac{-\log_2(\varepsilon)}{2\varepsilon}\left(e^{10\varepsilon\kappa}-1\right).$$

Plugging this estimate back into (51),

$$|\dot{\gamma}_k(t)| \le -\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa} \qquad \forall k \ge 1.$$

Trivially $|\dot{\gamma}_0| \le (1 - j_0)\kappa$, so if we define

$$C_{pth} = \max\left(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1 - j_0)\kappa\right)$$

then for all $k \ge 0$ and $t \in [-5, 0]$

$$|\dot{\gamma}_k(t)| \le C_{pth}.$$

Define

$$\theta_0(t,x) := \theta(t,x)$$

and for each $k \ge 0$, if $|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma_k(t))| \ge 2|B_4|$ then set

$$\theta_{k+1}(t,x) := \frac{2}{2-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) + \lambda\right].$$

Otherwise, set

$$\theta_{k+1}(t,x) := \frac{1}{1-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) - \lambda\right].$$

From Lemma 7.1, we know that θ_k and the calibrated function $\sum_{j \ge j_0} u_j(\varepsilon^k \cdot, \varepsilon^k \cdot)$ solve (2). By construction, θ_k, u_ℓ^k, u_h^k, Γ_k, and γ_k solve (21)

Since $|\theta_0| \le 2$ by assumption, we know in particular that

$$|\theta_k| \le 2 + 2^{-k_0}\left(|x - \Gamma_k(t)|^{1/4} - 2^{1/4}\right)_+ \tag{52}$$

holds for $k = 0$.

If (52) holds for k, then at least one of θ_k or $-\theta_k$ (depending on whether $|\{\theta_k \le 0\} \cap [-4,0] \times B_4(\Gamma_k(t))|$ is more or less than $2|B_4|$) will satisfy the assumptions of Proposition 6.2. In either case, we conclude that θ_{k+1} satisfies (52). By induction, this bound holds for all θ_k.

Each θ_k is between -2 and 2 on $[-5,0] \times B_2(\Gamma_k)$. But recall that each Γ_k is Lipschitz with constant kC_{pth}. Thus $|\Gamma_k(t)| \le 1$ for $t \in [-(kC_{pth})^{-1}, 0]$. On that time interval,

$$|\theta_k(t,x)| \le 2 \qquad \forall x \in B_1(0).$$

We conclude that

$$\left| \sup_{[-\varepsilon^k (kC_{pth})^{-1},0] \times B_{\varepsilon^k}(0)} \theta(t,x) - \inf_{[-\varepsilon^k (kC_{pth})^{-1},0] \times B_{\varepsilon^k}(0)} \theta(t,x) \right| \le 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In particular, for some positive constant C such that

$$\varepsilon^{Ck} \le (kC_{pth})^{-1} \qquad \forall k \ge 0,$$

we can say that

$$|t|^2 + |x|^2 \le \varepsilon^{(1+C)k}$$

implies that $(t,x) \in [-\varepsilon^k (kC_{pth})^{-1}, 0] \times B_{\varepsilon^k}(0)$ which in turn implies that

$$|\theta(t,x) - \theta(0,0)| \le 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In other words,

$$\begin{aligned}|\theta(t,x)-\theta(0,0)| &\leq 4\left(\frac{2}{2-\lambda}\right)^{-\frac{1}{1+C}\log_\varepsilon(|t|^2-|x|^2)+1}\\ &= 4\left(\frac{2}{2-\lambda}\right)\exp\left[\ln\left(\frac{2}{2-\lambda}\right)\frac{\ln(|t|^2+|x|^2)}{-(1+C)\ln(\varepsilon)}\right]\\ &= \frac{8}{2-\lambda}(|t|^2+|x|^2)^{-\frac{\ln(2)-\ln(2-\lambda)}{(1+C)\ln(\varepsilon)}}.\end{aligned}$$

This completes the proof of regularity. That $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ implies θ, u are a suitable pair is proven in Lemma **??**.

□

Appendix A. Technical Lemmas

In this appendix we state and prove a few technical lemmas.

Lemma A.1 (De Giorgi Iteration Argument)**.** *For any constant $\bar{C} \geq 0$, there exists a $\delta > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $f \in L^2([-2,0]\times\Omega)$ be a function with the property that for any positive constant a

$$\frac{d}{dt}\int (f-a)_+^2 + \int \left|\Lambda^{1/2}(f-a)_+\right|^2 \leq \bar{C}\left(\int \chi_{\{f\geq a\}} + \int (f-a)_+ + \int (f-a)_+^2\right). \tag{53}$$

Then

$$\int_{-2}^0 \int (f-0)_+^2\,dxdt \leq \delta$$

implies that

$$f(t,x) \leq 1 \qquad \forall t \in [-1,0], x \in \Omega.$$

Proof. Consider for $k \in \mathbb{N}$ the constants $t_k := -1-2^{-k}$ (so that $t_0 = -2$ and $t_\infty = -1$), and functions

$$f_k := (f-1+2^{-k})_+$$

(so that $f_0 = (f)_+$ and $f_\infty = (f-1)_+$).

Define

$$\mathcal{E}_k := \int_{t_k}^0 \int_\Omega f_k^2\,dxdt.$$

When $f_{k+1} > 0$, then in particular $f_k \geq 2^{-k-1}$. Thus for any finite p, there exists a constant C so

$$\chi_{\{f_{k+1}>0\}} \leq C^k f_k^p. \tag{54}$$

Let $k \geq 0$ and define $\eta : [-2,0] \to \mathbb{R}$ a continuous function

$$\eta(t) := \begin{cases} 0 & t \leq t_k \\ 2^{k+1}(t-t_k) & t_k \leq t \leq t_{k+1} \\ 1 & t_{k+1} \leq t. \end{cases}$$

Let $s \in (t_{k+1}, 0)$. Multiplying the inequality (53) with cutoff a_k by $\eta(t)$ and integrating in time from -2 to s, then integrating by parts, we obtain

$$\int f_k(s,x)^2\,dx - 2^{k+1}\int_{t_k}^{t_{k+1}}\int f_k(t,x)^2\,dxdt + \int_{t_{k+1}}^{s}\int \left|\Lambda^{1/2}f_k\right|^2 dxdt \leq \bar{C}\left(\int_{t_k}^{0}\int \chi_{\{f_k>0\}} + f_k + f_k^2\,dxdt\right)$$

By taking the supremum over all $s \in (t_{k+1}, 0)$, we obtain

$$\sup_{[t_{k+1},0]} \int f_k^2 \, dx + \int_{t_{k+1}}^0 \int \left|\Lambda^{1/2} f_k\right|^2 dxdt \le C \left(2^{k+1} \int_{t_k}^0 \int f_k^2 \, dxdt + \int_{t_k}^0 \int \chi_{\{f_k > 0\}} + f_k \, dxdt \right) \tag{55}$$

From Proposition 2.3 and Sobolev embedding,

$$\int_{t_{k+1}}^0 \left(\int f_k^4 \, dx \right)^{1/2} dt \le C \int_{t_{k+1}}^0 \int \left|\Lambda^{1/2} f_k\right|^2 dxdt.$$

Therefore by the Riesz-Thorin interpolation theorem,

$$\int_{t_{k+1}}^0 \int f_k^3 \, dxdt \le C \left(\sup_{[t_{k+1},0]} \int f_k^2 \, dx + \int_{t_{k+1}}^0 \int \left|\Lambda^{1/2} f_k\right|^2 \right)^{3/2}.$$

This estimate, along with (55) and (54), and the fact that $t_{k-1} < t_k$ and $f_{k-1} \ge f_k$, tell us that

$$\int_{t_{k+1}}^0 \int f_k^3 \, dxdt \le C^k \mathcal{E}_{k-1}^{3/2}.$$

Now we can estimate, using again (54) and the fact $f_k \ge f_{k+1}$,

$$\mathcal{E}_{k+1} \le C^k \int_{t_{k+1}}^0 \int f_k^3 \, dxdt \le C^k \mathcal{E}_{k-1}^{3/2}.$$

This nonlinear recursive inequality $\mathcal{E}_{k+1} \le C^k \mathcal{E}_{k-1}^{3/2}$, by a standard fact about nonlinear recursions (see [DG57] or [Vas16]), tells us that there exists a constant δ depending only on C (which in turn depends only on the constant $\bar{C}$ in (53))

$$\mathcal{E}_0 \le \delta \text{ implies } \lim_{k \to \infty} \mathcal{E}_k = 0.$$

By assumption

$$\mathcal{E}_0 = \int_{-2}^0 \int (f)_+ \le \delta.$$

Therefore $\mathcal{E}_k \to 0$ and, by the dominated convergence theorem,

$$\int_{-1}^0 \int (f-1)_+ \, dxdt = 0.$$

The result follows. □

Lemma A.2. *Let $\alpha \in (0,1)$. There exists a constant $C = C(\alpha)$ such that, for any set Ω and any $f \in C^{0,1}(\Omega)$,*

$$[f]_\alpha \le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha .$$

Proof. This simple lemma is a straightforward calculation:

$$\begin{aligned} \sup_{x,y \in \Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} &= \sup |f(x) - f(y)|^{1-\alpha} \left(\frac{|f(x) - f(y)|}{|x-y|} \right)^\alpha \\ &\le (2\|f\|_\infty)^{1-\alpha} \left(\sup \frac{|f(x) - f(y)|}{|x-y|} \right)^\alpha \\ &\le C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^\alpha . \end{aligned}$$

□

Lemma A.3. *Let $\alpha \in (0,1)$ and Ω a set that satisfies the cone condition. There exist constants $C = C(\alpha,\Omega)$ and $\ell = \ell(\Omega)$ such that, for any $f \in C^{1,\alpha}(\Omega)$*

$$\|\nabla f\|_\infty \leq C\left(\delta^{-1}\|f\|_\infty + \delta^\alpha [\nabla f]_\alpha\right)$$

for all $\delta < \ell$.

The idea of the proof is to average ∇f along an interval of length δ with endpoint x. The magnitude of the average will be small, since $f \in L^\infty$, and the average will differ not very much from $\nabla f(x)$ since $\nabla f \in C^{1,\alpha}$.

Proof. Since Ω satisfies the cone condition, there exist positive constants ℓ and $a < 1$ such that, at each point $x \in \bar{\Omega}$, there exist two unit vectors e_1 and e_2 such that $|e_1 \cdot e_2| \leq a$ and $x + \tau e_i \in \Omega$ for $i = 1,2$ and $0 < \tau \leq \ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

Consider the directional derivative $\partial_i f$ of f along the direction e_i, and observe that for any $0 < \delta \leq \ell$,

$$\left|\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau\right| = |f(x + \delta e_i) - f(x)| \leq 2\|f\|_\infty. \tag{56}$$

On the other hand, $\partial_i f$ is continous so, for any $\tau \in (0,\ell]$,

$$|\partial_i f(x) - \partial_i f(x + \tau e_i)| \leq [\nabla f]_\alpha \tau^\alpha.$$

From this, we obtain that

$$\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \leq \int_0^\delta \left(\partial_i f(x) + [\nabla f]_\alpha \tau^\alpha\right) d\tau = \delta\partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below. Thus

$$\left|\delta\partial_i f(x) - \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau\right| \leq [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}.$$

Combining this bound with (56), we obtain

$$|\partial_i f(x)| \leq \frac{2}{\delta}\|f\|_\infty + \frac{\delta^\alpha}{1+\alpha}[\nabla f]_\alpha.$$

This bound is independent of x and of $i = 1,2$. Since $e_1 \cdot e_2 \leq a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta \in (0,\ell]$,

$$\|\nabla f\|_\infty \leq \frac{C}{1-a^2}\left(\delta^{-1}\|f\|_\infty + \delta^\alpha [\nabla f]_\alpha\right).$$

□

Lemma A.4. *There exist constants $\bar{\lambda} > 0$ and $\alpha > 1$ such that, for any $0 < \varepsilon \leq 1/2$ and any $z \geq 1$*

$$\left(|\varepsilon^{-1}(z-1) + 3|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+ \geq \bar{\lambda}.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon = 1/2$, that is to show

$$f_\alpha(z) := \left(|2z+1|^{1/4} - 2^{1/4}\right)_+ - \alpha\left(|z|^{1/4} - 2^{1/4}\right)_+ \geq \bar{\lambda}$$

for all $z \geq 1$. Note that $f_\alpha(z) \geq f_\beta(z)$ if $\alpha < \beta$.

For $z \geq 2$,

$$f_\alpha(z) = (2z+1)^{1/4} - 2^{1/4} - \alpha z^{1/4} + \alpha 2^{1/4} = z^{1/4}\left((2+1/z)^{1/4} - \alpha\right) + (\alpha - 1)2^{1/4}.$$

For any $\alpha < 2^{1/4}$, clearly $f_\alpha(z)$ tends to ∞ as z increases. Therefore there exist N and $\alpha_0 > 1$ such that

$$f_\alpha(z) \geq 1 \qquad \forall z \geq N, \alpha \leq \alpha_0.$$

We can decompose $f_\alpha(z) = g_1(z) - (\alpha - 1)g_2(z)$ where

$$g_1(z) := \left(|2z+1|^{1/4} - 2^{1/4}\right)_+ - \left(|z|^{1/4} - 2^{1/4}\right)_+,$$
$$g_2(z) := \left(|z|^{1/4} - 2^{1/4}\right)_+.$$

Note that g_1, g_2 are both continuous, and $g_1(z)$ is strictly positive for $z \geq 1$. Therefore we can take $\alpha \in (1, \alpha_0]$ small enough that

$$\alpha - 1 < \frac{\inf_{[1,N]} g_1}{\sup_{[1,N]} g_2}.$$

For this α, $f_\alpha(z)$ is strictly positive on the compact interval $[1, N]$, and $f_\alpha(z) \geq 1$ on $[N, \infty)$. Therefore $f_\alpha(z)$ has a positive lower bound $\bar{\lambda}$ for all $z \geq 1$. □

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