The De Giorgi Method

Applications to Degenerate PDE

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Outline

1. Overview of the De Girgi Method
2. Superquadratic Hamilton-Jacobi Equations
3. Hypoelliptic Fokker-Planck Equation
4. L^2-stability of Conservation Laws
5. SQG on Bounded Domains

Overview of De Giorgi Method

Consider the toy problem

$$\partial_t u + \operatorname{div}(A \nabla u) = 0$$

Given $\Lambda^{-1} \leq A \leq \Lambda$ (in sense of inner product), parabolic, expect regularity
In fact, $\exists \alpha \in (0, 1)$ s.t. $\forall \varepsilon > 0$, $\exists C > 0$

$$\|u\|_{C^{\alpha}([\varepsilon,\infty)\times\mathbb{R}^n)} \leq C \|u(0, \cdot)\|_{L^2(\mathbb{R}^n)}$$

Energy Inequality

Let $A \subseteq \mathbb{R}^n$, $[a, b]$ an interval, $\varepsilon > 0$
Multiply by test function $\phi(t, x)(u - k)_+$, obtain

$$\sup_{[a,b]} \int_A (u-k)_+^2 + \int_a^b \int_A |\nabla (u-k)_+|^2 \lesssim \int_{a-\varepsilon}^b \int_{B_\varepsilon(A)} (u-k)_+^2$$

First De Giorgi Lemma

- L^2-to-L^∞ regularization
- local and nonlocal version
- proof by truncation, recursion

Consider $k \in \mathbb{N}$

$$[-2,0] \times B_2 =: Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq \cdots \supseteq [-1,0] \times B_1$$

and truncations

$$(u-0)_+ =: u_0 \geq u_1 \geq \cdots \geq u_k \geq \cdots \geq (u-1)_+$$

First De Giorgi Lemma

Lemma

Let u solve parabolic equation, there exist δ_0 small so

$$\iint_{Q_0} u_0^2 \leq \delta_0$$

implies

$$\iint_{[-1,0]\times B_1} (u-1)_+ = 0 \qquad \equiv \qquad u \leq 1 \ on \ [-1,0] \times B_1$$

Proof of First De Giorgi Lemma

Energy inequality says

$$\sup \int_{Q_k} u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \lesssim \iint_{Q_{k-1}} u_k^2$$

Sobolev embedding says, $\exists q > 2$

$$\iint_{Q_k} u_k^q \leq \left(\sup \int_{Q_k} u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \right)^{q/2}$$

Common sense says

$$\iint_{Q_k} u_k^2 \leq C_k \iint_{Q_k} u_{k-1}^q$$

Proof of First De Giorgi Lemma

Put together,

$$\iint_{Q_{k+1}} u_{k+1}^2 \leq C_k \left(\iint_{Q_k} u_k^2 \right)^{1+\varepsilon}$$

if u_0 is sufficiently small on Q_0, then $(u-1)_+$ vanishes on $[-1,0] \times B_1$

Second De Giorgi Lemma

- also called Isoperimetric Inequality
- quantitative version of "solutions to parabolic eqn have no jump discontinuities"
- often non-constructive (compactness)
- proof varies significantly between applications

Second De Giorgi Lemma

Lemma

$\exists\mu_0 > 0$ s.t., u sol'n parabolic eqn, following can't all be true:

$$u \leq 2 \qquad on\ [-1,4] \times B_3,$$

$$|\{u \geq 1\} \cap [2,4] \times B_1| \geq \delta_0,$$
$$|\{0 < u < 1\} \cap [0,4] \times B_2| \leq \mu_0,$$
$$|\{u \leq 0\} \cap [0,4] \times B_2| \geq \frac{1}{2}\,|[0,4] \times B_2|$$

Proof of Second De Giorgi Lemma

Assume false for all μ_0, take sequence u_k of counterexamples $\mu_0 = 1/k$

By first condition and energy inequality, u_k compact, has $L^2_t(H^1_x)$ limit u_∞

t fixed, $\forall x$ either $u_\infty \geq 1$ or $u_\infty \leq 0$

u_∞ is ≤ 0 on most of $[0, 4]$ but ≥ 1 on some of $[2, 4]$, so energy increases suddenly in time, impossible

Hölder Continuity

Proof of Hölder continuity uses recursion

apply two De Giorgi lemmas to rescaled solutions

Superquadratic Hamilton-Jacobi Equation

$$\partial_t u + |\nabla u|^p - \varepsilon \Delta u = 0, \qquad \varepsilon \in 0, 1, -1$$

- First considered by Lasry and Lions ('89), Schwab ('13) [homogenization]
- Best known results Cardaliaguet ('09), Cannarsa and Cardaliaguet ('10), Cardaliaguet and Silvestre ('12)
- $\varepsilon = 0$ cusps form, but solutions remain continuous
- For $p > 2$, continuous even for $\varepsilon = -1$ [first order drives regularization]
- Chan and Vasseur ('17) use De Giorgi for $\varepsilon = 0$

Superquadratic Hamilton-Jacobi

$$\partial_t u = H\left(t, x, u, \nabla u, D^2 u\right),$$

$$\Lambda^{-1}|\nabla u|^p - \mathrm{div}\,(A\nabla u) - f \leq H\left(t, x, u, \nabla u, D^2 u\right) \leq \Lambda|\nabla u|^p - \Lambda m^-(D^2 u) +$$

$p > 2$, A bounded unsigned matrix, $f \in L^q$, m^- returns lowest negative eigenvalue

Theorem

Solutions (in appropriate weak sense) regularize from $L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ into $C^\alpha([\varepsilon, \infty) \times \mathbb{R}^n)$

Proof of Superquadratic Hamilton-Jacobi

- De Giorgi method
- Adapted technique of Chan, Vasseur, overcome second-order term
- Combine divergence-form and non-divergence-form techniques
- Allow unbounded source term f, discontinuous A

Hypoelliptic Fokker-Planck

$$[\partial_t + v \cdot \nabla_x] f + (-\Delta_v)^s f = 0$$

- Rarefied gas, neutral particles in plasma
- Imbert and Silvestre ('16), Golse and Imbert and Mouhot and Vasseur ('16)
- Hypoelliptic: non-elliptic regularization, mixed elliptic/hyperbolic type
- Averaging Lemma (Golse et al '88): H^s theory of hypoellipticity, regularity of averages for kinetic equation

Hypoelliptic Fokker-Planck

$$[\partial_t + v \cdot \nabla_x] f = \int K[f(w) - f(v)]\, dw + \sigma$$

K symmetric (in $(v, w) \mapsto (w, v)$ and $(v, v + w) \mapsto (v, v - w)$ senses), $K \approx |v - w|^{-(n+2s)}$, $s \in (0, 1)$

Theorem

For f solution, $f \in L^\infty \cap L^2_{t,x}(H^s_v)$, $\sigma \in L^2 \cap L^r$ for $r >> 1$, there exists $\alpha \in (0, 1)$ depending on kernel, $C > 0$ depending on domain and kernel s.t.

$$\|f\|_{C^\alpha([\varepsilon,\infty)\times\mathbb{R}^n\times B_1)} \leq C\left(\|f\|_{L^\infty} + \|\sigma\|_{L^r}\right).$$

Proof of Hypoelliptic Fokker-Planck

- De Giorgi method
- No distinction between $s \geq 1/2$, $s < 1/2$
- Averaging Lemma

Global SQG

- Quasigeostrophic: perturbation of geostrophic wind, mid-latitude, atmospheric or ocean currents
- $\mathbb{R}^2$: Constantin, Majda, Tabak (93); Kiselev, Nazarov, Volberg (08); Caffarelli, Vasseur (10); Constantin, Vicol (12)
- Bounded domain: Kriventsov ('15); Novack, Vasseur ('18,19)
- New model by Constantin, Ignatova ('16); Constantin, Ignatova, Nguyen (various)

$$\partial_t \theta + \left(\nabla^{\perp} (-\Delta)^{-1/2} \theta \right) \cdot \nabla \theta + \nu (-\Delta)^s \theta = 0$$

Constantin, et al. Model

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = 0, \\ u = \nabla^{\perp}\Lambda^{-1}\theta. \end{cases} \tag{1}$$

$\Omega \subseteq \mathbb{R}^2$ smooth bounded open, $\Lambda := \sqrt{-\Delta_D}$ (defined spectrally), Δ_D the Dirichlet Laplacian on Ω

Theorem (S., Vasseur)

Let $\Omega \subseteq \mathbb{R}^2$ a bounded set, initial data $\theta_0 \in L^2(\Omega)$
There exists a global-in-time solution θ to SQG such that:
For any $S > 0$, there exists $\alpha \in (0,1)$ and a constant C so

$$\|\theta\|_{C^\alpha([S,\infty)\times\Omega)} \leq C\,\|\theta_0\|_{L^2(\Omega)}\,.$$

Proof of SQG on Bounded Domain

- De Giorgi method, Caffarelli, Vasseur (2010)
- Boundary issues: Laplacian & gradient don't commute, Caffarelli-Stinga ('16) kernel representation degenerates

Superquadratic Hamilton Jacobi

Begin with $p > 2$,

$$\partial_t u + |\nabla u|^p + \Delta u = 0.$$

Using $\varphi(t,x)(u-k)_+$ as test function, obtain energy inequality

$$\sup_{[-1,0]} \int_{B_1} (u-k)_+^2 + \int_{-1}^{0} \int_{B_1} (u-k)_+ \, |\nabla (u-k)_+|^p$$

$$\lesssim \int_{-2}^{0} \int_{B_2} (u-k)_+^2 + \int_{-2}^{0} \int_{B_2} |\nabla (u-k)_+|^2$$

Want to show, for some $q > 2$, on $Q_1 = [-1,0] \times B_1$,

$$\|(u-1)_+\|_{L^q(Q_1)} \lesssim \|(u)_+\|_{L^\infty(L^2)(Q_1)} + \|(u)_+|\nabla (u)_+|^p\|_{L^1(Q_1)}\,.$$

Works if we have coercivity!

Strategy: consider two regions, u small and u big

- u small $\Rightarrow$ L^q norm small
- u big $\Rightarrow$ coercivity $\Rightarrow$ L^q norm small

Implementation:

- $\|\nabla(u-1)_+\|_{L^p}^p \leq \|u_+|\nabla u_+|^p\|_{L^1}$
- $\|(u-1)_+\|_{L^q} \lesssim \|(u-1)_+\|_{L^\infty(L^2)} + \|\nabla(u-1)_+\|_{L^p}$

Hypoelliptic Equation

Begin with $\Lambda = (-\Delta_v)^{1/2}$, $s \in (0,1)$,

$$\partial_t f + v \cdot \nabla_x f + \Lambda^{2s} f = 0.$$

Note diffusion in v but not x!
Energy inequality will have

$$\|(f-\psi)_+\|_{L^\infty_t(L^2_{x,v})} + \|\Lambda^s (f-\psi)_+\|_{L^2_{t,x,v}}$$

on LHS

Averaging Lemma

Lemma (Bezard, 94): for $\alpha = 1/(2(1+m))$, $\Omega \Subset \bar{\Omega} \subseteq \mathbb{R}^+ \times \mathbb{R}^n$, and $f, g \in L^2(\bar{\Omega} \times \mathbb{R}^n)$, f compactly supported, we have

$$[\partial_t + v \cdot \nabla_x] f = g$$

implies

$$\left\| \int f \, dv \right\|_{H^\alpha(\Omega)} \lesssim \|f\|_{L^2(\bar{\Omega}\times\mathbb{R}^n)} + \left\| (1-\Delta_v)^{-m/2} g \right\|_{L^2(\bar{\Omega}\times\mathbb{R}^n)} .$$

Unfortunately: Can't apply lemma to $(f-\psi)_+$ due to truncation (nonlocal)

Barrier function

$$0 \le \varphi(t,x)(f-\psi)_+ \le F,$$

$$\|F\|_{L^2} + \left\|(1-\Delta_v)^{-m/2}\left[\partial_t + v\cdot\nabla_x\right]F\right\|_{L^2} \le C\,\|\varphi(f-\psi)_+\|_{L^2}\,.$$

Now:

$$\left\|\int F\,dv\right\|_{H^\alpha} \le \|\varphi(f-\psi)_+\|_{L^2}\,,$$

No regularity on f!

Solution: control of regularity is means to an end, control of integrability is the end

From energy inequality:

$$\|(f-\psi)_+\|_{L^\infty_t(L^2_{x,v})}+\|(f-\psi)_+\|_{L^2_{t,x}(L^{2+\varepsilon}_v)} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}}+\|(f-\psi)_+\|$$

From averaging lemma:

$$\|(f-\psi)_+\|_{L^{2+\varepsilon}_{t,x}(L^1_v)} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}}$$

Improvement in all three variables, for some $q>2$

$$\|(f-\psi)\|_{L^q_{t,x,v}} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}}$$

Localized Caccioppoli Inequality

θ solving transport-diffusion equation with velocity u
Then θ_+ satisfies

$$\frac{d}{dt}\int_\Omega \theta_+^2 + \int_\Omega \left|\Lambda^{1/2}\theta_+\right|^2 \leq \int_\Omega \theta_+^2 + \int_\Omega u\theta_+ \cdot \mathrm{d}\Psi + \dots$$

u bounded in:	L^∞	BMO	$B^0_{\infty,\infty}$
$\theta \in L^\infty \Rightarrow u \in \_\_$	$\times$	works on $\mathbb{R}^2$	complicated
$\int u\theta_+$ bounded	$\leq \int \theta_+$	John-Nirenberg	complicated
scaling invariant	$\checkmark$	$\checkmark$	$\checkmark$

Control on u: Littlewood Paley theory

Littlewood paley operators P_j, functional calculus for Λ,

Bernstein Inequalities

$$\|\Lambda^s P_j f\|_p \approx 2^{sj} \|f\|_p ,$$
$$\|\nabla \Lambda P_j f\|_p \approx 2^{(1+s)j} \|f\|_p .$$

Can say nothing about

$$P_j \nabla \Lambda^{-1} \theta,$$

but can say much about

$$\nabla \Lambda^{-1} P_j \theta.$$

Iwabuchi, Matsuyama, Taniguchi ("Bilinear estimates in Besov spaces generated by the Dirichlet Laplacian" 2017)

Control on u: high and low frequencies

To bound $u = \nabla\Lambda^{-1}\theta$, decompose as

$$u_{\text{low}} = \sum_{j=-\infty}^{0} \nabla\Lambda^{-1} P_j \theta$$

which is Lipschitz,

$$u_{\text{high}} = \sum_{j=0}^{\infty} \nabla\Lambda^{-1} P_j \theta$$

which should be in $W^{-\varepsilon,\infty}$.
Enough to control $\int u\theta_+ \cdot \mathrm{d}\Psi$

Control on u: high frequencies

How do we control

$$\left\|\Lambda^{-\varepsilon}\nabla\Lambda^{-1}P_j\theta\right\|_\infty?$$

Consider $P_\mu\nabla\Lambda^{-1}P_j f$, j fixed, $\mu \in \mathbb{Z}$

$$\int gP_\mu\nabla\Lambda^{-1}P_j f = \int (P_\mu g)\left(\nabla\Lambda^{-1}P_j f\right) \le \|g\|_1 2^{(1-1)j} \|f\|_\infty ,$$

$$\int gP_\mu\nabla\Lambda^{-1}P_j f = -\int (\nabla P_\mu g)\left(\Lambda^{-1}P_j f\right) \le 2^\mu \|g\|_1 2^{-j} \|f\|_\infty$$

Control on u: high frequencies

How do we control

$$\left\|\Lambda^{-\varepsilon}\nabla\Lambda^{-1}P_j\theta\right\|_\infty?$$

$$\Lambda^{-\varepsilon}\nabla\Lambda^{-1}P_j\theta = \sum_{\mu\in\mathbb{Z}}\Lambda^{-\varepsilon}P_\mu\nabla\Lambda^{-1}P_j\theta$$

- μ large ($> j$), like $2^{-\varepsilon\mu}$, summable
- μ small ($< j$), like $2^{-\varepsilon\mu}2^{\mu}$, summable

Control on u

- u is a sum of something Lipschitz and something $W^{-\varepsilon,\infty}$
- Either way, $\int u\theta \cdot \mathrm{d}\Psi$ controlled
- Independent of scale
- De Giorgi argument gives regularity

Thank you