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Applications of the De Giorgi method to degenerate parabolic PDE

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I wish to thank the multitudes of people who helped me. . . .

Applications of the De Giorgi method to degenerate parabolic PDE

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This is my abstract. I need to write the abstract.

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Introduction

1.1 Hilbert's Nineteenth Problem

In 1900, Hilbert laid out a list of 23 problems that he felt would guide the future of mathematics, similar to our modern Millennium Prize problems. The nineteenth such problem was a foundational question in the calculus of variations:

Problem 1. Let $\Omega \subseteq \mathbb{R}^d$, and let $X \subseteq L^2(\Omega)$ be the functions satisfying some boundary condition.

Let $F : \mathbb{R}^d \to \mathbb{R}$ be a smooth uniformly convex function with bounded derivative. Are elements of X for which the energy

$$E(u) := \int F(\nabla u)\, dx,$$

is minimized necessarily smooth?

It was already known that, for example, minimizers of $\int |\nabla u|^2$ are harmonic functions and hence analytic. The hypothesis was that for uniformly convex Lagrangians, which in particular satisfy $F(\xi) \approx |\xi|^2$, minimizers would be similarly regular.

The Euler-Lagrange equation for such an energy E is (using Einstein summation convention and subscripts representing derivatives) $\partial_i F_i(\nabla u) = 0$. By taking a derivative of this expression in the j^{th} direction, we obtain

$$\partial_i \left(F_{ik}(\nabla u)\partial_k u_j\right) = 0. \tag{1.1}$$

At this point we can consider the equation (1.1) as a linear equation in u_j. Simply define $A_{ik}(x) := F_{ik}(\nabla u(x))$ and consider

$$\operatorname{div}(A\nabla u_j) = 0.$$

This technique is known as "freezing" the coefficient or "freezing" the equation. Note that it is distinct from "linearizing" the equation, which involves expanding the equation around a given point using the first derivative and Taylor's theorem, though that term is often used colloquially to refer to both procedures.

Because F is smooth by assumption, the coefficient matrix A will have the same amount of regularity as ∇u. For example, if $u \in C^{k,\alpha}$ for some $k \in \mathbb{N}_{>0}$ and $\alpha \in (0,1)$, then $A \in C^{k-1,\alpha}$ by the chain rule and basic composition laws for Hölder continuous functions. Moreover, we know a priori that there exists a constant $\lambda \in (0,1)$ so $\lambda|\xi|^2 \leq \xi^\intercal A\xi \leq \lambda^{-1}|\xi|^2$ just by the uniform coercivity assumption on F.

These facts give us a simple strategy to prove the existence of smooth solutions.

1. By the direct method, a minimizer of E can be constructed in $H^1(\Omega)$.

2. Using the De Giorgi-Nash-Moser Theorem and the a priori bounds on A, we can show that any weak solution $w \in H^1(\Omega)$ of $\operatorname{div}(A\nabla w) = 0$ is necessarily Hölder continuous in $C^\alpha(\Omega)$.

3. By Schauder's Theorem, if $A \in C^{k,\alpha}$ and $u \in C^{k+1,\alpha}$ then $w = u_j \in C^{k+2,\alpha}$ as well (c.f. Gilbarg and Trudinger [GT01])

By induction, we find that $u \in C^{k,\alpha}$ for all $k \in \mathbb{N}$.

Historically, the De Giorgi-Nash-Moser theorem was the most difficult step to prove, and so Hilbert's Nineteenth Problem was first proven in De Giorgi's 1957 paper [DG57] (independently also by Nash [Nas58] in the same year, and later by Moser [Mos60] in 1960).

1.2 The De Giorgi Method

The method of De Giorgi was first applied to this elliptic problem, but in fact the core concept has wide-ranging applications. We will explore the method below, using the toy model of an inhomogeneous parabolic equation.

Let $d \geq 1$ a dimension and $\lambda \in (0,1)$ a coercivity parameter, let $A : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ a matrix satisfying $\lambda|\xi|^2 \leq \xi^\intercal A(t,x)\xi \leq \lambda^{-1}|\xi|^2$ for all $\xi \in \mathbb{R}^d$, and let $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ a scalar forcing term. Suppose that $u \in L^2(\mathbb{R}_+; H^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^d))$ satisfies the parabolic equation

$$\partial_t u = \operatorname{div}(A\nabla u) + f \qquad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{1.2}$$

The first step is to derive an energy inequality. Assuming u solves (1.2) in the sense of distributions, we can formally multiply the equation by a test function of the form $\varphi(t,x)(u-k)_+$ for φ a smooth cutoff function (i.e. which is identically one on some space-time region Q_1 and identically zero outside of some region Q_2) and k an arbitrary constant. This new equality can be manipulated by standard integration by parts and Hölder-type estimates into an energy inequality. Of course test functions must be smooth, and in general we cannot assume that $(u-k)_+$ is smooth. In this parabolic case, we can generally assume that a solution u exists in $L^2(H^1)$, which is sufficient to justify this formal calculation. In general, one may need to take the energy inequality so derived as an a priori assumption, and construct solutions which satisfy the energy inequality but not necessarily the regularity assumption needed to justify it (as in [SV19]).

The energy inequality, in the parabolic case, will have the general form

$$\int_A (u(t,\cdot)-k)_+^2\,dx+\int_s^t\int_A |\nabla(u-k)_+|^2\,dxdt \le C\left[\int_B (u(s,\cdot)-k)_+^2\,dx + \|f\|_{L^p}\left(\int_s^t\int_B (u-k)_+^{p^*}\,dxdt\right)^{1/p^*}\right]$$

where $A \subsetneq B$ are two bounded open sets with A compactly contained in B, $s < t$ are two times, $1/p + 1/p^* = 1$, and the constant C depends on the ellipticity constant λ and the distance between A and $B^{\complement}$. This is called an energy inequality because the energy $\int(u-k)_+^2$ at a time t is smaller than the energy at an earlier time s, where the dissipation term $\int|\nabla(u-k)_+|^2$ causes the energy to decrease or dissipate, and the source term corresponding to f puts more energy into the system, allowing the energy to increase. Note that the energy in a small region A is bounded by the energy in a larger region B, to account for energy that travels through space and enters through the boundary.

Though this form of the energy inequality makes the dynamics of the system clear, a more useful form is to consider intervals in time rather than points in time. For A a bounded open region in space and $[t,s]$ a time-interval, consider some parameter $\varepsilon > 0$ and define A_ε to be the ε-envelope around A (the points within distance ε of A). We call $[a-\varepsilon,b]\times A_\varepsilon$ a parabolic envelope of $[a,b]\times A$. Then

$$\sup_{t\in[a,b]}\int_A (u-k)_+^2 + \int_a^b\int_A |\nabla(u-k)_+|^2 \le C\left[\int_{a-\varepsilon}^b\int_{A_\varepsilon}(u-k)_+^2 + \left(\int_{a-\varepsilon}^b\int_{A_\varepsilon}(u-k)_+^{p^*}\right)^{1/p^*}\right]. \quad (1.3)$$

Armed with this energy inequality, the next step is to prove the so-called "first De Giorgi lemma." This result is sometimes called L^2 − to − L^∞ regularization. It comes in two flavors: a global-in-space variety which shows that any solution to (1.2) with L^2 initial data is necessarily in L^∞ locally in time, and a localized version which states that if $(u-k)_+$ has a sufficiently small L^2 norm in the parabolic envelope of some local region, then $(u-k)_+$ will satisfy an L^∞ bound in that region. For different PDE, either one of these results may or may not hold, but in general the former version is simpler so we shall concentrate on the latter version. It is the latter version which is useful in the proof of Hölder continuity.

Specifically, the first De Giorgi lemma states that

Lemma 1.2.1 (First De Giorgi Lemma). *There exists a constant δ_0 such that, for u a solution to (1.2) on $[0,2]\times B_2$ with $\|f\|_{L^p([0,2]\times B_2)} \le 1$ for some p sufficiently large, if*

$$\iint_{[0,2]\times B_2} (u-0)_+^2 \, dxdt \le \delta_0$$

then $u(t,x) \le \frac{1}{2}$ for $(t,x) \in [1,2]\times B_1$.

The proof of this lemma is by recursion. A sequence of functions $u_k := (u - \frac{1-2^{-k}}{2})_+$ and regions $Q_k := [1-2^{-k}, 2] \times B_{1+2^{-k}}$ are considered, and the goal is to prove that if

$$\iint_{Q_0} u_0^2 = \iint_{[0,2]\times B_2} (u-0)_+^2 \le \delta_0$$

then

$$\lim_{k\to\infty} \iint_{Q_k} u_k^2 = \iint_{[1,2]\times B_1} (u-1)_+^2 = 0.$$

To accomplish this, one constructs a recursive inequality comparing $\iint_{Q_k} u_k$ to $\iint_{Q_{k-1}} u_{k-1}$.

The main ingredient of this recursive inequality is the energy inequality (1.3), which compares the $L^\infty(L^2)\cap L^2(H^1)$ norm of u_k on Q_k to the $L^2 \cap L^{p^*}$ norm of u_k on Q_{k-1}.

By a variant of Sobolev's inequality, the $L^\infty(L^2)\cap L^2(H^1)$ norm of u on the left-hand-side of (1.3) controls the L^q norm of u, for some $q > 2$. As compared to the L^2 norm, the $L^\infty(L^2)$ norm has greater control on integrability in time, and the $L^2(H^1)$ norm has greater control on the integrability in space. By interpolation we obtain an improvement in both time and space:

$$\|u_k\|_{L^q(Q_{k-1})}^2 \le C\left[\|u_k\|_{L^2(H^1)}^2 + \|\nabla u_k\|_{L^2(Q_{k-1})}^2\right].$$

We now want to see that the right-hand-side of the energy inequality, the $L^2 \cap L^{p^*}$ norm, is controlled by the L^q norm of u_{k-1} on Q_{k-1}, with the same q as on the left-hand-side. This is true because u_{k-1} is bounded below on the support of u_k, meaning that in particular $u_k^a \le 2^{(k+1)(b-a)} u_{k-1}^b$ for any $0 \le a < b$. We have the non-linear bound, assuming $p^* \le q$,

$$\iint_{Q_{k-1}} u_k^2 + \left(\iint_{Q_{k-1}} u_k^{p^*} \right)^{1/p^*} \le \iint_{Q_{k-1}} u_{k-1}^q + \left(\iint_{Q_{k-1}} u_{k-1}^q \right)^{1/p^*}.$$

Notice that the exponent on u_{k-1} is always q, but the exponent on the integral itself varies. Combining this with the energy inequality and our bound on the left-hand-side of the energy inequality, we obtain

$$\|u_k\|_{L^q(Q_{k-1})} \le C^k \left[\|u_{k-1}\|_{L^q(Q_{k-1})}^{q/2} + \|u_{k-1}\|_{L^q(Q_{k-1})}^{q/(2p^*)} \right].$$

So long as the exponents $\frac{q}{2}$ and $\frac{q}{2p^*}$ are strictly greater than 1, this inequality is superlinear. In that case, if the initial value of the sequence is sufficiently small, the limit will be zero. This is sufficient to prove the lemma.

This proof method works because the energy inequality goes in the opposite direction of the conventional Sobolev and Hölder inequalities. The energy inequality has an order-1 quantity (meaning a single derivative) bounded by an order-0 quantity (meaning a quantity with no derivatives). After reducing the orders with Sobolev embedding, we find that the L^q norm is bounded by an L^2 norm and an L^{p^*} norm. Since $q > 2$, and assuming p is sufficiently large (specifically $1/p + 2/q < 1$), this defies our intuition that on bounded regions higher Lebesgue norms control lower Lebesgue norms. It is therefore not surprising that $L^2 - \text{to} - L^\infty$ regularization is possible. When applying this method to a given PDE, among the first questions one must ask is whether the natural exponent q is larger than any exponents which may appear on the right-hand-side of the energy inequality.

The next step in the De Giorgi argument is to prove the second De Giorgi lemma, also known as the isoperimetric inequality.

Lemma 1.2.2 (Second De Giorgi Lemma)**.** *There exists a constant $\mu_0 > 0$ such that, for u a solution to* (1.2) *on* $[0,4] \times B_3$ *with* $\|f\|_{L^p([0,4]\times B_3)} \leq 1$ *for some p sufficiently large, if*

$$u(t,x) \leq 2 \qquad \forall (t,x) \in [0,4] \times B_3 \tag{1.4}$$

and

$$|\{u \geq 1\} \cap [2,4] \times B_2| \geq \delta_0 \tag{1.5}$$

and

$$|\{u \leq 0\} \cap [1,4] \times B_2| \geq \frac{|[1,4] \times B_2|}{2} \tag{1.6}$$

then

$$|\{0 < u < 1\} \cap [1,4] \times B_2| \geq \mu_0. \tag{1.7}$$

This is called an isoperimetric inequality because, roughly speaking, if u were the indicator function of some set, then (1.5) would represent the measure of that set, (1.6) would represent the measure of its complement, and (1.7) would be analogous to the measure of its boundary. In this language, the lemma claims that solutions to (1.2) must have some minimum perimeter. The claim (1.5) uses the same δ_0 as in the statement of the first De Giorgi lemma 1.2.1. The intention is that, if (1.5) fails to be satisfied, then the first lemma can be applied to some translation of $(u-1)_+$.

Note that the assumption (1.4) must be satisfied on a larger region $[0,4] \times B_3$ than the rest of our assumptions. This is so that, by the energy inequality (1.3), $(u-0)_+$ will be in $L^2([1,4]; H^1(B_2))$. If we could say that, given assumption (1.4), $\|(u-0)_+\|_{H^1([1,4]\times B_2)}$ were uniformly bounded, meaning it is regular in both time and space, and we could set $\mu_0 = 0$, then the lemma would be trivial, simply because a function in H^1 cannot have a jump discontinuity.

Assuming still that $\|(u-0)_+\|_{H^1([1,4]\times B_2)}$ uniformly bounded, even for $\mu_0 > 0$ we could easily prove the result by contradiction. Take a sequence of solutions u_k satisfying (1.4)-(1.6) but such that

$$|\{0 < u_k < 1\} \cap [1,4] \times B_2| \leq \frac{1}{k}.$$

This sequence u_k would be uniformly bounded in H^1, and so it would have a strong L^2 limit u_∞. But $u_\infty \in H^1$ and u_∞ has a jump discontinuity, which gives us our contradiction.

Unfortunately, it is generally not the case that $(u-0)_+$ is H^1-regular in time. The proof of Lemma 1.2.2 relies on showing that the sequence u_k has enough uniform regularity in time (derived still from the assumption (1.4)) so that the strong-L^2 limit u_∞ exists, and cannot have a jump discontinuity. In general, it is easier to bound $\partial_t u_\infty$ from above than to bound it from below. This is why assumption (1.5) is phrased on the time-interval $[2,4]$ rather than $[1,4]$: to guarantee that a jump discontinuity in u_∞ will exist such that $\partial_t u_\infty \geq 0$ in the sense of distributions.

The actual technique for showing regularity-in-time is highly dependent on the specific PDE in question, and so it is not useful to give a more detailed outline.

Once the first and second De Giorgi lemmas are proven, the proof of Hölder continuity is typically similar across different applications of the method. One merely needs to apply the first and second lemmas to various translated and scaled copies of u. It is necessary therefore that the equation (1.2) be symmetric under a large family of transformations. In particular, we will consider transformations of the form $\bar{u}(t,x) := C + \alpha u(t,x)$ for *possibly negative* constants $C, \alpha \in \mathbb{R}$.

A common intermediate step between the De Giorgi lemmas and the proof of Hölder continuity is known as the oscillation lemma. Note that some sources use the name "second De Giorgi lemma" to refer to the oscillation lemma, rather than the isoperimetric inequality.

The oscillation of a function over a set S is defined by

$$\operatorname*{osc}_{S} f := \sup_{x \in S} f(x) - \inf_{x \in S} f(x).$$

The oscillation lemma then states that the oscillation of a solution u to (1.2) over a space-time region Q is bounded by the oscillation of the same u over a parabolic envelope of that region. We will present the lemma in a more rigid formulation, for clarity of presentation:

Lemma 1.2.3 (Oscillation Lemma)**.** *There exists a constant $\lambda_0 > 0$ such that, for u a solution to (1.2) on $[0,4] \times B_3$ with $\|f\|_{L^p([0,4]\times B_3)} \leq \lambda_0$ for p sufficiently large, if*

$$-2 \leq u(t,x) \leq 2 \qquad \forall (t,x) \in [0,4] \times B_3$$

then

$$\operatorname*{osc}_{[3,4]\times B_1} u \leq 4 - \lambda_0.$$

The oscillation will either decrease from above or below, depending whether $|\{u \leq 0\} \cap [1,4] \times B_2\}|$ is more or less than $\frac{|[1,4]\times B_2|}{2}$. We can assume without loss of generality that it is greater, and otherwise we can simply apply the following argument to $-u$.

Consider the sequence of functions $u_0 = u$ and $u_k := 2u_{k-1} - 2$. Note that each u_k will solve (1.2) (with source term $2^k f$) and satisfy assumptions (1.4) and (1.6) of the second De Giorgi lemma.

Assume that for some k_0, the function u_{k_0} satisfies the assumption (1.5). In particular, because of the way the sequence u_k is constructed, this means that u_k satisfies (1.5) for all $0 \le k \le k_0$. Therefore we can apply the second De Giorgi lemma and find that

$$|\{0 \le u_k \le 1\} \cap [1,4] \times B_2| \ge \mu_0 \qquad \forall 0 \le k \le k_0.$$

Because each of these k_0 sets are disjoint by construction, and because they are all contained in $[1,4] \times B_2$ which has finite measure, we find that our assumption on k_0 cannot hold for $k_0 = |[1,4] \times B_2| / \mu_0$. For this k_0, we conclude that u_{k_0} does not satisfy (1.5).

We can now apply the first De Giorgi lemma to $u_{k_0} - 1$ and conclude that $u_{k_0}(t,x) \le 3/2$ for $(t,x) \in [3,4] \times B_1$, or equivalently $u(t,x) \le 2 - 2^{-(k_0+1)}$.

Recall that a function g is Hölder continuous with exponent α at a point y_0 if and only if

$$\operatorname*{osc}_{|y-y_0|\le r} g(y) \le Cr^\alpha.$$

Therefore, by applying the oscillation lemma to dilations of u, we can easily conclude that u is Hölder continuous.

Because the constant μ_0 in the second De Giorgi lemma is obtained by a compactness argument, none of the constants obtained thereafter can be explicit. Most notably, the exponent α is not explicit. Therefore it is often desirable to obtain the second De Giorgi lemma by a more constructive argument, as in [Gue20].

1.3 Main Results and Outline

The remaining chapters of this dissertation will present will present various problems to which the De Giorgi method can be applied. Chapters 2, 4, 5, and 6 are based on the works [SV18], [Sto18], [SV19], and [Sto19] respectively, with only minor modifications.

Early applications of the De Giorgi method were to equations which were, in a sense, directly comparable to the Laplace or Heat Equations. For example, in the case of the equation (1.2) above, the behavior of the equation is primarily driven by the term $\mathrm{div}(-A\nabla u)$, which satisfies

$$\lambda \int u(-\Delta)u \leq -\int u \,\mathrm{div}(A\nabla u) \leq \lambda^{-1}\int u(-\Delta)u.$$

However, this direct comparability to the Laplacian is not necessary in order to use the De Giorgi method. In chapter 2, we apply the method to a problem in which the regularization is driven by a first-order term $|\nabla u|^p$ in which a more clearly elliptic term $\mathrm{div}(A\nabla u)$ is acts as an obstacle to regularity (due to its sign). In chapter 4, we apply the method to a hypoelliptic kinetic equation which has a Laplacian-like term $(-\Delta_v)^s f$ acting only in the v-direction and has no terms at all which directly regularize in the x-direction. In chapter 5, we apply the method to an equation which has both a first-order dissipation term $(-\Delta)^{1/2}\theta$ and also a first-order transport term. Finally, in chapter 3 we present a reformulation of an existing method for proving the isoperimetric lemma for a nonlocal heat equation with extremely low order dissipation, and in chapter 6 we apply a different elliptic method to show regularity for a hyperbolic conservation equation.

In chapter 2 we consider a Hamilton-Jacobi equation with superquadratic growth in its first-order term.

Hamilton-Jacobi equations are a class of highly nonlinear PDE. They are typically studied as a non-divergence form problem (using techniques such as Perron's method and maximum principles), as opposed to divergence form(using techniques such as distributional solutions and energy estimates), because they are not typically in the form of an Euler-Lagrange equation. The class of

Hamiltonian that we will study are of the form

$$\partial_t u = H(x, u, \nabla u, D^2 u) \approx -|\nabla u|^p + \text{error terms}$$

where $p > 2$ is a constant. Equations of this form appear in the work of Schwab [Sch13], in the study of homogenization for stochastic optimal control problems; taking the homogenous limit requires compactness, which comes in the form of a uniform regularity estimate. The first order term $|\nabla u|^p$ has a regularization effect, as proven by Cardaliaguet [Car09]. For $p > 2$, the first order term will dominate even when the "error" includes certain second order terms, as had been proven using probabilistic methods and comparison with sub- and supersolutions ([CC10], [CR11]), including the most comprehensive result by Cardaliaguet and Silvestre [CS12]. Using a modification of the De Giorgi method, the case without second-order error was tackled by Chan and Vasseur in [CV17]. The case with second-order error will be addressed in chapter 2 in which we will prove the following:

Theorem 1.3.1 (c.f. Theorem 2.1.1, first proven in [SV18])**.** *Let Ω an open subset of $\mathbb{R}^n$ and $[0, T]$ a time interval. Let $p > 2$ and $\Lambda \geq 1$ be constants, and $f \in L^q$ with q sufficiently large, and $A : [0, T] \times \Omega \to \mathbb{R}^{n\times n}$ satisfying $|\xi^\intercal A(t, x)\xi| \leq \Lambda |\xi|^2$ for all vectors ξ.*

Then if $u : [0, T] \times \Omega \to \mathbb{R}$ is a weak solution to

$$\partial_t u + \frac{1}{\Lambda} |\nabla u|^p \leq f + \operatorname{div}(A\nabla u),$$

$$\partial_t u + \Lambda |\nabla u|^p \geq -\Lambda + m^-(D^2 u)$$

for m^- the least-negative-eigenvalue function, u will be Hölder continuous uniformly on any compact subset of $(0, T] \times \Omega$. Its modulus of continuity will depend on p, Λ, and $\|u\|_{L^\infty}$.

Note that A is not assumed non-negative-definite.

This problem initially seems ill-suited for De Giorgi's method. It is nonlinear in an essential way and has no corresponding energy functional, which is why most previous investigations used maximum principles instead of energy methods which typically involve some form of linearization. De Giorgi method is an elliptic method, meaning it is typically driven by the coercivity of the second order term. This problem is not only non-elliptic, its second order term is actually a major obstacle to regularity. By tackling this problem using De Giorgi's method, however, we are able to expand the class of allowable errors beyond what was known in the literature, specifically allowing for discontinuous coefficients on the second order term and for source terms which are unbounded from above.

One major technical hurdle is that, unlike in the work of Chan and Vasseur [CV17], the Caccioppoli inequality that underlies the De Giorgi method will only be valid in regions where u is large. Another hurdle is the non-linearity of the equation; normally with De Giorgi we break nonlinear equations into a coupled linear (or at least variational) system and only treat these simpler equations, but in this problem the superquadratic growth is essential to the regularization. Lastly, as a consequence of the nonlinearity, the regularization from below happens backwards in time. In order to obtain the regularization that we desire, we must construct subsolutions which transport the regularity forwards in time and apply a maximum principle, thus mixing divergence-form and non-divergence-form methods.

In chapter 4, we consider a family of hypoelliptic kinetic equations.

Hypoelliptic equations are a class of degenerate elliptic equations with mixed elliptic and hyperbolic features. In particular, they include certain kinetic equations of the form

$$\partial_t f + v \cdot \nabla_x f - Q(f) = \sigma \tag{1.8}$$

for $f(t,x,v)$ a function of time, space and velocity, Q an elliptic operator in the velocity variable, and σ a source term. The idea is that such equations are elliptic in some variables (v) and hyperbolic in others (x), but due to mixing ($v \cdot \nabla_x$) regularization occurs in all variables.

General hypoelliptic equations were studied by Kolmogorov and by Hörmander [Hör67] as early as the 1930s. They originally considered only smooth solutions, but more recently a Sobolev H^s theory developed in the form of the study of averaging lemmas ([Ago84], [GLPS88]). An averaging lemma states that if the kinetic derivative of a function $[\partial_t + v \cdot \nabla_x] f$ is bounded in the Sobolev sense, then the velocity averages $\int f \, dv$ are regular. The theory of averaging lemmas had been developed into a full Sobolev hypoelliptic theory by Bouchut [Bou02], for Q a fractional Laplacian, and that theory in turn was used by Golse, Imbert, Mouhot and Vasseur [GIMV16] to prove Hölder continuity for Q a second order local elliptic operator in v.

I particularly studied the case of Q a uniformly elliptic singular integral operator. Such equations occur in the study of astrophysics to model particles interacting with a plasma ([Kan19], [Goy17], [LK74], [MR94]). They are also of interest for their relationship to the Boltzmann equation in the absence of the Grad cutoff assumption. Authors including Silvestre, Mouhot, Imbert and others ([IS16], [IMS18], [Mou18], [Sil16], [HST17], [HS17]) have recently been building an elliptic theory of the Boltzmann and Landau equations. In a landmark paper of this project, Silvestre and Imbert were able to prove Hölder continuity for a class of equations including both the Boltzmann equation and general uniformly elliptic singular integral operators as subcases.

In chapter 4 we study (1.8) with Q an integral operator in v given by

$$Q(f)(t,x,v) := \int K(t,x,v,w)[f(t,x,w) - f(t,x,v)] \, dw,$$

with K symmetric in v and w and $\kappa^{-1} \leq K(t,x,v,w)|v-w|^{n+2s} \leq \kappa$ for some $s \in (0,1)$ and constant κ, and σ an L^p function with finite but large p. We are able to show that solutions are C^α regular even with merely H^s initial data:

Theorem 1.3.2 (c.f. Theorem 4.1.1, frst proven in [Sto18])**.** *Let Ω and open subset of $\mathbb{R}^n$ and $[0,T]$ a time interval. Let $s \in (0,1)$ and $\kappa \geq 1$ be constants and Q as described above.*

Then there exists $p^* < \infty$ so that for any function $\sigma \in L^p \cap L^2([0,T] \times \Omega \times \mathbb{R}^n)$ and any weak solution $f \in L^\infty([0,T] \times \Omega \times \mathbb{R}^n) \cap L^2([0,T] \times \Omega; H^s(\mathbb{R}^n))$ to (1.8) will be Hölder continuous in space and time on any compact subset of $(0,T] \times \Omega$. The modulus of continuity depends only on σ and the L^∞ norm of the initial data.

The class of operators we consider is a special case of that considered by Imbert and Silvestre [IS16], who combined De Giorgi techniques and a Krylov approach to obtain regularity. Using the theory of averaging lemmas, we are able to apply solely energy-based techniques for a simplified proof and to allow for unbounded source terms. This work is inspired by that of Golse, Imbert, Mouhot and Vasseur [GIMV16], but requires a different approach due to the nonlocality of Q. In particular, we cannot use the full regularity result of Bouchut and need to work directly with a more standard averaging lemma [Béz94]. We are able to obtain $L^2 \to L^p$ $(p > 2)$ regularization from the regularization of averages, and then use the De Giorgi method to turn this into C^α regularization.

In chapter 5 we consider the Surface Quasi-Geostrophic equation (SQG) on a bounded domain.

The SQG equation is a special case of the equations describing large-scale atmospheric and oceanic currents ([Ped92], [Cha71]). In addition to its physical importance, SQG is of a mathematical form with interesting commonalities to the 3D Euler equations ([CMT94]), which explains its widespread study in pure and applied fluid mechanics. The form of SQG that we study, with critical dissipation, is

$$\partial_t\theta + \left[\nabla^\perp(-\Delta)^{-1/2}\theta\right]\cdot\nabla\theta + (-\Delta)^{1/2}\theta = 0. \tag{1.9}$$

Note that $(-\Delta)^{1/2}$ and $(-\Delta)^{-1/2}$ are both nonlocal operators.

Well-posedness for SQG on $\mathbb{R}^2$ has been known since 2010 ([CV10a], [KNV07], [CV12]), with multiple proofs from various perspectives. Physically motivated by, for example, air currents near land-sea boundaries, a few authors ([Kri15], [NV18b], [NV19]) have considered SQG on bounded domains. There are multiple ways to define the boundary behavior of such a system, however, so several different models have been proposed. In any case, the behavior of the nonlocal operator near the boundary complicates the analysis and demands new techniques.

Recently, Constantin and Ignatova [CI17] proposed a new model for SQG on bounded domains (specifically, defining $(-\Delta)^{1/2}$ as the spectral square-root of the Laplacian with homogeneous Dirichlet boundary conditions). They and Nguyen have published several papers on the topic, studying existence, uniqueness, regularity, and convergence for the equation with varying strengths of dissipation ([CI17], [CI16], [CIN18], [CN18a], [CN18b]). In particular, Constantin and Ignatova [CI16] showed that, for sufficiently regular initial data, solutions to the critical SQG are smooth in the interior of the domain. In said paper they identify boundary regularity, and specifically Hölder continuity up to the boundary, as a difficult open problem and an important step in the analysis of this equation.

In chapter (5) we obtain Hölder continuity up to the boundary:

Theorem 1.3.3 (c.f. Theorem 5.1.1, first proven in [SV19])**.** *Let Ω a bounded open subset of $\mathbb{R}^2$ with smooth boundary, $[0,T]$ a time interval, and $\theta : [0,T]\times\Omega \to \mathbb{R}$ a weak solution to critical SQG*

(1.9) *in* $L^\infty(0,T;L^4(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ *with* L^2 *initial data* θ_0.

Then θ *is Hölder continuous in time and space on* $[\varepsilon, T] \times \bar{\Omega}$ *for any* $\varepsilon > 0$. *The modulous of continuity depends only on the domain and* $\|\theta_0\|_{L^2}$.

Critical SQG has a dissipation term which is regularizing, and a transport term which has the potential to be deregularizing. Since the problem is critical, they are ostensibly equal in strength, so it is difficult to predict how solutions will behave. The transport term is particularly difficult to control near the boundary because the Dirichlet boundary conditions on $(-\Delta)$ are not translation invariant, and hence the commutator $[\nabla, (-\Delta)^{1/2}]$, is singular in this region. This proof is inspired by the previous work of Caffarelli and Vasseur [CV10a] on the global SQG to consider weaker norms in which the transport term may be bounded. In the case of global SQG, BMO is strong enough to constrain the regularity of the solution but weak enough that $\nabla^\perp(-\Delta)^{-1/2}$ is a bounded operator, while for the bounded-domain case, it was necessary to define a sophisticated and novel Banach space (somewhat analogous to $B^0_{\infty,\infty}(\Omega)$) adapted to this specific problem. This space is based on a generalized analogue of Littlewood-Paley theory (first studied by Iwabuchi, Matsuyama, and Taniguchi [IMT19], [IMT17], [IMT18]) in order to distinguish the low/high frequencies present in the commutator. The proof also utilizes a mixed Eulerian-Lagrangian approach with a moving reference frame adapted to counteract the transport term. Though utilized previously in the case of $\mathbb{R}^2$, this approach presents new difficulties in this case because, from a Lagrangian perspective, the domain is time-dependent.

In chapter 6 we consider the stability of shocks to a conservation law.

A shock is a special kind of traveling wave solution to a conservation law, i.e. of the form $s(x-\sigma t)$ for σ constant. A 1D inviscid shock is discontinuous at a single point and is constant, with two distinct values, on either side of that discontinuity; a viscous shock is a smoother approximation thereof. There is a significant literature devoted to the stability of shocks in both the L^1 and L^2 norms, so we shall concentrate on the case of scalar case. Many results give stability only for small perturbations ([FS98], [Kru70], [IO60]). Large perturbation L^2 stability of small inviscid shocks has been achieved by Vasseur and his group (e.g. [SV16], [LV11], [Leg11]) using the relative entropy method first introduced by DiPerna and Dafermos [Daf96] to study stability of Lipschitz solutions. L^2 stability will generally only hold up to a shift which depends on the solution:

$$\forall \text{ sol'n } u, \exists \gamma \text{ s.t. } \frac{d}{dt} \int |u(t,x) - s(x-\gamma(t))|^2 \, dx \leq 0.$$

In the dissipative case, meaning conservation laws of the form

$$\partial_t u + \operatorname{div} A(u) = \nu \Delta \eta'(u), \tag{1.10}$$

most of the time it is necessary to consider instead a weighted L^2 norm with a weight function a, called L^2-type stability. A recently developed relative-entropy technique has been able to obtain L^2-type stability with a arbitrarilly close to 1 in the L^∞ sense. This technique has been used in the case of 1D dissipative systems [KV19] (including 1D Navier-Stokes) and 1D scalar equations with near-constant dissipation [KV17] (e.g. $\eta'(u) = u$).

We prove in chapter 6 that 1D dissipative scalar conservation laws with uniformly convex flux and a nonlinear viscosity are L^2-type stable for sufficiently small shocks, independent of the dissipative parameter ν:

Theorem 1.3.4 (Theorem 1 in [Sto19])**.** *Let $\eta, A : \mathbb{R} \to \mathbb{R}$ be any uniformly convex functions with continuous third derivatives at 0. Then there exists $\varepsilon_0 > 0$ such that for any $\nu > 0$ and $\varepsilon \in (0, \varepsilon_0]$ the following holds:*

If $s : \mathbb{R} \to \mathbb{R}$ is a shock solution to (1.10) *with* $\|s\|_{L^\infty} \leq \varepsilon$*, then there exists a weight function* $a : \mathbb{R} \to (0, 2)$ *such that* s *is* L^2*-type stable up to a shift. Moreover,* $\|a - 1\|_{L^\infty}$ *can be made arbitrarilly small by taking* ε *sufficiently small.*

Because this result holds independently of the strength of dissipation ν, the result will apply also to vanishing viscosity limit solutions to the equivalent hyperbolic conservation law.

We use the relative entropy method. We are able to handle a wider variety of nonlinear viscosities by utilizing η (the function appearing in the dissipative term) as the entropy. This proof only uses one entropy for each equation, which is important if a technique is expected to generalize to systems; conservation systems typically only have a single entropy.

As in previous L^2-type estimates, we break up the solution u into a part which is L^∞ close to s and an error term which may be large in L^∞. The close part is handled similarly to the existing literature, while for the error term we need to make careful use of the relationship between the dissipative term and the mass density of the derivative of the weight function a.

Notation. Throughout this work, C will represent arbitrary constants which may change from line to line. The function space C_c^∞ contains smooth functions with compact support. We will use the notation $(x)_+ := \max(0, x)$. When the parentheses are ommited, the subscript $+$ is merely a label.

Chapter 2

Hamilton-Jacobi

2.1 Introduction

In this chapter, we will study C^γ regularization of solutions to a Hamilton-Jacobi evolution equation with viscosity:

$$\partial_t u + H(x,u,\nabla u) - \varepsilon \Delta u = 0, \qquad (t,x) \in (0,T) \times \Omega,$$

where $\Lambda > 0$, $\varepsilon \in [0,\Lambda]$, $\Omega \subseteq \mathbb{R}^n$, and the Hamiltonian has superquadratic growth in the gradient variable, uniform in x and t:

$$\frac{1}{\Lambda}|v|^p - f(x,t) \leq H(t,x,z,v) \leq \Lambda |v|^p + \Lambda, \qquad p > 2, f \in L^m, m > 1 + \frac{\max(n,2)}{p}.$$

We will show that solutions are uniformly Hölder continuous away from the boundary of Ω and after a positive time has elapsed.

Because $p > 2$, it is the first order term that will dominate at small scales. The second order term acts merely as a perturbation. In fact, although our motivation is a first-order Hamilton-Jacobi equation with viscosity, our techniques can handle much more general second order terms. Specifically, we will show the following theorem.

Theorem 2.1.1 (Main Theorem)**.** *Let constants $\Lambda > 0$, $\Lambda_0 \geq 0$, $p > 2$, $m > 1 + \frac{\max(n,2)}{p}$ be given, and let $\Omega \subseteq \mathbb{R}^n$ open and $T > 0$ be given, and let $f \in L^m([0,T] \times \Omega)$ with $\|f\|_m \leq \Lambda$ and a matrix $A \in L^\infty([0,T] \times \Omega; \mathbb{R}^{n \times n})$ with $\|A\|_\infty \leq \Lambda$ be given, and let $\bar{\Omega} \subset \Omega$ compact and $0 < s < T$ be given.*

There exists $0<\gamma<1$, depending on p, Λ, Λ_0, m, and n, such that any $u \in L^\infty((0,T)\times\Omega)$, $\nabla u \in L^p$, satisfying

$$\partial_t u + \Lambda^{-1}|\nabla u|^p - \operatorname{div}(A\nabla u) \leq f \tag{1.1}$$

in the sense of distributions, and satisfying

$$\partial_t u + \Lambda|\nabla u|^p - \Lambda_0 m^-(D^2 u) \geq -\Lambda \tag{1.2}$$

in the sense of viscosity, will have

$$u \in C^\gamma((s,T)\times\bar{\Omega})$$

with norm depending on $\|u\|_\infty$, p, Λ, Λ_0, m, n, s, and the distance between $\bar{\Omega}$ and $\mathbb{R}^n \setminus \Omega$.

Here m^- is a function that returns the lowest eigenvalue of a symmetric matrix, or 0 if all of the eigenvalues are positive. For a function to solve Inequality (1.2) in the sense of viscosity means, following the definition of Barles [Bar13], that the lower-semicontinuous envelope of that function is a viscosity supersolution of

$$\partial_t u + \Lambda|\nabla u|^p - \Lambda_0 m^-(D^2 u) = -\Lambda.$$

Hamilton-Jacobi equations of this general form, with a viscosity term and polynomial growth in the gradient, were studied by Lasry and Lions [LL89] in 1989, in connection with stochastic control problems. For the case $p<2$, this first-order-term can be viewed as a perturbation of a simple heat equation, and indeed solutions will be regular so long as the viscosity term is uniformly parabolic. However, in the superquadratic case $p>2$, it is the first order term which dominates at small scales, so standard parabolic theory does not apply.

Schwab [Sch13] studied homogenization problems for Hamilton-Jacobi equations with superquadratic growth, which required him to prove that the regularity of solutions to these equations is independent of the regularity of the Hamiltonian. His result still required, however, that the Hamiltonian be convex in Du. It was Barles [Bar10] and Dolcetta, Leoni, and Porretta [DLP10] who noticed that convexity was unnecessary in the time-independent case, and Cardaliaguet ([Car09], [CC10], [CS12]) for the time-dependent case.

In the case that f is bounded, Cardaliaguet and Silvestre ([CS12], Theorem 1.2) showed Hölder continuity, using a second order term $m^+(D^2u)$ instead of $\mathrm{div}(A\nabla u)$ in (1.1). In the case that f is not assumed bounded, they could only show Hölder regularity with second order term $\mathrm{tr}(AD^2u)$, $A \in C^1$. Our result requires no regularity on A, at the expense of requiring that $\nabla u \in L^p$ and u solve Inequality (1.1) in the sense of distribution. The motivation for considering f unbounded is from Lasry and Lions [LL07].

Most of the aforementioned results are proven by constructing super- and subsolutions. In [CV17], Hölder estimates are obtained, with f bounded and no second order term, using a variation of De Giorgi's method. The results in this chapter are a continuation of that project.

The proof will proceed mostly along the same lines as De Giorgi [DG57] and [CV17]. In the classical De Giorgi proof, in order to prove Hölder continuity one merely shows that if the function u is "mostly negative" in some range of time, then the upper bound is improved in a later range of time. If, alternatively, the function is not "mostly negative," it must be "mostly positive" and hence one can apply the original argument to $-u$, improving the lower bound on u in the same later range of time. Either way, the L^∞-bound of u is improved in the later time range.

In the sequel, the function $-u$ does not satisfy the same Inequality (1.1) as u. However, time-reversed $-u$ does satisfy Inequality (1.1) with A replaced by $-A$, since time reversal creates an extra minus sign on the ∂_t term. Thus unlike the classical De Giorgi proof, while the upper bound is improved in a later time range, the lower bound on u is improved in an earlier time range, because time was reversed. Note that while replacing A by $-A$ should ostensibly cause great difficulty, the second order term is here a perturbation, and the first order term is the driver of regularization, so we can handle negative viscosities so long as the solution is known to exist and to be bounded.

Next we must use the comparison principle in a small but crucial argument. Based on Inequality (1.2), a subsolution is constructed to show that a lower bound improvement in the early time range implies a smaller-but-still-positive improvement in the later time range. This is referred to as "flowing the improvement forward in time."

The key ingredient in improving the upper bound is an energy inequality. Because of the second order term, we must multiply (1.1) by u_+ to obtain the energy inequality (then we integrate by parts, and turn the second order term into a $|\nabla u|^2$ term). But the viscosity is a perturbation, and the true driver of the proof is the first order term. Multiplying the first order term by u_+ yields $u_+|\nabla u_+|^p$, which is difficult because u_+ acts like a coefficient which is not bounded below. Luckily, our goal is to bound u, and the difficulties only occur when u_+ is small.

Section 2.2 derives an energy inequality, which quantifies the ellipticity of our equation. Sections 2.3 and 2.4 use the energy inequalities to prove De Giorgi's two lemmas. Section 2.5 demonstrates how to flow the improvement forward in time, correcting for the necessary time reversal. Finally, in Section 2.6 we combine these lemmas to prove Hölder continuity. A reader unfamiliar with De Giorgi-style proofs might want to begin with Section 2.6, lest the former sections seem unmotivated.

Instead of proving continuity directly for u, it is preferable to consider

$$\bar{u} := u + \Lambda t, \qquad \bar{f} := f + \Lambda$$

which satisfies the inequality

$$\partial_t \bar{u} + \Lambda |\nabla \bar{u}|^p - \Lambda_0 m^-(D^2 \bar{u}) \geq 0. \tag{1.3}$$

Note also that, by scaling our solution appropriately, we can assume that Λ_0 is arbitrarily small.

Throughout this article, C will indicate a constant which varies from line to line. No two instances of the symbol should be assumed related to each other.

2.2 The Energy Inequalities

We begin by deriving the Energy Inequalities, which play an analogous role to the Cacciopoli inequality in De Giorgi's original paper. These inequalities serve to quantify the coercivity of the PDE in question. We actually consider an infinite family of Energy Inequalities, corresponding to different entropies, indexed by the parameter b. These inequalities must be valid even for non-positive matrices A.

The lemma below claims three different forms for the Energy Inequality. The first form will be used to compare distinct truncations of a solution in Section 2.3. The second and third forms are only valid for large values of b, the former being used in Section 2.3 and the latter being used in Section 2.4. Notice that the gradient of u appears in the right hand side of the first form, but not of the second or third forms.

Lemma 2.2.1 (Energy Inequality)**.** *Given u verifying Inequality* (1.1)*, with $\|A\|_\infty, \|f\|_m \le \Lambda$, on some domain $[S,0]\times\Omega$, given constants b, c and $S<T<0$, and given ϕ a smooth non-negative function constant in time and compactly supported in Ω, and defining $u_* = (u-c)_+$, then u_* satisfies the inequality*

$$\begin{aligned}&\sup_{t\in[T,0]} \int \phi^2 u_*^{b+1}(t) + \iint_T^0 \phi^2 u_*^b |\nabla u_*|^p \\ &\le C(\Lambda,b)\left(1+\frac{1}{T-S}\right)\left(\|\phi\|_\infty^2 + \|\nabla\phi\|_\infty^2\right)\left[\iint_S^0 (u_*^{b+1} + u_*^{b-1}|\nabla u_*|^2)\chi_{\{\phi\}} + \left(\iint_S^0 u_*^{bm^*}\chi_{\{\phi\}}\right)^{\frac{1}{m^*}}\right].\end{aligned} \tag{2.4}$$

Moreover, if $b > \sigma := \left(1-\frac{2}{p}\right)^{-1}$, then

$$\begin{aligned}&\sup_{t\in[T,0]} \int \phi^2 u_*^{b+1}(t) + \iint_T^0 \phi^2 u_*^b |\nabla u_*|^p \\ &\le C(\Lambda,b)\left(1+\frac{1}{T-S}\right)\left(\|\phi\|_\infty^2 + \|\nabla\phi\|_\infty^2\right)\left[\iint_S^0 (u_*^{b+1} + u_*^{b-\sigma})\chi_{\{\phi\}} + \left(\iint_S^0 u_*^{bm^*}\chi_{\{\phi\}}\right)^{\frac{1}{m^*}}\right].\end{aligned} \tag{2.5}$$

If $b>\sigma$ but ϕ is not necessarily constant in time, then still we have

$$\begin{aligned}&\langle \partial_t(u_*^{b+1}), \phi^2\rangle_{[S,0]\times\Omega} + \iint_S^0 \phi^2 u_*^b |\nabla u_*|^p \\ &\le C(\Lambda,b)\left(\iint_S^0 \phi^2 u_*^b f + \iint_S^0 u_*^{b+1}|\nabla\phi|^2 + \iint_S^0 \phi^2 u_*^{b-\sigma}\right).\end{aligned} \tag{2.6}$$

The integrals without limits are over all of Ω, $\chi_{\{\phi\}}$ means the indicator function of the support of ϕ, and m^* means the Hölder conjugate of m.

Proof. Formally, we want to integrate Inequality (1.1) against the test function $\phi^2 u_*^b$. Because our solution u is by assumption in $L^p(W^{1,p})$, the distributions $|\nabla u|^p$ and $\mathrm{div}(A\nabla u)$ both have enough regularity for this integration to make sense. To justify our calculations on $\partial_t u$, one can simply use the test function $\tau*(\phi^2(\tau*u_*)^b)$ for τ some approximation to the identity and $*$ meaning convolution in time and space, though for reasons of clarity we drop the mollifiers in the formal calculations below.

Multiply Inequality (1.1) by $\phi^2 u_*^b$, then integrate over all of space Ω:

$$\int \phi^2 u_*^b \partial_t u + \Lambda^{-1}\int \phi^2 u_*^b |\nabla u|^p + \int (\nabla(\phi^2 u_*^b))A(\nabla u) \leq \int \phi^2 u_*^b f.$$

Notice that $Du_* = \chi_{\{u_*>0\}} Du$ for any first order differential operator D, so in the above expression we may replace every instance of u with u_*. By the product rule, $(b+1)u_*^b\partial_t u_* = \partial_t(u_*^{b+1})$. Also, we can use the product rule and Young's Inequality to bound the A-term:

$$\begin{aligned}\nabla\left(\phi^2 u_*^b\right) A\nabla u_* &= b\phi^2 u_*^{b-1}(\nabla u_* A \nabla u_*) + 2\phi u_*^b(\nabla u_* A\nabla\phi)\\ &\leq b\Lambda\phi^2 u_*^{b-1}|\nabla u_*|^2 + 2\Lambda\left(\phi u_*^{\frac{b-1}{2}}|\nabla u_*|\right)\left(u_*^{\frac{b+1}{2}}|\nabla\phi|\right)\\ &\leq b\Lambda\phi^2 u_*^{b-1}|\nabla u_*|^2 + \Lambda\left(\phi u_*^{\frac{b-1}{2}}\nabla u_*\right)^2 + \Lambda\left(u_*^{\frac{b+1}{2}}\nabla\phi\right)^2\\ &= (b+1)\Lambda\phi^2 u_*^{b-1}|\nabla u_*|^2 + \Lambda u_*^{b+1}|\nabla\phi|^2.\end{aligned}$$

Putting all of these together, we arrive at

$$\frac{1}{b+1}\int \phi^2\partial_t(u_*^{b+1}) + \Lambda^{-1}\int\phi^2 u_*^b|\nabla u_*|^p \leq \int \phi^2 u_*^b f + \Lambda\int u_*^{b+1}|\nabla\phi|^2 + (b+1)\Lambda\int\phi^2 u_*^{b-1}|\nabla u_*|^2.$$

If $b>\sigma$, then using Young's Inequality with exponents $p/2$ and σ, and a small constant η, we can break up the final term of the above inequality:

$$\begin{aligned}u_*^{b-1}|\nabla u_*|^2 &\leq C(p)\left(\left(\eta u_*^{\frac{2b}{p}}|\nabla u_*|^2\right)^{p/2} + \left(\frac{1}{\eta}u_*^{b\left(1-\frac{2}{p}\right)-1}\right)^{\sigma}\right)\\ &\leq C(p)\left(\eta^{\frac{p}{2}}u_*^b|\nabla u_*|^p + \eta^{-\sigma}u_*^{b-\sigma}\right).\end{aligned}$$

By taking η sufficiently small (depending on p, b, Λ), the $u_*^b|\nabla u_*|^p$ term on the right can be absorbed by the same term with larger constant on the left. We use the shorthand

$$T(u_*,b) := \begin{cases} u_*^{b-1}|\nabla u_*|^2 & \text{if } b \le \sigma \\ u_*^{b-\sigma} & \text{if } b > \sigma \end{cases}$$

and write

$$\int \phi^2 \partial_t(u_*^{b+1}) + \int \phi^2 u_*^b |\nabla u_*|^p \le C(\Lambda,b)\left(\int \phi^2 u_*^b f + \int u_*^{b+1}|\nabla\phi|^2 + \int \phi^2 T(u_*,b)\right).$$

In the case that ϕ is time dependent, we can integrate the above in time to obtain (2.6). From now on, we assume that $\partial_t\phi = 0$, and hence $\int \phi^2 \partial_t(u_*^{b+1}) = \frac{d}{dt}\int \phi^2 u_*^{b+1}$.

For any times s,t satisfying $S \le s \le T \le t \le 0$, we can integrate the above inequality over $[s,t]$ (and apply Hölder's to remove dependence on f):

$$\begin{aligned} &\int \phi^2 u_*^{b+1}(t) + \iint_s^t \phi^2 u_*^b |\nabla u_*|^p \\ &\le C(\Lambda,b)\left(\int \phi^2 u_*^{b+1}(s) + \left(\iint_s^t (\phi^2 u_*^b)^{m^*}\right)^{\frac{1}{m^*}} + \iint_s^t u_*^{b+1}|\nabla\phi|^2 + \iint_s^t \phi^2 T(u_*,b)\right). \end{aligned}$$

Due to our choice of s,t, the above inequality implies that

$$\begin{aligned} &\int \phi^2 u_*^{b+1}(t) + \iint_T^t \phi^2 u_*^b |\nabla u_*|^p \\ &\le C(\Lambda,b)\left(\int \phi^2 u_*^{b+1}(s) + \left(\iint_S^0 (\phi^2 u_*^b)^{m^*}\right)^{\frac{1}{m^*}} + \iint_S^0 u_*^{b+1}|\nabla\phi|^2 + \iint_S^0 \phi^2 T(u_*,b)\right). \end{aligned}$$

Since the right side is independent of t, we can take a supremum of the left side over $T \le t \le 0$. Add to this the inequality with $t=0$ to obtain

$$\sup_{t\in[T,0]} \int \phi^2 u_*^{b+1}(t) + \iint_T^0 \phi^2 u_*^b |\nabla u_*|^p$$

$$\leq C(\Lambda,b)\left(\int \phi^2 u_*^{b+1}(s)+\left(\iint_S^0 (\phi^2 u_*^b)^{m^*}\right)^{\frac{1}{m^*}}+\iint_S^0 u_*^{b+1}|\nabla\phi|^2+\iint_S^0 \phi^2 T(u_*,b)\right).$$

Lastly, since this inequality holds for all $S\leq s\leq T$, it also holds if we average the right hand side over all values of s in that range,

$$\sup_{t\in[T,0]}\int \phi^2 u_*^{b+1}(t)+\iint_T^0 \phi^2 u_*^b|\nabla u_*|^p$$
$$\leq C(\Lambda,b)\left(\frac{1}{T-S}\iint_S^T \phi^2 u_*^{b+1}+\left(\iint_S^0 (\phi^2 u_*^b)^{m^*}\right)^{\frac{1}{m^*}}+\iint_S^0 u_*^{b+1}|\nabla\phi|^2+\iint_S^0 \phi^2 T(u_*,b)\right).$$

From here the result follows naturally. □

2.3 De Giorgi's first lemma

Now we present De Giorgi's first lemma. If we define

$$\overline{Q}_2:=[-2,0]\times B_2, \qquad Q_1:=[-1,0]\times B_1,$$

this lemma tells us that the supremum in Q_1 of solutions to (1.1) can be controlled by the measure of $\{u>0\}$ in $\overline{Q}_2$.

Proposition 2.3.1 (De Giorgi's First Lemma)**.** *There exists a constant $\delta_0>0$ depending only on Λ, p, m, and the dimension such that, for any u satisfying Inequality* (1.1) *on $\overline{Q}_2$ in the sense of distributions, the following implication holds:*

If

$$u(t,x)\leq 1 \qquad \forall (t,x)\in\overline{Q}_2$$

and

$$\left|\{u>0\}\cap\overline{Q}_2\right|\leq\delta_0,$$

then

$$u(t,x) \leq \frac{1}{2} \qquad \forall (t,x) \in Q_1.$$

De Giorgi's first lemma is proved by cutting off u at larger and larger values, and showing that as the cutoff value tends to $1/2$, some Lebesgue norm of the remainder tends to zero.

Proof. Let us specify the sequence of cutoffs. We'll consider

- heights $C_k = \frac{1}{2} - 2^{-k-1}$ from $C_0 = 0$ to $C_\infty = \frac{1}{2}$ with $C_k - C_{k-1} = 2^{-k-1}$;
- functions $u_k = \max(u - C_k, 0)$ from $u_0 = u_+$ to $u_\infty = (u - \frac{1}{2})_+$;
- balls B^k of radius $1 + 2^{-k}$ from $B^0 = B_2 = \{x : |x| < 2\}$ to $B^\infty = B_1 = \{x : |x| < 1\}$;
- times $T_k = -1 - 2^{-k}$ from $T_0 = -2$ to $T_\infty = -1$ with $T_k - T_{k-1} = 2^{-k}$;
- and smooth functions ϕ_k such that $\mathrm{supp}(\phi_k) = B^k$ and $\phi_k \restriction B^{k+1} \equiv 1$, with $0 \leq \phi_k \leq 1$ and $|\nabla \phi_k| \leq 2^{k+2}$.

Define the "energy" of the k^{th} level to be

$$\mathcal{E}_k := \sup_{t \in [T_{k+1}, 0]} \int (\phi_k u_k)^2(t) + \iint_{k+1} \phi_k^2 u_k |\nabla u_k|^p,$$

where $\iint_k$ means $\int_{T_k}^0 \int_{\mathbb{R}^n}$. First we will show via Sobolev's inequality that this energy term controls some $L^{(1+\beta)q}$ norm of $\phi_k u_k$. Then we will show via the Energy Inequality that the same $L^{(1+\beta)q}$ norm controls this energy term.

Step 1: Controlling $L^{(1+\beta)q}$ using $\mathcal{E}_k$

Before we can apply Sobolev's inequality, we have to deal with the inhomogeneity of the gradient term. We do this by "going up a level" from u_k to u_{k+1}.

$$
\begin{aligned}
\mathcal{E}_k &\geq \iint_{k+1} \phi_k^2 u_k \, |\nabla u_k|^p \\
&\geq \iint_{k+1} \phi_k^2 \left[2^{-(k+2)} \chi_{\{u_k \geq 2^{-k-2}\}}\right] |\nabla u_k|^p \\
&= 2^{-k-2} \iint_{k+1} \phi_k^2 \chi_{\{u_{k+1} \geq 0\}} \, |\nabla u_k|^p \\
&= 2^{-k-2} \iint_{k+1} \phi_k^2 \, |\nabla u_{k+1}|^p \\
&\geq 2^{-k-2} \iint_{k+1} \chi_{\{B^{k+1}\}} \, |\nabla u_{k+1}|^p \\
&= C^{-k} \int_{T_{k+1}}^0 \|\nabla u_{k+1}\|_{L^p(B^{k+1})}^p \\
&= C^{-k} \, \|\nabla u_{k+1}\|_{L^p([T_{k+1},0];L^p(B^{k+1}))}^p
\end{aligned}
$$

We introduce now a parameter $\beta \in (0,1]$, satisfying

$$
0 < \frac{1}{n} - \frac{\beta}{2} < \frac{1}{p}, \qquad n \geq 2
$$

or $\beta = 1$ if $n = 1$. We are going to apply Sobolev's Inequality to bound the $L^{p'}$ norm of $u_k^{1+\beta}$ by some Lebesgue norm of $\nabla u_k^{1+\beta}$.

Since

$$
\left\| u_{k+1}^\beta \right\|_{L^\infty([T_{k+1},0];L^{2/\beta}(B^{k+1}))}^{2/\beta} = \sup_{t \in [T_{k+1},0]} \|u_{k+1}(t)\|_{L^2(B^{k+1})}^2 \leq \sup_{t \in [T_{k+1},0]} \|\phi_k u_k(t)\|_{L^2(B^{k+1})}^2 \leq \mathcal{E}_k,
$$

we know by elementary properties of Lebesgue spaces that

$$
\begin{aligned}
\int_{T_{k+1}}^0 \left\| \nabla u_{k+1}^{\beta+1} \right\|_{L^{\frac{2p}{2+p\beta}}(B^{k+1})}^p &= \left\| u_{k+1}^\beta \nabla u_{k+1} \right\|_{L^p([T_{k+1},0];L^{\frac{2p}{2+p\beta}}(B^{k+1}))}^p \\
&\leq \left\| u_{k+1}^\beta \right\|_{L^\infty([T_{k+1},0];L^{2/\beta}(B^{k+1}))}^p \|\nabla u_{k+1}\|_{L^p([T_{k+1},0];L^p(B^{k+1}))}^p \qquad (3.7) \\
&\leq \left(\mathcal{E}_k^{\beta/2}\right)^p C^k \mathcal{E}_k = C^k \mathcal{E}_k^{1+\frac{p\beta}{2}}.
\end{aligned}
$$

If $n>1$, then let $\frac{1}{p'}=\frac{2+p\beta}{2p}-\frac{1}{n}=\frac{\beta}{2}+\frac{1}{p}-\frac{1}{n}$. If $n=1$, then take $p'=p$ (which renders some of the following calculations trivial). Sobolev Embedding yields

$$\begin{aligned}\left\|u_{k+1}^{1+\beta}\right\|_{L^{p'}\left(B^{k+1}\right)} &\leq \left\|u_{k+1}^{1+\beta}-⨍_{B^{k+1}} u_{k+1}^{1+\beta}\right\|_{L^{p'}\left(B^{k+1}\right)}+|B^{k+1}|^{\frac{1}{p'}-1}\int_{B^{k+1}} u_{k+1}^{1+\beta} \\ &\leq C\left(\left\|\nabla u_{k+1}^{1+\beta}\right\|_{L^{\frac{2p}{2+p\beta}}\left(B^{k+1}\right)}+\|u_{k+1}\|_{L^2\left(B^{k+1}\right)}^{1+\beta}\right).\end{aligned}$$

Remember that $⨍_E:=\frac{1}{|E|}\int_E$, and $1+\beta\leq 2$ so $L^{1+\beta}\subseteq L^2$.

With the above calculation and (3.7), we can estimate

$$\begin{aligned}\int_{T_{k+1}}^0 \left\|u_{k+1}^{1+\beta}\right\|_{L^{p'}(B_{k+1})}^p &\leq C\int_{T_{k+1}}^0 \left(\left\|\nabla u_{k+1}^{1+\beta}\right\|_{L^{\frac{2p}{2+p\beta}}(B^{k+1})}+\|u_{k+1}\|_{L^2(B^{k+1})}^{1+\beta}\right)^p \\ &\leq C\left(\int_{T_{k+1}}^0 \left\|\nabla u_{k+1}^{1+\beta}\right\|_{L^{\frac{2p}{2+p\beta}}(B^{k+1})}^p+T_{k+1}\sup_{t\in[T_{k+1},0]}\|u_{k+1}(t)\|_{L^2(B^{k+1})}^{p(1+\beta)}\right) \\ &\leq C\left(C^k\mathcal{E}_k^{1+\frac{p\beta}{2}}+\mathcal{E}_k^{p\frac{1+\beta}{2}}\right) \\ &\leq C^k\mathcal{E}_k^{1+\frac{p\beta}{2}}.\end{aligned}$$

This last estimate holds as long as $\mathcal{E}_k$ is less than one.

We wish to apply the Riesz-Thorin theorem to interpolate between the $L^p(L^{p'})$ and $L^\infty(L^{\frac{2}{1+\beta}})$ norms of $u_{k+1}^{1+\beta}$. First define

$$q=p+\left(1-\frac{p}{p'}\right)\frac{2}{1+\beta}. \tag{3.8}$$

Because $p'\geq p$ and hence $q\geq p$, we can let $\theta=\frac{p}{q}\in[0,1]$ and interpolate to obtain

$$(1-\theta)\frac{1}{\infty}+\theta\frac{1}{p}=0+\frac{1}{q}=\frac{1}{q}$$

and

$$(1-\theta)\frac{1+\beta}{2}+\theta\frac{1}{p'}=\left(\frac{q-p}{q}\right)\frac{1+\beta}{2}+\frac{1}{q}\left(\frac{p}{p'}\right)$$

$$
\begin{aligned}
&= \frac{1}{q}\left(1-\frac{p}{p'}\right)\left(\frac{2}{1+\beta}\right)\frac{1+\beta}{2}+\left(\frac{p}{p'}\right)\frac{1}{q}.\\
&= \frac{1}{q}.
\end{aligned}
$$

Therefore the Riesz-Thorin interpolation theorem yields

$$
\begin{aligned}
\left\|u_{k+1}^{1+\beta}\right\|_{L^q\left([T_{k+1},0]\times B^{k+1}\right)} &\le C\left[\left\|u_{k+1}^{1+\beta}\right\|_{L^\infty\left([T_{k+1},0];L^{\frac{2}{1+\beta}}(B^{k+1})\right)}\right]^{1-\theta}\left[\left\|u_{k+1}^{1+\beta}\right\|_{L^p\left([T_{k+1},0];L^{p'}(B^{k+1})\right)}\right]^{\theta}\\
&\le C\left[\sup_{t\in[T_{k+1},0]}\|\phi_k u_k\|_{L^2(B^k)}^{1+\beta}\right]^{1-\frac{p}{q}}\left[\left(C^k\mathcal{E}_k^{1+\frac{p\beta}{2}}\right)^{1/p}\right]^{\frac{p}{q}}\\
&\le C^k\left[\mathcal{E}_k^{\frac{1}{2}+\frac{\beta}{2}}\right]^{1-\frac{p}{q}}\mathcal{E}_k^{\frac{1}{q}+\frac{\beta}{2}\cdot\frac{p}{q}}\\
&= C^k\mathcal{E}_k^{\frac{1}{q}+\frac{1}{2}\left(1+\beta-\frac{p}{q}\right)}.
\end{aligned}
$$

Thus finally,

$$
\iint_{k+1}|\phi_{k+1}u_{k+1}|^{(1+\beta)q}\le\iint_{k+1}\chi_{\{B^{k+1}\}}(u_{k+1}^{1+\beta})^q\le C^k\mathcal{E}_k^{1+\frac{(1+\beta)q-p}{2}}. \tag{3.9}
$$

Step 2: A Recursive relation for the sequence $\mathcal{E}_k$

Recall from the definition (3.8) of q that $(1+\beta)q=2+(1+\beta)p-2\frac{p}{p'}$. If $n>1$, then by the definition of p' we have that $2\frac{p}{p'}=2+p\beta-2\frac{p}{n}$. If $n=1$, then $p'=p$ and $\beta=1$. Therefore,

$$
\begin{aligned}
(1+\beta)q &= p+2\frac{p}{n}, \qquad n>1\\
(1+\beta)q &= 2p, \qquad\quad\;\; n=1.
\end{aligned} \tag{3.10}
$$

The Energy Inequality (2.4), applied to u_k with $b=1$, ϕ_k, and times T_{k+1} and T_k, tells us that

$$
\mathcal{E}_k\le C2^{k+2}\left(\iint_k(u_k^2+|\nabla u_k|^2)\chi_{\{B^k\}}+\left(\iint_k u_k^{m^*}\chi_{\{B^k\}}\right)^{1/m^*}\right). \tag{3.11}
$$

Now that we have (3.9), we are ready to bound the three terms on this inequality's right hand side.

For the first and third terms on the right hand side, we can use a well known trick of De Giorgi [DG57]. For any $j \le (1+\beta)q$ we can apply Hölder's inequality followed by Chebyshev's inequality to obtain

$$\begin{aligned}
\iint_k u_k^j \chi_{\{B^k\}} &= \iint_k (\phi_{k-1} u_k)^j \chi_{\{B^k \cap \{u_{k-1} > 2^{-(k+1)}\}\}} \\
&\le \left(\iint_k (\phi_{k-1} u_k)^{(1+\beta)q} \right)^{j/[(1+\beta)q]} \left| \{\phi_{k-1} u_{k-1} > 2^{-(k+1)}\} \right|^{1-j/[(1+\beta)q]} \\
&\le \left(\iint_{k-1} (\phi_{k-1} u_{k-1})^{(1+\beta)q} \right)^{j/[(1+\beta)q]} \left| \{(\phi_{k-1} u_{k-1})^{(1+\beta)q} > 2^{-(k+1)(1+\beta)q}\} \right|^{1-j/[(1+\beta)q]} \\
&\le \left(\iint_{k-1} (\phi_{k-1} u_{k-1})^{(1+\beta)q} \right)^{j/[(1+\beta)q]} \left(2^{(k+1)(1+\beta)q} \iint_{k-1} (\phi_{k-1} u_{k-1})^{(1+\beta)q} \right)^{1-j/[(1+\beta)q]} \\
&\le 2^{(k+1)((1+\beta)q-j)} \iint_{k-1} (\phi_{k-1} u_{k-1})^{(1+\beta)q} \\
&\le C^k \mathcal{E}_{k-2}^{1+\frac{(1+\beta)q-p}{2}}.
\end{aligned}$$

We know from (3.10) that $2 < (1+\beta)q$ and $m* \le 1 + \frac{p}{n} \le (1+\beta)q$, so setting $j=2$ and $j=m^*$ gives us bounds on the first and third terms of (3.11), respectively.

For the second term of (3.11), calculate

$$\begin{aligned}
\iint_k |\nabla u_k|^2 \chi_{\{B^k\}} &\le \iint_k \phi_{k-1}^{4/p} \chi_{\{u_k>0\}} |\nabla u_{k-1}|^2 \chi_{\{\phi_k u_k > 0\}} \\
&\le \left(\iint_k \phi_{k-1}^2 \chi_{\{u_{k-1} > 2^{-(k+1)}\}} |\nabla u_{k-1}|^p \right)^{2/p} \left| \{\phi_{k-1} u_{k-1} > 2^{-(k+1)}\} \right|^{1-2/p} \\
&\le \left(2^{k+1} \iint_k \phi_{k-1}^2 u_{k-1} |\nabla u_{k-1}|^p \right)^{2/p} \left| \{(\phi_{k-1} u_{k-1})^{(1+\beta)q} > 2^{-(k+1)(1+\beta)q}\} \right|^{1-2/p} \\
&\le \left(2^{k+1} \mathcal{E}_{k-1} \right)^{2/p} \left(2^{(k+1)(1+\beta)q} \iint_{k-1} (\phi_{k-1} u_{k-1})^{(1+\beta)q} \right)^{1-2/p}
\end{aligned}$$

$$\leq \left(2^{k+1}\mathcal{E}_{k-2}\right)^{2/p}\left(2^{(k+1)(1+\beta)q}C^{k-2}\mathcal{E}_{k-2}^{1+\frac{(1+\beta)q-p}{2}}\right)^{1-2/p}$$
$$\leq C^k\mathcal{E}_{k-2}^{1+\left(1-\frac{2}{p}\right)\frac{(1+\beta)q-p}{2}}.$$

The second-to-last inequality used (3.9), and the fact that $\mathcal{E}_{k-1}\leq\mathcal{E}_{k-2}$.

Finally we have the recursive relation

$$\mathcal{E}_k\leq C^k\left(\mathcal{E}_{k-2}^{1+\frac{(1+\beta)q-p}{2}}+\mathcal{E}_{k-2}^{1+\left(1-\frac{2}{p}\right)\frac{(1+\beta)q-p}{2}}+\mathcal{E}_{k-2}^{\left(1+\frac{(1+\beta)q-p}{2}\right)\left(\frac{1}{m^*}\right)}\right). \tag{3.12}$$

From (3.10) and $p>2$, one sees that the first two of these exponents are strictly greater than 1. From (3.10) and $m^*<1+\frac{p}{n}$, one sees that the third exponent is strictly greater than 1.

Because we can assume wlog that all $\mathcal{E}_k$ are small, this simplifies for our purposes to

$$\mathcal{E}_k\leq C^k\mathcal{E}_{k-2}^{1+\varepsilon}.$$

Therefore the sequence $\mathcal{E}_{2n+1}$ is bounded by a sequence $a_{n+1}=c^na_n^{1+\varepsilon}$, $a_0=\mathcal{E}_1$. Because the exponent is greater than one, the bounding sequence will tend to zero as long as a_0 is sufficiently small.

But since $u\leq 1$ by assumption, we can calculate, for any $b>\sigma$,

$$\begin{aligned}
\mathcal{E}_1&=\sup_{[T_1,0]}\int\phi_1^2u_1^2+\iint_1\phi_1^2u_1|\nabla u_1|^p\\
&=2^{2(b-1)}\left(\sup_{[T_1,0]}\int\phi_1^2u_1^2\left(2^{-2}\chi_{\{u_0>2^{-2}\}}\right)^{b-1}+\iint_1\phi_1^2u_1\left(2^{-2}\chi_{\{u_0>2^{-2}\}}\right)^{b-1}|\nabla u_1|^p\right)\\
&\leq 2^{2(b-1)}\left(\sup_{[T_1,0]}\int\phi_1^2u_1^2u_0^{b-1}+\iint_1\phi_1^2u_1u_0^{b-1}|\nabla u_1|^p\right)\\
&\leq 2^{2(b-1)}\left(\sup_{[T_1,0]}\int\phi_0^2u_0^{b+1}+\iint_1\phi_0^2u_0^b|\nabla u_1|^p\right)
\end{aligned}$$

$$\leq C\left(\iint_0 (u_0^{b+1}+u_0^{b-\sigma})\chi_{\{B^0\}}+\left(\iint_0 u_0^{bm^*}\chi_{\{B^0\}}\right)^{\frac{1}{m^*}}\right)$$
$$\leq C\left(|\{u>0\}\cap\overline{Q}_2|+|\{u>0\}\cap\overline{Q}_2|+|\{u>0\}\cap\overline{Q}_2|^{1/m^*}\right).$$

Therefore there exists a $\delta_0>0$ sufficiently small that, if $\left|\{u>0\}\cap\overline{Q}_2\right|\leq\delta_0$, then $\mathcal{E}_1$ will be small enough that $\mathcal{E}_k\to 0$ as $k\to\infty$.

If $\mathcal{E}_k\to 0$, then

$$\|u_k\|_{L^q([-1,0]\times B_1)}\leq\|\phi_k u_k\|_{L^q([T_k,0]\times B^k)}\leq C^k\mathcal{E}_k^{\frac{1}{q}+\frac{q-p}{2q}}\to 0.$$

By the monotone convergence theorem, we conclude that $\|(u-1/2)_+\|_{L^q([-1,0]\times B_1)}=0$ and so

$$|\{u>\frac{1}{2}\}\cap[-1,0]\times B_1|=0.$$

□

2.4 De Giorgi's second lemma

The second De Giorgi lemma is a quantitative version of the statement "solutions to our PDE cannot have jump discontinuities."

Define the sets

$$Q_3=[-4,0]\times B_3,\qquad Q_2=[-4,0]\times B_2,$$

and remember that

$$\overline{Q}_2=[-2,0]\times B_2.$$

According to the next theorem, if a solution to (1.1) is negative in Q_2 on a set of large measure, and ≥ 1 in $\overline{Q}_2$ on a set of large measure, and it is bounded on all of Q_3, then that solution must be strictly between 0 and 1 on a set of large measure in $\overline{Q}_2$.

The proof is by compactness. Because the solution is bounded on Q_3, we can use the Energy nequality to bound its derivatives on Q_2. By a theorem of Aubin and Lions, which is an instance of the general principle "bounded derivatives imply compactness," we can conclude that the family of bounded solutions is precompact. Therefore, if the interstitial measure is not bounded below, there must be a limit function which would have both bounded derivatives and a jump discontinuity, a contradiction.

Because of the coefficient on $|\nabla u|$ in the Energy Inequality, the derivatives are not well controlled when u is near zero. This is solved by considering instead u raised to some power, whose derivatives are trivially controlled when u is near zero, and whose convergence implies the convergence of u.

Proposition 2.4.1 (De Giorgi's Second Lemma)**.** *There exists a positive constant μ_0 depending on Λ, p, m, δ_0, and the dimension, such that for any u satisfying Inequality* (1.1) *in the sense of distributions, with*

$$u(t,x) \leq 2 \qquad \forall (t,x) \in Q_3$$

and

$$|\{u \leq 0\} \cap Q_2| \geq \frac{|Q_2|}{2},$$

and, for δ_0 the quantity divined in Proposition 2.3.1,

$$\left|\{u \geq 1\} \cap \overline{Q}_2\right| \geq \delta_0,$$

it must be the case that

$$|\{0 < u < 1\} \cap Q_2| \geq \mu_0.$$

Proof. Suppose the proposition is false. Then we can consider a sequence u_i of functions which satisfy all the hypotheses of this proposition but for which

$$|\{0<u_i<1\}\cap Q_2|\le \frac{1}{i}.$$

Rather than seek a limit of the sequence u_i, we will actually seek a limit of $(u_i)_+^{\sigma+2}$, where $\frac{1}{\sigma}+\frac{2}{p}=1$ consistent with the notation in Lemma 2.2.1. First we need to bound the space and time derivatives of $(u_i)_+^{\sigma+2}$ uniformly in i.

Step 1: Bounding the derivatives

To bound the spatial derivatives, we use the Energy Inequality (2.5) with $b=(\sigma+1)p$, and choose a smooth cutoff function ϕ satisfying

$$\phi: B_3\to[0,1], \qquad \phi\ge 0, \qquad \mathrm{supp}(\phi) \text{ compact}, \qquad \psi(x)=1\ \forall x\in B_2.$$

By the Energy Inequality, we have

$$\begin{aligned}\iint_{B_2\times[-4,0]}|\nabla(u_i)_+^{\sigma+2}|^p &\le (\sigma+2)^p\iint_{-4}^{0}\psi(u_i)_+^{p(\sigma+1)}|\nabla(u_i)_+|^p\\ &\le C\iint_{-4}^{0}\left((u_i)_+^{p(\sigma+1)-\sigma}+(u_i)_+^{p(\sigma+1)+1}\right)\chi_{\{B_3\}}+C\left(\iint_{-4}^{0}(u_i)_+^{m^*p(\sigma+1)}\chi_{\{B_3\}}\right)^{1/m^*}\\ &\le C(\Lambda,p,n,m).\end{aligned}$$

Therefore the sequence $\nabla(u_i)_+^{\sigma+2}$ is bounded in $L^p([-4,0];L^p(B_2))$ uniformly in i.

Bounding the time derivative is much more involved. We will show that $\partial_t(u_i)_+^{\sigma+2}$ are uniformly bounded in $\mathcal{M}([-4,0];W^{-1,\infty})$, where $\mathcal{M}$ means the dual space to L^∞ and $W^{-1,\infty}$ is the dual of $\overline{C_0^\infty(B_2)\cap W^{1,\infty}(B_2)}$.

Using the Energy Inequality (2.6) with $b=\sigma+1$ and any test function $\varphi : Q_3 \to \mathbb{R}$ which is smooth and compactly supported in space (but not necessarily compactly supported in time), together with the fact that $\|f\|_1 \le \|f\|_m \le \Lambda$ and $u_i \le 2$, gives us the bound

$$\begin{aligned}\langle \partial_t (u_i)_+^{\sigma+2}, \varphi^2 \rangle_{[-4,0]\times B_3} &\le C(p,\Lambda)\left(\iint \varphi^2 (u_i)_+^{\sigma+1} f + \iint \varphi^2 (u_i)_+ + \iint (u_i)_+^{\sigma+2} |\nabla \varphi|^2 \right) \\ &\le C(p,\Lambda)\left(\|\varphi\|^2_{L^\infty(Q_3)} + \|\nabla\varphi\|^2_{L^\infty(Q_3)} \right).\end{aligned}$$

We must find a similar bound on $\langle \partial_t (u_i)_+^{\sigma+1}, \psi \rangle$ when ψ is not necessarily the square of a smooth function. Our strategy is to decompose ψ as a sum of a perfect square and a function independent of time. To this end, define $\sqrt{\phi}$ a specific smooth function (of space only) supported in B_3 and identically 1 on B_2. Then $\phi := \sqrt{\phi}^2$ will also be smooth, supported on B_3, and identically 1 on B_2.

Consider any $\psi \in C_0^\infty(Q_3)$, and set $K = \|\psi\|_\infty + \|\nabla \psi\|_\infty$. Here and in the sequel, $\|\cdot\|_\infty$ means $\|\cdot\|_{L^\infty(Q_3)}$. Note that $\psi + K\phi$ is non-negative, so we can define φ by the relation

$$\psi = \varphi^2 - K\phi.$$

Estimate

$$\begin{aligned}\iint_{Q_2} \psi \partial_t (u_i)_+^{\sigma+2} &= -K \iint_{Q_3} \phi \partial_t (u_i)_+^{\sigma+2} + \iint_{Q_3} \varphi^2 \partial_t (u_i)_+^{\sigma+2} \\ &\le K \left| \int_{-4}^0 \frac{d}{dt} \int \phi (u_i)_+^{\sigma+2} \right| + C\left(\|\varphi\|_\infty^2 + \|\nabla\varphi\|_\infty^2 \right) \\ &\le K \left[\int \phi (u_i)_+^{\sigma+2}(0,\cdot) + \int \phi (u_i)_+^{\sigma+2}(-4,\cdot) \right] + C\left(\|\psi + K\phi\|_\infty + \left\| \left(\nabla \sqrt{\psi + K\phi} \right)^2 \right\|_\infty \right).\end{aligned}$$

By the chain rule, this last term becomes

$$2\left\| (\nabla \sqrt{\psi + K\phi})^2 \right\|_\infty = \left\| \frac{|\nabla\psi + K\nabla\phi|^2}{\psi + K\phi} \right\|_\infty$$

$$
\begin{aligned}
&= \sup\left(\left\|\frac{|\nabla\psi + K\nabla\phi|^2}{\psi + K\phi}\right\|_{L^\infty(Q_2)}, \left\|\frac{|\nabla\psi + K\nabla\phi|^2}{\psi + K\phi}\right\|_{L^\infty(Q_3\setminus Q_2)}\right) \\
&= \sup\left(\left\|\frac{|\nabla\psi|^2}{\psi + K}\right\|_{L^\infty(Q_2)}, \left\|\frac{|K\nabla\phi|^2}{K\phi}\right\|_{L^\infty(Q_3\setminus Q_2)}\right) \\
&\le \sup\left(\frac{1}{\|\nabla\psi\|_\infty}\left\||\nabla\psi|^2\right\|_\infty, \frac{K^2}{K}\left\|\nabla\sqrt{\phi}\right\|_\infty^2\right) \\
&\le C_\phi K.
\end{aligned}
$$

In the above calculation, remember that ϕ is constant on Q_2 and $\psi = 0$ outside Q_2, that $\psi + K \ge \|\nabla\psi\|_\infty$ by the definition of K, and that $\sqrt{\phi}$ is smooth by assumption.

We see now that

$$\langle \psi, \partial_t (u_i)_+^{\sigma+2}\rangle \le C(\Lambda, p, n, \phi)\left(\|\psi\|_\infty + \|\nabla\psi\|_\infty\right)$$

and, by duality, $\partial_t (u_i)_+^{\sigma+2}$ is bounded in $\mathcal{M}([-4,0]; W^{-1,\infty}(B_2))$.

In order to apply our compactness lemma, we need $(u_i)_+^{\sigma+2}$ to be absolutely continuous in time (i.e. we want L^1, not $\mathcal{M}$). Therefore consider a family of mollifiers η_δ tending to a dirac measure as $\delta \to 0$. Convolving with respect to time, we obtain smooth-in-time functions.

$$\eta_\delta * (u_i)_+^{\sigma+2} \in L^p([-4,0]; W^{1,p}(B_2)), \qquad \partial_t\left[\eta_\delta * (u_i)_+^{\sigma+2}\right] \in L^1([-4,0]; W^{-1,\infty}(B_2))$$

are uniformly bounded independent of $\delta < 1$.

The Aubin-Lions Lemma indicates that the family $\eta_\delta * (u_i)_+^{\sigma+2}$ is compact in $L^1([-4,0]\times B_2)$. Choose a sequence $\delta_i \to 0$ such that

$$\left\|(u_i)_+^{\sigma+2} - \eta_{\delta_i} * (u_i)_+^{\sigma+2}\right\|_{L^1} \le \frac{1}{i}.$$

By compactness, the sequence $\eta_{\delta_i} * (u_i)_+^{\sigma+2}$ has a subsequential limit v, and

$$\left\|(u_i)_+^{\sigma+2} - v\right\|_1 \le \left\|(u_i)_+^{\sigma+2} - \eta_{\delta_i} * (u_i)_+^{\sigma+2}\right\|_1 + \left\|\eta_{\delta_i} * (u_i)_+^{\sigma+2} - v\right\|_1 \to 0.$$

That is to say, $(u_i)_+^{\sigma+2} \to v$ in $L^1(Q_2)$.

Step 2: Showing that the limit engenders a contradiction

By a measure-theoretic argument,

$$|\{v \le 0\} \cap Q_2| \ge \frac{|Q_2|}{2}, \tag{*}$$

$$|\{v \ge 1\} \cap \overline{Q}_2| \ge \delta_0, \text{ and} \tag{**}$$

$$|\{0 < v < 1\} \cap Q_2| = 0.$$

The map $f \mapsto \|\nabla f\|_{L^p(Q_2)}$ is lower-semi-continuous on $L^1(Q_2)$, and hence

$$\int_{-4}^{0} \|\nabla v\|_{L^p(B_2)}^p \, dt < \infty.$$

This implies that for almost every $t \in [-4,0]$, $\|\nabla v\|_p$ is finite; and for such t, v must have no spatial jump discontinuities. In other words, there are three kinds of $t \in [-4,0]$: those at which v is identically 0, those at which $v(t,x) \ge 1 \ \forall x \in B_2$, and the exceptions which have measure zero in $[-4,0]$.

If we define a new smooth cutoff ϕ on B_2, and set

$$H(t) = \left\|\phi^2(\cdot) v(t,\cdot)\right\|_{L^1(B_2)},$$

then for a.e. t, either $H(t) = 0$ or $H(t) \ge \|\phi^2\|_1$.

On the other hand, we know that H cannot have (certain kinds of) jump discontinuities. Because $(u_i)_+^{\sigma+2} \to v$ in $L^1(Q_2)$, we know that

$$H_i \equiv \left\|\phi^2 (u_i)_+^{\sigma+2}\right\|_1 \longrightarrow H \qquad \text{in } L^1([-4,0]).$$

And by the Energy Inequality (2.6), with cutoff ϕ and $b=\sigma+1$, the derivative of each H_i is bounded uniformly in i: notice that $\partial_t\phi=0$ and so for any time interval $[s,t]$ we have

$$\begin{aligned}H_i(t)-H_i(s) &= \int_s^t \frac{d}{dt}\int \phi^2 (u_i)_+^{\sigma+2} \\ &\le C(p,\Lambda,\phi)\iint_s^t \left((u_i)_+^{\sigma+1}+(u_i)_+^{\sigma+2}+(u_i)_+^1\right)\chi_{\{\mathrm{supp}(\phi)\}} \\ &\le [s-t]C(p,\Lambda,\phi).\end{aligned}$$

Therefore (again by lower-semi-continuity), $\frac{d}{dt}H$ is bounded above.

This means in particular that if $H(s)=0$, then $H(t)=0$ $\forall t\ge s$. And we know by (*) that $v=0$ on a set of large measure. In fact, necessarily $H(t)=0$ $\forall t\in(-2,0]$. This contradicts (**), and so the proposition is proven.

$\square$

2.5 Transporting improvement forwards in time

Using the propositions proven thus far, one can show, under the appropriate hypotheses, that if a solution to Inequality (1.1) is ≥ -2 in Q_3, then it is in fact $\ge -2+\varepsilon$ in $[-4,-3]\times B_\varepsilon$. This is not quite what we set out to prove; we want solutions to become regular after some time elapses, and hence the lower bound must be somewhere in the region $[-1,0]\times B_1$.

To bridge the gap, we use a barrier function to "flow" the improvement forward in time. Our solution will still be $\ge -2+\varepsilon'$ on a ball of radius ε' at the end of the time interval, and though ε' becomes smaller as time elapses, it never vanishes entirely.

This is the first time we use (1.3). This inequality is true only in a viscosity sense, so instead of energy methods, we must construct a barrier function which constitutes a subsolution to

$$\partial_t u + \Lambda |\nabla u|^p - \Lambda_0 m^-(D^2 u) = 0.$$

Proposition 2.5.1. *There exists a constant $0 < K_0 < 1$ depending only on p, Λ, and n such that the following holds: Let $0 < \lambda \leq K_0$ be a constant and u a viscosity supersolution to Inequality* (1.3) *on the interior of $[0,T] \times B_2$ with $T < 4$ and $\Lambda_0 \leq \lambda^2 K_0$. Suppose that*

$$u \geq -2 \quad on \quad [0,T] \times B_2,$$

$$u \geq -2 + \lambda^2 \quad on \quad 0 \times B_\lambda.$$

Then

$$u \geq -2 + \frac{\lambda^2}{2} \quad on \quad [0,T] \times B_{\lambda/2}.$$

Proof. We define the barrier function

$$\sigma(t,x) := -2 + \lambda^2 \beta\left(\frac{|x|}{\lambda}\right) - \frac{\lambda^2}{8} t,$$

where $\beta : \mathbb{R}^+ \to \mathbb{R}$ is a smooth function supported on $[0,1]$ and identically 1 on $[0,1/2]$.

If we can show that σ is a subsolution to (1.3), and that it is less than u on the parabolic boundary $0 \times B_2 \cup [0,T] \times \partial B_2$, then the standard theory of comparison principles tells us that $u \geq \sigma$ on the whole interior of $[0,T] \times B_2$. See [Cra97] for the elliptic version of the comparison principle, and [Juu01] for a treatment more specific to the parabolic case.

In particular, for $(t,x) \in [0,T] \times B_{\lambda/2}$ we have

$$\sigma(t,x) = -2 + \lambda^2(1 - t/8) \geq -2 + \lambda^2(1 - T/8) \geq -2 + \lambda^2/2.$$

Thus showing $u \geq \sigma$ will prove the proposition.

Step 1: Barrier is below u on the boundary

At $t=0$,

$$\sigma(0,x) \leq -2+\lambda^2 \leq u \qquad \forall x \in B_\lambda,$$

$$\sigma(0,x) \leq -2 \leq u \qquad \forall x \in B_2 \setminus B_\lambda;$$

and on the spatial boundary $|x|=2$,

$$\sigma(t,x) = -2 - \frac{\lambda^2}{8} t \leq -2 \leq u \qquad \forall t \in [0,T].$$

Thus on the parabolic boundary of $[0,T] \times B_2$, we have $\sigma \leq u$.

Step 2: Barrier is a subsolution

By construction

$$\partial_t \sigma(t,x) = -\lambda^2/8$$

and

$$|\nabla \sigma|(t,x) = \lambda \beta' \left(\frac{|x|}{\lambda} \right).$$

To compute $D^2\sigma$, notice that σ is radially symmetric in space, and so it suffices to compute the Hessian at the point $x=(|x|,0,\ldots,0)$. At this point, one can compute directly that

$$\begin{aligned} \partial_{11}\sigma(t,x) &= \frac{d^2}{dh^2}\bigg|_{h=0} \lambda^2 \beta \left(\frac{|x|+h}{\lambda} \right) \\ &= \beta'' \left(\frac{|x|}{\lambda} \right) \end{aligned}$$

and for $i \neq 0$

$$\partial_{ii}\sigma(t,x) = \frac{d^2}{dh^2}\bigg|_{h=0} \lambda^2 \beta \left(\frac{\sqrt{|x|^2+h^2}}{\lambda} \right)$$

$$= \frac{\lambda}{|x|}\beta'\left(\frac{|x|}{\lambda}\right).$$

For any $i \neq j$, assume without loss of generality that $i \neq 1$. Then $[\partial_i\sigma](x) = 0$ for any x in the hyperplane $x_i = 0$, by radial symmetry. Therefore $\partial_j[\partial_i\sigma] = 0$ at $(|x|,0,\ldots,0)$.

We conclude that the matrix $D^2\sigma(t,x)$ is a diagonal matrix with eigenvalues

$$\frac{\lambda}{|x|}\beta'\left(\frac{|x|}{\lambda}\right) \quad \text{and} \quad \beta''\left(\frac{|x|}{\lambda}\right),$$

and by symmetry it should have the same eigenvalues at generic x.

Therefore, to see if σ is a subsolution, calculate

$$\begin{aligned}
\partial_t\sigma + \Lambda|\nabla\sigma|^p - \Lambda_0 m^-(D^2\sigma) &= -\frac{\lambda^2}{8} + \Lambda\lambda^p(\beta')^p - \Lambda_0 \min\left(\beta'', \frac{\lambda}{|x|}\beta', 0\right) \\
&\leq \frac{-\lambda^2}{8} + \Lambda\lambda^p\|\beta'\|_\infty^p + \Lambda_0\|\beta''\|_\infty + \Lambda_0\frac{\lambda}{1/2}\|\beta'\|_\infty \\
&\leq \frac{-\lambda^2}{8} + \Lambda\lambda^p\|\beta'\|_\infty^p + \lambda^2 K_0\|\beta''\|_\infty + 2\lambda^3 K_0\|\beta'\|_\infty \\
&= \lambda^2\left(\Lambda\lambda^{p-2}\|\beta'\|_\infty^p + K_0\|\beta''\|_\infty + 2\lambda K_0\|\beta'\|_\infty - \frac{1}{8}\right) \\
&\leq \lambda^2\left(\Lambda K_0^{p-2}\|\beta'\|_\infty^p + K_0\|\beta''\|_\infty + K_0^2\frac{n-1}{1/2}\|\beta'\|_\infty - \frac{1}{8}\right).
\end{aligned}$$

This last quantity is negative provided K_0 sufficiently small, depending on Λ, p, the dimension, and the specific choice of β.

$\square$

2.6 Proof of the main theorem

Having completed the core of the proof, we now come to the final section. The pieces are all present, and we need only put them together. This section contains three lemmas before the proof. The first two (Lemmas 2.6.1 and 2.6.2) tell us which scalings constitute symmetries of our PDE. Lemma 2.6.3, the Oscillation Lemma, applies Propositions 2.3.1 and 2.4.1 iteratively in order to control the oscillation of solutions to our PDE. Finally the proof of the Main Theorem will show how the Oscillation Lemma is equivalent to interior Hölder continuity.

The proof of the Oscillation Lemma is slightly non-standard. The rest is technical, with no new ideas.

Lemma 2.6.1. *If u satisfies the two equations (1.1) and (1.3) on a cylinder $[T_0,0]\times\Omega$, and $\alpha,\beta>0$ are any two real numbers satisfying*

$$\beta\leq\alpha^{-1}\quad \beta\leq\alpha^{-\frac{p-1}{p-2}},\quad \beta\leq\alpha^{-\frac{p(m-1)+1}{p(m-1)-n}},$$

then the modified function

$$v(t,x):=\alpha u(\alpha^{p-1}\beta^p t,\beta x)$$

satisfies the equations

$$\partial_t v+\Lambda^{-1}|\nabla v|^p-\mathrm{div}(A'\nabla v)\leq f'$$

$$\partial_t v+\Lambda|\nabla v|^p-\Lambda_0' m^-(D^2 v)\geq 0$$

on $\left[\frac{T_0}{\alpha^{p-1}\beta^p},0\right]\times\frac{1}{\beta}\Omega$, with $\Lambda_0'=\alpha^{p-1}\beta^{p-2}\Lambda_0\leq\Lambda_0$, $\|A'\|_\infty\leq\|A\|_\infty$ and $\|f'\|_m\leq\|f\|_m$.

Proof. One must take

$$f'(t,x):=\alpha^p\beta^p f(\alpha^{p-1}\beta^p t,\beta x),$$

$$A'(t,x):=\alpha^{p-1}\beta^{p-2}A(\alpha^{p-1}\beta^p t,\beta x).$$

Applying our differential operator to v, we obtain

$$\begin{aligned}
\partial_t v + \Lambda^{-1}|\nabla v|^p - \operatorname{div}(A'\nabla v) &= (\alpha\beta)^p \partial_t u + (\alpha\beta)^p \Lambda^{-1}|\nabla u|^p - (\alpha\beta)^p \operatorname{div}(A\nabla u) \\
&= (\alpha\beta)^p \left[\partial_t u + \Lambda^{-1}|\nabla u|^p - \operatorname{div}(A\nabla u)\right] \\
&\leq f'
\end{aligned}$$

For the other inequality, similarly,

$$\begin{aligned}
\partial_t v + \Lambda|\nabla v|^p - \Lambda_0 m^-(D^2 v) &= (\alpha\beta)^p \partial_t u + (\alpha\beta)^p \Lambda|\nabla u|^p - \alpha\beta^2 \Lambda_0 m^-(D^2 u) \\
&= (\alpha\beta)^p \left[\partial_t u + \Lambda|\nabla u|^p - \Lambda m^-(D^2 u)\right] \\
&\geq 0.
\end{aligned}$$

That $\Lambda_0' \leq \Lambda_0$ and $\|A'\|_\infty \leq \|A\|_\infty$ follows immediately from our assumptions on α, β. For $\|f'\|_m$, we notice that $p - \frac{p+n}{m}$ is necessarily positive, and calculate

$$\begin{aligned}
\left\|\alpha^p\beta^p f(\alpha^{p-1}\beta^p t, \beta x)\right\|_m &= \alpha^p\beta^p(\alpha^{p-1}\beta^p\beta^n)^{-1/m}\|f\|_m \\
&= \alpha^{p-\frac{p-1}{m}}\beta^{p-\frac{p+n}{m}}\|f\|_m \\
&\leq \alpha^{p-\frac{p-1}{m}}\left(\alpha^{-\frac{p(m-1)+1}{p(m-1)-n}}\right)^{p-\frac{p+n}{m}}\|f\|_m = \|f\|_m.
\end{aligned}$$

$\square$

Lemma 2.6.2. *If u satisfies Inequality (1.1) on a cylinder $[T_0,0]\times\Omega$, there exist constants $e_1 \in (2,p)$ and $e_2 < 0$ dependent on n, m, p such that, for any two real numbers $0 < \beta \leq 1$ and $1 \leq \alpha \leq \beta^{e_2}$, the modified function*

$$v(t,x) := \alpha u(\beta^{e_1} t, \beta x)$$

also satisfies Inequality (1.1) on $[T_0,0]\times\Omega$ with parameters $\|f'\|_m \leq \|f\|_m$, $\|A'\|_\infty \leq \|A\|_\infty$ and the same Λ.

Proof. Since $\frac{n}{m-1} < p$ and $p > 2$, we can choose a constant $e_1 \in (\frac{n}{m-1}, p)$ such that $e_1 > 2$. Let

$$e_2 := \max\left(-\frac{p-e_1}{p-1}, \frac{n}{m} - e_1\frac{m-1}{m}\right)$$

so that

$$\alpha^{p-1}\beta^{p-e_1} = \left(\alpha\left(\frac{1}{\beta}\right)^{-\frac{p-e_1}{p-1}}\right)^{p-1} \leq \left(\alpha\left(\frac{1}{\beta}\right)^{e_2}\right)^{p-1} \leq 1$$

and $\alpha\beta^{e_1\frac{m-1}{m}-\frac{n}{m}} \leq 1$.

Define

$$\begin{aligned} A'(t,x) &:= \beta^{e_1-2}A(\beta^{e_1}t, \beta x), \\ f'(t,x) &:= \alpha\beta^{e_1}f(\beta^{e_1}t, \beta x). \end{aligned}$$

Applying our differential operator to v, we obtain

$$\begin{aligned} \partial_t v + \Lambda^{-1}|\nabla v|^p + \operatorname{div}(A'\nabla v) &= \alpha\beta^{e_1}\partial_t u + (\alpha\beta)^p\Lambda^{-1}|\nabla u|^p + \alpha\beta^{e_1}\operatorname{div}(A\nabla u) \\ &= \alpha\beta^{e_1}\left[\partial_t u + \left(\alpha^{p-1}\beta^{p-e_1}\right)\Lambda^{-1}|\nabla u|^p + \operatorname{div}(A\nabla u)\right] \\ &\leq \alpha\beta^{e_1}\left[\partial_t u + \Lambda^{-1}|\nabla u|^p + \operatorname{div}(A\nabla u)\right] \\ &\leq \alpha\beta^{e_1}f = f'. \end{aligned}$$

That $\|A'\|_\infty \leq \|A\|_\infty$ follows immediately from our assumption that $e_1 > 2$. It remains to calculate the norm of f':

$$\begin{aligned} \|f'\|_m &= \alpha\beta^{e_1}(\beta^{e_1}\beta^n)^{-1/m}\|f\|_m \\ &= \alpha\beta^{e_1(1-\frac{1}{m})-\frac{n}{m}}\|f\|_m \\ &\leq \|f\|_m. \end{aligned}$$

A priori, v will satisfy this inequality on $\left[\frac{T_0}{\beta^{e_1}},0\right]\times\frac{1}{\beta}\Omega$. Since we assume $\beta\leq 1$, this in particular means it is satisfied on $[T_0,0]\times\Omega$.

$\square$

At last we can prove the Oscillation Lemma. The oscillation of a function is the distance between its supremum and its infimum, and for solutions of (1.1) and (1.3), if the oscillation is finite on a region it will be strictly less on a strictly smaller region.

Lemma 2.6.3 (Oscillation Lemma)**.** *There exist constants $\lambda^*>0$, $r^*>0$, $T^*<0$ depending on Λ, p, n, μ_0 (from Proposition 2.4.1), δ_0 (from Proposition 2.3.1), K_0 (from Proposition 2.5.1), and e_1, e_2 (from Lemma 2.6.2) such that, for any solution u to Inequalities* (1.1) *and* (1.3) *on Q_3, with $\Lambda_0<(\lambda^*)^2K_0$, we have the following implication: If*

$$|u|\leq 2 \qquad \forall (t,x)\in Q_3,$$

then either

$$\sup_{[T^*,0]\times B_{r^*}(0)} u\leq 2-\frac{(\lambda^*)^2}{2}$$

or

$$\inf_{[T^*,0]\times B_{r^*}(0)} u\geq -2+\frac{(\lambda^*)^2}{2}.$$

The idea of the proof is to apply De Giorgi's First Lemma to some truncation of u. Remember that De Giorgi's First Lemma says that if the measure of $\{u_+>0\}$ is sufficiently small, then u_+ is L^∞-bounded on some smaller domain. This L^∞ bound is precisely what we wish to prove. We attempt to apply the lemma to each of $(u-C_k)_+$ for C_k an increasing series of constants. Obviously the measure shrinks as C_k increases; De Giorgi's Second Lemma allows us to quantify the decrease in measure, and find a precise k for which De Giorgi's First Lemma applies.

Proof. Let k_0 be the smallest integer greater than $|Q_2|/\mu_0$, where μ_0 is the constant in Proposition 2.4.1, and define

$$Q_{\text{small}} := [-4\cdot 2^{k_0 e_1/e_2}, 0] \times B_{2\cdot 2^{k_0/e_2}}.$$

There are two cases to consider: either we will upper-bound the supremum or we will lower-bound the infimum of u in the region $[T^*, 0] \times B_{r^*}(0)$. If

$$|\{u \le 0\} \cap Q_{\text{small}}| \ge \frac{|Q_{\text{small}}|}{2},$$

we are in the former case, so we call u "mostly negative" and define

$$v(t,x) := u(2^{k_0 e_1/e_2} t, 2^{k_0/e_2} x).$$

Otherwise, we are in the latter case, so we call u "mostly positive" and define

$$v(t,x) := -u(2^{k_0 e_1/e_2}(-4-t), 2^{k_0/e_2} x).$$

In either case,

$$|\{v \le 0\} \cap Q_2| \ge \frac{Q_2}{2}.$$

For integers $k \in [0, k_0]$ consider the functions

$$v_k = 2^k (v-2) + 2.$$

Notice that for all $k \le k_0$, $v_k \le 2$ on Q_3. By Lemma 2.6.2 with $\alpha = 2^k$ and $\beta = 2^{k_0/e_2}$ and domain Q_3, combined with the fact that Inequality (1.1) is preserved by translations, addition of constants, and the transformation $f(t,x) \mapsto -f(-t,x)$, each v_k satisfies Inequality (1.1) on Q_3.

We claim that $|\{v_{k_0} \ge 1\} \cap \overline{Q}_2| \le \delta_0$. If this were not the case, then in fact

$$|\{v_k \ge 1\} \cap \overline{Q}_2| > \delta_0,$$

for all $k \le k_0$, because the quantity is non-increasing as k increases. Similarly,

$$|\{v_k \le 0\} \cap Q_2| \ge \frac{|Q_2|}{2}$$

for all $k \le k_0$, because the same holds for v_0 and the quantity is non-decreasing.

This is enough for us to apply De Giorgi's Second Lemma to each v_k. By construction, the Lemma tells us that

$$|\{v_{k+1} \ge 0\} \cap Q_2| \le |\{v_k \ge 0\} \cap Q_2| - \mu_0.$$

This cannot possibly be true for all k between 0 and k_0, since $k_0\mu_0 > |Q_2|$. This is a contradiction.

Therefore $|\{v_{k_0} \ge 1\} \cap \overline{Q}_2| \le \delta_0$. We can apply De Giorgi's First Lemma to $v_{k_0} - 1$, and learn that $v_{k_0} \le 3/2$ on Q_1. In terms of v,

$$v(t,x) \le 2 - 2^{-k_0-1} \qquad \forall (t,x) \in Q_1.$$

In the case that u is mostly negative, this means

$$u(t,x) \le 2 - 2^{-k_0-1} \qquad \forall (t,x) \in [T,0] \times B_r(0), \qquad T = -2^{k_0 e_1/e_2},\ r = 2^{k_0/e_2}$$

and the proof is complete. So consider the case where u is mostly positive. We've shown that

$$u \ge -2 + 2^{-k_0-1} \qquad \forall (t,x) \in [-4 \cdot 2^{k_0 e_1/e_2}, -3 \cdot 2^{k_0 e_1/e_2}] \times B_r.$$

The problem here is the time interval; we want a lower bound on the infimum of u in a parabolic neighborhood of $(0,0)$. Define

$$\lambda^* = \min(K_0, \sqrt{2^{-k_0-1}}).$$

Proposition 2.5.1 applied to the lower-semicontinuous envelope of u tells us that, since we assumed $\Lambda_0 \le (\lambda^*)^2 K_0$,

$$u \ge -2 + \frac{(\lambda^*)^2}{2} \qquad \text{on } [4T,0] \times B_{\lambda^*/2}.$$

Letting $T^* = T$, $r^* = \min(r, \lambda^*/2)$, we see that either

$$\sup_{[T^*,0]\times B_{r^*}(0)} u \le 2 - \frac{(\lambda^*)^2}{2}$$

or

$$\inf_{[T^*,0]\times B_{r^*}(0)} u \ge -2 + \frac{(\lambda^*)^2}{2}.$$

□

Finally, we are ready to prove Theorem 2.1.1.

Proof. Instead of proving continuity directly for u, it is preferable to consider

$$\bar{u} \equiv u + \Lambda t,$$

which satisfies the Inequalities (1.1) and (1.3). Clearly $\bar{u}$ and u will have the same Hölder exponent.

Since $\bar{\Omega}$ is compact, there is a radius ρ such that $B_\rho(x) \subseteq \Omega$ for each $x \in \bar{\Omega}$.

Consider any two points $(t_0,x_0),(t_1,x_1) \in (s,T) \times \bar{\Omega}$, and assume wlog that $t_0 \ge t_1$. If these points are far away, then we can estimate the Hölder norm in a very rough way, using the L^∞ norm of $\bar{u}$. If the points are very close together, then we must use the Oscillation Lemma.

We want to rescale the function $\bar{u}$ to obtain w centered at (t_0,x_0) but solving the PDE on Q_3, with $\|w\|_\infty \le 2$, and with $\Lambda_0 \le (\lambda^*)^2 K_0$. To that end, choose α_w, β_w small enough that

$$\alpha_w \le \frac{2}{\|\bar{u}\|_{L^\infty([T,0]\times\Omega)}}, \qquad 3\beta_w \le \rho, \qquad 4\alpha_w^{p-1}\beta_w^p \le s, \qquad \alpha_w^{p-1}\beta_w^{p-2}\Lambda_0 \le (\lambda^*)^2 K_0,$$

and

$$\alpha_w\beta_w \leq 1, \qquad \alpha_w^{p-1}\beta_w^{p-2} \leq 1, \qquad \alpha^{p(m-1)+1}\beta^{p(m-1)-n} \leq 1.$$

Note that α_w and β_w depend on $\|u\|_{L^\infty}$.

Lemma 2.6.1 tells us that

$$w(t,x) := \alpha_w \bar{u}\left(t_0 + \alpha_w^{p-1}\beta_w^p t, x_0 + \beta_w x\right)$$

is a solution to Inequalities (1.1) and (1.3) on Q_3, with $\Lambda_0 \leq (\lambda^*)^2 K_0$. By construction $|w| \leq 2$ on Q_3.

Now that w is formatted correctly, the plan is to apply Lemma 2.6.3 iteratively, showing that the oscillation of w decreases as the distance to $(0,0)$ decreases.

Set

$$\alpha_1 = \frac{4}{4-(\lambda^*)^2/2},$$

and take β_1 small enough that $3\beta_1 \leq r^*$, and $4\alpha_1^{p-1}\beta_1^p \leq -T^*$, and small enough to satisfy the hypotheses of Lemma 2.6.1. Define $w_0 = w$ and iteratively define

$$w_{k+1}(t,x) := \alpha_1\left[w_k(\alpha_1^{p-1}\beta_1^p t, \beta_1 x) \pm \frac{(\lambda^*)^2}{4}\right],$$

with $\pm$ chosen as whichever sign minimizes $\|w_{k+1}\|_{L^\infty(Q_3)}$. By induction, $|w_k| \leq 2$ on Q_3 and w_k solves Inequalities (1.1) and (1.3) on Q_3 with $\Lambda_0 \leq (\lambda^*)^2 K_0$, and hence satisfies the hypotheses of Lemma 2.6.3.

Therefore, for all $k \geq 0$, we find that for $Q_k = [-(\alpha_1^{p-1}\beta_1^p)^k, 0] \times B_{\beta_1^k}$,

$$\sup_{Q_k} w(t,x) - \inf_{Q_k} w(t,x) \leq \frac{1}{\alpha_1^{k-1}}\left(4 - \frac{(\lambda^*)^2}{2}\right).$$

Remember that we are trying to bound the Hölder norm, the quantity

$$(*) = \frac{|\bar{u}(t_1,x_1) - \bar{u}(t_0,x_0)|}{|(t_0-t_1)^2 + |x_0-x_1|^2|^{\gamma/2}}.$$

If $\sqrt{(t_0-t_1)^2+|x_0-x_1|^2} \geq \alpha_w^{p-1}\beta_w^p$, then we can bound

$$(*) \leq \frac{2\|\bar{u}\|_\infty}{(\alpha_w^{p-1}\beta_w^p)^\gamma}.$$

Otherwise, we can use the control on the oscillation of w. Specifically, if

$$\sqrt{(t_0-t_1)^2+|x_0-x_1|^2} \leq \alpha_w^{p-1}\beta_w^p(\alpha_1^{p-1}\beta_1^p)^k$$

for any integer $k \geq 0$, then, because $\alpha_w\beta_w \leq 1$ and $\alpha_1\beta_1 \leq 1$,

$$\left(\frac{t_1-t_0}{\alpha_w^{p-1}\beta_w^p}, \frac{x_1-x_0}{\beta_w}\right) \in Q_k.$$

Therefore

$$\left| w\left(\frac{t_1-t_0}{\alpha_w^{p-1}\beta_w^p}, \frac{x_1-x_0}{\beta_w}\right) - w(0,0)\right| = \alpha_w\,|\bar{u}(t_1,x_1) - \bar{u}(t_0,x_0)| \leq \frac{4 - \frac{(\lambda^*)^2}{2}}{\alpha_1^{k-1}}.$$

This relationship implies that

$$\begin{aligned}|\bar{u}(t_1,x_1) - \bar{u}(t_0,x_0)| &\leq \left(4 - \frac{(\lambda^*)^2}{2}\right) \Bigg/ \left(\alpha_w \alpha_1^{\frac{\log\left(\sqrt{(t_0-t_1)^2+|x_0-x_1|^2}\big/(\alpha_w^{p-1}\beta_w^p)\right)}{\log(\alpha_1^{p-1}\beta_1^p)} - 2}\right) \\ &\leq \left(4 - \frac{(\lambda^*)^2}{2}\right)\frac{\alpha_1^2}{\alpha_w}\alpha_1^{\frac{\log(\alpha_w^{p-1}\beta_w^p)}{\log(\alpha_1^{p-1}\beta_1^p)}} \sqrt{(t_0-t_1)^2+|x_0-x_1|^2}^{\left(\frac{-\log(\alpha_1)}{\log(\alpha_1^{p-1}\beta_1^p)}\right)}.\end{aligned}$$

Hence if

$$\gamma = \frac{-\log(\alpha_1)}{\log(\alpha_1^{p-1}\beta_1^p)},$$

then

$$(*) \leq \left(4 - \frac{(\lambda^*)^2}{2}\right) \frac{\alpha_1}{\alpha_w} \alpha_1^{\frac{\log(\alpha_w^{p-1}\beta_w^p)}{\log(\alpha_1^{p-1}\beta_1^p)}}.$$

Note that the bound depends non-linearly on α_w and β_w, and hence on $\|u\|_\infty$, but γ depends only on n, p, m, Λ, and Λ_0.

This completes the proof.

$\square$

Chapter 3

Alternative Formulation of Isoperimetric Lemma

This chapter is based on chapter 4 of [CCV11a], in which is proven an isoperimetric inequality for the equation

$$\partial_t w + \int w(\cdot)w(y)K(\cdot,y)\,dy = 0 \tag{0.1}$$

where K satisfies

$$\chi_{\{|x-y|\leq 3\}}\frac{1-s/2}{\Lambda}|x-y|^{-(d+s)} \leq K(x,y) \leq (1-s/2)\Lambda|x-y|^{-(d+s)}. \tag{0.2}$$

This singular integral operator is comparable to $(-\Delta)^s$, where s is a parameter in $(0,1)$. The natural energy inequality for this equation shows that $(w-k)_+$ will generally be an element of $H^{s/2}$. However, because in general functions in $H^{s/2}$ are allowed to have jump discontinuities for $s\leq 1$, the method outlined in chapter 1 is not suitable. That method relies pivotally on the fact that H^1 functions cannot have jump discontinuities, and so fails whenever the dissipation of an equation is driven by $(-\Delta)^s$ with $s<1$.

Lemma 4.1 in [CCV11a] states

Lemma 3.0.1 (Isoperimetric Inequality for [CCV11a])**.** *Let Λ be the given constant in condition* (0.2) *and δ the constant defined in Corollary 3.3 of [CCV11a]. Then, there exists $\mu>0$, $\gamma>0$, and $\lambda\in(0,1)$, depending only on d, Λ, and s, such that for any solution $w:[-3,0]\times\mathbb{R}^d\to\mathbb{R}$ of* (0.1) *satisfying*

$$w(t,x)\leq 1+\psi_\lambda(x) \qquad \text{on } [-3,0]\times\mathbb{R}^d,$$

$$|\{w<\phi_0\}\cap(-3,-2)\times B_1|\geq\mu,$$

then we have either

$$\left|\{w>\phi_2\}\cap(-2,0)\times\mathbb{R}^d\right|\leq\delta,$$

or

$$\left|\{\phi_0<w<\phi_2\}\cap(-3,0)\times\mathbb{R}^d\right|\geq\gamma.$$

We will assume for contradiction that we have some function u satisfying all the hypotheses but such that

$$|u>\phi_2\cap((-2,0)\times R^N)|\geq\delta$$

and

$$|\phi_0<u<\phi_2\cap((-3,0)\times\mathbb{R}^N)|\leq\gamma,$$

where γ is a constant that we will set later.

Define functions

$$\begin{aligned}
m(t) &= |\{x\in B_2 : u(x,t)\leq 0\}|\\
g(t) &= |\{x\in B_2 : 0<u(x,t)<1-\lambda\}|\\
d(t) &= |\{x\in B_2 : 1-\lambda\leq u(x,t)\}|.
\end{aligned}$$

And also the functions

$$\begin{aligned}
H(t) &= \int u_+(t,x)^2\,dx\\
E(t) &= \iint u_+(t,x)w_-(t,y)K(x,y)\,dxdy.
\end{aligned}$$

The energy inequality easily gives us that $\frac{d}{dt}H$ and $\int_{-3}^{0} E\,dt$ are both bounded above by $C_1\lambda^2$. This calculation is in CCV.

In the classical case, we take a limit to make $g(t)=0$ for all t. Then $m(t)d(t)=0$ for all t because the energy $\int |Du_+|^2\,dx$ is finite at every t. In the less classical case, we still assume that g is very small at almost all t. It isn't true that $m(t)d(t)=0$ whenever $E(t)$ is finite, but it is true that if $m(t)$ and $d(t)$ are both bigger than some constant, then $E(t)$ must be bigger than some constant as well, which is similar.

We know that each function m,g,d is pointwise between 0 and $|B_2|$, and we have constraints on the integrals

$$\int_{-3}^{-2} m \geq \mu, \quad \int_{-2}^{0} d \geq \delta, \text{ and } \int_{-3}^{0} g \leq \gamma.$$

Therefore we can define constants μ_0, δ_0 such that

$$|[-3,-2] \cap \{m(t) \geq \mu_0\}| \geq 0.99$$
$$|[-2,0] \cap \{d(t) \geq \delta_0\}| \geq 1.99$$
$$|[-3,0] \cap \{g(t) \geq \gamma_0\}| \leq .01,$$

with the additional constraint (it'll make sense later)

$$\mu_0 + 4\delta_0 + 5\gamma_0 \leq |B_2|/2.$$

These constraints also inform our choice of γ, of course.

Now we argue that m and d are largely "disjoint." If at some time t, we have

$$m(t) \geq \mu_0, \qquad d(t) \geq \delta_0,$$

then

$$\begin{aligned}
E(t) &= \iint u_+(x)u_-(y)K(x,y)\,dxdy \\
&\geq \iint u_+(x)u_-(y)\chi_{\{B_2\}}(y)K(x,y)\,dxdy \\
&\geq \iint u_+(x)u_-(y)\chi_{\{B_2\}}(y)4^{-n-s} \\
&\geq C\int u_+(x)\,dx \int_{B_2} u_-(y)\,dy \\
&\geq C(\delta_0\lambda)(\mu_0(1-2\lambda)) \\
&\geq C_5\lambda,
\end{aligned}$$

where C_5 depends on μ_0 and δ_0. The moral is, because our energy is nonlocal, we don't need to rely on jump discontinuities. The above calculation is the equivalent of "no jump discontinuities with finite energy", but much more quantitative and not particularly difficult.

Remember that $\int_{-3}^{0} E\,dt \leq C_1\lambda^2$, so by Chebyshev,

$$|[-3,0]\cap\{E(t)\geq C_5\lambda\}|(C_5\lambda)\leq C_1\lambda^2.$$

If we take λ sufficiently small, then

$$|[-3,0]\cap\{E(t)\geq C_5\lambda\}|\leq .01.$$

But of course,

$$\{m\geq\mu_0\}\cap\{d\geq\delta_0\}\subseteq\{E\geq C_5\lambda\}.$$

Therefore we have the "early" and "late" sets

$$\big|[-3,-2]\cap\{d\leq\delta_0\}\cap\{g\leq\gamma_0\}\big| = \big|A_e\big|\geq 0.95,$$

$$\big|[-2,0]\cap\{m\le\mu_0\}\cap\{g\le\gamma_0\}\big| = \big|A_l\big| \ge 1.95.$$

These are the good sets, the sets where H is large and small respectively, and everything up till now was to show that these sets have non-zero measure.

Next, we'll want to show what good properties A_e and A_l have. We can calculate H directly,

$$H(t) \le (2\lambda)^2(\delta_0+\gamma_0) \qquad \forall t \in A_e,$$
$$H(t) \ge \lambda^2(|B_2|-\mu_0-\gamma_0) \qquad \forall t \in A_l.$$

What matters is the difference between these two values, and specifically it's very important that for $s\in A_e$, $t\in A_l$, we have $H(t)-H(s)\ge 0$.

$$H(t)-H(s) \ge \lambda^2(|B_2|-\mu_0-\gamma_0)-(2\lambda)^2(\delta_0+\gamma_0) = \lambda^2(|B_2|-\mu_0-4\delta_0-5\gamma_0) > \frac{|B_2|}{2}\lambda^2.$$

Since H increases some fixed amount and has a bounded slope, it must take time to do it. Namely, there is a set

$$D = \{t\in[-3,0] : (2\lambda)^2(\delta_0+\gamma_0) < H(t) < \lambda^2(|B_2|-\mu_0-\gamma_0)\}$$

of times where H is between its maximum value in A_e and its minimum value in A_l. This set has a positive measure, independent of λ, it has measure at least

$$|D| \ge \frac{|B_2|\lambda^2/2}{C_1\lambda^2} = \frac{|B_2|}{2C_1}.$$

So, at first d and g are small (in A_e), later g and m are small (in A_l), and these facts put strong constraints on H. In between (in D), H doesn't satisfy these constraints, so at all times in D either g must be big, or else neither d nor m can be small. But when d and m are both big then E is big as well. So at all times in D, either $g>\gamma_0$, or $E>C_5\lambda$.

So take λ and γ small enough that

$$|\{g > \gamma_0\}| + |\{E > C_5\lambda\}| < \frac{|B_2|}{2C_1}.$$

This is a contradiction.

Chapter 4

kinetic

4.1 Introduction

We study in this chapter the family of nonlocal kinetic equations

$$
\begin{gathered}
[\partial_t + v\cdot\nabla_x] f = \mathcal{L} f + a, \\
\mathcal{L} f := \int K(t,x,v,w)[f(w)-f(v)]\,dw.
\end{gathered}
\tag{1.1}
$$

The kernel K can be any measurable function which is symmetric in v and w and which satisfies a coercivity bound,

$$
\begin{gathered}
K(t,x,v,w) = K(t,x,w,v), \qquad K(t,x,v,v+w) = K(t,x,v,v-w) \\
\chi_{\{|v-w|\le 6\}} \frac{1}{\kappa}|v-w|^{-(n+2s)} \le K(t,x,v,w) \le \kappa |v-w|^{-(n+2s)}
\end{gathered}
\tag{1.2}
$$

for some constants $0<s<1$ and $\kappa>1$. The function a is a source term we take to be in some Lebesgue space, the variables t, x and v are taken in $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^n$ respectively, and we restrict ourselves to the case $2s<n$. The integral defining $\mathcal{L}$ is taken in the principle value sense.

These models are used extensively in nuclear- and astro-physics (c.f. Zaslavsky [Zas94], Goychuk [Goy17], and Haubold and Mathai [HM00]) to model the behavior of neutral particles moving through a plasma (c.f. Larsen and Keller [LK74]). They can also model two-species particle fields wherein the test particles are of a very dilute species ([Goy17]). The theory of anomalous diffusion (Mellet [Mel10] and Mellet, Mischler, and Mouhot [MMM11]) derives the small-mean-free-path limit of fractional kinetic equations such as (1.1) and shows that these equations represent the mesoscopic behavior of fat-tailed equilibrium distributions. These fat-tailed distributions appear in physical observations from astrophysics ([LK74] and Mendis and Rosenberg [MR94]).

One notable special case of (1.1) is the fractional kinetic Fokker-Planck Equation, corresponding to $\mathcal{L}=(-\Delta_v)^s$ or equivalently to a homogeneous kernel $K(t,x,v,w)=C_{n,s}|v-w|^{-n-2s}$. The (local) kinetic Fokker-Planck Equation is obtained in the limit $s\to 1$, correpsponding to $\mathcal{L}=-\Delta_v$.

If we think of f as a density function for a collection of particles, with t, x, and v being time, space, and velocity respectively, then the equation (1.1) states that these particles move freely through space with their velocities changing in a stochastic manner. If the velocity of a given particle varied according to the Weiner process, then f would obey a (local) kinetic Fokker-Planck Equation. However, when the velocity of each particle varies according to a Levy process (without drift), the density function obeys (1.1). A Levy process, unlike the Weiner process, allows individual particles to change velocity suddenly and discontinuously, which better approximates the effect of elastic collisions.

Another important model from the statistical mechanics of particles is the Boltzmann Equation

$$[\partial_t+v\cdot\nabla_x]\,f=Q(f,f).$$

In the non-cutoff case, the Boltzmann Equation sometimes enjoys a regularization effect similar the fractional Fokker-Planck equation (Alexandre, Morimoto, Ukai, Xu, and Yang [AMU+10]). Our equation (1.1) is closely related to the linear approximation of the bilinear collision operator $Q(\cdot,\cdot)$. If the mass, energy, and entropy of a solution are assumed to be uniformly bounded, then regularization due to hypoellipticity is observed for the Boltzmann Equation (Imbert and Silvestre [IS16]), and also for the closely related Landau Equation (Henderson and Snelson [HS17], Cameron, Silvestre, and Snelson [CSS18]). Note that [IS16] rewrites the Boltzmann equation in the form (1.1), but with kernel satisfying weaker constraints than (1.2). Their regularity results are discussed below. The most important assumption these papers require is that the mass is bounded away from the vacuum, which is connected to the coercivity of the collision operator. In [HST17], Henderson, Snelson, and Tarfulea show that this assumption really does hold for the Landau Equation. See Mouhot [Mou18] for a thorough review of the current state of research on this front.

Equation (1.1) is a typical hypoelliptic equation. Although regularization of the integral operator happens only in v, we will gain regularity in t, x thanks to the mixing property of the transport operator. This is reminiscent of the hypoelliptic theory based on C^∞ of Hörmander [Hör67] and Kolmogorov. Averaging lemmas such as [GLPS88] (Golse, Lions, Perthame, Sentis) can be seen as an H^s theory of hypoellipticity.

This H^s theory has already been applied specifically to the nonlocal kinetic Fokker-Planck Equation. Lerner, Morimoto, and Pravda-Starov [LMPS12] showed that solutions to certain fractional kinetic equations are in a Sobolev space H^σ in all three variables. This result was inspired by the work on hypoelliptic equations by Bouchut [Bou02], which is discussed in more detail below. The precise amount of Sobolev regularity is improved and expanded upon, for example, by Morimoto and Xu [MX07] and by Li [Li14]. In fact, [MX07] obtains C^∞ solutions in the case of no source term and $\mathcal{L}$ a specific operator similar to $(-\Delta)^s$.

This chapter extends a C^α hypoellipticity theory, as was first introduced for kinetic Fokker-Planck by Golse, Imbert, Mouhot, and Vasseur [GIMV16]. They show that solutions to the (local) kinetic Fokker-Planck Equation

$$[\partial_t + v \cdot \nabla_x] f = \Delta_v f$$

are Hölder continuous. In [IM18], Imbert and Mouhot show that, for certain initial data, the nonlinear Fokker-Planck Equation has smooth solutions for all time. They utilize the Hölder continuity of [GIMV16], as well as a Schauder-type estimate. In [IS16], Imbert and Silvestre obtain Hölder continuity for a class of nonlocal kinetic Fokker-Planck-type equations with operators $\mathcal{L}$ more general than those considered in the present chapter, and with uniformly bounded source terms.

The seminal work on averaging lemmas is by Golse, Lions, Perthame and Sentis in 1988 [GLPS88], which shows that solutions to $[\partial_t + v \cdot \nabla_x] f = g$ have their weighted velocity averages $\rho[f] = \int \eta f dv$ in $H^{1/2}$, assuming f and g are in L^2. This result had precursers in [GPS85] and Agoshkov [Ago84]. Many results followed, see for example DiPerna, Lions, and Meyer [DLM91] and DeVore and Petrova [DP01], which show various levels of regularity for $\rho[f]$ assuming different regularity measures of f and g.

Notable in the history of averaging lemmas is [Bou02], which showed that if f is regular in v (in the Sobolev sense) then not only is $\rho[f]$ regular but so is f itself. This powerful result was followed by generalizations in [LMPS12], [MX07], and [Li14] which are especially relevant to (1.1). We've used these results to establish the regularity needed to justify our calculations, as explained in Section 4.1, but we do not rely on their quantitative estimates.

Instead, the primary averaging lemma that we utilize is by Bezard [Béz94]. Like Golse et al. but unlike Bouchut, this lemma gives regularity only for the density $\rho[f]$. Bezard requires only that f and g lie in a negative Sobolev space H_v^{-s}, which gives us plenty of flexibility.

Our proof follows the De Giorgi method, pioneered by De Giorgi in [DG57] (c.f. also Vasseur [Vas16a], [SV18], Caffarelli and Vasseur [CV10b], Caffarelli, Chan, and Vasseur [CCV11b], and [GIMV16]). We are particularly inspired by [GIMV16], which applies De Giorgi's method to a kinetic equation, and [CCV11b], which applies the method to a nonlocal integro-differential operator.

For two functions $f, g \in H^s(\mathbb{R}^n)$, and $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, define the bilinear operator

$$B_{t,x}(f,g) = B(f,g) := \frac{1}{2}\int K(t,x,v,w)[f(w)-f(v)][g(w)-g(v)]\,dw dv,$$

and note that

$$\begin{aligned}
\int g(v)\mathcal{L}(f)(v)\,dv &= \int g(v)\int K[f(w)-f(v)]dw\,dv \\
&= \iint K[f(w)-f(v)]g(v)\,dwdv \\
&= \frac{1}{2}\left(\iint K[f(w)-f(v)]g(v)\,dwdv + \iint K[f(v)-f(w)]g(w)\,dvdw\right) \\
&= -\frac{1}{2}\left(\iint K[f(w)-f(v)][g(w)-g(v)]\,dwdv\right) \\
&= -B_{t,x}(f,g).
\end{aligned}$$

We call $f \in L^2(Q; H^s(\mathbb{R}^n))$ a weak solution to (1.1) on a domain $Q \subseteq \mathbb{R}\times\mathbb{R}^n$ when

$$-\iiint f[\partial_t + v\cdot\nabla_x]\phi\, dvdxdt = -\iint B(f,\phi)\,dxdt + \iiint a\phi\, dvdxdt \qquad \forall \phi \in L^2(Q; H^s(\mathbb{R}^n)).$$

Our main theorem is

Theorem 4.1.1 (Main theorem)**.** *Given constants $s \in (0,1)$, $\kappa > 1$, and $2s < n \in \mathbb{N}$, there exist exponents $\alpha \in (0,1)$ and $r_0 > 2$ such that for any open set $\Omega \subseteq \mathbb{R}^n$, $T > 0$, constant $r_0 < r \leq \infty$, and source term*

$$a \in L^r([0,T]\times\Omega\times\mathbb{R}^n) \cap L^2([0,T]\times\Omega\times\mathbb{R}^n),$$

there exists a constant such that the following is true:

If

$$f \in L^\infty([0,T)\times\Omega\times\mathbb{R}^n) \cap L^2([0,T)\times\Omega; H^s(\mathbb{R}^n)),$$

is a weak solution to (1.1) *subject to* (1.2)*, then* f *is in* $C^\alpha((0,T)\times\Omega\times\mathbb{R}^n)$.

Morover, for any $0<\bar{T}<T$ *and any compact set* $\bar{\Omega}\subset\Omega$*, there exists a constant* $C=C(n,s,\kappa,\Omega,\bar{\Omega},T,\bar{T})>0$ *independent of* f *such that the following bound holds:*

$$\|f\|_{C^\alpha\left([\bar{T},T]\times\bar{\Omega}\times B_1\right)}\leq C\left(\|f_0\|_{L^\infty([0,T]\times\Omega\times\mathbb{R}^n)}+\|a\|_{L^r([0,T]\times\Omega\times\mathbb{R}^n)}\right).$$

Although the assumption (1.2) on the kernel is a natural one for studying absolutely continuous kernels from an energy perspective, it is too strict to apply to e.g. the Boltzmann equation because the collision kernel may not be absolutely continuous or symmetric. As a result, in the case $a\in L^\infty$, our result is included in the result of [IS16]. Their proof does not use a averaging lemma, instead utilizing a careful study of the Green's function for the fractional Kolmogorov equation. They employ a Krylov approach to obtain a weak Harnack inequality. The advantage of our stronger assumtptions on the kernel is that our proof can be entirely energy based, which allows us to consider source terms which are not uniformly bounded. We are also able to take a unified variational approach to the cases $s<1/2$ and $s>1/2$ by adapting the technique of [CCV11b] to the kinetic context.

The assumption that solutions are in L^∞ will hold in particular when the initial data and source term are both in L^∞. In such a case, we could obtain a maximum principle by computing $\frac{d}{dt}\iint(f-C-t\|a\|_\infty)_+^2\,dvdx$.

With arbitrary source term, a more robust L^∞ bound can sometimes be obtained by adapting Proposition 4.3.1 below. As stated, this proposition requires an assumption of uniformly bounded growth for large values of v, to avoid interactions between high-velocity particles and the boundary $\partial\Omega$ of our spatial domain. Though outside the scope of the present work, this assumption could be removed with proper boundary conditions. For example, if we take $x \in \mathbb{T}^n$ the torus, then solutions will be L^∞ at any positive time.

In the case that K is homogeneous near the origin, meaning equal to $|v-w|^{-n-2s}$ for $|v-w|$ sufficiently small, we can obtain existence of an $L^2(H^s)$ weak solution from [MX07] Theorem 1.1 (by treating the difference between $\mathcal{L}f$ and $(-\Delta)^s f$ as a source term). When K is not homogeneous near the origin, our result is an a priori estimate. In particular, when a uniform L^∞ bound exists (as discussed above), we can obtain existence of continuous solutions through the method of continuity.

The symmetry assumption posed in (1.2) is actually two symmetry assumptions. The former, $K(t,x,v,w) = K(t,x,w,v)$, is crucial to the weak formulation of the problem and hence is used throughout this chapter. The latter assumption $K(t,x,v,v+w) = K(t,x,v,v-w)$ is really only used in the proof of Lemma 4.2.3. It is necessary because otherwise, in the case $s \geq 1/2$, the operator $\mathcal{L}$ might not be bounded even from C_c^∞ to L^∞. We list here a few alternative assumptions, any one of which could replace the latter symmetry assumption of (1.2) with no loss of generality.

- For any C^2 function ϕ, $\|\mathcal{L}\phi\|_\infty \leq C\|\phi\|_{C^2}$.
- The parameter s is strictly less than 1/2.
- For any $t,x,v \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, $\int_{B_1} wK(t,x,v,w)\,dw = 0$.
- The function $K(t,x,v,v+w)$ is independent of v.

The lower bound on the exponent r for the source term is

$$r_0 = \frac{n(1+s)(n+1)}{s}\left(2\frac{s}{n} + \frac{1}{2} + \frac{n}{2s}\right).$$

This bound is strictly greater than 2, and it is also strictly greater than $n+1+n/s$, which is the critical scaling exponent. This lower bound may not be sharp.

The remainder of this article is dedicated to the proof of Theorem 4.1.1. Section 4.2 contains a few preliminary lemmas. Sections 4.3 and 4.4 are dedicated to the proofs of the first and second De Giorgi lemmas, respectively. Section 4.5 combines the De Giorgi lemmas to obtain a Harnack inequality that proves Theorem 4.1.1.

In this chapter, a constant is called "universal" if it depends only on the dimension n, the order s of the operator $\mathcal{L}$, and the coercivity bound κ.

4.2 Preliminary Lemmas

This section contains three lemmas which will be relied upon extensively in the forthcoming sections.

The operator $\mathcal{L}$ behaves in many ways like the operator $-(-\Delta_v)^s = -\Lambda^{2s}$. The following lemma codifies the important similarities between the two operators, specifically the relationship between B and the H^s norm, and between $\mathcal{L}$ and the Bessell potential.

Lemma 4.2.1. *There exists a constan $C = C(n,s,\kappa)$ such that, for any function $f \in H^s(\mathbb{R}_n)$, we have the following bounds:*

$$\int |\Lambda^s f|^2 \, dv \leq \inf_{t,x} C\left(B_{t,x}(f,f) + \int f^2 \, dv\right),$$

and

$$\sup_{t,x} \left\| (1-\Delta_v)^{-s/2} \mathcal{L}_{t,x} f \right\|_{L^2(\mathbb{R}^n)} \leq C\left(\|\Lambda^s f\|_{L^2(\mathbb{R}^n)}\right).$$

Since these results are true irrespective of t and x, we will omit their mention in the sequel.

Proof. For the first inequality, simply calculate

$$\begin{aligned}
B(f,f) &= \iint K\,[f(w)-f(v)]^2\,dwdv \\
&\geq \frac{1}{\kappa}\iint_{|v-w|\leq 6} \frac{[f(w)-f(v)]^2}{|v-w|^{n+2s}}\,dwdv \\
&= \frac{1}{\kappa}\iint \frac{[f(w)-f(v)]^2}{|v-w|^{n+2s}}\,dwdv - \frac{1}{\kappa}\iint_{|v-w|\geq 6} \frac{[f(w)-f(v)]^2}{|v-w|^{n+2s}}\,dwdv \\
&\geq \frac{1}{\kappa}\iint \frac{[f(w)-f(v)]^2}{|v-w|^{n+2s}}\,dwdv - \frac{2}{\kappa}\int f(v)^2 \int \frac{\chi_{\{|u|\geq 6\}}}{|u|^{n+2s}}\,du\,dv - \frac{2}{\kappa}\int f(w)^2 \int \frac{\chi_{\{|u|\geq 6\}}}{|u|^{n+2s}}\,du\,dw \\
&= C(n,s,\kappa)\int |\Lambda^s f|^2\,dv - C'(n,s,\kappa)\int f^2\,dv.
\end{aligned}$$

For the second inequality, let g be any function in $H^s(\mathbb{R}^n)$. For t and x fixed, we have the following bound on inner products in v:

$$\begin{aligned}
|\langle \mathcal{L}f, g\rangle|_v &= \left| \iint [f(v+w)-f(v)][g(v+w)-g(v)]K(t,x,v,v+w)\,dwdv \right| \\
&= \left| \iint \left([f(v+w)-f(v)]|w|^{\frac{n+2s}{2}}\right) \frac{[g(v+w)-g(v)]}{|w|^{\frac{n+2s}{2}}} K\,dwdv \right| \\
&\leq \left(\iint [f(v+w)-f(v)]^2 K^2 |w|^{n+2s}\,dwdv\right)^{1/2} \left(\iint [g(v+w)-g(v)]^2 \frac{dwdv}{|w|^{n+2s}}\right)^{1/2} \\
&\leq \kappa \left(\iint [f(v+w)-f(v)]^2 \frac{dwdv}{|w|^{n+2s}}\right)^{1/2} \left(\iint [g(v+w)-g(v)]^2 \frac{dwdv}{|w|^{n+2s}}\right)^{1/2} \\
&= C(n,s,\kappa)\left(\int |\Lambda^s f|^2\,dv\right)^{1/2} \|g(t,x,\cdot)\|_{H^s(\mathbb{R}^n)}.
\end{aligned}$$

Therefore if ϕ is any $L^2(\mathbb{R}^n)$ test function, then

$$\begin{aligned}
\langle (1-\Delta_v)^{-s/2}\mathcal{L}f, \phi\rangle &= \langle \mathcal{L}f, (1-\Delta_v)^{-s/2}\phi\rangle \\
&\leq C(n,s,\kappa)\left(\int |\Lambda^s f|^2\,dv\right)^{1/2} \left\|(1-\Delta_v)^{-s/2}\phi\right\|_{H^s(\mathbb{R}^n)}
\end{aligned}$$

$$= C(n,s,\kappa)\left(\int |\Lambda^s f|^2\,dv\right)^{1/2} \|\phi\|_{L^2(\mathbb{R}^n)}.$$

The lemma follows by taking a supremum over all such ϕ.

$\square$

We now come to the energy inequality. An inequality of this type is to be expected due to the parabolic flavor of Equation (1.1), and it is in some ways the most important quality of our equation. Notice that the inequality gives control over the regularity in v, but not in t or x.

Lemma 4.2.2 (Energy Inequality)**.** *There exists a universal constant $C = C(n,s,\kappa)$ such that the following is true:*

Let $T < S < 0$ be times, and let $\Omega \subseteq \mathbb{R}^n$ be an open region in space and $\bar{\Omega} \subseteq \Omega$ a compact subset. Let $R > 0$ a radius and $\psi : \mathbb{R}^n \to \mathbb{R}$ a function of velocity. Denote $Q := (T,0] \times \Omega$ and $\bar{Q} := [S,0] \times \bar{\Omega}$, and define

$$\delta := \min\left(|T-S|, \operatorname{dist}(\bar{\Omega}, \Omega^{\mathsf{C}})\right).$$

Let $f \in L^2(\mathbb{Q}; H^s(\mathbb{R}^n))$ be any weak solution to (1.1) subject to (1.2) satisfying

$$f(t,x,v) \le \psi(v) \qquad \forall (t,x) \in Q, |v| \ge R,$$

and denote $f_+ := \max(f-\psi, 0)$ and $f_- := \max(\psi - f, 0)$ so that $f = f_+ + \psi - f_-$.

Then the following energy inequality holds:

$$\iint_{\bar{Q}} B(f_+, f_+)\,dxdt - \iint_{\bar{Q}} B(f_+, f_-)\,dxdt \le \\ \frac{C}{\delta}\left[R \iint_Q \int f_+^2\,dvdxdt + \left(\sup_{|v|<R} |\mathcal{L}\psi(v)|\right) \iint_Q \int f_+\,dvdxdt + \|a\|_{L^r(Q)} \|f_+\|_{L^{r^*}(Q)}\right].$$

The constant δ here is the distance from $\bar{Q}$ to the parabolic boundary of Q.

The quantity $B(f_+, f_+)$ is, as shown in Lemma 4.2.1, related to the fractional Dirichlet energy of f_+. We have an additional dissipation term $-B(f_+, f_-)$ which we call the cross term. Because f_+ and f_- have disjoint supports,

$$\begin{aligned} -B(f_+, f_-) &= -\iint K[f_+(w) - f_+(v)][f_-(w) - f_-(v)]\,dwdv \\ &= \iint K[f_+(w)f_-(v) + f_+(v)f_-(w)]\,dwdv \\ &= 2\iint K f_+(v) f_-(w)\,dwdv. \end{aligned}$$

In particular this means the cross term is non-negative. The cross term represents, in a sense, the energy which is lost when we localize f to create f_+. The bound on the cross term is critical to our proof in Section 4.4 of De Giorgi's second lemma.

Remark 4.2.1. The quantity f_- appears on the left but not the right hand side of the energy inequality. This means in particular that the growth and decay of any solution to (1.1) is constrained by the local behavior alone.

Proof. Define $\phi : \mathbb{R}^n \to [0,1]$ a function which equals 1 on $\bar{\Omega}$, which is supported on Ω, and which is Lipschitz with constant $\|\phi\|_{C^1} \le 2\delta^{-1}$.

Multiplying the left side of Equation (1.1) by the quantity $\phi^2 f_+$, we see that

$$\begin{aligned} \phi^2 f_+ [\partial_t + v\cdot\nabla_x] f &= \phi^2 f_+ [\partial_t + v\cdot\nabla_x](f_+ + \psi - f_-) \\ &= \phi^2 \frac{1}{2}[\partial_t + v\cdot\nabla_x] f_+^2 + \phi^2 f_+ [\partial_t + v\cdot\nabla_x]\psi - \phi^2 f_+ [\partial_t + v\cdot\nabla_x] f_- \\ &= \frac{\phi^2}{2}[\partial_t + v\cdot\nabla_x] f_+^2 \end{aligned}$$

because ψ is independent of x and t, and f_+ and f_- have disjoint supports.

Since $f_+ \in L^2(H^s)$, we can multiply Equation (1.1) by $2\phi^2 f_+$ and integrate with respect to x and v to obtain

$$\begin{aligned}\frac{d}{dt}\iint (\phi f_+)^2\,dvdx - \iint f_+^2 v\cdot\nabla_x(\phi^2)\,dvdx &= -2\int \phi^2 B(f_+, f_+ + \psi - f_-)dx + 2\iint \phi^2 a f_+\,dvdx\\ &= -2\int \phi^2 B(f_+,f_+)dx - 2\iint \phi^2 f_+\mathcal{L}\psi\,dvdx + 2\int \phi^2 B(f_+,f_-)dx + 2\iint \phi^2 a f_+\,dvdx.\end{aligned}$$

For any $S \le \tau \le T$, we integrate this equality from τ to 0 in time and then rearrange to obtain

$$\begin{aligned}&\iint (\phi f_+(0))^2\,dvdx + 2\int_\tau^0\int \phi^2 B(f_+,f_+)\,dxdt - 2\int_\tau^0\int \phi^2 B(f_+,f_-)\,dxdt\\ =&\int_\tau^0\iint \left(v\cdot\nabla_x\phi^2\right) f_+^2\,dvdxdt + 2\int_\tau^0\iint \phi^2\mathcal{L}(\psi) f_+\,dvdxdt + \int_\tau^0\iint \phi^2 a f_+\,dvdxdt + \iint (\phi f_+(\tau))^2\,dvdx.\end{aligned}$$

In particular,

$$\begin{aligned}&2\int_T^0\int \phi^2 B(f_+,f_+)\,dxdt - 2\int_T^0\int \phi^2 B(f_+,f_-)\,dxdt\\ &\qquad\le \int_S^0\iint \left|v\cdot\nabla_x\phi^2\right| f_+^2\,dvdxdt + 2\int_S^0\iint \phi^2\left(|\mathcal{L}(\psi)|+|a|\right) f_+\,dvdxdt + \iint (\phi f_+(\tau))^2\,dvdx.\end{aligned}$$

Now only one term depends on τ. If we take the average value over $\tau \in [S,T]$ for both sides of the inequality, we obtain

$$\begin{aligned}&2\int_T^0\int \phi^2 B(f_+,f_+)\,dxdt - 2\int_T^0\int \phi^2 B(f_+,f_-)\,dxdt\\ &\le \int_S^0\iint \left|v\cdot\nabla_x\phi^2\right| f_+^2\,dvdxdt + 2\int_S^0\iint \phi^2\left(|\mathcal{L}(\psi)|+|a|\right) f_+\,dvdxdt + \frac{1}{|S-T|}\int_S^T\iint (\phi f_+)^2\,dvdxdt.\end{aligned}$$

Our energy inequality follows. $\square$

The classical technique to localize a solution to a PDE is multiplication by a compactly supported cutoff function. This allows us to disregard the behavior of the solution outside a specified region, while the localized function usually solves the original PDE, modulo some sort of error term. One should not expect this technique to work for nonlocal PDE; the far-away behavior of the solution cannot be completely disregarded.

Instead, we must localize by a "soft cutoff," which is a fixed function ψ that vanishes in a specified local region but grows without bound outside that region. We have already seen soft cutoffs used in the statement and proof of Lemma 4.2.2 just above.

Throughout the following sections, we will utilize a few different soft cutoff functions. We will define all of our soft cutoff functions here and list all their relevant properties, then refer back to this lemma as we use them. These functions ψ^1 and ψ_θ are tailored to the required assumptions of Lemmas 4.3.1 and 4.4.1 respectively. They also must have certain specific relationships with eachother in order to prove Lemma 4.5.2, which is why we prefer to construct them here all at once.

Lemma 4.2.3. *Let $s \in (0,1)$ and $2s < n \in \mathbb{N}$ be specified constants. There exists a function $\psi^1 : \mathbb{R}^n \to \mathbb{R}^+$ and a family of functions $\psi_\theta : \mathbb{R}^n \to \mathbb{R}^+$ indexed by $\theta \in (0,1)$ with the following properties:*

(i) There exists a constant C_ψ such that for all $v \in \mathbb{R}^n$

$$\sup_{t,x} \left|\mathcal{L}_{t,x}\psi^1(v)\right| \le C_\psi, \qquad \sup_{t,x} |\mathcal{L}_{t,x}\psi_\theta(v)| \le C_\psi,$$

and for all $|v| \le 3$

$$\sup_{t,x} |\mathcal{L}_{t,x}\psi_\theta(v)| \le C_\psi \theta^{3s/2}.$$

(ii) For $|v| \le 1$,

$$\psi^1(v) = 0$$

and for $|v| \leq \theta^{-1}$,

$$\psi_\theta(v) = 0.$$

(iii) For any $\theta < \vartheta$, *and for all* $v \in \mathbb{R}^n$

$$\psi_\theta(v) \leq \psi_\vartheta(v) \leq \psi^1(v).$$

(iv) For all $|v| \geq 2$, *for any* $\theta \in (0,1)$,

$$1 + \psi_\theta(v) \leq \psi^1(v).$$

(v) For each θ, *there exists* $\varepsilon_0 = \varepsilon_0(s,\theta)$ *such that* $\varepsilon < \varepsilon_0$ *implies that for all* $|v| > \varepsilon^{-1}$,

$$\psi_\theta(v) \geq 2\psi_\theta(\varepsilon v) + 2.$$

Proof. First define a function $g:[0,\infty) \to [0,\infty)$ such that, for all $x > 1$,

$$g(x) = x^{s/2}$$

but $g(0) = g'(0) = 0$, and in the interval $[0,1]$ let g be defined so that g is smooth and non-decreasing, and $g(x) \leq x^{s/2}$.

Next define functions g_r for each $r > 0$ by

$$g_r(x) = \begin{cases} 0 & x < r \\ g(x-r) & x \geq r. \end{cases}$$

Then g_r is pointwise-decreasing in r and both $\|g_r''\|_{L^\infty}$ and the Hölder semi-norm $\|g_r\|_{\dot{C}^{s/2}}$ are finite and independent of r.

We'll define

$$\psi_\theta(v) := g_{\theta^{-1}}(|v|).$$

Let $C_1 > 1$ be a constant large enough that for any $\theta \in (0,1)$, for all $|v| \geq 2$

$$1 + \psi_\theta(v) \leq C_1 g_1(|v|).$$

Then define

$$\psi^1(v) = C_1 g_1(|v|).$$

Properties ((ii)), ((iii)), and ((iv)) all follow immediately from the construction. Notice also that all of these functions have uniformly bounded second derivatives and uniformly bounded $\dot{C}^{s/2}$ semi-norms.

Let ψ be either ψ^1 or any of the ψ_θ, and let $v \in \mathbb{R}^n$ be chosen. We wish to calculate $\mathcal{L}\psi(v)$, so let us break up the defining integral into the “near” part and the ”far” part.

$$\mathcal{L}\psi(v) = \int_{|w|<1} K(v, v+w)[\psi(v+w) - \psi(v)]\, dw + \int_{|w|\geq 1} K(v, v+w)[\psi(v+w) - \psi(v)]\, dw.$$

For the near part, we utilize the fact that ψ is smooth with bounded second derivative. We apply Taylor's theorom to find that

$$\psi(v+w) - \psi(v) = D\psi(v) \cdot w + D^2\psi(u) w \otimes w$$

for some u on the line segment between v and $v+w$. By the symmetry (1.2) of K,

$$\int_{|w|<1} K(v, v+w) D\psi(v) \cdot w\, dw = 0.$$

Note that this integral must be understood in the principal value sense.

The remainder is

$$\int_{|w|<1} K(v, v+w) D^2\psi(u) w \otimes w\, dw \leq C\kappa \int_{|w|<1} \frac{|w|^2}{|w|^{n+2s}}\, dw,$$

with C here being the bound on $\|D^2\psi\|_\infty$ which is independent of ψ. Since $n+2s-2<n$, the integral is finite.

Notice that if $\psi=\psi_\theta$ with $\theta<1/4$ and if $|v|\le 3$ then the near part of the integral is in fact zero.

For the far away part, we utilize the fact that ψ is Hölder continuous in $\dot{C}^{s/2}$ and estimate

$$|\psi(v+w)-\psi(v)|\le C|w|^{s/2}$$

with C independent of ψ. The integral of the far away part becomes

$$\int_{|w|\ge 1} K(v,v+w)[\psi(v+w)-\psi(v)]\,dw \le C\kappa \int_{|w|\ge 1} \frac{|w|^{s/2}}{|w|^{n+2s}}\,dw.$$

Since $n+2s-\frac{s}{2}>n$, the integral is finite.

In the case $\psi=\psi_\theta$ with $\theta<1/4$ and $|v|\le 3$, $\psi(v)=0$ so we can make the stronger estimate

$$|\psi(v+w)-\psi(v)|\le g_{\theta^{-1}}(|w|+3)\le \max(|w|+3-\theta^{-1},0)^{s/2}.$$

The integral of the far away part becomes

$$\int_{|w|\ge 1} K(v,v+w)[\psi(v+w)-\psi(v)]\,dw \le \kappa \int_{|w|\ge(\theta^{-1}-3)} \frac{(|w|+3-\theta^{-1})^{s/2}}{|w|^{n+2s}}\,dw \le C\int_{|w|>\frac{\theta^{-1}}{4}} \frac{dw}{|w|^{n+\frac{3}{2}s}}.$$

This integral is proportional to $\theta^{3s/2}$. The property ((i)) follows.

All that remains is to show ((v)), so fix some value of θ. We'll show the equivalent claim

$$\psi_\theta(v/\varepsilon)\ge 2\psi_\theta(v)+2 \qquad \forall |v|\ge 1. \tag{2.3}$$

For $|v|\ge\theta^{-1}+1$ and any $0<\varepsilon<1$, we can say

$$\psi_\theta(v/\varepsilon)=(|v|/\varepsilon-\theta^{-1})^{s/2}\ge(|v|/\varepsilon-\theta^{-1}/\varepsilon)^{s/2}=\varepsilon^{-s/2}(|v|-\theta^{-1})^{s/2}.$$

There exists $0<\varepsilon_1<1$ and $r_1>\theta^{-1}+1$ so that if $\varepsilon<\varepsilon_1$ and $|v|\geq r_1$ then

$$\varepsilon^{-s/2}(|v|-\theta^{-1})^{s/2}\geq 2\psi_\theta(v)+2.$$

Now take $\varepsilon_0<\varepsilon_1$ small enough that $\psi_\theta(1/\varepsilon_0)\geq 2\psi_\theta(r_1)+2$. Now for $1\leq|v|\leq r_1$ the inequality (2.3) holds because

$$\psi_\theta\left(\frac{v}{\varepsilon}\right)\geq\psi_\theta(1/\varepsilon_0)\geq 2\psi_\theta(r_1)+2\geq 2\psi_\theta(v)+2,$$

and for $|v|>r_1$ it holds by construction of r_1. This proves property (2.3). □

4.3 First De Giorgi Lemma

In this section we will prove De Giorgi's first lemma, which states that if a function solving (1.1) is bounded in some region in an integral sense, then it is pointwise bounded in a smaller region.

The function ψ^1 in the statement of this lemma is defined in Lemma 4.2.3.

Proposition 4.3.1 (De Giorgi's First Lemma)**.** *There exists a universal constant $\delta_0>0$ such that the following is true:*

For any $f\in L^2([-2,0]\times B_2;H^s(\mathbb{R}^n))$ a weak solution to (1.1) *subject to* (1.2) *with source term $\|a\|_{L^r([-2,0]\times B_2\times\mathbb{R}^n)}\leq 1$, if*

$$f(t,x,v)\leq\psi^1(v)\qquad\forall x\in B_2, t\in[-2,0], |v|\geq 2$$

holds and

$$\iiint_{[-2,0]\times B_2\times B_2}\max(f-\psi^1,0)^2\,dv\,dx\,dt\leq\delta_0$$

holds, then

$$f(t,x,v)\leq\frac{1}{2}\qquad\forall x\in B_1, t\in[-1,0], v\in B_1.$$

As in most De Giorgi-style proofs, we take a sequence of cutoffs of our function and show that their L^2 norm tends to zero. We show this by producing a non-linear recursive inequality. The key to the proof is the inequality (3.14), which is located at the end of the second step. This inequality tells us that our function cannot have very bad singularities, because any singularity which is L^2 integrable is also L^q integrable for some specific $q > 2$. Classically such an inequality is produced using the energy inequality and Sobolev embedding, but in this case we will also require an averaging lemma.

Our proof will proceed in three steps. In the first step, we will apply the averaging lemma to our cutoff function to show that it has higher integrability in the t and x variables. Actually we will apply the averaging lemma to a barrier function, because our solution itself has certain negative measures in its derivatives. This is fine, since higher integrability for the barrier function trivially implies higher integrability for the original function. In the second step, we will obtain higher integrability in the v variable using the usual technique (with the energy inequality and Sobolev embedding). Then we use Riesz-Thorin interpolation to combine our integrability in t, x and v. In the third and final step, we produce the standard nonlinear recursion and argue that our cutoffs tend to zero in the limit.

Proof. We begin by specifying the sequence of cutoff functions. For $k \in \mathbb{N}$, consider soft cutoffs

$$\psi_k := \psi^1 + \frac{1}{2} - 2^{-k-1}$$

so that $\psi_0 = \psi^1$ and in the limit $\psi_\infty = \psi^1 + \frac{1}{2}$. Then we have a sequence of cutoff functions

$$f_k := \max(f - \psi_k, 0).$$

We'll make frequent use of the fact that for any k,

$$\chi_{\{f_k>0\}} \leq 2^{k+1} f_{k-1}. \tag{3.4}$$

We also must specify a sequence of space-time regions. Define

$$T_k := -1-2^{-k}, \qquad B^k := \{x \in \mathbb{R}^n : |x| \leq 1+2^{-k}\}, \qquad Q_k := [T_k, 0] \times B^k$$

so that $Q_0 = [-2,0] \times B_2$ and in the limit $Q_\infty = [-1,0] \times B_1$. Notice that the distance from the interior of B^k to the boundary of B^{k-1} is 2^{-k}, and that $T_k - T_{k-1} = 2^{-k}$.

For brevity, we will use $\int_k$ to denote an integral with bounds $[T_k, 0]$ or B^k or Q_k, as shall be clear from context. We also frequently will use C^k to mean $[C(n,s,\kappa)]^k$, a quantity which grows geometrically in k for n, s, and κ held constant.

Step 1: Higher integrability in t, x

Define $\eta_{k,\varepsilon}$ a smooth function which is supported on $[T_{k-1}, 0]$ and equal to 1 on $[T_k, -\varepsilon]$. Then define $\mu_\varepsilon(t) = \chi_{\{[-\varepsilon,0]\}} \partial_t \eta_{k,\varepsilon}$ the derivative of $\eta_{k,\varepsilon}$ near 0, and assume without loss of generality that $\mu_\varepsilon \leq 0$. The derivative of $\eta_{k,\varepsilon}$ will be bounded uniformly in ε *except* for the blowup near 0 which is captured by μ_ε. In symbols, $\sup_\varepsilon \|\partial_t \eta_{k,\varepsilon} - \mu_\varepsilon\|_\infty \leq C^k$.

In addition, let $\phi_k(x)$ be a smooth function supported on B^{k-1} and equal to 1 on B^k, with derivative $\|\nabla_x \phi_k\|_\infty \leq C^k$.

We want to apply the averaging lemma to $\eta_{k,\varepsilon}\phi_k f_k$, so let's apply the transport operator to this function.

$$\begin{aligned}
[\partial_t + v\cdot\nabla_x](\eta_{k,\varepsilon}\phi_k f_k) &= f_k[\partial_t + v\cdot\nabla_x](\eta_{k,\varepsilon}\phi_k) + \eta_{k,\varepsilon}\phi_k[\partial_t + v\cdot\nabla_x] f_k \\
&= f_k[\partial_t + v\cdot\nabla_x](\eta_{k,\varepsilon}\phi_k) + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}[\partial_t + v\cdot\nabla_x](f-\psi_k) \\
&= f_k[\partial_t + v\cdot\nabla_x](\eta_{k,\varepsilon}\phi_k) + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}\mathcal{L}f + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}} a
\end{aligned}$$

$$= f_k\,[\partial_t + v\cdot\nabla_x]\,(\eta_{k,\varepsilon}\phi_k) + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}\mathcal{L}\psi_k + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}a + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}\mathcal{L}(f-\psi_k).$$

By a well known pointwise inequality (c.f. Caffarelli and Sire [CS17]),

$$\chi_{\{f>\psi_k\}}\mathcal{L}(f-\psi_k) \le \mathcal{L} f_k.$$

Also $\mu_\varepsilon \le 0$. Therefore if we define

$$F_k := \eta_{k,\varepsilon} f_k(v\cdot\nabla_x\phi_k) + \phi_k f_k(\partial_t\eta_{k,\varepsilon} - \mu_\varepsilon) + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}\mathcal{L}\psi_k + \eta_{k,\varepsilon}\phi_k\chi_{\{f>\psi_k\}}a,$$

then

$$[\partial_t + v\cdot\nabla_x]\,(\eta_{k,\varepsilon}\phi_k f_k) \le F_k + \mathcal{L}(\eta_{k,\varepsilon}\phi_k f_k).$$

The source term F_k is in $L^2(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$. From (3.4), Lemma 4.2.3 property ((i)), and the definitions of ϕ_k, $\eta_{k,\varepsilon}$ and μ_ε,

$$\begin{aligned}
\iiint\limits_{\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n} F_k^2 &\le \iiint_{k-1}\left[\eta_{k,\varepsilon}^2(v\cdot\nabla_x\phi_k)^2 + \phi_k^2(\partial_t\eta_{k,\varepsilon}-\mu_\varepsilon)^2\right] f_k^2 + \iiint_{k-1}(\eta_{k,\varepsilon}\phi_k)^2\left[(\mathcal{L}\psi_k)^2 + a^2\right]\chi_{\{f_k>0\}} \\
&\le C^k\iiint_{k-1} f_k^2 + C^k\iiint_{k-1} f_{k-1}^2 + C^k\left(\iiint_{k-1} f_{k-1}^2\right)^{1-\frac{2}{r}} \\
&\le C^k\left(\iiint_{k-1} f_{k-1}^2\right)^{1-\frac{2}{r}}.
\end{aligned} \tag{3.5}$$

Because the averaging lemma requires equality, not the inequality that we have, we'll construct a barrier function g_k. Define g_k as some solution to the PDE

$$\begin{cases} [\partial_t + v\cdot\nabla_x]\,g_k = F_k + \mathcal{L}g_k & \forall t,x,v\in(T_{k-1},\infty)\times\mathbb{R}^n\times\mathbb{R}^n \\ g_k = \eta_{k,\varepsilon}\phi_k f_k = 0 & t = T_{k-1} \\ g_k = 0 & t < T_{k-1}. \end{cases} \tag{3.6}$$

Since $F_k\in L^2(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)$, a solution $g_k\in L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^n;H^s(\mathbb{R}^n))$ exists by [MX07] (see Section 4.1 for more detail).

Moreover, $g_k \geq \eta_{k,\varepsilon}\phi_k f_k \geq 0$ by a maximum principle: the function $\max(\eta_{k,\varepsilon}\phi_k f_k - g_k, 0)$ is a subsolution to $[\partial_t + v\cdot\nabla_x]h = \mathcal{L}h$ so it has non-increasing energy, and it vanishes at $t = T_{k-1}$ so it must be identically zero.

We'll now produce some bounds on g_k. Take the PDE (3.6) and multiply it by g_k, then integrate over $x \in \mathbb{R}^n$, $v \in \mathbb{R}^n$.

$$\frac{d}{dt}\frac{1}{2}\iint g_k^2\, dv dx = -\int B(g_k, g_k)\, dx + \iint g_k F_k\, dv dx.$$

Now applying Lemma 4.2.1 and Hölder's inequality,

$$\frac{d}{dt}\frac{1}{2}\iint g_k^2\, dv dx + \frac{1}{\kappa}\iint |\Lambda^s g_k|^2\, dv dx \leq C\iint g_k^2\, dv dx + \frac{1}{2}\iint F_k^2\, dv dx. \tag{3.7}$$

If we define

$$G(t) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} g_k^2(t)\, dv dx$$

we see from (3.7) that G satisfies

$$\frac{d}{dt}G(t) \leq CG(t) + \iint F_k^2(t).$$

Also, by construction, $G(T_{k-1}) = 0$. Thus by Gronwall's inequality, for all $t > T_{k-1}$:

$$\begin{aligned} G(t) &\leq e^{C(t-T_{k-1})}\int_{T_{k-1}}^{t}\iint F_k^2(\tau)\, dv dx\, d\tau \\ &\leq e^{C(t-T_{k-1})}\iiint_{\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n} F_k^2\, dv dx d\tau. \end{aligned}$$

This means that for any compact interval K in $\mathbb{R}$,

$$\|g_k\|_{L^\infty(K;L^2(\mathbb{R}^n\times\mathbb{R}^n))} \leq C_K \|F_k\|_{L^2(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)}. \tag{3.8}$$

Armed with this inequality, and the fact that $\partial_t \chi_K$ is in the dual space of $L^\infty(t)$, we integrate (3.7) over K:

$$\iiint_{K\times\mathbb{R}^n\times\mathbb{R}^n} |\Lambda^s g_k|^2 \, dvdxdt \leq C(n,s,\kappa)\left(\iiint_{K\times\mathbb{R}^n\times\mathbb{R}^n} g_k^2 + \iiint_{K\times\mathbb{R}^n\times\mathbb{R}^n} F_k^2 \, dvdxdt + \iiint_{\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n} g_k^2 \partial_t \chi_K\right)$$

$$\leq C_K \iiint_{\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n} F_k^2 \, dvdxdt. \tag{3.9}$$

We can now apply Lemma 0.1.1, the Averaging Lemma, to g_k. Let $\eta(v)$ be a $C_c^\infty(\mathbb{R}^n)$ function which is identically 1 on $v \in B_2$ and non-negative for all v, and choose any set, for example $[-3,1]\times B_3$, which compactly contains $[-2,0]\times B_2$. The lemma yields that

$$\left\|\int \eta g_k \, dv\right\|_{H^\beta([-2,0]\times B_2)} \leq C\left(\|g_k\|_{L^2([-3,1]\times B_3\times\mathbb{R}^n)} + \left\|(1-\Delta_v)^{-s/2}(F_k+\mathcal{L}g_k)\right\|_{L^2([-3,1]\times B_3\times\mathbb{R}^n)}\right)$$

with $\beta = (2(1+s))^{-1} < 1$.

Therefore, by the bounds (3.5) and (3.8), and by Lemma 4.2.1 and the bound (3.9),

$$\left\|\int \eta g_k \, dv\right\|_{H^\beta([-2,0]\times B_2)} \leq C^k \left(\iiint_{k-1} f_{k-1}^2\right)^{\frac{1}{2}-\frac{1}{r}}. \tag{3.10}$$

Define p_1 by

$$\frac{1}{p_1} = \frac{1}{2} - \frac{\beta}{n+1} = \frac{1}{2} - \frac{1}{2(1+s)(n+1)} \in (0,1/2).$$

By Sobolev embedding,

$$\left\|\int \eta g_k \, dv\right\|_{L^{p_1}(t,x)} \leq C\left\|\int \eta g_k \, dv\right\|_{H^\beta(t,x)}. \tag{3.11}$$

Since f_k is supported where $\eta \equiv 1$, the integral $\int \eta f_k \, dv$ is just the $L^1(v)$ norm of f_k. Recall also that $\eta_{k,\varepsilon}\phi_k f_k \leq g_k$. Therefore we can bound the $L^{p_1,p_1,1}$ norm of f_k:

$$\int_{T_k}^{-\varepsilon}\int_{B^k}\left(\int f_k \, dv\right)^{p_1} dxdt \leq \iint \left(\int \eta\,[\eta_{k,\varepsilon}\phi_k f_k]\, dv\right)^{p_1} dxdt$$

$$\le \iint \left(\int \eta g_k \, dv \right)^{p_1} dx dt$$

Since this inequality is true for all ε, we can chain it with (3.10) and (3.11) to conclude that

$$\|f_k\|_{L^{p_1,p_1,1}(Q_k \times \mathbb{R}^n)} \le C^k \|f_{k-1}\|_{L^2(Q_{k-1}\times\mathbb{R}^n)}^{1-\frac{2}{r}}. \tag{3.12}$$

Step 2: Higher integrability in all three variables

Since each f_k is supported on $|v| \le 2$, and $\|\mathcal{L}\psi_k\|_\infty \le C_\psi$ by Lemma 4.2.3, property ((i)), we can apply the energy inequality from Lemma 4.2.2 to obtain

$$\iint_{k+1} B(f_k, f_k) \le C^k \iiint_k f_k^2 + C^k \iiint_k f_k + C^k \|f_k\|_{L^{r^*}(Q_k)}.$$

From this inequality, Lemma 4.2.1, and (3.4):

$$\iiint_{k+1} |\Lambda^s f_k|^2 \le C^k \iiint_k f_{k-1}^2 + C^k \left(\iiint_k f_{k-1}^2 \right)^{1/r^*}.$$

When $\|f_k\|_2 < 1$, as we assume without loss of generality, the second term on the right-hand-side will dominate.

Therefore, letting p_2 be defined by $\frac{1}{p_2} = \frac{1}{2} - \frac{s}{n}$, we have by Sobolev embedding

$$\|f_k\|_{L^{2,2,p_2}(Q_{k+1}\times\mathbb{R}^n)} \le C^k \|f_{k-1}\|_{L^2(Q_k\times\mathbb{R}^n)}^{1/r^*}. \tag{3.13}$$

Now we wish to utilize Riesz-Thorin interpolation to interpolate between this inequality and (3.12).

Consider $\theta \in [0,1]$ and the function

$$\theta \mapsto \left[\frac{\theta}{2} + \frac{1-\theta}{p_1} \right] - \left[\frac{\theta}{p_2} + \frac{1-\theta}{1} \right].$$

Because this function is negative at $\theta = 0$ and positive at $\theta = 1$, it must equal zero at some point θ^*, and at this point we can define q by

$$\frac{1}{q} := \frac{\theta^*}{2} + \frac{1-\theta^*}{p_1} = \frac{\theta^*}{p_2} + \frac{1-\theta^*}{1}.$$

Moreover, since $1/q$ is a nontrivial convex combination of $1/2$ and $1/p_1$, it must be the case that $q > 2$. Riesz-Thorin tells us that

$$\|f_k\|_{L^{q,q,q}(Q_k \times \mathbb{R}^n)} \leq \|f_k\|_{L^{2,2,p_2}(Q_k \times \mathbb{R}^n)}^{\theta^*} \|f_k\|_{L^{p_1,p_1,1}(Q_k \times \mathbb{R}^n)}^{1-\theta^*}.$$

Combining this with the bounds (3.12) and (3.13),

$$\|f_k\|_{L^q(Q_k \times \mathbb{R}^n)} \leq C^k \|f_{k-2}\|_{L^2(Q_{k-2} \times \mathbb{R}^n)}^{1-\frac{2}{r}+\frac{\theta^*}{r}}. \tag{3.14}$$

This bound is the key to De Giorgi's first lemma.

Step 3: The recursion

This step is standard to all De Giorgi arguments. It does not depend on the specifics of our PDE (1.1) in any way, except through the bound (3.14).

For any k, by (3.4),

$$\begin{aligned} \iiint_k f_k^2 &= \iiint_k f_k^2 \chi_{\{f_k > 0\}}^{q-2} \\ &\leq 2^{(k+1)(q-2)} \iiint_k f_k^2 f_{k-1}^{q-2} \\ &\leq C^k \iiint_{k-1} f_{k-1}^q. \end{aligned}$$

From this and (3.14),

$$\iiint_k f_k^2 \leq C^k \left(\iiint_{k-3} f_{k-3}^2 \right)^{\frac{q}{2}\left(1-\frac{2}{r}+\frac{\theta^*}{r}\right)}.$$

Since q and θ^* are independent of r and $q>2$, the exponent on this recursive inequality will be greater than 1 for r sufficiently large. Specifically, the exponent exceeds 1 precisely when $r>r_0$, with r_0 as defined in Section 4.1, though we omit the explicit calculation.

Since the exponent is greater than one, and the sequence

$$k \mapsto \iiint_k f_k^2 \tag{3.15}$$

is monotone decreasing, by a standard fact about sequences (c.f. [Vas16a]) we can now say that this sequence limits to 0 as $k \to \infty$, provided the initial value

$$\iiint_{[-2,0]\times B_2 \times \mathbb{R}^n} \max(f-\psi^1,0)^2 \, dv dx dt \leq \delta_0$$

is sufficiently small.

Lastly, since the limit of that sequence (3.15) is zero, by the Lebesgue's monotone convergence theorem

$$\iiint_{[-1,0]\times B_1 \times \mathbb{R}^n} (f-\psi^1-\frac{1}{2})_+^2 \, dv dx dt = 0.$$

Since $\psi^1 = 0$ on B_1, the proposition is proven.

$\square$

4.4 Second De Giorgi Lemma

In this section we will prove the second De Giorgi lemma, the intermediate value lemma. It says that solutions to our PDE cannot have, in a small region, very much measure above a certain value and also very much measure below another value unless the solution also has sufficient measure between the two values. The lemma is sometimes called an isoperimetric inequality.

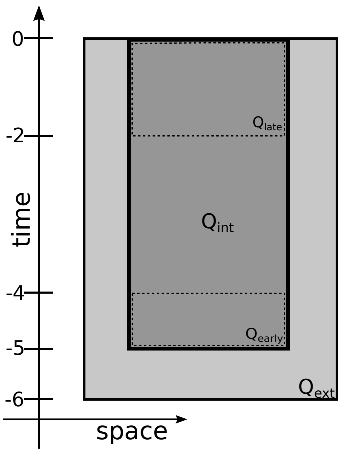

Figure 4.1: Four overlapping cylinders described in Proposition 4.4.1.

To state Proposition 4.4.1, we must define four cylindrical regions in space-time:

$$Q_{\text{ext}} := [-6,0] \times B_3$$

$$Q_{\text{int}} := [-5,0] \times B_2$$

$$Q_{\text{early}} := [-5,-4] \times B_2$$

$$Q_{\text{late}} := [-2,0] \times B_2.$$

The constant δ_0 in the statement of this proposition is defined in Proposition 4.3.1.

Proposition 4.4.1 (Second De Giorgi Lemma)**.** *There exist universal constants $\gamma_0 > 0$ and $0 < \theta_0 < 1/3$ such that the following is true:*

For any $f \in L^2(Q_{ext}; H^s(\mathbb{R}^n))$ a weak solution to (1.1) *subject to* (1.2) *with*

$$\|a\|_{L^r(Q_{ext} \times \mathbb{R}^n)} \leq \theta_0$$

satisfying

$$|f(t,x,v)| \le 1+\psi_{\theta_0}(v) \qquad \forall (t,x,v) \in Q_{ext} \times \mathbb{R}^n,$$

if

$$|\{f \le 0\} \cap Q_{early} \times B_2| \ge \frac{|Q_{early}| \cdot |B_2|}{2} \tag{4.16}$$

and

$$|\{f \ge 1-\theta_0\} \cap Q_{late} \times B_2| \ge \delta_0 \tag{4.17}$$

then

$$|\{0 < f < 1-\theta_0\} \cap Q_{int} \times B_3| \ge \gamma_0. \tag{4.18}$$

As in other applications of De Giorgi's method, the idea of the proof is to produce a sequence of solutions to our PDE with smaller and smaller intermediate measure, show that they are compact and have a discontinuous limit, and then show that said limit function inherits enough regularity from the PDE to result in a contradiction.

Our version of the proof is divided into four steps. In the first step, we show that our sequence is uniformly differentiable in v. We then use the averaging lemma to show that, in some very specific sense, our sequence is uniformly differentiable in t and x. In the second step, we combine the results of step one to obtain compactness in all three variables, thus producing our limit. In the third step, we show that this limit function is regular in v. The limit is constant in v for $|v|$ small, and behaves like an indicator function depending only on t and x. In the fourth and final step, we show that certain t- and x-derivatives of our limit function are bounded, and that this contradicts what we know about the jump discontinuities in our limit.

Proof. Assume that the theorem is false. Then there must exist a sequence f_i of solutions to our equation (1.1) with operators $\mathcal{L}_i$ subject to (1.2) and source terms

$$\|a_i\|_{L^r(Q_{\text{ext}}\times\mathbb{R}^n)} \leq 1/i$$

such that

$$|f_i(t,x,v)| \leq 1+\psi_{1/i} \qquad \forall (t,x,v)\in Q_{\text{ext}}\times\mathbb{R}^n$$

while

$$|\{f_i\leq 0\}\cap Q_{\text{early}}\times B_2| \geq \frac{|Q_{\text{early}}|\cdot|B_2|}{2},$$
$$|\{f_i\geq 1-\frac{1}{i}\}\cap Q_{\text{late}}\times B_2| \geq \delta_0,$$
$$|\{0<f_i<1-\frac{1}{i}\}\cap Q_{\text{int}}\times B_3| \leq \frac{1}{i}.$$

We wish to take a limit of these functions f_i.

Step 1: Regularity in v and regularity in t,x

Let $F:\mathbb{R}^n\to\mathbb{R}$ be a smooth radially-increasing function of v which is identically -1 on B_2 and identically 0 outside of B_3. Since $F\in C_c^\infty$, it is trivial to show that

$$\|\mathcal{L}_i F\|_\infty \leq C(n,s,\kappa). \tag{4.19}$$

To obtain compactness, we use a very blunt cutoff function $\bar{\psi}$ defined by

$$\bar{\psi}(v) := \psi_{\frac{1}{3}}(v)+1+F(v),$$
$$f_i^+ := \max\left(f-\bar{\psi},0\right),$$
$$f_i^- := \max\left(\bar{\psi}-f,0\right).$$

Because $\psi_{1/3} \geq \psi_\theta$ for all $\theta < 1/3$ by Lemma 4.2.3, property ((iii)), each f_i^+ for i sufficiently large will be supported on $v \in B_3$. In fact

$$0 \leq f_i^+(t,x,v) \leq -F(v) \qquad \forall (t,x,v) \in Q_{\text{ext}} \times \mathbb{R}^n. \tag{4.20}$$

Each f_i is a solution to (1.1), so we can apply Lemma 4.2.2 on the regions Q_{ext} and Q_{int} with cutoff $\bar{\psi}$. From (4.19) and Lemma 4.2.3, property ((i)) we know that $\left\|\mathcal{L}_i \bar{\psi}\right\|_\infty$ is bounded by a finite universal constant. The right hand side of this energy inequality is then universally bounded by (4.20) so

$$\iint_{Q_{\text{int}}} B_i(f_i^+, f_i^+)\, dxdt - \iint_{Q_{\text{int}}} B_i(f_i^+, f_i^-)\, dxdt \leq C(n,s,\kappa). \tag{4.21}$$

In particular, by Lemma 4.2.1,

$$\iint_{Q_{\text{int}}} \int \left|\Lambda^s f_i^+\right|^2 dvdxdt \leq C(n,s,\kappa). \tag{4.22}$$

Critically, the constant $C(n,s,\kappa)$ does not depend on i.

Unfortunately the energy inequality does not give us regularity in the t and x variables. In order to obtain compactness, therefore, we must rely on an averaging lemma. To that end, apply the transport operator to ${f_i^+}^2$ and obtain

$$\begin{aligned}
[\partial_t + v \cdot \nabla_x]\, {f_i^+}^2 &= 2 f_i^+ [\partial_t + v \cdot \nabla_x]\, f_i \\
&= 2 f_i^+ \mathcal{L}_i f_i + 2 f_i^+ a_i \\
&= 2 f_i^+ \mathcal{L}_i \left(f_i - \bar{\psi}\right) + 2 f_i^+ \mathcal{L}_i \bar{\psi} + 2 f_i^+ a_i.
\end{aligned}$$

For any function g and operator $\mathcal{L}$ satisfying (1.2), and $g_+ := \max(g, 0)$, it is true that, for any t, x fixed,

$$2 g_+ \mathcal{L} g = \int 2[g_+(v) g(w) - g_+(v)^2] K(t,x,v,w)\, dw$$

$$
\begin{aligned}
&= \int [g_+(w)^2 - g_+(v)^2] K\,dw + \int [2g_+(v)g(w) - g_+(v)^2 - g_+(w)^2] K\,dw \\
&= \int [g_+(w)^2 - g_+(v)^2] K\,dw - \int [g_+(w) - g_+(v)]^2 K\,dw + \int 2g_+(v)[g(w) - g_+(w)] K\,dw \\
&= \mathcal{L} g_+^2 - \int [g_+(w) - g_+(v)]^2 K\,dw - 2\int g_+(v) g_-(w) K\,dw.
\end{aligned}
$$

Thus

$$
[\partial_t + v \cdot \nabla_x]\, f_i^{+2} = H := H_1 + H_2 + H_3 + H_4
$$

where

$$
\begin{aligned}
H_1 &:= \mathcal{L}_i \left(f_i^{+2} \right), \\
H_2 &:= -\int [f_i^+(w) - f_i^+(v)]^2 K(v,w)\,dw, \\
H_3 &:= -2\int f_i^+(v) f_i^-(w) K(v,w)\,dw, \\
H_4 &:= 2 f_i^+ \mathcal{L}_i \bar{\psi} + 2 f_i^+ a_i.
\end{aligned}
$$

We proceed to bound H, term by term, independent of i.

We begin with an H^s bound on f_i^{+2}:

$$
\begin{aligned}
\int \left| \Lambda^s (f_i^{+2}) \right|^2 dv &= \iint \frac{|f_i^{+2}(w) - f_i^{+2}(v)|^2}{|v-w|^{n+2s}}\, dwdv \\
&= \iint \left[f_i^+(w) + f_i^+(v) \right]^2 \frac{|f_i^+(w) - f_i^+(v)|^2}{|v-w|^{n+2s}}\, dwdv \\
&\le 2^2 \left\| f_i^+ \right\|_{L^\infty}^2 \int \left| \Lambda^s (f_i^+) \right|^2 dv. \qquad (4.23)
\end{aligned}
$$

From this, the bounds (4.20) and (4.22), and Lemma 4.2.1, we obtain

$$
\left\| (1-\Delta_v)^{-s/2} H_1 \right\|_{L^2(Q_{\mathrm{int}} \times \mathbb{R}^n)} \le C(n,s,\kappa). \qquad (4.24)
$$

The terms H_2 and H_3 are strictly negative, so their total variations as measures are simply the absolute values of their integrals. Thus their norms in $\mathcal{M}(Q_{\text{int}} \times \mathbb{R}^n)$ are

$$\left| \iint_{Q_{\text{int}}} \int H_2 \, dv dx dt \right| = \iint_{Q_{\text{int}}} B_i(f_i^+, f_i^+) \, dx dt,$$

$$\left| \iint_{Q_{\text{int}}} \int H_3 \, dv dx dt \right| = - \iint_{Q_{\text{int}}} B_i(f_i^+, f_i^-) \, dx dt.$$

These are of course universally bounded by (4.21).

Recall that $(1-\Delta_v)^{-\left(s+\frac{n}{2}\right)/2}$ can be represented as convolution with a Green's function $G_{s+n/2}(v)$ (see e.g. Stein [Ste70]). The function $G_{s+n/2}$ decays exponentially as $|v| \to \infty$ and has a singularity like $\frac{1}{|v|^{\frac{n}{2}-s}}$ near zero. Therefore $G_{s+n/2}$ is in L^2. By Young's Inequality, convolution of a measure and an L^2 function is bounded by the product of their $\mathcal{M}$ and L^2 norms respectively, so

$$\left\| (1-\Delta_v)^{-\left(s+\frac{n}{2}\right)/2} H_2 \right\|_{L^2(Q_{\text{int}} \times \mathbb{R}^n)} \leq C(n,s,\kappa), \tag{4.25}$$

$$\left\| (1-\Delta_v)^{-\left(s+\frac{n}{2}\right)/2} H_3 \right\|_{L^2(Q_{\text{int}} \times \mathbb{R}^n)} \leq C(n,s,\kappa). \tag{4.26}$$

Lastly, from (4.20) and since $r \geq 2$ we know

$$\|H_4\|_{L^2(Q_{\text{int}} \times \mathbb{R}^n} \leq C(n,s,\kappa). \tag{4.27}$$

Finally we are ready to apply Lemma 0.1.1 to f_i^{+2}, which says for any $\eta \in C_c^\infty(\mathbb{R}^n)$ and any subset $\bar{\Omega}$ compactly contained in the interior of Q_{ext},

$$\left\| \int \eta f_i^{+2} \, dv \right\|_{H^\alpha(\bar{\Omega})} \leq C(\eta, \bar{\Omega}) \left(\left\| f_i^{+2} \right\|_{L^2(Q_{\text{int}} \times \mathbb{R}^n)} + \left\| (1-\Delta_v)^{-\left(s+\frac{n}{2}\right)/2} H \right\|_{L^2(Q_{\text{int}} \times \mathbb{R}^n)} \right)$$

where

$$\alpha = \left(2\left(s+\frac{n}{2}\right)\right)^{-1}.$$

From (4.24), (4.25), (4.26), and (4.27), we can say that in fact

$$\left\| \int \eta f_i^{+2}\,dv \right\|_{H^\alpha(\bar{\Omega})} \le C(n,s,\kappa,\eta,\bar{\Omega}). \tag{4.28}$$

Step 2: Producing a strong L^2 limit

Since all the f_i^+ are bounded by (4.20), $\{f_i^{+2}\}_i$ is a bounded subset of $L^2(Q_{\text{int}} \times \mathbb{R}^n)$. By Banach-Alaoglu, there exists a function f^+ such that, along some subsequence,

$$f_i^{+2} \rightharpoonup f^{+2}$$

weakly in $L^2(Q_{\text{int}} \times \mathbb{R}^n)$.

Our goal is to show that this limit converges also strongly in $L^2_{\text{loc}}(Q_{\text{int}}; L^2(\mathbb{R}^n))$. To that end, fix some compact subset $\bar{\Omega}$ of Q_{int}.

Strong and weak limits, when both exist, must be equal, so with the bound (4.28) we apply Rellich-Kondrachov to prove that

$$\int \eta(v) f_i^{+2}\,dv \to \int \eta(v) f^{+2}\,dv$$

strongly in $L^2(\bar{\Omega})$, without passing to a further subsequence, for any $\eta \in C_c^\infty(\mathbb{R}^n)$.

In particular, if we fix some η such that $\eta_\varepsilon(v) = \varepsilon^{-n}\eta(v/\varepsilon)$ is an approximation to the identity, then for $\varepsilon > 0$ and $v \in \mathbb{R}^n$ fixed,

$$\iint_{\bar{\Omega}} \left[\int f_i^{+2}(w)\eta_\varepsilon(v-w)\,dw - \int f^+(w)^2 \eta_\varepsilon(v-w)\,dw \right]^2 dx\,dt \xrightarrow{i\to\infty} 0.$$

Note that this is pointwise (in v) convergence of convolutions.

Since the f_i^+ are all bounded by (4.20), and by weak convergence so is f^+, we can apply the Lebesgue dominated convergence theorem to conclude that not only do these convolutions converge pointwise in v, but they converge in integral as well. That is,

$$\int\iint_{\bar{\Omega}} \left[\left(f_i^{+2} *_v \eta_\varepsilon\right)(v) - \left(f^{+2} *_v \eta_\varepsilon\right)(v)\right]^2 dxdtdv \to 0. \tag{4.29}$$

It is known (see Lemma 0.1.3 in the appendix for a proof) that for any $g \in H^s(\mathbb{R}^n)$,

$$\|g - g * \eta_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C(n,s,\eta)\,\|g\|_{H^s(\mathbb{R}^n)}\,\varepsilon^s.$$

Therefore for our functions f_i^{+2},

$$\iint_{\bar{\Omega}}\int \left(f_i^{+2} - f_i^{+s} *_v \eta_\varepsilon\right)^2 dvdxdt \leq C(n,s,\eta)\varepsilon^{2s} \iint_{Q_{\text{int}}}\int |\Lambda^s f_i^+|^2\, dvdxdt.$$

Remember that $\left\|f_i^{+2}\right\|_{L^2(Q_{\text{int}};H^s(\mathbb{R}^n))}$ is bounded by (4.23) and (4.22), and, since the H^s norm is weakly lower-semi-continuous, $\left\|f^{+2}\right\|_{L^2(Q_{\text{int}};H^s(\mathbb{R}^n))}$ will be bounded as well.

Therefore we can bound

$$\begin{aligned}\left\|f_i^{+2} - f_+^2\right\|_2 &\leq \left\|f_i^{+2} - \eta_\varepsilon *_v f_i^{+2}\right\|_2 + \left\|\eta_\varepsilon *_v f_i^{+2} - \eta_\varepsilon *_v f^{+2}\right\|_2 + \left\|f^{+2} - \eta_\varepsilon *_v f^{+2}\right\|_2 \\ &\leq C\varepsilon^s + \left\|\eta_\varepsilon *_v f_i^{+2} - \eta_\varepsilon *_v f^{+2}\right\|_2.\end{aligned}$$

By $\|\cdot\|_2$ we mean $\|\cdot\|_{L^2(\bar{\Omega}\times\mathbb{R}^n)}$. For any $\delta > 0$, we take ε small enough that $C\varepsilon^s \leq \delta/2$. Then with ε fixed, we choose i large enough that (by (4.29)) $\left\|\eta_\varepsilon *_v f_i^{+2} - \eta_\varepsilon *_v f^{+2}\right\|_2 \leq \delta/2$. This proves that $\left\|f_i^{+2} - f_+^2\right\|_2$ goes to 0 as $i \to \infty$.

Since this is true for any $\bar{\Omega}$ compactly contained in the interior of Q_{int}, we can say that $f_i^{+2} \to f^{+2}$ in $L^2_{\text{loc}}(Q_{\text{int}}; L^2(\mathbb{R}^n))$.

In fact, since our domain is compact, this convergence happpens in $L^1_{\text{loc}}(Q_{\text{int}};L^2(\mathbb{R}^n))$ as well. Since f_i^+ and f_+ are non-negative, and since $(x-y)^2 \le |x^2-y^2|$ for any non-negative real numbers x and y, we can say that

$$f_i^+ \to f^+ \qquad \text{in } L^2_{\text{loc}}(Q_{\text{int}};L^2(\mathbb{R}^n)).$$

Step 3: The limit is constant in v

We'll denote

$$f = f^+ + 1 + F.$$

Because $f_i^+ \to f^+$ strongly in L^2_{loc}, we know that

$$\begin{aligned} |\{f=0\}\cap Q_{\text{early}}\times B_2| &\ge \frac{|Q_{\text{early}}|\cdot|B_2|}{2}, \\ |\{f=1\}\cap Q_{\text{late}}\times B_2| &\ge \delta_0, \\ |\{1+F<f<1\}\cap Q_{\text{int}}\times B_3| &= 0. \end{aligned} \tag{4.30}$$

Remark 4.4.1. If $s\ge 1/2$, we can use the fact that the H^s_v norm of f is known to be finite for almost every t,x fixed and obtain (4.33) immediately, making the remainder of Step 3 unnecessary. It is only in the case $s<1/2$ that this regularity in v is insufficient to rule out jump discontinuities. Therefore we follow the technique used in [CCV11b] and by Bass and Kassmann in [FBK05] to exploit the energy inequality's cross term.

For $0<\lambda\ll 1$, define the functions

$$\begin{aligned} f^+_{i,\lambda} &:= (f_i - \psi_\lambda - 1 - \lambda F)_+, \\ f^-_{i,\lambda} &:= (f_i - \psi_\lambda - 1 - \lambda F)_-. \end{aligned}$$

From the the energy inequality of Lemma 4.2.2, we see that for all i the cross term is bounded

$$-\iint_{Q_{\text{int}}} B\left(f_{i,\lambda}^+, f_{i,\lambda}^-\right) \leq C(n,s,\kappa)\left[\iint_{Q_{\text{ext}}}\int {f_{i,\lambda}^+}^2 + \sup_{v\in B_3}\mathcal{L}_i(\psi_\lambda+\lambda F)\iint_{Q_{\text{ext}}}\int f_{i,\lambda}^+ + \|a_i\|_r \left\|f_{i,\lambda}^+\right\|_{r^*}\right]. \tag{4.31}$$

For $v\in B_3$, Lemma 4.2.3, property ((i)) says that $\mathcal{L}_i\psi_\lambda(v)\leq C_\psi\lambda^{3s/2}$. Moreover by (4.19), $|\mathcal{L}_i\lambda F(v)|\leq C\lambda$ for some universal constant C.

For λ fixed and i sufficiently large,

$$f_i \leq 1+\psi_{1/i}\leq 1+\psi_\lambda$$

so

$$0\leq f_{i,\lambda}^+\leq \lambda F.$$

Therefore, for λ fixed and i sufficiently large, the inequality (4.31) yields

$$\iint_{Q_{\text{int}}} -B\left(f_{i,\lambda}^+, f_{i,\lambda}^-\right) \leq C(n,s,\kappa)\left[\lambda^2+(\lambda+\lambda^{3s/2})\lambda+(1/i)\lambda\right].$$

The cross term in turn bounds the integral of $f_{i,\lambda}^+$ and $f_{i,\lambda}^-$. For any t,x fixed

$$\begin{aligned}
-B_i(f_{i,\lambda}^+, f_{i,\lambda}^-) &= \iint K(v,w) f_{i,\lambda}^+(v) f_{i,\lambda}^-(w)\,dw dv\\
&\geq \frac{1}{\kappa}\iint_{|v-w|\leq 6} \frac{f_{i,\lambda}^+(v) f_{i,\lambda}^-(w)}{|v-w|^{n+2s}}\,dwdv\\
&\geq \frac{1}{\kappa}\iint_{|v|\leq 3, |w|\leq 3} \frac{f_{i,\lambda}^+(v) f_{i,\lambda}^-(w)}{6^{n+2s}}\,dwdv\\
&= C\int_{B_3} f_{i,\lambda}^+\,dv \int_{B_3} f_{i,\lambda}^-\,dv.
\end{aligned}$$

Since $f_i\to f$ strongly in $L^2_{\text{loc}}(Q_{\text{int}};L^2(\mathbb{R}^n))$, these upper- and lower-bounds on the cross term hold in the limit:

$$\iint_{Q_{\text{int}}}\left[\int_{B_3}(f-\psi_\lambda-1-\lambda F)_+\,dv\int_{B_3}(f-\psi_\lambda-1-\lambda F)_-\,dv\right]dxdt \leq C(n,s,\kappa)(\lambda^2+\lambda^{1+3s/2}). \tag{4.32}$$

This bound on the limit f is very strong, because by (4.30) we have either $f(t,x,v)=1$ or $f(t,x,v)=1+F(v)$ for almost all $(t,x,v)\in Q_{\text{int}}\times B_3$. For such (t,x,v), also $\psi_\lambda(v)=0$ and so

$$f-\psi_\lambda-1-\lambda F=[-\lambda F]\chi_{\{f=1\}}+[(1-\lambda)F]\chi_{\{f=1+F\}}.$$

The function $-\lambda F$ is non-negative, while $(1-\lambda)F$ is non-positive, so at any point $t,x\in Q_{\text{int}}$,

$$\int_{B_3}(f-\psi_\lambda-1-\lambda F)_+\,dv=-\lambda\int F\chi_{\{f=1\}}\,dv$$
$$\int_{B_3}(f-\psi_\lambda-1-\lambda F)_-\,dv=-(1-\lambda)\int F\chi_{\{f=1+F\}}\,dv.$$

Plugging this into (4.32) and moving all the λ to one side, we obtain

$$\iint_{Q_{\text{int}}}\int F\chi_{\{f=1\}}\,dv\int F\chi_{\{f=1+F\}}\,dv\,dx\,dt\leq C(n,s,\kappa)\frac{\lambda^2+\lambda^{1+3s/2}}{\lambda(1-\lambda)}.$$

The left-hand side is independent of λ, and the right side tends to 0 as $\lambda\to 0$, so we conclude that the left-hand side is in fact 0. In particular, this means that for almost every $t,x\in Q_{\text{int}}$, either

$$|\{v:f(t,x,v)=1\}\cap B_3|=0\qquad\text{or}\qquad|\{v:f(t,x,v)=1+F\}\cap B_3|=0.\tag{4.33}$$

Step 4: The limit has bounded derivative, which is a contradiction

What remains is to argue that f increases from 0 to 1, without taking intermediate values along the way, despite having bounded derivative. Moreover, it is not enough to bound the derivatives in any weak sense, because jump discontinuities can hide in sets of measure zero.

Since f is only defined up to an a.e.-equivalence class, we can assume without loss of generality that, for every (not a.e.) $t,x\in Q_{\text{int}}$, either $f(t,x,v)\equiv 1$ or $f(t,x,v)\equiv 1+F$.

For each i, since $\bar{\psi}$ is constant in t and x, it is true that

$$[\partial_t + v \cdot \nabla_x] \left(f_i - \bar{\psi}\right) = \mathcal{L}_i \left(f_i - \bar{\psi}\right) + \mathcal{L}_i \bar{\psi} + a_i.$$

Multiplying by $\chi_{\{f_i \geq \bar{\psi}\}}$ and recalling the standard pointwise inequality for integral operators (c.f. [CS17]),

$$[\partial_t + v \cdot \nabla_x] f_i^+ \leq \mathcal{L}_i f_i^+ + \chi_{\{f_i \geq \bar{\psi}\}} \mathcal{L}_i \bar{\psi} + \chi_{\{f_i \geq \bar{\psi}\}} a_i.$$

By (4.19) and Lemma 4.2.3, property ((i)), the term $\chi_{\{f_i \geq \bar{\psi}\}} \mathcal{L}_i \bar{\psi}$ is less than a universal constant $C(n,s,\kappa)$, and of course the L^r norm of $\chi_{\{f_i \geq \bar{\psi}\}} a_i$ is less than $1/i$ so this term will vanish in the limit. Let $\phi \in C_c^\infty(Q_{\text{int}} \times \mathbb{R}^n)$ be a non-negative test function and consider

$$-\langle f_i^+, [\partial_t + v \cdot \nabla_x] \phi \rangle \leq \langle f_i^+, \mathcal{L}_i \phi \rangle + \langle C, \phi \rangle + \frac{1}{i} \|\phi\|_{r^*}.$$

For $\phi \in C_c^\infty$ fixed, the functions $\mathcal{L}_i \phi$ will be uniformly bounded in L^∞ and decay like $|v|^{-n-2s}$. In particular they are uniformly bounded in $L^2(Q_{\text{int}} \times \mathbb{R}^n)$. Therefore

$$\langle f_i^+ - f^+, \mathcal{L}_i \phi \rangle \to 0$$

so in little-o notation

$$-\langle f_i^+, [\partial_t + v \cdot \nabla_x] \phi \rangle \leq \langle \mathcal{L}_i f^+, \phi \rangle + \langle C, \phi \rangle + o(1).$$

By (4.33) and (4.19),

$$\mathcal{L}_i f^+ = -\chi_{\{t,x: f \equiv 1\}} \mathcal{L}_i F \leq C(n,s,\kappa).$$

Thus for some universal constant $C_1 = C_1(n,s,\kappa)$ we have, in the sense of distributions,

$$[\partial_t + v \cdot \nabla_x](f - 1 - F) \leq C_1.$$

To make the remaining calculation rigorous, let $\eta_\varepsilon(t,x)$ be an approximation to the identity and define

$$f_\varepsilon = \eta_\varepsilon *_{t,x} f.$$

These functions f_ε are smooth and $f_\varepsilon \to f$ pointwise a.e. as $\varepsilon \to 0$. For $(t,x) \in Q_{\text{int}}$ fixed, f_ε, like f, is constant over all $v \in B_2$. Because the transport operator commutes with convolution in t and x,

$$[\partial_t + v \cdot \nabla_x] f_\varepsilon = \eta_\varepsilon *_{t,x} [\partial_t + v \cdot \nabla_x] f \leq C_1.$$

This inequality is true not only in the sense of distributions but also pointwise because the functions are smooth.

Define two sets

$$M_1 = \{t,x \in Q_{\text{late}} : f(t,x,v) = 1\},$$
$$M_0 = \{t,x \in Q_{\text{early}} : f(t,x,v) = 1 + F(v)\}.$$

By (4.30) we know that $|M_0| \geq \frac{|Q_{\text{early}}|}{2}$ and $|M_1| \geq \frac{\delta_0}{|B_2|}$. By Egorov's theorem, for ε sufficiently small,

$$\begin{aligned} |M_1^\varepsilon| &:= |\{t,x \in Q_{\text{late}} : f_\varepsilon(t,x,v) > 0.9 \, \forall v \in B_2\}| \geq \frac{\delta_0}{2|B_2|}, \\ |M_0^\varepsilon| &:= |\{t,x \in Q_{\text{early}} : f_\varepsilon(t,x,v) < 0.1 \, \forall v \in B_2\}| \geq \frac{|Q_{\text{early}}|}{4}. \end{aligned} \tag{4.34}$$

Fixing ε, choose a point $(t_0,x_0) \in M_0^\varepsilon$.

For any $(t_1,x_1) \in M_1^\varepsilon$, we can define the velocity $\bar{v} := \frac{x_1 - x_0}{t_1 - t_0}$ and see that $|\bar{v}| \leq 2$. Then the function

$$\tau \mapsto f_\varepsilon\big((1-\tau)t_0 + \tau t_1, (1-\tau)x_0 + \tau x_1, \bar{v}\big)$$

is equal to 0 at $\tau = 0$ and equal to 1 at $\tau = 1$, and its derivative is less than $(t_1 - t_0)C_1$. Therefore

$$\mathcal{H}^1(\text{segment}[(t_0,x_0),(t_1,x_1)] \cap \{t,x : 0.1 < f(t,x,v) < 0.9 \forall v \in B_2\}) \geq \frac{.8\sqrt{1+|\bar{v}|^2}}{C_1} \geq \frac{2}{C_1}. \tag{4.35}$$

The facts (4.34) and (4.35) tell us, by the elementary geometric argument of Lemma 0.1.2, that the cone with vertex (t_0, x_0) and base M_1^ε must intersect $\{t,x : 0.1 < f(t,x,v) < 0.9 \forall v \in B_2\}$ on a set with measure $(\delta_0/2|B_2|)(2/C_1)^2/80$.

In particular,

$$|\{0.1 < f_\varepsilon < 0.9\} \cap Q_{\text{int}} \times B_2| \geq \frac{2\delta_0}{80C_1^2|B_2|} > 0.$$

This bound holds for all ε sufficiently small, but we know from (4.30) that it is not true for f. By Egorov's theorem, this is a contradiction.

Therefore our sequence f_i must not exist, and the proposition must be true.

□

4.5 Hölder Continuity

In this section, we explain how Proposition 4.3.1 and Proposition 4.4.1 together lead to Hölder regularity of our solution. We begin by showing that the PDE (1.1) is scaling invariant. We then show, in Lemma 4.5.2, how to combine Proposition 4.3.1 and Proposition 4.4.1 to create a sort of Harnack's inequality. The ideas here are not new, in particular we follow [CCV11b] very closely.

Lemma 4.5.1 (Scaling)**.** *If f solves (1.1) on some region $Q \times \mathbb{R}^n \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, then for any constant $\varepsilon < 1$,*

$$\bar{f}(t,x,v) := f(\varepsilon^{2s}t, \varepsilon^{1+2s}x, \varepsilon v)$$

will solve

$$\partial_t \bar{f} + v \cdot \nabla_x \bar{f} = \int [\bar{f}(w) - \bar{f}(v)] \bar{K}(t,x,v,w)\, dw + \bar{a}$$

on the appropriate region $Q_\varepsilon \times \mathbb{R}^n$ *with* $\bar{K}$ *symmetric and satisfying* (1.2), *and with*

$$\|\bar{a}\|_{L^r(Q_\varepsilon \times \mathbb{R}^n)} \le \varepsilon^{2s\left(1 - \frac{n+1+n/s}{r}\right)} \|a\|_{L^r(Q \times \mathbb{R}^n)}.$$

Proof. Denote

$$p = (t,x,v), \qquad \bar{p} = (\varepsilon^{2s} t, \varepsilon^{1+2s} x, \varepsilon v).$$

Evaluate the equality (1.1) at the point $\bar{p}$, so that

$$(\partial_t f)(\bar{p}) + \varepsilon v \cdot (\nabla_x f)(\bar{p}) = (\mathcal{L} f)(\bar{p}) + a(\bar{p}). \tag{5.36}$$

For our modified function $\bar{f}$ evaluated at p,

$$\partial_t \bar{f}(p) = \varepsilon^{2s} (\partial_t f)(\bar{p}), \tag{5.37}$$

$$\nabla_x \bar{f}(p) = \varepsilon^{1+2s} (\nabla_x f)(\bar{p}). \tag{5.38}$$

Define

$$\bar{K}(t,x,v,w) := \varepsilon^{n+2s} K(\varepsilon^{2s} t, \varepsilon^{1+2s} x, \varepsilon v, \varepsilon w).$$

It's clear that $\bar{K}$ is still symmetric. Since

$$\bar{K}(t,x,v,w) \ge \varepsilon^{n+2s} \chi_{\{\varepsilon|v-w| \le 6\}} \frac{1}{\kappa} (\varepsilon|v-w|)^{-(n+2s)} \ge \chi_{\{|v-w| \le 6\}} \frac{1}{\kappa} |v-w|^{-(n+2s)}$$

and

$$\bar{K}(t,x,v,w) \le \varepsilon^{n+2s} \kappa (\varepsilon|v-w|)^{-(n+2s)} = \kappa |v-w|^{-(n+2s)},$$

$\bar{K}$ satisfies the bound (1.2).

For this $\bar{K}$,

$$\int [\bar{f}(w)-\bar{f}(v)]\bar{K}(p,w)\,dw = \varepsilon^{n+2s}\int [f(\varepsilon w)-f(\varepsilon v)]K(\bar{p},\varepsilon w)\,dw$$
$$= \varepsilon^{n+2s}\frac{1}{\varepsilon^n}\int [f(\varepsilon w)-f(\varepsilon v)]K(\bar{p},\varepsilon w)\,d(\varepsilon w)$$
$$= \varepsilon^{2s}(\mathcal{L}f)(\bar{p}). \tag{5.39}$$

Define

$$\bar{a}(t,x,v) := \varepsilon^{2s}a(\varepsilon^{2s}t,\varepsilon^{1+2s}x,\varepsilon v). \tag{5.40}$$

Then the L^r norm of $\bar{a}$ is

$$\|\bar{a}\|_r = \varepsilon^{2s}\varepsilon^{-\frac{2s+n(1+2s)+n}{r}}\left(\iiint a(\varepsilon^{2s}t,\varepsilon^{1+2s}x,\varepsilon v)^r\,d(\varepsilon^{2s}t)\,d(\varepsilon^{1+2s}x)\,d(\varepsilon v)\right)^{1/r}.$$

Plugging (5.37), (5.38), (5.39), and (5.40) into (5.36) yields

$$\varepsilon^{-2s}\partial_t\bar{f}(p)+\varepsilon\varepsilon^{-1-2s}v\cdot\nabla_x\bar{f}(p) = \varepsilon^{-2s}\int [\bar{f}(w)-\bar{f}(v)]\bar{K}(p)\,dw+\varepsilon^{-2s}\bar{a}(p).$$

Multiply both sides by ε^{2s} to obtain our desired result. $\square$

Remark 4.5.1. In addition to scaling, we can also translate solutions of (1.1). If f is a solution and (t_0,x_0,v_0) is a point in its domain, then

$$\bar{f}(t,x,v) := f(t_0+t,x_0+x+v_0t,v_0+v)$$

will be a solution to (1.1) with similarly adjusted source term and kernel. This translation invariance is necessary for the proof of Hölder continuity, though we omit any further detail.

The following lemma should be thought of as a Harnack inequality, except that it keeps track also of the growth in v.

In the sequel, θ_0 and γ_0 refer to the constant defined in the statement of Proposition 4.4.1, and δ_0 refers to the constant defined in Proposition 4.3.1 which is used again in the statement of Proposition 4.4.1.

Lemma 4.5.2 (Oscillation Lemma)**.** *There exists a universal constant* $0<\lambda<1$ *such that the following is true:*

If $f \in L^2(Q_{ext}; H^s(\mathbb{R}^n))$ *is a weak solution to* (1.1) *subject to* (1.2) *with source term*

$$\|a\|_{L^r(Q_{ext})} \leq \lambda\theta_0$$

and satisfying

$$|f(t,x,v)| \leq 1 + \lambda\psi_{\theta_0}(v) \tag{5.41}$$

for all $t,x \in Q_{ext}$, $v \in \mathbb{R}^n$, *then*

$$\left[\sup_{[-1,0]\times B_1\times B_1} f\right] - \left[\inf_{[-1,0]\times B_1\times B_1} f\right] \leq 2-\lambda.$$

Moreover, at least one of the two functions

$$\bar{f}_1(t,x,v) = \left(1+\frac{\lambda}{2}\right)\left[f(\lambda^{2s}t, \lambda^{1+2s}x, \lambda v) + \lambda/2\right]$$

or

$$\bar{f}_2(t,x,v) = \left(1+\frac{\lambda}{2}\right)\left[f(\lambda^{2s}t, \lambda^{1+2s}x, \lambda v) - \lambda/2\right]$$

will also solve (1.1) *subject to* (1.2) *in the weak sense with source term smaller than* $\lambda\theta_0$ *and satisfy*

$$|\bar{f}_i(t,x,v)| \leq 1 + \lambda\psi_{\theta_0}(v)$$

for all $t,x \in Q_{ext}$, $v \in \mathbb{R}^n$.

Proof. Choose $k_0 \in \mathbb{N}$ such that

$$\gamma_0 k_0 > |Q_{\text{int}} \times B_3|.$$

Take λ small enough that

$$\lambda \leq \frac{\theta_0^{k_0+1}}{2}, \qquad 3\lambda^{1+2s} < 1, \qquad 6\lambda^{2s} < 1, \qquad \lambda < \varepsilon_0, \qquad \text{and } \left(1+\frac{\lambda}{2}\right)\lambda^{2s\left(1-\frac{n+1+n/s}{r}\right)} \leq 1 \tag{5.42}$$

where $\varepsilon_0 = \varepsilon_0(s,\theta_0)$ is defined in Lemma 4.2.3 property ((v)).

Assume without loss of generality that

$$|\{f \leq 0\} \cap Q_{\text{early}} \times B_2| \geq |Q_{\text{early}}| \cdot |B_2|/2. \tag{5.43}$$

If this were not true, then we could simply discuss $-f$ instead. This proposition holds for f if and only if it holds for $-f$.

With this assumption, we will assert that the proposition's result is true for

$$\bar{f}(t,x,v) = \left(1+\frac{\lambda}{2}\right)\left[f(\lambda^{2s}t, \lambda^{1+2s}x, \lambda v) + \lambda/2\right].$$

It is clear by Lemma 4.5.1 and linearity of Equation (1.1) that $\bar{f}$ will solve (1.1) subject to (1.2) with source term $\bar{a}$ smaller than $\lambda\theta_0$ by (5.42). We must show that $\bar{f}$ is also bounded as desired.

Consider the sequence of functions

$$\begin{aligned} f_0 &= f \\ f_k &= \frac{f_{k-1}-1}{\theta_0} + 1 = \frac{f-1}{{\theta_0}^k} + 1. \end{aligned}$$

Since equation (1.1) is linear, all f_k will also be solutions with source terms $\frac{1}{\theta_0^k}a$.

For each $0 \leq k \leq k_0+1$ and any $(t,x,v) \in Q_{\text{ext}} \times \mathbb{R}^n$,

$$|a(t,x,v)| \leq \frac{\lambda\theta_0}{\theta_0^k} \leq \theta_0$$

by the assumption (5.42), and by (5.41) and (5.42),

$$f_k = \frac{f-1}{{\theta_0}^k} + 1 \leq \frac{\lambda}{{\theta_0}^k}\psi_{\theta_0} + 1 \leq \psi_{\theta_0} + 1. \tag{5.44}$$

We wish to show that f_{k_0} satisfies

$$|\{f_{k_0} \geq 1 - \theta_0\} \cap Q_{\text{late}} \times B_2| \leq \delta_0. \tag{5.45}$$

Therefore assume, for contradiction, that (5.45) does not hold. Then by construction, each f_k will satisfy (4.17) for $0 < k \leq k_0$. Moreover, all f_k will satisfy (4.16) since f_0 does by (5.43). Therefore we can apply Proposition 4.4.1 and conclude that each f_k for k from 0 to k_0 must satisfy (4.18). That means that the set

$$S_k := |\{f_k \leq 0\} \cap Q_{\text{int}} \times B_3|$$

must increase in measure by at least γ_0 with each increment of k. By choice of k_0, this would be a contradiction. We conclude that (5.45) holds.

Due to (5.44) and Lemma 4.2.3, property ((iv)), we say that for all $t, x \in Q_{\text{late}}$ and all $|v| \geq 2$

$$f_{k_0+1}(t,x,v) \leq 1 + \psi_{\theta_0}(v) \leq \psi^1(v).$$

By (5.44), $f_{k_0+1}(t,x,v) \leq 1$ for all $(t,x,v) \in [-2,0] \times B_2 \times B_2$, so we can say by (5.45) that

$$\iiint_{Q_{\text{late}} \times B_2} \max(f_{k_0+1} - \psi^1, 0)^2 \, dv dx dt \leq \delta_0.$$

This is sufficent to apply Proposition 4.3.1 to f_{k_0+1} and conclude that $f_{k_0+1} \leq 1/2$ on $[-1,0] \times B_1 \times B_1$. Thus for the original f,

$$-1 \leq f \leq 1 - \frac{1}{2}{\theta_0}^{k_0+1} \leq 1 - \lambda \qquad \forall (t,x,v) \in [-1,0] \times B_1 \times B_1. \tag{5.46}$$

This proves the lemma's first claim.

We now know from (5.46), the definition of $\bar{f}$, and (5.42) that for all $t,x \in Q_{\text{ext}}$ and $|v| \leq \lambda^{-1}$

$$\begin{aligned}\bar{f}(t,x,v) &\leq \left(1+\frac{\lambda}{2}\right)[1-\lambda+\lambda/2] \leq 1,\\ \bar{f}(t,x,v) &\geq \left(1+\frac{\lambda}{2}\right)[-1+\lambda/2] \geq -1.\end{aligned}$$

For $t,x \in Q_{\text{ext}}$ and $|v| \geq \lambda^{-1}$, since $\lambda < \varepsilon_0$, we know by Lemma 4.2.3, property ((v)) that

$$2\psi_{\theta_0}(\lambda v)+2 \leq \psi_{\theta_0}(v).$$

Therefore

$$\begin{aligned}\left|\bar{f}(t,x,v)\right| &\leq \left(1+\frac{\lambda}{2}\right)[1+\lambda\psi_{\theta_0}(\lambda v)+\lambda/2]\\ &\leq \left(1+\frac{\lambda}{2}\right)\left[1+\frac{\lambda}{2}\psi_{\theta_0}(v)-\lambda+\lambda/2\right]\\ &\leq 1+\lambda\psi_{\theta_0}(v).\end{aligned}$$

This completes the proof.

□

Theorem 4.1.1 is proven by iteratively applying this Lemma 4.5.2 to an appropriately scaled function.

Chapter 5

SQG

5.1 Preliminaries

The surface quasigeostrophic equation (SQG) is a special case of the quasi-geostrophic system (QG) with uniform potential vorticity. The QG model is used extensively in meteorology and oceanography (e.g. Charney [Cha71]). These models are described in Pedlosky [Ped92]. The SQG model was popularized by Constantin, Majda and Tabak in [CMT94], due to its similarities with the Euler and Navier-Stokes equation. They proposed it as a toy model for the study of 3D Fluid equations (see also Held, Garner, Pierrehumbert, and Swanson [HPGS95]).

We consider in this chapter critical SQG on a bounded domain. We will focus on the following model, which was introduced by Constantin and Ignatova in [CI17] and [CI16]. Consider Ω a connected bounded domain in $\mathbb{R}^2$ with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and the Laplacian with homogeneous Dirichlet boundary conditions $-\Delta_D$. If $(\eta_k)_{k\in\mathbb{N}}$ is the sequence of L^2-normalized eigenfunctions of $-\Delta_D$ with corresponding eigenvalues λ_k listed in non-decreasing order, define

$$\Lambda f := \sum_{k=0}^{\infty} \sqrt{\lambda_k} \langle f, \eta_k \rangle_{L^2(\Omega)} \eta_k.$$

The critical SQG problem on Ω with initial data $\theta_0 \in L^2(\Omega)$ is

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0 & (0,T) \times \Omega, \\ u = \nabla^{\perp} \Lambda^{-1} \theta & [0,T] \times \Omega, \\ \theta = \theta_0 & \{0\} \times \Omega. \end{cases} \tag{1.1}$$

In the model, the dissipation $\Lambda = (-\Delta_D)^{1/2}$ is due to the Ekman pumping, while the nonlinear velocity u comes from the geostrophic and hydrostatic balance (see [Ped92]).

The main result of this chapter is the following:

Theorem 5.1.1. *There exists a universal constant $C_1 > 0$ such that the following holds:*

For any $\Omega \subseteq \mathbb{R}^2$ open and bounded with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, there exists for any $S > 0$ a constant $C_S > 0$ (depending also on Ω), and for any $k > 0$ a constant $\alpha_{k,S} \in (0,1)$ (depending also on Ω) so the following holds:

For any $\theta_0 \in L^2(\Omega)$ there exists a global-in-time weak solution $\theta \in L^\infty([0,\infty); L^2(\Omega)) \cap L^2([0,\infty); \mathcal{H}^{1/2})$ to (1.1) verifying $\theta(t,x) = 0$ on $(0,\infty) \times \partial\Omega$ and $\lim_{t\to 0} \theta(t,\cdot) = \theta_0$ in the L^2-weak sense. For $k \geq \|\theta_0\|_{L^2(\Omega)}$ and for every $S > 0$

$$\theta \in C^{\alpha_{k,S}}([S,\infty) \times \bar{\Omega})$$

where $\bar{\Omega}$ denotes the closure of Ω.

Moreover,

$$\|\theta\|_{L^\infty([S,\infty)\times\bar{\Omega})} \leq \frac{C_1}{S} \|\theta_0\|_{L^2(\Omega)}$$

and

$$\|\theta\|_{C^{\alpha_{k,S}}([S,\infty)\times\bar{\Omega})} \leq C_S \|\theta_0\|_{L^2(\Omega)}.$$

This model was first thoroughly studied in the cases without boundaries (either $\mathbb{R}^2$ or the torus $\mathbb{T}^2$). Global weak solutions were first constructed in Resnick [Res95]. Global regularity was first shown with small initial values by Constantin, Cordoba, and Wu [CCW01], or extra C^α regularity on the velocity in Constantin and Wu [CW08] and Dong and Pavlović [DP09]. In

[KNV07], Kiselev, Nazarov and Volberg showed the propagation of C^∞ regularity. The global C^∞ regularity for any L^2 initial values was first proved in [CV10b] (see also Kiselev and Nazarov [KN09] and Constantin and Vicol [CV12]).

In the presence of boundaries, there are several distinct ways to define SQG. This can be attributed to alternative generalizations of the fractional Laplacian. Kriventsov [Kri15] considered a two-phase problem which satisfies critical SQG only in part of the domain, and was able to prove Hölder regularity in the time-independent case. This problem, intended to model air currents over a region containing both land and water, contains a half-Laplacian and a Riesz transform defined, not spectrally, but in terms of extension. In [NV18b], the authors consider the Euler-Coriolis-Boussinesq model and derive the full 3D inviscid quasigeostrophic system in an impermeable cylinder (see also [NV19] for the construction of small time smooth solutions to the model). They obtain natural boundary conditions for SQG distinct from the homogeneous conditions introduced in [CI17], [CI16] and described above. However, due to the complexity of the model described in [NV18b], we focus in this chapter only on the homogenous case.

Existence of weak solutions for (1.1) is proven in [CI17], and local existence and uniqueness for strong solutions with sufficiently smooth initial data is proven by Constantin and Nguyen in [CN18b] (see also Constantin and Nguyen [CN18a] and Constantin, Ignatova, and Nguyen [CIN18] for the inviscid case). The interior regularity of solutions is proven in [CI16] (together with propagation of L^∞ bounds). The method of proof for interior regularity uses nonlinear maximum principles, introduced by Constantin and Vicol [CV12]. However, the bounds obtained in [CI16] blow up near the boundary and do not provide global regularity. In [CI16] Remark 1, questions about global regularity are suggested as open problems. Both the $C^\alpha(\bar{\Omega})$ regularity, and bootstrapping to $C^\infty(\bar{\Omega})$

regularity, are indentified as interesting problems. Our result answers the first question, by showing that solutions θ to (1.1) are globally Hölder continuous. Bootstrapping to C^∞ involves different techniques, and will be studied in a forthcoming work [SV].

Our proof is based on the De Giorgi method pioneered by De Giorgi in [DG57]. The method was applied to the SQG problem first in [CV10b]. The method is powerful for showing C^α regularity of elliptic- and parabolic-type equations. It has been applied in a variety of situations for non-local problems, such as the fractional heat equation in [CCV11a], the time-fractional case in [ACV16], the 3D Quasigeostrophic problem in [NV18a], or the kinetic setting by Imbert and Silvestre [IS16] or in [Sto18]. The method has also been applied in more exotic, non-elliptic situations such as Hamilton-Jacobi equations (see [CV17], [SV18]).

The De Giorgi method involves rescaling our equation by zooming in iteratively, and applying regularity results at each scale. Therefore it is important that certain results be proven independently of the domain Ω. The particular dependence on Ω will be made clear in each lemma of this chapter. As a general overview, in the proof of Theorem 5.1.1 we will apply the results of Sections 5.3 and 5.4 only on a single fixed domain, while the results of Sections 5.5 and 5.6 must be applied at each level of zoom with a different rescaled domain each time.

The first broad idea of our proof consists in decoupling the velocity u from θ to work on a linear equation, and prove alternating regularity results for θ and u independently. We can show that θ is in L^∞ without any assumption on u (see Section 5.3). Using that L^∞ bound, we will need to obtain scaling invariant controls on the drift $u = \nabla\Lambda^{-1}\theta$. By scaling invariant, we mean that the bound, once proven on Ω fixed, will remain true of the scaled function $u(\varepsilon\cdot, \varepsilon\cdot)$ for all ε. Unfortunately, although the Riesz transform is bounded from L^p to L^p for all p finite, it is not bounded for $p=\infty$. The usual technique, therefore, is to consider BMO (as in [CV10b] and [NV18a]), but in the case of bounded domains the Riesz transform is not known to be bounded in this space either. Our solution is to use extensions of the Littlewood-Paley theory to bounded domains.

The adaptation of Fourier analysis and Littlewood-Paley theory to Schrodinger operators is a well-studied subject (e.g. Zheng [Zhe06], Benedetto and Zheng [BZ10]). As an application of this theory, Iwabuchi, Matsuyama, and Taniguchi [IMT19], [IMT18], and Bui, Duong, and Yang [BDY12] have considered operators defined on open subsets of $\mathbb{R}^n$, which includes as a special case the operator $-\Delta_D$ (a Schrodinger operator with zero potential). In particular, in [IMT17], Iwabuchi, Matsuyama, and Taniguchi derive many important results, including the Bernstein inequalities, for Besov spaces adapted to the operator $-\Delta_D$ on bounded open subsets of $\mathbb{R}^n$ with smooth boundary. This theory turns out to greatly improve our understanding of the Riesz transform $\nabla\Lambda^{-1}$ on bounded domains.

Using the results of [IMT17], we will be able to show that the Riesz transform of an L^∞ function whose Fourier decomposition $f=\sum f_k\eta_k$ is supported on high frequencies $k>N$ will be bounded in the weak sobolev space $W^{-1/4,\infty}$, and the Riesz transform of an L^∞ function whose Fourier decomposition is supported on low frequencies $k<N$ will have bounded Lipschitz constant. The cutoff N for dividing high frequencies from low frequencies must depend however on the size of the domain Ω. In the case of $\mathbb{R}^2$, where ∇ and Λ^{-1} commute, this is equivalent to the observation that the Riesz transform is bounded from L^∞ to the Besov space $B^0_{\infty,\infty}$. In the case of bounded domains, the argument must be more subtle. We must decompose θ into its Littlewood-Paley projections, individually bound the Riesz transform of each projection in multiple spaces, and then recombine these infinitely-many functions into a low-frequency collection and a high-

frequency collection depending on the scale of oscillation we are trying to detect (see Section 5.4 and Lemma 5.5.1).

We make this notion precise with the following definition:

Definition 5.1.1 (Calibrated sequence)**.** Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set and $0 < T \in \mathbb{R}$. We call a function $u \in L^2([0,T] \times \Omega)$ **calibrated** if it can be decomposed as the sum of a calibrated sequence

$$u = \sum_{j \in \mathbb{Z}} u_j$$

with each $u_j \in L^2([0,T] \times \Omega)$ and the infinite sum converging in the sense of L^2.

We call a sequence $(u_j)_{j \in \mathbb{Z}}$ **calibrated** for a constant κ and a center N if each term of the sequence satisfies the following bounds.

$$\begin{aligned} \|u_j\|_{L^\infty([0,T]\times\Omega)} &\leq \kappa, \\ \|\nabla u_j\|_{L^\infty([0,T]\times\Omega)} &\leq 2^j 2^{-N} \kappa, \\ \left\|\Lambda^{-1/4} u_j\right\|_{L^\infty([0,T]\times\Omega)} &\leq 2^{-j/4} 2^{N/4} \kappa. \end{aligned}$$

In Section 5.7 we will show that a calibrated velocity remains calibrated at all scales (specifically, with fixed constant κ but a changing center N). Therefore we can consider, for any domain Ω and time T, the system of linear equations

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, & [-T,0] \times \Omega \\ \operatorname{div} u = 0 & [-T,0] \times \Omega. \end{cases} \tag{1.2}$$

In Section 5.3 we show that solutions to (1.1) with L^2 initial data exist and regularize instantly into L^∞, and in Section 5.4 we show that the Riesz transform of L^∞ data is calibrated.

Then in Sections 5.5 and 5.6 we will show that solutions to (1.2) with calibrated velocity have decreasing oscillation between scales. By iteratively applying this oscillation lemma and scaling our equation, we show in Section 5.7 that θ is Hölder continuous.

The low-freqency component of a calibrated velocity u will be uniformly Lipschitz, which means it is only bounded up to a constant. This is similar to the case of BMO velocity functions in [CV10b] and [NV18a], which by the John-Nirenberg inequality are also bounded up to a constant. As in these cases, we consider a moving reference frame, denoted $\Gamma:[0,T]\to\mathbb{R}^2$, in which our velocity is shifted by a constant, making the low-frequency component of u bounded. There are two differences between our implementation of this technique and the implementation in [CV10b] and [NV18a]: firstly, we subtract off the value of the low-frequency part of u at a point, rather than subtracting off the average of u on a ball. Secondly, rather than applying the standard De Giorgi argument to $\tilde{\theta}(t,x):=\theta(t,x+\Gamma(t))$, we must reformulate the De Giorgi argument to "follow" the path $\Gamma(t)$ explicitly. This is a purely notational difference, but it is necessary because otherwise Ω would be time-dependent.

At each scale, there will be a natural Lagrangian path Γ_ℓ corresponding to the low-frequency part of u. However, the low-frequency part of u changes non-trivially as we zoom, so Γ_ℓ will be different at each scale. Throughout Sections 5.5 and 5.6, we will use Γ_ℓ to denote the "current" Lagrangian path and Γ to denote the Lagrangian path at the previous scale. In the proof of Theorem 5.1.1 in Section 5.7, these are denoted $\Gamma_k(t)$ and $\varepsilon^{-1}\Gamma_{k-1}(\varepsilon t)$ respectively. In Lemmas 5.5.2, 5.5.3, 5.5.4 and 5.6.1, we will make assumptions about θ which are centered on $x\approx\Gamma(t)$ and obtain conclusions which are similarly centered on $x\approx\Gamma(t)$, conditioned on $\gamma:=\Gamma_\ell-\Gamma$ being small in Lipschitz norm. Finally in Lemma 5.6.2, we will show that, given bounds on θ for $x\approx\Gamma(t)$, we can bound θ for $x \approx \Gamma_\ell(t)$ for t sufficiently small, again conditioned on $\gamma := \Gamma_\ell - \Gamma$ being small in Lipschitz norm. Controlling γ amounts to controlling the change in Γ_ℓ between consecutive scales, which is much easier to obtain than scale-independent bounds on Γ_ℓ.

Previous applications of the De Giorgi method to non-local equations such as (1.2) generally make extensive use of either an extension representation (c.f. [CV10b]) or a singular integral representation (c.f. [NV18a]). In this chapter, we use the singular integral representation for the Dirichlet fractional Laplacian derived by Caffarelli and Stinga [CS16]. It is based on the results of Stinga and Torrea [ST10] which generalize the extension representation of Caffarelli and Silvestre [CS07]. This theory is pivotal in translating the existing non-local De Giorgi techniques to the problem at hand (see Section 5.2).

In order to apply De Giorgi's method to weak solutions of (1.2), we will need to assume a certain a priori estimate which holds, in particular, for $L^2(H^1_0)$ weak solutions. However, such solutions are only known to exist for short time and for H^2 initial data, as shown by Constantin and Nguyen in [CN18b]. We call weak solutions in $L^2(\mathcal{H}^{1/2})$ which happen to verify this a priori estimate "suitable solutions," by analogy to suitable solutions to Navier-Stokes as in [CKN82]. We give the formal definition of suitable solutions in Section 5.3, where we also construct global-in-time suitable solutions using the vanishing viscosity method. Compared to [CI17], our solutions verify a full family of localized energy inequalities which allow us to apply the De Giorgi method.

The chapter is organized as follows. Section 5.2 is dedicated to basic properties of the operator Λ and the corresponding Sobolev spaces $\mathcal{H}^s$. In Section 5.3 we construct weak solutions which verify the suitability conditions. In Section 5.4 we prove that the Riesz transform of the L^∞ function θ is callibrated. Section 5.5 contains the De Giorgi Lemmas. Section 5.6 is dedicated to the local decrease in oscillation through an analog of the Harnack inequality. Finally in Section 5.7 we prove the main theorem, Theorem 5.1.1. In the Appendix 0.2 we prove a few technical lemmas which are needed in the main chapter.

Notation. Throughout the chapter, we will use the following notations. By η_k and λ_k we mean the eigenfunctions and eigenvalues of $-\Delta_D$, with $\lambda_0 \leq \lambda_1 \leq \dots$ and $\|\eta_k\|_2 = 1$ for all k. If $f = \sum_k f_k \eta_k$ then

$$\begin{aligned} \|f\|_{\mathcal{H}^s} &:= \left(\sum_k \lambda_k^s f_k^2 \right)^{1/2} \\ &= \int |\Lambda^s f|^2. \end{aligned}$$

We suppress the dependence on Ω, though in fact Λ, λ_k, and η_k are defined in terms of the domain Ω. The relevant domain will be clear from context. The norm on $\mathcal{H}^s$ is in fact a norm, not a seminorm, since $\|f\|_{L^2(\Omega)} \leq \lambda_0^{-s/2} \|f\|_{\mathcal{H}^s}$.

For a set A and a function $f: A \to \mathbb{R}$, denote

$$\begin{aligned} [f]_{\alpha;A} &:= \sup_{x,y \in A, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, && \alpha \in (0,1], \\ \|f\|_{C^\alpha(A)} &:= \|f\|_{L^\infty(A)} + [f]_{\alpha;A}, && \alpha \in (0,1], \\ \|f\|_{C^{k,\alpha}(A)} &:= \sum_{n=0}^{k} \|D^n f\|_{L^\infty(A)} + \left[D^k f\right]_{\alpha;A}, && \alpha \in (0,1], k \in \mathbb{N}. \end{aligned}$$

When the domain A is ommited, the relevant spatial domain Ω is implied.

Throughout this chapter, if an integral sign is written $\int$ without a specified domain, the domain is implied to be Ω, with Ω defined in context.

For any vector $v = (v_1, v_2)$, by $v^\perp$ we mean $(-v_2, v_1)$. By $\nabla^\perp$ we mean $(-\partial_y, \partial_x)$.

In the remainder of this chapter, the differential operator D^2 refers to the Hessian in space, excluding time derivatives. By abuse of notation, if $\Gamma:[a,b]\to\mathbb{R}^2$, we write $[a,b]\times B_R(\Gamma)$ to denote $\{(t,x)\in[a,b]\times\mathbb{R}^2:|x-\Gamma(t)|\le R\}$.

5.2 Properties of the Fractional Dirichlet Laplacian

In this section we will investigate the basic properties of the operator Λ and the space $\mathcal{H}^s$ on a general domain Ω.

We begin by stating a result of [CS16] which gives us a singular integral representation of the $\mathcal{H}^s$ norm.

Proposition 5.2.1 (Caffarelli-Stinga Representation). *Let $s\in(0,1)$ and $f,g\in\mathcal{H}^s$ on a bounded $C^{2,\beta}$ domain $\Omega\subseteq\mathbb{R}^2$. Then*

$$\int_\Omega \Lambda^s f\Lambda^s g\,dx = \iint_{\Omega^2}[f(x)-f(y)][g(x)-g(y)]K_{2s}(x,y)\,dxdy+\int_\Omega f(x)g(x)B_{2s}(x)\,dx \tag{2.3}$$

for kernels K_{2s} and B_{2s} which depend on the parameter s and the domain Ω.

There exists a constant $C=C(s)$ independent of Ω such that

$$0\le K_{2s}(x,y)\le\frac{C(s)}{|x-y|^{2+2s}}$$

for all $x\ne y\in\Omega$ and

$$0\le B_{2s}(x)$$

for all $x\in\Omega$.

Moreover, for any $s,t\in(0,2)$ there exists a constant $c=c(s,t,\Omega)$ such that for all $x\ne y\in\Omega$

$$K_t(x,y)\le c|x-y|^{s-t}K_s(x,y). \tag{2.4}$$

Proof. See [CS16] Theorems 2.3 and 2.4.

Theorem 2.4 in [CS16] does not explicitly state the result (2.4). However, it does state that for each kernel K_s there exists a constant c_s dependent on s and Ω such that

$$\frac{1}{c_s}|x-y|^{2+s}K_s(x,y) \leq \min\left(1, \frac{\eta_0(x)\eta_0(y)}{|x-y|^2}\right) \leq c_s|x-y|^{2+s}K_s(x,y).$$

Since the middle term does not depend on s, we can say that

$$|x-y|^{2+t}K_t(x,y) \leq c_t c_s |x-y|^{2+s}K_s(x,y)$$

from which (2.4) follows. □

From the explicit formulae given in [CS16], we see that K_{2s} is approximately equal to the standard kernel for the $\mathbb{R}^2$ fractional Laplacian $(-\Delta)^s$ when both x and y are in the interior of Ω or when x and y are extremely close together, but decays to zero when one point is in the interior and the other is near the boundary. The kernel B_{2s} is well-behaved in the interior but has a singularity at the boundary $\partial\Omega$. This justifies our thinking of the K_{2s} term as the interior term and B_{2s} as a boundary term.

When comparing the computations in this chapter to corresponding computations on $\mathbb{R}^2$, one finds that the interior term behaves nearly the same as in the unbounded case, while the boundary term behaves roughly like a lower order term (in the sense that it is easily localized).

Many useful results can be derived from Caffarelli-Stinga representation formula. We summarize them in the following lemma.

Lemma 5.2.2. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$.*

(a) *Let* $s \in (0,1)$*. If* f *and* g *are non-negative functions with disjoint support (i.e.* $f(x)g(x) = 0$ *for all* x*), then*

$$\int \Lambda^s f \Lambda^s g \, dx \leq 0.$$

(b) *Let* $s \in (0,1)$*. If* $g \in C^{0,1}(\Omega)$ *then for some constant* $C = C(s)$ *independent of* Ω

$$\|fg\|_{\mathcal{H}^s} \leq 2\|g\|_\infty \|f\|_{\mathcal{H}^s} + C\|f\|_2 \sup_y \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}} dx.$$

(c) *Let* $s \in (0,1)$*. If* $g \in C^{0,1}(\Omega)$ *then for some constant* $C = C(s)$ *independent of* Ω

$$\|fg\|_{\mathcal{H}^s} \leq C\|g\|_{C^{0,1}(\Omega)} (\|f\|_2 + \|f\|_{\mathcal{H}^s}).$$

(d) *Let* $s \in (0,1/2)$*. Let* g *an* $L^\infty(\Omega)$ *function and* $f \in \mathcal{H}^{2s}$ *be non-negative with compact support. Let* C_{dmn} *be a constant such that*

$$K_s(x,y) \leq C_{dmn}|x-y|^{3s} K_{4s}(x,y). \tag{2.5}$$

Then there exists a constant C *depending only on* s *and* C_{dmn} *such that*

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} (\|f\|_2 + \|f\|_{\mathcal{H}^{2s}}).$$

(e) *Let* g *an* $L^\infty(\Omega)$ *function and* $f \in \mathcal{H}^{1/2}$ *be non-negative with compact support. Let* C_{dmn} *be a constant such that*

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4} K_1(x,y).$$

Then there exists a constant C *depending only on* C_{dmn} *such that*

$$\int g \Lambda^{1/4} f \leq C\|g\|_\infty |\mathrm{supp}(f)|^{1/2} (\|f\|_2 + \|f\|_{\mathcal{H}^{1/2}}).$$

Proof. We prove these corollaries one at a time.

Proof of ((a))**:** From Proposition 5.2.1

$$\int \Lambda^s f \Lambda^s g\, dx = \iint [f(x)-f(y)][g(x)-g(y)]K(x,y)\, dx dy + \int f(x)g(x)B(x)\, dx.$$

Since f and g are non-negative and disjoint, the B term vanishes. Moreover, the product inside the K term becomes

$$[f(x)-f(y)][g(x)-g(y)] = -f(x)g(y) - f(y)g(x) \le 0.$$

Since K is non-negative, the result follows.

Proof of ((b))**:** From Proposition 5.2.1

$$\begin{aligned}\int |\Lambda^s(fg)|^2 &= \iint (g(x)[f(x)-f(y)] + f(y)[g(x)-g(x)])^2 K + \int f^2 g^2 B \\ &\le 2\|g\|_\infty^2 \|f\|_{\mathcal{H}^s}^2 + C(s)\int f(y)^2 \int \frac{|g(x)-g(y)|^2}{|x-y|^{2+2s}}\, dx dy.\end{aligned}$$

Proof of ((c))**:** This follows immediately from ((b)), since

$$|g(x)-g(y)| \le (\|g\|_\infty) \wedge (\|\nabla g\|_\infty |x-y|)$$

and

$$\int \frac{1 \wedge |x-y|^2}{|x-y|^{2+2s}}\, dx$$

is bounded uniformly in y.

Proof of ((d))**:** From Proposition 5.2.1 we can decompose

$$\int \Lambda^{s/2} g \Lambda^{s/2} f = I_< + I_\ge + II$$

where

$$
\begin{aligned}
I_< &:= \iint_{|x-y|<1} [g(x)-g(y)][f(x)-f(y)]K_s, \\
I_\geq &:= \iint_{|x-y|\geq 1} [g(x)-g(y)][f(x)-f(y)]K_s, \\
II &:= \int fgB_s.
\end{aligned}
$$

First we estimate $I_<$. From (2.5) and from the symmetry of the integrand and the fact that $[f(x)-f(y)]$ vanishes unless at least one of $f(x)$ or $f(y)$ is non-zero,

$$|I_<| \leq 2 \iint_{|x-y|<1} \chi_{\{f>0\}}(x)\,|g(x)-g(y)| \cdot |f(x)-f(y)| \cdot |x-y|^{3s} K_{4s}.$$

We can break this up by Hölder's inequality

$$|I_<| \leq 2 \left(\iint_{|x-y|<1} \chi_{\{f>0\}}(x)[g(x)-g(y)]^2 |x-y|^{6s} K_{4s} \right)^{1/2} \left(\iint [f(x)-f(y)]^2 K_{4s} \right)^{1/2}.$$

The kernel $|x-y|^{6s}K_{4s}\chi_{\{|x-y|<1\}}$ is integrable in y for x fixed. Therefore

$$|I_<| \leq 2 \left((2\,\|g\|_\infty)^2 \int C\chi_{\{f>0\}}(x)\,dx \right)^{1/2} \left(\|f\|^2_{\mathcal{H}^{2s}} \right)^{1/2}. \tag{2.6}$$

For the term $I_\geq$, by the symmetry of the integrand we have

$$|I_\geq| \leq 2\,\|g\|_\infty 2 \int |f(x)| \int_{|x-y|\geq 1} K_s(x,y)\,dy\,dx.$$

Since $K_s\chi_{\{|x-y|\geq 1\}}$ is integrable in y for x fixed,

$$|I_\geq| \leq C\,\|g\|_\infty \|f\|_1. \tag{2.7}$$

For the boundary term II,

$$|II| \leq \|g\|_\infty \int \chi_{\{f>0\}} f B_s.$$

Since $f \geq 0$, $[f(x)-f(y)][\chi_{\{f>0\}}(x)-\chi_{\{f>0\}}(y)] \geq 0$. Therefore

$$\int \chi_{\{f>0\}} f B_s \leq \int \Lambda^{s/2} \chi_{\{f>0\}} \Lambda^{s/2} f = \int \chi_{\{f>0\}} \Lambda^s f.$$

Applying Hölder's inequality, we arrive at

$$|II| \leq \|g\|_\infty |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s}.$$

This combined with (2.6) and (2.7) gives us

$$\int \Lambda^{s/2} g \Lambda^{s/2} f \leq C \|g\|_\infty \left(|\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^{2s}} + \|f\|_1 + |\mathrm{supp}(f)|^{1/2} \|f\|_{\mathcal{H}^s} \right).$$

The lemma follows since $\|f\|_1 \leq |\mathrm{supp}(f)|^{1/2} \|f\|_2$ and since $\|f\|_{\mathcal{H}^s} \leq \|f\|_{L^2} + \|f\|_{\mathcal{H}^{2s}}$.

Proof of ((e))**:** This is an immediate application of part ((d)).

$\square$

Let us consider the relationship between the norm $\mathcal{H}^s$ and the H^s norm on $\mathbb{R}^2$.

It is known (see [CI16] and [CS16]) that for $s \in (0,1)$ the spaces $\mathcal{H}^s$ are equivalent to certain subsets of $H^s(\Omega)$ spaces defined in terms of the Gagliardo semi-norm. In particular, we know that smooth functions with compact support are dense in $\mathcal{H}^s$ for $s \in [0,1]$ and that elements of $\mathcal{H}^s$ have trace zero for $s \in [\frac{1}{2}, 1]$.

The most important fact for us is that the fractional Sobolev norms defined in terms of extension are dominated by our $\mathcal{H}^s$ norm with a constant that is independent of Ω.

We do not claim that this result is new, but we present a detailed proof because the result is crucial to the De Giorgi method. The De Giorgi lemmas require Sobolev embeddings and Rellich-Kondrachov embeddings which are independent of scale.

Define the extension-by-zero operator $E: L^2(\Omega) \to L^2(\mathbb{R}^2)$

$$Ef(x) = \begin{cases} f(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \setminus \Omega. \end{cases}$$

Proposition 5.2.3. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. For any $s \in [0,1]$ and function $f \in \mathcal{H}^s$,*

$$\int_{\mathbb{R}^2} \left| (-\Delta)^{s/2} Ef \right|^2 \leq \int_{\Omega} |\Lambda^s f|^2.$$

Here $(-\Delta)^s$ is defined in the fourier sense.

We will prove this proposition by interpolating between $s=0$ and $s=1$. Before we can do this, we must prove the same in the $s=1$ case. This result is known (see e.g. Jerison and Kenig [JK95]) but we include the proof for completeness.

Lemma 5.2.4. *Let $\Omega \subseteq \mathbb{R}^2$ be any bounded open set with Lipschitz boundary. For all functions f in $\mathcal{H}^1$,*

$$\int_{\Omega} |\nabla f|^2 = \int_{\Omega} |\Lambda f|^2.$$

Proof. Let η_i and η_j be two eigenfunctions of the Dirichlet Laplacian on Ω. Note that these functions are smooth in the interior of Ω and vanish at the boundary, so we can apply the divergence theorem and find

$$\int \nabla \eta_i \cdot \nabla \eta_j = -\int \eta_i \Delta \eta_j = \lambda_j \int \eta_i \eta_j = \lambda_j \delta_{i=j}.$$

Consider a function $f = \sum f_k \eta_k$ which is an element of $\mathcal{H}^1$, by which we mean $\sum \lambda_k f_k^2 < \infty$. Since $\|\nabla \eta_k\|_{L^2(\Omega)} = \sqrt{\lambda_k}$, the following sums all converge in $L^2(\Omega)$ and hence the calculation is justified:

$$\int |\nabla f|^2 = \int \left(\sum_i f_i \nabla \eta_i \right) \left(\sum_j f_j \nabla \eta_j \right)$$

$$= \int \sum_{i,j} (f_i f_j) \nabla \eta_i \cdot \nabla \eta_j$$

$$= \sum_{i,j} (f_i f_j) \int \nabla \eta_i \cdot \nabla \eta_j$$

$$= \sum_j \lambda_j f_j^2.$$

From this the result follows. $\square$

We come now to the proof of Proposition 5.2.3. The proof is by complex interpolations using the Hadamard three-lines theorem.

Proof. Let g be any Schwartz function in $L^2(\mathbb{R}^2)$, and let f be a function in $\mathcal{H}^s$. Define the function

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{z/2} g E \Lambda^{s-z} f, \qquad z \in \mathbb{C}, \mathrm{Re}(z) \in [0,1].$$

Recall (see e.g. [JK95]) that when $t \in \mathbb{R}$, $(-\Delta)^{it}$ is a unitary transformation on $L^2(\mathbb{R}^2)$, and Λ^{it} is a unitary transformation on $L^2(\Omega)$.

When $\mathrm{Re}(z) = 0$, then $\left\|(-\Delta)^{z/2} g\right\|_2 = \|g\|_2$ and $\|\Lambda^{s-z} f\|_2 = \|f\|_{\mathcal{H}^s}$. Hence

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 0.$$

When $\mathrm{Re}(z) = 1$, integrate by parts to obtain

$$\Phi(z) = \int_{\mathbb{R}^2} (-\Delta)^{(z-1)/2} g (-\Delta)^{1/2} E \Lambda^{s-z} f.$$

Then $\left\|(-\Delta)^{(z-1)/2} g\right\|_2 = \|g\|_2$, while $\|\Lambda^{s-z} f\|_{\mathcal{H}^1} = \|f\|_{\mathcal{H}^s}$. As an $\mathcal{H}^1$ function, $\Lambda^{s-z} f$ has trace zero so

$$\left\|\nabla E \Lambda^{s-z} f\right\|_{L^2(\mathbb{R}^2)} = \left\|\nabla \Lambda^{s-z} f\right\|_{L^2(\Omega)} = \|f\|_{\mathcal{H}^s}.$$

Of course $\left\|(-\Delta)^{1/2}\cdot\right\|_{L^2(\mathbb{R}^2)} = \|\nabla\cdot\|_{L^2(\mathbb{R}^2)}$ in general so

$$\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}, \qquad \mathrm{Re}(z) = 1.$$

In order to apply the Hadamard three-lines theorem, we must show that Φ is differentiable in the interior of its domain.

Rewrite the integrand of Φ as

$$\mathcal{F}^{-1}(|\xi|^z \hat{g})\, E \sum_k \lambda_k^{\frac{s-z}{2}} f_k.$$

The derivative $\frac{d}{dz}$ commutes with linear operators like $\mathcal{F}^{-1}$ and E, so the derivative is

$$\mathcal{F}^{-1}(\ln(|\xi|)|\xi|^z \hat{g})\, E \sum_k \lambda_k^{\frac{s-z}{2}} f_k + \mathcal{F}^{-1}(|\xi|^z \hat{g})\, E \sum_k \frac{-1}{2} \ln(\lambda_k) \lambda_k^{\frac{s-z}{2}} f_k. \tag{2.8}$$

Fix some $z \in \mathbb{C}$ with $\mathrm{Re}(z) \in (0,1)$. Since g is a Schwartz function, $\ln(|\xi|)|\xi|^z \hat{g}$ is in L^2. Moreover, for any $\varepsilon > 0$ we have $\ln(\lambda_k)\lambda_k^{\frac{s-z}{2}} \le C \lambda_k^{\frac{s-z+\varepsilon}{2}}$ for some C independent of k but dependent on z, ε. Take $\varepsilon < \mathrm{Re}(z)$ and, since $f \in \mathcal{H}^s$, this sum will converge in L^2.

The differentiated integrand (2.8) is therefore a sum of two products of L^2 functions. In particular it is integrable, which means we can interchange the integral sign and the derivative $\frac{d}{dz}$ and prove that $\Phi'(z)$ is finite for all $0 < \mathrm{Re}(z) < 1$.

By the Hadamard three-lines theorem, for any $z \in (0,1)$ we have $\Phi(z) \le \|g\|_2 \|f\|_{\mathcal{H}^s}$. Evaluating $\Phi(s)$, we see

$$\int_{\mathbb{R}^2} (-\Delta)^{s/2} g E f \le \|g\|_{L^2(\mathbb{R}^2)} \|f\|_{\mathcal{H}^s}.$$

This inequality holds for any Schwartz function $g \in L^2(\mathbb{R}^n)$ and any $f \in \mathcal{H}^s$.

Since Schwartz functions are dense in $L^2(\mathbb{R}^2)$ and $(-\Delta)^{s/2}$ is self-adoint, the proof is complete.

$\square$

5.3 Existence of suitable solutions

In this section, we define the needed notion of suitable solutions. This involves two families of localized energy inequalities. The first family (3.11) concerns the time evolution of $\int_\Omega (\theta-\Psi)_+^2$ for generic cutoff functions Ψ. We need also to control the time derivative $\partial_t(\theta-\Psi)_+^2$ in the sense of distributions for the second De Giorgi lemma (see Proposition 5.5.4, step 2). This control comes in the family of inequalities (3.12).

It is important that the universal constant C^* appearing in the suitability conditions (3.11) and (3.12) is independent of Ω. The De Giorgi argument requires that we apply the same bound iteratively as we rescale the solution, so our bounds must be scale independent. For this reason, we will define the constant through Proposition 5.3.1 before stating the definition of suitable solutions. As with the Navier-Stokes equations, it is not obvious that weak solutions constructed directly from the Galerkin scheme are suitable. Therefore we will construct our weak solutions as vanishing viscosity limits of

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \Lambda\theta = \varepsilon\Delta\theta & (0,\infty)\times\Omega, \\ u = \nabla^\perp\Lambda^{-1}\theta & [0,\infty)\times\Omega, \\ \theta=\theta_0 & \{0\}\times\Omega. \end{cases} \tag{3.9}$$

The construction of solutions to (3.9) will follow the Galerkin method (as in [CI17]).

We begin by defining the universal constant C^* and simultaneously showing that the inequalities (3.11) and (3.12) are valid for sufficiently smooth solutions to the linear equation

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = \varepsilon \Delta \theta, \\ \operatorname{div} u = 0, \end{cases} \tag{3.10}$$

uniformly with respect to $\varepsilon \in [0,1]$. This smoothness requirement will be shown to be valid when $\varepsilon > 0$.

Proposition 5.3.1 (Energy Inequalities)**.** *There exists a universal constant C^* such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be bounded and open with $C^{2,\beta}$ boundary, $\beta \in (0,1)$, and let $0 < T < \infty$ a time, and let $\varepsilon \in [0,1]$. Let θ, u be a solution to (3.10) *on $\Omega \times [0,T]$, with $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^4(0,T;L^4(\Omega))$.*

Then for any smooth non-negative function $\Psi \in C^\infty([0,\infty) \times \mathbb{R}^2)$ satisfying $\|\nabla \Psi\|_{L^\infty([0,\infty)\times\mathbb{R}^2)} \leq k$ and the Hölder seminorm $\sup_{[0,\infty)} [\Psi(t,\cdot)]_{1/4;\mathbb{R}^2} \leq k$ for some constant k, any time $S \in (0,T)$, and any smooth non-negative $\varphi \in C_c^\infty(S,T;C^\infty(\Omega))$, the function $\theta_+ := (\theta - \Psi)_+$ satisfies

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \leq C^* \left(k^2 \int \chi_{\{\theta \geq \Psi\}} + \left| \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) \right| \right) \qquad \forall t \in [0,T] \tag{3.11}$$

in the sense of distributions and

$$\begin{aligned} \frac{-1}{2}\int_S^T \int \theta_+^2 \partial_t \varphi \leq \frac{1}{2}\int_S^T \int \theta_+^2 u \cdot \nabla \varphi - \int_S^T \int \varphi \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) + C^* \|\varphi\|_{C^0(S,T;C^2)} \bigg(\left(1 + \frac{1}{S}\right) \int_0^T \int \theta_+^2 \\ + k^2 \int_0^T \int \chi_{\{\theta \geq \Psi\}} + \int_0^T \left| \int \theta_+ (\partial_t \Psi + u \cdot \nabla \Psi) \right| \bigg). \end{aligned} \tag{3.12}$$

Remark 5.3.1. Note that since C^* is universal, Proposition 5.3.1 does not depend on the values of $\|\theta\|_{L^\infty(L^2)}$, $\|\theta\|_{L^2(H_0^1)}$, $\|u\|_{L^\infty(L^2)}$, or $\|u\|_{L^4(L^4)}$, but only on the fact that these quantities are finite.

Therefore, using the natural scaling of (3.10), if (θ,u) verify the assumptions of Proposition 5.3.1 on $[0,T]\times\Omega$, then so does $\big(\lambda\theta(\mu\cdot,\mu\cdot),u(\mu\cdot,\mu\cdot)\big)$ on $[0,\mu^{-1}T]\times\mu^{-1}\Omega$, for any $\lambda\in\mathbb{R}$ and $\mu>0$ such that $\mu^{-1}\varepsilon\in[0,1]$. Therefore the proposition applies also to these scaled functions, with the same universal constant C^*.

Proof of Proposition 5.3.1. Since $\theta_+\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, we can multiply (3.10) by θ_+ and integrate in space to obtain

$$0=\int\theta_+[\partial_t+u\cdot\nabla+\Lambda-\varepsilon\Delta](\theta_++\Psi-\theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. We analyze them one at a time.

Firstly,

$$\begin{aligned}\int\theta_+[\partial_t+u\cdot\nabla+\Lambda-\varepsilon\Delta]\theta_+&=\left(\frac{1}{2}\right)\frac{d}{dt}\int\theta_+^2+\left(\frac{1}{2}\right)\int\operatorname{div}u\,\theta_+^2+\int\left|\Lambda^{1/2}\theta_+\right|^2+\varepsilon\int|\nabla\theta_+|^2\\&=\left(\frac{1}{2}\right)\frac{d}{dt}\int\theta_+^2+\int\left|\Lambda^{1/2}\theta_+\right|^2+\varepsilon\int|\nabla\theta_+|^2.\end{aligned}$$

The Ψ term produces important error terms:

$$\begin{aligned}\int\theta_+[\partial_t+u\cdot\nabla+\Lambda-\varepsilon\Delta]\Psi&=\int\theta_+\partial_t\Psi+\int\theta_+u\cdot\nabla\Psi+\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi+\varepsilon\nabla\theta_+\cdot\nabla\Psi\\&=\int\theta_+(\partial_t\Psi+u\cdot\nabla\Psi)+\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi+\varepsilon\nabla\theta_+\cdot\nabla\Psi.\end{aligned}$$

Since θ_+ and θ_- have disjoint support, the θ_- term is nonnegative by Lemma 5.2.2 part ((a)):

$$\int\theta_+[\partial_t+u\cdot\nabla+\Lambda]\theta_-=\left(\frac{1}{2}\right)\int\theta_+\partial_t\theta_-+\int\theta_+u\cdot\nabla\theta_-+\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_-+\varepsilon\int\nabla\theta_+\nabla\theta_-\leq 0.$$

Put together, we arrive at

$$\left(\frac{1}{2}\right)\frac{d}{dt}\int\theta_+^2+\int\left|\Lambda^{1/2}\theta_+\right|^2+\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi\leq-\int\theta_+(\partial_t\Psi+u\cdot\nabla\Psi)-\varepsilon\left[\int\nabla\theta_+\cdot\nabla\Psi+\int|\nabla\theta_+|^2\right]. \tag{3.13}$$

The ε term is bounded, using the fact that $\nabla\theta_+=\chi_{\{\theta_+>0\}}\nabla\theta_+$ and $\varepsilon\in[0,1]$, by

$$\begin{aligned} -\varepsilon\left[\int|\nabla\theta_+|^2+\int\nabla\theta_+\cdot\nabla\Psi\right] &\leq\frac{-\varepsilon}{2}\int|\nabla\theta_+|^2+\frac{\varepsilon}{2}\int|\nabla\Psi|^2\chi^2_{\{\theta_+>0\}}\\ &\leq\frac{k^2}{2}\int\chi_{\{\theta_+>0\}}. \end{aligned} \tag{3.14}$$

At this point we break down the $\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi$ term using the formula from Proposition 5.2.1.

$$\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi=\iint[\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y)+\int\theta_+\Psi B.$$

Since $B\geq 0$ and Ψ is non-negative by assumption, the B term is non-negative and so

$$\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi\geq\iint[\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y). \tag{3.15}$$

The remaining integral is symmetric in x and y, and the integrand is only nonzero if at least one of $\theta_+(x)$ and $\theta_+(y)$ is nonzero. Hence

$$\left|\iint[\theta_+(x)-\theta_+(y)][\Psi(x)-\Psi(y)]K(x,y)\right|\leq 2\iint\chi_{\{\theta_+>0\}}(x)\,|\theta_+(x)-\theta_+(y)|\cdot|\Psi(x)-\Psi(y)|\,K(x,y).$$

Now we can break up this integral using Young's inequality, and since $\iint[\theta_+(x)-\theta_+(y)]^2K\leq\|\theta_+\|^2_{\mathcal{H}^{1/2}}$ the inequality (3.15) becomes

$$\int\Lambda^{1/2}\theta_+\Lambda^{1/2}\Psi+\frac{1}{2}\int\left|\Lambda^{1/2}\theta_+\right|^2\geq-2\iint\chi_{\{\theta_+>0\}}(x)[\Psi(x)-\Psi(y)]^2K(x,y). \tag{3.16}$$

It remains to bound the quantity $[\Psi(x)-\Psi(y)]^2K(x,y)$. By Proposition 5.2.1, there is a universal constant C such that

$$K(x,y)\leq\frac{C}{|x-y|^3}.$$

The cutoff Ψ is locally Lipschitz, and Hölder continuous with exponent $1/4$, by assumption. Therefore

$$[\Psi(x)-\Psi(y)]^2K(x,y) \le Ck^2|x-y|^{-1}\wedge|x-y|^{-2.5}.$$

Since $1<2<2.5$, this quantity is integrable. Thus

$$\int \chi_{\{\theta_+>0\}}(x)\int[\Psi(x)-\Psi(y)]^2K(x,y)\,dydx \le Ck^2\int \chi_{\{\theta_+>0\}}\,dx.$$

Combining this with (3.13), (3.16), and (3.14) we obtain (3.11).

We begin now the proof of (3.12). Since $\theta_+\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$, by interpolation we can further conclude $\theta_+,u\in L^4(0,T;L^4(\Omega))$. Therefore we can multiply (3.10) by $\varphi\theta_+$ and integrate in space to obtain

$$0=\int \varphi\theta_+[\partial_t+u\cdot\nabla+\Lambda](\theta_++\Psi-\theta_-)$$

which decomposes into three terms, corresponding to θ_+, Ψ, and θ_-. After rearranging and integrating by parts, this becomes

$$\frac{1}{2}\int\varphi\partial_t\theta_+^2=\frac{1}{2}\int\theta_+^2u\cdot\nabla\varphi-\int\varphi\theta_+(\partial_t\Psi+u\cdot\nabla\Psi)-\int\varphi\theta_+\Lambda\theta_+-\int\varphi\theta_+\Lambda\Psi+\varepsilon\int\varphi\theta_+\Delta(\theta_++\Psi). \tag{3.17}$$

The ε term decomposes as

$$\begin{aligned}\varepsilon\int\varphi\theta_+\Delta(\theta_++\Psi)&=-\varepsilon\int\varphi\nabla\theta_+\cdot\nabla(\theta_++\Psi)-\varepsilon\int\theta_+\nabla\varphi\cdot\nabla(\theta_++\Psi)\\&=-\varepsilon\int\varphi|\nabla\theta_+|^2-\varepsilon\int\varphi\nabla\theta_+\cdot\nabla\Psi+\frac{\varepsilon}{2}\int\theta_+^2\Delta\varphi-\varepsilon\int\theta_+\nabla\varphi\cdot\nabla\Psi\\&\le\frac{\varepsilon}{2}\int\varphi|\nabla\Psi|^2\chi_{\{\theta_+>0\}}+\frac{\varepsilon}{2}\int\theta_+^2\Delta\varphi+\frac{\varepsilon}{2}\int\theta_+^2|\nabla\varphi|+\frac{\varepsilon}{2}\int\chi_{\{\theta_+>0\}}|\nabla\varphi||\nabla\Psi|^2\\&\le k^2\|\varphi\|_{C^1}\int\chi_{\{\theta_+>0\}}+\|\varphi\|_{C^2}\int\theta_+^2.\end{aligned} \tag{3.18}$$

The $\int \varphi\theta_+\Lambda\theta_+$ term is bounded by Lemma 5.2.2 part ((c))

$$-\int \varphi\theta_+\Lambda\theta_+ \le C\,\|\varphi\|_{C^1}\left(\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2\right) \tag{3.19}$$

and the $\int \varphi\theta_+\Lambda\Psi$ term is bounded, just as for the $\int \theta_+\Lambda\Psi$ term in the previous family of inequalities but with the addition of Lemma 5.2.2 part ((c)),

$$\begin{aligned}
-\int \varphi\theta_+\Lambda\Psi &\le \iint [\varphi(x)\theta_+(x)-\varphi(y)\theta_+(y)][\Psi(x)-\Psi(y)]K_1\\
&\le 2\iint \chi_{\{\theta_+>0\}}|\varphi(x)\theta_+(x)-\varphi(y)\theta_+(y)|\cdot|\Psi(x)-\Psi(y)|K_1\\
&= 2\iint \left(\|\varphi\|_{C^1}^{-1/2}|\varphi(x)\theta_+(x)-\varphi(y)\theta_+(y)|\right)\left(\|\varphi\|_{C^1}^{1/2}\chi_{\{\theta_+>0\}}|\Psi(x)-\Psi(y)|\right)K_1\\
&\le \|\varphi\|_{C^1}^{-1}\|\varphi\theta_+\|_{\mathcal{H}^{1/2}}^2 + \|\varphi\|_{C^1}\iint \chi_{\{\theta_+>0\}}[\Psi(x)-\Psi(y)]^2K_1\\
&\le C\,\|\varphi\|_{C^1}\left(\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2\right) + Ck^2\,\|\varphi\|_{C^1}\left(\int_{\mathbb{R}^2}|y|^{-1}\wedge|y|^{-2.5}\,dy\right)\int \chi_{\{\theta_+>0\}}.
\end{aligned} \tag{3.20}$$

From the inequality (3.11) already proven, we can obtain by a standard argument

$$\int_S^T\int \left|\Lambda^{1/2}\theta_+\right|^2 \le \frac{1}{S}\int_0^T\int \theta_+^2 + k^2\int_0^T\int \chi_{\{\theta_+>0\}} + \int_0^T\left|\int \theta_+\left(\partial_t\Psi + u\cdot\nabla\Psi\right)\right|. \tag{3.21}$$

By combining (3.17) with (3.18), (3.19), and (3.20) we obtain

$$\frac{1}{2}\int \varphi\partial_t\theta_+^2 = \frac{1}{2}\int \theta_+^2 u\cdot\nabla\varphi - \int \varphi\theta_+\left(\partial_t\Psi + u\cdot\nabla\Psi\right) + C\,\|\varphi\|_{C^2}\left(\int \theta_+^2 + k^2\int \chi_{\{\theta_+>0\}} + \int \left|\Lambda^{1/2}\theta_+\right|^2\right).$$

Integrating this inequality from S to T and applying (3.21), we obtain (3.12). □

We will construct global-in-time solutions to (1.2) (equivalently (3.10) with $\varepsilon = 0$) for any initial value $\theta_0 \in L^2$ which verify these energy inequalities (3.11) and (3.12) with the same universal constant C^* at all scales, but which may not be in $L^2(H_0^1)$.

Definition 5.3.1. A pair θ, u is called a **suitable solution** to (1.2) on $[0,\infty)\times\Omega$ if $\Omega\subseteq\mathbb{R}^2$ open and bounded, $\theta, u\in L^\infty(\mathbb{R}_+;L^2(\Omega))$, $\theta\in L^2(\mathbb{R}_+;\mathcal{H}^{1/2}(\Omega))$, $u\in L^3(\mathbb{R}_+;L^3(\Omega))$ and θ, u is a suitable solution on $[0,T]\times\Omega$ for all $0<T<\infty$.

A pair θ, u is called a **suitable solution** to (1.2) on a space time domain $[0,T]\times\Omega$ if $T<\infty$, $\Omega\subseteq\mathbb{R}^2$ open and bounded, $\theta, u\in L^\infty(0,T;L^2(\Omega))$, $\theta\in L^2(0,T;\mathcal{H}^{1/2}(\Omega))$, $u\in L^3(0,T;L^3(\Omega))$ and

1. θ, u solve (1.2) in the sense of distributions on $[0,T]\times\Omega$,

2. θ, u satisfy (3.11) and (3.12) at all scales with the same universal constant C^* defined in Proposition 5.3.1. More specifically, the following holds:

   Let $\lambda\in\mathbb{R}$ and $\mu\in(0,1)$ be given and let $\Psi\in C^\infty([0,\infty)\times\mathbb{R}^2)$ be any smooth non-negative function satisfying $\|\nabla\Psi\|_{L^\infty([0,\infty)\times\mathbb{R}^2)}\leq k$ and $\sup_{[0,\infty)}[\Psi(t,\cdot)]_{1/4;\mathbb{R}^2}\leq k$ for some constant k.

   Define $\tilde{\Omega}:=\{x\in\mathbb{R}^2:\mu x\in\Omega\}$, $\tilde{T}:=\mu^{-1}T$, $\tilde{\theta}_+(t,x):=(\lambda\theta(\mu t,\mu x)-\Psi(t,x))_+$, and $\tilde{u}(t,x):=u(\mu t,\mu x)$. Let $S\in(0,\tilde{T})$ and let $\varphi\in C_c^\infty(S,T;C^\infty(\tilde{\Omega}))$ be non-negative.

   Then $\tilde{\theta}_+$ and $\tilde{u}$ and φ and Ψ satisfy (3.11) and (3.12) on $\tilde{\Omega}$ with times 0, S and $\tilde{T}$, with the same universal constant C^*.

The rest of this section is dedicated to the proof of the following proposition:

Proposition 5.3.2 (Existence of global suitable solutions)**.** *There exists a universal constant $C>0$ such that the following holds:*

Given an open, bounded domain $\Omega\subseteq\mathbb{R}^2$ with $C^{2,\beta}$ boundary, $\beta\in(0,1)$, and initial data $\theta_0\in L^2(\Omega)$, there exists a global-in-time weak solution θ to (1.1) such that, for any $0<T<\infty$, θ and $u:=\nabla^\perp\Lambda^{-1}\theta$ are a suitable solution to (1.2) on $[0,T]\times\Omega$.

Moreover, $\theta \in L^\infty([0,\infty);L^2(\Omega)) \cap L^2([0,\infty);\mathcal{H}^{1/2}(\Omega))$, *and* $\theta(t,\cdot) \to \theta_0(\cdot)$ *weakly in* $L^2(\Omega)$ *as* $t \to 0$, *and for any* $S > 0$

$$\|\theta\|_{L^\infty([S,\infty)\times\Omega)} \leq \frac{C}{S} \|\theta_0\|_{L^2(\Omega)}.$$

To construct global suitable solutions, we will use the vanishing viscosity method. First, we must prove existence of global weak solutions to (3.9).

Lemma 5.3.3 (Existence for viscous equation)**.** *There exists a universal constant C such that the following holds:*

Given an open, bounded domain $\Omega \subseteq \mathbb{R}^2$, initial data $\theta_0 \in L^2(\Omega)$ and a constant $\varepsilon > 0$, there exists a global-in-time weak solution θ to (3.9)*.*

In particular, $\theta \in C^0([0,\infty);L^2(\Omega)) \cap L^2([0,\infty);H^1_0(\Omega))$, $u \in C^0([0,\infty);L^2(\Omega)) \cap L^4([0,\infty)\times\Omega)$, *and* $\partial_t \in L^2([0,\infty);H^{-1}(\Omega))$, *and* $\theta(t,\cdot) \to \theta_0(\cdot)$ *weakly in* $L^2(\Omega)$ *as* $t \to 0$, *and for any* $S > 0$

$$\|\theta\|_{L^\infty([S,\infty)\times\Omega)} \leq \frac{C}{S} \|\theta_0\|_{L^2(\Omega)}.$$

The proof of existence is by Galerkin's method, while the L^∞ bound uses a De Giorgi argument.

Proof. Recall that η_j are the eigenfunctions of $-\Delta_D$. Let N be an integer parameter, and $W_N := \text{span}(\eta_0,\ldots,\eta_N)$, which consists only of smooth functions which vanish on $\partial\Omega$. We seek first a solution $\theta_N \in W_N$ to the weak equation

$$\int \varphi \partial_t \theta_N + \int \varphi \nabla^\perp \Lambda^{-1} \theta_N \cdot \nabla \theta_N + \int \varphi \Lambda \theta_N + \varepsilon \int \nabla \theta_N \nabla \varphi = 0, \qquad \forall t \in \mathbb{R}_{\geq 0}, \varphi \in W_N. \tag{3.22}$$

If we write

$$\theta_N(t,x) := \sum_{i=0}^N \alpha_{i,N}(t) \eta_i(x)$$

and choose $\varphi = \eta_i$ as a test function, then θ_N solves (3.22) if and only if, for all $i \le N$,

$$\alpha'_{i,N}(t) + \sum_{j=0}^{N}\sum_{k=0}^{N} \alpha_{j,N}(t)\alpha_{k,N}(t) B_{ijk} + \lambda_i^{1/2}\alpha_{i,N}(t) + \varepsilon\lambda_i\alpha_{i,N}(t) = 0$$

with

$$B_{ijk} = \lambda_j^{-1/2} \int \eta_i \nabla^{\perp}\eta_j \cdot \nabla\eta_k$$

a constant tensor.

By Peano's existence theorem for ODEs, solutions to this system exist on some interval $[0,T]$ where T depends on Ω and N and (since W_N is finite dimensional and all norms are equivalent) the L^2 norm of the initial data.

Since $\theta_N \in W_N$ we can take θ_N as a test function and obtain, for any solution θ_N to (3.22),

$$\frac{d}{dt}\int \theta_N^2 + \int \left|\Lambda^{1/2}\theta_N\right|^2 + \varepsilon \int |\nabla\theta_N|^2 = 0.$$

Therefore in particular $\|\theta_N\|_{L^2(\Omega)}$ is non-increasing in time and we conclude that θ_N exists for all time. Moreover, θ_N is uniformly bounded in $L^\infty(L^2(\Omega))$ and $L^2(H_0^1(\Omega))$.

To take a limit in N, we need uniform regularity in time. From (3.22) we can bound

$$\int_0^\infty \int \partial_t\theta_N\varphi \le \|\theta_N\|_{L^4(L^4)}^2 \|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(L^2)}\|\varphi\|_{L^2(H_0^1)} + \|\theta_N\|_{L^2(H_0^1)}\|\varphi\|_{L^2(H_0^1)}.$$

Note that θ_N is uniformly bounded in $L^4(L^4)$ by interpolation and $L^2(L^2)$ by Poincaré's inequality. Therefore $\iint \varphi\partial_t\theta_N \le C\|\varphi\|_{L^2(H_0^1)}$ for all $\varphi \in L^2(W_N)$ for a constant C independent of N. Since $\partial_t\theta_N \in W_N$, this is sufficient to show that $\|\partial_t\theta_N\|_{L^2(H^{-1})}$ is uniformly bounded.

By Aubin-Lions, we conclude that θ_N is a compact sequence in $L^2([0,\infty)\times\Omega)$ and so it has an L^2 limit θ. This limit θ is in $L^\infty(L^2(\Omega))$ and $L^2(H_0^1(\Omega))$ and $\partial_t\theta \in L^2(H^{-1}(\Omega))$.

We must prove that θ is a weak solution to (3.9). Let $\varphi \in C_c^\infty(W_M)$ for some M. For $N \geq M$,

$$-\iint \theta_N \partial_t \varphi - \iint \theta_N \nabla^\perp \Lambda^{-1} \theta_N \cdot \nabla \varphi + \iint \theta_N \Lambda \varphi - \varepsilon \iint \theta_N \Delta \varphi = 0.$$

This expression is continuous for $\theta_N \in L^2(L^2)$, so by taking $N \to \infty$ we obtain

$$-\int \theta \partial_t \varphi - \int \theta \nabla^\perp \Lambda^{-1} \theta \cdot \nabla \varphi + \int \theta \Lambda \varphi + \varepsilon \int \nabla \theta \cdot \nabla \varphi = 0$$

for any $\varphi \in C_c^\infty(W_M)$ for any $M \in \mathbb{N}$. By density, θ solves (3.9) in the sense of distributions.

Since $\partial_t \theta_N$ is uniformly bounded in $L^2(H^{-1})$, we know $\theta_N(t, \cdot) \to \theta_0$ weakly in L^2 uniformly in N and so the same holds for θ.

Lastly, for any constant $a \geq 0$, the function $(\theta - a)_+$ satisfies

$$\begin{aligned}
\frac{d}{dt} \int \frac{1}{2} (\theta - a)_+^2 + \int \left| \Lambda^{1/2} (\theta - a)_+ \right|^2 &= \frac{-1}{2} \int u \cdot \nabla (\theta - a)_+^2 - \int a \Lambda (\theta - a)_+ - \varepsilon \int |\nabla (\theta - a)_+|^2 \\
&= -\int a (\theta - a)_+ B_1 - \varepsilon \int |\nabla (\theta - a)_+|^2 \\
&\leq 0.
\end{aligned}$$

This inequality is scaling-invariant, so the same holds for $\lambda \theta(\mu t, \mu x)$ for any $\lambda, \mu > 0$.

By the standard De Giorgi argument (see Lemma 0.2.1 in the Appendix for details), there exists a universal constant δ such that $\int_0^2 \int (\lambda \theta(\mu t, \mu x)_+^2 \, dx dt \leq \delta$ implies $\theta \leq \lambda^{-1}$ on $[\mu 1, \mu 2]$. In fact, by comparison with a constant super-solution, $\theta \leq \lambda^{-1}$ on $[\mu, \infty)$. Taking $\lambda = \sqrt{\frac{\delta}{2\mu^{-2} \|\theta_0\|_{L^2(\Omega)}^2}}$, we find $\theta(t, \cdot) \leq C t^{-1} \|\theta_0\|_{L^2(\Omega)}$ for a universal constant C. Applying the same argument to $-\theta$ gives the L^∞ bound. □

Now that we have global existence of solutions to (3.9) for $\varepsilon > 0$, we can prove Proposition 5.3.2 by taking a limit as $\varepsilon \to 0$.

Proof of Proposition 5.3.2. For any parameter $\varepsilon > 0$, define $\theta_\varepsilon \in L^2(H_0^1)$ the weak solution to (3.9) constructed in Lemma 5.3.3. The θ_ε are uniformly bounded in $L^\infty(L^2)$ and $L^2(\mathcal{H}^{1/2})$ by the standard energy argument, so by interpolation they are also uniformly bounded in $L^4(L^{8/3})$. Recall that θ_ε are uniformly bounded in $L^\infty(L^\infty)$ after any positive time.

For any smooth φ, we have

$$\int_0^\infty \int \partial_t \theta_\varepsilon \varphi \leq \|\theta_\varepsilon\|_{L^4(L^{8/3})}^2 \|\varphi\|_{L^2(W^{1,4})} + \|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})} \|\varphi\|_{L^2(\mathcal{H}^{1/2})} + \varepsilon \|\theta_\varepsilon\|_{L^2(\mathcal{H}^{1/2})} \|\varphi\|_{L^2(\mathcal{H}^{3/2})}.$$

Therefore $\partial_t \theta_\varepsilon$ is uniformly bounded in $L^2(\mathcal{H}^{-3/2})$. By Aubin-Lions, the sequence θ_ε, up to a subsequence, has a strong limit in $L^2(L^2)$. Call this limit θ.

Since $\partial_t \theta_\varepsilon$ is uniformly bounded and $\theta(t,\cdot) \to \theta_0$ weakly in L^2, the same holds for θ.

Define $u_\varepsilon := \nabla^\perp \Lambda^{-1} \theta_\varepsilon$, and by continuity of the Riesz transform we have $u_\varepsilon \to u$ in $L^2([0,\infty) \times \Omega)$ where $u := \nabla^\perp \Lambda^{-1} \theta$.

It remains only to prove that θ and u satisfy the energy inequalities (3.11) and (3.12). Recall that θ_ε and u_ε satisfy (3.11) and (3.12) by Proposition 5.3.1, so we need only show that these inequalities hold also in the limit. The details of this calculation are given below.

Let $0 < T < \infty$ be a constant, and let λ, μ, Ψ, S and φ be as in the definition of suitable solutions. Define

$$\begin{aligned} \tilde{\theta}_\varepsilon(t,x) &:= \lambda \theta_\varepsilon(\mu t, \mu x), & \tilde{\theta}(t,x) &= \lambda \theta(\mu t, \mu x), \\ \tilde{u}_\varepsilon(t,x) &:= u_\varepsilon(\mu t, \mu x), & \tilde{u}(t,x) &:= u(\mu t, \mu x), \\ \tilde{\theta}_{\varepsilon,+}(t,x) &:= \left(\tilde{\theta}_\varepsilon(t,x) - \Psi(t,x)\right)_+, & \tilde{\theta}_+(t,x) &:= \left(\tilde{\theta}(t,x) - \Psi(t,x)\right)_+, \end{aligned}$$

and let $\tilde{\Omega} := \{x \in \mathbb{R}^2 : \mu x \in \Omega\}$, $\tilde{T} := \mu^{-1} T$, $\tilde{\varepsilon} := \mu^{-1} \varepsilon$.

Note that $\tilde{\theta}_\varepsilon$ and $\tilde{u}_\varepsilon$ are weak solutions to (3.10) with viscosity $\tilde{\varepsilon}$. Therefore, if $\varepsilon \le \mu$ then $\tilde{\theta}_{\varepsilon,+}$ and $\tilde{u}_\varepsilon$ satisfy (3.11). The terms $\int \tilde{\theta}_{\varepsilon,+}(\partial_t \Psi + \tilde{u}_\varepsilon \cdot \nabla \Psi)$ and $\int \chi_{\{\tilde{\theta}_\varepsilon \ge 0\}}$ are continuous under L^2 limits, and the quantities $\frac{d}{dt}\int \tilde{\theta}_{\varepsilon,+}^2$ and $\int \left|\Lambda^{1/2}\tilde{\theta}_{\varepsilon,+}\right|^2$ are lower-semicontinuous under L^2 limits, so we conclude that $\tilde{\theta}_+$ and $\tilde{u}$ satisfy (3.11).

Similarly, $\tilde{\theta}_{\varepsilon,+}$ and $\tilde{u}_\varepsilon$ satisfy (3.12) if $\varepsilon \le \mu$. On $[S,\tilde{T}]$ we have a uniform $L^\infty(L^\infty)$ bound for $\tilde{\theta}_\varepsilon$. Therefore $\tilde{\theta}_{\varepsilon,+}$ converges in $L^3(L^3)$, and so $\int_S^{\tilde{T}} \int \tilde{\theta}_{\varepsilon,+}^2 \tilde{u}_\varepsilon \cdot \nabla \varphi$ is conserved in the limit $\varepsilon \to 0$. The remaining terms in (3.12) are $L^2(L^2)$ continuous, so $\tilde{\theta}_+$ and $\tilde{u}$ satisfy (3.12). □

5.4 Littlewood-Paley Theory

In this section we will prove that, because θ is uniformly bounded in L^∞, the velocity $u = \nabla^\perp \Lambda^{-1}\theta$ is calibrated (see Definition 5.1.1). The proof will utilize a Littlewood-Paley theory adapted to a bounded set Ω.

Because the Littlewood-Paley theory depends in an essential way on the domain Ω, any results proven in this way will also be domain-dependent. Therefore, in the proof of Hölder continuity in Section 5.7, we will apply the following Proposition only to the unscaled function θ on the unscaled domain Ω. As we zoom in, the velocity will remain calibrated, so there will be no further need for this result.

Proposition 5.4.1. *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $\theta \in L^\infty(\Omega)$. Then there exists an integer $j_0 = j_0(\Omega)$ and a sequence of divergence-free functions $(u_j)_{j \ge j_0}$ calibrated for some constant $\kappa = \kappa(\Omega, \|\theta\|_\infty)$ with center 0 (see Definition 5.1.1) such that*

$$\nabla^\perp \Lambda^{-1}\theta = \sum_{j \ge j_0} u_j$$

with the infinite sum converging in the sense of L^2.

Before we can prove this, we define the Littlewood-Paley projections and prove some of their properties:

Let ϕ be a Schwartz function on $\mathbb{R}$ which is suited to Littlewood-Paley decomposition. Specifically, ϕ is non-negative, supported on [1/2,2], and has the property that

$$\sum_{j\in\mathbb{Z}}\phi(2^j\xi)=1 \qquad \forall\xi\neq 0.$$

For any $f=\sum f_k\eta_k$ in $L^2(\Omega)$, we define the Littlewood-Paley projections

$$P_jf:=\sum_k \phi(2^j\lambda_k^{1/2})f_k\eta_k.$$

Note that P_j depends strongly on the domain Ω.

Recall that $-\Delta_D$ has some smallest eigenvalue λ_0 (depending on Ω) so if we define $j_0=\log_2(\lambda_0)-1$ then $P_j=0$ for all $j<j_0$.

The Bernstein Inequalities adapted for a bounded domain are proved in [IMT17]. We restate their result here:

Lemma 5.4.2 (Bernstein Inequalities)**.** *Let $1\leq p\leq\infty$ and $\Omega\subset\mathbb{R}^2$ a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$, and let $(P_j)_{j\in\mathbb{Z}}$ be the Littlewood-Paley decomposition defined above.*

There exists a constant C depending on p and Ω such that the following hold for any $f\in L^p(\Omega)$:

For any $\alpha\in\mathbb{R}$ and $j\in\mathbb{Z}$,

$$\|\Lambda^\alpha P_jf\|_{L^p(\Omega)}\leq C2^{\alpha j}\|f\|_{L^p(\Omega)}.$$

For any $\alpha \in \mathbb{R}$ and $j \geq j_0$

$$\|\nabla \Lambda^\alpha P_j f\|_{L^p(\Omega)} \leq C 2^{(1+\alpha)j} \|f\|_{L^p(\Omega)}.$$

Proof. The first claim is Lemma 3.5 in [IMT17]. It is also an immediate corollary of [IMT18] Theorem 1.1.

The second claim is similar to Lemma 3.6 in [IMT17]. A hypothesis of Lemma 3.6 is that

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \leq \frac{C}{\sqrt{t}} \qquad 0 < t \leq 1$$

(a property of Ω). The result of Lemma 3.6 only covers the case $j > 0$.

In [FMP04] it is proved that that if Ω is $C^{2,\beta}$ then

$$\left\|\nabla e^{-t\Delta_D}\right\|_{L^\infty \to L^\infty} \leq \frac{C}{\sqrt{t}} \qquad 0 < t \leq T$$

which, by taking some T depending on j_0, is enough to prove the desired result for $j \geq j_0$ by a trivial modification of the proof in [IMT17]. $\square$

The following lemma is a simple but crucial result which can be thought of as describing the commutator of the gradient operator and the projection operators. In the case of $\mathbb{R}^2$, the Littlewood-Paley projections commute with the gradient so $P_i \nabla P_j = 0$ unless $|i-j| \leq 1$. On a bounded domain, this is not the case; the gradient does not maintain localization in frequency-space. However, the following lemma formalizes the observation that $P_i \nabla P_j \approx 0$ when $i << j$.

Lemma 5.4.3. *Let $1 \leq p \leq \infty$. There exists a constant C depending on p and Ω such that or any function $f \in L^p(\Omega)$,*

$$\|P_i \nabla P_j f\|_p \leq C \min(2^j, 2^i) \|f\|_p.$$

Proof. Let q be the Hölder conjugate of p and g be an L^q function. Then since P_i is self-adjoint

$$\int gP_i\nabla P_j f = \int (P_i g)\nabla P_j f \leq C2^j \|g\|_q \|f\|_p$$

by Lemma 5.4.2.

Further integrating by parts,

$$\int gP_i\nabla P_j f = -\int (\nabla P_i g)P_j f \leq C2^i \|g\|_q \|f\|_p .$$

This also follows from Lemma 5.4.2.

The result follows. □

We are now ready to prove Proposition 5.4.1.

Proof of 5.4.1. For each integer $j \geq j_0$, we define u_j to be the $\frac{\pi}{2}$-rotation of the Riesz transform of the j^{th} Littlewood-Paley projection of θ:

$$u_j := \nabla^{\perp}\Lambda^{-1}P_j\theta.$$

Qualitatively, we know that $\theta \in L^2$ and hence $u_j \in L^2$. In fact, $u = \sum u_j$ in the L^2 sense.

We must bound u_j, $\Lambda^{-1/4}u_j$, and ∇u_j all in $L^\infty(\Omega)$.

By straightforward application of Lemma 5.4.2,

$$\|u_j\|_\infty \leq C\|\theta\|_\infty . \tag{4.23}$$

Since $u_j \in L^2$, we know that

$$\Lambda^{-1/4}u_j = \sum_{i\in\mathbb{Z}} P_i\Lambda^{-1/4}u_j.$$

Define $\bar{P}_k := P_{k-1} + P_k + P_{k+1}$. Then $\bar{P}_k P_k = P_k$, and since the projections P_k are spectral operators, they commute with Λ^s and each other. We therefore rewrite

$$\left(P_i \Lambda^{-1/4} u_j\right)^{\perp} = \left(\Lambda^{-1/4} \bar{P}_i\right)\left(P_i \nabla P_j\right)\left(\Lambda^{-1} \bar{P}_j\right)\theta.$$

On the right hand side we have three bounded linear operators applied sequentially to $\theta \in L^\infty$. The first operator has norm $C2^{-j}(2^1 + 2^0 + 2^{-1})$ by Lemma 5.4.2. The second operator has norm $C\min(2^j, 2^i)$ by Lemma 5.4.3. The third operator has norm $C2^{-i/4}(2^{1/4} + 2^0 + 2^{-1/4})$ by Lemma 5.4.2. Therefore

$$\left\|P_i \Lambda^{-1/4} u_j\right\|_\infty \leq C2^{-i/4} \min(2^j, 2^i) 2^{-j} \|\theta\|_\infty.$$

Summing these bounds on the projections of $\Lambda^{-1/4} u_j$, and noting that

$$\sum_{i \in \mathbb{Z}} 2^{-j} 2^{-i/4} \min(2^j, 2^i) = 2^{-j} \sum_{i \leq j} 2^{i3/4} + \sum_{i > j} 2^{-i/4} \leq C2^{-j/4},$$

we obtain

$$\left\|\Lambda^{-1/4} u_j\right\|_\infty \leq C2^{-j/4} \|\theta\|_\infty. \tag{4.24}$$

Lastly, we must show that ∇u_j is in L^∞. Equivalently, we will show that $\Lambda^{-1} P_j \theta$ is $C^{1,1}$. The method of proof is Schauder theory.

For convenience, define

$$F := \Lambda^{-1} P_j \theta.$$

Notice that F is a linear combination of Dirichlet eigenfunctions, so in particular it is smooth and vanishes at the boundary. Therefore

$$-\Delta F = \Lambda^2 F = \Lambda P_j \theta.$$

We apply the standard Schauder estimate from Gilbarg and Trudinger [GT01] Theorem 6.6 to bound some $C^{2,\alpha}$ semi-norm of F by the L^∞ norm of F and the C^α norm of its Laplacian. By assumption there exists $\beta \in (0,1)$ such that Ω is $C^{2,\beta}$, and for this β we have by the Schauder estimate

$$\left[D^2 F\right]_\beta \leq C \left\|\Lambda^{-1} P_j \theta\right\|_\infty + C \left\|\Lambda P_j \theta\right\|_\infty + C \left[\Lambda P_j \theta\right]_\beta. \tag{4.25}$$

By Lemma 5.4.2,

$$\left\|\Lambda^{-1} P_j \theta\right\|_\infty \leq C 2^{-j} \left\|\theta\right\|_\infty,$$
$$\left\|\Lambda P_j \theta\right\|_\infty \leq C 2^{j} \left\|\theta\right\|_\infty,$$
$$\left\|\nabla \Lambda P_j \theta\right\|_\infty \leq C 2^{2j} \left\|\theta\right\|_\infty.$$

By Lemma 0.2.2 (see Appendix 0.2) we can interpolate these last two bounds to obtain

$$\left[\Lambda P_j \theta\right]_\beta \leq C 2^{j(1+\beta)} \left\|\theta\right\|_\infty.$$

Plugging these estimates into (4.25) yields

$$\left[D^2 F\right]_\beta \leq C \left(2^{-j} + 2^j + 2^{j(1+\beta)}\right) \left\|\theta\right\|_\infty.$$

Recall that without loss of generality we can assume $j \geq j_0$. Therefore up to a constant depending on j_0, the term $2^{j(1+\beta)}$ bounds 2^j and 2^{-j} so we can write

$$\left[D^2 F\right]_\beta \leq C 2^{j(1+\beta)} \left\|\theta\right\|_\infty.$$

Using this estimate and the fact that $\left\|\nabla F\right\|_\infty = \left\|\nabla \Lambda^{-1} P_j \theta\right\|_\infty \leq C \left\|\theta\right\|_\infty$ (see (4.23)), we can interpolate to obtain an L^∞ bound on $D^2 F$. Lemma 0.2.3 states that since $F \in C^{2,\beta}$ and Ω is sufficiently regular, there exist a constant $\ell = \ell(\Omega)$ such that for any $\delta \in [0, \ell]$ we have

$$\left\|D^2 F\right\|_\infty \leq C \left(\delta^{-1} \left\|\nabla F\right\|_\infty + \delta^\beta \left[D^2 F\right]_\beta\right)$$

$$\leq C\left(\delta^{-1}+\delta^{\beta}2^{j(1+\beta)}\right)\|\theta\|_{\infty}.$$

Set $\delta=2^{-j}(2^{j_0}\ell)\leq\ell$. Then

$$\left\|D^2F\right\|_{\infty}\leq C\left(2^j+2^{-j\beta}2^{j(1+\beta)}\right)\|\theta\|_{\infty}=C(\Omega)2^j\|\theta\|_{\infty}.$$

Since $D^2F=\nabla u_j$, this estimate together with (4.23) and (4.24) complete the proof. $\square$

5.5 De Giorgi Estimates

Our goal in this section is to prove De Giorgi's first and second lemmas for suitable solutions to (1.2) with u uniformly calibrated. The De Giorgi lemmas will eventually be applied iteratively to various rescalings of the solution θ, so the following results must be independent of the size of the domain Ω. Any properties we do assume for the domain, such as the regularity of the boundary, must be scaling invariant.

Rather than working directly with the calibrated sequence, we will decompose u into just two terms, a low-pass term and a high-pass term. The construction is described in the following lemma. Note that we make no assumption on the center of calibration, which means this result is indendent of scale.

Lemma 5.5.1. *Let*

$$u=\sum_{j_0}^{\infty}u_j$$

with the sum converging in the L^2 sense. Assume that $(u_j)_{j\in\mathbb{Z}}$ is a calibrated sequence with constant κ and some center, and that $\operatorname{div}(u_j)=0$ for all j.

Then

$$u = u_\ell + u_h$$

with

$$\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)} \le 2\kappa,$$

$$\left\|\Lambda^{-1/4} u_h\right\|_{L^\infty([-T,0]\times\Omega)} \le 6\kappa.$$

and $\operatorname{div}(u_\ell) = \operatorname{div}(u_h) = 0$.

We call u_ℓ the low-pass term, and u_h the high-pass term.

Proof. Let N be the center to which $(u_j)_{j\in\mathbb{Z}}$ is calibrated.

We define

$$u_h = \sum_{j=N+1}^{\infty} u_j$$

and bound

$$\left\|\Lambda^{-1/4} u_h\right\|_\infty \le \sum_{j>N} \left\|\Lambda^{-1/4} u_j\right\|_\infty \le \kappa \frac{2^{-1/4}}{1-2^{-1/4}}.$$

We define

$$u_\ell = \sum_{j=j_0}^{N} u_j$$

and bound

$$\|\nabla u_\ell\|_\infty \le \sum_{j\le N} \|\nabla u_j\|_\infty \le \kappa \frac{1}{1-2^{-1}}.$$

□

In order to prove the De Giorgi lemmas, we must derive an energy inequality for the function $(\theta-\Psi)_+$ where $\Psi(t,x)$ grows sublinearly in $|x|$. Considering the suitability condition (3.11), we see that control can only be gained if the quantity $\partial_t\Psi+u\cdot\nabla\Psi$ is bounded. This requires a barrier function which is moving in space along a Lagrangian path Γ_ℓ of u_ℓ.

To that end, we shall consider, for any domain Ω and time T, functions $\theta:[-T,0]\times\Omega\to\mathbb{R}$, L^2 functions u_ℓ and $u_h:[-T,0]\times\Omega\to\mathbb{R}^2$, and a Lipschitz path $\Gamma_\ell:[-T,0]\to\Omega$ which satisfy

$$\begin{cases}\theta,(u_\ell+u_h) \text{ suitable solution to (1.2)} & \text{on } [-T,0]\times\Omega,\\ \operatorname{div}(u_\ell)=\operatorname{div}(u_h)=0 & \text{on } [-T,0]\times\Omega,\\ \dot\Gamma_\ell(t)=u_\ell(t,\Gamma_\ell(t)) & \text{on } [-T,0].\end{cases} \tag{5.26}$$

Because Γ_ℓ depends on u_ℓ which depends on N, the path Γ_ℓ will change significantly between scales. In particular, though $\Gamma_\ell\in\operatorname{Lip}([-T,0];\mathbb{R}^2)$, we cannot assume any uniform bound on it Lipschitz constant. We can bound, however, the difference between Γ_ℓ at consecutive scales. Therefore we must consider in the following lemmas an arbitrary Lipschitz path Γ, which was produced at a previous scale, and denote $\gamma:=\Gamma_\ell-\Gamma$ which will be uniformly bounded.

Now we prove an energy inequality for solutions to (5.26). Though this lemma is independent of the size of the domain, it depends on the geometry of the domain in a way encoded by the constant C_{dmn}. We will later show that this constraint on Ω is scaling invariant.

Lemma 5.5.2 (Energy inequality)**.** *Let κ, C_{dmn}, C_{pth}, T, and R be positive constants, and let $\psi:\mathbb{R}^2\to\mathbb{R}$ be a function such that $\|\nabla\psi\|_\infty$, $\|D^2\psi\|_\infty$, and $\sup_t[\psi(t,\cdot)]_{1/4}$ are all finite. Then there exists a constant $C>0$ such that the following holds:*

Let $\Omega\subseteq\mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$, and let $\Gamma:[-T,0]\to$

$\mathbb{R}^2$ *be Lipschitz. Assume that on* Ω *the functions* $K_{1/4}$ *and* K_1 *(defined in* (2.3)*) satisfy the relation*

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4}K_1(x,y) \qquad \forall x \neq y \in \Omega.$$

Let θ, u_ℓ, u_h, *and* Γ_ℓ *solve* (5.26) *on* $[-T,0]\times\Omega$, *and satisfy* $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-T,0]\times\Omega)} \leq 6\kappa$ *and* $\|\nabla u_\ell\|_{L^\infty([-T,0]\times\Omega)} \leq 2\kappa$. *Denote* $\gamma := \Gamma_\ell - \Gamma$ *and assume* $\|\dot\gamma\|_{L^\infty([-T,0])} \leq C_{pth}$ *and* $\gamma(0)=0$.

Consider the functions

$$\theta_+ := (\theta - \psi(\cdot - \Gamma))_+, \qquad \theta_- := (\psi(\cdot - \Gamma) - \theta)_+.$$

If θ_+ *is supported on* $x \in \Omega \cap B_R(\Gamma(t))$ *then* θ_+ *and* θ_- *satisfy the inequality*

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 - \int \Lambda^{1/2}\theta_+\Lambda^{1/2}\theta_- \leq C\left(\int \chi_{\{\theta\geq\psi\}} + \int \theta_+ + \int \theta_+^2\right).$$

Proof. Define

$$\Psi(t,x) := \psi(x - \Gamma(t))$$

so that

$$\partial_t\Psi + (u_\ell + u_h)\cdot\nabla\Psi = (u_\ell - \dot\Gamma + u_h)\cdot\nabla\psi(x-\Gamma(t)).$$

Applying (3.11) we arrive at

$$\frac{d}{dt}\int \theta_+^2 + \int \left|\Lambda^{1/2}\theta_+\right|^2 \leq C\left(\int \chi_{\{\theta\geq\psi\}} + \left|\int \theta_+(u_\ell - \dot\Gamma(t) + u_h)\cdot\nabla\psi(x-\Gamma(t))\right|\right). \tag{5.27}$$

Consider first the high-pass term $\int \theta_+ u_h \cdot \nabla\psi$. This term is equal to $\int \Lambda^{1/4}(\theta_+\nabla\Psi)\Lambda^{-1/4}u_h$, as can be calculated by first decomposing $\theta_+\nabla\Psi$ and u_h as sums of eigenfunctions. The operations on these infinite sums are justified because $\theta_+\nabla\Psi$, $\Lambda^{1/4}(\theta_+\nabla\Psi)$, u_h, and $\Lambda^{-1/4}u_h$ are all in L^2. Therefore we can apply Lemma 5.2.2 parts ((e)) and ((c)) to obtain

$$\int \Lambda^{-1/4}u_h\Lambda^{1/4}(\theta_+\nabla\psi) \leq C\left\|\Lambda^{-1/4}u_h\right\|_\infty \left(\|\nabla\psi\|_\infty + \left\|D^2\psi\right\|_\infty\right)|\mathrm{supp}(\theta_+)|^{1/2}\left(\|\theta_+\|_{L^2} + \|\theta_+\|_{\mathcal{H}^{1/2}}\right).$$

We apply Young's inequality to find that for any constant $\varepsilon > 0$ there exists $C = C(\psi, \kappa, C_{dmn}, \varepsilon)$ such that

$$\int u_h\theta_+\nabla\psi(x-\Gamma(t))\,dx \leq C\left(|\mathrm{supp}(\theta_+)| + \int \theta_+^2\right) + \varepsilon\int\left|\Lambda^{1/2}\theta_+\right|^2. \tag{5.28}$$

Consider now the low-pass term. By (5.26)

$$u_\ell(t,x) - \dot\Gamma(t) = u_\ell(t,x) - u_\ell(t,\Gamma+\gamma) + \dot\gamma. \tag{5.29}$$

Since u_ℓ is has derivative bounded by 2κ,

$$\begin{aligned}|u_\ell(t,x) - u_\ell(t,\Gamma+\gamma)| &\leq |u_\ell(t,x) - u_\ell(t,\Gamma)| + |u_\ell(t,\Gamma) - u_\ell(t,\Gamma+\gamma)| \\ &\leq 2\kappa|x-\Gamma| + 2\kappa|\gamma|.\end{aligned}$$

By assumption $|\dot\gamma| \leq C_{pth}$ and $\gamma(0) = 0$, and so for $t \in [-T, 0]$ we have $|\gamma(t)| \leq TC_{pth}$.

Plugging these bounds into (5.29) we obtain

$$\left|u_\ell(t,x) - \dot\Gamma(t)\right| \leq 2\kappa|x-\Gamma| + 2\kappa TC_{pth} + C_{pth}.$$

Now we can bound the low pass term

$$\int (u_\ell - \dot\Gamma)\theta_+\nabla\psi(x-\Gamma) \leq (2\kappa T + 1)C_{pth}\|\nabla\psi\|_\infty\int\theta_+\,dx + \|\nabla\psi\|_\infty 2\kappa\int|x-\Gamma|\theta_+\,dx.$$

By assumption, $|x-\Gamma|\theta_+ \leq R\theta_+$, so from this, (5.28), and (5.27) the result follows. □

This energy inequality is sufficient to prove the De Giorgi Lemmas.

The first lemma is a local version of the L^2 to L^∞ regularization, stating that solutions with small L^2 norm in a region will have small L^∞ norm in a smaller region.

Proposition 5.5.3 (First De Giorgi Lemma)**.** *Let κ, C_{dmn}, and C_{pth}, be positive constants. Then there exists a constant $\delta_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma : [-2,0] \to \mathbb{R}^2$ be Lipschitz. Assume that on Ω the functions $K_{1/4}$ and K_1 (defined in (2.3)) satisfy the relation

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4} K_1(x,y) \qquad \forall x \neq y \in \Omega.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (5.26) on $[-2,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4} u_h\right\|_{L^\infty([-2,0]\times\Omega)} \leq 6\kappa$ and $\|\nabla u_\ell\|_{L^\infty([-2,0]\times\Omega)} \leq 2\kappa$. Denote $\gamma := \Gamma_\ell - \Gamma$ and assume $\|\dot\gamma\|_{L^\infty([-2,0])} \leq C_{pth}$ and $\gamma(0) = 0$.

If

$$\theta(t,x) \leq 2 + \left(|x-\Gamma(t)|^{1/4} - 2^{1/4}\right)_+ \qquad \forall t \in [-2,0], x \in \Omega \setminus B_2(\Gamma(t))$$

and

$$\int_{-2}^0 \int_{\Omega\cap B_2(\Gamma(t))} (\theta)_+^2 \, dx dt \leq \delta_0$$

then

$$\theta(t,x) \leq 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma(t)).$$

Proof. Let ψ be such that $\psi = 0$ for $|x| \leq 1$ and $\psi(x) = 2 + \left(|x|^{1/4} - 2^{1/4}\right)_+$ for $|x| > 2$, and let $\nabla\psi$ and $D^2\psi$ be bounded.

For any constant $a>0$, we can apply Lemma 5.5.2 to the function

$$\theta_a := (\theta(t,x) - \psi(x-\Gamma(t)) - a)_+$$

and obtain

$$\frac{d}{dt}\int \theta_a^2 + \int \left|\Lambda^{1/2}\theta_a\right|^2 \leq C\left(\int \chi_{\{\theta\geq\psi+a\}} + \int \theta_a + \int \theta_a^2\right).$$

Thus $\theta - \psi(x-\Gamma)$ satisfies the assumptions of Lemma 0.2.1. There exists a constant, which we call δ_0, so that if

$$\int_{-2}^{0}\int (\theta(t,x) - \psi(x-\Gamma(t)))_+ \, dxdt \leq \delta_0$$

then

$$\theta(t,x) \leq 1 + \psi(x-\Gamma(t)) \qquad \forall t\in[-1,0], x\in\Omega.$$

By construction of ψ, our result follows immediately.

□

Next, we will prove De Giorgi's second lemma, a quantitative analog of the isoperimetric inequality.

Proposition 5.5.4 (Second De Giorgi Lemma)**.** *Let κ, C_{dmn}, C_{pth}, and $\beta\in(0,1)$ be positive constants. Then there exists a constant $\mu>0$ such that the following holds:*

Let $\Omega\subseteq\mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta\in(0,1)$, and let $\Gamma:[-5,0]\to\mathbb{R}^2$ be Lipschitz. Assume that on Ω the functions $K_{1/4}$ and K_1 (defined in (2.3)*) satisfy the relation*

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4}K_1(x,y) \qquad \forall x\neq y\in\Omega.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (5.26) on $[-5,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-5,0]\times\Omega)}\leq 6\kappa$ and $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)}\leq 2\kappa$. Denote $\gamma:=\Gamma_\ell-\Gamma$ and assume $\|\dot\gamma\|_{L^\infty([-5,0])}\leq C_{pth}$ and $\gamma(0)=0$.

Suppose that for $t\in[-5,0]$ and any $x\in\Omega$,

$$\theta(t,x)\leq 2+\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+.$$

Then the three conditions

$$|\{\theta\geq 1\}\cap[-2,0]\times B_2(\Gamma)|\geq \delta_0/4, \tag{5.30}$$

$$|\{0<\theta<1\}\cap[-4,0]\times B_4(\Gamma)|\leq \mu,$$

$$|\{\theta\leq 0\}\cap[-4,0]\times B_4(\Gamma)|\geq 2|B_4| \tag{5.31}$$

cannot simultaneously be met.

Here δ_0 is the constant from Proposition 5.5.3, which of course depends on κ, C_{pth}, and C_{dmn}.

Proof. Suppose that the proposition is false. Then there must exist, for each $n\in\mathbb{N}$, a bounded open set Ω_n with C^{2,β_n} boundary for $\beta_n\in(0,1)$, a Lipschitz path $\Gamma_n:[-5,0]\to\mathbb{R}^2$, a function $\theta_n:[-5,0]\times\Omega_n\to\mathbb{R}$, functions $u_\ell^n,u_h^n:[-5,0]\times\Omega_n\to\mathbb{R}^2$, and paths $\Gamma_\ell^n=\Gamma_n+\gamma_n:[-5,0]\to\mathbb{R}^2$ which solve (5.26) and satisfy all of the the assumptions of our proposition (with the same constants κ, C_{pth}, and C_{dmn}), except that

$$|\{0<\theta_n<1\}\cap[-4,0]\times B_4(\Gamma_n)|\leq 1/n. \tag{5.32}$$

Let $\psi:\mathbb{R}^2\to\mathbb{R}$ be a smooth function which vanishes on B_2 such that $\psi(x)=2+\left(|x|^{1/4}-2^{1/4}\right)_+$ for $|x|>3$.

Fix n and define

$$\theta_+ := (\theta_n - \psi(x-\Gamma_n))_+ .$$

Then θ_+ is supported on $\Omega \cap B_3(\Gamma_n)$ and is less than $2+3^{1/4}-2^{1/4} \leq 3$ everywhere.

Our goal is to bound the derivatives of θ_+^2 so that we can apply a compactness argument to the sequence θ_n. (It is the calculations in Step 2 below in which it becomes necessary to consider θ_+^2 instead of θ_+.)

The remainder of the proof is divided in three steps. First we show that the sequence of θ_+ is compact in space, then we show that it is compact in time, and finally we show that the limiting function implies a contradiction.

Step 1: Compactness in space

Apply the energy inequality Lemma 5.5.2 to θ and $\psi(x-\Gamma_n)$, and find that for some C independent of n

$$\frac{d}{dt}\int \theta_+^2 \leq C. \tag{5.33}$$

Moreover, by integrating Lemma 5.5.2 in time from -5 to $s \in [-4,0]$ and taking a supremum over s, we find

$$\sup_{[-4,0]} \int \theta_+^2 + \int_{-4}^{0}\int \left|\Lambda^{1/2}\theta_+\right|^2 + \int_{-4}^{0}\int \Lambda^{1/2}\theta_+ \Lambda^{1/2}\theta_- \leq C. \tag{5.34}$$

This proves in particular that $\theta_+ \in L^2(-4,0;\mathcal{H}^{1/2}(\Omega))$ is uniformly bounded.

Furthermore, $\left\|\theta_+^2\right\|_{L^2(-4,0;\mathcal{H}^{1/2}(\Omega_n))}$ is uniformly bounded because

$$\begin{aligned}\left\|\Lambda^{1/2}(\theta_+^2)\right\|_2^2 &= \iint [\theta_+(x)^2-\theta_+(y)^2]^2 K + \int \theta_+^4 B \\ &\leq 2\iint \theta_+(x)^2[\theta_+(x)-\theta_+(y)]^2 K + 2\iint \theta_+(y)^2[\theta_+(x)-\theta_+(y)]^2 K + \|\theta_+\|_\infty^2 \int \theta_+^2 B\end{aligned}$$

$$\le C\|\theta_+\|_\infty^2\|\theta_+\|_{\mathcal{H}^{1/2}}^2.$$

By Proposition 5.2.3, for E the extension-by-zero operator from $L^2(\Omega_n)$ to $L^2(\mathbb{R}^2)$,

$$\left\|E\theta_+^2\right\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2))}\le C \tag{5.35}$$

where C does not depend on n.

Step 2: Compactness in time

Let $\varphi\in C_0^\infty([-4,0];C^\infty(\Omega))$ a test function. Since each θ_n and $u_h^n+u_\ell^n$ is a suitable solution to (1.2) on $[-5,0]\times\Omega_n$ by assumption, we can apply the inequality (3.12) to find that, for some constant C independent of n and of φ, on $[-4,0]\times\Omega_n$

$$\begin{aligned}\iint\varphi\partial_t\theta_+^2+\iint\varphi\dot\Gamma_n\cdot\nabla\theta_+^2\le\iint\theta_+^2\left(u_\ell^n-\dot\Gamma_n+u_h^n\right)\cdot\nabla\varphi-2\iint\varphi\theta_+\left(u_\ell^n-\dot\Gamma_n+u_h^n\right)\cdot\nabla\psi\\+C\|\varphi\|_{C^0(C^2)}\left(1+\int_{-5}^0\left|\int\theta_+\left(u_\ell^n-\dot\Gamma_n+u_h^n\right)\cdot\nabla\psi\right|\right).\end{aligned}\tag{5.36}$$

For the low pass terms, as in the proof of Lemma 5.5.2, we have $\left|u_\ell^n(t,x)-\dot\Gamma_n(t)\right|\le(1+8\kappa)C_{pth}+6\kappa$ for $t\in[-4,0]$ and $x\in\operatorname{supp}(\theta_+)\subseteq B_3(\Gamma_n(t))$. Thus for $t\in[-4,0]$ we have for C independent of n and φ

$$\begin{aligned}\int\left(u_\ell^n-\dot\Gamma_n\right)\cdot(\theta_+^2\nabla\varphi)&\le C\|\nabla\varphi\|_{L^\infty(\Omega)},\\ \int\left(u_\ell^n-\dot\Gamma_n\right)\cdot(\theta_+\varphi\nabla\psi)&\le C\|\varphi\|_{L^\infty(\Omega)},\\ \int\left(u_\ell^n-\dot\Gamma_n\right)\cdot(\theta_+\nabla\psi)&\le C.\end{aligned}\tag{5.37}$$

For the high pass terms, we have u_h^n uniformly bounded in $\dot W^{-1/4,\infty}$. From step 1, we know θ_+^2 is uniformly bounded in $L^2(-4,0;\mathcal{H}^{1/2})$ so, by Lemma 5.2.2 parts (e) and (c), there is a constant

C independent of n and φ such that

$$\begin{aligned}
\iint u_h^n\cdot(\theta_+^2\nabla\varphi) &\leq C\left(\|\nabla\varphi\|_{C^0(-4,0;L^\infty(\Omega))}+\|\varphi\|_{C^0(-4,0;C^2(\Omega))}\right),\\
\iint u_h^n\cdot(\theta_+\varphi\nabla\psi) &\leq C\left(\|\varphi\|_{C^0(-4,0;L^\infty(\Omega))}+\|\varphi\|_{C^0(-4,0;C^1(\Omega))}\right),\\
\int_{-5}^0\left|\int u_h^n\cdot(\theta_+\nabla\psi)\right| &\leq C.
\end{aligned}\tag{5.38}$$

Plugging these six bounds into (5.36), for a constant C independent of n and φ, for any φ nonnegative we have

$$\int_{-4}^0\int_{\Omega_n}\left(\partial_t\theta_+^2+\dot{\Gamma}_n\cdot\nabla\theta_+^2\right)\varphi\,dxdt\leq C\,\|\varphi\|_{C^0(-4,0;C^2(\Omega_n))}.\tag{5.39}$$

Note that

$$\int_{-4}^0\int_{\Omega_n}\left(\partial_t\theta_+^2+\dot{\Gamma}_n\cdot\nabla\theta_+^2\right)dxdt=\theta_+(0,\Gamma_n(0))^2-\theta_+(-4,\Gamma_n(-4))^2$$

is uniformly bounded above and below. Therefore, by decomposing $\varphi=(\varphi+\|\varphi\|_{C^0})-\|\varphi\|_{C_0}$ into a non-negative smooth function plus a constant, we can see that (5.39) holds for general φ.

Step 3: Taking the limit

We wish to analyze the limiting behavior of θ_+^2 in the vicinity of Γ_n. First we shift these functions following Γ_n and define new functions on $[-4,0]\times\mathbb{R}^2$ by

$$v_n(t,x):=\begin{cases}\theta_+(t,x+\Gamma_n(t))^2, & x+\Gamma_n(t)\in\Omega_n,\\ 0, & x+\Gamma_n(t)\notin\Omega_n.\end{cases}$$

Each v_n is supported on $|x|\leq 3$, and

$$v_n(t,x)=(\theta_n(t,x+\Gamma_n(t))-\psi(x))_+^2\tag{5.40}$$

whenever the right hand side is defined.

Note that

$$\partial_t v_n(t,x) = \partial_t \theta_+^2(t, x+\Gamma_n) + \dot{\Gamma}_n \cdot \nabla \theta_+^2(t, x+\Gamma_n).$$

For C independent of n, we know from (5.35) that

$$\|v_n\|_{L^2(-4,0;H^{1/2}(\mathbb{R}^2)} \leq C$$

and from (5.39) that

$$\|\partial_t v_n\|_{\mathcal{M}(-4,0;C^{-2}(\Omega))} \leq C$$

where $\mathcal{M}$ is the space of Radon measures with total-variation norm and $C^{-2}(\Omega)$ is the dual of $C^2(\Omega)$.

Therefore, by the Aubin-Lions Lemma, the set $\{v_n\}_n$ is compactly embedded in $L^2([-4,0]\times\mathbb{R}^2)$. Up to a subsequence, there is a function $v \in L^2([-4,0]\times\mathbb{R}^2)$ such that

$$v_n \xrightarrow{L^2} v.$$

We know that $v \in L^\infty$, $\text{supp}(v) \subseteq [-4,0]\times B_3(0)$, and $v \in L^2(H^{1/2})$ because these properties hold uniformly on v_n.

By (5.33)

$$\frac{d}{dt}\int_{\mathbb{R}^2} v_n \, dx = \frac{d}{dt}\int_{\Omega_n} \theta_+^2 \, dx \leq C \tag{5.41}$$

so the same must be true of v, for $\frac{d}{dt}$ interpreted in the sense of distributions.

By (5.30), (5.32), and (5.31) applied to v_n (recalling the relation (5.40)), we conclude that

$$\begin{cases} |\{v \geq 1\} \cap [-2,0]\times B_2(0)| \geq \delta_0/4, \\ \left|\{0 < v < [1-\psi]^2\} \cap [-4,0]\times B_4(0)\right| \leq 0, \\ |\{v \leq 0\} \cap [-4,0]\times B_4(0)| \geq 2|B_4|. \end{cases} \tag{5.42}$$

For any $(t,x) \in [-4,0] \times B_4(0)$, either $v(t,x) \geq [1-\psi(x)]^2$ or else $v(t,x) = 0$. In fact, since $\|v(t,\cdot)\|_{H^{1/2}} < \infty$ for almost every t and $H^{1/2}$ does not contain functions with jump discontinuities, the function v is either identically 0 or else $\geq [1-\psi(x)]^2$ at each t.

Thus $\int v(t,x)\,dx$ is either 0 or else $\geq \int [1-\psi(x)]^2\,dx > 0$ at each t. By (5.41) and (5.42), v must be identically zero for all $t > -2$ but also must be non-zero for some $t > -2$, which is a contradiction.

Our assumption that the sequence θ_n exists must have been false, and the proposition must be true.

□

5.6 A Decrease in Oscillation

We combine the two De Giorgi lemmas (Propositions 5.5.3 and 5.5.4) to produce an oscillation lemma. This result is similar to the weak Harnack inequality for harmonic functions. As in the previous section, all of the following results must be independent of the size of Ω, and any assumptions made on Ω must be scaling invariant.

Lemma 5.6.1 (Oscillation Lemma)**.** *Let κ, C_{dmn}, and C_{pth}, be positive constants. Then there exists a constant $k_0 > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma : [-5,0] \to \mathbb{R}^2$ be Lipschitz. Assume that on Ω the functions $K_{1/4}$ and K_1 (defined in (2.3)*) satisfy the relation*

$$K_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4} K_1(x,y) \qquad \forall x \neq y \in \Omega.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (5.26) on $[-5,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4}u_h\right\|_{L^\infty([-5,0]\times\Omega)}\leq 6\kappa$ and $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)}\leq 2\kappa$. Denote $\gamma:=\Gamma_\ell-\Gamma$ and assume $\|\dot\gamma\|_{L^\infty([-5,0])}\leq C_{pth}$ and $\gamma(0)=0$.

Suppose that for all $t\in[-5,0]$ and any $x\in\Omega$

$$\theta(t,x)\leq 2+2^{-k_0}\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+, \tag{6.43}$$

and that

$$|\{\theta\leq 0\}\cap[-4,0]\times B_4(\Gamma)|\geq 2|B_4|.$$

Then for all $t\in[-1,0]$, $x\in\Omega\cap B_1(\Gamma)$ we have

$$\theta(t,x)\leq 2-2^{-k_0}.$$

Proof. Let μ and δ_0 as in Proposition 5.5.4, and take k_0 large enough that $(k_0-1)\mu>4|B_4|$.

Consider the sequence of functions,

$$\theta_k(t,x):=2+2^k(\theta(t,x)-2).$$

That is, $\theta_0=\theta$ and as k increases, we scale vertically by a factor of 2 while keeping height 2 as a fixed point. Note that since θ satisfies (6.43), each θ_k for $k\leq k_0$ and $(t,x)\in[-5,0]\times\Omega$ satisfies

$$\theta_k(t,x)\leq 2+\left(|x-\Gamma(t)|^{1/4}-2^{1/4}\right)_+.$$

This is precisely the assumption in Proposition 5.5.4.

Note also that

$$|\{\theta_k\leq 0\}\cap[-4,0]\times B_4(\Gamma)| \tag{6.44}$$

is an increasing function of k, and hence is greater than $2|B_4|$ for all k.

Assume, for means of contradiction, that

$$|\{1 \le \theta_k\} \cap [-2,0] \times B_2(\Gamma)| \ge \delta_0/4 \tag{6.45}$$

for $k = k_0 - 1$. Since this quantity is decreasing in k, it must then exceed $\delta_0/4$ for all $k < k_0$ as well.

Applying Proposition 5.5.4 to each θ_k, we conclude that

$$|\{0 < \theta_k < 1\} \cap [-4,0] \times B_4(\Gamma)| \ge \mu.$$

In particular, this means that the quantity (6.44) increases by atleast μ every time k increases by 1. By choice of k_0 and the fact that quantity (6.44) is trivially bounded by $4|B_4|$, we obtain a contradiciton. Therefore, the assumption (6.45) must fail for $k = k_0 - 1$.

Therefore θ_{k_0} must satisfy the assumptions of Proposition 5.5.3. In particular, we conclude that

$$\theta_{k_0}(t,x) \le 1 \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

For the original function θ, this means that

$$\theta(t,x) \le 2 - 2^{-k_0} \qquad \forall t \in [-1,0], x \in \Omega \cap B_1(\Gamma).$$

□

By assuming that θ is small near $x = \Gamma(t)$, we have shown that the oscillation of θ is decreased in a smaller neighborhood of $\Gamma(t)$. However, our goal is to control the oscillation near $x = \Gamma_\ell(t))$. Therefore we will prove the following proposition:

Proposition 5.6.2 (Oscillation Lemma with shift)**.** *Let κ, C_{dmn}, and C_{pth}, be positive constants, and let k_0 be as in Lemma 5.6.1. Then there exists a constant $\lambda > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $\Gamma : [-5,0] \to \mathbb{R}^2$ be Lipschitz. Assume that on Ω the functions $K_{1/4}$ and K_1 (defined in (2.3)) satisfy the relation

$$K_{1/4}(x,y) \leq C_{dmn} |x-y|^{3/4} K_1(x,y) \qquad \forall x \neq y \in \Omega.$$

Let θ, u_ℓ, u_h, and Γ_ℓ solve (5.26) on $[-5,0]\times\Omega$, and satisfy $\left\|\Lambda^{-1/4} u_h\right\|_{L^\infty([-5,0]\times\Omega)} \leq 6\kappa$ and $\|\nabla u_\ell\|_{L^\infty([-5,0]\times\Omega)} \leq 2\kappa$. Denote $\gamma := \Gamma_\ell - \Gamma$ and assume $\|\dot{\gamma}\|_{L^\infty([-5,0])} \leq C_{pth}$ and $\gamma(0) = 0$.

Suppose that for all $t \in [-5,0]$ and any $x \in \Omega$

$$|\theta(t,x)| \leq 2 + 2^{-k_0} \left(|x - \Gamma(t)|^{1/4} - 2^{1/4} \right)_+ \tag{6.46}$$

and that

$$|\{\theta \leq 0\} \cap [-4,0] \times B_4(\Gamma)| \geq 2|B_4|.$$

Then for any $\varepsilon \in (0,1/5]$ such that

$$5C_{pth} \leq \varepsilon^{-1} - 3 \tag{6.47}$$

we have

$$\left| \frac{2}{2-\lambda} [\theta(\varepsilon t, \varepsilon x) + \lambda] \right| \leq 2 + 2^{-k_0} \left(|x - \varepsilon^{-1}\Gamma_\ell(\varepsilon t)|^{1/4} - 2^{1/4} \right)_+ .$$

for all $t \in [-5,0]$ and x such that $\varepsilon x \in \Omega$.

The idea of the proof is to consider a small enough time interval that $\Gamma(t)$ is very close to $\Gamma_\ell(t)$. This is possible because $\Gamma_\ell - \gamma$ is uniformly Lipschitz by assumption.

If, in this proposition, we only wished to show the existence of some $\varepsilon = \varepsilon(k_0, C_{pth})$ satisfying the proposition's conclusion, then a simpler non-constructive proof would suffice. However, in Section 5.7 we will apply this proposition with parameters k_0 and C_{pth} depending on ε. To avoid circularity, we must prove the result for all ε satisfying (6.47).

Proof. Let $\bar{\lambda}>0$ and $\alpha>1$ be the universal constants defined in Lemma 0.2.4. Take $\lambda>0$ such that

$$2\lambda \le 2^{-k_0}, \qquad (2+\lambda)\left(\frac{2}{2-\lambda}\right) \le 2+2^{-k_0}\bar{\lambda}, \qquad \frac{2}{2-\lambda} \le \alpha. \tag{6.48}$$

Denote

$$\bar{\theta}(t,x) := \frac{2}{2-\lambda}\left[\theta(\varepsilon t, \varepsilon x)+\lambda\right]$$

defined for $t\in[-5/\varepsilon,0]$ and

$$x\in\Omega_\varepsilon := \{x\in\mathbb{R}^2 : \varepsilon x\in\Omega\}$$

and denote

$$\phi(x) := \left(|x|^{1/4}-2^{1/4}\right)_+.$$

We proved in Lemma 5.6.1 that $\theta(t,x)\le 2-2^{-k_0}$ for $t\in[-1,0]$ and $x\in\Omega\cap B_1(\Gamma)$. On this same set, $\theta(t,x)\ge -2$ by assumption. By the definition of $\bar{\theta}$ and by (6.48), for all $t\in[-1/\varepsilon,0]$ and $x\in\Omega\cap B_{1/\varepsilon}(\varepsilon^{-1}\Gamma(\varepsilon t))$ we have therefore

$$\begin{cases} \bar{\theta}(t,x) & \le \frac{2}{2-\lambda}\left[2-2^{-k_0}+\lambda\right] \le \frac{2}{2-\lambda}\left[2-\lambda\right]=2. \\ \bar{\theta}(t,x) & \ge \frac{2}{2-\lambda}\left[-2+\lambda\right]=-2. \end{cases} \tag{6.49}$$

Similarly, the bound (6.46) on θ becomes the equivalent bounds on $\bar{\theta}$, for all $(t,x)\in[-5/\varepsilon,0]\times\Omega_\varepsilon$

$$\bar{\theta}(t,x) \le \frac{2}{2-\lambda}\left[2+2^{-k_0}\phi(|\varepsilon x-\Gamma(\varepsilon t)|)+\lambda\right] \tag{6.50}$$

and

$$\bar{\theta}(t,x) \ge \frac{2}{2-\lambda}\left[-2-2^{-k_0}\phi(|\varepsilon x-\Gamma(\varepsilon t)|)+\lambda\right]. \tag{6.51}$$

Let $t\in[-5,0]$ and $x\in\Omega_\varepsilon$, and define

$$y := x-\varepsilon^{-1}\Gamma(\varepsilon t).$$

From (6.50) and the assumptions (6.48), we can bound

$$\begin{aligned}\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2+\lambda+2^{-k_0}\phi(\varepsilon|y|)\right]\\ &\leq 2+2^{-k_0}\bar{\lambda}+2^{-k_0}\alpha\phi(\varepsilon|y|)\\ &= 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right].\end{aligned}$$

From (6.51) and the assumptions (6.48), we can bound

$$\begin{aligned}-\bar{\theta}(t,x) &\leq \frac{2}{2-\lambda}\left[2-\lambda+2^{-k_0}\phi(\varepsilon|y|)\right]\\ &\leq 2+2^{-k_0}\alpha\phi(\varepsilon|y|)\\ &\leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right].\end{aligned}$$

Therefore

$$\left|\bar{\theta}(t,x)\right| \leq 2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right]. \tag{6.52}$$

If $|y|\leq\varepsilon^{-1}$ then from (6.49) we have

$$\left|\bar{\theta}(t,x)\right| \leq 2 \leq 2+2^{-k_0}\phi(x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t))$$

which is our desired result. Therefore assume without loss of generality that $|y|\geq\varepsilon^{-1}$. In this case we can apply Lemma 0.2.4 which states that, since $\varepsilon<1/2$ and $\varepsilon|y|\geq 1$, it is a property of ϕ, α, and $\bar{\lambda}$ that

$$2+2^{-k_0}\left[\bar{\lambda}+\alpha\phi(\varepsilon|y|)\right] \leq 2+2^{-k_0}\left[\phi(|y|-\varepsilon^{-1}+3)\right].$$

For $t\in[-5,0]$, we have by assumption (6.47)

$$|y|-\varepsilon^{-1}+3 \leq |y|-5C_{pth} \leq |y-\varepsilon^{-1}\gamma(\varepsilon t)|.$$

The estimate (6.52) becomes

$$\left|\bar{\theta}(t,x)\right| \le 2+2^{-k_0}\phi(|x-\varepsilon^{-1}\Gamma(\varepsilon t)-\varepsilon^{-1}\gamma(\varepsilon t)|).$$

This concludes the proof. $\square$

5.7 Hölder Continuity

In this section we shall prove the main theorem, Theorem 5.1.1. We will demonstrate Hölder continuity by iteratively applying Proposition 5.6.2 and rescaling.

We begin with a lemma to describe the scaling properties of (1.2).

Lemma 5.7.1 (Scaling)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$, and let $j_0 \in \mathbb{Z}$ and $\varepsilon > 0$ be constants.*

Suppose that $\theta : [-T,0] \times \Omega \to \mathbb{R}$ and $u : [-T,0] \times \Omega \to \mathbb{R}^2$ are a suitable solution to (1.2) *and u is calibrated by a sequence $(u_j)_{j \ge j_0}$ with constant κ and center N.*

Suppose that on Ω the functions $K_{1/4}$ and K_1 (defined in (2.3)*) satisfy the relation*

$$K_{1/4}(x,y) \le C_{dmn}|x-y|^{3/4}K_1(x,y) \qquad \forall x \ne y \in \Omega.$$

Then

$$\bar{\theta}(t,x) := \theta(\varepsilon t, \varepsilon x)$$

and

$$\bar{u}(t,x) := \sum_{j=j_0}^{\infty} \bar{u}_j(t,x), \qquad \bar{u}_j(t,x) := u_j(\varepsilon t, \varepsilon x)$$

are also a suitable solution to (1.2) *on* $[-T/\varepsilon, 0] \times \Omega_\varepsilon$ *where* $\Omega_\varepsilon = \{x \in \mathbb{R}^2 : \varepsilon x \in \Omega\}$.

Moreover, $(\bar{u}_j)_{j \geq j_0}$ *is calibrated with the same constant* κ *but with center* $N - \log_2(\varepsilon)$*, and the relation*

$$\bar{K}_{1/4}(x,y) \leq C_{dmn}|x-y|^{3/4}\bar{K}_1(x,y) \qquad \forall x \neq y \in \Omega_\varepsilon \tag{7.53}$$

holds.

Proof. Denote by $\bar{\Lambda}$ the square root of the Laplacian with Dirichlet boundary conditions on Ω_ε. One can calculate (see e.g. [CS16] Section 2.4) that for $(t,x) \in [-T/\varepsilon, 0] \times \Omega_\varepsilon$

$$\Lambda\theta(\varepsilon t, \varepsilon x) = \varepsilon \bar{\Lambda}\bar{\theta}(t,x).$$

Similarly, in the Caffarelli-Stinga representation from Proposition 5.2.1 the operator $\bar{\Lambda}^s$ will have kernel

$$\bar{K}_s(x,y) = \varepsilon^{s-2}K_s(\varepsilon x, \varepsilon y).$$

From these facts it is clear that the scaled functions satisfy (1.2) and (7.53).

To show that $(\bar{u}_j)_{j\in\mathbb{Z}}$ is calibrated, we must translate the three bounds on u_j to corresponding bounds on $\bar{u}_j$. Each of the calculations are similar, so we show only one:

$$\|\nabla \bar{u}_j\|_\infty = \varepsilon\|\nabla u_j\|_\infty \leq 2^{\log_2(\varepsilon)}2^j 2^{-N}\kappa = 2^j 2^{-(N-\log_2(\varepsilon))}\kappa.$$

□

The next lemma demonstrates Hölder continuity of suitable solutions. The proof method is to consruct a sequence of rescaled functions all of which, by induction, satisfy the assumptions of

Proposition 5.6.2. We will assume that the velocity u is the Riesz transform of an L^∞ function Θ, which will in practice typically be θ itself, up to scaling and translation.

Lemma 5.7.2 (Continuity of suitable solutions)**.** *There exists a universal constant C such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be an open, bounded domain with $C^{2,\beta}$ boundary, $\beta \in (0,1)$. Let $\Theta \in L^\infty([-5,0] \times \Omega)$. Then there exists a constant $\alpha \in (0,1)$ depending on Ω and $\|\Theta\|_{L^\infty}$ such that the following holds:

Let $\theta : [-5,0] \times \Omega \to \mathbb{R}$ and $u : [-5,0] \times \Omega \to \mathbb{R}^2$ be a suitable solution to (1.2). Assume that $\|\theta\|_{L^\infty([-5,0]\times\Omega)} \leq 2$ and that $u = \nabla^\perp \Lambda^{-1} \Theta$.

Then for any point $P \in \bar{\Omega}$, θ is Hölder continuous at $(0,P)$ and

$$\sup_{(t,x)\in[-5,0]\times\Omega} \frac{|\theta(t,x) - \theta(0,P)|}{(|t|^2 + |x-P|^2)^{\alpha/2}} \leq C.$$

Proof. By relabelling our coordinate system, we can assume without loss of generality that $P = 0$ is the origin in $\mathbb{R}^2$.

From Proposition 5.4.1, we know that

$$u = \nabla^\perp \Lambda^{-1} \Theta = \sum_{j=j_0}^{\infty} u_j$$

for a sequence $(u_j)_{j \geq j_0}$ of divergence-free functions calibrated with some constant $\kappa = \kappa(\Omega, \|\Theta\|_{L^\infty})$ and center 0. Assume without loss of generality that $j_0 < 0$.

Choose a constant $0 < \varepsilon < 1/5$ such that

$$5 \max\left(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1-j_0)\kappa\right) \leq \varepsilon^{-1} - 3. \tag{7.54}$$

For integers $k \geq 0$ consider the domains

$$\Omega_k := \{x \in \mathbb{R}^2 : \varepsilon^k x \in \Omega\}.$$

If K_s^k are the kernels defined in Proposition 5.2.1 corresponding to the operators Λ^s on Ω_k, then by Proposition 5.2.1 and Lemma 5.7.1 the relation

$$K_{1/4}^k(x,y) \leq C_{dmn}|x-y|^{3/4} K_1^k(x,y) \qquad \forall x \neq y \in \Omega_k$$

holds for some constant C_{dmn} independent of k.

For notational convenience, denote

$$\sum_k = \sum_{j > -k\log_2(\varepsilon)}, \qquad \sum^k = \sum_{j \leq -k\log_2(\varepsilon)}$$

and define the following functions on $[-5,0] \times \Omega_k$:

$$u_\ell^k(t,x) := \sum^k u_j(\varepsilon^k t, \varepsilon^k x),$$
$$u_h^k(t,x) := \sum_k u_j(\varepsilon^k t, \varepsilon^k x).$$

By Lemmas 5.7.1 we know the sequence $(u_j(\varepsilon^k \cdot, \varepsilon^k \cdot))_j$ is calibrated with constant κ and center $-k\log_2(\varepsilon)$, and hence by 5.5.1 we know that, independently of k,

$$\left\|\Lambda^{-1/4} u_h^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \leq 6\kappa$$

and

$$\left\|\nabla u_\ell^k\right\|_{L^\infty([-5,0]\times\Omega_k)} \leq 2\kappa.$$

Each u_ℓ^k is a finite sum of L^∞ functions, hence L^∞ itself, though not uniformly in k.

Define $\Gamma_k, \gamma_k : [-5,0] \to \mathbb{R}^2$ by the following recursive formulae and ODEs:

$$\begin{aligned}
&\Gamma_0(t) := 0, && t \in [-5,0],\\
&\gamma_k(0) := 0, && k \geq 0,\\
&\dot{\gamma}_k(t) := u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t), && k \geq 0, t \in [-5,0)\\
&\Gamma_k(t) := \varepsilon^{-1}\gamma_{k-1}(\varepsilon t) + \varepsilon^{-2}\gamma_{k-2}(\varepsilon^2 t) + \cdots + \varepsilon^{-k}\gamma_0(\varepsilon^k t), && k \geq 1, t \in [-5,0].
\end{aligned}$$

Since each u_ℓ^k is L^∞ in space-time and Lipschitz in space, these γ_k exist by a version of the Cauchy-Lipschitz theorem. For example, Theorem 3.7 of Bahouri, Chemin, and Danchin [BCD11] proves existence and uniqueness in our case. In particular, since u_ℓ^k is a vector field which is tangential to the boundary of Ω_k and has unique flows, the Lagrangian path

$$\Gamma_\ell^k(t) := \Gamma_k(t) + \gamma(k)$$

for u_ℓ^k must remain inside $\bar{\Omega}_k$ for all time and so our expressions remain well-defined.

The quantity γ_k here corresponds to the frequency packets u_j with $-(k-1)\log_2(\varepsilon) < j \leq -k\log_2(\varepsilon)$. These frequencies are included in the definition of u_ℓ^k but not the definition of u_ℓ^{k-1} (they would instead be included in u_h^{k-1}).

By construction, for $k \geq 0$ we have $\Gamma_{k+1}(t) = \varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)$. Therefore

$$\begin{aligned}
\dot{\Gamma}_{k+1}(t) &= \partial_t\left[\varepsilon^{-1}\gamma_k(\varepsilon t) + \varepsilon^{-1}\Gamma_k(\varepsilon t)\right]\\
&= \dot{\gamma}_k(\varepsilon t) + \dot{\Gamma}_k(\varepsilon t)\\
&= u_\ell^k(\varepsilon t, \gamma_k(\varepsilon t) + \Gamma_k(\varepsilon t))\\
&= u_\ell^k(\varepsilon t, \varepsilon\Gamma_{k+1}(t)).
\end{aligned}$$

With this in hand, we can bound the size of γ_k. Namely, for $k \geq 1$,

$$\begin{aligned}
\dot{\gamma}_k(t) &= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - \dot{\Gamma}_k(t) \\
&= u_\ell^k(t, \Gamma_k(t) + \gamma_k(t)) - u_\ell^{k-1}(\varepsilon t, \varepsilon \Gamma_k(t)) \\
&= \sum^k u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t) + \varepsilon^k \gamma_k(t)) - \sum^{k-1} u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t)) \\
&= \sum^{k-1} \left[u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t) + \varepsilon^k \gamma_k(t)) - u_j(\varepsilon^k t, \varepsilon^k \Gamma_k(t)) \right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k \ldots) \\
&= \left[u_\ell^{k-1}\left(\varepsilon t, \varepsilon \Gamma_k(t) + \varepsilon \gamma_k(t)\right) - u_\ell^{k-1}(\varepsilon t, \varepsilon \Gamma_k(t)) \right] + \sum_{k-1}^{k} u_j(\varepsilon^k t, \varepsilon^k \ldots).
\end{aligned}$$

The function $x \mapsto u_\ell^{k-1}(\varepsilon t, x)$ is Lipschitz, with Lipschitz constant less than 2κ. Moreover, each u_j has $\|u_j\|_\infty \leq \kappa$. Thus from the above calculation we can bound

$$|\dot{\gamma}_k(t)| \leq 2\kappa\varepsilon |\gamma_k(t)| - \kappa \log_2(\varepsilon). \tag{7.55}$$

Applying Gronwall's inequality, we find that for $t \in [-5, 0]$

$$|\gamma_k(t)| \leq \frac{-\log_2(\varepsilon)}{2\varepsilon} \left(e^{10\varepsilon\kappa} - 1\right).$$

Plugging this estimate back into (7.55),

$$|\dot{\gamma}_k(t)| \leq -\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa} \qquad \forall k \geq 1.$$

Trivially $|\dot{\gamma}_0| \leq (1 - j_0)\kappa$, so if we define

$$C_{pth} = \max\left(-\kappa \log_2(\varepsilon) e^{10\varepsilon\kappa}, (1 - j_0)\kappa\right)$$

then for all $k \geq 0$ and $t \in [-5, 0]$

$$|\dot{\gamma}_k(t)| \leq C_{pth}.$$

Moreover, the assumption (6.47) then follows from (7.54).

Define

$$\theta_0(t,x) := \theta(t,x)$$

and for each $k \geq 0$, if $|\{\theta_k \leq 0\} \cap [-4,0] \times B_4(\Gamma_k(t))| \geq 2|B_4|$ then set

$$\theta_{k+1}(t,x) := \frac{2}{2-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) + \lambda\right].$$

Otherwise, set

$$\theta_{k+1}(t,x) := \frac{1}{1-\lambda}\left[\theta_k(\varepsilon t, \varepsilon x) - \lambda\right].$$

From Lemma 5.7.1, we know that θ_k and the calibrated function $\sum_{j \geq j_0} u_j(\varepsilon^k \cdot, \varepsilon^k \cdot)$ solve (1.2). By construction, θ_k, u_ℓ^k, u_h^k, and Γ_ℓ^k solve (5.26).

Since $|\theta_0| \leq 2$ by assumption, we know in particular that

$$|\theta_k| \leq 2 + 2^{-k_0}\left(|x - \Gamma_k(t)|^{1/4} - 2^{1/4}\right)_+ \tag{7.56}$$

holds for $k = 0$.

If (7.56) holds for k, then at least one of θ_k or $-\theta_k$ (depending on whether $|\{\theta_k \leq 0\} \cap [-4,0] \times B_4(\Gamma_k(t))|$ is more or less than $2|B_4|$) will satisfy the assumptions of Proposition 5.6.2. In either case, we conclude that θ_{k+1} satisfies (7.56). By induction, this bound holds for all θ_k.

Each θ_k is between -2 and 2 on $[-5,0] \times B_2(\Gamma_k)$. But recall that each Γ_k is Lipschitz with constant kC_{pth}. Thus $|\Gamma_k(t)| \leq 1$ for $t \in [-(kC_{pth})^{-1}, 0]$. On that time interval,

$$|\theta_k(t,x)| \leq 2 \qquad \forall x \in B_1(0).$$

We conclude that

$$\left| \sup_{[-\varepsilon^k (kC_{pth})^{-1},0]\times B_{\varepsilon^k}(0)} \theta(t,x) - \inf_{[-\varepsilon^k (kC_{pth})^{-1},0]\times B_{\varepsilon^k}(0)} \theta(t,x) \right| \leq 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In particular, for some positive constant C such that

$$\varepsilon^{Ck} \leq (kC_{pth})^{-1} \qquad \forall k \geq 0,$$

we can say that

$$|t|^2 + |x|^2 \leq \varepsilon^{(1+C)k}$$

implies that $(t,x) \in [-\varepsilon^k (kC_{pth})^{-1}, 0] \times B_{\varepsilon^k}(0)$ which in turn implies that

$$|\theta(t,x) - \theta(0,0)| \leq 4\left(\frac{2}{2-\lambda}\right)^{-k}.$$

In other words,

$$\begin{aligned}
|\theta(t,x) - \theta(0,0)| &\leq 4\left(\frac{2}{2-\lambda}\right)^{-\frac{1}{1+C}\log_\varepsilon(|t|^2-|x|^2)+1} \\
&= 4\left(\frac{2}{2-\lambda}\right)\exp\left[\ln\left(\frac{2}{2-\lambda}\right)\frac{\ln(|t|^2+|x|^2)}{-(1+C)\ln(\varepsilon)}\right] \\
&= \frac{8}{2-\lambda}(|t|^2+|x|^2)^{-\frac{\ln(2)-\ln(2-\lambda)}{(1+C)\ln(\varepsilon)}} \\
&\leq 8(|t|^2+|x|^2)^{-\frac{\ln(2)-\ln(2-\lambda)}{(1+C)\ln(\varepsilon)}}
\end{aligned}$$

□

We are now able to prove the main result, Theorem 5.1.1.

Proof of Theorem 5.1.1. Recall that Ω, S, k, and θ_0 are given.

In Proposition 5.3.2 we construct global-in-time solutions to (1.1). By construction, there is a universal constant C_1 so $\|\theta(t,\cdot)\|_{L^\infty(\Omega)} \le C_1 t^{-1} \|\theta_0\|_{L^2(\Omega)}$.

Consider a point (t_0, x_0) with $t_0 > S$. Consider arbitrary constants $\lambda, \mu \in (0,1]$ and note that

$$\tilde{\theta}(t,x) := \lambda\theta(t_0+\mu t, \mu x), \qquad \tilde{u}(t,x) := u(t_0+\mu t, \mu x)$$

is a suitable solution to (1.2) on $[-t_0/\mu, \infty) \times \tilde{\Omega}$ where $\tilde{\Omega} := \{x \in \mathbb{R}^2 : \mu x \in \Omega\}$.

If $S + \mu(-5) = \frac{S}{2}$, or equivalently if $\mu = S/10$, then then we have

$$\left\|\tilde{\theta}\right\|_{L^\infty([-5,0]\times\tilde{\Omega})} \le \lambda 2 C_1 \frac{k}{S}.$$

Take $\lambda = S/(C_1 k)$.

On $[-5,0] \times \tilde{\Omega}$ we have $\tilde{\theta}$ and $\tilde{u}$ a suitable solution to (1.2) satisfying $\left\|\tilde{\theta}\right\|_{L^\infty} \le 2$ and $\tilde{u} = \nabla^\perp \Lambda^{-1}\Theta$ with $\|\Theta\|_{L^\infty} \le 2C_1 k/S$. Therefore we can apply Lemma 5.7.2 to $\tilde{\theta}$, $\tilde{u}$ and find that $\tilde{\theta}$ satisfies, for $\alpha = \alpha(k,S)$ and C universal,

$$\sup_{(t,x)\in[-5,0]\times\tilde{\Omega}} \frac{|\tilde{\theta}(t,x) - \tilde{\theta}(t_0,x_0)|}{\left(|t-t_0|^2 + |x-x_0|^2\right)^{\alpha/2}} \le C.$$

For the original unscaled θ, we have

$$\sup_{(t,x)\in[S/2,t_0]\times\Omega} \frac{|\theta(t,x) - \theta(t_0,x_0)|}{\left(|t-t_0|^2 + |x-x_0|^2\right)^{\alpha/2}} \le C\lambda^{-1}\mu^{-\alpha} \le C(\lambda\mu)^{-1} = C\frac{10C_1}{S^2}k.$$

$\square$

Chapter 6

conservation

This paper will consider 1D scalar dissipative conservation equations of the form

$$\partial_t u + \partial_x [Q(u)] = \nu \partial_{xx} \eta'(u), \tag{0.1}$$

where Q and η are uniformly convex functions, meaning that for some constant $\Lambda \geq 1$,

$$\frac{1}{\Lambda} \leq \eta''(x), Q''(x) \leq \Lambda \tag{0.2}$$

holds for all $x \in \mathbb{R}$. This bound on η'' is natural because η'' measures the coercivity of the dissipation in divergence form.

Equations of this form admit a class of traveling wave solutions known as shocks. Shocks are monotone decreasing and exponentially constant at $\pm\infty$. Given any two values $s_- > s_+$, there exists a shock $s : \mathbb{R} \to (s_+, s_-)$ such that

$$\lim_{s \to \infty} s(x) = s_+,$$
$$\lim_{s \to -\infty} s(x) = s_-,$$

and $s(x - t\sigma)$ is a solution to (0.1) with constant

$$\sigma := \frac{Q(s_-) - Q(s_+)}{s_- - s_+}.$$

This formula for σ is known as the Rankine-Hugoniot condition. Viscous shocks are a generalization of inviscid shocks, which are piece-wise constant with a single jump discontinuity. Inviscid shocks are recovered in the limit as $\nu \to 0$.

We will show in this paper that sufficiently small shock solutions are L^2-stable. Since even small perturbations in L^2 can significantly affect the travelling speed of a shock, we will show stability only up to a Lipschitz shift which depends on the perturbation. This limitation is not present in the L^1 theory (see Kruzkhov [Kru70]), but is well known in the theory of L^2 shock stability (see Leger [Leg11]).

We will prove the following:

Theorem 6.0.1 (Main Theorem)**.** *Let $\Lambda \geq 1$ be a constant, and let $\eta, Q : \mathbb{R} \to \mathbb{R}$ be satisfy* (0.2) *on the interval $(-R, R)$ for $R \in (0, +\infty]$, and let η''', Q''' continuous at 0. Then there exists a constant ε_0 such that the following holds:*

Let $\nu > 0$ be any constant. Let $s : \mathbb{R} \times [0, \infty) \to [s_+, s_-]$ be a shock solution to (0.1) *with $|s_+ - s_-| = 2\varepsilon \leq 2\varepsilon_0$, and let $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be a solution to* (0.1) *such that $\|u(\cdot,0) - s(\cdot,0)\|_{L^2(\mathbb{R})} < \infty$. If $R < \infty$, assume $\|u(\cdot,0)\|_{L^\infty(\mathbb{R})} < R$.*

Then there exists a Lipschitz function $\gamma : [0, \infty) \to \mathbb{R}$ such that for any $t \in [0, \infty)$ we have

$$\int |u(x,t) - s(x - \gamma(t), t)|^2 \, dx \leq 4\Lambda^2 \int |u(x,0) - s(x,0)|^2 \, dx.$$

The quantity $\|\gamma'\|_{L^\infty}$ depends only on ε, Λ, and $\|u(\cdot,0) - s(\cdot,0)\|_{L^2(\mathbb{R})}$.

Notice that this result is independent of the strength ν of the dissipation.

We prove this result using the method of relative entropy, first introduced by DiPerna and Dafermos [Daf96] to study stability of Lipschitz solutions of conservation laws. This method has since been applied by Vasseur, Serre, Leger, and others ([SV16], [LV11], [Leg11]) to show L^2 stability of shocks under large perturbations.

For an entropy function f, we denote the relative entropy between two solutions u_1 and u_2 by

$$f(u_1|u_2) := f(u_1) - f(u_2) - f'(u_2)[u_1 - u_2].$$

In this paper, we will use the function η, the antiderivative of the dissipative term, as our entropy function. Our proof involves taking the time-derivative of the relative entropy of u relative to the shock s. Because of the assumption (0.2), the integral of the relative entropy is essentially equal to the L^2 norm. However, this quantity will not decrease in general, as shown by Vasseur and Kang in [KV17]. We supplement the method by considering a weighted psuedo-norm, as in [Vas16b] and [Vas08]. The weight function a is independent of solution u, and is approximately constant.

We will show the following result, from which Theorem 6.0.1 follows as a corollary:

Theorem 6.0.2. *Let $\Lambda \geq 1$ be a constant, and let $\eta, Q : \mathbb{R} \to \mathbb{R}$ satisfy (0.2) for all $x \in \mathbb{R}$, and let η''', Q''' continuous at 0. Let $\nu = 1$. Then there exists a constant ε_0 such that the following holds:*

Let $0 < \varepsilon < \varepsilon_0$ be a constant and let $s : \mathbb{R} \to [s_+, s_-]$ be a stationary shock solution to (0.1) with $s_\pm = \mp\varepsilon$. Then there exists a weight function $a : \mathbb{R} \to [1/2, 2]$ such that the following holds:

For any $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ solving (0.1) such that $\|u(\cdot,0) - s(\cdot)\|_{L^2(\mathbb{R})} < \infty$, there exists a Lipschitz function $\gamma : [0,\infty) \to \mathbb{R}$ such that for any $t \in [0,\infty)$ we have

$$\frac{d}{dt}\int a(x+\gamma(t))\eta(u(x,t)|s(x+\gamma(t)))\,dx \leq -\varepsilon_0 \int a(x+\gamma(t))\left|\partial_x(\eta'(u) - \eta'(s))\right|^2 dx.$$

The quantity $\|\gamma'\|_{L^\infty}$ depends only on ε, Λ, and $\|u(\cdot,0) - s(\cdot)\|_{L^2(\mathbb{R})}$, and $\|a-1\|_\infty$ tends to 0 as $\varepsilon \to 0$.

The theory of L^2 stability of shocks is contrasted with the L^1 theory, as in the work of Kruzkov [Kru70]. See also Ilyin and Oleinik [IO60] and Freistuhler and Serre [FS98]. Unlike Kruzkov's result, we only need one entropy. Though 1D scalar laws have infinitely many entropies in general, systems of conservation laws typically only have one entropy so methods which rely on multiple entropies are more difficult to generalize, though such generalizations exist, see for example Bressan, Liu, and Yang [BLY99]. The L^p stability theory has also been studied by Adimurthi, Ghoshal, and Veerappa Gowda [AGVG14]. L^2 stability has been studied outside the context of relative entropy, as by Goodman [Goo86], though wish stronger assumptions on the perturbation.

Since our result is independent of the strength ν of dissipation, it is well suited to taking an inviscid limit.

The technique used in this paper has previously been applied by Kang and Vasseur to certain 1D dissipative systems in [KV19] (including 1D isotropic Navier-Stokes) and 1D scalar equations with constant dissipation in [Kan19] (i.e. $\eta'(u) = u$). We are able to consider arbitrary convex dissipation by utilizing η as an entropy.

As in [KV19], the proof proceeds by braking up the solution u into a part which is L^∞ close to s and an error term which may be large in L^∞. The close part is handled similarly to the existing literature, while for the error term we must make careful use of the relationship between the dissipative term and the derivative of the weight function a.

The paper is structured as follows: in Section 6.1 we compute the time derivative of the relative entropy. In Section 6.2 we present a number of lemmas which will be used throughout the paper. In Section 6.3 we show that our expression for the derivative of the relative entropy is non-positive under a number of special assumptions. Finally in Section 6.4 we prove Theorems 6.0.1

and 6.0.2.

6.1 Time derivative

For any function f, define

$$f(x|y) := f(x) - f(y) - f'(y)(x-y),$$

In particular, for η our entropy the quantity $\eta(u|s)$ is called the relative entropy of u relative to s.

We call η an entropy function if there exists a function G such that

$$G'(x) = Q'(x)\eta'(x).$$

In the 1D case, such a G trivially exists.

We also define

$$F(x;y) := G(x) - G(y) - \eta'(y)\left[Q(x) - Q(y)\right].$$

We begin by computing the time derivative of the relative entropy with arbitrary shift and arbitrary weight.

Proposition 6.1.1 (Time Derivative)**.** *Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and $s : \mathbb{R} \to \mathbb{R}$ be solutions to* (0.1) *with $\nu = 1$ and s a stationary solution. Assume $|u(\cdot,t) - s(\cdot)| \in L^2(\mathbb{R})$ for all t.*

Then for any differentiable function $\gamma : [0,\infty) \to \mathbb{R}$ and weight function $a \in L^\infty(\mathbb{R})$, we have

$$\frac{d}{dt}\int a(x+\gamma(t))\eta\big(u(x)\big|s(x+\gamma(t))\big)\,dx = R(u) := \dot{\gamma}Y(u) + B(u) - D(u)$$

where

$$Y(u) := \int a'\eta(u|s)\,dx - \int as'\eta''(s)(u-s)dx,$$

$$D(u) := \int a|\partial_x(\eta'(u) - \eta'(s))|^2 dx,$$

$$B(u) := \int a' F(u;s) dx - \int a\eta''(s) s' Q(u|s) dx + \int \frac{a''}{2}(\eta'(u) - \eta'(s))^2 dx + \int a\eta'(u|s)\partial_x Q(s) dx.$$

Here it is understood that u is evaluated always at (x,t) while a and s are evaluated at $x+\gamma(t)$.

The expressions $Y(u)$, $B(u)$, $D(u)$ will be referenced throughout this paper with the definitions given above, and they will be abbreviated as Y, B, D when the input u is clear from context.

Proof. Initially, we have

$$\frac{d}{dt}\int a(x+\gamma(t))\eta(u(x)|s(x+\gamma(t))\, dx = \int a'\dot{\gamma}\eta(u|s)\, dx + \int a[\eta'(u) - \eta'(s)]\,\partial_t u\, dx$$
$$+ \int a[-\eta''(s)(u-s)]\,\partial_t s\, dx.$$

Since $\partial_t s = \dot{\gamma}s'$, we have, with Y as defined in the theorem statement,

$$\frac{d}{dt}\int a\eta(u|s)\, dx = \dot{\gamma}[Y] + \int a[\eta'(u) - \eta'(s)]\,\partial_t u\, dx \tag{1.3}$$

Note that

$$\partial_t u = \partial_{xx}\eta'(u) - \partial_x Q(u)$$

and that because s is a shock with zero drift,

$$\partial_x Q(s) = \partial_{xx}\eta'(s).$$

Therefore, writing $w = \eta'(u) - \eta'(s)$,

$$\begin{aligned}\int aw\partial_t u\,dx &= \int aw\left[\partial_{xx}\eta'(u) - \partial_x Q(u)\right]dx + \int a\left[-\eta''(s)(u-s)\right]\left[\partial_{xx}\eta'(s) - \partial_x Q(s)\right]dx \\ &= \int aw\partial_{xx}w\,dx + \int a\eta'(u|s)\partial_{xx}\eta'(s)\,dx - \int a\left(w\partial_x Q(u) - \eta''(s)(u-s)\partial_x Q(s)\right)dx\end{aligned} \tag{1.4}$$

Now, notice that

$$\begin{aligned}\partial_x F(u;s) + \eta''(s)Q(u|s)s' &= \left[\eta'(u)Q'(u) - \eta'(s)Q'(u)\right]\partial_x u + \left[-\eta''(s)(Q(u) - Q(s))\right]s' \\ &\qquad + \eta''(s)Q(u|s)s' \\ &= \left[\eta'(u) - \eta'(s)\right]Q'(u)\partial_x u + \eta''(s)s'\left[Q(u|s) - (Q(u) - Q(s))\right] \\ &= \left[\eta'(u) - \eta'(s)\right]\partial_x Q(u) - \eta''(s)(u-s)Q'(s)s' \\ &= w\partial_x Q(u) - \eta''(s)(u-s)\partial_x Q(s).\end{aligned} \tag{1.5}$$

Combining (1.3), (1.4), and (1.5) we obtain

$$\frac{d}{dt}\int a\eta(u|s)\,dx = \dot{\gamma}[Y] + \int aw\partial_{xx}w\,dx + \int a\eta'(u|s)\partial_x Q(s)\,dx - \int a\left[\partial_x F(u;s) + \eta''(s)s'Q(u|s)\right]dx.$$

Integrating by parts, we have

$$\begin{aligned}\int aw\partial_{xx}w\,dx &= \frac{1}{2}\int a''w^2\,dx - \int a|\partial_x w|^2\,dx \\ &= \frac{1}{2}\int a''w^2\,dx - D\end{aligned}$$

and

$$\int a\partial_x F(u;s)\,dx = -\int a'F(u;s)\,dx.$$

The proposition follows.

$\square$

Notice that each term in Y and B contain either a derivative of s or a derivative of a. This inspires us to choose our weight function a to be a linear transformation of s. We can then perform a change of variables and simplify the expression even further. The new variable $y = \eta'(s(x))$ is known as the entropic variable.

Lemma 6.1.2. *Under the same assumptions as Proposition 6.1.1, if $a := 1 - \frac{\lambda}{\varepsilon}\eta'(s)$ for some $\lambda > 0$ then, in terms of the variable $y := \eta'(s)$, we have*

$$
\begin{aligned}
Y &= \frac{\lambda}{\varepsilon}\int \eta(u|s)\,dy + \int a(u-s)\,dy,\\
D &= -\int a\eta''(s)s'|\partial_y w|^2\,dy,\\
B &= \frac{\lambda}{\varepsilon}\int F(u;s)\,dy + \int aQ(u|s)\,dy + \frac{\lambda}{\varepsilon}\int \frac{Q'(s)}{2\eta''(s)}w^2\,dy - \int a\frac{Q'(s)}{\eta''(s)}\eta'(u|s)\,dy.
\end{aligned}
$$

Proof. Notice first that $x \mapsto \eta'(s)$ is a monotone-decreasing differentiable bijection, so u is a well-defined function of y. The integrating factor for this new variable is

$$dy = -\eta''(s)s'\,dx.$$

Note the minus sign because s' is negative so the direction of integration is reversed.

The derivatives of a are

$$\partial_x a = -\frac{\lambda}{\varepsilon}\eta''(s)s'$$

and

$$\partial_{xx} a = -\frac{\lambda}{\varepsilon}\partial_{xx}\eta'(s) = -\frac{\lambda}{\varepsilon}\partial_x Q(s).$$

The derivative of $Q(s)$ is

$$\partial_x Q(s) = Q'(s)s' = -\frac{Q(s)}{\eta''(s)}\eta''(s)s'.$$

From here, the form of Y and B are trivial to compute.

For D, we must simply compute

$$\partial_x w = \eta''(s) s' \partial_y w.$$

$\square$

6.2 Lemmas

This section consists of a series of lemmas which will be necessary throughout the rest of the paper.

We begin by applying Taylor's formula to each of the quantities appearing in the expressions $Y(u)$, $B(u)$, and $D(u)$ defined in Lemma 6.1.2. These estimates, together with the bounds on the derivatives of η and Q, will be the basis of all our control on the quantities Y, B, D.

Lemma 6.2.1. *Let x_1 and x_2 be real numbers. Then the following estimates hold:*

(a) There exists a point z_0 between x_1 and x_2 such that

$$x_1 - x_2 = \frac{1}{\eta''(z_0)}(\eta'(x_1) - \eta'(x_2)).$$

(b) There exists a point z_1 between x_1 and x_2 such that

$$Q(x_1|x_2) = \frac{Q''(z_1)}{2\eta''(z_0)^2}(\eta'(x_1) - \eta'(x_2))^2.$$

(c) There exists a point z_2 between x_1 and x_2 such that

$$\eta(x_1|x_2) = \frac{\eta''(z_2)}{2\eta''(z_0)^2}(\eta'(x_1) - \eta'(x_2))^2.$$

(d) There exists a point z_3 between x_1 and x_2 such that

$$\eta'(x_1|x_2) = \frac{\eta'''(z_3)}{2\eta''(z_0)^2}[\eta'(x_1) - \eta'(x_2)]^2.$$

(e) There exists a point z_4 between x_1 and x_2 such that

$$\eta'(x_1|x_2) = \left(1 - \frac{\eta''(x_2)}{\eta''(z_4)}\right)[\eta'(x_1) - \eta'(x_2)].$$

(f) There exists a point z_5 between x_1 and x_2 such that

$$F(x_1;x_2) = \frac{1}{2}\eta''(z_5)\frac{Q'(z_5)}{\eta''(z_0)^2}(\eta'(x_1) - \eta'(x_2))^2 + \frac{1}{2}\eta'(z_5) - \eta'(x_2)]\frac{Q''(z_5)}{\eta''(z_0)^2}(\eta'(x_1) - \eta'(x_2))^2.$$

(g) If s is a stationary shock solution to (0.1) *with $\nu = 1$, and $\varsigma \in (s_+, s_-)$ is a real number, then there exist points $z_6, z_7, z_8 \in (s_+, s_-)$ such that*

$$-\eta''(s)s'\restriction_{s=\varsigma} = \frac{Q''(z_6)}{2\eta''(z_7)\eta''(z_8)}[\eta'(\varsigma) - \eta'(s_+)][\eta'(s_-) - \eta'(\varsigma)].$$

Proof. Claim (a) follows immediately from Taylor's theorem:

$$\eta'(x_1) = \eta'(x_2) + \eta''(z_0)(x_1 - x_2).$$

Applying Taylor's theorem to Q,

$$Q(x_1) = Q(x_2) + Q'(x_2)(x_1 - x_2) + \frac{Q''(z_1)}{2}(x_1 - x_2)^2.$$

Therefore

$$Q(x_1|x_2) = \frac{Q''(z_1)}{2}(x_1 - x_2)^2$$

and (b) follows from (a).

Claims (c) and (d) follow by the same logic as (b).

Apply (a) to the definition of $\eta'(x_1|x_2)$ to obtain

$$\eta'(x_1|x_2)=[\eta'(x_1)-\eta'(x_2)]-\frac{\eta'(x_2)}{\eta'(z_0)}[\eta'(x_1)-\eta'(x_2)]$$

and (e) follows.

For (f), we can calculate, by Taylor's theorem,

$$\begin{aligned}F(x_1;x_2)&=F(x_2;x_2)+\frac{d}{dx_1}F(x_2;x_2)(x_1-x_2)+\frac{1}{2}\frac{d^2}{dx_1^2}F(t_5;x_2)(x_1-x_2)^2\\&=0+0+\frac{1}{2}[\eta''(t_5)Q'(t_5)-Q''(t_5)[\eta'(t_5)-\eta'(x_2)]](x_1-x_2)^2.\end{aligned}$$

From this and (a), the claim (f) follows.

Since s is a shock solution, $\partial_x Q(s)=\partial_{xx}\eta'(s)$. Moreover $\partial_x\eta'(s)\restriction_{s=s_-}=0$. Therefore

$$-\eta''(\varsigma)s'\restriction_{s=\varsigma}=-\partial_x\eta'(\varsigma)\restriction_{s=\varsigma}=Q(s_-)-Q(y).$$

Now since $Q(s_+)=Q(s_-)$ by the Rankine-Hugoniot condition, there exists a point $z_6\in(s_+,s_-)$ such that

$$Q(s_-)-Q(\varsigma)=Q''(z_6)(\varsigma-s_+)(s_--\varsigma).$$

Applying (a) a final time, the proof is complete. □

The following lemma is Proposition 3.3 in [KV19]. It is a Poincaré type inequality.

Lemma 6.2.2 (Poincaré)**.** *Given a constant C_1, there exists a constant $\delta_0>0$, such that for any $\delta\le\delta_0$ the following holds:*

For any $W \in L^2(0,1)$ such that $\sqrt{x(1-x)}\partial_x W \in L^2(0,1)$ with $\|W\|_2^2 \le C_1$, the quantity

$$\frac{-1}{\delta}\left(\int_0^1 W^2 dx + 2\int_0^1 W\,dx\right)^2 + (1+\delta)\int_0^1 W^2 dx + \frac{2}{3}\int_0^1 W^3 dx + \delta\int_0^1 |W|^3\,dx - (1-\delta)\int_0^1 x(1-x)|\partial_x W|^2 dx$$

is non-positive.

The following lemma is a kind of weighted Gagliardo-Nirenberg interpolation. The quantity $D(u)$ defined in Lemma 6.1.2 controls the second derivative of w but that control degenerates near the endpoints. The lemma interpolates between D and the L^2 norm to control arbitrary L^p norms.

Lemma 6.2.3 (Gagliardo-Nirenberg)**.** *Let $h>0$, $p\ge 1$, $L>0$, and $\bar{C} \le 2h^2 L$ be constants. For any $w \in L^2([-L,L])$ with*

$$\int_{-L}^{L} w^2\,dy \le \bar{C},$$

define

$$\tilde{D} := \int_{-L}^{L} (y-L)(L-y)\chi_{\{|w|>h\}}\left|\partial_y w\right|^2 dy.$$

Then for any $q \in (0,1)$ there exists a constant C_q depending only on q such that

$$\int (w-h)_+^p\,dy \le C_q\left(h^{-2}\bar{C}\right)^q |L|^{-p/2}\tilde{D}^{p/2}.$$

Proof. By Chebyshev's inequality, $|\{|w|>h\}| \le h^{-2}\bar{C}$ so since $h^{-2}\bar{C} \le L$ there exists a point $y_0 \in [-L/2, L/2]$ such that $(w-h)_+(y_0) = 0$.

For any other point y_1, we can calculate

$$\begin{aligned}|(w-h)_+(y_1)| &= |(w-h)_+(y_1) - (w-h)_+(y_2)| \\ &\le \int_{y_0}^{y_1} |\partial_y (w-h)_+|\,dy\end{aligned}$$

$$\le \left(\int_{y_0}^{y_1} [(L+y)(L-y)]^{-1}\,dy \right)^{1/2} \left(\int_{y_0}^{y_1} [(L+y)(L-y)]\,|\partial_y (w-h)_+|^2\,dy \right)^{1/2}$$
$$\le \left(\frac{1}{2L} [\ln(L+y) - \ln(L-y)]_{y_0}^{y_1} \right)^{1/2} \tilde{D}^{1/2}$$
$$= \frac{\tilde{D}^{1/2}}{(2L)^{1/2}} [\ln(L+y_1) - \ln(L-y_1) - \ln(L+y_0) + \ln(L-y_0)]^{1/2}$$

Since $y_0 \in [-L/2, L/2]$, we can estimate $\frac{L+y_0}{L-y_0} \in [1/3, 3]$ so $\ln(L+y_0) - \ln(L-y_0)$ is bounded. The expression $\ln(L+y_1) - \ln(y_1 - L)$ is similarly bounded for $|y_1| < L/2$. For $|y_1| > L/2$, the $\ln(L-y_1)$ term will dominate for y_1 positive and the $\ln(y_1 + L)$ term will dominate for y_1 negative, so for some constant C we have the bound

$$|(w-h)_+(y_1)| \le C \left(\frac{\tilde{D}}{L} \right)^{1/2} \max(1, |\ln(L - |y_1|)|)^{1/2}.$$

Let $\mu \le h^{-2}\bar{C}$ be the measure of the set $\{|w| > h\}$. Without loss of generality we assume that this region is concentrated near $\pm L$, and so

$$\begin{aligned} \int (w-h)_+^p\,dy &\le 2 \int_{L-\mu/2}^{L} C^p \left(\frac{\tilde{D}}{L} \right)^{p/2} |\ln(L - |y_1|)|^{p/2}\,dy \\ &\le C \left(\frac{\tilde{D}}{L} \right)^{p/2} \int_0^{\mu/2} |\ln(x)|^{p/2}\,dx \\ &\le C_q \left(\frac{\tilde{D}}{L} \right)^{p/2} \mu^q. \end{aligned}$$

Here we have used an estimate of the integral of $\ln(x)$ near the origin which uses the fact that $\ln(x)$ grows slower than any power of x.

Since $\mu \le h^{-2}\bar{C}$, the lemma follows. $\square$

The following final lemma shows that the quantity Y bounds the L^2 norm.

Lemma 6.2.4. *There exists a constant $C = C(\Lambda)$ so that the following holds:*

Let η and Q as in Theorem 6.0.2 and u,s be any functions such that $u - s \in L^2(\mathbb{R})$. Let $Y(u)$ be as in Lemma 6.1.2. Then the function $w := \eta'(u) - \eta'(s)$ satisfies

$$\int w^2 \, dy \le C(\Lambda) \left[\frac{\varepsilon}{\lambda} |Y(u)| + \frac{\varepsilon^3}{\lambda^2} \right].$$

Proof. From the definition of Y, we know that

$$\frac{\lambda}{\varepsilon} \int \eta(u|s) \, dy \le |Y| + \int a(u-s) \, dy.$$

The right-hand side is of course non-negative since η convex.

Recall the notation $w = \eta'(u) - \eta'(s)$. From Lemma 6.2.1 (c) and (a) we know that $\eta(u|s) \ge \Lambda^{-3} w^2$ and $|u - s| \le \Lambda |w|$. Of course $|a| \le 2$. Therefore

$$\int w^2 \, dy \le \Lambda^3 \frac{\varepsilon}{\lambda} |Y| + \Lambda^3 \frac{\varepsilon}{\lambda} 2\Lambda \int |w| \, dy.$$

By Hölder's inequality, $2 \int |w| \, dy \le \frac{\lambda}{2\Lambda^4 \varepsilon} \int w^2 \, dy + \frac{2\Lambda^4 \varepsilon}{\lambda} \int 1 \, dy$. Thus

$$\int w^2 \, dy \le \Lambda^3 \frac{\varepsilon}{\lambda} |Y| + \frac{1}{2} \int w^2 \, dy + 2\Lambda^8 \frac{\varepsilon^2}{\lambda^2} \int 1 \, dy.$$

Since

$$\int 1 \, dy = \eta'(s_-) - \eta'(s_+) \le 2\Lambda\varepsilon,$$

the lemma follows. ∎

6.3 Functional Estimates

In this section, we consider the quantity $-Y(u)^2+B(u)-D(u)$ under certain assumptions on u. Note that we do not need to assume u is a solution of (0.1) in this section at all, only that u and s are in some sense small functions.

Proposition 6.3.1 (Decrease for small perturbations)**.** *Let η and Q satisfy* (0.2) *for all $x\in\mathbb{R}$ and have η''', Q''' continuous at 0. For any positive constant $\bar{C}$, there exist constants $h_1>0$ and $\varepsilon_1>0$, such that the following holds:*

Let s be a stationary shock solution to (0.1) *with $\nu=1$ and $s_\pm=\mp\varepsilon$ with $0<\varepsilon<\varepsilon_1$, and let $\bar{u}\in L^\infty(\mathbb{R})$ be such that $|\bar{w}|:=|\eta'(\bar{u})-\eta'(s)|\leq h$ for some $0<h<h_1$. Let $0<\lambda<\varepsilon_1$ and $a:=1-\frac{\lambda}{\varepsilon}\eta'(s)$ such that $1/2\leq a\leq 2$. Assume*

$$\int \bar{w}^2\leq \bar{C}\frac{\varepsilon^3}{\lambda^2}.$$

Then

$$\bar{R}:=\frac{-1}{2\varepsilon^2}Y(\bar{u})^2+B(\bar{u})-(1-h)D(\bar{u})$$

is non-positive.

In the case that η and Q are quadratic polynomials, for example if $\Lambda=1$, this theorem would hold by a straightforward application of Lemma 6.2.2. Since η and Q have continuous second derivatives, for small inputs their second derivatives will be nearly constant and we can treat them as polynomials. We will use Taylor's theorem, specifically in the form of Lemma 6.2.1, to formalize this observation.

Proof. Let δ_0 be the constant indicated by Lemma 6.2.2 corresponding to constant $\Lambda\bar{C}$, and consider arbitrary $0<\delta\leq\delta_0$.

We will estimate Y, B, and D using the formulae provided in Lemma 6.1.2. Notice that, since η''' and Q''' exist and are continuous at 0, η'' and Q'' must also be continuous at 0.

First we analyze the term Y. Define

$$Y_1 := \frac{\lambda}{\varepsilon}\int \eta(\bar{u}|s)\,dy.$$

By Lemma 6.2.1 (c), there exist $t_1,\, t_2 \in [-\varepsilon_1 - h_1, \varepsilon_1 + h_1]$ so

$$\left|Y_1 - \frac{1}{2\eta''(0)}\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy\right| = \left|\frac{\eta''(t_1)}{2\eta''(t_2)^2} - \frac{1}{2\eta''(0)}\right|\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy.$$

Since η'' is continuous at 0, for $\varepsilon_1 + h_1$ sufficiently small we can say

$$\left|Y_1 - \frac{1}{2\eta''(0)}\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy\right| \le \delta\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy.$$

Define

$$Y_2 = \int a(u-s)\,dy$$

and, by applying Lemma 6.2.1 (a), we can argue as above that for $\varepsilon_1 + h_1$ sufficiently small we have

$$\begin{aligned}
\left|Y_2 - \eta''(0)^{-1}\int \bar{w}\,dy\right| &= \int \left[\eta''(t_1)^{-1}a - \eta''(0)^{-1}\right]\bar{w}\,dy \\
&= \int \eta''(t_1)^{-1}(a-1)\bar{w}\,dy + \int \left[\eta''(t_1)^{-1} - \eta''(0)^{-1}\right]\bar{w}\,dy \\
&\le \int \left(\lambda\Lambda + \left|\eta''(t_1)^{-1} - \eta''(0)^{-1}\right|\right)|\bar{w}|\,dy \\
&\le C(\lambda+\delta)\int |\bar{w}|\,dy.
\end{aligned}$$

Since $Y = Y_1 + Y_2$, assuming without loss of generality $\varepsilon < \delta$, we can apply the general formula $-(a+b)^2 \le -\left(\frac{\varepsilon}{2\delta}\right)a^2 + \frac{\varepsilon}{\delta}b^2$ for $a,b \in \mathbb{R}$ and $\varepsilon/\delta \in (0,1]$ to obtain

$$-Y^2 \le \frac{\varepsilon}{8\delta}\eta''(0)^{-2}\left(\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy + 2\int \bar{w}\,dy\right)^2 + C\frac{\varepsilon}{\delta}\left((\lambda+\delta)\int |\bar{w}|\,dy + \delta\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy\right)^2$$

Since $\int |\bar{w}|\,dy \le C\varepsilon^{1/2}\sqrt{\int \bar{w}^2\,dy}$ and $\int \bar{w}^2\,dy \le \bar{C}\varepsilon^3/\lambda^2$,

$$\begin{aligned}-Y^2 &\le \frac{\varepsilon}{\delta}\frac{\Lambda^2}{8}\left(\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy + 2\int \bar{w}\,dy\right)^2 + C(\lambda+\delta)^2\frac{\varepsilon}{\delta}\left(\int |\bar{w}|\,dy\right)^2 + C\frac{\varepsilon}{\delta}\delta^2\frac{\lambda^2}{\varepsilon^2}\frac{\varepsilon^3}{\lambda^2}\int \bar{w}^2\,dy \\ &\le \frac{\varepsilon}{\delta}\frac{\Lambda^2}{8}\left(\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy + 2\int \bar{w}\,dy\right)^2 + C\left(\frac{\varepsilon^2\lambda^2}{\delta} + \varepsilon^2\delta + \delta\right)\int \bar{w}^2\,dy.\end{aligned} \tag{3.6}$$

Now we analyze B.

For the relative flux term, we estimate by Lemma 6.2.1 (b) and continuity of η'' and Q''

$$\begin{aligned}\left|\int aQ(\bar{u}|s)\,dy - \frac{Q''(0)}{2\eta''(0)^2}\int \bar{w}^2\,dy\right| &\le \int \left|a\frac{Q''(t_1)}{2\eta''(t_2)} - \frac{Q''(0)}{2\eta''(0)^2}\right|\bar{w}^2\,dy \\ &\le \int \left(\frac{Q''(t_1)}{2\eta''(t_2)}|a-1| + \left|\frac{Q''(t_1)}{2\eta''(t_2)} - \frac{Q''(0)}{2\eta''(0)^2}\right|\right)\bar{w}^2\,dy \\ &\le \int \left(\lambda\frac{Q''(t_1)}{2\eta''(t_2)} + \left|\frac{Q''(t_1)}{2\eta''(t_2)} - \frac{Q''(0)}{2\eta''(0)^2}\right|\right)\bar{w}^2\,dy \\ &\le C(\lambda+\delta)\int \bar{w}^2\,dy\end{aligned} \tag{3.7}$$

for ε_1 and h_1 sufficiently small.

The $\bar{w}^2$ term is an error term:

$$\frac{\lambda}{\varepsilon}\int \frac{Q'(s)}{2\eta''(s)}\bar{w}^2\,dy \le \lambda\frac{\Lambda^2}{2}\int \bar{w}^2\,dy, \tag{3.8}$$

as is the $\eta'(\bar{u}|s)$ term: by Lemma 6.2.1 (d), for ε_0 and h_0 sufficiently small

$$\int a\frac{Q'(s)}{\eta''(s)}\eta'(\bar{u}|s)\,dy \le C\varepsilon\int \bar{w}^2\,dy. \tag{3.9}$$

Note that C here depends on $\eta'''(0)$.

To bound the F term of B, we utilize the formula, valid for any f with $f(0)=f'(0)=0$,

$$f(x) = \int_0^x f''(t)(x-t)\,dt.$$

Since

$$\frac{d^2}{dx^2}F(x;s) = Q''(x)[\eta'(x)-\eta'(s)] + Q'(x)\eta''(x),$$

we have, letting $c_Q \in [-\varepsilon,\varepsilon]$ be the unique point such that $Q'(c_Q)=0$,

$$\begin{aligned}
F(x;s) &= \int_s^x (x-\tau)[\eta'(\tau)-\eta'(s)]Q''(\tau) + (x-\tau)\eta''(\tau)Q'(\tau)\,d\tau \\
&= \int_s^x (x-\tau)(\tau-s)\eta''(t_1)Q''(\tau)\,d\tau + \int_s^x \eta''(\tau)Q''(t_2)(x-\tau)(\tau-c_Q)\,d\tau \\
&= \int_s^x \eta''(t_1)Q''(\tau)(x-\tau)(\tau-s)\,d\tau + \int_s^x \eta''(\tau)Q''(t_2)(x-\tau)(\tau-s)\,d\tau \\
&\qquad + \int_s^x \eta''(\tau)Q''(t_2)(x-\tau)(s-c_Q)\,d\tau
\end{aligned} \tag{3.10}$$

for some points $t_1 \in [s,\tau]$ and $t_2 \in [c_Q,\tau]$ depending on τ.

We can estimate each of these three integrals:

$$\begin{aligned}
&\left|\int_s^x \eta''(t_1)Q''(\tau)(x-\tau)(\tau-s)\,d\tau - \eta''(0)Q''(0)\frac{(x-s)^3}{6}\right| \le \sup_\tau |\eta''(t_1)Q''(\tau)-\eta''(0)Q''(0)|\,\frac{|x-s|^3}{6}, \\
&\left|\int_s^x \eta''(\tau)Q''(t_2)(x-\tau)(\tau-s)\,d\tau - \eta''(0)Q''(0)\frac{(x-s)^3}{6}\right| \le \sup_\tau |\eta''(\tau)Q''(t_2)-\eta''(0)Q''(0)|\,\frac{|x-s|^3}{6}, \\
&\left|\int_s^x \eta''(\tau)Q''(t_2)(x-\tau)(s-c_Q)\,d\tau\right| \le 2\varepsilon\Lambda^2\int_x^s |x-\tau|\,d\tau = \varepsilon\Lambda^2(x-s)^2.
\end{aligned} \tag{3.11}$$

Therefore, if ε_1 and h_1 are sufficiently small then from (3.10) and (3.11) we obtain

$$\left|\frac{\lambda}{\varepsilon}\int F(\bar{u};s)\,dy - \frac{\lambda}{\varepsilon}\frac{Q''(0)}{3\eta''(0)^2}\int \bar{w}^3\,dy\right| \le C\frac{\lambda}{\varepsilon}\delta\int |\bar{w}|^3\,dy + C\lambda\int \bar{w}^2\,dy. \tag{3.12}$$

Combining (3.7), (3.8), (3.9), and (3.12),

$$B \le \frac{\lambda}{\varepsilon}\frac{Q''(0)}{3\eta''(0)^2}\int \bar{w}^3\,dy + \delta\frac{\lambda}{\varepsilon}C\int |\bar{w}|^3\,dy + \frac{Q''(0)}{2\eta''(0)^2}\int \bar{w}^2\,dy + C(\lambda+\delta+\varepsilon)\int \bar{w}^2\,dy. \tag{3.13}$$

Lastly, we bound the quantity D. Define $y_\pm := \eta'(\mp\varepsilon)$. Applying Lemma 6.2.1 (g),

$$\begin{aligned}(1-h)D(\bar{u}) &\geq \frac{Q''(t_1)}{2\eta''(t_2)\eta''(t_3)}(1-h)\int [y-y_-][y_+-y]|\partial_y \bar{w}|^2\,dy \\ &\geq \frac{Q''(0)}{2\eta''(0)^2}(1-\delta)\int [y-y_-][y_+-y]|\partial_y \bar{w}|^2\,dy\end{aligned} \tag{3.14}$$

so long as ε_1 and h_1 are sufficiently small.

We can now bound the quantity $\bar{R}$. By combining the bounds (3.6), (3.13), and (3.14) on Y, B, and D respectively,

$$\begin{aligned}\bar{R} \leq & \frac{-C}{\varepsilon\delta}\left(\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy + 2\int \bar{w}\,dy\right)^2 + \frac{\lambda}{\varepsilon}\frac{Q''(0)}{3\eta''(0)^2}\int \bar{w}^3 + \frac{Q''(0)}{2\eta''(0)^2}\int \bar{w}^2 \\ & - \frac{Q''(0)}{2\eta''(0)^2}(1-\delta)\int [y-y_-][y_+-y]|\partial_y \bar{w}|^2\,dy \\ & + C\left(\frac{\lambda^2}{\delta} + \lambda + \delta + \varepsilon\right)\int \bar{w}^2\,dy + C\frac{\lambda\delta}{\varepsilon}\int |\bar{w}|^3\,dy \\ = & \frac{Q''(0)}{2\eta''(0)^2}\Bigg[\frac{-C}{\varepsilon\delta}\left(\frac{\lambda}{\varepsilon}\int \bar{w}^2\,dy + 2\int \bar{w}\,dy\right)^2 + \frac{\lambda}{\varepsilon}\frac{2}{3}\int \bar{w}^3 + \int \bar{w}^2 \\ & - (1-\delta)\int [y-y_-][y_+-y]|\partial_y \bar{w}|^2\,dy \\ & + C\left(\frac{\varepsilon^2\lambda^2}{\delta} + \lambda + \delta + \varepsilon\right)\int \bar{w}^2\,dy + C\frac{\lambda\delta}{\varepsilon}\int |\bar{w}|^3\,dy\Bigg]\end{aligned} \tag{3.15}$$

We will now perform a change of coordinates. Let $L := \eta'(s_+) - \eta'(s_-)$. Consider $z \in [0,1]$ and define

$$\begin{aligned} y &:= \eta'(s_+) + zL, \\ dy &= L\,dz, \\ W(z) &:= \frac{\lambda}{\varepsilon}\bar{w}(y) = \frac{\lambda}{\varepsilon}\bar{w}\left(\eta'(s_+) + zL\right), \\ \partial_z W(z) &= \frac{\lambda}{\varepsilon}L\partial_y \bar{w}(y).\end{aligned}$$

Note that $z=0$ corresponds to $y=\eta'(s_+)$ and $z=1$ to $y=\eta'(s_-)$.

In these coordinates,

$$\int \bar{w}\,dy = \frac{\varepsilon}{\lambda} L \int W\,dz,$$
$$\int \bar{w}^2\,dy = \frac{\varepsilon^2}{\lambda^2} L \int W^2\,dz,$$
$$\int \bar{w}^3\,dy = \frac{\varepsilon^3}{\lambda^3} L \int W^3\,dz,$$
$$\int [y-y_-][y_+-y]|\partial_y \bar{w}|^2\,dy = \frac{\varepsilon^2}{\lambda^2} L \int z(1-z)|\partial_z W|^2\,dz.$$

In terms of z and W, (3.15) becomes

$$\bar{R} \leq \frac{LQ''(0)}{2\eta''(0)^2}\frac{\varepsilon^2}{\lambda^2}\left[\frac{-C_2 L}{\varepsilon\delta}\left(\int W^2\,dz + 2\int W\,dz\right)^2 + \frac{2}{3}\int W^3\,dz + \int W^2\,dz - (1-\delta)\int z(1-z)|\partial_z W|^2\,dz\right.$$
$$\left. + C_3\left(\frac{\varepsilon^2\lambda^2}{\delta} + \lambda + \delta + \varepsilon\right)\int W^2\,dz + C\delta\int |W|^3\,dz\right]$$

Fixing now δ so that $\delta < \frac{\delta_0}{3C_3}$ and $\delta < C_2\Lambda\delta_0$, then taking ε_1 small enough that $C_3(\frac{\lambda^2}{\delta}+\delta+\varepsilon+\lambda+\delta)\leq\delta_0$ and $\varepsilon_1<\delta$, and recalling $L/\varepsilon\leq\Lambda$, we can bound

$$\bar{R} \leq C\frac{\varepsilon^2}{\lambda^2}\left[\frac{-1}{\delta_0}\left(\int W^2\,dz + 2\int W\,dz\right)^2 + \frac{2}{3}\int W^3\,dz + \delta_0\int |W|^3\,dz + (1+\delta_0)\int W^2\,dz\right.$$
$$\left. - (1-\delta_0)\int z(1-z)|\partial_z W|^2\,dz\right].$$

We can now apply Lemma 6.2.2 and the proof is complete. □

Now that we know $-\varepsilon^{-2}Y^2 + B - D$ is non-negative for u sufficiently close to s, we can bound the same quantity for u large by decomposing into a part near s and a part far away.

Proposition 6.3.2 (Decrease for large perturbations)**.** *Let η and Q satisfy* (0.2) *for all $x\in\mathbb{R}$ and have η''', Q''' continuous at 0. For any positive constant $\bar{C}$, there exists a constant $\varepsilon_2>0$ such that the following holds:*

Let s be a stationary shock solution to (0.1) *with $\nu=1$ and $s_\pm=\mp\varepsilon$ with $0<\varepsilon<\varepsilon_2$. There exists a $\lambda>0$ such that for all $u:\mathbb{R}\to\mathbb{R}$ such that $w:=\eta'(u)-\eta'(s)$ satisfies*

$$\int w^2\leq\bar{C}\frac{\varepsilon^3}{\lambda^2},$$

u and $a:=1-\frac{\lambda}{\varepsilon}\eta'(s)$ satisfy

$$R(u):=\frac{-1}{2\varepsilon^2}Y(u)^2+B(u)-D(u)\leq-\varepsilon_2 D(u).$$

Proof. Let h_1 and ε_1 be the parameters defined by Proposition 6.3.1, and define $\bar{u}$ for a parameter $0<h<h_1$ such that

$$\begin{cases}\bar{u}=u & |\eta'(u)-\eta'(s)|\leq h,\\ \eta'(u)-\eta'(s)=h\,\mathrm{sign}(u-s) & \text{else.}\end{cases}$$

Then we can define $\bar{w}:=\eta'(\bar{u})-\eta'(s)$, $\tilde{w}:=w-\bar{w}$, $\tilde{Y}:=Y(u)-Y(\bar{u})$, $\tilde{B}:=B(u)-B(\bar{u})$, and $\tilde{D}:=D(u)-D(\bar{u})$. For $\tilde{D}$ we have

$$\tilde{D}=\int a\chi_{\{|w|>h\}}|\partial_x(\eta'(u)-\eta'(s))|^2dx. \tag{3.16}$$

We will bound $\tilde{Y}$, $\tilde{B}$, and $\tilde{D}$ one at a time.

To bound $\tilde{Y}$, we calculate

$$\begin{aligned}\eta(u|s)-\eta(\bar{u}|s)&=\int_{\bar{u}}^{u}\eta'(t)-\eta'(s)\,dt\\ &\leq\Lambda\int_{\bar{u}}^{u}[t-s]\,dt\end{aligned}$$

$$\begin{aligned}
&= \Lambda\left[\frac{t^2}{2} - ts\right]_{\bar{u}}^{u} \\
&= \Lambda\left[\frac{(u-\bar{u})^2}{2} + (u-\bar{u})(\bar{u}-s)\right] \\
&\leq C\left(\tilde{w}^2 + h\tilde{w}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{Y} &\leq C\frac{\lambda}{\varepsilon}\left(\int \tilde{w}^2 + h\int \tilde{w}\right) + \int \tilde{w} \\
&\leq C\frac{\lambda}{\varepsilon}\int \tilde{w}^2\,dy + C\left(\frac{\lambda h}{\varepsilon} + 1\right)\int \tilde{w}\,dy.
\end{aligned}$$

Since

$$-Y(u)^2 \leq -Y(\bar{u})^2/2 + \tilde{Y}^2,$$

and $\int \tilde{w} \leq \varepsilon^{1/2}\left(\int \tilde{w}^2\right)^{1/2}$, and $\int \tilde{w}^2 \leq \bar{C}\varepsilon^3/\lambda^2$, we can bound

$$\frac{-1}{\varepsilon^2}Y(u)^2 \leq \frac{-1}{2\varepsilon^2}Y(\bar{u})^2 + C\varepsilon\int \tilde{w}^2\,dy + C\left(\varepsilon + \frac{\lambda^2 h^2}{\varepsilon}\right)\int \tilde{w}\,dy. \tag{3.17}$$

For the B term, we must assume without loss of generality that $2\Lambda\varepsilon \leq h_1$ (so that Q' does not change sign between $\bar{u}$ and u). Then we can calculate

$$\begin{aligned}
F(u;s) - F(\bar{u};s) &= \int_{\bar{u}}^{u} Q'(t)[\eta'(t) - \eta'(s)]\,dt \\
&\leq \Lambda^2\left|\int_{\bar{u}}^{u} t[t-s]\,dt\right| \\
&= \Lambda^2\left|\frac{t^3}{3} - \frac{t^2 s}{2}\right|_{\bar{u}}^{u} \\
&= \Lambda^2\left|\frac{(u-\bar{u})^3}{3} + (u-\bar{u})^2\bar{u} - \frac{(u-\bar{u})^2 s}{2} + (u-\bar{u})\bar{u}^2 - (u-\bar{u})\bar{u}s\right| \\
&\leq C\left(|\tilde{w}|^3 + h\tilde{w}^2 + h^2|\tilde{w}| + \varepsilon\tilde{w}^2 + \varepsilon h|\tilde{w}|\right).
\end{aligned} \tag{3.18}$$

Similarly,

$$
\begin{aligned}
Q(u|s)-Q(\bar{u}|s) &= \int_{\bar{u}}^{u} [Q'(t)-Q'(s)]\,dt \\
&\le \Lambda \int_{\bar{u}}^{u} [t-s]\,dt \\
&= \Lambda \left[\frac{(u-\bar{u})^2}{2} + (\bar{u}-s)(u-\bar{u}) \right] \\
&\le C\left(\tilde{w}^2 + h\tilde{w}\right),
\end{aligned} \tag{3.19}
$$

and

$$
\begin{aligned}
\eta'(u|s)-\eta'(\bar{u}|s) &= \int_{\bar{u}}^{u} \eta''(x)-\eta''(s)\,dx \\
&\le 2\Lambda \left| \int_{\bar{u}}^{u} dx \right| \\
&\le 2\Lambda^2 |\tilde{w}|,
\end{aligned} \tag{3.20}
$$

and trivially

$$
\begin{aligned}
w^2-\bar{w}^2 &= \tilde{w}^2 + 2\bar{w}\tilde{w} \\
&\le \tilde{w}^2 + 2h|\tilde{w}|.
\end{aligned} \tag{3.21}
$$

Combining (3.18), (3.19), (3.20), and (3.21), we can bound $\tilde{B}$

$$
\begin{aligned}
|\tilde{B}| &\le C\left(\frac{\lambda}{\varepsilon}|\tilde{w}|^3 + \left[\frac{\lambda}{\varepsilon}(h+\varepsilon)+1+\lambda\right]\tilde{w}^2 + \left[\frac{\lambda}{\varepsilon}(h^2+\varepsilon h)+(\varepsilon+h)+\varepsilon+\lambda h\right]|\tilde{w}| \right) \\
&= C\left(\frac{\lambda}{\varepsilon}|\tilde{w}|^3 + \left[\frac{\lambda h}{\varepsilon}+1+\lambda\right]\tilde{w}^2 + \left[\frac{\lambda h^2}{\varepsilon}+\varepsilon+h+\lambda h\right]|\tilde{w}| \right).
\end{aligned} \tag{3.22}
$$

Using (3.16), (3.17), and (3.22), we can decompose the quantity $R(u)$ as

$$
\begin{aligned}
R(u) \le & \left[\frac{-1}{2\varepsilon^2} Y(\bar{u})^2 - B(\bar{u}) - (1-h)D(\bar{u}) \right] \\
&+ \left[\frac{1}{\varepsilon^2}\left(\frac{\lambda}{\varepsilon}\int \tilde{w}^2\,dy + \left(1+\frac{\lambda h}{\varepsilon}\right)\int \tilde{w}\,dy \right)^2 + \frac{\lambda}{\varepsilon}\int \tilde{w}^3\,dy + \frac{\lambda h}{\varepsilon}\int \tilde{w}^2\,dy - (1-h)\tilde{D} \right] \\
&+ \left[\frac{\lambda h^2}{\varepsilon}\int \tilde{w}\,dy - \frac{h}{2}D(u) \right] - \frac{h}{2}D(u) \\
&:= R_1 + R_2 + R_3 - \frac{h}{2}D(u).
\end{aligned} \tag{3.23}
$$

By Proposition 6.3.1, we know $R_1 \leq 0$. It remains to show the same for R_2 and R_3.

Using the fact that $\int \tilde{w}^2\,dy \leq \bar{C}\varepsilon^3/\lambda^2$, we can bound the quantity R_2

$$R_2 \leq \frac{1}{\varepsilon^2}\left[\frac{\lambda^2}{\varepsilon^2}\left(\int \tilde{w}^2\,dy\right)\int \tilde{w}^2\,dy + \left(1+\frac{h\lambda}{\varepsilon}\right)^2\left(\int \tilde{w}\,dy\right)^2\right] + \frac{h\lambda}{\varepsilon}\int \tilde{w}^2\,dy + \frac{\lambda}{\varepsilon}\int \tilde{w}^3\,dy - (1-h)\tilde{D}$$
$$\leq \left(\frac{1}{\varepsilon}+\frac{h\lambda}{\varepsilon}\right)\int \tilde{w}^2\,dy + \left(\frac{1}{\varepsilon^2}+\frac{h^2\lambda^2}{\varepsilon^4}\right)\left(\int \tilde{w}\,dy\right)^2 + \frac{\lambda}{\varepsilon}\int \tilde{w}^3\,dy - (1-h)\tilde{D}. \tag{3.24}$$

By Lemma 6.2.3, we know that for any exponent $q \in (0,1)$ we have

$$\int \tilde{w}\,dy \leq C_q\left(\frac{\varepsilon^3}{h^2\lambda^2}\right)^q \varepsilon^{-1/2}\tilde{D}^{1/2},$$
$$\int \tilde{w}^2\,dy \leq C_q\left(\frac{\varepsilon^3}{h^2\lambda^2}\right)^q \varepsilon^{-1}\tilde{D},$$
$$\int \tilde{w}^3\,dy \leq C_q\left(\int \tilde{w}^2\,dy\right)^{1/2}\left(\int \tilde{w}^4\,dy\right)^{1/2} \leq \left(\frac{\varepsilon^3}{h^2\lambda^2}\right)^{q/2}\frac{\varepsilon^{1/2}}{h\lambda}\tilde{D}.$$

From these estimates with the appropriate q, we find that if ε, λ and h are appropriately small (specifically if $\varepsilon \leq C_h\lambda^3$ for constant C_h depending on h) then

$$\left(\frac{1}{\varepsilon}+\frac{h\lambda}{\varepsilon}\right)\int \tilde{w}^2\,dy + \left(\frac{1}{\varepsilon^2}+\frac{h^2\lambda^2}{\varepsilon^4}\right)\left(\int \tilde{w}\,dy\right)^2 + \frac{\lambda}{\varepsilon}\int \tilde{w}^3\,dy \leq \frac{1}{2}\tilde{D}.$$

Plugging this estimate into (3.24), and assuming without loss of generality $h < 1/2$, the quantity R_2 will be non-positive.

It remains to bound the quantity R_3.

Let $f := \left(|w| - \frac{h}{2}\right)_+$. Then $\tilde{w} = \left(f - \frac{h}{2}\right)_+$.

By Lemma 6.2.3 with exponent 3/4,

$$\int f^2\,dy \leq C\frac{\varepsilon^{5/4}}{h^{3/2}\lambda^{3/2}}D(u).$$

By Chebyshev's inquality,

$$\int |\tilde{w}|\,dy = \int (|w|-h)_+\,dy \le \frac{2}{h}\int f^2\,dy.$$

Therefore,

$$R_3 \le \left(C\frac{\lambda h^2}{\varepsilon}\frac{2}{h}\frac{\varepsilon^{5/4}}{h^{3/2}\lambda^{3/2}} - \frac{h}{2}\right) D(u) = \left(C\frac{\varepsilon^{1/4}}{h^{3/2}\lambda^{1/2}} - 1\right)\frac{h}{2}D(u).$$

So long as $\varepsilon < \left(C^{-1}h^{3/2}\lambda^{1/2}\right)^4$, the quantity is non-positive.

Since R_1, R_2, and R_3 are all non-positive, by (3.23) we know $R(u) \le -h/2D(u) \le -\varepsilon_2 D(u)$.

□

6.4 Proof of Main Theorem

We will now prove Theorem 6.0.2. The idea of the proof is to define the shift function γ such that when $|Y(u)|$ is large, the $\dot{\gamma}Y$ term is negative and dominant, while when $|Y(u)|$ is small we can apply Proposition 6.3.2.

Proof. Take ε_0 to be the constant ε_2 defined in Proposition 6.3.2.

We must construct a shift function γ, so we begin by making elementary bounds on the term B. Note that $u(x)-s(x)$ is guaranteed to be in L^2 for short time by the basic existence theorems of, for example, [Ser99]. Moreover,

$$\int |s(x)-s(x-\xi)|^2\,dx \le C(1+\sqrt{(\xi)}). \tag{4.25}$$

From the estimates of Lemma 6.2.1, we know that for some constant C,

$$|B(u)| \le C(\varepsilon,\lambda,\Lambda)\left(\int w^3\,dy + \int w^2\,dy + \int w\,dy\right).$$

Moreover, since by Hölder's inequality $\int w^3\,dy \le \left(\int w^2\,dy\right)^{3/4}\left(\int w^6\,dy\right)^{1/4}$, we can further say by Lemma 6.2.3, by taking $h^2 = \frac{2\Lambda}{\varepsilon}\int w^2\,dy$, that

$$\begin{aligned}\int w^3\,dy &\le C\left(\int w^2\,dy\right)^{3/4}\left(\Lambda h\varepsilon + \varepsilon^{-1}D^3\right)^{1/4}\\ &\le C(\varepsilon)\left(\int w^2\,dy\right)^{7/8} + C(\varepsilon)\left(\int w^2\,dy\right)^{3/4} D^{3/4}.\end{aligned}$$

It follows that

$$2|B| - (1-\varepsilon_0)D \le C(\varepsilon)\left[1 + \left(\int w^2\,dy\right)^3\right]. \tag{4.26}$$

Of course, $\int w^2\,dy$ depends on γ.

Define

$$\Phi_\varepsilon(y) := \begin{cases} 1 & y \le -\varepsilon^2 \\ \frac{-y}{\varepsilon^2} & |y| \le \varepsilon^2 \\ -1 & y \ge \varepsilon^2. \end{cases}$$

We define $\gamma(t)$ as the solution of the nonlinear ODE:

$$\begin{cases} \dot\gamma(t) &= \Phi_\varepsilon(Y(u^\gamma))\left(\frac{1}{\varepsilon^2}\left(2|B_\gamma(u)| - (1-\varepsilon_1)D_\gamma(u)\right)_+ + 1\right) \\ \gamma(0) &= 0 \end{cases}$$

From (4.25) and (4.26), we know that

$$\left(2|B_\gamma(u)| - (1-\varepsilon_1)D_\gamma(u)\right)_+ \le C(\varepsilon, \int |u(x) - s(x)|^2\,dx)\left[1 + |\gamma(t)|^{3/2}\right].$$

Therefore the quantity γ exists for a short time.

If $|Y| \ge \varepsilon^2$ then

$$\begin{aligned}\dot\gamma Y + B - D &\le -2\left(2|B| - (1-\varepsilon_0)D\right)_+ + 1 + B - D\\ &\le -2|B| + (1-\varepsilon_0)D - \varepsilon^2 + B - D < -\varepsilon_0 D.\end{aligned}$$

Alternatively, if $|Y| \leq \varepsilon^2$, then

$$\dot{\gamma}Y \leq -\frac{1}{\varepsilon^2}Y^2.$$

We can therefore apply Proposition 6.3.2 and conclude that

$$\dot{\gamma}Y + B - D < -\varepsilon_0 D.$$

It follows, from Proposition 6.1.1, that $\int |u(x) - s(x-\gamma(t))|^2 \, dx$ is uniformly bounded so long as γ exists.

Now that we have a uniform bound on $\int w^2 \, dy$, the bound (4.26) shows that γ exists and is Lipschitz for all time. □

Lastly we prove Theorem 6.0.1.

Proof. The proof is by application of Theorem 6.0.2.

If s is not of the form required by Theorem 6.0.2, we can replace Q by

$$\tilde{Q}(x) := Q(x-a) + bx + c$$

for suitable constants a, b, and c so that s is stationary and centered about 0. Recall that by the Rankine-Hugoniot condition, if $Q(s_+) = Q(s_-)$ then s is stationary.

If η and Q only satisfy the bound (0.2) on a compact interval $[-R, R]$ then, so long as $\|u\|_\infty < R$, we can modify η and Q outside this region and u will solve the modified (0.1).

If $\nu \neq 1$, we merely consider

$$\tilde{u}(t,x) := u(x/\nu, t/\nu)$$

and

$$\tilde{s}(x) := s(x/\nu).$$

Then $\tilde{u}$ solves (and $\tilde{s}$ is a shock solution to)

$$\frac{1}{\nu}\partial_t u + \frac{1}{\nu}\partial_x Q(u) = \frac{1}{\nu^2}\nu\partial_{xx}\eta'(u)$$

which is equivalent to (0.1) with $\nu = 1$. $\square$

Appendix

0.1 Technical Lemmas for Chapter 4

We prove here the averaging lemma used throughout this chapter. This lemma is an immediate corollary of [Béz94] Theorem 6. It is merely a localization of that result.

Lemma 0.1.1 (Averaging Lemma)**.** *Let Ω be an open subset of space-time $\mathbb{R}\times\mathbb{R}^n$, and $\bar{\Omega}$ a compact subset of Ω.*

For any smooth function $\eta\in C_c^\infty(\mathbb{R}^n)$ and any $m\in\mathbb{R}^+$, there exists a constant $C=C(n,m,\eta,\bar{\Omega},\Omega)$ and a constant

$$\alpha=\frac{1}{2(1+m)}$$

such that the following is true:

For any two functions f and g in $L^2(\Omega\times\mathbb{R}^n)$ satisfying

$$[\partial_t+v\cdot\nabla_x]f=g,$$

it is true that

$$\left\|\int \eta f\,dv\right\|_{H^\alpha(\bar{\Omega})}\leq C\left(\|f\|_{L^2(\Omega\times\mathbb{R}^n)}+\left\|(1-\Delta_v)^{-m/2}g\right\|_{L^2(\Omega\times\mathbb{R}^n)}\right).$$

By $\|g\|_{H^\alpha(\bar{\Omega})}$, we mean the infimum of $\|\tilde{g}\|_{H^\alpha(\mathbb{R}^{n+1})}$ over all extensions $\tilde{g}$ of g to $\mathbb{R}^{n+1}$.

Proof. Let $\phi(t,x)$ be a smooth function supported on Ω and identically equal to 1 on $\bar{\Omega}$. Then

$$[\partial_t+v\cdot\nabla_x](\phi f)=\phi g+f[\partial_t+v\cdot\nabla_x]\phi.$$

By [Béz94] Theorem 6,

$$\left\|\phi\int \eta f\,dv\right\|_{H^\alpha(\mathbb{R}\times\mathbb{R}^n)}\leq C\left(\|\phi f\|_{L^2(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)}+\left\|(1-\Delta_v)^{-m/2}(\phi g+f[\partial_t+v\cdot\nabla_x]\phi)\right\|_{L^2(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n)}\right).$$

Because $(1-\Delta_v)^{-m/2}$ is a bounded operator from L^2 to L^2, and because ϕ is a smooth function supported on Ω and depending only on t and x,

$$\left\|(1-\Delta_v)^{\frac{-m}{2}}(\phi g + f[\partial_t + v\cdot\nabla_x]\phi)\right\|_{L^2(\mathbb{R}^{1+n+n})} \leq C(\phi)\left\|(1-\Delta_v)^{\frac{-m}{2}} g\right\|_{L^2(\Omega\times\mathbb{R}^n)} + C(m,\phi)\|f\|_{L^2(\Omega\times\mathbb{R}^n)}.$$

The result follows. □

The following is a technical lemma about the geometry of cones. We use it at the very end of the proof of Proposition 4.4.1.

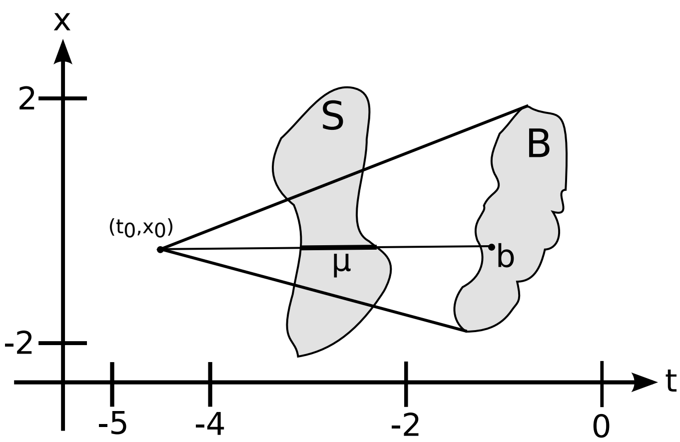

Figure 1: A diagram showing the assumptions of Lemma 0.1.2.

Lemma 0.1.2. *Let $\mathcal{C}\subseteq\mathbb{R}\times\mathbb{R}^n$ be a cone from a vertex $(t_0,x_0)\in[-5,-4]\times B_2$ to a base set $B\subseteq[-2,0]\times B_2$. Let S be a subset of $\mathbb{R}\times\mathbb{R}^n$ such that for each $b\in B$, the line segment connecting (t_0,x_0) to b intersects S on a set with Hausdorff $\mathcal{H}^1$ measure at least μ.*

Then

$$|\mathcal{C}\cap S| \geq \frac{|B|\mu^2}{80}.$$

Proof. Let $A(t)$ be the cross-sectional area of our cone at time slice t. If $\mathcal{H}^n$ is the Hausdorff measure of dimension n, we write

$$A(t) = \mathcal{H}^n(\mathcal{C} \cap [\{t\} \times \mathbb{R}^n]).$$

By the nature of cones, $A(t_0) = 0$, A is affine for $t_0 < t < -2$, then sub-affine for $-2 < t < 0$, and $A(t) = 0$ for $t > 0$. Specifically,

$$A(t) = \frac{A(-2)}{-2-t_0}(t-t_0) \qquad t_0 < t < -2,$$

$$A(t) \leq \frac{A(-2)}{-2-t_0}(t-t_0) \qquad -2 \leq t.$$

Since B is contained in $\mathcal{C} \cap [-2,0] \times \mathbb{R}^n$,

$$|B| \leq \int_{-2}^{0} A(t)\,dt \leq \int_{-2}^{0} \frac{A(-2)}{-2-t_0}(t-t_0)\,dt = \frac{A(-2)}{-2-t_0}\left[t_0^2 - (2+t_0)^2\right]/2 \leq 4A(-2).$$

This means that

$$A(-2) \geq \frac{|B|}{4}.$$

Now we have a lower bound on the size of the cone, so for $t_0 \leq t \leq -2$

$$A(t) \geq \frac{|B|}{4(-2-t_0)}(t-t_0). \tag{1.27}$$

Consider the map from B to $\{0\} \times \mathbb{R}^n$ given by stereographic projection from the point (t_0, x_0), and let db be a probability measure on B proportional to the pullback of $\mathcal{H}^n\lceil_{\{0\}\times\mathbb{R}^n}$ under this projection. Then db represents the proportion of any time-slice of $\mathcal{C}$ generated by rays through a given portion of B.

To find the measure of $\mathcal{C}\cap S$, we must ask how much each time slice intersects S, or in integral form

$$|\mathcal{C}\cap S| = \int_{t_0}^{0} A(t) \int_{b\in B} \chi_{\{(t,x)\in\mathcal{C}\cap S\}}\, db\, dt.$$

By Fubini, this becomes

$$|\mathcal{C}\cap S| = \int_{b\in B} \int_{t_0}^{0} A(t) \chi_{\{(t,x)\in\mathcal{C}\cap S\}}\, dt\, db. \tag{1.28}$$

From the definition of μ and the arc length formula,

$$\mu \le \int_{t_0}^{0} \chi_{\{(t,x)\in\mathcal{C}\cap S\}} \sqrt{1+|b-x_0|^2/(-2-t_0)^2}\, dt \le \sqrt{5}\int_{t_0}^{0} \chi_{\{(t,x)\in\mathcal{C}\cap S\}}.$$

Because $A(t)$ is increasing and $\chi_{\{(t,x)\in\mathcal{C}\cap S\}}$ integrates to at least $\mu/\sqrt{5}$,

$$\int_{t_0}^{0} A(t)\chi_{\{(t,x)\in\mathcal{C}\cap S\}}\, dt \ge \int_{t_0}^{t+\mu/\sqrt{5}} A(t)\, dt.$$

From this bound, (1.28), and (1.27) we can at last compute

$$|\mathcal{C}\cap S| \ge \frac{|B|}{4(-2-t_0)} \int_{t_0}^{t_0+\mu/\sqrt{5}} (t-t_0)\, dt = \frac{|B|}{4(-2-t_0)} \frac{\mu^2}{10} \ge \frac{|B|\mu^2}{80}.$$

□

The following lemma is a commonly known fact about mollifiers. Despite being known, a proof is surprisingly difficult to find in the existing literature. Therefore, in the interest of completeness, we prove it here.

Lemma 0.1.3. *Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that the sequence $\eta_\varepsilon(v) = \varepsilon^{-n}\eta(v/\varepsilon)$ is an approximation to the identity. There exists a constant $C = C(n,s,\eta)$ such that, for any $g \in H^s(\mathbb{R}^n)$,*

$$\|g - g*\eta_\varepsilon\|_{L^2(\mathbb{R}^n)} \le C\|g\|_{H^s(\mathbb{R}^n)}\varepsilon^s.$$

Proof. The bound is easy to compute by taking the Fourier transform and using Plancharel's theorem:

$$\begin{aligned}\|g-g*\eta_\varepsilon\|_{L^2}^2 &= \int \hat{g}^2\,(1-\hat{\eta}_\varepsilon)^2\,d\xi \\ &\le \int (1+\xi^2)^s \hat{g}^2\,d\xi\,\sup_\xi \frac{|1-\hat{\eta}_\varepsilon(\xi)|^2}{(1+\xi^2)^s} \\ &= \|g\|_{H^s(\mathbb{R}^n)}^2 \sup_\xi \frac{|1-\hat{\eta}_\varepsilon(\xi)|^2}{(1+\xi^2)^s}.\end{aligned}$$

Since $\eta \in C_c^\infty$, the fourier transform $\hat{\eta}$ is Lipschitz with some constant $\bar{C}$. Thus $\hat{\eta}_\varepsilon(\xi) = \hat{\eta}(\varepsilon\xi)$ is Lipschitz with constant $\bar{C}\varepsilon$. Since η_ε is an approximation to the identity, $\hat{\eta}_\varepsilon(0) = 1$ and $|\hat{\eta}_\varepsilon(\xi)| \le 1$ for all ξ. Thus

$$|1-\hat{\eta}_\varepsilon(\xi)| \le \min(2, \bar{C}\varepsilon|\xi|).$$

The function $\frac{\min(2,\bar{C}\varepsilon\xi)^2}{(1+\xi^2)^s}$ achieves its maxumum value at the critical point $\bar{C}\varepsilon|\xi| = 2$, and that maximum value is

$$\frac{2^2}{\left(1+\left(\frac{2}{\bar{C}\varepsilon}\right)^2\right)^s} = \frac{4\varepsilon^{2s}}{\left(\varepsilon^2+4/\bar{C}^2\right)^s} \le C\varepsilon^{2s}.$$

□

0.2 Technical Lemmas for Chapter 5

In this appendix we state and prove a few technical lemmas.

Lemma 0.2.1 (De Giorgi Iteration Argument)**.** *For any constant $\bar{C} \ge 0$, there exists a $\delta > 0$ such that the following holds:*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary for some $\beta \in (0,1)$. Let $f \in L^2([-2,0] \times \Omega)$ be a function with the property that for any positive constant a

$$\frac{d}{dt}\int (f-a)_+^2 + \int \left|\Lambda^{1/2}(f-a)_+\right|^2 \leq \bar{C}\left(\int \chi_{\{f\geq a\}} + \int (f-a)_+ + \int (f-a)_+^2\right). \tag{2.29}$$

Then

$$\int_{-2}^{0}\int (f-0)_+^2 \, dxdt \leq \delta$$

implies that

$$f(t,x) \leq 1 \qquad \forall t \in [-1,0], x \in \Omega.$$

Proof. Consider for $k \in \mathbb{N}$ the constants $t_k := -1-2^{-k}$ (so that $t_0 = -2$ and $t_\infty = -1$), and functions

$$f_k := (f-1+2^{-k})_+$$

(so that $f_0 = (f)_+$ and $f_\infty = (f-1)_+$).

Define

$$\mathcal{E}_k := \int_{t_k}^{0}\int_\Omega f_k^2 \, dxdt.$$

When $f_{k+1} > 0$, then in particular $f_k \geq 2^{-k-1}$. Thus for any finite p, there exists a constant C so

$$\chi_{\{f_{k+1}>0\}} \leq C^k f_k^p. \tag{2.30}$$

Let $k \geq 0$ and define $\eta : [-2,0] \to \mathbb{R}$ a continuous function

$$\eta(t) := \begin{cases} 0 & t \leq t_k \\ 2^{k+1}(t-t_k) & t_k \leq t \leq t_{k+1} \\ 1 & t_{k+1} \leq t. \end{cases}$$

Let $s\in(t_{k+1},0)$. Multiplying the inequality (2.29) with cutoff a_k by $\eta(t)$ and integrating in time from -2 to s, then integrating by parts, we obtain

$$\int f_k(s,x)^2\,dx - 2^{k+1}\int_{t_k}^{t_{k+1}}\int f_k(t,x)^2\,dxdt + \int_{t_{k+1}}^{s}\int \left|\Lambda^{1/2} f_k\right|^2 dxdt \le \bar{C}\left(\int_{t_k}^{0}\int \chi_{\{f_k>0\}} + f_k + f_k^2\,dxdt\right)$$

By taking the supremum over all $s\in(t_{k+1},0)$, we obtain

$$\sup_{[t_{k+1},0]}\int f_k^2\,dx + \int_{t_{k+1}}^{0}\int \left|\Lambda^{1/2} f_k\right|^2 dxdt \le C\left(2^{k+1}\int_{t_k}^{0}\int f_k^2\,dxdt + \int_{t_k}^{0}\int \chi_{\{f_k>0\}} + f_k\,dxdt\right) \tag{2.31}$$

From Proposition 5.2.3 and Sobolev embedding,

$$\int_{t_{k+1}}^{0}\left(\int f_k^4\,dx\right)^{1/2} dt \le C\int_{t_{k+1}}^{0}\int \left|\Lambda^{1/2} f_k\right|^2 dxdt.$$

Therefore by the Riesz-Thorin interpolation theorem,

$$\int_{t_{k+1}}^{0}\int f_k^3\,dxdt \le C\left(\sup_{[t_{k+1},0]}\int f_k^2\,dx + \int_{t_{k+1}}^{0}\int \left|\Lambda^{1/2} f_k\right|^2\right)^{3/2}.$$

This estimate, along with (2.31) and (2.30), and the fact that $t_{k-1}<t_k$ and $f_{k-1}\ge f_k$, tell us that

$$\int_{t_{k+1}}^{0}\int f_k^3\,dxdt \le C^k \mathcal{E}_{k-1}^{3/2}.$$

Now we can estimate, using again (2.30) and the fact $f_k\ge f_{k+1}$,

$$\mathcal{E}_{k+1} \le C^k\int_{t_{k+1}}^{0}\int f_k^3\,dxdt \le C^k\mathcal{E}_{k-1}^{3/2}.$$

This nonlinear recursive inequality $\mathcal{E}_{k+1}\le C^k\mathcal{E}_{k-1}^{3/2}$, by a standard fact about nonlinear recursions (see [DG57] or [Vas16a]), tells us that there exists a constant δ depending only on C (which in turn depends only on the constant $\bar{C}$ in (2.29))

$$\mathcal{E}_0\le\delta \text{ implies } \lim_{k\to\infty}\mathcal{E}_k = 0.$$

By assumption

$$\mathcal{E}_0 = \int_{-2}^{0}\int (f)_+ \leq \delta.$$

Therefore $\mathcal{E}_k \to 0$ and, by the dominated convergence theorem,

$$\int_{-1}^{0}\int (f-1)_+ \, dxdt = 0.$$

The result follows. $\square$

Lemma 0.2.2. *Let $\alpha \in (0,1)$. There exists a constant $C = C(\alpha)$ such that, for any set Ω and any $f \in C^{0,1}(\Omega)$,*

$$[f]_\alpha \leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^{\alpha}.$$

Proof. This simple lemma is a straightforward calculation:

$$\begin{aligned}\sup_{x,y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha} &= \sup |f(x)-f(y)|^{1-\alpha} \left(\frac{|f(x)-f(y)|}{|x-y|}\right)^{\alpha} \\ &\leq (2\|f\|_\infty)^{1-\alpha} \left(\sup \frac{|f(x)-f(y)|}{|x-y|}\right)^{\alpha} \\ &\leq C \|f\|_\infty^{1-\alpha} \|\nabla f\|_\infty^{\alpha}.\end{aligned}$$

$\square$

Lemma 0.2.3. *Let $\alpha \in (0,1)$ and Ω a set that satisfies the cone condition. There exist constants $C = C(\alpha,\Omega)$ and $\ell = \ell(\Omega)$ such that, for any $f \in C^{1,\alpha}(\Omega)$*

$$\|\nabla f\|_\infty \leq C\left(\delta^{-1}\|f\|_\infty + \delta^\alpha [\nabla f]_\alpha\right)$$

for all $\delta < \ell$.

The idea of the proof is to average ∇f along an interval of length δ with endpoint x. The magnitude of the average will be small, since $f \in L^\infty$, and the average will differ not very much from $\nabla f(x)$ since $\nabla f \in C^{1,\alpha}$.

Proof. Since Ω satisfies the cone condition, there exist positive constants ℓ and $a<1$ such that, at each point $x \in \bar{\Omega}$, there exist two unit vectors e_1 and e_2 such that $|e_1 \cdot e_2| \leq a$ and $x + \tau e_i \in \Omega$ for $i = 1,2$ and $0 < \tau \leq \ell$. In other words, Ω contains rays at each point that extend for length ℓ, end at x, and are non-parallel with angle at least $\cos^{-1}(a)$.

Consider the directional derivative $\partial_i f$ of f along the direction e_i, and observe that for any $0 < \delta \leq \ell$,

$$\left| \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \right| = |f(x + \delta e_i) - f(x)| \leq 2\,\|f\|_\infty. \tag{2.32}$$

On the other hand, $\partial_i f$ is continous so, for any $\tau \in (0, \ell]$,

$$|\partial_i f(x) - \partial_i f(x + \tau e_i)| \leq [\nabla f]_\alpha \tau^\alpha.$$

From this, we obtain that

$$\int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \leq \int_0^\delta (\partial_i f(x) + [\nabla f]_\alpha \tau^\alpha)\, d\tau = \delta \partial_i f(x) + [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}$$

and a similar bound holds from below. Thus

$$\left| \delta \partial_i f(x) - \int_0^\delta \partial_i f(x + \tau e_i)\, d\tau \right| \leq [\nabla f]_\alpha \frac{\delta^{1+\alpha}}{1+\alpha}.$$

Combining this bound with (2.32), we obtain

$$|\partial_i f(x)| \leq \frac{2}{\delta} \|f\|_\infty + \frac{\delta^\alpha}{1+\alpha} [\nabla f]_\alpha.$$

This bound is independent of x and of $i=1,2$. Since $e_1\cdot e_2\le a$ by assumption, by a little linear algebra we can bound ∇f in terms of the $\partial_i f$ and obtain that, for all $\delta\in(0,\ell]$,

$$\|\nabla f\|_\infty\le\frac{C}{1-a^2}\left(\delta^{-1}\|f\|_\infty+\delta^\alpha[\nabla f]_\alpha\right).$$

$\square$

Lemma 0.2.4. *There exist constants $\bar\lambda>0$ and $\alpha>1$ such that, for any $0<\varepsilon\le 1/2$ and any $z\ge 1$*

$$\left(|\varepsilon^{-1}(z-1)+3|^{1/4}-2^{1/4}\right)_+-\alpha\left(|z|^{1/4}-2^{1/4}\right)_+\ge\bar\lambda.$$

Proof. For z fixed, this function is increasing as ε decreases, so it will suffice to show the lemma when $\varepsilon=1/2$, that is to show

$$f_\alpha(z):=\left(|2z+1|^{1/4}-2^{1/4}\right)_+-\alpha\left(|z|^{1/4}-2^{1/4}\right)_+\ge\bar\lambda$$

for all $z\ge 1$. Note that $f_\alpha(z)\ge f_\beta(z)$ if $\alpha<\beta$.

For $z\ge 2$,

$$f_\alpha(z)=(2z+1)^{1/4}-2^{1/4}-\alpha z^{1/4}+\alpha 2^{1/4}=z^{1/4}\left((2+1/z)^{1/4}-\alpha\right)+(\alpha-1)2^{1/4}.$$

For any $\alpha<2^{1/4}$, clearly $f_\alpha(z)$ tends to ∞ as z increases. Therefore there exist N and $\alpha_0>1$ such that

$$f_\alpha(z)\ge 1\qquad\forall z\ge N,\alpha\le\alpha_0.$$

We can decompose $f_\alpha(z)=g_1(z)-(\alpha-1)g_2(z)$ where

$$g_1(z):=\left(|2z+1|^{1/4}-2^{1/4}\right)_+-\left(|z|^{1/4}-2^{1/4}\right)_+,$$

$$g_2(z) := \left(|z|^{1/4} - 2^{1/4}\right)_+.$$

Note that g_1, g_2 are both continuous, and $g_1(z)$ is strictly positive for $z \geq 1$. Therefore we can take $\alpha \in (1, \alpha_0]$ small enough that

$$\alpha - 1 < \frac{\inf_{[1,N]} g_1}{\sup_{[1,N]} g_2}.$$

For this α, $f_\alpha(z)$ is strictly positive on the compact interval $[1, N]$, and $f_\alpha(z) \geq 1$ on $[N, \infty)$. Therefore $f_\alpha(z)$ has a positive lower bound $\bar{\lambda}$ for all $z \geq 1$. □

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