The De Giorgi Method

Applications to Degenerate PDE

Logan F. Stokols

Department of Mathematics
The University of Texas at Austin

Thesis Defense, 4 May, 2020

TEXAS
The University of Texas at Austin

Outline

1. Overview of the De Giorgi Method
2. Superquadratic Hamilton-Jacobi Equations
3. Hypoelliptic Fokker-Planck Equation
4. L^2-stability of Shocks
5. SQG on Bounded Domains

TEXAS
The University of Texas at Austin

Overview of De Giorgi Method

Consider the toy problem

$$\partial_t u - \operatorname{div}(A \nabla u) = 0$$

Given $\lambda I \leq A \leq \Lambda I$ (in sense of positive matrices), parabolic, expect regularity

In fact, $\exists \alpha \in (0,1)$ s.t. $\forall \varepsilon > 0$, $\exists C > 0$

$$\|u\|_{C^\alpha([\varepsilon,\infty)\times\mathbb{R}^n)} \leq C \|u(0,\cdot)\|_{L^2(\mathbb{R}^n)}$$

c.f. De Giorgi ['57]

Energy Inequality

Let $A \subseteq \mathbb{R}^n$, $[a,b]$ an interval, $\varepsilon > 0$

Multiply by test function $\phi(t,x)(u-k)_+$, obtain

$$\sup_{[a,b]} \int_A (u-k)_+^2 + \int_a^b \int_A |\nabla (u-k)_+|^2 \lesssim \int_{a-\varepsilon}^b \int_{B_\varepsilon(A)} (u-k)_+^2$$

First De Giorgi Lemma

- L^2-to-L^∞ regularization
- global and local version
- proof by truncation, recursion

Consider $k \in \mathbb{N}$

$$Q_0 := [-2,0] \times B_2 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq \cdots \supseteq [-1,0] \times B_1$$

and truncations

$$u_0 := (u-0)_+ \geq u_1 \geq \cdots \geq u_k \geq \cdots \geq (u-1)_+$$

First De Giorgi Lemma

Lemma

Let u solve parabolic equation, there exist δ_0 small so

$$\iint_{Q_0} u_0^2 \leq \delta_0$$

implies

$$\iint_{[-1,0]\times B_1} (u-1)_+ = 0 \qquad \equiv \qquad u \leq 1 \; on \; [-1,0] \times B_1$$

Proof of First De Giorgi Lemma

Energy inequality says

$$\sup \int_{Q_k} u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \lesssim \iint_{Q_{k-1}} u_k^2$$

Sobolev embedding says, $\exists q > 2$

$$\iint_{Q_k} u_k^q \lesssim \left(\sup \int_{Q_k} u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \right)^{q/2}$$

Basic Arithmetic says

$$\iint_{Q_k} u_k^2 \lesssim \iint_{Q_k} u_{k-1}^q$$

TEXAS
The University of Texas at Austin

Proof of First De Giorgi Lemma

Put together,

$$\iint_{Q_{k+1}} u_{k+1}^2 \leq C_k \left(\iint_{Q_k} u_k^2 \right)^{1+\varepsilon}$$

if u_0 is sufficiently small on Q_0, then $(u-1)_+$ vanishes on $[-1,0] \times B_1$

Second De Giorgi Lemma

- also called Isoperimetric Inequality
- quantitative version of "solutions to parabolic eqn have no jump discontinuities"
- can be non-constructive (compactness)
- proof varies significantly between applications

Second De Giorgi Lemma

Lemma

$\exists \mu_0 > 0$ *s.t., u solving parabolic equation, if*

$$u \leq 2 \qquad on \ [-1,4] \times B_3,$$

$$
\begin{aligned}
&and && |\{u \geq 1\} \cap [2,4] \times B_2| \geq \delta_0, \\
&and && |\{u \leq 0\} \cap [0,4] \times B_2| \geq \frac{1}{2}|[0,4] \times B_2| \\
&then && |\{0 < u < 1\} \cap [0,4] \times B_2| \geq \mu_0.
\end{aligned}
$$

TEXAS
The University of Texas at Austin

Proof of Second De Giorgi Lemma

Assume false for all μ_0, take sequence u_k of counterexamples $\mu_0 = 1/k$

By first condition and energy inequality, u_k compact, has $L_t^2(H_x^1)$ limit u_∞

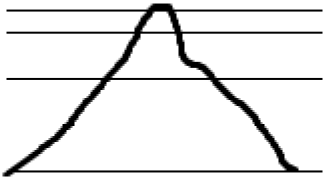
a.e. t fixed, $\forall x$ either $u_\infty \geq 1$ or $u_\infty \leq 0$

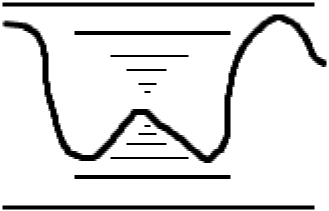
u_∞ is ≤ 0 on most of $[0,4]$ but ≥ 1 on some of $[2,4]$, so energy increases suddenly in time, impossible

Hölder Continuity

Proof of Hölder continuity uses recursion

apply two De Giorgi lemmas to rescalings of solution





Superquadratic Hamilton-Jacobi Equation

$$\text{e.g.} \qquad \partial_t u + |\nabla u|^p - \varepsilon \Delta u = 0, \qquad \varepsilon \in \{0, 1, -1\}$$

- First considered by Lasry and Lions ['89]; Schwab ['13] (homogenization)
- Strongest known results Cardaliaguet ['09], Cannarsa and Cardaliaguet ['10], Cardaliaguet and Silvestre ['12]
- $\varepsilon = 0$ singularities can form, but solutions always continuous
- For $p > 2$, continuous even for $\varepsilon = -1$ [first order drives regularization]
- Chan and Vasseur ['17] use De Giorgi for $\varepsilon = 0$

TEXAS
The University of Texas at Austin

Superquadratic Hamilton-Jacobi Equation

$$\partial_t u = H\left(t, x, u, \nabla u, D^2 u\right),$$

$$\Lambda^{-1}|\nabla u|^p - \operatorname{div}(A\nabla u) - f \leq H\left(t, x, u, \nabla u, D^2 u\right) \leq \Lambda|\nabla u|^p - \Lambda m^-(D^2 u) + \Lambda$$

$p > 2$, A bounded unsigned matrix, $f \in L^q$, m^- returns lowest negative eigenvalue

Theorem (S., Vasseur [CMS, '18])

Solutions (in appropriate weak sense) regularize from $L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ into $C^\alpha([\varepsilon, \infty) \times \mathbb{R}^n)$

TEXAS
The University of Texas at Austin

Superquadratic Hamilton-Jacobi: Proof

- De Giorgi method
- Adapted technique of Chan, Vasseur, overcome second-order term
- Combine divergence-form and non-divergence-form techniques
- Allow discontinuous A

TEXAS
The University of Texas at Austin

Superquadratic Hamilton-Jacobi: Proof excerpt

Consider

$$\partial_t u + |\nabla u|^p + \Delta u = 0.$$

Using $\varphi(t,x)(u-k)_+$ as test function, obtain energy inequality

$$\sup_{[-1,0]} \int_{B_1} (u-k)_+^2 + \int_{-1}^{0} \int_{B_1} (u-k)_+ |\nabla (u-k)_+|^p$$

$$\lesssim \int_{-2}^{0} \int_{B_2} (u-k)_+^2 + \int_{-2}^{0} \int_{B_2} |\nabla (u-k)_+|^2$$

Want to show, on $Q_1 = [-1,0] \times B_1$, for some $q > 2$

$$\|(u-1)_+\|_{L^q(Q_1)} \lesssim \|(u)_+\|_{L_t^\infty(L_x^2)(Q_1)} + \|(u)_+ |\nabla (u)_+|^p\|_{L^1(Q_1)}.$$

Superquadratic Hamilton-Jacobi: Proof excerpt

Coercivity when u large

Strategy: consider two regions, u small and u big

- u small $\Rightarrow L^q$ norm small
- u big $\Rightarrow$ coercivity $\Rightarrow L^q$ norm small

Implementation:

- $\|\nabla(u-1)_+\|_{L^p}^p \leq \|u_+|\nabla u_+|^p\|_{L^1}$
- $\|(u-1)_+\|_{L^q} \lesssim \|(u-1)_+\|_{L^\infty(L^2)} + \|\nabla(u-1)_+\|_{L^p}$

Hypoelliptic Fokker-Planck

$$\text{e.g.} \qquad [\partial_t + v \cdot \nabla_x] f + (-\Delta_v)^s f = 0$$

- Rarefied gas, neutral particles in plasma
- Imbert and Silvestre ['16]; Golse and Imbert and Mouhot and Vasseur ['16]
- Hypoelliptic: non-elliptic regularization, mixed elliptic/hyperbolic type
- Averaging Lemma (Golse et al '88): H^s theory of hypoellipticity, regularity of averages for kinetic equation

TEXAS
The University of Texas at Austin

Hypoelliptic Fokker-Planck

$$[\partial_t + v \cdot \nabla_x] f = \int K[f(w) - f(v)]\, dw + \sigma$$

$K \approx |v - w|^{-(n+2s)}$, $s \in (0,1)$, K symmetric in $(v,w) \mapsto (w,v)$ and in $(v, v+y) \mapsto (v, v-y)$

Theorem (S. [SIMA, '19])

For f solution, $f \in L^\infty \cap L^2_{t,x}(H^s_v)$, $\sigma \in L^2 \cap L^r$ for $r >> 1$, there exists $\alpha \in (0,1)$ depending on kernel, $C > 0$ depending on domain and kernel s.t.

$$\|f\|_{C^\alpha([\varepsilon,\infty)\times\mathbb{R}^n\times B_1)} \le C\left(\|f\|_{L^\infty} + \|\sigma\|_{L^r}\right).$$

Hypoelliptic Fokker-Planck: Proof excerpt

Consider $\Lambda = (-\Delta_v)^{1/2}$, $s \in (0,1)$,

$$\partial_t f + v \cdot \nabla_x f + \Lambda^{2s} f = 0.$$

Note diffusion in v but not x!
Energy inequality will have

$$\|(f-\psi)_+\|_{L^\infty_t(L^2_{x,v})} + \|\Lambda^s (f-\psi)_+\|_{L^2_{t,x,v}}$$

on LHS

Averaging Lemma

Lemma (Bézard, '94): for $\alpha = 1/(2(1+m))$, $\Omega \Subset \bar{\Omega} \subseteq \mathbb{R}^+ \times \mathbb{R}^n$, and $f, g \in L^2(\bar{\Omega} \times \mathbb{R}^n)$, f compactly supported, we have

$$[\partial_t + v \cdot \nabla_x] f = g$$

implies

$$\left\| \int f \, dv \right\|_{H^\alpha(\Omega)} \lesssim \|f\|_{L^2(\bar{\Omega}\times\mathbb{R}^n)} + \left\| (1-\Delta_v)^{-m/2} g \right\|_{L^2(\bar{\Omega}\times\mathbb{R}^n)}.$$

TEXAS
The University of Texas at Austin

Hypoelliptic Fokker-Planck: Proof excerpt

Unfortunately: Can't apply lemma to $(f-\psi)_+$ due to truncation (nonlocal)

Barrier function

$$0 \le \varphi(t,x)(f-\psi)_+ \le F,$$

$$\|F\|_{L^2} + \left\| (1-\Delta_v)^{-m/2} \left[\partial_t + v\cdot\nabla_x\right] F \right\|_{L^2} \le C \left\| \varphi(f-\psi)_+ \right\|_{L^2}.$$

Now:

$$\left\| \int F \, dv \right\|_{H^\alpha} \le \left\| \varphi(f-\psi)_+ \right\|_{L^2},$$

No regularity on f!

Hypoelliptic Fokker-Planck: Proof excerpt

From averaging lemma:

$$\|(f-\psi)_+\|_{L^{2+\varepsilon}_{t,x}(L^1_v)} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}}$$

From energy inequality:

$$\|(f-\psi)_+\|_{L^\infty_t(L^2_{x,v})} + \|(f-\psi)_+\|_{L^2_{t,x}(L^{2+\varepsilon}_v)} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}} + \|(f-\psi)_+\|_{L^1_{t,x,v}}$$

Improvement in all three variables, for some $q>2$

$$\|(f-\psi)\|_{L^q_{t,x,v}} \lesssim \|(f-\psi)_+\|_{L^2_{t,x,v}}$$

Control of regularity is means to an end, control of integrability is the end

TEXAS
The University of Texas at Austin

L^2 stability of Shocks

Consider 1D scalar viscous conservation law

$$\partial_t u + \partial_x[Q(u)] = \varepsilon \partial_{xx} \eta'(u)$$

where η, Q uniformly convex and $\varepsilon > 0$ arbitrary.

Theorem (S. [Submitted])

For u a solution and s a sufficiently small shock solution, there exists Lipschitz $\gamma(t)$ such that

$$\|u(\cdot,t) - s(\cdot - \gamma(t))\|_2$$

stable in time, up to a constant factor. Result is independent of ε.

c.f. Kang ['19], Kang & Vasseur ['19]

SQG in Bounded Domains

$$\text{e.g.} \qquad \partial_t \theta + \left(\nabla^{\perp} (-\Delta)^{-1/2} \theta \right) \cdot \nabla \theta + \nu (-\Delta)^s \theta = 0$$

- atmospheric or ocean currents, used in weather modelling
- $\mathbb{R}^2$: Constantin, Majda, Tabak ['93]; Kiselev, Nazarov, Volberg ['08]; Caffarelli, Vasseur ['10]; Constantin, Vicol ['12]
- Bounded domain: Kriventsov ['15]; Novack, Vasseur ['18,'19]
- Best studied model by Constantin, Ignatova ['16]; Constantin, Ignatova, Nguyen [various]
- Boundary issues: Laplacian & gradient don't commute, Caffarelli-Stinga ['16] kernel representation degenerates

TEXAS
The University of Texas at Austin

SQG in Bounded Domains

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ u = \nabla^\perp \Lambda^{-1} \theta. \end{cases}$$

$\Omega \subseteq \mathbb{R}^2$ smooth bounded open, $\Lambda := \sqrt{-\Delta_D}$ (defined spectrally), Δ_D the Dirichlet Laplacian on Ω

Theorem (S., Vasseur [ARMA, '20])

Let $\Omega \subseteq \mathbb{R}^2$ a bounded set, initial data $\theta_0 \in L^2(\Omega)$
There exists a global-in-time solution θ to SQG such that:
For any $\varepsilon > 0$, there exists $\alpha \in (0,1)$ and a constant C so

$$\|\theta\|_{C^\alpha([\varepsilon,\infty)\times\Omega)} \leq C \|\theta_0\|_{L^2(\Omega)}.$$

SQG in Bounded Domains: Proof excerpt

Velocity u, energy inequality with cutoff Ψ has drift term on RHS

$$\int_\Omega u(\theta - \Psi)_+ \cdot \mathrm{d}\Psi$$

recall u is Riesz transform of $\theta \in L^\infty$

u bounded in:	L^∞	BMO	$B^0_{\infty,\infty}$
$\theta \in L^\infty \Rightarrow u \in$ __	$\times$	works on $\mathbb{R}^2$	complicated
$\int u\theta_+$ bounded	$\leq \int \theta_+$	John-Nirenberg	complicated
scaling invariant	$\checkmark$	$\checkmark$	$\checkmark$

TEXAS
The University of Texas at Austin

Control on u: Littlewood Paley theory

Littlewood paley operators $P_j = P_j(\Lambda)$, functional calculus for Λ,

Bernstein Inequalities

$$\|\Lambda^s P_j f\|_p \approx 2^{sj} \|f\|_p ,$$
$$\|\nabla \Lambda^s P_j f\|_p \approx 2^{(1+s)j} \|f\|_p .$$

Commutation Relation

$$\|P_i \nabla P_j f\|_p \lesssim \min(2^j, 2^i) \|f\|_p .$$

Bernstein: Iwabuchi, Matsuyama, Taniguchi (“Bilinear estimates in Besov spaces generated by the Dirichlet Laplacian” 2017)

TEXAS
The University of Texas at Austin

Control on u: high and low frequencies

Instead of Besov norm $\sup_j \left\|P_j \nabla \Lambda^{-1}\theta\right\|_\infty$ consider

$$\nabla \Lambda^{-1} P_j \theta.$$

To bound $\int u\theta_+$, decompose as

$$u_{\text{low}} = \sum_{j=-\infty}^{0} \nabla \Lambda^{-1} P_j \theta$$

which is Lipschitz,

$$u_{\text{high}} = \sum_{j=0}^{\infty} \nabla \Lambda^{-1} P_j \theta$$

which is in $W^{-\varepsilon,\infty}$.

TEXAS
The University of Texas at Austin

Thank you