

The De Giorgi Method

Applications to Degenerate PDE

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- 2 Superquadratic Hamilton-Jacobi Equations
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Overview of De Giorgi Method

Consider the toy problem

$$\partial_t u - \operatorname{div}(A \nabla u) = 0$$

Given $\lambda I \leq A \leq \Lambda I$ (in sense of positive matrices), parabolic, expect regularity

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In fact, $\exists \alpha \in (0, 1)$ s.t. $\forall \varepsilon > 0, \exists C > 0$

$$\|u\|_{C^\alpha([\varepsilon, \infty) \times \mathbb{R}^n)} \leq C \|u(0, \cdot)\|_{L^2(\mathbb{R}^n)}$$

c.f. De Giorgi ['57]

Energy Inequality

Let $A \subseteq \mathbb{R}^n$, $[a, b]$ an interval, $\varepsilon > 0$

Multiply by test function $\phi(t, x)(u - k)_+$, obtain

$$\sup_{[a, b]} \int_A (u - k)_+^2 + \int_a^b \int_A |\nabla (u - k)_+|^2 \lesssim \int_{a-\varepsilon}^b \int_{B_\varepsilon(A)} (u - k)_+^2$$

First De Giorgi Lemma

- L^2 -to- L^∞ regularization
- global and local version
- proof by truncation, recursion

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Consider $k \in \mathbb{N}$

$$Q_0 := [-2, 0] \times B_2 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq \cdots \supseteq [-1, 0] \times B_1$$

and truncations

$$u_0 := (u - 0)_+ \geq u_1 \geq \cdots \geq u_k \geq \cdots \geq (u - 1)_+$$

First De Giorgi Lemma

Lemma

Let u solve parabolic equation, there exist δ_0 small so

$$\iint_{Q_0} u_0^2 \leq \delta_0$$

implies

$$\iint_{[-1,0] \times B_1} (u - 1)_+^2 = 0 \quad \equiv \quad u \leq 1 \text{ on } [-1,0] \times B_1$$

Proof of First De Giorgi Lemma

Energy inequality says

$$\sup_{Q_k} \int u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \lesssim \iint_{Q_{k-1}} u_k^2$$

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Sobolev embedding says, $\exists q > 2$

$$\iint_{Q_k} u_k^q \lesssim \left(\sup_{Q_k} \int u_k^2 + \iint_{Q_k} |\nabla u_k|^2 \right)^{q/2}$$

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Basic Arithmetic says

$$\iint_{Q_k} u_k^2 \lesssim \iint_{Q_k} u_k^q$$

Proof of First De Giorgi Lemma

Put together,

$$\iint_{Q_{k+1}} u_{k+1}^2 \leq C_k \left(\iint_{Q_k} u_k^2 \right)^{1+\varepsilon}$$

if u_0 is sufficiently small on Q_0 , then $(u - 1)_+$ vanishes on $[-1, 0] \times B_1$

Second De Giorgi Lemma

- also called Isoperimetric Inequality
- quantitative version of “solutions to parabolic eqn have no jump discontinuities”
- can be non-constructive (compactness)
- proof varies significantly between applications

Second De Giorgi Lemma

Lemma

$\exists \mu_0 > 0$ s.t., u solving parabolic equation, if

$$u \leq 2 \quad \text{on } [-1, 4] \times B_3,$$

and $|\{u \geq 1\} \cap [2, 4] \times B_2| \geq \delta_0,$

and $|\{u \leq 0\} \cap [0, 4] \times B_2| \geq \frac{1}{2} |[0, 4] \times B_2|$

then $|\{0 < u < 1\} \cap [0, 4] \times B_2| \geq \mu_0.$

Proof of Second De Giorgi Lemma

Assume false for all μ_0 , take sequence u_k of counterexamples $\mu_0 = 1/k$

By first condition and energy inequality, u_k compact, has $L_t^2(H_x^1)$ limit u_∞

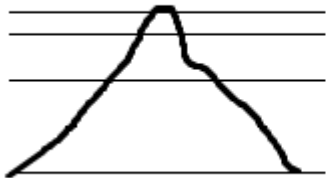
a.e. t fixed, $\forall x$ either $u_\infty \geq 1$ or $u_\infty \leq 0$

u_∞ is ≤ 0 on most of $[0, 4]$ but ≥ 1 on some of $[2, 4]$, so energy increases suddenly in time, impossible

Hölder Continuity

Proof of Hölder continuity uses recursion

apply two De Giorgi lemmas to rescalings of solution



Superquadratic Hamilton-Jacobi Equation

Superquadratic Hamilton-Jacobi Equation

e.g.
$$\partial_t u + |\nabla u|^p - \varepsilon \Delta u = 0, \quad \varepsilon \in \{0, 1, -1\}$$

- First considered by Lasry and Lions [’89]; Schwab [’13] (homogenization)
- Strongest known results Cardaliaguet [’09], Cannarsa and Cardaliaguet [’10], Cardaliaguet and Silvestre [’12]
- $\varepsilon = 0$ singularities can form, but solutions always continuous
- For $p > 2$, continuous even for $\varepsilon = -1$ [first order drives regularization]
- Chan and Vasseur [’17] use De Giorgi for $\varepsilon = 0$

Superquadratic Hamilton-Jacobi Equation

$$\partial_t u = H(t, x, u, \nabla u, D^2 u),$$

$$\Lambda^{-1} |\nabla u|^p - \operatorname{div}(A \nabla u) - f \leq H(t, x, u, \nabla u, D^2 u) \leq \Lambda |\nabla u|^p - \Lambda m^-(D^2 u) + \Lambda$$

$p > 2$, A a bounded unsigned matrix, $f \in L^q$, m^- returns lowest negative eigenvalue

Theorem (S., Vasseur [CMS, '18])

Solutions (in appropriate weak sense) regularize from $L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$ into $C^\alpha([\varepsilon, \infty) \times \mathbb{R}^n)$

Superquadratic Hamilton-Jacobi: Proof

- De Giorgi method
- Adapted technique of Chan, Vasseur, overcome second-order term
- Combine divergence-form and non-divergence-form techniques
- Allow discontinuous A

Superquadratic Hamilton-Jacobi: Proof excerpt

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Consider

$$\partial_t u + |\nabla u|^p + \Delta u = 0.$$

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$$\partial_t u + |\nabla u|^p + \Delta u = 0.$$

Using $\varphi(t, x)(u - k)_+$ as test function, obtain energy inequality

$$\begin{aligned} & \sup_{[-1,0]} \int_{B_1} (u - k)_+^2 + \int_{-1}^0 \int_{B_1} (u - k)_+ |\nabla(u - k)_+|^p \\ & \lesssim \int_{-2}^0 \int_{B_2} (u - k)_+^2 + \int_{-2}^0 \int_{B_2} |\nabla(u - k)_+|^2 \end{aligned}$$

Superquadratic Hamilton-Jacobi: Proof excerpt

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Want to show, on $Q_1 = [-1, 0] \times B_1$, for some $q > 2$

$$\|(u - 1)_+\|_{L^q(Q_1)} \lesssim \|(u)_+\|_{L_t^\infty(L_x^2)(Q_1)} + \|(u)_+ |\nabla(u)_+|^p\|_{L^1(Q_1)}.$$

Superquadratic Hamilton-Jacobi: Proof excerpt

Coercivity when u large

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Coercivity when u large

Strategy: consider two regions, u small and u big

- u small $\Rightarrow L^q$ norm small
- u big \Rightarrow coercivity $\Rightarrow L^q$ norm small

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Coercivity when u large

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Implementation:

- $\|\nabla(u-1)_+\|_{L^p}^p \leq \|u_+|\nabla u_+|^p\|_{L^1}$
- $\|(u-1)_+\|_{L^q} \lesssim \|(u-1)_+\|_{L^\infty(L^2)} + \|\nabla(u-1)_+\|_{L^p}$

Hypoelliptic Fokker-Planck

Hypoelliptic Fokker-Planck

e.g.
$$[\partial_t + v \cdot \nabla_x]f + (-\Delta_v)^s f = 0$$

- Rarefied gas, neutral particles in plasma
- Imbert and Silvestre ['16]; Golse and Imbert and Mouhot and Vasseur ['16]
- Hypoelliptic: non-elliptic regularization, mixed elliptic/hyperbolic type
- Averaging Lemma (Golse et al '88): H^s theory of hypoellipticity, regularity of averages for kinetic equation

Hypoelliptic Fokker-Planck

$$[\partial_t + v \cdot \nabla_x]f = \int K[f(w) - f(v)] dw + \sigma$$

$K \approx |v - w|^{-(n+2s)}$, $s \in (0, 1)$, K symmetric in $(v, w) \mapsto (w, v)$ and in $(v, v + y) \mapsto (v, v - y)$

Theorem (S. [SIMA, '19])

For f solution, $f \in L^\infty \cap L^2_{t,x}(H^s_v)$, $\sigma \in L^2 \cap L^r$ for $r \gg 1$, there exists $\alpha \in (0, 1)$ depending on kernel, $C > 0$ depending on domain and kernel s.t.

$$\|f\|_{C^\alpha([\varepsilon, \infty) \times \mathbb{R}^n \times B_1)} \leq C (\|f\|_{L^\infty} + \|\sigma\|_{L^r}).$$

Hypoelliptic Fokker-Planck: Proof excerpt

Hypoelliptic Fokker-Planck: Proof excerpt

Consider $\Lambda = (-\Delta_v)^{1/2}$, $s \in (0, 1)$,

$$\partial_t f + v \cdot \nabla_x f + \Lambda^{2s} f = 0.$$

Note diffusion in v but not x !

Energy inequality will have

$$\|(f - \psi)_+\|_{L_t^\infty(L_{x,v}^2)} + \|\Lambda^s(f - \psi)_+\|_{L_{t,x,v}^2}$$

on LHS

Averaging Lemma

Lemma (Bézar, '94): for $\alpha = 1/(2(1+m))$, $\Omega \in \bar{\Omega} \subseteq \mathbb{R}^+ \times \mathbb{R}^n$, and $f, g \in L^2(\bar{\Omega} \times \mathbb{R}^n)$, f compactly supported, we have

$$[\partial_t + v \cdot \nabla_x]f = g$$

implies

$$\left\| \int f \, dv \right\|_{H^\alpha(\Omega)} \lesssim \|f\|_{L^2(\bar{\Omega} \times \mathbb{R}^n)} + \left\| (1 - \Delta_v)^{-m/2} g \right\|_{L^2(\bar{\Omega} \times \mathbb{R}^n)}.$$

Hypoelliptic Fokker-Planck: Proof excerpt

Unfortunately: Can't apply lemma to $(f - \psi)_+$ due to truncation (nonlocal)

Barrier function

$$0 \leq \varphi(t, x)(f - \psi)_+ \leq F,$$

$$\|F\|_{L^2} + \left\| (1 - \Delta_v)^{-m/2} [\partial_t + v \cdot \nabla_x] F \right\|_{L^2} \leq C \|\varphi(f - \psi)_+\|_{L^2}.$$

Now:

$$\left\| \int F dv \right\|_{H^\alpha} \leq \|\varphi(f - \psi)_+\|_{L^2},$$

No regularity on f !

Hypoelliptic Fokker-Planck: Proof excerpt

From averaging lemma:

$$\|(f - \psi)_+\|_{L_{t,x}^{2+\varepsilon}(L_v^1)} \lesssim \|(f - \psi)_+\|_{L_{t,x,v}^2}$$

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From energy inequality:

$$\|(f - \psi)_+\|_{L_t^\infty(L_{x,v}^2)} + \|(f - \psi)_+\|_{L_{t,x}^2(L_v^{2+\varepsilon})} \lesssim \|(f - \psi)_+\|_{L_{t,x,v}^2} + \|(f - \psi)_+\|_{L_{t,x,v}^1}$$

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Improvement in all three variables, for some $q > 2$

$$\|(f - \psi)\|_{L_{t,x,v}^q} \lesssim \|(f - \psi)_+\|_{L_{t,x,v}^2}$$

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Improvement in all three variables, for some $q > 2$

$$\|(f - \psi)\|_{L_{t,x,v}^q} \lesssim \|(f - \psi)_+\|_{L_{t,x,v}^2}$$

Control of regularity is means to an end, control of integrability is the end

L^2 stability of Shocks

L^2 stability of Shocks

Consider 1D scalar viscous conservation law

$$\partial_t u + \partial_x [Q(u)] = \varepsilon \partial_{xx} \eta'(u)$$

where η , Q uniformly convex and $\varepsilon > 0$ arbitrary.

Theorem (S. [Submitted])

For u a solution and s a sufficiently small shock solution, there exists Lipschitz $\gamma(t)$ such that

$$\|u(\cdot, t) - s(\cdot - \gamma(t))\|_2$$

stable in time, up to a constant factor. Result is independent of ε .

c.f. Kang ['19], Kang & Vasseur ['19]

SQG in Bounded Domains

SQG in Bounded Domains

e.g.
$$\partial_t \theta + \left(\nabla^\perp (-\Delta)^{-1/2} \theta \right) \cdot \nabla \theta + \nu (-\Delta)^s \theta = 0$$

- atmospheric or ocean currents, used in weather modelling
- \mathbb{R}^2 : Constantin, Majda, Tabak [’93]; Kiselev, Nazarov, Volberg [’08]; Caffarelli, Vasseur [’10]; Constantin, Vicol [’12]
- Bounded domain: Kriventsov [’15]; Novack, Vasseur [’18, ’19]
- Best studied model by Constantin, Ignatova [’16]; Constantin, Ignatova, Nguyen [various]
- Boundary issues: Laplacian & gradient don’t commute, Caffarelli-Stinga [’16] kernel representation degenerates

SQG in Bounded Domains

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta + \Lambda \theta = 0, \\ u = \nabla^\perp \Lambda^{-1} \theta. \end{cases}$$

$\Omega \subseteq \mathbb{R}^2$ smooth bounded open, $\Lambda := \sqrt{-\Delta_D}$ (defined spectrally), Δ_D the Dirichlet Laplacian on Ω

Theorem (S., Vasseur [ARMA, '20])

Let $\Omega \subseteq \mathbb{R}^2$ a bounded set, initial data $\theta_0 \in L^2(\Omega)$

There exists a global-in-time solution θ to SQG such that:

For any $\varepsilon > 0$, there exists $\alpha \in (0, 1)$ and a constant C so

$$\|\theta\|_{C^\alpha([\varepsilon, \infty) \times \Omega)} \leq C \|\theta_0\|_{L^2(\Omega)}.$$

SQG in Bounded Domains: Proof excerpt

SQG in Bounded Domains: Proof excerpt

Velocity u , energy inequality with cutoff Ψ has drift term on RHS

$$\int_{\Omega} u(\theta - \Psi)_+ \cdot d\Psi$$

recall u is Riesz transform of $\theta \in L^\infty$

u bounded in:	L^∞	BMO	$B_{\infty,\infty}^0$
$\theta \in L^\infty \Rightarrow u \in __$	\times	works on \mathbb{R}^2	complicated
$\int u\theta_+$ bounded	$\leq \int \theta_+$	John-Nirenberg	complicated
scaling invariant	\checkmark	\checkmark	\checkmark

Control on u : Littlewood Paley theory

Littlewood paley operators $P_j = P_j(\Lambda)$, functional calculus for Λ ,
Bernstein Inequalities

$$\begin{aligned}\|\Lambda^s P_j f\|_p &\approx 2^{sj} \|f\|_p, \\ \|\nabla \Lambda^s P_j f\|_p &\approx 2^{(1+s)j} \|f\|_p.\end{aligned}$$

Commutation Relation

$$\|P_i \nabla P_j f\|_p \lesssim \min(2^j, 2^i) \|f\|_p.$$

Bernstein: Iwabuchi, Matsuyama, Taniguchi (“Bilinear estimates in Besov spaces generated by the Dirichlet Laplacian” 2017)

Control on u : high and low frequencies

Instead of Besov norm $\sup_j \|P_j \nabla \Lambda^{-1} \theta\|_\infty$ consider

$$\nabla \Lambda^{-1} P_j \theta.$$

To bound $\int u \theta_+$, decompose as

$$u_{\text{low}} = \sum_{j=-\infty}^0 \nabla \Lambda^{-1} P_j \theta$$

which is Lipschitz,

$$u_{\text{high}} = \sum_{j=0}^{\infty} \nabla \Lambda^{-1} P_j \theta$$

which is in $W^{-\varepsilon, \infty}$.

Thank you