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Modifying the Black-Scholes Option Pricing Model for Alternative Underlying Instruments

The Black-Scholes option pricing model may be used to evaluate options on various types of underlying instruments, but significant modifications are necessary. In addition to financing costs, for example, the formula for commodity options must incorporate storage costs over the option's life, whereas the formula for securities must consider expected dividends or interest income. Conventional options on futures contracts, on the other hand, entail no holding costs. And the formula for "futures-style" options on futures must recognize that neither holding costs nor short-term rates are a factor in pricing.

These adjustments have implications for the "put-call parity" relationship, which provides information about the relative time values associated with puts and calls. For securities, put time value will exceed call time value if expected dividends or interest income exceed financing costs. In the case of conventional commodity options, call time value always exceeds put time value. The time value of a conventional put (call) on a futures contract will exceed call (put) time value if the call (put) options is in-the-money. Put and call time values are always equal for futures-style options on futures.

The authors provide a computer program, written in BASIC, for calculating "fair market" put and call option premiums for options on securities, commodities and futures.

MODELS FOR THE PRICING of options have proliferated as new types of option contracts have been introduced.[1] The common ancestor of all these models is the Black-Scholes stock option pricing model.[2] The Black-Scholes model, although based on assumptions that may be deemed overly restrictive for practical applications, is nonetheless the most widely known and referenced method for estimating "fair market" option premiums.

1. Footnotes appear at end of article

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This article demonstrates how the Black-Scholes option pricing formula is adapted for the valuation of four classes of options—(1) conventional (prepaid premium) options on securities (such as stocks and bonds), (2) conventional options on storable commodities (such as metals and grains), (3) conventional options on futures contracts, and (4) "futures-style" (margined premium) options on futures contracts. The modifications necessary to adapt the classic formula illustrate some of the basic differences between the instruments that may underlie an option contract. Put-call parity and option price sensitivity to changes in underlying parameters are considered within the framework of this discussion. We also provide a computer program, written in BASIC, that may be used to apply the model variations presented.[3]

Modifications of the Black-Scholes Formula

The Black-Scholes option pricing formula relies on a number of restrictive assumptions. These include (1) a constant short-term interest rate, (2) zero transaction costs, (3) a lognormal distribution of the underlying security, commodity or futures price, (4) continuously accruing holding costs and interest and dividend payments, and (5) an inability to exercise the option prior to the expiration date. These assumptions are frequently violated in practice. This has not, however, seriously impaired the usefulness of the option pricing formula for many practical applications.

The formulas for the prices of a conventional call, C, and a conventional put, P, may be expressed as:

$$C = e^{-rT}[(S + H)N(d_1) - EN(d_2)], \quad (1)$$

$$P = e^{-rT}[EN(-d_2) - (S + H)N(-d_1)], \quad (2)$$

where

$$d_1 = [\log (S/E + H/E) + 0.5v^2T] / v\sqrt{T},$$

$$d_2 = d_1 - v\sqrt{T},$$

and

- S = the price of the underlying security, commodity or futures contract,
- H = all holding costs associated with the underlying security, commodity or futures contract, less any cash dividends or interest payments during the life of the option,
- E = the exercise price,
- T = the term to option expiration (in years),
- r = the interest rate applicable over the option's life,
- v = price "volatility" measured as the annualized standard deviation of the percentage change in the price of the underlying stock, bond, commodity or futures contract, and
- N(x) = the value of the normal cumulative distribution function evaluated at x.

The value of the parameter d_1 is such that $N(d_1)$ represents the approximate probability that a call option will be in-the-money at expiration. An interesting feature of the Black-Scholes formula is that it does not rely on an estimate of the expected increase or decrease in the price of the underlying security, commodity or futures contract; it requires only the standard deviation of the percentage change in this price and the absolute price level.

To modify Equations (1) and (2) for the evaluation of options on different types of underlying instruments, it is necessary to pay close attention to the parameter H. This variable represents the holding costs associated with the underlying instrument during the life of the option. These include financing, storage, insurance and spoilage costs less any gains due to cash dividends or interest income.

For securities, the parameter H may be calculated as follows:

$$H = S(e^{rT} - 1) - D, \quad (3)$$

where D equals expected dividend or interest income over the option's life. For commodities, the calculation is:

$$H = S(e^{rT} - 1) + G, \quad (4)$$

where G equals expected storage costs over the option's life, including warehousing, rental, insurance, spoilage and demurrage. For futures contracts, H equals zero.

Dividends, interest and storage costs are frequently incorporated as a multiplicative factor of the security or commodity price, rather than an additive factor, as in Equations (3) and (4). The difference in treatment does not affect the theoretical premium itself, but it does slightly affect the expected change in the option premium as a percentage of a minute fluctuation in the price of the underlying instrument (the theoretical "delta," discussed below). Inasmuch as dividends, interest and storage costs typically are not determined by reference to the value of the underlying instrument, the additive process will generally result in a more precise value.

When the underlying security or commodity has an active futures market, the option pricing formula may rely on the futures price, rather than on the spot price adjusted to reflect holding costs. Futures prices reflect the holding costs associated with the underlying cash instrument and may yield superior results because liquid futures markets are generally more efficient than their cash counterparts. Furthermore, arbitrage may be conducted more readily against a liquid futures, rather than a cash, market; options exercisable in cash against spot values of stock indexes are, for the most part, driven by stock index futures prices, rather than the spot value of the index itself.[4]

A futures-style option on a futures contract is similar to a futures contract in that its premium would initially be margined in short-term securities. There is therefore no opportunity cost associated with the purchase of such an option; nor is there an opportunity cost associated with the underlying futures contract. The net result is that short-term rates are not a factor in the pricing of futures-style options on futures contracts. The theoretical prices of a futures-style call option, C^*, and put option, P^*, are derived as:[5]

$$C^* = SN(d_1) - EN(d_2), \quad (5)$$

$$P^* = EN(-d_2) - SN(-d_1). \quad (6)$$

Put-Call Parity

The combination of a long call and short put, where both "legs" of the combination share a common strike price and expiration, is tantamount to holding a levered long position in the underlying instrument. This "put-call parity" relationship may be verified directly from the Black-Scholes model and expressed as follows:

$$C - P = e^{-rT}(S + H - E). \quad (7)$$

The put-call parity relationship provides information about the relative time values associated with puts and calls. The time value associated with an option is the excess of its premium over its in-the-money, or intrinsic, value. If the option is out-of-the-money, it has no intrinsic value; its premium therefore reflects only time value.

If the put and call have equivalent time values, then $C - P$ must equal $S - E$. In the case of an option on a security, however, combining Equations (3) and (7) reveals the following:

$$C - P <> S - E \text{ if } D >< E(e^{rT} - 1). \quad (8)$$

That is, if expected dividends and interest income (D) are sufficiently large, the put time value will exceed the call time value. The put and call time values will be equal when:

$$D = E(e^{rT} - 1),$$

that is, when expected dividends and interest income equal the interest on E dollars over the option's life or, roughly, the financing cost associated with the purchase of the underlying security.

This analysis is not precise in the case of a stock option, because stock dividends do not accrue to the seller at the time the stock is sold. Calls on stock are therefore sometimes exercised prior to expiration, so that a dividend payment on the underlying stock may be collected. Nevertheless, options on stocks paying large dividends tend to have larger put time values and smaller call time values than options on comparable stocks paying smaller dividends. Generally, stock dividends do not approach financing costs (that is, stocks exhibit "negative carry"); stock call options therefore have smaller time values than stock put options.

The analysis of time value for conventional put and call options on bonds is similar to the analysis for stock options. If the bond interest income exceeds the short-term interest on E dollars over the option's life, then the put time value exceeds the call time value. This is the general case in a positive yield curve environment, when short-term rates are less than long-term rates and the bond market exhibits "positive carry."

For a conventional option on a commodity, combining Equations (4) and (7) results in the following:

$$C - P = S - e^{-rT}(E - G). \quad (9)$$

Again, if the put and call have equal time values, then $C - P$ must equal $S - E$. However, Equation (9) indicates that, in the case of options on commodities, $C - P$ always exceeds $S - E$, hence the call time value always exceeds the put time value.

For conventional options on futures, Equations (5) and (7) may be combined for:

$$C - P = e^{-rT}(S - E). \quad (10)$$

If S exceeds E (i.e., the call option is in-the-money), then Equation (10) indicates that $C - P$ is less than $S - E$, hence the put time value exceeds the call time value. Conversely, if E exceeds S (i.e., the put option is in-the-money), then $C - P$ exceeds $S - E$, hence the call time value exceeds the put time value. When S equals E (i.e., both options are at-the-money), then the call and put options have equal time values.

The put-call parity relationship for futures-style options on futures is:

$$C^* - P^* = S - E. \quad (11)$$

Table I Put Versus Call Time Values

Option Class	*Put and Call Time Value Relationship*
Security	Put time value exceeds call time value only if $D > E(e^{rT} - 1)$ where D is the dividend or interest income over the option life.
Commodity	Call time value is always greater than put time value.
Futures Contract	Call is in-the-money: Put time value exceeds call time value. Call and put are at-the-money: Time values are equal. Put is in-the-money: Call time value exceeds put time value.
Futures-Style Option on Futures	Put and call time values are always equal.

Table II Parameter Effects on Option Premiums*

Option Class	*D*	*G*	*r*	*v*	*T*	*S*	*E*	*S-E*
	Call Premium							
Security	–		+	+	+	+	–	+
Commodity		+	+	+	+	+	–	+
Futures Contract			–	+	+	+	–	+
Futures Contract, Futures-Style			∅	+	+	+	–	+
	Put Premium							
Security	+		–	+	+	–	+	–
Commodity		–	–	+	+	–	+	–
Futures Contract			–	+	+	–	+	–
Futures Contract, Futures-Style			∅	+	+	–	+	–

* Given an increase in a parameter, the theoretical option premium will generally increase (+), decrease (–) or remain stable (∅). *Key*: D = expected cash dividends or interest income during option life; G = storage costs during option life; r = applicable interest rate over option life; v = standard deviation of percentage change in underlying price ("volatility"); T = time to maturity; S = price of underlying security, commodity or futures contract; E = exercise price.

The put and call time values of such options are always equal. Futures-style options on futures are priced "symmetrically" because the only difference between the call and put premiums is the difference between their intrinsic values.

Table I summarizes the put and call time value relationships discussed above.

Effects of Parameter Changes

Table II summarizes the effects of changes in various model parameters on the absolute levels of theoretical option premiums. The results, derived from Equations (5) and (6), represent the direction of change in the option premium—increase (+), decrease (–) or no change (0)—when a given parameter is assumed to increase.

The parameter having the most obvious effect on the option premium is the underlying security, commodity or futures contract price. The derivative of the theoretical premium with respect to the underlying price is known as "delta." Option deltas are widely computed and referenced by option traders.

The delta associated with calls on securities and commodities and futures-style options on futures is equal to $N(d_1)$. The put option delta is equal to $N(d_1) - 1$. For conventional options on futures contracts, the call and put deltas take the following forms:

$$e^{-rT}N(d_1)$$

and

$$e^{-rT}N(d_1) - e^{-rT},$$

respectively. Thus the delta of a conventional option on a futures contract is slightly smaller in absolute value than the corresponding delta of an option on the security or commodity that may underlie the futures contract.

A Program for Option Premiums

Table III provides a user-interactive program, written in BASIC, that may be used to calculate fair market put and call option premiums and associated deltas for options on securities, commodities and futures contracts. The program requires the user to input the following:

(1) the type of instrument underlying the option (whether a futures contract, a security or a commodity),
(2) the type of option (whether a put or a call) to be evaluated,
(3) the underlying instrument's price,
(4) the exercise price,
(5) the term until expiration in days,
(6) price volatility as measured by the annualized standard deviation of percentage price changes, and
(7) the applicable short-term interest rate.

In the case of an option on a security, the user will have to input, in addition, the annual dividend or coupon. In the case of an option on a commodity, the user must input storage costs.

The program does not explicitly accommodate a futures-style option on a futures contract. These options may be easily evaluated, however, by following the path applicable for a futures contract and entering zero for the short-term interest rate. This program expects all entries and provides all results in decimals. Inasmuch as many underlying instruments and associated

Table III Option Evaluation Program Listing

```
10 PRINT "OPTION EVALUATION PROGRAM"
20 PRINT
30 INPUT "    Underlying type (F/S/C)            "; UT$
40 IF  UT$ <> "F" AND UT$ <> "S" AND UT$ <> "C" THEN 30
50 INPUT "    Option type (P/C)                  "; OT$
60 IF  OT$ <> "P" AND OT$ <> "C" THEN 50
70 INPUT "    Exercise price                     "; E
80 INPUT "    Days to expiration                 "; T
90 T = T / 365!
100 IF UT$ = "F" THEN INPUT "    Futures price                "; U
110 IF UT$ = "S" THEN INPUT "    Security price               "; U
120 IF UT$ = "C" THEN INPUT "    Commodity price              "; U
130 IF UT$ = "S" THEN INPUT "    Annual dividend or coupon "; H
140 IF UT$ = "C" THEN INPUT "    Annual holding cost          "; H
150 INPUT "    Price volatility                   "; V
160 INPUT "    Short-term interest rate           "; R
170 PRINT
180 GOSUB 1000
190 IF OT$ = "P" THEN 230
200 PRINT USING "  Theoretical call premium  = #.##"; CPRE
210 PRINT USING "  Delta                     = #.##"; CDEL
220 PRINT : PRINT  : END
230 PRINT USING "  Theoretical put premium  = #.##"; PPRE
240 PRINT USING "  Delta                    = #.##"; PDEL
250 PRINT : PRINT  : END
1000 ' ****************************************************************
1010 ' *** OPTION THEORETICAL PREMIUM AND DELTA ***
1020 ' ****************************************************************
1030 IF T <= 0! THEN 1500
1040 ENRT = EXP (-R * T / 100!)
1050 VRT = V * SQR (T) / 100!
1060 IF UT$ = "F" THEN U1 = U
1070 IF UT$ = "S" THEN U1 = U / ENRT - H * T
1080 IF UT$ = "C" THEN U1 = U / ENRT + H * T
1090 D1 = LOG (U1 / E) / VRT + .5 * VRT
1100 D2 = D1 - VRT
1110 X = D1
1120 GOSUB 2000
1130 ND1 = NX
1140 X = D2
1150 GOSUB 2000
1160 ND2 = NX
1170 CPRE = ENRT * (U1 * ND1 - E * ND2)
1180 PPRE = CPRE - ENRT * (U1 - E)
1190 CDEL = ND1
1200 PDEL = 1! - ND1
1210 IF UT$ = "F" THEN CDEL = CDEL * ENRT
1220 IF UT$ = "F" THEN PDEL = PDEL * ENRT
1230 IF CPRE < (U - E) THEN CPRE = U - E : CDEL = 1!
1240 IF PPRE < (E - U) THEN PPRE = E - U : PDEL = 1!
1250 RETURN
1500 IF U > E THEN 1600
1510 IF U < E THEN 1700
1520 CPRE = 0! : PPRE = 0!
1530 CDEL = 0! : PDEL = 0!
1540 RETURN
1600 CPRE = U - E : PPRE = 0!
1610 CDEL = 1! : PDEL = 0!
1620 RETURN
1700 CPRE = 0! : PPRE = E - U
1710 CDEL = 0! : PDEL = 1!
1720 RETURN
2000 '****************************************
2010 ' *** NORMAL CDF ROUTINE ***
2020 '****************************************
2030 Z = .3989423 * EXP (-.5 * X * X)
2040 Y = 1! / (1! + .2316419 * ABS (X))
2050 NX = ((((1.330274 * Y - 1.821256) * Y + 1.781478) * Y - .3565638) * Y + .3193815) * Y * Z
2060 IF X > 0! THEN NX = 1! - NX
2070 RETURN
```

options are quoted in terms of sixteenths, thirty-seconds or sixty-fourths, the user must alter the program as appropriate. ■

Footnotes

1. See, for example, F. Black, "The Pricing of Commodity Contracts," *Journal of Financial Economics* 3 (1976), pp. 167–179; J. Cox and S.A. Ross, "A Survey of Some New Results in Financial Option Pricing Theory," *Journal of Finance* 31 (1976), pp. 383–402; Cox and Ross, "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics* 3 (1976), pp. 145–166; R. Geske, "The Valuation of Compound Options," *Journal of Financial Economics* 7 (1979), pp. 63–81; J.W. Hoag, "An Introduction to the Valuation of Commodity Options" (Working Paper No. 110, Institute of Business and Economic Research, University of California, Berkeley, 1978); A. Madansky, "Debt Options" (Working paper, Graduate School of Business, University of Chicago, 1982); R.C. Merton, "Option Pricing When Underlying Stock Returns Are Discontinuous," *Journal of Financial Economics* 3 (1976), pp. 125–144; M. Rubinstein, "Nonparametric Tests of Alternative Option Models" (Working Paper No. 117, Institute of Business and Economic Research, University of California, Berkeley, 1981); and C.W. Smith, "Option Pricing: A Review," *Journal of Financial Economics* 3 (1976), pp. 3–52.
2. F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy*, May/June 1973, pp. 637–654.
3. This article assumes familiarity with the fundamental concepts associated with the option markets. For helpful intuitive derivations of option pricing models, the reader is referred to R.M. Bookstaber, *Option Pricing and Strategies in Investing* (Reading, Mass.: Addison-Wesley Publishing Company, 1981) and J.C. Cox, S.A. Ross and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics* 7 (1979), pp. 229–263.
4. See M. Asay, "Pricing Options: Stock Index Futures and Physicals," *Market Perspectives*, May 1983, pp. 1–5.
5. See M. Asay, "A Note on the Design of Commodity Option Contracts," *The Journal of Futures Markets* 3 (1982), pp. 1–7, for further discussion of these formulas and J. Meisner, "Advantages of Futures-Style Options" (Chicago Board of Trade, 1983), for a comparison of futures-style options with conventional options.