THE LETTER BOX

Ed Weinberger **explains how to build one of those "black box" implied volatility calculators**

To modern option traders, the most important letter of the Greek alphabet is σ (sigma), a measure of the volatility of the security underlying their options. It is both a fundamental determinant of an option premium and an essential input to the computation of the risk management parameters delta, vega, and gamma. Sigma is not priced explicitly by the market, so it must either be estimated from historical data or inferred from market option premia. As many authors have noted, the volatility implied by the option price has the advantage of being a prospective volatility estimate, and thus represents the market's best guess at future volatilites. For this reason, most practitioners prefer implied volatility estimates to the retrospective estimates obtained from historical data.

If we knew the volatility, we could compute option prices via the Black-Scholes formulas for European options or numerical methods, such as binary trees, for American options. Unfortunately, the relationship between premium and volatility for both types of options is too complicated for the volatility to be worked out by simple algebraic manipulation. Iterative methods must be used.

Implementing an appropriate method is an exercise in numerical risk management. The most obvious approach (the bisection method) brackets the true implied volatility between a series of successively tighter upper and lower bounds, replacing the upper or lower bound at each stage by the average of the bounds, depending on the option premium predicted by this average value. Bisection represents the safest possible "investment", in that it is guaranteed to find the right answer. As in finance, however, safety has its price: each additional significant digit of accuracy is obtained at the cost of slightly more than three iterations ($\log_2 10$, to be exact), which makes bisection among the slowest known methods. This might not be a problem if we are only interested in a single option, but large portfolios can contain thousands of options, each of which must be processed accurately to avoid cumulative errors in computing portfolio hedge ratios. Furthermore, some of these options are likely to be American, so that each iteration involves at least one re-evaluation of an entire binary tree.

Alternatively, we can take a flier on a much faster method, which doubles the

The volatility implied by the option price has the advantage of being a prospective volatility estimate, and thus represents the market's best guess at future volatilities

number of correct digits in each volatility estimate after each iteration – when it works. The method, invented by Sir Isaac Newton and obvious, no doubt, to the father of calculus, is illustrated in figure 1. The solid curve is P(σ), the Black-Scholes estimate of the option premium as a function of σ; the horizontal lines are actual market premia. For a given market premium, P_{market}, the implied volatility is the value at which the P(σ) curve intersects the horizontal line $P = P_{market}$. Newton's idea was that the diagonal dotted line, the tangent to P(σ) at σ_1, intersects the market price line at a volatility σ_2 near the true implied volatility, and that a tangent drawn at σ_2 yields a still better estimate, σ_3. More generally, given the estimate σ_i, the improved estimate, σ_{i+1}, is given by

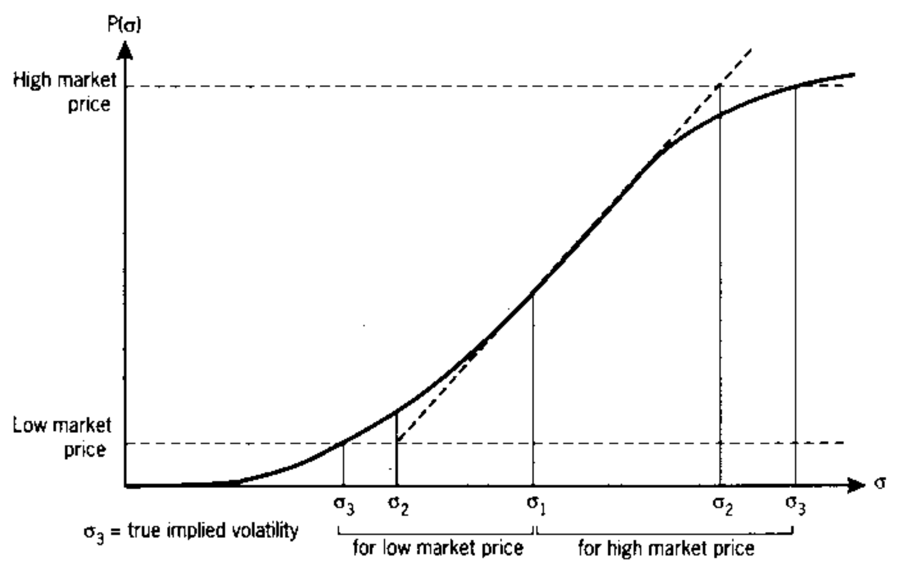
$$\sigma_{i+1} = \sigma_i - \frac{P(\sigma_i) - P_{market}}{P'(\sigma_i)}. \qquad (1)$$

Newton's method is especially convenient for finding implied volatilities for European options because $P'(\sigma_i)$, the slope of the tangent line at volatility σ_i, can then be computed explicitly via the formula

$$P'(\sigma_i) = S\sqrt{\frac{T}{2\pi}}\exp\left\{-\frac{\left[\ln(S\ X)+\left(r-q+\sigma_i^2\ 2\right)T\right]^2}{2\sigma_i^2 T} - qT\right\}, \qquad (2)$$

where S is the underlying market variable, X is the option strike, T is the time to option maturity, r is the risk-free rate, and q is the payout rate, if any, of the underlying secu-

1. Newton's method



rity. (For non-dividend paying securities $q = 0$, for single-currency financial futures, $q = r$, and for foreign currencies q is the foreign risk-free rate.) For both European and American options, $P'(\sigma_i)$ is also the vega of the option at that volatility, so that we get the option vega for free when σ_i converges to the true implied volatility.

Figure 2 shows what can go wrong with Newton's method when applied to a put option on one-year Libor with a strike at 5%. Given the current (March 1993) market, one year Libor is 3.625%, and $\sigma = 26\%$, so that the option premium should be 5.51% of the notional principal. However, if $\sigma_i = 10\%$, a seemingly reasonable value, subsequent iterates rapidly move *away* from the true implied volatility. Volatilities soon reach such ridiculous levels that the corresponding vega values are effectively zero, resulting in a nasty "division by zero" error when the computer evaluates (1).

This risk management tale has a happy ending, though, because Newton's method is guaranteed to converge from the starting point[1]

$$\sigma_1 = \sqrt{\frac{2\left|\ln(S/X) + (r-q)T\right|}{T}}. \qquad (3)$$

At this sigma value, the corresponding vega value – and thus the slope of the curve in figure 1 – is a maximum. If the estimated option premium using (3) as the volatility is larger than the actual market premium, the volatility iterates generated by Newton's method will be a decreasing sequence bounded below by the true volatility. Such a sequence must converge, and since (1) will continue to generate ever-decreasing iterates until the true implied volatility is reached, (1) must converge to that value. As figure 1 suggests, a similar argument applies when the volatility estimate (3) is smaller than the market volatility.

Implied volatilities for American options can be found efficiently via a scheme similar to (1). We start by finding the implied volatility and vega, σ^{EURO} and $vega^{EURO}$, of the European option with the same terms as the American option, as above. Because σ^{EURO} and $vega^{EURO}$ are good approximations to the corresponding volatility and vega for the American option, (1) can then be used to obtain a second implied volatility estimate, provided that $P(\sigma)$ now represents the American option premium as a function of σ. This second estimate is often good enough, but, if not, improved estimates can be found via the secant method, ie the iteration scheme

$$\sigma_{i+1} = \sigma_i - \left[P(\sigma_i) - P_{market}\right]\frac{\sigma_i - \sigma_{i-1}}{P(\sigma_i) - P(\sigma_{i-1})} \qquad (4)$$

Note that (4) is just a variant of (1) with the approximation

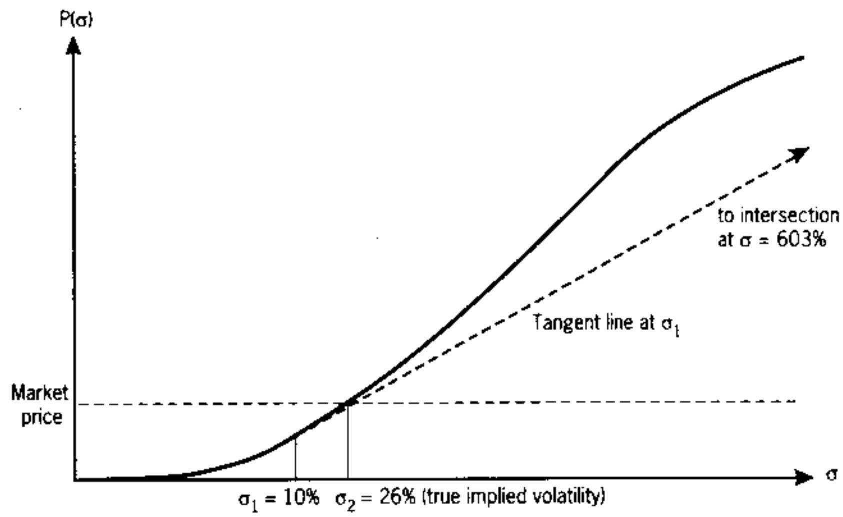
$$P'(\sigma_i) \approx \frac{P(\sigma_i) - P(\sigma_{i-1})}{\sigma_i - \sigma_{i-1}}.$$

I cannot assert with mathematical certainty that (4) will always work, but it can be shown[2] that

$$\left(\sigma_{i+1} - \sigma^{AMER}\right) = M\left(\sigma_i - \sigma^{AMER}\right)\left(\sigma_{i-1} - \sigma^{AMER}\right), \qquad (5)$$

where σ^{AMER} is the true implied volatility, and M is a function of S, X, T, r, and q that, for reasonable values of these quantities, has a numerical value of no more than 10 or so. Thus, if σ^{EURO} is within a few percent of σ^{AMER}, the error in σ_i will each be at least a factor of 10 smaller than σ_{i-1} for $i > 2$. In fact, (5) can be used to show[3] that each iteration of (4) increases the number of correct digits by a factor of at least 1.6.■

2. Newton's method applied to put option on one-year Libor with strike of 5%



[1] See Manaster, S, and G Koehler, Journal of Finance, 37 (1), 1982, page 227 for a derivation of this formula in the special case when q = 0
[2] Stoer, J, and R Bulirsch, 1980, *Introduction to numerical analysis*, Springer Verlag, New York, page 293
[3] Stoer and Bulirsch, page 293

Ed Weinberger is assistant vice-president and manager, quantitative analysis, in the interest rate group of Midland Bank in New York