by Mark Schroder

Adapting the Binomial Model to Value Options on Assets with Fixed-Cash Payouts

The binomial model has proved to be an efficient technique for approximating the value of an option on a zero-dividend or fixed-yield-dividend stock. With some small modifications, this model can also be used to approximate the value of an option on a stock that pays fixed-cash dividends. If one assumes ex-dividend stock prices to be lognormally distributed, stock price can be partitioned into two parts—a riskless part representing the present value of all dividends to be paid over the option's life and a risky part representing the remaining net assets of the firm. The stock price net of the present value of the escrowed dividends is used as the starting point of the binomial process. Then, a small number of formulas are used recursively to compute option prices at all nodes.

It may be more realistic, however, to assume lognormal cum-dividend prices with known but uncertain dividends. Dividends, which are not escrowed, may not be paid in full if the value of the firm falls far enough. In this case, one can construct the binomial tree ignoring dividends and interpolating option prices at ex-dividend dates.

Previously, solutions to the fixed-cash-dividend problem could be obtained only by using a cumbersome and computationally expensive version of the binomial model, or by using more complicated finite-difference techniques. The proposed binomial models offer good results from much less complex computations. Furthermore, the binomial models cost little more (and in some cases cost even less) than the fixed-yield-dividend binomial model.

THE BINOMIAL OPTION-PRICING technique has gained wide popularity as a result of its simplicity, intuitive appeal and ability to handle a variety of stock price processes. Geske and Shastri have found the binomial model to be generally efficient computationally, and concluded that the "binomial approximation appears to dominate all the finite difference schemes when either there are no payouts or a small number of options are being valued."[1] When valuing options on assets with fixed-cash dividends (as opposed to fixed-yield dividends), however, the efficiency of the binomial model quickly deteriorates. This article presents two techniques for adapting the binomial model to value call and put options on assets with fixed-cash payouts. The first technique assumes that the asset price minus the present value of payouts over the life of the option follows a lognormal distribution. The second technique assumes that the cum-dividend price is lognormally distributed.

1. Footnotes appear at end of article.

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Problems in Applying the Binomial Model to Fixed-Cash-Dividend Stocks

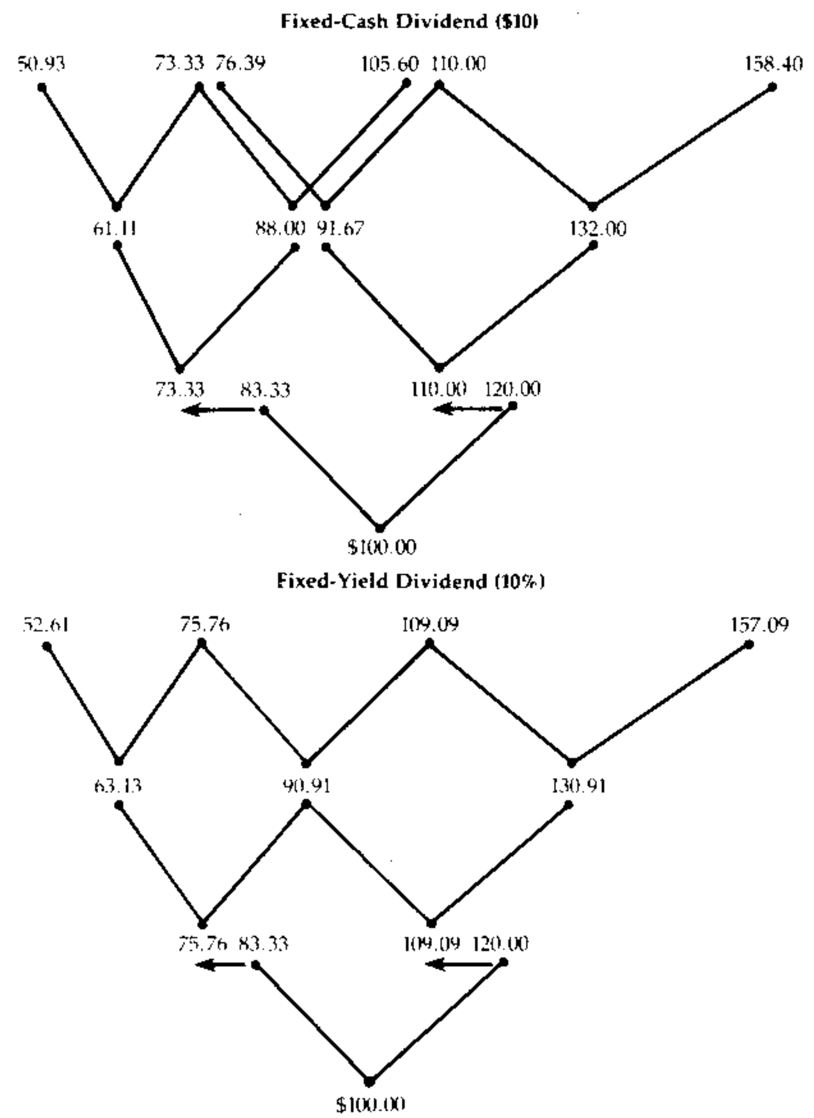
The binomial option pricing technique, while relatively efficient and simple to implement in the case of fixed-yield dividends, becomes cumbersome and inefficient in the case of fixed-cash dividends. With no payouts, or payouts proportional to the stock price, the stock price after an up-down movement always coincides with the price after a down-up movement. Consequently, the number of nodes per period increases by only one each period. With fixed-cash payouts, however, the number of nodes per period doubles from, say, n to 2n after an ex-dividend date; thereafter, the number of nodes increases by 2n each period. Consequently, the tree tends to "explode" with even a small number of dividends. A tree of 60 periods would contain a total of 1,891 nodes for a zero-dividend or fixed-yield-dividend stock. The number of nodes would grow to 15,841, 106,491, 589,816 and 2,784,691 for one, two, three and four equally spaced fixed-cash dividends.[2]

The problem is often avoided by assuming fixed-yield dividends. Figure A compares fixed-yield and fixed-cash-dividend binomial trees over three periods, with an ex-dividend date falling at the end of the first period. The assumption of fixed-yield dividends may be unrealistic, however. Dividend policies, as is often pointed out, may be approximated over the long term by a fixed dividend yield, but over the short term, which is relevant for most option contracts, firms tend to pay fixed-cash dividends.

Analytic Models for Options on Fixed-Cash-Dividend Stocks

Roll derived an analytic formula for pricing American call options on stocks with one certain

Figure A Binomial Trees



fixed-cash dividend before expiration by demonstrating that the option's payoff can be replicated by forming a portfolio of two European options and a compound option; all three options have known analytic pricing formulas.[3] Geske outlined an extension of the Roll model to value calls on stocks with any number of fixed-cash payments before expiration; the precise formula was derived by Selby and Hodges.[4]

Geske and Johnson derived an analytic formula for the value of American puts on fixed-cash-dividend and non-dividend-paying stocks by considering the option as an infinite sum of compound options, each exercisable at discrete points in time.[5] They used the Richardson extrapolation method to estimate the option price by valuing puts with one, two and three exercise points. Their method (using four exercise points) was found to be accurate and more efficient than numerical techniques, including the fixed-cash-dividend binomial model.

These models all assume that the stock price less the discounted value of the dividends to be paid over the life of the option (this will sometimes be referred to as the net, or ex-dividend, price) follows a lognormal process. This assumption makes the analytic solutions possible and ensures that the known dividends are certain (i.e., the money escrowed will cover the dividends no matter what happens to the remaining value of the firm). Numerical analysis indicates that solutions based on this assumption closely approximate the solutions derived by assuming instead that cum-dividend prices are lognormally distributed.[6] (Further evidence comparing these approaches is presented below.)

Rubinstein has considered the more general case of a firm comprised of risky and riskless assets.[7] (In the models above, only the escrowed dividends are riskless.) Fixed-cash dividends are paid out of the riskless portion of the firm's assets and fixed-yield dividends are paid out of the risky portion of the value of the firm. The firm's capital structure includes both stock and debt. While Rubinstein derived an analytical formula for a European call option on the firm's stock, an American formula could be derived by employing the compound option approach used in the above models.

From a practical standpoint, direct solution of these analytical American option formulas is limited to calls or puts with few exercise points. The valuation of an option with N possible exercise points (including expiration) requires the computation of an N-dimensional multivariate normal integral.[8] To price a call on a stock with two dividends, for example, a trivariate normal integral must be evaluated. The high computational costs of accurately computing multivariate integrals prompted Whaley to exclude options with more than one dividend date before expiration from his empirical study of the Roll model.[9]

Adapting the Binomial Model to Lognormal Ex-Dividend Prices

The binomial model can be easily adapted to approximate the analytical models described above. Because these models assume that ex-dividend (not cum-dividend) stock prices are lognormally distributed, the stock price must be partitioned into two parts—a riskless part, representing the present value of all dividends to be paid over the option's life, and a risky part, representing the remaining net assets of the firm. The risky part of the stock price is assumed to follow a lognormal distribution and is unaffected by dividend payments (i.e., dividends are paid out of the riskless part); consequently, the evolution of the stock price can be approximated without the number of nodes exploding.

The stock price net of the present value of escrowed dividends is used as the starting point of the binomial process. The up and down parameters suggested by Cox, Ross and Rubinstein may be used without modification:[10]

$$u = e^{\sigma\sqrt{t/n}},$$
$$d = 1/u,$$
$$p = (u - r)/(u - d),$$

where t is the number of years to expiration, σ is the annualized volatility of the stock, r is one plus the rate of interest over one period, and n is the number of periods in the binomial model. The exercisable proceeds of the call (as always, early exercise of a call is optimal only at ex-dividend dates) are the net stock price plus the current balance of the escrowed dividend account minus the exercise price. The value of the escrowed dividend account at any point in time is the present value of all dividends remaining in the life of the option.

The call price at each node is calculated recursively using the following formula

$$C = \text{Max}\,[(pC_u + (1 - p)C_d)/r,\; S^{net} + D - K] \qquad (1)$$

at nodes corresponding to ex-dividend dates, or simply

$$C = (pC_u + (1 - p)C_d)/r \tag{2}$$

at all other nodes. C_u and C_d are the call prices after upward and downward stock price movements; D is the current balance in the escrowed dividend account; S^{net} is the stock price net of escrowed dividends; and r is one plus the risk-free return over one binomial period. Equation (2) represents a discounted weighted average of the possible call prices next period. More specifically, it is the discounted expected value of the option if investors are risk-neutral. At expiration, D will equal the dividend per share, if any, during the last period of the binomial tree, and the call price at each terminal node will be:

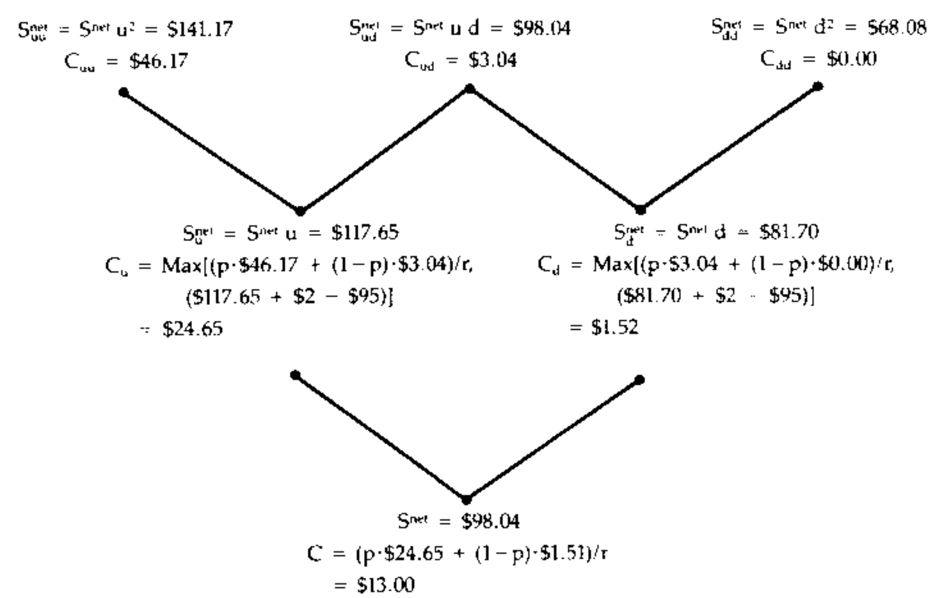
$$C = \text{Max}\,[S^{net} + D - K, 0].$$

For puts, early exercise is generally always possible. The following formula is thus used to compute prices at all nodes:

$$P = \text{Max}\,[(pP_u + (1 - p)P_d)/r,\ K - S^{net} - D].$$

Figure B illustrates the application of the proposed model to a call option with 182 days to expiration, on a stock paying a single \$2 dividend. The ex-dividend date, in 91 days, corresponds to the net of the first of the two binomial periods. The stock price net of escrowed dividends at the beginning of the first period is \$98.04 (= 100 − 1.96). (The present value of the dividend is the current, or end-of-period-0, balance in the escrowed dividend account; the

Figure B Fixed-Cash Dividend: Lognormal Ex-Dividend Stock Prices*



*A \$2.00 dividend is paid in 91 days. The current stock price, S, is \$100 and the strike price is \$95. The option expires in 182 days; σ is 36.51%, and the annual risk-free interest rate is 8%. The binomial parameters, following Cox, Ross and Rubinstein, are as follows:

$$u = e^{\sigma\sqrt{t/n}} = e^{.3651\sqrt{(182/365)/2}} = 1.12$$

$$d = 1/u = 1/1.2$$

$$p = (u - r)/(u - d) = 0.50738$$

$$r = 1.08^{\sqrt{(182/365)/2}} = 1.0194$$

At the end of Period 0, the escrowed dividend balance (D) will be $\$2.00\cdot 1.08^{-182/365} = \$1.96$; it will be \$2.00 at the end of Period 1 and \$0.00 at the end of Period 2.

account balance each period appears at the bottom of the figure.) The call prices are calculated recursively from expiration. The intrinsic value of the option at the ex-dividend date is the current net stock price plus \$2 (the balance in the escrowed dividend account) minus the strike price of \$95.

Advantages of the Proposed Technique

When applied to American calls on stocks with one ex-dividend date, the proposed numerical procedure will yield results identical, in the limit, to the Roll model. More generally, it is identical in the limit to the Geske, Selby and Hodges call and Geske and Johnson put models, which are applicable to options on stocks with any number of fixed-cash dividends. The binomial model, however, has several advantages. It is intuitive and easy to program; industry practitioners may find it difficult to implement models requiring the accurate evaluation of multivariate normal integrals. The model is as easy to apply to puts as it is to calls; only the boundary condition needs to be changed. (Also, the boundary condition must be applied throughout the life of the option, rather than at ex-dividend dates only, as is the case for calls.) The Geske and Johnson approximation method for American put options becomes more complicated with the introduction of dividends, as the exercise points must be chosen carefully to obtain accurate results. If nothing else, the modified binomial model can serve as a useful check on the solutions of these more complicated models.

The last three columns of Table I compare binomial approximations with the exact call option price computed using the analytical model. The prices of the 150-iteration binomial model are generally within one cent of the exact solution. The estimated deltas are all within 0.001 of their exact values.

Adapting the Binomial Model to Lognormal Cum-Dividend Prices

As mentioned above, the analytic solutions to the fixed-cash-dividend problem assume that dividends to be paid over the life of the option are known with certainty and that the stock price minus the present value of these dividends is lognormally distributed. This may be considered an ad hoc and unrealistic description of the structure of the firm. For example, one theoretical problem with this model is that two options that differ only by maturity will have two different assumed stochastic processes for the same underlying stock price if the present values of dividends over the lives of the two options differ.[11]

An alternative, and possibly more accurate, stock price model is lognormal cum-dividend prices with known but uncertain dividends. Because dividends are not escrowed, they may not be paid in full (or at all) if the value of the firm falls far enough. No analytic solution exists for options (American or European) on stocks with this distribution.

Rather than modeling the possible stock price paths while incorporating the impact of dividends, one can construct the binomial tree ignoring dividends and interpolating option prices at ex-dividend dates. This is the same approach used for handling fixed-cash dividends in finite-difference methods (both implicit and explicit). Given the similarity of the binomial and finite-difference methods, this approach seems reasonable.

Figure C illustrates this technique applied to an option with the same terms as the example in Figure B. The starting point of the tree is the gross stock price, \$100. Working recursively from expiration, one computes call prices at the end of period 1 (the ex-dividend date) as follows:

$$C_u(\$120 + \$2) = \text{Max}\,[(p \cdot \$49.00 + (1 - p) \cdot \$5.00)/r,\ \$120 + \$2 - \$95],$$
$$C_d(\$83.33 + \$2) = \text{Max}\,[(p \cdot \$5.00 + (1 - p) \cdot \$0.00)/r,\ \$83.33 + \$2 - \$95].$$

Implicit in the computations is the assumption that the ex-dividend prices at the end of period 1 are \$120 and \$83.33 and the cum-dividend prices are \$122 and \$85.33. However, in order to compute prices at the beginning of period 1, it is necessary to have end-of-period option values at *cum*-dividend stock prices of \$120 and \$83.33 [i.e., C_u (\$120) and C_d (\$83.33) are needed]. These call prices may be interpolated as follows:[12]

$$C_u(\$120) = C(\$122) - (\$122 - \$120)$$

$$\frac{C(\$122) - C(\$85.33)}{\$122 - \$85.33} = \$25.66,$$

$$C_u(\$83.33) = C(\$85.33) - (\$85.33 - \$83.33)$$

Table I Binomial American Call Prices*

$\sigma = 30\%$

Days to Exp.	K	FYD Exact	FCD 50 Iter.	FCD 150 Iter.	FCD Exact	FCDex 50 Iter.	FCDex 150 Iter.	FCDex Exact
90	90	12.22	12.21	12.20	12.19	12.16	12.16	12.16
		(.806)	(.807)	(.808)	(.808)	(.811)	(.810)	(.810)
	100	6.00	5.99	6.00	6.01	5.99	5.98	5.97
		(.539)	(.541)	(.541)	(.542)	(.541)	(.541)	(.541)
	110	2.48	2.52	2.51	2.51	2.49	2.48	2.48
		(.288)	(.291)	(.292)	(.292)	(.290)	(.291)	(.291)
180	90	14.10	14.12	14.10	14.10	14.03	14.00	13.99
		(.736)	(.745)	(.745)	(.745)	(.748)	(.748)	(.748)
	100	8.48	8.51	8.52	8.53	8.45	8.41	8.40
		(.548)	(.556)	(.557)	(.557)	(.558)	(.557)	(.557)
	110	4.74	4.85	4.82	4.81	4.68	4.70	4.69
		(.368)	(.376)	(.376)	(.376)	(.373)	(.374)	(.374)
360	90	16.82	16.90	16.88	16.89	16.64	16.56	16.57
		(.687)	(.704)	(.705)	(.706)	(.710)	(.711)	(.711)
	100	11.77	11.89	11.92	11.94	11.63	11.60	11.60
		(.554)	(.572)	(.574)	(.575)	(.575)	(.575)	(.576)
	110	8.02	8.24	8.22	8.23	7.91	7.90	7.90
		(.428)	(.446)	(.447)	(.449)	(.445)	(.445)	(.446)

$\sigma = 50\%$

Days to Exp.	K	FYD Exact	FCD 50 Iter.	FCD 150 Iter.	FCD Exact	FCDex 50 Iter.	FCDex 150 Iter.	FCDex Exact
90	90	15.25	15.27	15.25	15.24	15.20	15.18	15.17
		(.716)	(.717)	(.718)	(.718)	(.710)	(.719)	(.719)
	100	9.87	9.86	9.89	9.90	9.85	9.84	9.83
		(.551)	(.554)	(.555)	(.555)	(.555)	(.555)	(.555)
	110	6.10	6.20	6.18	6.16	6.14	6.10	6.09
		(.397)	(.401)	(.402)	(.402)	(.401)	(.401)	(.401)
180	90	18.75	18.76	18.79	18.79	18.64	18.61	18.59
		(.682)	(.691)	(.691)	(.691)	(.694)	(.694)	(.693)
	100	13.87	13.90	13.94	13.95	13.80	13.77	13.75
		(.568)	(.577)	(.577)	(.577)	(.578)	(.578)	(.578)
	110	10.11	10.19	10.24	10.22	10.06	10.02	10.01
		(.460)	(.468)	(.469)	(.469)	(.468)	(.468)	(.468)
360	90	23.50	23.68	23.66	23.66	23.19	23.14	23.11
		(.664)	(.683)	(.684)	(.684)	(.688)	(.688)	(.688)
	100	19.15	19.34	19.38	19.41	18.94	18.86	18.85
		(.585)	(.604)	(.605)	(.606)	(.607)	(.607)	(.607)
	110	15.57	15.98	15.90	15.91	15.32	15.33	15.34
		(.510)	(.531)	(.531)	(.532)	(.530)	(.530)	(.531)

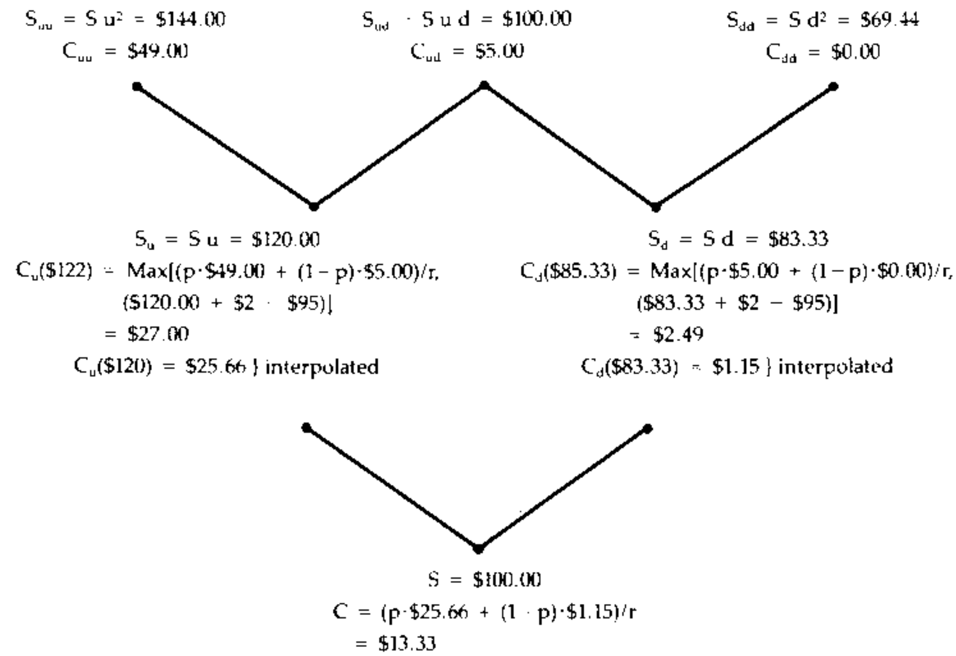
* The three models are (1) fixed-yield dividends, lognormally distributed cum-dividend stock prices (FYD), (2) fixed-cash dividends, lognormally distributed cum-dividend stock prices (FCD) and (3) fixed-cash dividends, lognormally distributed ex-dividend stock prices (FCDex). The current (gross) stock price is $100 and the interest rate is 7.0%. Deltas appear in parentheses below the prices. The stock is assumed to pay $1.50 dividends in 45, 135, 225 and 315 days. The fixed-yield model assumes dividend payments of 1.50% ($1.50/$100) of the stock price at each ex-dividend date. The exact prices for the (fixed-cash) lognormal cum-dividend and the fixed-yield models are computed using the binomial model with 1,000 iterations. The exact prices for the (fixed-cash) lognormal ex-dividend model are computed using the Geske, Selby and Hodges n-dividend extension of the Roll model. Newton's interpolation method (for quadratic interpolation) is used to compute option prices at ex-dividend dates for the (fixed-cash) lognormal cum-dividend model. The fixed-yield prices were also computed by setting the percentage payments to equate the present value of each fixed-yield dividend with the present value of each fixed-cash dividend. These prices were found to be very close to the prices reported in the table.

$$\frac{C(\$122) - C(\$85.33)}{\$122 - \$85.33} = \$1.15.$$

In general, on ex-dividend dates, the option prices are computed at cum-dividend stock prices of S + D:

$$C(S + D) = \text{Max}\ [(pC_u + (1 - p)C_d)/r,\ S + D - K].$$

Figure C Fixed-Cash Dividend: Lognormal Cum-Dividend Stock Prices*



*A \$2.00 dividend is paid in 91 days. The current stock price, S, is \$100 and the strike price is \$95. The option expires in 182 days; σ is 36.51%, and the annual risk-free interest rate is 8%. The binomial parameters are the same as those given in Figure B.

Then one interpolates C(S), the option price corresponding to a cum-dividend stock price of S. Equation (2) is used to compute option prices at all other dates. For puts, the procedure is the same, except that exercise must be checked at all nodes. Finally, European option prices are computed by using the formula.

$$C(S + D) = (pC_u + (1 - p)C_d)/r$$

at ex-dividend dates, again interpolating C(S), then using Equation (2) for all other periods.

As the number of binomial periods increases, option prices computed using this proposed technique converge to those obtained using the fixed-cash-dividend binomial model described above. The computational cost is generally far less, however, because the cost of interpolating prices at ex-dividend dates is usually much smaller than the cost of evaluating the much larger number of nodes required by the former model.

Advantages over Finite-Difference Methods

The binomial model offers a number of advantages over finite-difference methods for valuing options on stocks with fixed-cash dividends. Explicit finite-difference techniques, in general, are subject to stability problems; therefore, the size of the grid must be chosen carefully to ensure convergence.[13] The binomial model, on the other hand, always converges as the number of periods approaches infinity. Implicit finite-difference techniques are stable, but they are time-consuming because they require the solution of a set of simultaneous equations.

Both proposed binomial models are likely to run considerably faster than finite-difference models. Geske and Shastri found the computational cost of the fixed-dividend-yield binomial method to be considerably less than that of competing finite-difference methods when calculating a single option price.[14] The computational cost of the first proposed binomial method for valuing fixed-cash dividends (assuming lognormal ex-dividend stock prices) will be similar to that of the binomial fixed-yield method. The second proposed binomial model will be slower because it requires the interpolation of option prices at ex-dividend dates. However, this is likely to erode the binomial model's large cost advantage only slightly.

Finally, either proposed model is easily adapted to more complicated valuation problems, such as risky corporate liabilities on firms with fixed-cash distributions (including coupon payments), compound options on assets with fixed-cash payouts, or even long-term options on stocks that pay fixed-cash dividends in the near term and fixed-yield dividends in the future.

Comparing the Models

Below, we compare the option prices for three stock price models—lognormal cum-dividend prices with fixed-cash dividends (FCD); lognormal cum-dividend prices with fixed-yield dividends (FYD); and lognormal ex-dividend prices with fixed-cash dividends (FCDex).

A European call or put option on a FCD stock will always be worth at least as much as a European option on a FYD stock if the present values of the dividends are equal. In contrast to fixed-cash dividends, fixed-yield dividends depend on the stock price at the ex-dividend date. On one hand, higher dividends associated with higher stock prices diminish the value of the call. On the other hand, the call is made more valuable by the lower dividends associated with lower stock prices. The full benefit of lower dividends will not be realized, however, as long as there is some probability that the stock will expire out of the money.[15] In other words, an option on the FYD stock is worth less because of the asymmetrical payoff of the option. The difference between the FCD and FYD prices will increase with the variance and time to expiration of the stock, and the difference will tend to be largest for (not too far) out-of-the-money options.

The possibility of early exercise in American options complicates the comparison. Though American options on FCD stocks are usually more valuable than those on FYD stocks, this is not always true. Table I contains two examples (90 days to expiration; $K = \$90$; and $\sigma = 30\%$, 50%) where the call on the FYD stock is slightly more valuable. In both cases, the option is in the money and the stock price is near the critical price above which it is optimal to exercise at the ex-dividend date. The holder of the option on the FYD stock is unaffected by the higher dividend caused by a rising stock price, because it will be optimal to exercise before the dividend is paid. He will benefit, however, from a lower dividend payment associated with a falling stock price.

As stated earlier, it is often assumed that the true stock price model is FCD, but FYD or FCDex models are used as approximations. The evidence in the table shows that American FYD and FCD prices are usually very close. For the 90-day options, the maximum difference is only 6 cents. The differences increase with volatility and time to expiration, and the differences are generally largest for out-of-the-money and smallest for in-the-money options. The possibility of early exercise tends to reduce the differences between FCD and FYD option prices; consequently, the reported differences between the American option prices are often substantially smaller than the differences between European options. (Results for European options are not shown.)

Option prices on FCDex stocks appear in the last column of the table. Because of the escrowing of the dividends (i.e., only the ex-dividend price follows a lognormal process), the FCDex prices are always less than the FCD prices. The FCDex prices are also almost always less than the FYD prices. The differences between the FCDex and FCD option prices are small for the 90-day options; the maximum difference is only 9 cents. The differences become as high as 57 cents for the out-of-the-money, one-year option on the 50 per cent volatility stock. ■

Footnotes

1. R. Geske and K. Shastri, "Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques," *Journal of Financial and Quantitative Analysis*, March 1985, pp. 45–71.
2. If $n_1, n_2, \ldots, n_{i-1}$ represent the number of periods between each of i-1 dividends, and n_i represents the number of periods between the last ex-dividend and expiration, the total number of nodes in the fixed-cash-dividend tree is:

$$0.5\{n_1(n_1 + 1) + n_2(n_1 + 1)(n_2 + 1) + \ldots + n_{i-1}(n_1 + 1) \cdot \ldots \cdot (n_{i-1} + 1) + (n_i + 2)(n_1 + 1)(n_2 + 1) \cdot \ldots \cdot (n_i + 1)\}.$$

3. R. Roll, "An Analytic Valuation Formula for Unprotected American Call Options on Stocks with Known Dividends," *Journal of Financial Economics*, November 1977, pp. 251–258.
4. Geske, "The Valuation of Compound Options," *Journal of Financial Economics*, March 1979, pp. 63–81 and J. P. M. Selby and S. D. Hodges, "On the Evaluation of Compound Options," *Management Science* 1987, pp. 347–355. Geske examined the closely related problem of valuing a corporate

bond with n coupon payments as a compound option in "The Valuation of Corporate Liabilities as Compound Options," *Journal of Financial and Quantitative Analysis*, November 1977, pp. 541–552.

5. Geske and H. E. Johnson, "The American Put Option Valued Analytically," *Journal of Finance*, December 1984, pp. 1511–1524.
6. Geske, "Valuation of Compound Options," *op. cit.*
7. M. Rubinstein, "Displaced Diffusion Option Pricing," *Journal of Finance*, March 1983, pp. 213–217.
8. In "The Valuation of Corporate Liabilities," *op. cit.*, Geske applies an integral-reduction algorithm developed in R. N. Curnow and C. W. Dunnett, "The Numerical Evaluation of Certain Multivariate Normal Integrals," *Annals of Mathematical Statistics* 33 (1962), pp. 571–579. This reduces an N-dimensional integral to an integral whose dimension is the integral part of N/2. A more general integral reduction algorithm can reduce an N-dimensional integral to an integral whose dimension is the integral part of the base 2 logarithm of N (if $N > 5$, this is less than the integral part of N/2). See M. Schroder, "A Reduction Method Applicable to Compound Option Formulas," *Management Science*, forthcoming. These methods can be applied to any multivariate integral arising from compound option problems because of the simple form that the correlation matrix takes as a result of the sequential nature of the exercise points.
9. R. W. Whaley, "Valuation of American Call Options on Dividend Paying Stocks: Empirical Tests," *Journal of Financial Economics*, March 1982, pp. 29–58.
10. J. C. Cox, S. A. Ross and M. Rubinstein, "Option Pricing: A Simplified Approach," *Journal of Financial Economics*, September 1979, pp. 229–263.
11. This problem was pointed out to me by Mark Rubinstein.
12. Linear interpolation is shown here. However, quadratic interpolation (which fits a second-degree polynomial to three points) is used in Table 1. Note that the call price at the lowest stock price is actually extrapolated.
13. For a discussion of explicit and implicit finite-difference methods and their application to pricing American stock options, see M. J. Brennan and E. S. Schwartz, "Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis," *Journal of Financial and Quantitative Analysis*, September 1978, pp. 461–474.
14. "Valuation by Approximation," *op. cit.* For American calls on stocks paying two dividends, the cost of executing the binomial method was roughly 1/9th the cost of the least expensive finite-difference method. For American puts on non-dividend stocks, the binomial method was around 2/17th the cost, and for American puts on stocks paying two dividends, the binomial method was around 1/5th the cost. If option prices corresponding to a range of stock prices are desired (finite-difference methods always calculate a range of option prices), the binomial method need not be re-executed for each stock price. For example, to calculate option prices for initial net stock prices of S^{net}, uS^{net} and dS^{net}, the binomial tree can be widened at both sides. Redundant calculations will thereby be avoided. Direct solution of the analytic models described in the section on prior analytic models, however, requires re-executions for each stock price considered.
15. Consider, for example, an out-of-the-money call on a stock going ex-dividend shortly before the option expires. The option will finish in the money only if the stock price, and therefore the dividend payment, rises. If the stock price falls, so will the dividend, but the option expires worthless nevertheless.