Debt Maturity Management

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Abstract

This paper studies how a borrower issues long- and short-term debt in response to shocks to the fundamental value. Short-term debt protects creditors from future dilution and incentivizes the borrower to reduce leverage after small negative shocks. Long-term debt postpones default and allows the borrower time to recover after large negative shocks. When borrowers are in distress, they rely on short-term debt; however, they issue both types of debt during more normal periods. Our model generates novel implications for the dynamic adjustment of debt maturities.

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1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in several aspects; an important one is its maturity. Borrowing can be short, as in the case of trade credit, or long, as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing following shocks to their fundamental value?

Traditional capital structure models in dynamic corporate finance – which build upon Fischer et al. (1989) and Leland (1994) – typically assume that (1) all debt has the same (expected) maturity and (2) the borrower either commits to the total leverage or may only increase leverage after retiring all existing debt and paying some exogenous issuance cost.[1] Although these assumptions simplify the analysis, they are inconsistent with the ample empirical evidence that borrowers often issue a mix of long- and short-term debt simultaneously and that adjusting the outstanding debt's maturity profile can take some time to accomplish.

This paper introduces a simple and tractable framework to address these questions. Our model features two types of debt: short- and long-term. The central tension arises from the trade-off between commitment and the option value to postpone default. Without a commitment to future issuance, a borrower would have incentives to dilute existing long-term debt. In contrast, short-term debt cannot be diluted as it matures before the borrower can borrow again, providing a commitment mechanism. Meanwhile, long-term debt offers the option to postpone default: following a large negative shock, long-term debt shares the losses to the firm's value with equity holders, potentially avoiding immediate default. Even though the borrower is risk-neutral, the costs associated with bankruptcy imply that the option to delay bankruptcy can be advantageous, creating a demand for long-term debt. The optimal composition of long-term and short-term debt balances the dilution costs of long-term debt and the option value it provides.

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss later.

More specifically, a risk-neutral borrower has assets that generate an income flow that follows a geometric Brownian motion (GBM) whose drift switches between two different regimes representing the upturn and downturn. The expected growth rate of the income is high in an upturn but low in a downturn. A transition from the upturn to the downturn is a large negative shock, interpreted as the downside risk. Creditors are competitive, risk-neutral, and more patient than the borrower. The difference in patience motivates the borrower to issue debt. Two types of debt are available. The short-term debt matures instantaneously (i.e., has zero maturity). Long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing both types of debt at any time to adjust the maturity profile of the outstanding debt.

Our paper shares with the existing literature (Fama and Miller, 1972; Black and Scholes, 1973; Admati et al., 2018) that an uncommitted borrower with outstanding long-term debt is always tempted to issue more debt to dilute long-term creditors. In equilibrium, creditors anticipate future dilution and long-term debt prices adjust downwards to the level where the borrower cannot capture any benefit (DeMarzo and He, 2021). By contrast, short-term debt matures before the borrower can issue debt again and, therefore, does not suffer from dilution. Following small negative shocks, short-term debt incentivizes the borrower to reduce the leverage, even though she has not committed to doing so. This happens because short-term debt constantly resets, so the borrower reaps all the benefits from delevering, whereas in the case of long-term debt, the delevering benefits are shared with existing long-term creditors.

Given short-term debt's advantage in addressing the commitment problem, the borrower naturally benefits from issuing it. Indeed, our results show that during downturns, the borrower can costlessly commit to not issuing long-term debt by issuing a large amount of short-term debt, fully exhausting the borrowing capacity. However, during upturns, this commitment via short-term debt comes at a cost. The potential arrival of a future downturn introduces an interesting trade-off in issuing short-term debt. On one hand, the borrower can issue enough short-term debt to exhaust the borrowing capacity, fully capturing the benefits of debt issuance. But this makes the short-term debt *risky*, leading to immediate default if a downturn occurs before the debt matures. Alternatively, the borrower can issue a *safe* level of short-term debt, preserving some borrowing capacity to avoid default even if the state switches to a downturn. However, this safe level means the borrower can no longer fully capture the benefits of debt issuance. Our model shows that when a borrower has a substantial amount of outstanding long-term debt, she opts for the risky level of short-term debt. In contrast, if the outstanding long-term debt is low, she chooses the safe level of short-term debt. The rationale is that the incremental increase in default probability represents the marginal cost of issuing short-term debt. When there is a high level of outstanding long-term debt, this marginal cost is low, and vice versa.

The issuance of long-term debt during upturns is directly related to the choice of short-term debt level. When the borrower optimally issues the risky (and higher) level of short-term debt, there is no additional value from borrowing long-term. In these cases, the firm relies entirely on short-term financing. This is because any long-term debt issued is exposed to the same downside risk as the short-term debt and therefore there is no option value to postpone default. Conversely, when the borrower optimally issues the safe (and lower) level of short-term debt, there is additional value from borrowing long-term. This occurs because compared to short-term debt, long-term debt gives equity holders stronger incentives to inject cash into the firm by providing the option to postpone default. This feature is particularly valuable when short-term debt does not cause immediate default upon a state switch, as a small amount of additional long-term debt would not trigger default during regime changes, while the same increment in short-term debt would. Notably, the borrower values this option from long-term debt despite being risk-neutral, because the costs associated with bankruptcy, as acknowledged in previous literature (Smith and Stulz, 1985).[2] Finally, without the commitment to not dilute creditors, the value of long-term debt gets dissipated, but nevertheless, the firm may still issue long-term debt due to a Coasian logic:

[2]The literature on risk management highlights that due to the cost of bankruptcy, even a risk-neutral borrower can benefit from risk management, (Smith and Stulz, 1985; Bolton et al., 2011; Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To our knowledge, no previous work has established the link between maturity management and risk management in a corporate finance setting.

creditors, anticipating future issuance, price debt to reflect dilution, offsetting borrowing benefits and leaving equity holders indifferent.

Our model generates several empirically testable predictions. First, we predict equity holders' willingness to inject capital varies with debt structure and shock types: after small, frequent negative shocks, they generally provide capital unless the long-term debt-to-cash-flow ratio is very high, while after large negative shocks, the willingness for recapitalization increases with the proportion of long-term debt. Second, we predict countercyclical market leverage and procyclical debt maturity, consistent with findings by Halling et al. (2016) and Chen et al. (2021). Third, firms with greater exposure to large negative systematic risks should maintain higher proportions of long-term debt, supported by evidence that firms with longer debt maturities have countercyclical risk exposure (Chaderina et al., 2022) and higher-beta firms tend to have longer debt maturity (Chen et al., 2021). Fourth, firms with higher proportions of long-term debt are more resilient to large negative shocks, consistent with Almeida et al. (2011)'s findings from the 2007 crisis. Finally, our model predicts firms in countries with weak creditor protections will rely more heavily on short-term debt financing, consistent with Aghaee et al. (2024).

Related literature

Our paper builds on the literature of dynamic corporate finance in the tradition of Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019). Important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the related problem when the borrower can only issue zero-maturity debt (see also Bolton et al. (2021), who further model costly equity issuance). In these papers, the borrower can only issue one type of debt, so the tradeoff between borrowing long and short is not explicitly studied. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed, and the borrower can choose between two types of exponentially maturing debt. Another related paper is Chen et al. (2021), which examines debt maturity management across business cycles. Their model, which builds on Goldstein et al. (2001), assumes that all outstanding debt must be recalled before any adjustments to its level or maturity can be made. Additionally, they assume that all debt matures simultaneously.

The insight that long-term debt can be diluted has been recognized by Fama and Miller (1972), Black and Scholes (1973), and formalized by Admati et al. (2018). Brunnermeier and Oehmke (2013) show equity and short-term debt can dilute long-term debt's recovery value in bankruptcy. Our paper focuses on dilution outside the bankruptcy, which comes from the borrower's lack of commitment to issuance. As DeMarzo (2019) shows, the borrower's problem when she only issues long-term debt is related to the Coase conjecture on the durable-goods monopoly (Coase, 1972). In our context, the borrower is the monopolist, and long-term debt is the durable goods. Short-term debt resembles the leasing solution (Bulow, 1982) to the durable-goods monopoly problem. Whereas we emphasize the use of short-term debt to deal with the dilution problem, Malenko and Tsoy (2020) emphasizes an alternative mechanism based on the role of reputation. Their model, featuring a single debt type, predicts an interior optimal debt maturity.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. The insight that short-term debt resolves the lack of commitment is also present in another related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runnable feature of short-term debt. Our paper shows how the repricing feature of short-term debt (independent of run externalities) resolves the commitment issue in the context of dynamic capital structure.[3]

The benefit of long-term debt is related to the literature on fiscal policy and sovereign debt that considers the hedging benefit of long-term debt. For example, Angeletos (2002) shows that the ex-post variations in the market value of public debt hedge the government against bad fiscal conditions. Aguiar et al. (2019) show that without hedging motives, commitment issues would

[3] Also see Hu and Varas (2021) on this feature of short-term debt in the context of financial intermediaries.

lead to a borrower only issuing short-term debt. Through a simple example, they demonstrate that hedging could incentivize long-term debt issuance.[4] Arellano and Ramanarayanan (2012) calibrate a quantitative model of sovereign borrowing with two debt maturities. Our paper differs by providing a tractable continuous-time model with closed-form solutions that reveals maturity choice dynamics.

2 The Model

2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower, often interpreted as a firm. The relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk-neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate ρ, which exceeds r, the discount rate of creditors.

The borrower's asset generates earnings at a rate X_t, which evolves according to:

$$\frac{dX_t}{X_{t-}} = \mu_{\theta_t} dt + \sigma dB_t - \mathbb{1}_{\{\theta_t = L\}} dN_t, \tag{1}$$

where B_t is a standard Brownian motion, N_t is a Poisson process with arrival rate η, and $\theta_t \in \{H, L\}$ represents the regime with $\theta_0 = H$. At a random time τ_λ, which arrives with intensity λ, the regime switches to L and stays unchanged.[5] The drift μ_{θ_t} differs across the two regimes with $\mu_L < \mu_H$, so that the high state H is associated with a higher expected growth rate in the borrower's cash flow. Below, we refer to the high state as the *upturn*, the low state as the *downturn*, and the regime switch as the *downside* risk.

[4] Niepelt (2014) studies a related question, but default decisions on different maturities can be independent. Bigio et al. (2021) study debt maturity management under liquidity costs but without dilution. In their model, the borrower's choice depends on the bond demand curve, which is micro-founded via search (Duffie et al., 2005).

[5] In Internet Appendix B.4, we study the model in which state L is non-absorbing and show that our main results carry over.

In addition, in the downturn, the firm is exposed to a *disaster* shock upon which the cash flow X_t permanently drops to zero. This disaster arrives at a random time τ_η that is exponentially distributed with mean arrival rate η.[6] The disaster shock plays a role when we consider the incentives of an initially unlevered borrower to issue long-term debt (see subsection 4.3 for a detailed discussion).

2.2 Debt Maturity Structure

The difference between the discount rates $\rho - r$ offers benefits for the borrower to issue debt.[7] Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the outstanding debt maturity structure. In particular, we do not restrict the borrower to commit to a particular issuance path but instead let the issuance decisions be made at each instant.

All *short-term debt* matures instantaneously. This implies that the borrower does not have the chance to issue new debt before the existing short-term debt matures. We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt \downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time t and let y_{t-} be the associated short rate. *Long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing bonds with coupon rate r and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. Let F_t be the aggregate face value of long-term debt outstanding at time t.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume the bankruptcy cost is 100%. In other words, creditors cannot recover any value once the borrower defaults.

[6] If we consider a borrower that starts with an exogenous amount of outstanding long-term debt, then the results are unchanged if we assume that $\eta = 0$.

[7] The difference can be related to liquidity differences, contracting costs, or market segmentation. An alternative setup is to introduce tax shields, and the results are similar.

2.3 Valuation

Let τ_b be the endogenous time the borrower chooses to default. We define p_t as the price per unit of the face value. The break-even condition implies that for $t < \tau_b$,

$$p_t = \mathbb{E}_t \left[\int_t^{\tau_\xi \wedge \tau_b} e^{-r(s-t)} r ds + e^{-r(\tau_\xi - t)} \mathbb{1}_{\{\tau_b > \tau_\xi\}} \right], \tag{2}$$

where τ_ξ is long-term debt's (stochastic) maturing date. The two components in the expression correspond to the coupon and final payments. The short rate y_{t-} depends on the borrower's equilibrium default decisions, and the break-even condition suggests:

$$y_{t-} = r + \lim_{dt \downarrow 0} \frac{\Pr_{t-dt}\left(\tau_b \leq t | \tau_b > t - dt\right)}{dt}, \tag{3}$$

where the second term on the right-hand side is the credit spread compensating creditors for the hazard rate of default. According to (3), y_{t-} compensates the creditors for the probability of default occurring between $t - dt$ and t. For example, if in the upturn, short-term creditors expect default only upon a transition to the downturn, then $y_{t-} = r + \lambda$. Similarly, if in the downturn, short-term debt only defaults when the disaster shock hits, then $y_{t-} = r + \eta$.

Over a short time interval $[t, t + dt)$, the net cash flow to the borrower is

$$\left[X_t - (r + \xi) F_t - y_{t-} D_{t-} \right] dt + p_t dG_t + dD_t, \tag{4}$$

where $(r + \xi) F_t$ is the interest and principal payments to long-term creditors, $y_{t-} D_{t-}$ the interest payments to short-term creditors. The remaining two terms, $p_t dG_t$ and dD_t are the proceeds from issuing long- and short-term debt.[8]

Define V_t as the continuation value of the borrower, which we sometimes refer to as the equity value at time t. The borrower chooses the endogenous time of default as well as the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate

[8] One can think of dD_t as the net issuance of short-term debt. Specifically, $dD_t = D_t - D_{t-}$ if there is a jump at t.

function as given. Once again, let us emphasize that all these decisions, default and issuance, are made without commitment.

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \geq t\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-\rho(s-t)} \left\{ \left[X_s - (r+\xi)F_s - y_{s-}D_{s-} \right] ds + p_s dG_s + dD_s \right\} \right]. \tag{5}$$

To guarantee the valuations are finite, we follow the literature and make the following assumption.

Assumption 1. $r + \lambda > \mu_H$, $r + \eta > \mu_L$.

2.4 Smooth Equilibrium

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state θ_t, the cash-flow level X_t, and the amount of outstanding debt $\{D_{t-}, F_t\}$. The equilibrium requires the following. First, creditors break even; that is, p_t follows equation (2) and y_{t-} follows equation (3). Second, the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t \geq 0$. Finally, an MPE is *smooth* if no jump occurs in long-term debt issuance, in which case we write $dG_t = g_t F_t dt$. In a smooth equilibrium, the aggregate face value of long-term debt evolves according to

$$dF_t = (g_t - \xi) F_t dt. \tag{6}$$

Let us define J_t as the joint (maximized) continuation value of the borrower and short-term creditors if default does not occur at time t. The following result motivates us to work with J_t for the remainder of this paper.

Proposition 1. *The equity value is* $V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_{\theta_t}(X_t, F_t) - D_{t-}, 0\}$*, where the joint* *continuation value J_t is given by the value function $J_{\theta_t}(X_t, F_t)$ of the following problem:*

$$
\begin{aligned}
J_H(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ & X_s - (r+\xi)F_s + p_s g_s F_s \\
& + (\rho + \lambda - y_{s-})D_{s-} + \lambda \max\{J_L(X_s, F_s) - D_{s-}, 0\} \Big\} ds \Bigg] \qquad (7) \\
J_L(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \Big\{ & X_s - (r+\xi)F_s + p_s g_s F_s + (\rho + \eta - y_{s-})D_{s-} \Big\} ds \Bigg],
\end{aligned}
$$

where the maximization is subject to the limited liability constraint $D_{s-} \le J_{\theta_{s-}}(X_s, F_s)$.

The terms in (7) are related to those in (5). Here, $(\rho + \lambda - y_s)D_{s-}$ reflects the gains from issuing short-term debt. The last term in (7) stands for the event of regime-shifting, upon which the borrower would rather default and renege on the payments if the amount of outstanding short-term debt exceeds the maximum joint value without an immediate default; that is if $D_{s-} > J_L(X_s, F_s)$. The terms in $J_L(X, F)$ can be interpreted similarly.

Proposition 1 generates an interesting economic insight: even though the borrower makes decisions on debt issuance, these decisions are made to maximize the joint value of the borrower and short-term debt. This is because any issuance decisions will be immediately reflected in the credit risk faced by short-term creditors, affecting the price of short-term debt and the proceeds from issuing it. Note that the payoff to existing long-term creditors is ignored in the maximization problem because their debts have been issued in the past, and the borrower ignores how new debt issuance affects their valuation.[9] This result relates to Aguiar et al. (2019) in the context of sovereign debt, where the equilibrium issuance decisions can be characterized by the solution to a planner's problem that ignores the payoff to existing long-term creditors. Meanwhile, the max operator in (7) shows that the borrower and short-term creditors still have conflicts on whether to default immediately.

Proposition 1 implies that the state variable D_{t-} only enters the problem by affecting whether

[9]The payoff to new long-term creditors is at dt order in the smooth equilibrium.

the borrower defaults immediately at time t. Following this result, we can suppress the problem's dependence on D_{t-} and treat it as a decision variable. A smooth MPE is characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F,D)$, $D_\theta(X,F)$, and $g_\theta(X,F)$. By exploiting the homogeneity of the problem, we can further reduce the problem's dimension and write $J_\theta(X,F) = Xj_\theta(f)$, $D_\theta(X,F) = Xd_\theta(f)$, where $f = F/X$. The functions g_θ, p_θ, and y_θ are homogeneous of degree zero, so we can write them as $g_\theta(f)$, $p_\theta(f)$, and $y_\theta(f,d)$. For simplicity, we refer to $J_\theta(X,F)$ and $j_\theta(f)$ as the unscaled and scaled *value function* for the rest of this paper.

2.5 Model Discussion

Risk Structure. The model incorporates three sources of risk: a Brownian motion capturing small, frequent fluctuations in daily operating cash flows, a binary state system with upturn (H) and downturn (L) states representing large, infrequent shocks, and a disaster shock that captures very large and negative shocks that lead to immediate business closure. The regime-shift risk from upturn to downturn represents a downside risk, interpretable as industry or macroeconomic shocks. Whereas this risk creates a role for long-term debt, there are other specifications that yield similar results. For example, we obtain similar qualitative results if the cash flow process follows a jump-diffusion process with negative jumps, a case that we consider in one of the extensions in subsection 5.3. Under some conditions on parameters, the equilibrium in the model with disaster risk in the low state is similar to the equilibrium in a model without disaster risk but with more than two regimes.[10]

Debt Structure. The model's debt structure is motivated by discrete-time microfoundations. There, short-term debt would last for one period and need to be repaid before the borrower issues again. In the continuous-time setup, this feature is captured by zero-maturity debt. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing it each period would lead to a staggered structure. This feature is well captured by exponentially maturing debt in the continuous-time setup.

[10] See Internet Appendix B.5 for details.

We would like to acknowledge that it is an unrealistic assumption – in reality, no debt has zero maturity, and no debt of positive maturity can be frictionlessly diluted. These assumptions are meant to capture a general idea that shorter-term debt is more difficult to dilute than longer-term debt because shorter-term debt comes due earlier.[11] In our model, this difference takes a stark form (no dilution at all for short-term debt, easy dilution for debt of any finite maturity). This is a feature of a continuous-time setup, and it comes with the advantage of analytic solutions.

Zero recovery in default. The assumption that creditors do not recover any value once the borrower defaults is made for simplicity and does not affect our mechanism. It implies that debt seniority becomes irrelevant, ruling out the theoretical channel highlighted in Brunnermeier and Oehmke (2013) whereby the equity holder dilutes existing creditors' recovery value in bankruptcy through issuing new debt. In subsection 5.2, we assume instead that the borrower can restructure her debt after defaulting, in which case creditors still recover some positive value.

3 Equilibrium with One-Time Long-Term Debt Issuance

Before proceeding to solve the model, we study a simpler case in which the borrower can always issue short-term debt but only has one chance to issue long-term debt at $t = 0$.[12] This exercise serves two purposes. First, it illustrates some of the trade-offs involved in the choice of long-term debt abstracting from the problem of debt dilution. Second, the value functions in this exercise turn out identical to the ones in which the borrower has full flexibility in issuing both types of debt.

[11] For instance, in a model in which the search for creditors takes some time, we suspect that dilution will be finite and increasing in maturity.

[12] To highlight the benefits and costs of short-term debt, we study the model in which the borrower is only allowed to issue short-term debt in Internet Appendix A.1.

3.1 Short-Term Debt and Value Function After $t = 0$

We solve this problem backward. Let F_0 be the level of long-term debt issued at $t = 0$. Without any new issuance, the outstanding long-term debt evolves according to

$$dF_t = -\xi F_t dt.$$

We will work with the scaled value function after $t = 0$, denoted as $j_\theta^0(f)$, where the superscript 0 highlights that we are imposing no further long-term debt issuance, i.e., $g_\theta(f) = 0$, after $t = 0$. By considering the change in the value function in Proposition 1 over a small interval and substituting $J_\theta^0(X, F) = X j_\theta^0(f)$, we get the following Hamilton-Jacobi-Bellman (HJB) equation in the continuation region:

$$(\rho + \eta - \mu_L) j_L^0(f) = \max_{d_L^0 \in [0, j_L^0(f)]} 1 - (r + \xi) f + \left(\rho + \eta - y_L^0\right) d_L^0 - (\mu_L + \xi) f j_L^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_L^{0\prime\prime}(f) \quad (8)$$

Short-term debt only defaults upon the disaster shock, so $y_L^0 = r + \eta$. Given $\rho > r$, results on the short-term debt are straightforward: the borrower can exhaust her borrowing capacity by fully levering up so $d_L^0 = j_L^0(f)$. In this case, the borrower effectively "sells" the entire firm to creditors whose discount rate is r.

With outstanding long-term debt, the borrower endogenously defaults when the fundamental X_t deteriorates sufficiently compared to the outstanding long-term debt F_t, or equivalently the ratio of long-term debt to earnings f_t hits an endogenous boundary f_L^b, where f_L^b satisfies the value matching condition $j_L^0(f_L^b) = 0$ and the smooth pasting condition $j_L^{0\prime}(f_L^b) = 0$.

Following a similar analysis, we arrive at the HJB for the scaled value function $j_H^0(f)$ in the continuation region:

$$(\rho+\lambda-\mu_H)\, j_H^0(f) = \max_{d_H^0\in[0,j_H^0(f)]} 1-(r+\xi)\, f + \left(\rho+\lambda-y_H^0\right) d_H^0 + \lambda \max\left\{j_L^0(f)-d_H^0, 0\right\}$$
$$-\,(\mu_H+\xi) f j_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f). \quad (9)$$

The additional term in (9) relative to (8) represents the state transition from high to low. As usual, the value function satisfies the value matching and smooth pasting conditions $j_H^0(f_H^b)=0$ and $j_H^{0\prime}(f_H^b)=0$ at the default boundary $f=f_H^b$.

In the upturn, the choice of short-term debt entails a more interesting tradeoff. Default does not occur if $j_L^0(f)\geq d_H^0$; hence, given the potential of a regime switch, the safe level of short-term debt cannot exceed $j_L^0(f)$. Meanwhile, the borrower can issue risky short-term debt up to $j_H^0(f)$. Therefore, short-term creditors demand a short rate

$$y_H^0\left(f, d_H^0\right) = \begin{cases} r & \text{if } d_H^0\in[0, j_L^0(f)] \\ r+\lambda & \text{if } d_H^0\in(j_L^0(f), j_H^0(f)]. \end{cases} \quad (10)$$

Substituting (10) into (9), we find that short-term debt issuance in the upturn follows a bang-bang solution between the risky level $j_H^0(f)$ and the safe level $j_L^0(f)$. The former has the benefits of higher leverage but carries the risk of default and a loss in firm value upon the regime switch from H to L. Specifically, a choice of $j_H^0(f)$ is optimal when

$$(\rho-r)j_H^0(f)\geq(\rho+\lambda-r)j_L^0(f) \implies \underbrace{(\rho-r)\left(j_H^0(f)-j_L^0(f)\right)}_{\text{benefit from higher leverage}} \geq \underbrace{\lambda j_L^0(f)}_{\text{additional bankruptcy cost.}} \quad (11)$$

Our analysis focuses on parameter values where both risk-free and risky short-term debt can be optimal, depending on the level of outstanding long-term debt f. This economically more interesting case arises under the following condition:

Assumption 2. *The intensity of the regime switch from upturn to downturn satisfies:*

$$\lambda > \bar{\lambda} \equiv \sqrt{\left(\frac{\rho - \mu_H}{2}\right)^2 + (\rho - r)(\mu_H + \eta - \mu_L)} - \left(\frac{\rho - \mu_H}{2}\right). \tag{12}$$

Under this condition, there exists a threshold $f_\dagger > 0$. When $f < f_\dagger$, (11) fails, and the borrower optimally issues risk-free short-term debt $j_L^0(f)$ to avoid the additional bankruptcy cost $\lambda j_L^0(f)$. When $f \geq f_\dagger$, (11) holds, and the benefit of higher leverage outweighs this cost, making risky short-term debt $j_H^0(f)$ optimal. This threshold policy follows from a single-crossing property proven in Lemma 2 in the appendix.[13]

Proposition 2 (Short-term debt Issuance)**.** *When the borrower has outstanding long-term debt and can only issue short-term debt, the optimal short-term debt issuance is as follows:*

- *In state $\theta = L$, the borrower issues short-term debt $d_L^0(f) = j_L^0(f)$ and pays a short rate $y_L^0\left(f, d_L^0(f)\right) = r + \eta$.*
- *In state $\theta = H$, the borrower issues short-term debt*

$$d_H^0(f) = \begin{cases} j_L^0(f) & \text{if } f < f_\dagger \\ j_H^0(f) & \text{if } f \geq f_\dagger \end{cases}$$

  and pays a short rate given by (10)*.*
- *Under Assumption 2, there exists a unique $f_\dagger \in (0, f_L^b)$ such that* (11) *holds (fails) for $f \geq f_\dagger$ ($f < f_\dagger$).*

Expressions for the value function $j_\theta^0(f)$ can be found in Proposition 10 in Internet Appendix A.3.

[13] When $\lambda \leq \bar{\lambda}$, (11) holds for any $f > 0$, allowing us to set $f_\dagger = 0$ without loss of generality.

3.2 Long-term Debt Issuance at $t = 0$

Now, we turn to the problem of initial issuance at $t = 0$. Let $p_\theta^0(f)$ be the price of long-term debt. The borrower chooses the initial amount of short-term debt D_0 and long-term debt F_0 to maximize the firm value, which includes equity and the proceeds from debt issuance. Given $\theta_0 = H$ and the homogeneity of the borrower's problem, we can write this problem as

$$\max_{f_0} \; j_H^0(f_0) + p_H^0(f_0) f_0. \tag{13}$$

We have the following result.

Proposition 3 (Equilibrium with short-term debt and one-time long-term debt issuance)**.** *Under Assumption 2, if the borrower can only issue long-term debt once at* $t = 0$*, the solution to equation (13) is strictly positive, i.e.,* $f_0^* > 0$*.*

Given the flexibility of short-term debt issuance, why issue any long-term debt at $t = 0$? The key benefit is the embedded option to postpone default. In the context of American options, the holder has stronger incentives to wait if the expiration date is far because the option's exercise value may increase over time. In our context, if the maturity of long-term debt gets longer, the borrower is less likely to exercise the default option on long-term debt upon the regime shift from high to low. This happens because the default option of long-term debt remains active for an extended period, unlike the option on short-term debt, which expires immediately. Consequently, with long-term debt, the borrower has reduced incentives to default, as there is a greater possibility that conditions may improve before the debt matures.

The agency problem associated with exercising the default option can be framed in terms of the severity of the debt overhang problem. Although our model does not consider investment, debt overhang can be broadly defined as any firm value-enhancing activities undertaken by equity holders that are reduced because of leverage. Hence, equity holders' injections of funds can be interpreted as a value-enhancing activity because they reduce expected bankruptcy costs. The reduction in equity holders' incentives to inject funds into the firm corresponds to the debt overhang for recapitalization considered by Admati et al. (2018).

Our analysis reveals that in bad times, when default is more likely, the longer expiration of the default option has limited value, so there is a modest difference in debt overhang between short- and long-term debt. However, in good times, when default is unlikely, the longer expiration of the default option becomes valuable. In this scenario, long-term debt has the potential to generate less debt overhang, incentivizing the borrower to issue it.

This effect is related to previous results by Gertner and Scharfstein (1991) and Diamond and He (2014). Diamond and He (2014) identify a potential benefit of long-term debt in reducing debt-overhang when they assert that "the long-term debt holders – due to less frequent repricing – share more losses with equity holders when assets-in-place deteriorate" (p. 740). Similarly, in their discussion about the disadvantage of short-term debt, they note that "not sharing losses in bad times pushes equity holders to default" (p. 741). This last point resembles an observation made by Gertner and Scharfstein (1991), who show that short-term debt leads to more ex-post debt overhang during bankruptcy reorganizations because it has higher market leverage than long-term debt for the same face value.

4 Equilibrium with Continuous Long-Term Debt Issuance

In this section, we proceed to solve the model in section 2, where the borrower has the flexibility to issue both long- and short-term debt at all times.

4.1 Long-Term Debt Issuance

Our first result shows that the lack of commitment to future issuance policies fully erodes the value of long-term debt. In equilibrium, the joint payoff to the borrower and short-term creditors is identical to that in a scenario where no long-term debt is issued after $t = 0$. This result is analogous to Coase (1972) conjecture and has a similar counterpart in DeMarzo and He (2021) (see

Proposition 2 there).

Proposition 4. *Suppose the borrower can flexibly issue long-term debt, and there is no commitment to future debt issuance. In any smooth equilibrium, the joint valuation of equity and short-term debt $j_\theta(f)$ equals the one without new issuance of long-term debt. That is, $j_\theta(f) = j_\theta^0(f)$.*

Proposition 4 implies that the evolution of $j_\theta(f)$ is identical to that of $j_\theta^0(f)$ described by (8) and (9), respectively. In equilibrium, long-term debt's price must satisfy

$$p_\theta(X,F) = -\frac{\partial J_\theta(X,F)}{\partial F} \Rightarrow p_\theta(f) = -j_\theta'(f), \tag{14}$$

where $p_\theta\,(X,F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, and $\frac{\partial J_\theta(X,F)}{\partial F}$ is the associated drop in the continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, the marginal proceeds must be fully offset by the drop in continuation value. Given so, in equilibrium, the borrower is indifferent and gains no marginal benefit from adjusting long-term debt, so her equilibrium payoff is the same as if she were never to issue any debt going forward.

Even though the borrower is indifferent between issuing long-term debt and not, this result does not imply she never borrows long on the equilibrium path. In fact, the equilibrium long-term debt issuance policy needs to be consistent with the price of long-term debt.[14] Let us now derive the equilibrium issuance policies.

It follows from Itô's lemma that, before the disaster $dN_t = 1$ in the low state, f_t evolves according to[15]

$$df_t = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right) f_t dt - \sigma f_t dB_t. \tag{15}$$

[14] The reason is analogous to the logic behind mixed strategies: a player needs to be indifferent to her choice of action, yet equilibrium strategies are uniquely determined to maintain that indifference. In our context, the equilibrium long-term debt issuance needs to be consistent with the price of long-term debt.

[15] We omit the disaster shock dN_t when $\theta_t = L$. Upon the disaster shock $dN_t = 1$, X_t gets absorbed at 0, so f_t jumps to ∞. The borrower then defaults immediately, and the price of the long-term debt jumps to zero.

In the downturn $\theta_t = L$, the price satisfies the following HJB equation:

$$(r+\xi+\eta)\,p_L(f) = \underbrace{r+\xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f)-\xi-\mu_L+\sigma^2\right)fp_L'(f)+\frac{1}{2}\sigma^2 f^2 p_L''(f)}_{\text{expected change in bond price}}. \tag{16}$$

To derive the issuance function g_L, we plug $d_L = j_L(f)$ into (8) (replace $j_L^0(f)$ with $j_L(f)$), differentiate the resulting equation once, add (16) on both sides, and apply equation (14). Turning to the upturn $\theta_t = H$, the price $p_H(f)$ satisfies the following HJB equation:

$$(r+\xi+\lambda)\,p_H(f) = r+\xi+\mathbb{1}_{\{f<f_\dagger\}}\lambda p_L(f)+\left(g_H(f)-\xi-\mu_H+\sigma^2\right)fp_H'(f)+\frac{1}{2}\sigma^2 f^2 p_H''(f). \tag{17}$$

Compared with (16), (17) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \le f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The derivation of the issuance policy $g_H(f)$ follows the same steps as the one in the low state.

Proposition 5 (Long-term debt issuance)**.** *The equilibrium price of long-term debt is* $p_\theta(f) = -j_\theta'(f)$ *for* $\theta \in \{H, L\}$ *and* $f \in [0, f_\theta^b)$*. The issuance policies are as follows.*[16]

1. *Downturn* $\theta = L$*:* $\forall f \in [0, f_L^b)$*, the issuance policy is*

$$g_L(f) = 0.$$

2. *Upturn* $\theta = H$*: there exists a unique* $f_\dagger \in (0, f_L^b)$ *such that*

$$g_H(f) = \begin{cases} \frac{(\rho-r)(p_H(f)-p_L(f))}{-fp_H'(f)} & \text{if } f \in [0, f_\dagger) \\ 0 & \text{if } f \in [f_\dagger, f_H^b). \end{cases} \tag{18}$$

According to Proposition 5, the borrower does not issue any long-term debt in the low state

[16] In the proof of Proposition 4, we show the value function is convex under the smooth issuance policy. This implies that it is indeed optimal for the firm to issue long-term debt smoothly. In the proof, we also verify that the equity holders cannot benefit from issuing an atom Δ.

and in the high state when $f \geq f_\dagger$. By contrast, in the high state when $f < f_\dagger$, the borrower issues long-term debt. The numerator in equation (18) captures the benefit of long-term debt coming from the incremental borrowing capacity provided by long-term debt, whereas the denominator captures the cost of issuance due to price impact. We can provide a heuristic derivation of the equilibrium issuance policy based on a local perturbation approach that illustrates this cost-benefit analysis.

Let's consider the upturn state, $\theta = H$, and an existing long-term debt level $f = f_0 \in [0, f_\dagger)$. Suppose the borrower deviates for "one period" by issuing an extra amount of long-term debt Δ at time t and buying it back at $t + dt$. The proceeds from additional long-term debt issuance is $p_H(f_0 + \Delta) \cdot \Delta$. Meanwhile, this adjustment reduces the proceeds from issuing riskless short-term debt by $j_L(f_0) - j_L(f_0 + \Delta)$. Note that the difference in patience implies that each dollar proceeds from total debt issuance results in a flow benefit of $\rho - r$. Combining the proceeds from long- and short-term debt, the total marginal benefit from this adjustment is

$$(\rho - r)\left[\underbrace{p_H(f_0 + \Delta) \cdot \Delta}_{\approx p_H(f_0)\Delta} - \underbrace{(j_L(f_0) - j_L(f_0 + \Delta))}_{\approx -j_L'(f_0)\Delta = p_L(f_0)\Delta}\right] \approx (\rho - r)\left(p_H(f_0) - p_L(f_0)\right) \cdot \Delta. \qquad (19)$$

The crux of the matter is that the proceeds of long-term debt issuance depend on its price in the *high* state, whereas the impact on short-term borrowing is determined by long-term debt's price in the *low* state. The cost of this adjustment depends on the price impact of such trade, which is

$$\text{Cost of one period adjustment} \approx -p_H(f_0 + \Delta)\Delta + p_H(f_0)\Delta = -p_H'(f_0)\,(\Delta)^2\,.$$

In equilibrium, the marginal benefit is equal to the marginal cost, so

$$(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)\Delta \approx -p_H'(f_0)\,(\Delta)^2 \Longrightarrow \Delta = \frac{(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)}{-p_H'(f_0)}.$$

The issuance function in the other cases of Proposition 5 can be derived using similar heuristic arguments. When the borrower fully exhausts borrowing capacity, the increment in long-term debt's market value is fully offset by the reduction in short-term debt, so no long-term debt is issued. For example, in the high state, when $f > f_{\dagger}$, the reduction in short-term debt becomes $j_H(f_0) - j_H(f_0 + \Delta)$, which is approximately $p_H(f_0) \cdot \Delta$. In this case, the total marginal benefit from this adjustment (the counterpart of (19)) becomes zero.

We end this subsection by discussing the result when the regime-switching intensity is low, i.e., when $\lambda \leq \bar{\lambda}$. In this case, the borrower optimally chooses to issue no long-term debt at all, both in the one-time issuance setting (as in section 3) and with flexible issuance opportunities (as in this section), i.e., $g_\theta(f) = 0, \forall f \in [0, f_\theta^b), \theta \in \{H, L\}$. This result holds because when the risk of regime shift is relatively low, the borrower finds it optimal to take the risk and exhaust her borrowing capacity by fully levering up with short-term debt.

4.2 Long-Term Debt Buyback

An interesting feature in equation (18) is that the optimal issuance of long-term debt could be negative, implying that the borrower actually buys back long-term debt. This result differs from the literature on the leverage-ratchet effect, which predicts a borrower without commitment to debt issuance would never actively buy back the outstanding debt (DeMarzo and He, 2021; Admati et al., 2018). The reason is as follows. Once $f > f_{\dagger}$, the borrower will take too much short-term debt, generating a rollover risk whereby default will occur following a regime shift. For f sufficiently close to (but still below) $f_{\dagger}$, the borrower could find it optimal to buy back some long-term debt and reduce the chances that f rises above $f_{\dagger}$. Note that equation (18) implies buyback occurs whenever $p_H(f) < p_L(f)$, suggesting that long-term debt is riskier in the upturn compared to the downturn. Intuitively, for f sufficiently close to (but still below) $f_{\dagger}$, a few negative Brownian shocks could push f above $f_{\dagger}$, which leads to an immediate default following a regime switch. By contrast, under the same f in the low state, a default may only occur after a long sequence of negative Brownian shocks that push f to f_L^b. Therefore, the default risk can be higher in the high state than in the low state when f is close to $f_{\dagger}$.

Moreover, we provide a sufficient condition under which the borrower never buys back long-term debt. Our results in Proposition 5 still goes through under this condition.

Corollary 1. *The borrower never repurchases its long-term debt if $\eta \geq \lambda$.*

4.3 Initial Debt Issuance

So far, our analysis has shown that for a given level $f > 0$, the borrower could issue/repurchase some long-term debt. However, it remains a question whether an initially unlevered borrower would issue any long-term debt. The next proposition provides the necessary conditions under which this would be the case.

Proposition 6. *Suppose that $\lambda > \bar{\lambda}$ and $\eta > 0$. If*

$$\frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda - \mu_H\right)}}{\sigma^2} \geq 2$$
$$\frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \eta - \mu_L\right)}}{\sigma^2} \geq 2,$$

then there exists an equilibrium in which an initially unlevered borrower will issue long-term debt, that is $\lim_{f \to 0} g_H(f)f > 0$.

Without the disaster shock, an unlevered borrower would never issue long-term debt. Intuitively, if there is no disaster, starting from $f = 0$, a marginal unit of long-term debt is riskless for the unlevered borrower in both the upturn and the downturn; that is, $p_H(0) = p_L(0) = 1$. According to (18), there is no long-term debt issuance. The other parametric conditions lead to $p'_H(0) > -\infty$, which ensures that the price impact of issuing an additional unit of long-term debt is not too large to deter the unlevered borrower from issuing it.

Let us conclude this subsection by pointing out that there exists another Markov Perfect Equilibrium in which the unlevered borrower only issues short-term debt, and the reasons are similar to the zero-leverage equilibrium in (DeMarzo and He, 2021, p. 35).

5 Extensions, Robustness, and Empirical Implications

Subsection 5.1 explores an extension that allows the borrower to issue financial instruments to hedge against the regime shift. In subsection 5.2, we introduce the possibilities of short-term debt renegotiation. Results show that long-term debt is not issued under either perfect hedging or frictionless renegotiation. In subsection 5.3, we show that results are similar if there are no regime shifts but cash flows have downside jumps. In subsection 5.4, we extend the benchmark to allow the states are transitory and quantify the model. Finally, subsection 5.5 discusses the model's empirical relevance.

5.1 Hedging the regime-shift shock

One advantage of having a tractable framework is to allow us to explore alternative risk-sharing mechanisms. We introduce derivative contracts that provide borrowers with hedging against regime shifts and show that these derivatives can serve as (partial) substitutes for long-term debt.

Suppose a short-term derivative contract is written on a variable $\hat{\theta}_t$ that is correlated with θ_t. If there is a regime switch, then $\hat{\theta}_t$ switches with probability $q \in [0, 1]$ and remains constant otherwise. The buyers of this derivative pay a spread $\varsigma \cdot dt$ over the period $[t, t + dt)$, in exchange for a payment of \$1 at time $t + dt$ if there is a change in $\hat{\theta}_t$. No arbitrage implies $\varsigma = \lambda q$. We have the following result.

Proposition 7. *The firm never issues long-term debt if markets are complete. That is, if $q = 1$, then $g_\theta(f) = 0$ for all f. If $f_\dagger > 0$, then $f_\dagger$ is decreasing in q.*

This result shows that long-term debt becomes unnecessary when the borrower can fully insure against regime switches using derivatives. Moreover, the substitution between derivative contracts and long-term debt is monotonic: better hedging effectiveness leads to less long-term debt issuance.[17]

[17] While we use the term "hedging," our mechanism is more akin to insurance as it specifically protects against downside risk rather than eliminating all value fluctuations.

5.2 Restructuring of Short-Term Debt

We have established in subsection 3.2 that long-term debt has option value because with the regime shift, the adjustment in short-term debt is not always possible, and the borrower may default. Now, we show that if, instead, short-term debt can be frictionlessly restructured to postpone default, it crowds out reasons to use long-term debt.

To do this, we need to distinguish between default and bankruptcy. When the borrower announces a default with outstanding short-term debt, restructuring may occur with probability α. Since the amount of short-term debt is zero when the borrower defaults at f_θ^b in either state, restructuring is only relevant upon a regime shift from H to L. In the restructuring process, the equity holder makes the offer with probability β and short-term creditors with probability $1 - \beta$. Details of the restructuring game and its extensive form are provided in Online Appendix B.2 and illustrated in Figure 1.

Proposition 8. *The firm never issues long-term debt if the short-term debt can be renegotiated without friction. That is, if $\alpha = 1$, then $g_\theta(f) = 0$ for all f. If $f_\dagger > 0$, then $f_\dagger$ is decreasing in α.*

The intuition behind this result is similar to the one with derivative contracts. The benefit of long-term debt is to delay bankruptcy upon regime shift. However, this no longer needed if the short-term debt can be restructured.

5.3 Jump Risk

This subsection shows that our mechanism remains robust when large negative shocks are modeled as downward jumps in cash flows. Specifically, we assume the cash flow follows a jump-diffusion process

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \omega^{-1}\right) X_{t-} dN_t, \tag{20}$$

where jumps are governed by a Poisson process with intensity λ, and the size of the jump ω is deterministic. We construct an equilibrium characterized by thresholds $f_\dagger$ and f^b, and derive the scaled value function $j(f)$, which satisfies a delay differential equation with value matching and smooth pasting conditions $j(f^b) = j'(f^b) = 0$. The optimal short-term debt policy is given by $d(f) = \frac{j(\omega f)}{\omega}$ for $f \in [0, f_\dagger)$ and $d(f) = j(f)$ for $f \in [f_\dagger, f^b]$, where $f_\dagger$ is determined by the condition $(\rho+\lambda-r)\frac{j(\omega f_\dagger)}{\omega} = (\rho-r)j(f_\dagger)$. Long-term debt issuance occurs only when outstanding debt is low relative to cash flow, and its issuance rule resembles the baseline model, adjusted for the impact of jumps. Further details are provided in Internet Appendix B.3.

5.4 Transitory Shocks and Model Quantification

In the benchmark model, we have assumed that the low state $\theta_t = L$ is absorbing. If we interpret the changes in the regime as business cycles, it is natural to assume that states are transitory. We can extend the model to consider this situation. Such a model is solved in Internet Appendix B.4.

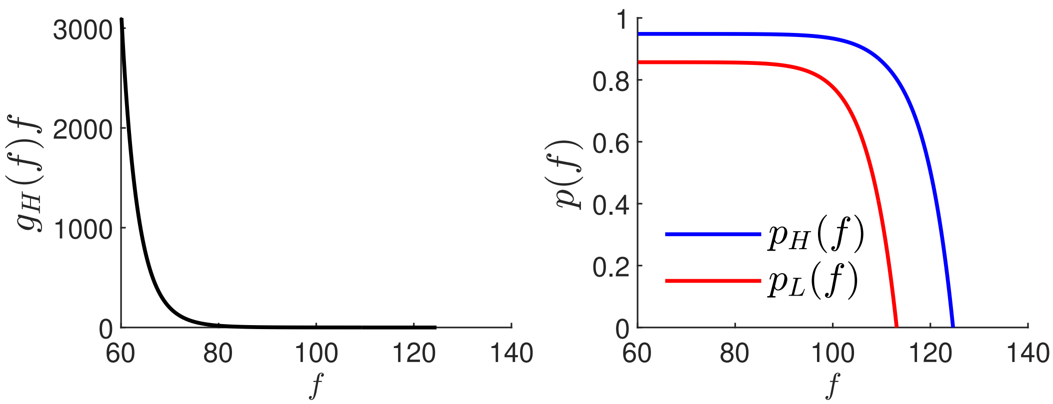

Figure 1: Long-term debt issuance policy and price

This figure plots the long-term debt issuance policy when $\theta = H$ and the corresponding long-term debt price as a function of f, where the parameter values are provided in Table 1. Here $f_\dagger = 109.34$, $f_H^b = 113.10$, and $f_L^b = 124.64$.

We parametrize the model following Chen et al. (2021). Table 1 in the Internet Appendix lists the value of parameters. In Figure 1, we plot the long-term debt issuance policy when $\theta = H$ and the corresponding long-term debt price. In the upturn, when f is high, $p_H(f)$ is low according to the right panel, which implies credit risk is high when the borrower has a significant amount of long-term debt outstanding (i.e., f is very close to f_H^b). Thus, a positive relationship exists between credit risk and the maturity of outstanding debt. However, as illustrated by the left panel, close to the default boundary, newly issued debt is exclusively short-term, so there is a negative relation between credit risk and the maturity of the newly issued debt. This difference necessitates the need for a dynamic model, which differentiates between the stock and the flow of debt.

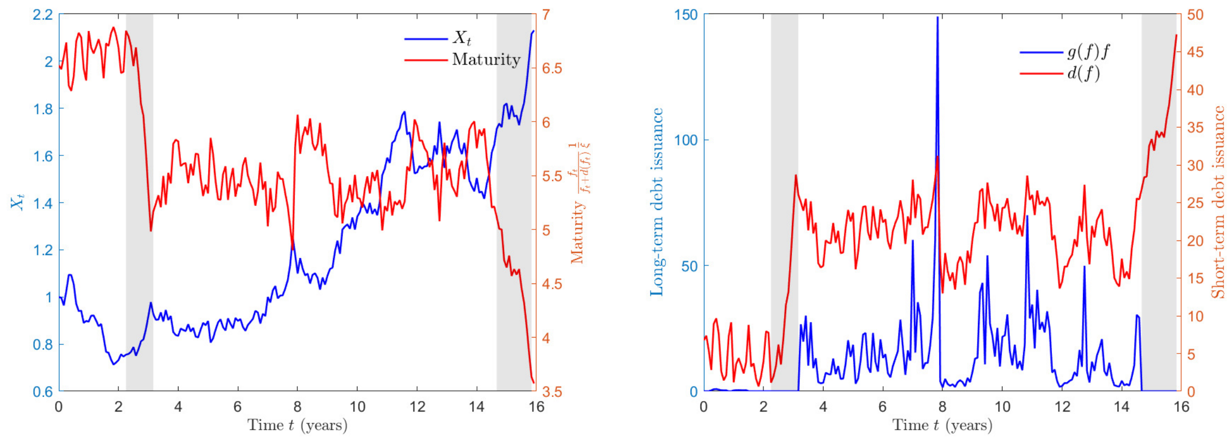

Figure 2: Sample path of earnings, maturity, and debt issuance

This figure simulates the sample path of one firm and plots the time series of earnings X_t, maturity, and debt issuance of both long-term debt and short-term debt, where the parameter values are provided in Table 1.

In Figure 2, we simulate a sample path and plot the time series of the cash-flow rate, debt maturity, and debt issuance for this parameter values. Here, debt maturity is defined as the average maturity of total debt outstanding weighted by their book value: $\text{Maturity}_t := \frac{F_t}{F_t+D_t}\frac{1}{\xi}$. In the absence of a regime shift, the maturity of debt seems to move in the opposite direction to cash flows. In other words, the borrower expands the average debt maturity following a negative Brownian shock to X_t. Intuitively, this pattern holds because, after a negative Brownian shock to X_t, the borrower immediately rolls over less short-term debt, whereas she only reduces long-term debt outstanding gradually over time. Meanwhile, when the regime shifts and the downturn arrives, the borrower exclusively borrows short-term debt and the average maturity goes down.[18] Therefore, our model implies that within a regime, cash flows and debt maturity negatively co-move with each other. The correlation between the two time series in the left panel in Figure 2 is -0.6. Across regimes, the higher cash-flow growth regime has a longer average debt maturity (5.7 years) than the lower-growth regime (5.2 years).

5.5 Empirical Relevance

Our model differs from the sovereign debt literature by focusing on risk-neutral borrowers with access to equity financing, rather than risk-averse borrowers who borrow to smooth consumption. This distinction leads to novel predictions about equity holders' willingness to inject capital under various debt structures and shock types. After small, frequent negative shocks (modeled as Brownian motion), equity holders are generally willing to inject capital unless the long-term debt-to-cash-flow ratio is very high. In contrast, the borrower is more willing to recapitalize after large negative shocks (modeled as regime shifts) when debt maturity gets longer.

Our model implies countercyclical market leverage, consistent with Halling et al. (2016), and procyclical debt maturity, aligning with Chen et al. (2021)'s aggregate-level findings. At the firm level, Mian and Santos (2018) show that firms, particularly those with investment-grade ratings, actively manage maturity to hedge refinancing risk during good times. Our model predicts that firms more exposed to large and negative systematic risks optimally maintain higher proportions of long-term debt. This prediction is supported by several empirical studies: Chaderina et al. (2022) find firms with longer debt maturities have countercyclical systematic risk exposure; Chen et al. (2021) document that higher-beta firms tend to have longer debt maturity; and González (2015) shows corporate bond maturity is procyclical across 39 countries from 1995 to 2012.

A key testable implication of our model is that firms with higher proportions of long-term

[18] This result depends on the binary-state setup, where no additional downside risk exists in the low state. With more than two states, the borrower may still issue long-term debt in the low state. The broader message is the transition to a worse state, the borrower may only issue short-term debt for a while.

debt should demonstrate greater resilience to large negative shocks. This prediction aligns with evidence from Almeida et al. (2011), who study firm outcomes during the 2007 credit crisis. Using a differences-in-differences matching estimator to control for firm heterogeneity, they find that firms with large fractions of short-term debt maturing immediately after the crisis experienced significantly worse outcomes. Specifically, these firms reduced quarterly investment (normalized by capital) by 2.5% more than similar firms whose debt matured well after the crisis - approximately one-third of their pre-crisis investment levels. Our model also generates novel predictions about debt issuance dynamics following large negative shocks. Specifically, we predict that firms will shift toward shorter-term debt issuance after experiencing significant negative shocks. This pattern reflects the optimal response to increased uncertainty about future prospects, as shorter-term debt allows for more flexible adjustment to changing conditions while potentially reducing immediate financing costs. Gorton et al. (2021); Brunnermeier (2009); Krishnamurthy (2010) documented that financial firms shortened their debt maturity structure during the 2007–2008 crisis.

Our paper emphasizes that short-term debt serves as a mechanism to mitigate creditor dilution, particularly when creditor protections are weak. Consequently, we predict that firms operating in countries with debtor-friendly legal systems or weak contract enforcement will rely more heavily on short-term debt financing. Aghaee et al. (2024) provide consistent evidence.[19]

As argued by Guedes and Opler (1996), empirical research on debt maturity should differentiate between new debt issuance and existing balance sheet liabilities. While static models treat these similarly, one merit of constructing a dynamic model is to offer separate and novel predictions. According to our model, credit risk could be high when the borrower has a significant amount of long-term debt outstanding (that is, when f_t is close to $f^b_{\theta_t}$). However, close to the default boundary, newly issued debt is exclusively short-term. These patterns are illustrated by Figure 1 and discussed there.

Our model highlights two distinct paths to firm default, each tied to a fundamental area of

[19] Also see Gopalan et al. (2016), which documents an increase in long-term debt after a reduction in debt enforcement cost in the context of India.

corporate finance. The first type, primarily driven by long-term debt, occurs gradually as firm fundamentals deteriorate relative to long-term debt burden and, therefore, is relatively predictable. The second type, primarily driven by short-term financing, occurs suddenly when firms face large negative shocks, and, therefore, is less predictable. This difference maps to the capital structure decisions and working capital management in business finance.[20]

6 Final Remarks

Our paper offers a theory of debt maturity. Short-term debt mitigates the lack of commitment problem and incentivizes the borrower to reduce leverage after small negative shocks. Long-term debt offers the borrower the option to delay default after large negative shocks.

Covenants are a widely used commitment device in practice: they typically restrict new borrowing by linking it to financial ratios such as the interest coverage ratio. They help mitigate one key problem with long-term debt—the borrower's incentive to issue new debt and dilute existing claims. However, they do little to address the problem that the borrower is reluctant to repurchase or retire long-term debt after negative shocks. This is especially true for incurrence covenants, which are common in bond contracts and mainly restrict actions at the time new debt is issued. In the context of our model, we could consider a covenant that limits additional long-term debt issuance when the long-term debt to cash flow ratio exceeds a specified threshold. This is similar to the debt ceiling policy analyzed in DeMarzo et al. (2021). Moreover, covenants often rely on imperfect measures of firm fundamentals and are difficult to renegotiate, especially in bond markets. These limitations are further amplified by the rise of "cov-lite" debt, which weakens creditor protections. As a result, the main trade-off between short- and long-term debt emphasized in our model remains relevant even when covenants are present.

[20] For example, Ross et al. (2017) organize business finance around three fundamental questions: capital budgeting, capital structure, and working capital management. The textbook refers to capital structure decisions as obtaining long-term financing for investments, while working capital management addresses the firm's day-to-day financial activities and short-term financing needs. This classical framework aligns naturally with our model's distinction between defaults driven by long-term versus short-term debt.

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Appendix

A Proofs

Proof of Proposition 1

Proof. We prove the result for J_H, and the one for J_L follows similar steps. Let $\theta_t = H$ and $\tau_\lambda \geq t$ be the time that the state switches from H to L. By the principle of dynamic programming,

$$\begin{aligned}
V_{t-} = & \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b \wedge \tau_\lambda} e^{-\rho(s-t)} \bigg((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \bigg) \\
& \qquad + e^{-\rho(\tau_b \wedge \tau_\lambda - t)} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
= & \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \bigg((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \bigg) \\
& \qquad + e^{-\rho(\tau_\lambda - t)} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
= & \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \bigg((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \bigg) \\
& \qquad + e^{-\rho(\tau_\lambda - t)} \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
= & \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \bigg((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \\
& \qquad + \lambda \max\{J_s - D_{s-}, 0\}\, ds \bigg) \Bigg],
\end{aligned}$$

where we have used the definition $V_{\tau_\lambda} = \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\}$. Using the integration by parts formula for semi-martingales (Corollary 2 in Section 2.6 of Protter (2005)), we get

$$\mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} dD_s \right] = \mathbb{E}_t \left[e^{-(\rho+\lambda)(\tau_b - t)} D_{\tau_b} \right] - D_{t-} + \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} (\rho+\lambda) D_{s-} ds \right].$$

At the time of default, $D_{\tau_b} = 0$. Hence

$$V_{t-} = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ X_s - (r+\xi)F_s + p_s g_s F_s + (\rho + \lambda - y_{s-}) D_{s-} + \lambda \max\left\{ J_L(X_s, F_s) - D_{s-}, 0 \right\} \Big\} ds \right] - D_{t-}.$$

Limited liability requires that $V_{t-} \geq 0$, so the amount of short-term borrowing must satisfy the constraint $D_{t-} \leq J_{\theta_t}(X_t, F_t)$. □

Proof of Proposition 2

First, we derive an equation for the scaled value function. Second, we prove the existence and uniqueness of a solution to the scaled value function and a single-crossing property in $\theta_t = H$.

Low state $\theta_t = L$. Let $g_\theta(f) = 0$, we can derive the following equation in the continuation region:

$$(\rho+\eta)J_L^0(X,F) = \max_{D_L^0 \in [0, J_L^0(X,F)]} X - (r+\xi)F + \left(\rho + \eta - y_L^0\right) D_L^0 - \xi F \frac{\partial J_L^0(X,F)}{\partial F} + \mu_L X \frac{\partial J_L^0(X,F)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 J_L^0(X,F)}{\partial X^2}. \quad \text{(A.1)}$$

Note that the choice of short-term debt D_L^0 is capped by the level of value function J_L^0, and the choice of D_L^0 also affects J_L^0. Substituting $J_\theta^0(X,F) = X j_\theta^0(f)$ in equation (A.1), we get the following HJB for the scaled value function $j_L^0(f)$:

$$(\rho + \eta - \mu_L) j_L^0(f) = \max_{d_L^0 \in [0, j_L^0(f)]} 1 - (r+\xi) f + \left(\rho + \eta - y_L^0\right) d_L - (\mu_L + \xi) f j_L^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_L^{0\prime\prime}(f). \quad \text{(A.2)}$$

Given that the coefficient in front of d_L^0 satisfies $\rho + \eta - y_L^0 = \rho - r > 0$, it is always optimal for the borrower to issue as much short-term debt as possible, which leads to $d_L^0(f) = j_L^0(f)$. The rest of the problem becomes standard. The borrower defaults if $f \geq f_L^b$, where f_L^b satisfies the value matching condition $j_L^0(f_L^b) = 0$ and the smooth pasting condition $j_L^{0'}(f_L^b) = 0$.

High state $\theta_t = H$. We arrive at the HJB in the continuation region:

$$(\rho+\lambda)J_H^0\,(X,F) = \max_{D_H^0 \in \left[0, J_H^0(X,F)\right]} X - (r+\xi)\,F + \left(\rho+\lambda - y_H^0\right) D_H^0 + \lambda \max\left\{J_L^0(X,F) - D_H^0, 0\right\}$$
$$- \xi F \frac{\partial J_H^0(X,F)}{\partial F} + \mu_H X \frac{\partial J_H^0(X,F)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 J_H^0(X,F)}{\partial X^2}. \quad \text{(A.3)}$$

The scaled value function $j_H^0(f)$ therefore satisfies

$$(\rho+\lambda-\mu_H)\, j_H^0(f) = \max_{d_H^0 \in [0, j_H^0(f)]} 1 - (r+\xi)\, f + \left(\rho+\lambda - y_H^0\right) d_H^0 + \lambda \max\left\{j_L^0(f) - d_H^0, 0\right\}$$
$$- (\mu_H + \xi) f j_H^{0'}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0''}(f). \quad \text{(A.4)}$$

As usual, the value function satisfies the value matching and smooth pasting conditions $j_H^0(f_H^b) = 0$ and $j_H^{0'}(f_H^b) = 0$ at the default boundary f_H^b. The formulas for the value function $j_\theta^0(f)$ can be found in Proposition 10 in Internet Appendix A.3.

The rest of the proof consists of characterizing the optimal short-term debt policy in the high state. In the first part, we show the existence and uniqueness of a solution. In the second part, we prove a single-crossing property and therefore show that it is optimal for the borrower to issue riskless short-term debt $d_H^0 = j_L^0\,(f)$ if $f \leq f_\dagger$.

Existence and Uniqueness: For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t}\left(1 - (r+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f\right]$$

subject to $\quad dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t,$

where

$$\nu(z,\tilde{j}) \equiv \max_{d_H^0 \in [0,\tilde{j}]} \left(\rho+\lambda-y_H^0(z,d_H^0)\right) d_H^0 + \lambda \max\left\{j_L^0(z)-d_H^0,0\right\} = \max\{(\rho+\lambda-r)\, j_L^0(z),(\rho-r)\,\tilde{j}\},$$

and $\hat{\rho} \equiv \rho+\lambda-\mu_H$. It follows from the HJB equation that the value function j_H^0 is a fixed point $j_H^0(f)=\Phi(j_H^0)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f,\tilde{j}_1) \geq \nu(f,\tilde{j}_0)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\nu(z,\tilde{j}+a) = \max\{(\rho+\lambda-r)\, j_L^0(z),(\rho-r)\,(\tilde{j}+a)\} \leq (\rho-r)a+\nu(z,\tilde{j}),$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)+a)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right] \\
&\quad + \frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right] \\
&\quad + \frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f) + \frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \leq \Phi(\tilde{j})(f) + \frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}$$

Thus, the operator Φ is monotone and satisfies discounting, it follows then by Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H^0(f)=\Phi(j_H^0)(f)$.

The result on short-term debt issuance is established in the following two lemmas, and the proofs are supplemented in the Internet Appendix.

Lemma 1. *The condition*

$$(\rho + \lambda - r)\, j_L^0\,(0) > (\rho - r)\, j_H^0\,(0)\,.$$

is satisfied if and only if $\lambda > \bar{\lambda}$.

Next, the following result shows that it is optimal for the borrower to issue $d_H^0 = j_L^0\,(f)$ when $f \leq f_\dagger$ and $d_H^0 = j_H^0\,(f)$ otherwise.

Lemma 2 (Single-crossing)**.** *There exists a unique* $f_\dagger \in \left(0, f_L^b\right)$ *such that* $(\rho + \lambda - r)\, j_L^0\,(f) \geq (\rho - r)\, j_H^0\,(f)$ *if and only if* $f \leq f_\dagger$.

Proof of Proposition 3

Proof. When the borrower can commit to not issuing any long-term debt after $t = 0$, i.e., $g_H\,(f) = g_L\,(f) \equiv 0$. The debt price in the low state satisfies the following asset pricing equation

$$(r + \xi + \eta)\, p_L^0(f) = (r + \xi) + \left(-\xi - \mu_L + \sigma^2\right) f p_L^{0'}\,(f) + \frac{1}{2}\sigma^2 f^2 p_L^{0''}\,(f)\,, \tag{A.5}$$

where the default boundary f_L^b is determined from Internet Appendix A.3. We guess the solution of the debt price takes the form

$$p_L^0(f) = A_1^0 + A_2^0 f^{\gamma_1 - 1} + A_3^0 f^{\gamma_2 - 1}.$$

Combining with $p_L^0(f_L^b) = 0$ and $\lim_{f \to 0} p_L^0(f) < \infty$, we know

$$p_L^0(f) = \frac{r + \xi}{r + \eta + \xi} - \frac{r + \xi}{r + \eta + \xi} \left(\frac{f}{f_L^b}\right)^{\gamma_1 - 1}. \tag{A.6}$$

Under $\lambda > \bar{\lambda}$, the price of long-term debt in the high state satisfies

$$(r+\xi+\lambda)\, p_H^0(f) = (r+\xi) + \lambda p_L^0(f) \mathbf{1}_{\{d_H^0(f) \le j_L^0(f)\}} + \left(-\xi - \mu_H + \sigma^2\right) f p_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 p_H^{0\prime\prime}(f), \tag{A.7}$$

where the default boundary f_H^b and the threshold $f_\dagger$ are determined from Internet Appendix A.3.

When $f \in [0, f_\dagger]$, we guess a solution of the form

$$p_H^0(f) = B_1^0 + B_2^0 f^{\gamma_1 - 1} + B_3^0 f^{\beta_1 - 1} + B_4^0 f^{\beta_2 - 1}. \tag{A.8}$$

Plugging into equation (A.7), we can get

$$B_1^0 = \frac{(r+\xi)(r+\eta+\xi+\lambda)}{(r+\xi+\lambda)(r+\eta+\xi)};$$
$$B_2^0 = \frac{-\lambda \frac{r+\xi}{r+\eta+\xi}\left(f_L^b\right)^{1-\gamma_1}}{r+\xi+\lambda - (-\xi-\mu_H+\sigma^2)(\gamma_1 - 1) - \frac{1}{2}\sigma^2(\gamma_1-1)(\gamma_1-2)};$$

The condition $\lim_{f\to 0} p_H^0(f) < \infty$ implies that $B_4^0 = 0$, which implies B_3^0 is the only unknown in equation (A.8).

When $f \in \left(f_\dagger, f_H^b\right]$, we guess a solution of the form

$$p_H^0(f) = C_1^0 + C_2^0 f^{\beta_1 - 1} + C_3^0 f^{\beta_2 - 1}, \tag{A.9}$$

Plugging into equation (A.7), we can get $C_1^0 = \frac{r+\xi}{r+\xi+\lambda}$. $\left\{C_2^0, C_3^0\right\}$ are the unknowns in equation (A.9). In the end, we solve the three unknowns $\left\{B_3^0, C_2^0, C_3^0\right\}$ from the following boundary conditions:

$$p_H^0(f_H^b) = 0,$$
$$p_H^0(f_\dagger^-) = p_H^0(f_\dagger^+),$$
$$p_H^{0\prime}(f_\dagger^-) = p_H^{0\prime}(f_\dagger^+).$$

Having determined the price of long-term debt, we can analyze the issuance decision at $t = 0$. Let $H^0(f_0) = j_H^0(f_0) + p_H^0(f_0) f_0$ denote the value of the firm given long-term debt over cash flows f_0. Under $\lambda > \bar{\lambda}$, $H^{0'}(f_0) = j_H^{0'}(f_0) + p_H^0(f_0) + p_H^{0'}(f_0) f_0$. From the closed-form solutions of $j_H^0(f)$ and $p_H^0(f)$, we can get

$$H^{0'}(f_0)|_{f_0=0} = \frac{r+\xi}{r+\eta+\xi} \frac{\eta(\rho-r)}{(r+\xi+\lambda)(\rho+\xi+\lambda)} > 0. \tag{A.10}$$

It implies there exists a $f_0^* > 0$ such that

$$H^0(f_0^*) = j_H^0(f_0^*) + p_H^0(f_0^*) f_0^* > H^0(0) = \frac{J_H^s(X_t)}{X_t}.$$

Therefore, the borrower optimally issues a positive amount of long-term debt f_0^* at $t = 0$. □

Proof of Proposition 4

First, we demonstrate that the joint value $j_\theta(f)$ under no commitment is equivalent to the value $j_\theta^0(f)$ without long-term debt issuance. Second, we prove $j_\theta(f)$ is strictly convex so that it is optimal for the borrower to issue long-term debt smoothly.

From the value function (7) and substituting $J_\theta(X, F) = X j_\theta(f)$, we get the following HJB equations for the scaled value function $j_\theta(f)$:

$$\begin{aligned}
(\rho+\eta-\mu_L) j_L(f) = & \max_{g_L, d_L \in [0, j_L(f)]} 1-(r+\xi) f + \left(p_L(f) + j_L'(f)\right) g_L f + (\rho+\eta-y_L) d_L \\
& - (\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 j_L''(f) \\
(\rho+\lambda-\mu_H) j_H(f) = & \max_{g_H, d_H \in [0, j_H(f)]} 1-(r+\xi) f + \left(p_H(f) + j_H'(f)\right) g_H f + (\rho+\lambda-y_H) d_H \\
& + \lambda \max\{j_L(f) - d_H, 0\} - (\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).
\end{aligned} \tag{A.11}$$

If the borrower finds it optimal to adjust debt smoothly, then it must be the case that this coefficient equals zero, or equivalently,

$$p_\theta(f) = -j_\theta^{'}(f). \tag{A.12}$$

Given any smooth equilibrium with value function $j_\theta(f)$, (A.11) and (A.12) imply (8) and (9), and as a result setting the issuance policy to $g_\theta = 0$ does not change the equity value $j_\theta(f)$. Hence, the value $j_\theta(f)$ under this equilibrium could be obtained under no long-term debt issuance, which corresponds to $j_\theta^0(f)$.

We also need to verify that the equity holders cannot benefit from issuing an atom Δ, in which case they get a payoff $j_\theta(f+\Delta) + p_\theta(f+\Delta)\Delta$. The first order condition with respect to Δ is $j_\theta'(f+\Delta) + p_\theta(f+\Delta) + p_\theta'(f+\Delta)\Delta = p_\theta'(f+\Delta)\Delta = 0$. Which means that either $\Delta = 0$ or $p_\theta'(f+\Delta) = 0$. The second order condition is $p_\theta''(f+\Delta)\Delta + p_\theta'(f+\Delta) \leq 0$, which evaluated at $\Delta = 0$ yields $p_\theta'(f) \leq 0$. It follows from the first and second-order conditions that a smooth equilibrium requires that $p_\theta'(f) = -j_\theta^{''}(f) < 0$.

From the closed-form solution for $j_L(f)$ we immediately obtain that $j_L^{''}(f) > 0$. Next, we verify that $j_H(f)$ is also a strictly convex function on $\left[0, f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly.

Strict convexity of $j_H(f)$ on $\left[0, f_H^b\right]$. The proof relies on a few auxiliary lemmas, and proofs for all three are supplemented in the Internet Appendix.

Lemma 3.

$$j_H^{'}(f) \geq -1, \qquad \forall f \in \left[0, f_H^b\right],$$

Lemma 4.

$$f_H^b > \frac{1}{r+\xi} \qquad \textit{and} \qquad \min\left\{j_H^{''}(0), j_H^{''}(f_H^b)\right\} > 0,$$

Lemma 5.

$$j_H^{'''}(f_\dagger^-) > j_H^{'''}(f_\dagger^+).$$

Now we are ready to verify that the solution to the HJB equation is convex. We differentiate the HJB twice and let $\tilde{u} \equiv -f j_H^{''}$ to get

$$\frac{1}{2}\sigma^2 f^2 \tilde{u}'' - \left(\mu_H + \xi - \sigma^2\right) f \tilde{u}' - \left(\rho + \lambda + \xi\right) \tilde{u} = \left(\rho + \lambda - r\right) f j_L^{''} \qquad f \in (0, f_\dagger) \tag{A.13}$$

$$\frac{1}{2}\sigma^2 f^2 \tilde{u}'' - \left(\mu_H + \xi - \sigma^2\right) f \tilde{u}' - \left(r + \lambda + \xi\right) \tilde{u} = 0 \qquad f \in (f_\dagger, f_H^b). \tag{A.14}$$

Applying the maximum principle in Theorem 1 to function $\tilde{u}$ in the two equations above (note that $j_L^{''}(f) > 0$), we can conclude that $\tilde{u}$ cannot have an interior nonnegative local maximum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. More specifically, when we map equation (A.13) to equation (A.24),

$$u\left(f\right) \equiv \tilde{u}\left(f\right), \, g\left(f\right) \equiv \frac{-\left(\mu_H + \xi - \sigma^2\right) f}{\frac{1}{2}\sigma^2 f^2}, \, h\left(f\right) \equiv \frac{-\left(\rho + \lambda + \xi\right)}{\frac{1}{2}\sigma^2 f^2},$$

and when we map equation (A.14) to equation (A.24),

$$u\left(f\right) \equiv \tilde{u}\left(f\right), \, g\left(f\right) \equiv \frac{-\left(\mu_H + \xi - \sigma^2\right) f}{\frac{1}{2}\sigma^2 f^2}, \, h\left(f\right) \equiv \frac{-\left(r + \lambda + \xi\right)}{\frac{1}{2}\sigma^2 f^2},$$

Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonnegative maximum is that $\tilde{u}(f_\dagger) > 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are negative, this requires that -$j_H^{''}(f_\dagger-) - f_\dagger j_H^{'''}(f_\dagger-) = \tilde{u}'(f_\dagger-) > \tilde{u}'(f_\dagger+) = -j_H^{''}(f_\dagger+) - f_\dagger j_H^{'''}(f_\dagger+)$. From the HJB equation it follows that j_H is twice continuously differentiable at $f_\dagger$, so such a kink would require $j_H^{'''}(f_\dagger^-) < j_H^{'''}(f_\dagger^+)$, which is ruled out by Lemma 5. We can conclude that $\tilde{u}$ does not have an interior nonnegative maximum, so it follows that $\tilde{u}(f) = -f j_H^{''}(f) < 0$ on $(0, f_H^b)$ from Corollary 2 as $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are negative. This implies $j_H^{''}(f) > 0$ on $[0, f_H^b]$.

Proof of Proposition 5

Proof. First, we consider the low state. The debt price satisfies the asset pricing equation

$$\left(r + \xi + \eta\right) p_L(f) = \left(r + \xi\right) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'\left(f\right) + \frac{1}{2}\sigma^2 f^2 p_L''\left(f\right). \tag{A.15}$$

The indifference condition (14) implies that the price of debt is $p_L(f) = -j_L'(f)$. Substituting in (A.15) we get

$$(r+\xi+\eta)\, j_L'(f) = -(r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f)\,. \tag{A.16}$$

We differentiate the HJB at $\theta = L$, which leads to

$$(r+\eta+\xi)\, j_L^{'}(f) + (r+\xi) + \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f) = 0. \tag{A.17}$$

Combining (A.17) and (A.16) we get

$$g_L(f) = 0.$$

In the high state, the debt price follows

$$(r+\xi+\lambda)\, p_H(f) = (r+\xi) + \lambda p_L(f)\mathbf{1}_{\{d_H(f)\le j_L(f)\}} + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)\,.$$

Under $\lambda > \bar{\lambda}$, we first consider the case when $f \in (0, f_\dagger)$. The debt price is

$$(r+\xi+\lambda)\, p_H(f) = (r+\xi) + \lambda p_L(f) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)\,. \tag{A.18}$$

Differentiating the HJB equation for $j_H(f)$ we obtain,

$$\begin{aligned}(r+\lambda+\xi)\, j_H'(f) + (r+\xi) - \lambda j_L'(f) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) \\ = (\rho - r)\, j_L'(f) - (\rho - r)\, j_H'(f).\end{aligned} \tag{A.19}$$

Combining equations (A.18) and (A.19) with the indifference condition (14) – which requires that $p_H(f) = -j_H'(f)$ – we find that

$$g_H(f) = \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-f p_H'(f)}.$$

Finally, for $f \in (f_\dagger, f_H^b)$, the debt price is

$$(r+\xi+\lambda)\, p_H(f) = (r+\xi) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)\,. \tag{A.20}$$

We differentiate the HJB equation (A.4) to obtain,

$$(r+\lambda+\xi)\, j_H'(f) + (r+\xi) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0, \tag{A.21}$$

which combined with equation (A.20) and the optimality condition $p_H(f) = -j_H'(f)$ yields

$$g_H(f) = 0.$$

□

Proof of Corollary 1

Proof. We differentiate the HJB at $\theta = L$, which leads to

$$(r+\eta+\xi)\, j_L^{'}(f) + (r+\xi) + \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f) = 0, \quad \forall f \in \left[0, f_L^b\right].$$

When $f \in [0, f_\dagger]$, we differentiate the HJB at $\theta = H$:

$$(\rho+\lambda+\xi)\, j_H'(f) + (r+\xi) - (\rho+\lambda-r)\, j_L'(f) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0.$$

The difference is

$$\begin{aligned}\frac{1}{2}\sigma^2 f^2 \left[j_H'''(f) - j_L'''(f)\right] - \left(\mu_H + \xi - \sigma^2\right) f \left(j_H''(f) - j_L''(f)\right) - (\rho+\lambda+\xi)\left(j_H'(f) - j_L'(f)\right)&\\ = (\mu_H - \mu_L)\, f j_L''(f) - \eta j_L'(f) > 0&\end{aligned} \tag{A.22}$$

since $j_L'(f) \leq 0$ and $j_L''(f) > 0$ from the strict convexity. Applying the maximum principle in Theorem 1 to equation (A.22), $\triangle(f) \equiv j_H'(f) - j_L'(f)$ can not have a nonnegative maximum in the region $f \in (0, f_\dagger)$. More specifically, when we map equation (A.22) to equation (A.24),

$$u(f) \equiv \triangle(f), \; g(f) \equiv \frac{-\left(\mu_H + \xi - \sigma^2\right) f}{\frac{1}{2}\sigma^2 f^2}, \; h(f) \equiv \frac{-(\rho + \lambda + \xi)}{\frac{1}{2}\sigma^2 f^2}.$$

When $f \in \left(f_\dagger, f_L^b\right]$, we differentiate the HJB at $\theta = H$:

$$(r + \lambda + \xi) j_H'(f) + (r + \xi) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0.$$

The difference is

$$\begin{aligned} &\frac{1}{2}\sigma^2 f^2 \left(j_H'''(f) - j_L'''(f)\right) - \left(\mu_H + \xi - \sigma^2\right) f \left(j_H''(f) - j_L''(f)\right) \\ &- (r + \lambda + \xi)\left(j_H'(f) - j_L'(f)\right) = (\mu_H - \mu_L) f j_L''(f) - (\eta - \lambda) j_L'(f) > 0 \end{aligned} \tag{A.23}$$

where we assume $\eta \geq \lambda$, and have shown $j_L'(f) \leq 0$ as well as $j_L''(f) > 0$ from the strict convexity. By the maximum principle, $\triangle(f) \equiv j_H'(f) - j_L'(f)$ can not have a nonnegative maximum in the region $f \in \left(f_\dagger, f_L^b\right)$. More specifically, when we map equation (A.23) to equation (A.24),

$$u(f) \equiv \triangle(f), \; g(f) \equiv \frac{-\left(\mu_H + \xi - \sigma^2\right) f}{\frac{1}{2}\sigma^2 f^2}, \; h(f) \equiv \frac{-(r + \lambda + \xi)}{\frac{1}{2}\sigma^2 f^2}.$$

Since both $j_H'(f)$ and $j_L'(f)$ are continuous for all $f \in \left[0, f_L^b\right]$, $\triangle(f)$ is continuous for all $f \in \left[0, f_L^b\right]$. It implies that $\triangle(f)$ can not have a nonnegative maximum in the region $f \in \left(0, f_L^b\right)$.

In addition, given that

$$\begin{aligned} &\triangle(0) = \frac{r + \xi}{r + \eta + \xi} - \frac{(r + \xi)(\eta + \rho + \lambda + \xi)}{(\rho + \lambda + \xi)(r + \eta + \xi)} = -\frac{r + \xi}{r + \eta + \xi}\frac{\eta}{\rho + \lambda + \xi} < 0, \\ &\triangle\left(f_L^b\right) = j_H'(f_L^b) < 0, \end{aligned}$$

we know $\triangle(f) < 0$ for any $f \in \left[0, f_L^b\right]$ from Corollary 2.

From $p_L(f) = -j_L'(f)$, $p_H(f) = -j_H'(f)$, we know $p_H(f) > p_L(f)$ for any $f \in \left[0, f_L^b\right]$. Furthermore, when $f \in \left[0, f_L^b\right]$, the firm never repurchases the long-term debt since

$$g(f) = \frac{(\rho - r)(p_H(f) - p_L(f))}{-fp_H'(f)} = \frac{(\rho - r)(p_H(f) - p_L(f))}{fj_H''(f)} > 0.$$

$\square$

Proof of Proposition 6

Proof. Under $\lambda > \bar{\lambda}$ and $\theta = H$, in the region $f < f_\dagger$, the issuance function is given by

$$g_H(f)f = \frac{(\rho - r)(j_L'(f) - j_H'(f))}{-p_H'(f)}.$$

As $\gamma > 1$ and $\phi > 1$, we have that

$$\lim_{f \to 0} j_L'(f) + 1 = \frac{\eta}{r + \eta + \xi},$$
$$\lim_{f \to 0} \left(j_L'(f) - j_H'(f)\right) = \frac{\eta}{\rho + \lambda + \xi} \frac{r + \xi}{r + \eta + \xi},$$

which are strictly positive if $\eta > 0$. Therefore, $\lim_{f \to 0}(g_H(f)f)$ is strictly positive as long as $p_H'(0) > -\infty$, which requires that $\gamma \geq 2$ and $\phi \geq 2$.

$\square$

Internet Appendix for "Debt Maturity Management"

This Internet Appendix contains additional analysis to accompany the manuscript. Section A provides the remaining proofs for the analysis, including all technical lemmas. Section B provides the details for Section 5.

A Remaining proofs

Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

Theorem 1 (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{A.24}$$

in an interval $(0, b)$ with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Theorem 2 (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that u is a nonconstant solution of the differential inequality* (A.24) *having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

Corollary 2. *If u satisfies* (A.24) *in an interval (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u\,(a) \leq 0$, $u\,(b) \leq 0$, then $u\,(x) < 0$ in (a, b) unless $u \equiv 0$.*

A.1 Equilibrium with Only Short-Term Debt

In this section, we explore the model where the borrower can only issue short-term debt, which relates to Abel (2018). Heuristically, short-term debt is repaid after each "dt" so that the borrower always renews with zero leverage. The dilution problem therefore no longer exists. The choice of short-term debt follows the standard trade-off theory whereby the equity holder balances cheap debt against costly bankruptcy. Given there is no long-term debt, we work with the unscaled value function, denoted as $J^s_\theta(X_t)$. We also use $D^s_\theta(X_t)$ and $y^s_\theta(X_t)$ for the value and rates of short-term debt.

Note that short-term debt is immune to Brownian shocks and may only default after jump shocks, including regime shifts and disaster shocks. In the low state, default occurs following the disaster, which does not depend on the leverage level. Therefore, the borrower can exhaust her borrowing capacity by fully levering up. In this case, the borrower effectively "sells" the entire firm to creditors whose discount rate is r. The firm value becomes

$$J^s_L(X_t) = \mathbb{E}_t\left[\int_t^\infty e^{-(r+\eta)(s-t)}X_s ds\right] \Rightarrow J^s_L(X_t) = \frac{X_t}{r+\eta-\mu_L}. \tag{A.25}$$

In the high state, the issuance decision involves a more interesting tradeoff. Default does not occur if $J^s_t \geq D^s_{t-}$; hence, given the potential of a regime switch, the safe level of short-term debt cannot exceed $J^s_L(X_t)$. Meanwhile, the borrower can also exhaust her borrowing capacity by issuing short-term debt $J^s_H(X_t)$, but this becomes risky. The borrower prefers risky short debt $J^s_H(X_t)$ if the benefits from higher leverage exceed the additional bankruptcy cost due to the regime switch, i.e.,

$$\underbrace{(\rho - r)\left(J^s_H(X_t) - J^s_L(X_t)\right)}_{\text{benefit from higher leverage}} > \underbrace{\lambda J^s_L(X_t).}_{\text{additional bankruptcy cost}} \tag{A.26}$$

Similar to (A.25), the associated high-state firm value when the borrower chooses $J^s_H(X_t)$ is $\frac{X_t}{r+\lambda-\mu_H}$.

If the borrower chooses riskless short-term debt $J_L^s(X_t)$, the firm value becomes

$$\mathbb{E}_t\left[\int_t^\infty e^{-(\rho+\lambda)(s-t)}\Big\{X_s+\underbrace{(\rho-r)J_L^s(X_s)}_{\text{leverage benefits}}+\underbrace{\lambda J_L^s(X_s)}_{\text{firm value in } L}\Big\}ds\right]=\frac{X_t}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right).$$

Proposition 1 implies without long-term debt, the borrower chooses short-term debt to maximize the firm value. We have the following results.

Proposition 9 (Equilibrium with only short-term debt)**.** *If only short-term debt is allowed, the unique equilibrium is the following.*

1. *In the low state L, the value function and short-term debt issuance are*

$$D_L^s(X_t)=J_L^s(X_t)=\frac{X_t}{r+\eta-\mu_L}.$$

The borrower only defaults upon the disaster shock, so the short rate is $y_L^s(X_t)=r+\eta$.

2. *In the high state H, the value function is*

$$J_H^s(X_t)=\max\left\{\frac{X_t}{r+\lambda-\mu_H},\frac{X_t}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right)\right\}.$$

- *If $\lambda\le\bar{\lambda}$ ($\bar{\lambda}$ is defined in equation (12)), short-term debt is*

$$D_H^s(X_t)=\frac{X_t}{r+\lambda-\mu_H}$$

The borrower defaults upon the regime switch, and the short rate is $y_H^s(X_t)=r+\lambda$.

- *If $\lambda>\bar{\lambda}$,*

$$D_H^s(X_t)=\frac{X_t}{r+\eta-\mu_L}.$$

The borrower does not default upon the regime switch, and the short rate is $y_H^s(X_t)=r$.

Proof of Proposition 9

Proof. When $\theta_t = H$, we arrive at the HJB:

$$(\rho+\lambda)J_H^s(X) = \max_{D_H^s \in \left[0, J_H^s(X)\right]} X + (\rho+\lambda-y_H^s) D_H^s + \lambda \max\left\{J_L^s(X) - D_H^s, 0\right\} + \mu_H X \frac{\partial J_H^s(X)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 J_H^s(X)}{\partial X^2}. \quad \text{(A.27)}$$

Given the potential of a regime switch, the borrower can issue risk-free short-term debt up to $J_L^s(X)$ and risky short-term debt up to $J_H^s(X)$. Therefore, short-term creditors demand a short rate

$$y_H^s(X, D_H^s) = \begin{cases} r & \text{if } D_H^s \in [0, J_L^s(X)] \\ r+\lambda & \text{if } D_H^s \in (J_L^s(X), J_H^s(X)]. \end{cases} \quad \text{(A.28)}$$

Substituting (A.28) into (A.27), we find that short-term debt issuance in the upturn follows a bang-bang solution between the risky level $J_H^s(X)$ and the safe level $J_L^s(X)$. The former has the benefits of higher leverage but carries the risk of default and a loss in firm value upon the regime switch from H to L. Specifically, a choice of $J_H^s(X)$ if and only if

$$(\rho - r) J_H^s(X) \geq (\rho+\lambda-r) J_L^s(X). \quad \text{(A.29)}$$

Now we solve the value functions. In the low state, the borrower chooses short-term debt $D_L = J_L^s$ and only defaults upon the disaster shock. So the short rate is $y_L^s(X) = r+\eta$, which implies the value of the firm is

$$J_L^s(X) = \frac{X}{r+\eta-\mu_L}.$$

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage, in which case

$$J_H^s(X) = \frac{X}{r + \lambda - \mu_H}.$$

On the other hand, if she borrows riskless debt up to $J_L^s(X)$, the firm value is

$$J_H^s(X) = \frac{X}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - r}{r + \eta - \mu_L}\right).$$

From here, we get that the value of the firm is

$$J_H^s(X) = X \max\left\{\frac{1}{r + \lambda - \mu_H}, \frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - r}{r + \eta - \mu_L}\right)\right\}.$$

As in Lemma 1, equation (A.29) is satisfied if and only if $\lambda > \bar{\lambda}$, where $\bar{\lambda}$ is defined in equation (12).

□

A.2 Proofs of Auxiliary Lemmas

Proof of Lemma 1

Proof. The proof of Proposition 9 makes it clear that the condition $\lambda > \bar{\lambda}$ guarantees that

$$(\rho + \lambda - r)\, j_L^0(0) > (\rho - r)\, j_H^0(0).$$

This inequality is satisfied only if

$$\frac{\rho + \lambda - r}{\rho - r} > \frac{\rho + \lambda + \eta - \mu_L}{\rho + \lambda - \mu_H}.$$

Combining terms, we can write this as the following quadratic inequality

$$\lambda^2 + (\rho - \mu_H)\,\lambda - (\rho - r)\,(\mu_H - \mu_L + \eta) > 0.$$

The left hand side is positive if and only if λ is greater than the unique positive root of the quadratic equation for $\bar{\lambda}$

$$\bar{\lambda}^2 + (\rho - \mu_H)\,\bar{\lambda} - (\rho - r)\,(\mu_H - \mu_L + \eta) = 0.$$

□

Proof of Lemma 2

Proof. Define $a \equiv 1 + \frac{\lambda}{\rho - r}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function u let,

$$\begin{aligned}
L^{0\dagger}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\,fu' - (\rho + \lambda - \mu_H)\,u \\
L^{\dagger b}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\,fu' - (r + \lambda - \mu_H)\,u.
\end{aligned}$$

The HJB in state $\theta = H$ can be written as

$$L^{0\dagger} j_H^0 + 1 - (r + \xi)\,f + (\rho + \lambda - r)\,j_L^0 = 0,\ \ f \in (0, f_\dagger) \tag{A.30}$$

$$L^{\dagger b} j_H^0 + 1 - (r + \xi)\,f = 0,\ \ f \in (f_\dagger, f_H^b). \tag{A.31}$$

Similarly, the HJB in state $\theta = L$ can be written as

$$L^{0\dagger} a j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} + a\,(\rho + \lambda - (r + \eta) + \mu_L - \mu_H)\,j_L^0 + a\,(1 - (r + \xi)\,f) = 0 \tag{A.32}$$

$$L^{\dagger b} a j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} - a\,(\mu_H - \mu_L + \eta - \lambda)\,j_L^0 + a\,(1 - (r + \xi)\,f) = 0. \tag{A.33}$$

Let's define

$$H(f) \equiv a(\mu_H - \mu_L) f j_L^{0'} - a\left(\mu_H - \mu_L + \eta - \lambda\right) j_L^0 + (a-1)\left(1 - (r+\xi) f\right),$$

Then, we can show that

$$L^{0\dagger}\left(a j_L^0 - j_H^0\right) + H(f) = 0,\ \ f \in (0, f_\dagger) \tag{A.34}$$

$$L^{\dagger b}\left(a j_L^0 - j_H^0\right) + H(f) = 0,\ \ f \in (f_\dagger, f_H^b). \tag{A.35}$$

To see why this is the case, notice that when $f \in (0, f_\dagger)$, combining equation (A.30) and equation (A.32), we can get

$$\begin{aligned}
0 &= L^{0\dagger} a j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} + a\left(\rho + \lambda - (r+\eta) + \mu_L - \mu_H\right) j_L^0 + a\left(1 - (r+\xi) f\right) \\
&= L^{0\dagger}\left(a j_L^0 - j_H^0\right) - 1 + (r+\xi) f - (\rho + \lambda - r) j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} \\
&+ a\left(\rho + \lambda - (r+\eta) + \mu_L - \mu_H\right) j_L^0 + a\left(1 - (r+\xi) f\right) \\
&= L^{0\dagger}\left(a j_L^0 - j_H^0\right) + a(\mu_H - \mu_L) f j_L^{0'} + (a-1)\left(1 - (r+\xi) f\right) \\
&+ a\left(\rho + \lambda - (r+\eta) + \mu_L - \mu_H\right) j_L^0 - (\rho + \lambda - r) j_L^0 \\
&= L^{0\dagger}\left(a j_L^0 - j_H^0\right) + a(\mu_H - \mu_L) f j_L^{0'} + (a-1)\left(1 - (r+\xi) f\right) \\
&- a\left(\mu_H - \mu_L + \eta - \lambda\right) j_L^0
\end{aligned}$$

where the last line comes from

$$\begin{aligned} a\left(\rho+\lambda-(r+\eta)+\mu_L-\mu_H\right)-(\rho+\lambda-r) &= \frac{\rho-r+\lambda}{\rho-r}\left(\rho-r+(\lambda-\eta)+\mu_L-\mu_H\right)-(\rho+\lambda-r) \\ &= \frac{\rho-r+\lambda}{\rho-r}\left((\lambda-\eta)+\mu_L-\mu_H\right) \\ &= a\left((\lambda-\eta)+\mu_L-\mu_H\right) \\ &= -a\left(\mu_H-\mu_L+\eta-\lambda\right). \end{aligned}$$

On the other hand, when $f \in (f_\dagger, f_H^b)$, combining equation (A.31) and equation (A.33), we can get

$$\begin{aligned} 0 &= L^{\dagger b} a j_L^0 + a(\mu_H-\mu_L) f j_L^{0'} - a\left(\mu_H-\mu_L+\eta-\lambda\right) j_L^0 + a\left(1-(r+\xi) f\right) \\ &= L^{\dagger b}\left(a j_L^0 - j_H^0\right) - 1 + (r+\xi) f + a(\mu_H-\mu_L) f j_L^{0'} - a\left(\mu_H-\mu_L+\eta-\lambda\right) j_L^0 + a\left(1-(r+\xi) f\right) \\ &= L^{\dagger b}\left(a j_L^0 - j_H^0\right) + a(\mu_H-\mu_L) f j_L^{0'} - a\left(\mu_H-\mu_L+\eta-\lambda\right) j_L^0 + (a-1)\left(1-(r+\xi) f\right) \end{aligned}$$

We show single crossing by applying Theorem 1 to the function $a j_L^0 - j_H^0$ on equations (A.34) and (A.35). We will discuss the sign of $H(f)$ in the cases below (at the end of this proof). Before doing that, when we map equation (A.34) to equation (A.24),

$$u(f) \equiv a j_L^0 - j_H^0,\ g(f) \equiv \frac{-\left(\mu_H+\xi\right) f}{\frac{1}{2}\sigma^2 f^2},\ h(f) \equiv \frac{-\left(\rho+\lambda-\mu_H\right)}{\frac{1}{2}\sigma^2 f^2},$$

and when we map equation (A.35) to equation (A.24),

$$u(f) \equiv a j_L^0 - j_H^0,\ g(f) \equiv \frac{-\left(\mu_H+\xi\right) f}{\frac{1}{2}\sigma^2 f^2},\ h(f) \equiv \frac{-\left(r+\lambda-\mu_H\right)}{\frac{1}{2}\sigma^2 f^2}.$$

Now we need to determine the sign of $H(f)$. From Lemma 1 and the hypothesis $\lambda > \bar{\lambda}$, we get that

$$\left(\rho+\lambda-\mu_H\right) a - \left(\rho+\lambda-\eta-\mu_L\right) > 0.$$

Evaluating $H(f)$ at the two boundaries, we get

$$\begin{aligned} H(0) &= -a\left(\mu_H - \mu_L + \eta - \lambda\right)\frac{1}{r+\eta-\mu_L} + (a-1) \\ &= \frac{1}{r+\eta-\mu_L}\left(\left(\rho+\lambda-\mu_H\right)a - \left(\rho+\lambda-\eta-\mu_L\right)\right) > 0, \\ H(f_L^b) &= (a-1)\left(1-(r+\xi)\, f_L^b\right) < 0. \end{aligned}$$

If $\lambda \geq \eta - (\mu_H - \mu_L)(\gamma - 1)$, the second derivative of $H(f)$, which is given by

$$\begin{aligned} H''(f) &= \Big[(\mu_H-\mu_L)a\frac{f j_L^{0'''}}{j_L^{0''}} + (\mu_H-\mu_L)a + a\left(\lambda-\eta\right)\Big] j_L^{0''} \\ &= \Big[(\mu_H-\mu_L)(\gamma-1) + \lambda - \eta\Big] a j_L^{0''}, \end{aligned} \tag{A.36}$$

is non-negative, so $H(f)$ is convex and its maximum on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. If $\lambda < \eta - (\mu_H - \mu_L)(\gamma - 1)$ then $H(f)$ is concave, so it can cross zero from above only once. In either case, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.
  - On $f \in [0, f_\dagger]$, we know $H(f) > 0$ and $L^{0\dagger}\left(a j_L^0 - j_H^0\right) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $a j_L^0(f) - j_H^0(f)$ cannot have a negative interior minimum on $[0, f_\dagger]$. Given $a j_L^0(0) - j_H^0(0) > 0$, we know that $a j_L^0(f) - j_H^0(f) > 0$, $\forall f \in [0, f_\dagger)$. Moreover, Theorem 2 and Corollary 2 imply $a j_L^{0'}(f_\dagger) - j_H^{0'}(f_\dagger) < 0$.
  - On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}\left(a j_L^0 - j_H^0\right) \geq 0$. Using Theorem 1, we know that $a j_L^0(f) - j_H^0(f)$ cannot have a nonnegative interior maximum. Given that $a j_L^0\left(f_L^b\right) - j_H^0\left(f_L^b\right) < 0$, $a j_L^0(f) - j_H^0(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
  - On $f \in [f_\dagger, f']$. Suppose there exists a $f'' \in (f_\dagger, f')$ such that $a j_L^0(f'') - j_H^0(f'') > 0$. Given that $a j_L^0(f_\dagger) - j_H^0(f_\dagger) = 0$ and $a j_L^{0'}(f_\dagger) - j_H^{0'}(f_\dagger) < 0$, it must be that $a j_L^0(f) - j_H^0(f)$ has a nonpositive interior minimum on $[f_\dagger, f'']$. Meanwhile, from $L^{\dagger b}(a j_L^0(f) -$

$j_H^0(f)) \leq 0$ for $f \in (f_\dagger, f'')$, we know from Theorem 1 that $aj_L^0(f) - j_H^0(f)$ cannot have a nonpositive interior minimum on $(f_\dagger, f'')$, which constitutes a contradiction.

- Case 2: $f' \leq f_\dagger$.
  - On $f \in [f_\dagger, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}\left(aj_L^0 - j_H^0\right) \leq 0$. From Theorem 1 and 2, we know $aj_L^0(f) - j_H^0(f) \leq 0$ and $aj_L^{0'}(f_\dagger) - j_H^{0'}(f_\dagger) \leq 0$.
  - On $f \in [f', f_\dagger]$, $L^{0\dagger}\left(aj_L^0 - j_H^0\right) \geq 0$ so that $aj_L^0(f) - j_H^0(f)$ cannot have a nonnegative interior maximum. Together with $aj_L^{0'}(f_\dagger) - j_H^{0'}(f_\dagger) \leq 0$, this shows $aj_L^0(f) - j_H^0(f) \geq 0$.
  - On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}\left(aj_L^0 - j_H^0\right) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L^0(f) - j_H^0(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L^0(0) - j_H^0(0) > 0$, we know that $aj_L^0(f) - j_H^0(f) > 0$, $\forall f \in [0, f')$.

□

Proof of Lemma 3

Proof. Let $\hat{u} = j_H^{'}(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (A.51) that $\hat{u}(0) \geq 0$ and (A.47) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\hat{u} = -(\rho + \lambda - r)(j_L^{'} + 1) < 0, \qquad f \in [0, f_\dagger]$$
$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\hat{u} = -\lambda < 0 \qquad f \in [f_\dagger, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. □

Proof of Lemma 4

Proof. For any $f \leq \frac{1}{r+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1 - (r + \xi) f + (\rho + \lambda - y) d > 0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1}{r+\xi}$. Plugging (A.46) and (A.47) into the HJB equation for $j_H(f)$, we get $j_H^{''}\left(f_H^b\right)$ whenever $f_H^b > \frac{1}{r+\xi}$.

Next, let us turn to prove that $j_H^{''}(0) \geq 0$. Let us define $u \equiv j_H^{'}$ and differentiate the HJB equation once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H + \xi - \sigma^2\right) f u' - (\rho + \lambda + \xi)\, u = (r+\xi) - (\rho + \lambda - r)\, j_L^{'}.$$

Moreover, let z be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H + \xi - \sigma^2\right) f z' - (\rho + \lambda + \xi)\, z = (r+\xi) - (\rho + \lambda - r)\, j_L^{'}(0)$$

with boundary conditions

$$\lim_{f \downarrow 0} z(f) < \infty$$
$$z(f_\dagger) = u(f_\dagger) = j_H^{'}(f_\dagger).$$

The solution is

$$z(f) = -\frac{r+\xi}{\rho+\lambda+\xi} + \frac{(\rho+\lambda-r)\, j_L^{'}(0)}{\rho+\lambda+\xi} + \left(j_H^{'}(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-r)\, j_L^{'}(0)}{\rho+\lambda+\xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1 = \frac{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho+\lambda+\xi\right)}}{\sigma^2} > 0.$$

Let $\delta(f) = z - u$. It is easily verified that $\delta(0) = 0$ and $\delta(f_\dagger) = 0$. Moreover, δ satisfies

$$\frac{1}{2}\sigma^2 f^2 \delta'' - \left(\mu_H + \xi - \sigma^2\right) f \delta' - (\rho+\lambda+\xi)\, \delta = (\rho+\lambda-r)\left(j_L^{'}(f) - j_L^{'}(0)\right) \geq 0.$$

By Theorem 1, δ cannot have an interior nonnegative maximum, and the maximum is attained at $f=0$. Theorem 2 further implies $\delta'(0)<0$ so $u'(0)>z'(0)$. Finally, we know that

$$z'(f)=\omega_1\left(j_H'(f_\dagger)+\frac{r+\xi}{\rho+\lambda+\xi}-\frac{(\rho+\lambda-r)\,j_L'(0)}{\rho+\lambda+\xi}\right)f_\dagger^{-\omega_1}f^{\omega_1-1}=\omega_1\left(j_H'(f_\dagger)+1\right)f_\dagger^{-\omega_1}f^{\omega_1-1},$$

which implies $z'(f)\geq 0$ given that $j_H'(f_\dagger)\geq -1$. Therefore, $u'(0)=j_H''(0)>0$. $\square$

Proof of Lemma 5

Proof. We differentiate the HJB (A.4) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2(j_H'''(f_\dagger+)-j_H''''(f_\dagger-))=(\rho-r)\left[aj_L'(f_\dagger)-j_H'(f_\dagger)\right],$$

where $a\equiv 1+\frac{\lambda}{\rho-r}$ The proof of Proposition 2 shows $aj_L'(f_\dagger)-j_H'(f_\dagger)<0$ so that $j_H''''(f_\dagger-)>j_H'''(f_\dagger+)$. $\square$

A.3 Detailed Solutions of the Joint Value $j_\theta^0(f)$

Proposition 10 (Value function)**.** *In state $\theta=L$, the joint continuation value is*

$$j_L^0(f)=\underbrace{\frac{1}{r+\eta-\mu_L}-\frac{r+\xi}{r+\eta+\xi}f}_{\textit{no default value}}+\underbrace{\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\textit{default option}},\tag{A.37}$$

where $\gamma>1$ is provided in equation (A.42)*, and the default boundary f_L^b in* (A.43)*.*

In state $\theta=H$:

- *If $\lambda > \bar{\lambda}$, the joint continuation value is*

$$j_H^0(f) = \begin{cases} u_0(f) + \left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^\phi & f \in [0, f_\dagger) \\ u_1(f) + \left(j_H^0(f_\dagger) - u_1(f_\dagger)\right)h_0(f) - u_1(f_H^b)h_1(f) & f \in \left[f_\dagger, f_H^b\right], \end{cases} \tag{A.38}$$

where $\phi > 1$ is provided in (A.50), *and*

$$u_0(f) = \underbrace{\frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - r}{r + \eta - \mu_L}\right) - \frac{r + \xi}{\rho + \lambda + \xi}\left(1 + \frac{\rho + \lambda - r}{r + \eta + \xi}\right)f}_{\textit{no default value}} + \underbrace{\delta\frac{1}{\gamma - 1}\frac{1}{r + \eta - \mu_L}\left(\frac{f}{f_L^b}\right)^\gamma}_{\textit{default option in low state}} \tag{A.39}$$

$$u_1(f) = \underbrace{\frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi}f}_{\textit{no default value}}. \tag{A.40}$$

The discount factors δ, $h_0(\cdot)$, and $h_1(\cdot)$ are defined in equations (A.52) *and* (A.55). *The boundaries $f_\dagger$ and f_H^b are determined using the boundary conditions equation* (A.45) *and equation*(A.47).

- *If $\lambda \leq \bar{\lambda}$, the joint continuation value is*

$$j_H^0(f) = \underbrace{\frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi}f}_{\textit{no default value}} + \underbrace{\frac{r + \xi}{r + \lambda + \xi}\frac{f_H^b}{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_1}}_{\textit{default option}} \tag{A.41}$$

where $\beta_1 > 1$ is provided in equation (A.53), *and the default boundary f_H^b in* (A.60).

Solution to the HJB equation in the low state. Equation (8) is a second-order ODE, and a standard solution takes the form

$$j_L^0(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$
\begin{aligned}
A_0 &= \frac{1}{r+\eta-\mu_L} \\
A_1 &= \frac{r+\xi}{r+\eta+\xi} \\
\gamma_1 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\eta-\mu_L\right)}}{\sigma^2} > 1 \\
\gamma_2 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\eta-\mu_L\right)}}{\sigma^2} < 0.
\end{aligned}
\tag{A.42}
$$

The condition $\lim_{f\to 0} j_L^0(f) < \infty$ implies $A_3 = 0$. We define $\gamma \equiv \gamma_1$. Combining with value-matching and smooth-pasting condition, we get the default boundary is

$$
f_L^b = \frac{\gamma}{\gamma-1}\frac{r+\eta+\xi}{\left(r+\eta-\mu_L\right)(r+\xi)}. \tag{A.43}
$$

Therefore, the joint value at $\theta = L$ is

$$
j_L^0(f) = \frac{1}{r+\eta-\mu_L} - \frac{r+\xi}{r+\eta+\xi}f + \frac{r+\xi}{r+\eta+\xi}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^{\gamma}.
$$

Solution to the HJB equation in the high state for $f_\dagger > 0$. The value function satisfies equation (A.4) together with the boundary conditions

$$
j_H^0(f_\dagger-) = j_H^0(f_\dagger+) \tag{A.44}
$$

$$
j_H^{0\prime}(f_\dagger-) = j_H^{0\prime}(f_\dagger+) \tag{A.45}
$$

$$
j_H^0(f_H^b) = 0 \tag{A.46}
$$

$$
j_H^{0\prime}(f_H^b) = 0 \tag{A.47}
$$

$$
\lim_{f\to 0} j_H^0(f) < \infty \tag{A.48}
$$

$$
j_H^0\left(f_\dagger\right) = \frac{\rho+\lambda-r}{\rho-r} j_L^0\left(f_\dagger\right). \tag{A.49}
$$

First, we consider the solution for $f \in [0, f_\dagger]$, in which region the value function satisfies the equation

$$(\rho+\lambda-\mu_H)\, j_H^0(f) = 1-(r+\xi)\, f + (\rho+\lambda-r)\, j_L^0(f) - (\mu_H+\xi)\, f j_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f)\,.$$

The unique solution to this ODE satisfying condition (A.48) takes the form

$$j_H^0(f) = u_0(f) + B f^{\phi},$$

where the coefficient ϕ is given by

$$\phi = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(\rho+\lambda-\mu_H\right)}}{\sigma^2} > 1, \tag{A.50}$$

and a particular solution u_0 is given by

$$u_0(f) = \underbrace{\frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right) - \frac{r+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-r}{r+\eta+\xi}\right) f}_{\text{no default value}} + \underbrace{\delta\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\text{default option in low state}} \tag{A.51}$$

where the discount factor δ is

$$\delta \equiv \frac{\rho+\lambda-r}{\rho+\lambda-r-\eta+(\mu_H-\mu_L)(\gamma-1)} \in (0,1). \tag{A.52}$$

The coefficient B is pinned down from the value at $j_H^0(f_\dagger)$

$$B = f_\dagger^{-\phi}\left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H^0(f) = u_0(f) + \left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^\phi, \qquad \forall f \in [0, f_\dagger],$$

where $j_H^0(f_\dagger) = \frac{\rho+\lambda-r}{\rho-r} j_L^0(f_\dagger)$. The solution on the interval $\left[f_\dagger, f_H^b\right]$ can be obtained in a similar way. Let us define

$$\mathcal{D}^H \varphi \equiv -(\mu_H + \xi)\varphi' + \frac{1}{2}\sigma^2 f^2 \varphi''.$$

In this interval, the value function satisfies the equation

$$(r + \lambda - \mu_H)\, j_H^0(f) = 1 - (r + \xi)\, f + \mathcal{D}^H j_H^0(f).$$

The homogeneous equation

$$(r + \lambda - \mu_H)\, \varphi = \mathcal{D}^H \varphi$$

has two solution f^{β_1} and f^{β_2}, where

$$\begin{aligned}
\beta_1 &= \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r + \lambda - \mu_H)}}{\sigma^2} > 1 \\
\beta_2 &= \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r + \lambda - \mu_H)}}{\sigma^2} < 0.
\end{aligned} \tag{A.53}$$

Hence, the value function takes the form

$$j_H^0(f) = u_1(f) + D_1 f^{\beta_1} + D_2 f^{\beta_2}.$$

The particular solution is

$$u_1(f) = \frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi} f. \tag{A.54}$$

Finally, by combining equations (A.44) and (A.46), we get

$$D_1 = \frac{j_H^0(f_\dagger) + u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}$$
$$D_2 = (f_H^b)^{-\beta_2}\left(-u_1(f_H^b) - D_1(f_H^b)^{\beta_1}\right).$$

It follows that the solution to the value function on this interval is given by

$$j_H^0(f) = u_1(f) + \big(j_H^0(f_\dagger) - u_1(f_\dagger)\big)h_0\left(f, f_\dagger, f_H^b\right) - u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right),$$

where

$$\begin{aligned} h_0\left(f\middle|f_\dagger, f_H^b\right) &= \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \\ h_1\left(f\middle|f_\dagger, f_H^b\right) &= \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}. \end{aligned} \tag{A.55}$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions (A.45) and (A.47). These two conditions lead to the two-variable, non-linear equation system below

$$u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = u_1'(f_H^b)f_H^b + \big(j_H^0(f_\dagger) - u_1(f_\dagger)\big)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \tag{A.56}$$

$$\begin{aligned} &\big(u_0'(f_\dagger) - u_1'(f_\dagger)\big)f_\dagger + \phi\big(j_H^0(f_\dagger) - u_0(f_\dagger)\big) = \\ &u_1(f_H^b)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2} + \big(j_H^0(f_\dagger) - u_1(f_\dagger)\big)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}. \end{aligned} \tag{A.57}$$

Solution to the HJB equation in the high state if $\lambda \leq \bar{\lambda}$. The value function satisfies

$$(r + \lambda - \mu_H)\, j_H^0(f) = 1 - (r + \xi)\, f - (\mu_H + \xi)\, f j_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f)\,. \tag{A.58}$$

We guess the solution of the value function takes the form

$$j_H^0(f) = D_0^0 - D_1^0 f + D_2^0 f^{\beta_1} + D_3^0 f^{\beta_2}.$$

Plugging into equation (A.58), we can get

$$D_0^0 = \frac{1}{r + \lambda - \mu_H},$$
$$D_1^0 = \frac{r + \xi}{r + \lambda + \xi}.$$

The condition $\lim_{f \to 0} j_H^0(f) < \infty$, we know $D_3^0 = 0$. Since $j_H^0(f_H^b) = 0$ and $j_H^{0\prime}(f_H^b) = 0$, we know

$$j_H^0(f) = \frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi} f + \frac{r + \xi}{r + \lambda + \xi} \frac{f_H^b}{\beta_1} \left(\frac{f}{f_H^b} \right)^{\beta_1}, \tag{A.59}$$

where

$$f_H^b = \frac{\beta_1}{\beta_1 - 1} \frac{r + \lambda + \xi}{(r + \xi)\,(r + \lambda - \mu_H)}. \tag{A.60}$$

B Analysis of Extensions

B.1 Subsection 5.1 with Derivative Contract

In the high state, the HJB equation is

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = \max_{d \in [0, j_H(f)], z \geq 0} \; & 1 - (r + \xi)\, f - q\lambda z + (\rho + \lambda - y_H)\, d - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ & + \lambda q \max\{j_L(f) + z - d, 0\} + \lambda(1 - q) \max\{j_L(f) - d, 0\}\,. \end{aligned} \tag{A.61}$$

The following Lemma characterizes the solution to the maximization problem in equation (A.61)

Lemma 6. *The optimal short-term debt and hedging policy* $d_H(f), z(f)$ *is*

$$d_H(f) = \begin{cases} j_L(f) & \text{if } j_L(f) \geq \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) & \text{Otherwise.} \end{cases}$$

$$z(f) = \begin{cases} 0 & \text{if } j_L(f) \geq \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) - j_L(f) & \text{Otherwise.} \end{cases}$$

Proof. In equation (A.61), d_H and z are chosen to maximize

$$-q\lambda z + (\rho + \lambda - y_H)\, d_H + \lambda q \max\left\{j_L(f) + z - d_H, 0\right\} + \lambda(1-q) \max\left\{j_L(f) - d_H, 0\right\}.$$

There are three situations that we need to consider:

1. If $d_H \leq j_L(f)$, the objective becomes

$$-q\lambda z + (\rho + \lambda - r)\, d_H + \lambda q\left(j_L(f) + z - d_H\right) + \lambda(1-q)\left(j_L(f) - d_H\right) = (\rho - r)\, d_H + \lambda j_L(f),$$

which is maximized at $d_H = j_L(f)$ with the maximum value

$$(\rho + \lambda - r)\, j_L(f).$$

2. If $d_H \in \big(j_L(f), j_L(f) + z\big]$, the objective becomes

$$-q\lambda z + (\rho + \lambda q - r)\, d_H + \lambda q\left(j_L(f) + z - d_H\right) = (\rho - r)\, d_H + \lambda q j_L(f),$$

which is maximized at $d_H = j_L(f) + z$ with the maximum value

$$(\rho - r + \lambda q) j_L(f) + (\rho - r) z.$$

Given that $d_H = j_L(f) + z \leq j_H(f)$, we know $z \leq j_H(f) - j_L(f)$. The maximized $z = j_H(f) - j_L(f)$, and the maximum value is

$$(\rho - r)j_H(f) + \lambda q j_L(f).$$

3. If $d_H > j_L(f) + z$, the objective becomes $-q\lambda z + (\rho - r)\, d_H$, which is clearly maximized at $z = 0$ and $d_H = j_H(f)$, with a maximum value

$$(\rho - r)\, j_H(f).$$

Clearly, the last one is dominated, so the borrower's choice is

- If $(\rho - r)j_H(f) + \lambda q j_L(f) \leq (\rho + \lambda - r)j_L(f)$, then $d_H = j_L(f)$, and z is irrelevant so without loss of generality set as zero.
- Otherwise, then $d_H = j_H(f)$ and $z = j_H(f) - j_L(f)$.

$\square$

When $d_H(f) = j_L(f)$, the firm will survive the regime switch anyway, so insurance is unnecessary. By contrast, when $d_H(f) = j_H(f)$ so that short-term debt is risky, the equity holder buys enough derivative contracts to insure against the regime shift. The equilibrium takes a similar form as the one in section 4. The amount of short term debt is $d_H(f) = j_L(f)$ when $f < f_\dagger$, and $d_H(f) = j_H(f)$ if $f > f_\dagger$, with the threshold $f_\dagger$ given by the indifference condition

$$f_\dagger = \min\left\{f \geq 0 : (\rho + \lambda(1-q) - r)j_L(f) \leq (\rho - r)j_H(f)\right\}. \tag{A.62}$$

Given the optimal policy in Lemma 6, we can write the HJB equation in simpler form

$$\begin{aligned}(\rho+\lambda-\mu_H)\, j_H(f) &= 1-(r+\xi)\, f+(\rho+\lambda-r)\, j_L(f)-(\mu_H+\xi) f j_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f\in(0,f_\dagger)\\ (r+\lambda-\mu_H)\, j_H(f) &= 1-(r+\xi)\, f+q\lambda j_L(f)-(\mu_H+\xi) f j_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f\in(f_\dagger,f_H^b).\end{aligned} \tag{A.63}$$

The price of debt is now given by the solution to the asset pricing equation.

$$\begin{aligned}(r+\xi+\lambda)\, p_H(f) &= r+\xi+\lambda p_L(f)+\left(g_H(f)+\sigma^2-\mu_H-\xi\right) f p_H'(f)+\frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f\in(0,f_\dagger)\\ (r+\xi+\lambda)\, p_H(f) &= r+\xi+\lambda q p_L(f)+\left(g_H(f)+\sigma^2-\mu_H-\xi\right) f p_H'(f)+\frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f\in(f_\dagger,f_H^b).\end{aligned}$$

From, here, together with the indifference condition $j_H'(f)=-p_H(f)$, we can obtain the equilibrium issuance function. We omit the details, but a similar calculations to the ones in the absence of hedging show that the equilibrium issuance policy is given by (18). Notice that although the form of the issuance function does not change the total issuance of long-term debt does change as the price of long-term debt is now different. The main impact of hedging though is on the value of the threshold $f_\dagger$.

It immediately follows from this indifference condition that when $q=1$, the threshold $f_\dagger$ is equal to zero as $(\rho-r)(j_H(f)-j_L(f))>0$ from the definition of $f_\dagger$ in equation (A.62). In other words, the borrower does not issue any long-term debt if she can perfectly insure against the regime shift. The solution in Lemma 6 becomes $d_H(f)=j_H(f)$ and $z(f)=j_H(f)-j_L(f)$ for all $f\in[0,f_H^b]$. This provides a proof of Proposition 7.

Solution HJB Equation

The solution to the HJB equation takes the same form as in the baseline model. The function u_0 is still given by (A.51), but the expression for u_1 is different to the one in equation (A.54) because it includes the term $q\lambda j_L(f)$ capturing the continuation value after the regime switch.

For $f\in(0,f_\dagger)$, there is no change in the differential equation, so the solution remains the same.

On $f \in (f_\dagger, f_H^b)$ the HJB equation becomes

$$(r + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f + q\lambda j_L(f) + \mathcal{D}^H j_H(f)$$

If $f \geq f_L^b$, the continuation value $j_L(f)$ and the particular solution is

$$u_1(f) = \frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi} f.$$

When $f < f_L^b$, the particular solution takes the form

$$u_1(f) = \frac{1}{r + \lambda - \mu_H}\left(1 + q\frac{\lambda}{r + \eta - \mu_L}\right) - \frac{r + \xi}{r + \lambda + \xi}\left(1 + q\frac{\lambda}{r + \eta + \xi}\right) f + C\left(\frac{f}{f_L^b}\right)^{\gamma},$$

Substituting in the previous the ODE, we find that the constant C is given by

$$C = \frac{\lambda q}{\lambda - \eta + (\mu_H - \mu_L)(\gamma - 1)} \frac{1}{\gamma - 1} \frac{1}{r + \eta - \mu_L}.$$

The solution then to the HJB equation is

$$j_H(f) = \begin{cases} u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi} & f \in [0, f_\dagger] \\ u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right) h_0\left(f \middle| f_\dagger, f_L^b\right) + \left(j_H(f_L^b) - u_1(f_L^b)\right) h_1\left(f \middle| f_\dagger, f_L^b\right) & f \in \left(f_\dagger, f_L^b\right) \\ u_1(f) + \left(j_H(f_L^b) - u_1(f_L^b)\right) h_0\left(f \middle| f_L^b, f_H^b\right) - u_1(f_H^b) h_1\left(f \middle| f_L^b, f_H^b\right) & f \in \left[f_L^b, f_H^b\right] \end{cases}$$

where

$$u_0(f) = \mathcal{A}\frac{1}{\rho + \lambda - \mu_H} - \mathcal{B}\frac{r + \xi}{\rho + \lambda + \xi} f + \delta\frac{1}{\gamma - 1}\frac{1}{r + \eta - \mu_L}\left(\frac{f}{f_L^b}\right)^{\gamma}$$

$$u_1(f) = \begin{cases} \frac{1}{r+\lambda-\mu_H}\left(1 + \frac{\lambda q}{r+\eta-\mu_L}\right) - \frac{r+\xi}{r+\lambda+\xi}\left(1 + \frac{\lambda q}{r+\eta+\xi}\right) f + C\left(\frac{f}{f_L^b}\right)^{\gamma} & \text{if } f < f_L^b \\ \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi} f & \text{if } f \geq f_L^b, \end{cases}$$

and the constants $\mathcal{A}, \mathcal{B}, \delta$ are given by

$$\mathcal{A} = \frac{\rho + \lambda + \eta - \mu_L}{r + \eta - \mu_L} \qquad \mathcal{B} = \frac{\rho + \lambda + \eta + \xi}{r + \eta + \xi} \qquad \delta = \frac{\rho + \lambda - r}{\rho + \lambda - r - \eta + (\mu_H - \mu_L)(\gamma - 1)}.$$

The functions $h_0(\cdot)$ and $h_1(\cdot)$ are defined in equation (A.55).

Finally, to show that $f_\dagger$ is decreasing in q when $f_\dagger > 0$, it suffices to show that $j_H(f)$ is increasing in q.

Lemma 7. *If $(\rho + \lambda(1 - q) - r) j_L(0) > (\rho - r) j_H(0)$, then the value function $j_H(f)$ is strictly increasing in q.*

Proof. For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t} \left(1 - (r + \xi)z_t + \nu(z_t, \tilde{j}(z_t)|q)\right) dt \middle| z_0 = f\right]$$
$$\text{subject to} \quad dz_t = -(\xi + \mu_H)z_t dt - \sigma z_t dB_t,$$

where

$$\nu(z, \tilde{j}|q) \equiv \max\{(\rho + \lambda - r) j_L(z), q\lambda j_L(z) + (\rho - r)\tilde{j}\}$$

and $\hat{\rho} \equiv \rho + \lambda - \mu_H$. It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f, \tilde{j}_1|q) \geq \nu(f, \tilde{j}_0|q)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\begin{aligned}\nu(z, \tilde{j} + a|q) &= \max\{(\rho + \lambda - r) j_L(z), q\lambda j_L(z) + (\rho - r)(\tilde{j} + a)\} \\ &\leq \max\{(\rho + \lambda - r) j_L(z) + (\rho - r)a, q\lambda j_L(z) + (\rho - r)(\tilde{j} + a)\} = (\rho - r)a + \nu(z, \tilde{j}|q),\end{aligned}$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)+a|q)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f)+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \leq \Phi(\tilde{j})(f)+\frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}$$

As Φ is monotone and satisfies discounting, it follows from Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f)=\Phi(j_H)(f)$.

For any pair of parameters $q_1 \geq q_0$, the inequality $\nu(f,\tilde{j}|q_1) \geq \nu(f,\tilde{j}|q_0)$ implies that the operator Φ is increasing q. It follows from Theorem 1 in Villas-Boas (1997) that the fixed point $j_H(f)=\Phi(j_H)(f)$ increases in q. □

B.2 Subsection 5.2 with Restructuring

In this section, we provide the analysis of the equilibrium when short-term debt can be restructured in section 5.2. Following the state transition, the borrower receives $j_L(f)-d_H$ if she does not default. If there is a default, with probability $1-\alpha$, there is no renegotiation, and she receives zero. With probability α, there is renegotiation. In this case, if short-term creditors make an offer, they receive $j_L(f)$ while the borrower receives 0. If the equity holder makes an offer, they offer 0 and obtain $j_L(f)$, while the short-term creditors get 0; however, such an offer is credible only if $j_L(f) < d_H$. If $j_L(f) \geq d_H$, then the only credible offer is d_H.[1] It is easy to verify that renegotiation

[1]When there is indifference, we break ties in favor of the efficient outcome of continuation.

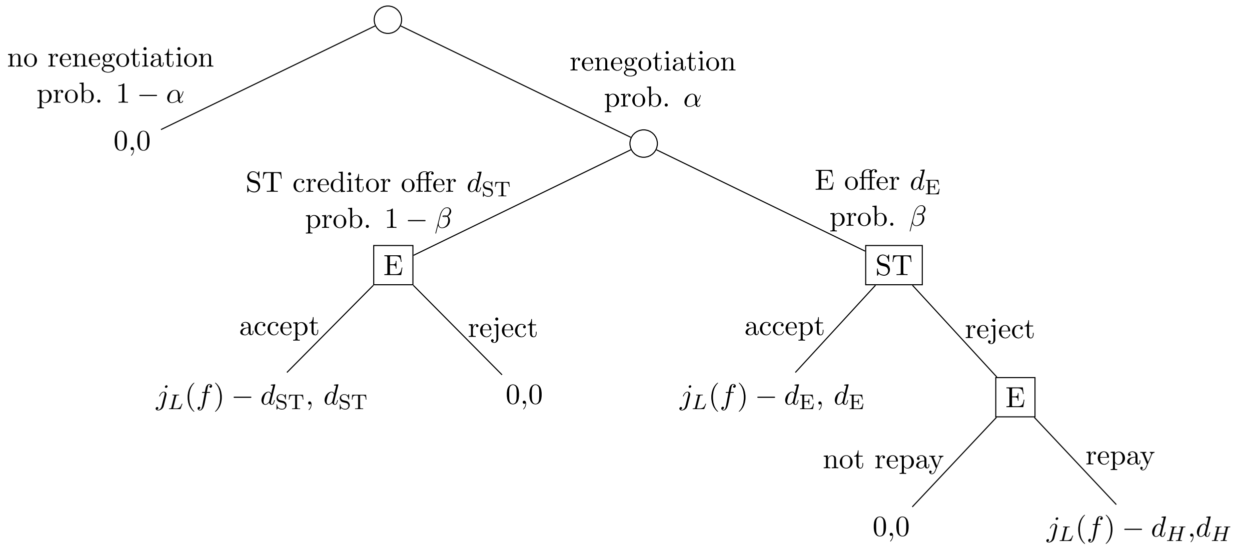

Figure 1: Renegotiation Process

The game tree illustrates the renegotiation process. If the firm defaults, renegotiation is triggered with probability α; otherwise, there is bankruptcy. In the event of renegotiation, the equity holder gets to make an offer with probability β, in which case she offers d_{E}. Otherwise, the offer is made by short-term creditors, in which case they offer d_{ST}. In the tree, E indicates nodes where the equity holder moves and ST nodes where short-term creditors move. At the end of the tree, the first coordinate indicates the payoff to the equity holder, and the second coordinate indicates the payoff to short-term creditors.

is triggered only after the regime switch and $j_L(f_t) < d_H(f_t)$. The firm goes bankrupt only if the restructuring process fails, which happens with probability $1-\alpha$. To determine the interest rate, we need to analyze the expected recovery. If $d_H(f) \leq j_L(f)$, there is no default, and short-term creditors are paid in full. The equity holder does not have incentives to default because no credible offer would allow paying less than d_H. If $d_H > j_L(f)$, the equity holder default so the expected payoff is $(1-\alpha\beta) \times 0 + \alpha\beta \times j_L(f)$. In the event of default, each creditor gets zero with probability $1-\alpha(1-\beta)$; that is, if either renegotiation is not possible or if it is possible, but equity holder is the one to make the offer. With probability $\alpha(1-\beta)$, the short-term debt recovery per dollar is

$j_L(f)/d_H$. Hence, the short rate is given by

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \le j_L(f) \\ r + \lambda\left(1 - \alpha(1-\beta)\frac{j_L(f)}{d_H}\right) & \text{if } d_H > j_L(f). \end{cases}$$

The analysis in state L is unchanged. In state H, we construct an equilibrium similar to the one without restructuring. The HJB in the high state follows

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = &\max_{d_H \in [0, j_H(f)]} (1-\pi) - (r+\xi)\, f + (\rho + \lambda - y_H)\, d_H \\ &+ \lambda\Big(\big(j_L(f) - d_H\big)\mathbb{1}_{d_H \le j_L(f)} + \alpha\beta j_L(f)\mathbb{1}_{d_H > j_L(f)}\Big) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \quad \text{(A.64)}\end{aligned}$$

The optimal solution for short-term debt is

$$d_H(f) = \begin{cases} j_L(f) & \text{if } j_L(f) \ge \frac{\rho - r}{\rho + \lambda(1-\alpha) - r} j_H(f) \\ j_H(f) & \text{Otherwise.} \end{cases}$$

Note that when $\alpha = 0$, we are back to the benchmark model. Interestingly, α and β serve different purposes: the former leads to efficiency loss, and the latter is only about how to redistribute the surplus across the coalition. The threshold now is determined by the indifference condition

$$f_\dagger = \min\big\{f \ge 0 : (\rho + \lambda(1-\alpha) - r) j_L(f) \le (\rho - r) j_H(f)\big\}. \quad \text{(A.65)}$$

The HJB equation (A.64) can be written as

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) &= 1 - (r+\xi)\, f + (\rho + \lambda - r) j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (0, f_\dagger) \\ (r + \lambda - \mu_H)\, j_H(f) &= 1 - (r+\xi)\, f + \lambda\alpha j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (f_\dagger, f_H^b).\end{aligned} \quad \text{(A.66)}$$

We see that the only difference with the original equation is that now, when the firm is fully levered, there is a term $\lambda\alpha j_L(f)$ capturing the continuation value after the regime shift. Notice that the HJB equation (A.66) takes the same form as the one with hedging in equation (A.63), so hedging and renegotiation serve a similar economic purpose in the model. The asset pricing equation for bond prices becomes

$$(r+\xi+\lambda)\,p_H\,(f) = r+\xi+\lambda p_L\,(f) + \left(g_H(f)+\sigma^2-\mu_H-\xi\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f\in(0,f_\dagger)$$
$$(r+\xi+\lambda)\,p_H\,(f) = r+\xi+\lambda\alpha p_L + \left(g_H(f)+\sigma^2-\mu_H-\xi\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f\in(f_\dagger,f_H^b).$$

Thus, together with the indifference condition $j_H'(f) = -p_H(f)$, we obtain the equilibrium issuance function in the high state is given by the same expression as the one in equation (18). The HJB and price equations are identical to the ones with hedging, so the proof of Proposition 8 follows the one in Online Appendix B.1.

B.3 Subsection 5.3 with Cash Flow Jumps

Suppose that

$$dX_t = \mu X_t dt + \sigma X_t dB_t - (1-\omega^{-1})X_{t-}dN_t,$$

where N_t is a Poisson process with intensity λ and $\omega > 1$. Using Ito's Lemma, f_t solves

$$df_t = (g_t - \mu - \xi + \sigma^2) f_t dt - \sigma f_t dB_t + (\omega - 1) f_{t-} dN_t$$

Thus, the scaled value function satisfies the delay differential equation

$$(\rho+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f)$$
$$+\max\left\{(\rho-r)\frac{j(\omega f)}{\omega}+\lambda\frac{j(\omega f)}{\omega},(\rho-r)j(f)\right\}$$

We guess and verify that the optimal short-term debt policy is given by

$$d(f) = \begin{cases} \frac{j(\omega f)}{\omega} & \text{if } f \in [0, f_\dagger] \\ j(f) & \text{if } f \in (f_\dagger, f^b]. \end{cases}$$

The HJB equation can be written as

$$(\rho + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + (\rho + \lambda - r)\frac{j(\omega f)}{\omega},\ f \in (0, f_\dagger)$$
$$(r + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f),\ f \in (f_\dagger, f^b).$$

The default boundary solves the value matching and smooth pasting conditions $j(f^b) = j'(f_b) = 0$.

Long-term bonds satisfy the asset pricing equation

$$(r + \xi + \lambda)\, p(f) = 1 - (r + \xi)\, f + \left(g(f) - \mu - \xi + \sigma^2\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + \lambda p(\omega f),\ f \in (0, f_\dagger)$$
$$(r + \xi + \lambda)\, p(f) = 1 - (r + \xi)\, f + \left(g(f) - \mu - \xi + \sigma^2\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f),\ f \in (f_\dagger, f^b).$$

Finally, we derive the issuance policy $g(f)$ combining the asset pricing equation with the indifference condition $p(f) = -j'(f)$. This yields

$$g\,(f) = \frac{(\rho - r)\,(p\,(f) - p\,(\omega f))}{f j''\,(f)}.$$

Numerical Computation: For computational purposes, it is easier to work with the the state variable $x = \log(1/f) = -\log f$. Let $\tilde{j}(x) \equiv j(e^{-x})$ and $\delta = \log \omega$. Then, we get

$$\tilde{j}'(x) = -j'(e^{-x})e^{-x}$$
$$\tilde{j}''(x) = j''(e^{-x})e^{-2x} + j'(e^{-x})e^{-x} = j''(e^{-x})e^{-2x} - \tilde{j}'(x)$$

Substituting in the HJB equation we get

$$(r+\lambda-\mu)\,\tilde{j}(x) = 1-(r+\xi)\,e^{-x}+\left(\mu+\xi+\frac{1}{2}\sigma^2\right)\tilde{j}'(x)+\frac{1}{2}\sigma^2\tilde{j}''(x)$$
$$-(\rho-r)\min\left\{\tilde{j}(x)-a\frac{\tilde{j}(x-\delta)}{e^{\delta}},0\right\},$$

where

$$a \equiv \frac{\rho+\lambda-r}{\rho-r}.$$

We write this as a system of two first order equations. Letting $y_0(x)=\tilde{j}(x)$ and $y_1(x)=\tilde{j}'(x)$, we can reduce the second order equation to the following system of first order equations

$$y_0'(x) = y_1(x)$$
$$y_1'(x) = \frac{2}{\sigma^2}\left[(r+\xi)\,e^{-x}-1+(r+\lambda-\mu)\,y_0(x)-\left(\mu+\xi+\frac{1}{2}\sigma^2\right)y_1(x)\right.$$
$$\left.+(\rho-r)\min\left\{y_0(x)-a\frac{y_0(x-\delta)}{e^{\delta}},0\right\}\right].$$

The previous equation is a system of two first order delay differential equations with constant coefficient that can be solved using standard numerical routines. The value matching and smooth pasting conditions at the default boundary $x^b = -\log f^b$ are $y_0(x^b) = y_1(x^b) = 0$. The only remaining step is to specify the transversality condition. From the HJB equation we get $j(0) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}$, so we have the transversality condition

$$\lim_{x\to\infty} y_0(x) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}.$$

To incorporate this transversality condition, we approximate the value function for $f=\epsilon$. This corresponds to a value of x given by $x_\epsilon = -\log\epsilon$. Differentiating the HJB we get

$$(\rho+\lambda+\xi)\,j'(f) = -(r+\xi)-\left(\mu+\xi-\sigma^2\right)fj''(f)+(\rho+\lambda-r)j'(\omega f),$$

and evaluating at $f = 0$, we get

$$j'(0) = -1.$$

Hence, for ϵ close to zero

$$j(\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - \epsilon,$$

which means that

$$y_0(x_\epsilon) = \tilde{j}(x_\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - e^{-x_\epsilon}.$$

Finally, we can write the price and issuance function in terms of the functions $y_0(x), y_1(x)$. The price of long-term bonds is given by

$$p(f) = -j'(f) = \tilde{j}'(x)e^x = y_1(x)e^x.$$

Letting $x_\dagger = -\log f_\dagger$, we get that on the $[x_\dagger, \infty)$, the issuance policy is

$$g(x) = \frac{(\rho-r)\big(p\left(e^{-x}\right) - p\left(e^{-(x-\delta)}\right)\big)}{-e^{-x}p'\left(e^{-x}\right)} = \frac{(\rho-r)\big(y_1(x) - y_1(x-\delta)e^{-\delta}\big)}{-e^{-x}(y_1'(x)+y_1(x))}$$

B.4 Subsection 5.4 with Transitory Shocks

In this section, we extend the model to consider some further empirical implications. In the main model, we have assumed that the state $\theta_t = L$, is absorbing. If we interpret the changes in regime as business-cycles, it is natural to assume that these are transitory. We can extend the model to consider this situation. We denote the transition rate from the high state to the low state by λ_{HL}, and the transition rate from the low state to the high state by λ_{LH}. The stationary distribution of the process θ_t is then given by $\Pr(\theta = H) = \lambda_{LH}/(\lambda_{LH} + \lambda_{HL})$.

We quantify the model following Chen et al. (2021). As in their paper, the transition probabilities between states are set to $\lambda_{HL} = 0.1$ and $\lambda_{LH} = 0.5$, yielding unconditional probabilities of $\frac{5}{6}$ for the high (H) stage and $\frac{1}{6}$ for the low (L) stage. Using their reported state-dependent interest rates (5.6% in H, 2.6% in L), we calculated the unconditional rate as $r = 5.1\%$, then adjusted for their 20% tax rate to derive $\rho = (1+20\%) \times r = 6.12\%$. Debt maturities (7 years in H, 6.5 years in L) were averaged to 6.91 years unconditionally, implying a amortization rate $\xi = 1/6.91 \approx 0.145$. Cash flow parameters include growth rates $\mu_H = 6.2\%$ and $\mu_L = 1.6\%$, and unconditional volatility $\sigma = 13.93\%$. As estimated by Petrosky-Nadeau et al. (2018), the disaster shock's arrival rate is $\eta = 8.6\%$. The parameters are summarized in Table 1.

Table 1: Model Parameters

	$\theta = H$	$\theta = L$
State transition intensities: $\lambda_{\theta\theta'}$	0.1	0.5
Cash flow expected growth rate: μ_θ	6.2%	1.6%
Risk-free interest rate: r	5.1%	
Borrower's discount rate: ρ	6.12%	
Cash flow volatility: σ	13.93%	
Amortization rate: ξ	0.145	
Disaster shock's arrival rate: η	8.6%	

The equilibrium has the same qualitative features. The only changes is that in the HJB equation (A.2) for $j_L(f)$ and in the asset pricing equation (16) for $p_L(f)$, we have to add additional terms $\lambda_{LH}\big(j_H(f) - j_L(f)\big)$ and $\lambda_{LH}\big(p_H(f) - p_L(f)\big)$, respectively. The issuance policy takes the general form provided in equation (18). When shocks are transitory, the HJB equation for $j_L(f)$ becomes

$$(\rho + \eta - \mu_L)\, j_L\,(f) = 1 - (r + \xi)\, f + (\rho - r)\, j_L(f) + \mathcal{D}^L j_L(f) + \lambda_{LH}\big(j_H(f) - j_L(f)\big).$$

The indifference condition for the issuance of short-term debt in high state remains the same and is given by

$$(\rho + \lambda_{HL} - r)\, j_L(f_\dagger) \geq (\rho - r)\, j_H(f_\dagger).$$

We solve the equation in this region, and the combines the solution using smooth pasting and value matching conditions at the threshold $f_\dagger$. The default boundary are determined using the same value matching and smooth pasting conditions as in the main version of the model.

Solution for $f \in (0, f_\dagger)$. The characteristic equation of the associated homogeneous equation is now a quartic equation instead of a quadratic one. Hence the solution takes the general form:

$$
\begin{aligned}
j_L(f) &= A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2} + A_4 f^{\gamma_3} + A_5 f^{\gamma_4} \\
j_H(f) &= B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2} + B_4 f^{\gamma_3} + B_5 f^{\gamma_4}.
\end{aligned}
$$

Substituting the conjecture in the ODE, we get the linear system

$$
\begin{aligned}
\left(r + \lambda_{LH} + \eta - \mu_L\right) A_0 &= 1 + \lambda_{LH} B_0 \\
\left(\rho + \lambda_{HL} - \mu_H\right) B_0 &= 1 + \left(\rho + \lambda_{HL} - r\right) A_0 \\
\left(r + \xi + \lambda_{LH} + \eta\right) A_1 &= \left(r + \xi\right) + \lambda_{LH} B_1 \\
\left(\rho + \xi + \lambda_{HL}\right) B_1 &= \left(r + \xi\right) + \left(\rho + \lambda_{HL} - r\right) A_1.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{\left(\rho + \lambda_{HL} - \mu_H\right)\left(r + \eta - \mu_L\right) + \lambda_{LH}\left(r - \mu_H\right)} \\
A_1 &= \frac{\left(\rho + \xi + \lambda_{HL} + \lambda_{LH}\right)\left(r + \xi\right)}{\left(\rho + \xi + \lambda_{HL}\right)\left(r + \xi + \eta\right) + \lambda_{LH}\left(r + \xi\right)} \\
B_0 &= \frac{1 + \left(\rho + \lambda_{HL} - r\right) A_0}{\rho + \lambda_{HL} - \mu_H} \\
B_1 &= \frac{\left(r + \xi\right) + \left(\rho + \lambda_{HL} - r\right) A_1}{\rho + \xi + \lambda_{HL}}
\end{aligned}
$$

In addition, for any $i = 1, \ldots, 4$

$$
\begin{aligned}
\left(r + \lambda_{LH} + \eta - \mu_L\right) A_{i+1} &= -\left(\mu_L + \xi\right) A_{i+1}\gamma_i + \lambda_{LH} B_{i+1} + \frac{1}{2}\sigma^2 A_{i+1}\gamma_i\left(\gamma_i - 1\right) \\
\left(\rho + \lambda_{HL} - \mu_H\right) B_{i+1} &= \left(\rho + \lambda_{HL} - r\right) A_{i+1} - \left(\mu_H + \xi\right) B_{i+1}\gamma_i + \frac{1}{2}\sigma^2 B_{i+1}\gamma_i\left(\gamma_i - 1\right)
\end{aligned}
$$

If we multiply the equation for A_2 by γ_1, we get

$$(r + \lambda_{LH} + \eta - \mu_L)\,\gamma_1 A_2 = -\left(\mu_L + \xi\right) A_2 \gamma_1^2 + \lambda_{LH} B_2 \gamma_1 + \frac{1}{2}\sigma^2 A_2 \gamma_1^2 \left(\gamma_1 - 1\right)$$

when $\lambda_{LH} \neq 0$, from the equation for A_2 we have

$$\lambda_{LH} B_2 = \left[(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\,\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 \left(\gamma_1 - 1\right) \right] A_2.$$

Substituting in the equation for B_2 we obtain an expression for $B_2\gamma_1$ that can be then substituted back in the equation for A_2 (multiplied by γ_1). Canceling A_2, we obtain the characteristic equation for the homogeneous equation

$$\begin{aligned}
&\frac{1}{4}\sigma^4 \gamma_1^4 + \frac{1}{2}\sigma^2 \left(\mu_L + \mu_H + 2\xi + \sigma^2\right) \gamma_1^3 \\
&\quad + \left[\frac{1}{4}\sigma^4 - \frac{1}{2}\sigma^2 \left(\rho + \lambda_{HL} + r + \lambda_{LH} + \eta - 2\left(\mu_L + \mu_H + \xi\right)\right) + \left(\mu_L + \xi\right)\left(\mu_H + \xi\right) \right] \gamma_1^2 \\
&\quad + \left[\left(\mu_H + \xi + \frac{1}{2}\sigma^2 \right) \left(r + \lambda_{LH} + \eta - \mu_L\right) + \left(\rho + \lambda_{HL} - \mu_H\right) \left(\mu_L + \xi + \frac{1}{2}\sigma^2 \right) \right] \gamma_1 \\
&\qquad + \left(\rho + \lambda_{HL} - \mu_H\right)\left(r + \eta - \mu_L\right) + \left(r - \mu_H\right)\lambda_{LH} = 0
\end{aligned}$$

This equation has four roots.

Solution for $f \in (f_\dagger, f_H^b)$. In this case, we guess a solution of the form

$$\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4} \\
j_H(f) &= D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.
\end{aligned}$$

From the HJB equation for $j_H(f)$, we get that β_1 and β_2 are the roots for the quadratic equation

$$\frac{1}{2}\sigma^2 \beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2 \right) \beta + \mu_H - r - \lambda_{HL} = 0,$$

which are given by

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2},$$
$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2}.$$

From the equation for j_L, we get that β_3 and β_4 are given by the roots to the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\beta - \left(r + \lambda_{LH} + \eta - \mu_L\right) = 0,$$

which are

$$\beta_3 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} > 1,$$
$$\beta_4 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} < 0.$$

Matching coefficients, we get that

$$\begin{aligned}
D_0 &= \frac{1}{r + \lambda_{HL} - \mu_H} \\
D_1 &= \frac{r + \xi}{r + \xi + \lambda_{HL}} \\
C_0 &= \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L} \\
C_1 &= \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta} \\
C_2 &= \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + \left(\mu_L + \xi\right)\beta_1 - \frac{1}{2}\sigma^2\left(\beta_1 - 1\right)\beta_1} \\
C_3 &= \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + \left(\mu_L + \xi\right)\beta_2 - \frac{1}{2}\sigma^2\left(\beta_2 - 1\right)\beta_2}.
\end{aligned}$$

Boundary Conditions. We still need to determine the coefficients (A_i, B_i) for $i = 2, \ldots, 5$, the coefficients D_2, D_3, and C_4, C_5, as well as the thresholds $f_\dagger, f_H^b, f_L^b$.

We start considering $f \in (0, f_{\dagger})$. Under reasonable parameters, we have found that all four roots of the quartic characteristic equation are real, and that two of them are positive (let the positive roots be γ_1 and γ_2). If this is the case, the transversality conditions

$$\lim_{f \to 0} j_H(f) < \infty,$$
$$\lim_{f \to 0} j_L(f) < \infty,$$

imply that $A_4 = A_5 = B_4 = B_5 = 0$. Thus, we can write the value function as

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}$$
$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2},$$

where the coefficients A_0, A_1, B_0, B_1 have already been determined. Moreover, from the previous analysis we already have that for $i = 2, 3$

$$\Big[(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\, \gamma_{i-1} - \frac{1}{2}\sigma^2 \gamma_i\, (\gamma_{i-1} - 1) \Big] A_i = \lambda_{LH} B_i,$$

so the coefficients $\{B_2, B_3\}$ are immediately determined by the 2 free coefficients $\{A_2, A_3\}$.

Next, we consider the intervals $(f_{\dagger}, f_H^b)$ and $(f_{\dagger}, f_L^b)$. Here we have that $j_\theta(f)$ takes the form

$$j_L(f) = C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4}$$
$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.$$

where we have 4 free coefficients $\{C_2, C_3, C_4, C_5\}$ since $\{D_2, D_3\}$ are fully determined by $\{C_2, C_3\}$. Thus, we have to determine $(A_2, A_3, C_2, C_3, C_4, C_5)$ in addition to the free boundary $(f_{\dagger}, f_L^b, f_H^b)$; hence, we need 9 boundary conditions. The first boundary condition is the indifference condition

$$(\rho + \lambda_{HL} - r)\, j_L(f_{\dagger}) = (\rho - r)\, j_H(f_{\dagger}).$$

The value function must be continuously differentiable at $f_\dagger$ se we have the value matching and smooth pasting conditions at $f_\dagger$

$$
\begin{aligned}
j_H(f_\dagger -) &= j_H(f_\dagger +)\\
j_L(f_\dagger -) &= j_L(f_\dagger +)\\
j_H'(f_\dagger -) &= j_H'(f_\dagger +)\\
j_L'(f_\dagger -) &= j_L'(f_\dagger +).
\end{aligned}
$$

Finally, we have the value matching and smooth pasting conditions at the default boundary

$$
\begin{aligned}
j_L(f_L^b) &= 0\\
j_H(f_H^b) &= 0\\
j_L'(f_L^b) &= 0\\
j_H'(f_H^b) &= 0.
\end{aligned}
$$

Substituting the value function in these conditions, we get

$$
\begin{aligned}
A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= \frac{\rho - r}{\rho + \lambda_{HL} - r} \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right)\\
B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} &= D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2}\\
A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4}\\
-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} &= -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1}\\
-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} &= -C_1 + \beta_1 C_1 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1}\\
&\quad + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}
\end{aligned}
$$

and

$$
\begin{aligned}
C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} &= 0 \\
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0 \\
-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} &= 0 \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0.
\end{aligned}
$$

We can simply the above 9 equations into 3 equations and only solve the three unknowns $(f_\dagger, f_L^b, f_H^b)$:

From

$$
\begin{aligned}
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0, \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0,
\end{aligned}
$$

we know

$$
\begin{aligned}
D_2 &= \frac{\frac{\beta_2}{r + \lambda_{HL} - \mu_H} - (\beta_2 - 1) \frac{r + \xi}{r + \xi + \lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_1} (\beta_1 - \beta_2)} \\
D_3 &= \frac{\frac{\beta_1}{r + \lambda_{HL} - \mu_H} - (\beta_1 - 1) \frac{r + \xi}{r + \xi + \lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_2} (\beta_2 - \beta_1)}.
\end{aligned}
$$

Then we know C_2, C_3 from

$$
\begin{aligned}
C_2 &= \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi) \beta_1 - \frac{1}{2}\sigma^2 (\beta_1 - 1) \beta_1} \\
C_3 &= \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi) \beta_2 - \frac{1}{2}\sigma^2 (\beta_2 - 1) \beta_2}
\end{aligned}
$$

From

$$B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} = D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2}$$

$$-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} = -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1},$$

we know

$$B_2 = \frac{-D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} + B_1 - \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1}}{\gamma_1 (f_\dagger)^{\gamma_1 - 1}},$$

$$B_3 = \frac{D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} - \left(B_0 - B_1 f_\dagger + \frac{1}{\gamma_1}\left(-D_1 f_\dagger + \beta_1 D_2 (f_\dagger)^{\beta_1} + \beta_2 D_3 (f_\dagger)^{\beta_2} + B_1 f_\dagger\right)\right)}{\left(1 - \frac{\gamma_2}{\gamma_1}\right)(f_\dagger)^{\gamma_2}}$$

Then we know A_2 and A_3 where

$$A_2 = \frac{\lambda_{LH} B_2}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1)}.$$

$$A_3 = \frac{\lambda_{LH} B_3}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma_2 - \frac{1}{2}\sigma^2 \gamma_2 (\gamma_2 - 1)}$$

From

$$C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} = 0$$

$$-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} = 0$$

we know

$$C_4 = \frac{C_1 - \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} - \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}},$$

$$C_5 = \frac{-C_0 + C_1 f_L^b - C_2 \left(f_L^b\right)^{\beta_1} - C_3 \left(f_L^b\right)^{\beta_2} - \frac{C_1 f_L^b - \beta_1 C_2 \left(f_L^b\right)^{\beta_1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2}}{\beta_3}}{\left(1 - \frac{\beta_4}{\beta_3}\right)\left(f_L^b\right)^{\beta_4}}$$

Therefore, we only need to solve $(f_\dagger, f_L^b, f_H^b)$ from the following 3 equations:

$$\begin{aligned}
A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= \frac{\rho - r}{\rho + \lambda_{HL} - r} \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right) \\
A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4} \\
-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} &= -C_1 + \beta_1 C_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1} \\
&\quad + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}
\end{aligned}$$

Limit when $\sigma \to 0$

As in the case where the low state is absorbing, we can obtain a more explicit solution for the equilibrium in the limit when $\sigma \to 0$.

Proposition 11 (Limit long-term debt issuance policy)**.** *Suppose that $\mu_L + \xi < 0$, $\mu_H + \xi > 0$, and*

$$(r + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH} \geq 0.$$

In the limit when $\sigma \to 0$, the issuance policy is

$$g_\theta(f) = \frac{\rho - r}{\gamma - 1} \left[g_0 + g_1 \left(\frac{f}{f_\dagger} \right)^{-(\gamma - 1)} \right] \mathbb{1}_{\{f < f_\dagger, \theta = H\}}$$

where g_0 and g_1 are positive coefficients and $\gamma > 1$ is the unique positive root of

$$\gamma^2 + \left(\frac{\rho + \lambda_{HL} - \mu_H}{\mu_H + \xi} - \frac{r + \lambda_{LH} + \eta - \mu_L}{-(\mu_L + \xi)} \right) \gamma - \frac{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH}}{-(\mu_L + \xi)(\mu_H + \xi)} = 0.$$

Consider the case when $\mu_L + \xi < 0 < \mu_H + \xi$. The characteristic equation for γ_i converges to the quadratic equation

$$\begin{aligned}
(\mu_L + \xi)(\mu_H + \xi)\gamma_1^2 + \left[(\mu_H + \xi)(r + \lambda_{LH} + \eta - \mu_L) + (\rho + \lambda_{HL} - \mu_H)(\mu_L + \xi) \right] \gamma_1 \\
+ (\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH} = 0
\end{aligned}$$

The present value of cash flow is finite given the creditors' discount rate only if

$$(r+\lambda_{HL}-\mu_H)(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH}>0,$$

which implies that the quadratic equation has one negative and one positive root. Let γ be the positive root, which can be verified to always be greater than 1. Similarly, the roots β_i converge to

$$\begin{aligned}
\beta_1 &= \infty \\
\beta_2 &= -\frac{r+\lambda_{HL}-\mu_H}{\mu_H+\xi} \\
\beta_3 &= \frac{r+\lambda_{LH}+\eta-\mu_L}{-(\mu_L+\xi)} \\
\beta_4 &= -\infty.
\end{aligned}$$

Thus the solution to the HJB equation on $(0, f_\dagger)$, becomes

$$\begin{aligned}
j_L(f) &= A_0 - A_1 f + A_2 f^\gamma \\
j_H(f) &= B_0 - B_1 f + B_2 f^\gamma,
\end{aligned}$$

while the solution for $(f_\dagger, f_H^b)$ and $(f_\dagger, f_L^b)$ becomes

$$\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_3 f^{\beta_2} + C_4 f^{\beta_3} \\
j_H(f) &= D_0 - D_1 f + D_3 f^{\beta_2}.
\end{aligned}$$

The coefficient A_2, B_2 are given by

$$A_2 = \frac{\lambda_{LH}}{(r+\lambda_{LH}+\eta-\mu_L)+(\mu_L+\xi)\gamma}\frac{B_1-D_1}{\gamma\,(f_\dagger)^{\gamma-1}}.$$

From the continuity and smoothness of $j_H(f)$ at $f_\dagger$, we know

$$\begin{aligned}
B_0 - B_1 f_\dagger + B_2 f_\dagger^{\gamma} &= D_0 - D_1 f_\dagger + D_3 f_\dagger^{\beta_2} \\
-B_1 + \gamma B_2 f_\dagger^{\gamma-1} &= -D_1 + \beta_2 D_3 f_\dagger^{\beta_2-1},
\end{aligned}$$

which implies that

$$\begin{aligned}
B_2 &= \frac{\beta_2 (D_0 - B_0) + (1-\beta_2)(D_1 - B_1) f_\dagger}{(\beta_2 - \gamma) f_\dagger^{\gamma}} \\
D_3 &= \frac{(1-\gamma)(D_1 - B_1) f_\dagger + \gamma (D_0 - B_0)}{(\beta_2 - \gamma) f_\dagger^{\beta_2}}.
\end{aligned}$$

The coefficients C_3, C_4 are

$$\begin{aligned}
C_3 &= \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\beta_2} \\
C_4 &= \frac{C_1 - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}}.
\end{aligned}$$

Substituting in the HJB equation, we can write the solution for $f \in [0, f_\dagger]$ as

$$\begin{aligned}
j_L(f) &= A_0 - A_1 f + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\gamma} \frac{\beta_2 (D_0 - B_0) + (1-\beta_2)(D_1 - B_1) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\gamma} \\
j_H(f) &= B_0 - B_1 f + \frac{\beta_2 (D_0 - B_0) + (1-\beta_2)(D_1 - B_1) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\gamma}.
\end{aligned}$$

For $f > f_\dagger$, we can write

$$\begin{aligned}
j_H(f) &= D_0 - D_1 f + \left[\frac{(1-\gamma)(D_1 - B_1) f_\dagger + \gamma (D_0 - B_0)}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2} \\
&= D_0 \left[1 + \frac{\gamma}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\beta_2}\right] - D_1 f \left[1 - \frac{1-\gamma}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\beta_2 - 1}\right] - \left[\frac{(1-\gamma) B_1 f_\dagger + \gamma B_0}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2}
\end{aligned}$$

and

$$
\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_3 f^{\beta_2} + C_4 f^{\beta_3} \\
&= C_0 - C_1 f \left[1 - \frac{1}{\beta_3}\left(\frac{f}{f_L^b}\right)^{\beta_3 - 1}\right] + C_3 f^{\beta_2}\left[1 - \frac{\beta_2}{\beta_3}\left(\frac{f}{f_L^b}\right)^{\beta_3 - \beta_2}\right].
\end{aligned}
$$

where we recollect that the constant $A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1$ are

$$
\begin{aligned}
A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + \lambda_{LH}(r - \mu_H)} \\
A_1 &= \frac{(\rho + \xi + \lambda_{HL} + \lambda_{LH})(r + \xi)}{(\rho + \xi + \lambda_{HL})(r + \xi + \eta) + \lambda_{LH}(r + \xi)} \\
B_0 &= \frac{1 + (\rho + \lambda_{HL} - r) A_0}{\rho + \lambda_{HL} - \mu_H} \\
B_1 &= \frac{(r + \xi) + (\rho + \lambda_{HL} - r) A_1}{\rho + \xi + \lambda_{HL}} \\
D_0 &= \frac{1}{r + \lambda_{HL} - \mu_H} \\
D_1 &= \frac{r + \xi}{r + \xi + \lambda_{HL}} \\
C_0 &= \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L} \\
C_1 &= \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta}.
\end{aligned}
$$

Finally, we get the equations determining the thresholds $f_\dagger, f_H^b, f_L^b$ which are given now by

$$
\begin{aligned}
j_L(f_\dagger -) &= j_L(f_\dagger +) \\
j_L(f_L^b) &= 0 \\
j_H(f_H^b) &= 0
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
A_0 - A_1 f_\dagger + A_2 f_\dagger^{\gamma} &= C_0 - C_1 f_\dagger + C_3 {f_\dagger}^{\beta_2} + C_4 {f_\dagger}^{\beta_3} \\
C_0 - C_1 f_L^b + C_3 {f_L^b}^{\beta_2} + C_4 {f_L^b}^{\beta_3} &= 0 \\
D_0 - D_1 f_H^b + D_3 {f_H^b}^{\beta_2} &= 0.
\end{aligned}
$$

Issuance Function: Before deriving the issuance function we need to derive the debt price, which is given by $p_\theta(f) = -j'_\theta(f)$. Taking derivatives for $f \in (0, f_\dagger)$ we get

$$
\begin{aligned}
p_L(f) &= A_1 + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\gamma} \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1} \\
p_H(f) &= B_1 + \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}.
\end{aligned}
$$

Finally, we compute the issuance function. We need to find $p'_H(f)$ for $f \in (0, f_\dagger)$. This expression is given by

$$
-f p'_H(f) = -\frac{\gamma(\gamma - 1)}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}.
$$

Next, we compote $p_H(f) - p_L(f)$ which is

$$
p_H(f) - p_L(f) = B_1 - A_1 + \frac{r + \eta - \mu_L + (\mu_L + \xi)\,\gamma}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\gamma} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \frac{\gamma}{\gamma - \beta_2} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}
$$

From here we get that

$$
\begin{aligned}
g_H(f) &= \frac{(\rho - r)(p_H(f) - p_L(f))}{-f p'_H(f)} \\
&= \frac{\rho - r}{\gamma - 1} \left[g_0 + g_1 \left(\frac{f}{f_\dagger}\right)^{-(\gamma - 1)} \right]
\end{aligned}
$$

where

$$g_0 = -\frac{r+\eta-\mu_L+(\mu_L+\xi)\gamma}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\gamma}$$
$$g_1 = -\left(1-\frac{\beta_2}{\gamma}\right)\frac{(B_1-A_1)f_\dagger}{\beta_2\left(D_0-B_0\right)+\left(1-\beta_2\right)\left(D_1-B_1\right)f_\dagger}$$

Finally, noting that $p_L(0)=A_1$, $p_H(0)=B_1$, and $B_0=j_H(0)$, an substituting the relations

$$(\rho-r)j_H(0)-\left(\rho+\lambda_{HL}-r\right)j_L(0)=1-\left(r+\lambda_{HL}-\mu_H\right)j_H(0)$$
$$(\rho-r)p_H(0)-\left(\rho+\lambda_{HL}-r\right)p_L(0)=(r+\xi)-\left(r+\xi+\lambda_{HL}\right)p_H(0),$$

we can write

$$g_1=\left(\frac{1}{\gamma}+\frac{\mu_H+\xi}{r+\lambda_{HL}-\mu_H}\right)\frac{(r+\lambda_{HL}-\mu_H)(p_H(0)-p_L(0))f_\dagger}{(\rho-r)j_H(0)-\left(\rho+\lambda_{HL}-r\right)j_L(0)-\Big[(\rho-r)p_H(0)-\left(\rho+\lambda_{HL}-r\right)p_L(0)\Big]f_\dagger}$$

Letting

$$\varphi(f)\equiv(\rho-r)j_H(f)-\left(\rho+\lambda_{HL}-r\right)j_L(f)$$

we can write

$$(\rho-r)j_H(0)-\left(\rho+\lambda_{HL}-r\right)j_L(0)-\Big[(\rho-r)p_H(0)-\left(\rho+\lambda_{HL}-r\right)p_L(0)\Big]f_\dagger=\varphi(0)+\varphi'(0)f_\dagger.$$

For $f\in(0,f_\dagger)$,

$$\varphi''(f)=\left(\rho+\lambda_{HL}-r\right)\left[\frac{\rho-r}{\rho+\lambda_{HL}-r}-\frac{\lambda_{LH}}{r+\lambda_{LH}+\eta-\mu_L+\left(\mu_L+\xi\right)\gamma}\right]\gamma(\gamma-1)\mathcal{Q}f^{\gamma-2},$$

where

$$\mathcal{Q}\equiv\frac{\gamma}{\gamma-\beta_2}\frac{\beta_2\left(D_0-B_0\right)+\left(1-\beta_2\right)\left(D_1-B_1\right)f_\dagger}{f_\dagger}\left(\frac{1}{f_\dagger}\right)^{\gamma-1}.$$

$j_\theta(f)$ is strictly convex only if $\mathcal{Q}>0$. Hence, the sign of φ'' is determined by the sign of the term within the parenthesis. The coefficient γ is the positive root of the quadratic equation

$$\gamma^2 + \left(\frac{\rho + \lambda_{HL} - \mu_H}{\mu_H + \xi} - \frac{r + \lambda_{LH} + \eta - \mu_L}{-(\mu_L + \xi)} \right) \gamma - \frac{(\rho + \lambda_{HL} - \mu_H)\,(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH}}{-(\mu_L + \xi)(\mu_H + \xi)} = 0.$$

This equation can be rewritten more conveniently as

$$\frac{\lambda_{LH}}{r + \lambda_{LH} + \eta + (\gamma - 1)\mu_L + \gamma\xi} = \frac{\rho + \lambda_{HL} + (\gamma - 1)\mu_H + \xi\gamma}{\rho + \lambda_{HL} - r} > \frac{\rho - r}{\rho + \lambda_{HL} - r},$$

which implies that the term within the parenthesis in $\varphi''(f)$ is negative. Thus, we conclude that $\varphi(f)$ is concave on $[0, f_\dagger]$, so $\varphi(f_\dagger) \leq \varphi(0) + \varphi'(0) f_\dagger$. By construction, $\varphi(f_\dagger) = 0$, so it follows that $\varphi(0) + \varphi'(0) f_\dagger \geq 0$, which means that $g_1 > 0$. Moreover, from the previous equation for γ we also get that $r + \lambda_{LH} + \eta + (\gamma - 1)\mu_L + \gamma\xi > 0$ and

$$r + \eta + (\gamma - 1)\mu_L + \gamma\xi = -\lambda_{LH} \frac{r + (\gamma - 1)\mu_H + \xi\gamma}{\rho + \lambda_{HL} + (\gamma - 1)\mu_H + \xi\gamma} < 0,$$

so it follows that $g_0 > 0$.

B.5 The microfoundation of the disaster shock

Now we show that the disaster shock can be microfounded by a model with three states, high (H), low (L), and disaster (ℓ), where $\mu_H > \mu_L > \mu_\ell$. In other words, the low state can still get worse. As before, let λ be the transition intensity from H to L and η be the one from L to ℓ. We are interested in the condition such that in the low state L, the borrower optimally choose to issue risky short-term debt. In other words, the corresponding $f_\dagger$ is zero in the low state L. To do so, we only need to study the value functions in $\theta = L$ and $\theta = \ell$.

When $\theta = L$, the HJB is

$$(\rho + \eta) \, j_L(f) = \max_{\{0 \le d_L \le j_L\}} 1 - (r + \xi) \, f + (\rho + \eta - y) \, d_\theta(f) + \eta \, (j_\ell(f) - d_L(f))^+$$
$$+\mu_L \left(j_L(f) - j_L'(f) \, f \right) + \frac{1}{2}\sigma^2 f^2 j_L''(f) - \xi f j_L'(f)$$
$$\Rightarrow (\rho + \eta - \mu_L) \, j_L(f) = \max_{\{0 \le d_L \le j_L\}} 1 - (r + \xi) \, f + (\rho + \eta - y_L) \, d_L(f)$$
$$+\eta \, (j_\ell(f) - d_L(f))^+ - (\mu_L + \xi) \, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)$$

The HJB when $\theta = \ell$ is

$$(\rho - \mu_\ell) \, j_\ell(f) = \max_{\{0 \le d_\ell \le j_\ell\}} 1 - (r + \xi) \, f + (\rho - y_\ell) \, d_\ell(f) - (\mu_\ell + \xi) \, f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \, .$$
$$= 1 - (r + \xi) \, f + (\rho - r) \, j_\ell(f) - (\mu_\ell + \xi) \, f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f) \, ,$$

which implies that

$$(r - \mu_\ell) \, j_\ell(f) = 1 - (r + \xi) \, f - (\mu_\ell + \xi) \, f j_\ell'(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''(f)$$

The short rate is

$$y_L(d, f) = r + \eta \mathbb{1}_{d > j_\ell(f)}$$

and

$$y_\ell(d, f) = r.$$

In state ℓ, we have $d = j_\ell(f)$. In state L we have

1. If $d_L = j_\ell(f)$, the flow benefit of issuing short-term debt is

$$(\rho + \eta - r) \, j_\ell(f)$$

2. If $d_L = j_L(f)$, the flow benefit of issuing short-term debt is

$$(\rho - r)\, j_L(f)$$

3. Hence, $d = j_\ell$ is optimal if

$$(\rho + \eta - r)\, j_\ell(f) \geq (\rho - r) j_L(f)$$

We can conclude that

- If $(\rho + \eta - r)\, j_\ell(f) \geq (\rho - r)\, j_L(f)$ the HJB equation is

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L\,(f) &= 1 - (r + \xi)\, f + (\rho + \eta - r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi)\, f j_L'\,(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

  which can be reduced to

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L(f) &= 1 - (r + \xi)\, f + (\rho + \eta - r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

- If $(\rho + \eta - r)\, j_\ell(f) < (\rho - r)\, j_H(f)$ the HJB equation is

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L\,(f) &= 1 - (r + \xi)\, f + (\rho - r)\, j_L(f) \\
&\quad - (\mu_L + \xi)\, f j_L'\,(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

which can be reduced to

$$\left(\left(r+\eta\right)-\mu_\ell\right)j_\ell\left(f\right)=1-\left(r+\xi\right)f-\left(\mu_\ell+\xi\right)fj_\ell'\left(f\right)+\frac{1}{2}\sigma^2f^2j_\ell''\left(f\right)$$
$$\left(\left(r+\eta\right)-\mu_L\right)j_L(f)=1-\left(r+\xi\right)f-\left(\mu_L+\xi\right)fj_L'(f)+\frac{1}{2}\sigma^2f^2j_L''\left(f\right)$$

- In state ℓ, the default boundary is f_ℓ^b, satisfying

$$j_\ell(f_\ell^b)=0$$
$$j_\ell'(f_\ell^b)=0.$$

Since $j_L\left(0\right)=\frac{1}{\rho+\eta-\mu_L}\frac{\rho+\eta-\mu_\ell}{r-\mu_\ell}$ and $j_\ell\left(0\right)=\frac{1}{r-\mu_\ell}$, to let $f_\dagger=0$ in the low state, it is necessary and sufficient to have

$$\left(\rho+\eta-r\right)j_\ell(0)\leq\left(\rho-r\right)j_L(0).$$

That is

$$\left(\rho+\eta-r\right)\frac{1}{\left(r+\eta\right)-\mu_\ell}\leq\left(\rho-r\right)\frac{1}{\rho+\eta-\mu_L}\frac{\rho+\eta-\mu_\ell}{r-\mu_\ell}.$$

From here, we get

$$\eta^2+\left(\rho-\mu_L\right)\eta-\left(\rho-r\right)\left(\mu_L-\mu_\ell\right)\leq 0.$$

This implies that

$$0<\eta\leq\bar{\eta}=\frac{-\left(\rho-\mu_L\right)+\sqrt{\left(\rho-\mu_L\right)^2+4\left(\rho-r\right)\left(\mu_L-\mu_\ell\right)}}{2}. \tag{A.67}$$