Debt Maturity Management

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Abstract

This paper studies how a borrower issues long- and short-term debt in response to shocks to the fundamental value. Our theory highlights the tradeoff between commitment and hedging. Short-term debt protects creditors from future dilution and incentivizes the borrower to reduce leverage after negative shocks. Long-term debt postpones default and allows the borrower time to recover after large negative shocks, thereby providing hedging. When borrowers are in distress, they rely on short-term debt; however, they issue both types of debt during more normal periods. Our model generates novel implications for the dynamic adjustment of debt maturities.

Keywords: debt maturity; capital structure; risk management; durable-goods monopoly.

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1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in several aspects; an important one is its maturity. Borrowing can be short, as in the case of trade credit, or long, as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing following shocks to their fundamental value?

The academic literature falls behind in providing a useful and tractable framework to study these questions despite their obvious importance. For example, the Leland model (Leland, 1994) and the vast follow-up literature typically assume that (1) all debt has the same (expected) maturity and (2) the borrower either commits to the total leverage or may only increase leverage after retiring all existing debt and paying some exogenous issuance cost.[1] Although these assumptions simplify the analysis, they are inconsistent with the ample empirical evidence that borrowers often issue a mix of long- and short-term debt simultaneously and that adjusting the outstanding debt's maturity profile can take some time to accomplish.

This paper introduces a simple and tractable framework to address these questions. Our model features two types of debt: short- and long-term. The central tension arises from the trade-off between commitment and hedging. Without a commitment to future issuances, a borrower would have incentives to dilute existing long-term debt. In contrast, short-term debt cannot be diluted as it matures before the borrower can borrow again, providing a commitment mechanism. Meanwhile, long-term debt offers crucial hedging benefits: following a large negative shock, long-term debt allows the losses to the firm's value to be shared with equity holders, potentially avoiding immediate default. Even for a risk-neutral borrower, the costs associated with bankruptcy imply that hedging through delayed bankruptcy can be advantageous, creating a demand for long-term debt. The optimal composition of long-term and short-term debt balances the dilution costs of long-term debt and the hedging benefits it provides.

More specifically, a risk-neutral borrower has assets that generate an income flow that follows a geometric Brownian motion (GBM) whose drift switches between two different regimes representing the upturn and downturn. The expected growth rate of the income is high in an upturn but low in a downturn. A transition from the upturn to the downturn is a large negative shock, interpreted as the downside risk. Creditors are competitive, risk-neutral, and more patient than the borrower. The difference in patience motivates the borrower to issue debt. Two types of debt are available. The short-term debt matures instantaneously (i.e., has zero maturity) and needs to be continuously

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss later.

rolled over. Long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing both types of debt at any time to adjust the maturity profile of the outstanding debt.

Our paper shares with the existing literature (Fama and Miller, 1972; Black and Scholes, 1973; Admati et al., 2018) that an uncommitted borrower with outstanding long-term debt is always tempted to issue more debt to dilute long-term creditors. Should the borrower be solely financed by long-term debt, she would never reduce leverage after negative shocks, even if such a reduction could enhance the firm value. In equilibrium, creditors anticipate future dilution and long-term debt prices adjust downwards to the level where the borrower cannot capture any benefit (DeMarzo and He, 2021). By contrast, short-term debt matures before the borrower can issue debt again and, therefore, does not suffer from dilution. Following negative shocks, short-term debt incentivizes the borrower to reduce the leverage, even though she has not committed to doing so. This happens because short-term debt constantly resets, so the borrower reaps all the benefits from delevering, whereas in the case of long-term debt, the delevering benefits are shared with existing long-term creditors.

Given short-term debt's advantage in addressing the commitment problem, the borrower naturally benefits from issuing it. Indeed, our results show that during downturns, the borrower can costlessly commit to not issuing long-term debt by issuing a large amount of short-term debt, fully exhausting their borrowing capacity. However, during upturns, this commitment via short-term debt comes at a cost. The potential arrival of a future downturn introduces an interesting trade-off in issuing short-term debt. On one hand, the borrower can issue enough short-term debt to exhaust their borrowing capacity, fully capturing the benefits of debt issuance. But this makes the short-term debt *risky*, leading to immediate default if a downturn occurs before the debt matures. Alternatively, the borrower can issue a *safe* level of short-term debt, preserving some borrowing capacity to avoid default even if the state switches to a downturn. However, this safe level means the borrower can no longer fully capture the benefits of debt issuance. Our model shows that when a borrower has a substantial amount of outstanding long-term debt, she opts for the risky level of short-term debt. In contrast, if the outstanding long-term debt is low, she chooses the safe level of short-term debt. The rationale is that the incremental increase in default probability represents the marginal cost of issuing short-term debt. When there is a high level of outstanding long-term debt, this marginal cost is low, and vice versa.

The issuance of long-term debt during upturns is directly related to the choice of short-term debt level. When the borrower optimally issues the risky (and higher) level of short-term debt, any long-term debt issued is exposed to the same downside risk as the short-term debt. In this scenario, long-term debt offers no hedging benefit but additionally suffers from dilution costs, so the borrower has no incentive to issue it. Conversely, when the borrower optimally issues the safe (and lower) level of short-term debt, she can benefit from also issuing long-term debt. Intuitively, if a downturn arrives, the resulting loss in the firm's fundamental value is shared between the borrower and long-term creditors only; short-term creditors' claims must be fully paid unless the borrower defaults. Therefore, by allowing the borrower to share losses from an economic regime shift, long-term debt provides valuable hedging benefits before a downturn occurs. Notably, the borrower values these hedging benefits from long-term debt despite being risk-neutral, because the costs associated with bankruptcy make such hedging advantageous, as acknowledged in previous literature (Smith and Stulz, 1985).[2]

Our model generates several novel predictions about the dynamics of corporate debt maturity structure. First, we find that firms increase their use of short-term debt after negative shocks, as they seek to mitigate commitment problems. However, firms return to issuing long-term debt as economic conditions normalize to regain the hedging benefits. This prediction aligns with evidence from the financial crisis of firms issuing shorter-maturity debt when nearing default (Brunnermeier (2009); Krishnamurthy (2010)).

Second, our model predicts that debt maturity is countercyclical – the average maturity of new issuances is longer during economic expansions than recessions. This is consistent with empirical findings that market leverage is countercyclical (Halling et al. (2016)), while debt maturity structure is procyclical (Mian and Santos (2018); Chen et al. (2021)).

Third, the model implies that default risk among outstanding debt increases with the amount of long-term debt, while default risk is negatively related to the maturity of newly issued debt. This highlights the importance of differentiating between the stock of outstanding debt and the flow of new issuances when examining the link between debt maturity and credit risk. The stock of outstanding debt reflects cumulative historical financing decisions, while the flow of new issuances reflects current financing conditions and incentives. Our model suggests that these two aspects of debt maturity can have opposing relationships with default risk. Almeida et al. (2011) find that, during the financial crisis, borrowers with a significant amount of long-term debt maturing in the near future were more likely to be pushed closer to default and to reduce their real economic activity.

Finally, we show that firms more exposed to large downside risks are predicted to have longer debt maturities on average, as they seek to hedge against these risks. This insight highlights the key role of debt maturity in firms' overall risk management strategies, a central theme of our model.

[2]The literature on risk management highlights that due to the cost of bankruptcy, even a risk-neutral borrower can benefit from hedging, (Smith and Stulz, 1985; Bolton et al., 2011; Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To our knowledge, no previous work has established the link between maturity management and risk management in a corporate finance setting.

We discuss how these predictions compare to existing empirical evidence and suggest directions for future empirical work to test our model.

Related literature

Our paper builds on the literature of dynamic corporate finance pioneered by Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019). Important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the related problem when the borrower can only issue zero-maturity debt (see also Bolton et al. (2021), who further model costly equity issuance). In these papers, the borrower can only issue one type of debt, so the tradeoff between borrowing long and short is not explicitly studied. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed, and the borrower can choose between two types of exponentially maturing debt. Our paper differs in two aspects. First, we allow for flexibility in adjusting total leverage. Second, we model short-term debt as debt that matures instantaneously. The different approaches in modeling short-term debt render the mechanisms of the two papers drastically different. Whereas we emphasize the tradeoff between commitment and hedging, their paper focuses on rollover losses and dilution. Malenko and Tsoy (2020) study the role of reputation and predict interior optimal debt maturity. Brunnermeier and Yogo (2009) also study debt maturity in the context of liquidity risk, and they show long-term debt is optimal if the firm is close to default (or close to debt restructuring as in their paper). Our results are the opposite: the borrower will issue exclusively short if she is close to default.

The insight that long-term debt can be diluted has been recognized by Fama and Miller (1972) and Black and Scholes (1973). More recently, Admati et al. (2018) formalized this argument and showed that the borrower financed by long-term debt never voluntarily reduces leverage even after negative shocks to the fundamentals. Brunnermeier and Oehmke (2013) show equity and short-term debt can dilute long-term debt's recovery value in bankruptcy. Our paper rules out this mechanism by assuming zero recovery value in the benchmark model. Instead, we focus on dilution outside the bankruptcy, which comes exclusively from the borrower's lack of commitment to issuance and default.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. One advantage of a fully-dynamic setup is that it allows us to make empirical predictions regarding the stock (existing debt) and the flow (new issuance) of debt maturity. The insight that short-term debt resolves the lack of commitment is also present in another related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runnable feature of short-term debt. As DeMarzo (2019) shows, the borrower's problem when she only issues long-term debt is related to the Coase conjecture on the durable-goods monopoly (Coase, 1972). In our context, the borrower is the monopolist, and long-term debt is the durable goods. Short-term debt resembles the leasing solution (Bulow, 1982) to the durable-goods monopoly problem. Relatedly, Gertner and Scharfstein (1991) show that conditional on financial distress, short-term debt has a higher market leverage for the same face value than long-term debt and therefore leads to more ex-post debt overhang (also see Diamond and He (2014)).

The hedging benefit of long-term debt is also related to the literature on fiscal policy and sovereign debt. For example, Angeletos (2002) shows that the ex-post variations in the market value of public debt hedge the government against bad fiscal conditions. Buera and Nicolini (2004) shows that state-contingent debt can be synthetically constructed by a rich maturity structure of non-contingent debt (also see Gale (1990)).

Aguiar et al. (2019) argue that in the absence of hedging motives, a lack of commitment would make long-term debt so expensive that a borrower would never actively issue it. However, they demonstrate through a simple example that the presence of hedging could incentivize a borrower to issue long-term debt. In contrast, we develop a fully dynamic model incorporating hedging, in which the borrower issues a combination of long- and short-term debt. We show that during periods with a high risk of default, the force described in Aguiar et al. (2019) dominates, and the borrower relies exclusively on short-term financing. However, in normal periods with a lower default risk, other considerations such as hedging become more influential, and borrowers actively issue long-term debt.[3]

Arellano and Ramanarayanan (2012) calibrate a quantitative model of sovereign borrowing with two debt maturities. Our paper differs from their analysis in several key aspects. First, we develop a tractable model that provides a closed-form characterization of the equilibrium and highlights the fundamental economic forces driving the choice between different maturities. The tractability of our continuous-time setting allows us to fully characterize the dynamics of debt maturity and its adjustment to shocks, as well as to study the impact of alternative risk-sharing mechanisms, such as renegotiation and insurance contracts, on the choice of debt maturity. Second, our framework identifies the specific type of risk – large downside risk – that borrowers seek to hedge against by issuing long-term debt. In contrast to their paper, where the representative agent is risk-averse and

[3] Niepelt (2014) studies a related question, but default decisions on different maturities can be independent. Bigio et al. (2021) study debt maturity management under liquidity costs but without dilution. In their model, the borrower's choice depends on the bond demand curve, which is micro-founded via search (Duffie et al., 2005).

can benefit from future fluctuations in bond spreads, our paper emphasizes that long-term debt's ability to delay default provides better hedging against large downside risks, which is particularly valuable during good economic times.

Throughout the paper, we use the term hedging to refer generically to any position that the borrower could take to manage its risks. However, it is worth noting that the literature sometimes distinguishes between hedging and insurance as two different approaches to risk management (Bodie and Merton, 2000, Ch. 10). Specifically, hedging involves eliminating all fluctuations in the value of an asset or portfolio, effectively locking in a fixed return. This is typically achieved through the use of financial instruments that offset any potential gains or losses. In contrast, insurance is considered to be a risk management practice that focuses on covering potential losses while still allowing to benefit from potential gains. In this context, while a risk-averse borrower, such as the one described in Arellano and Ramanarayanan (2012), benefits from eliminating all fluctuations (that is, has incentives to fully hedge the shocks), our risk-neutral borrower only benefits from insuring against downside risk, as they seek to avoid states of financial distress.

2 The Model

2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower, often interpreted as a firm. The relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk-neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate ρ, which exceeds r, the discount rate of creditors.

The borrower's asset generates earnings at a rate X_t, which evolves according to:

$$\frac{dX_t}{X_{t-}} = \mu_{\theta_t} dt + \sigma dB_t - \mathbb{1}_{\{\theta_t = L\}} dN_t, \tag{1}$$

where B_t is a standard Brownian motion, N_t is a Poisson process with arrival rate η, and $\theta_t \in \{H, L\}$ represents the regime with $\theta_0 = H$. At a random time τ_λ, which arrives with intensity λ, the regime switches to L and stays unchanged.[4] The drift μ_{θ_t} differs across the two regimes with $\mu_L < \mu_H$, so that the high state H is associated with a higher expected growth rate in the borrower's cash flow. Below, we refer to the high state as the *upturn*, the low state as the *downturn*, and the regime switch as the *downside* risk. In addition, a *disaster* shock hits in the downturn at a random time

[4] In Internet Appendix B.5, we study the model in which state L is non-absorbing and show that our main results carry over.

τ_η that arrives at a Poisson intensity η, upon which the cash flow X_t permanently drops to zero. This disaster shock does not play a significant role if we consider a borrower that starts with an outstanding amount of long-term debt. However, we introduce it so an initially unlevered borrower has incentives to issue long-term debt (see section 3.3 for details). Under some conditions, the model with the disaster is similar to one with more than two regimes.[5]

2.2 Debt Maturity Structure

The difference between the discount rates $\rho - r$ offers benefits for the borrower to issue debt.[6] Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the outstanding debt maturity structure. In particular, we do not restrict the borrower to commit to a particular issuance path but instead let the issuance decisions be made at each instant.

All *short-term debt* matures instantaneously and, therefore, needs to be continuously rolled over. The fact that short-term debt matures instantaneously implies that the borrower does not have the chance to issue new debt before the existing short-term debt matures. We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt\downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time t and let y_{t-} be the associated short rate. *Long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing bonds with coupon rate r and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. Let F_t be the aggregate face value of long-term debt outstanding at time t.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume the bankruptcy cost is 100%. In other words, creditors cannot recover any value once the borrower defaults.

2.3 Valuation

Let τ_b be the endogenous time the borrower chooses to default. We define p_t as the price per unit of the face value. The break-even condition implies that for $t < \tau_b$,

$$p_t = \mathbb{E}_t\left[\int_t^{\tau_\xi \wedge \tau_b} e^{-r(s-t)} r ds + e^{-r(\tau_\xi - t)} \mathbb{1}_{\{\tau_b > \tau_\xi\}}\right], \tag{2}$$

where τ_ξ is long-term debt's (stochastic) maturing date. The two components in the expression correspond to the coupon and final payments. The short rate y_{t-} depends on the borrower's

[5]See Internet Appendix B.6 for details.

[6]The difference can be related to liquidity differences, contracting costs, or market segmentation. An alternative setup is to introduce tax shields, and the results are similar.

equilibrium default decisions, and the break-even condition suggests:

$$y_{t-} = r + \lim_{dt \downarrow 0} \frac{\Pr_{t-dt}\left(\tau_b \leq t | \tau_b > t - dt\right)}{dt}, \tag{3}$$

where the second term on the right-hand side is the risk premium compensating creditors for the hazard rate of default. According to (3), y_{t-} compensates the creditors for the probability of default occurring between $t - dt$ and t. For example, if in the upturn, short-term creditors expect default only upon a transition to the downturn, then $y_{t-} = r + \lambda$. Similarly, if in the downturn, short-term debt only defaults when the disaster shock hits, then $y_{t-} = r + \eta$.

Over a short time interval $[t, t + dt)$, the net cash flow to the borrower is

$$\Big[X_t - (r + \xi) F_t - y_{t-} D_{t-}\Big] dt + p_t dG_t + dD_t, \tag{4}$$

where $(r + \xi) F_t$ is the interest and principal payments to long-term creditors, $y_{t-} D_{t-}$ the interest payments to short-term creditors. The remaining two terms, $p_t dG_t$ and dD_t are the proceeds from issuing long- and short-term debt.[7]

Define V_t as the continuation value of the borrower, which we sometimes refer to as the equity value at time t. The borrower chooses the endogenous time of default as well as the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate function as given. Once again, let us emphasize that all these decisions, default and issuance, are made without commitment.

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \geq t\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-\rho(s-t)} \left\{ \left[X_s - (r+\xi) F_s - y_{s-} D_{s-}\right] ds + p_s dG_s + dD_s \right\} \right]. \tag{5}$$

To guarantee the valuations remain finite, we follow the literature and assume both $r + \lambda > \mu_H$ and $r + \eta > \mu_L$ hold.

2.4 Smooth Equilibrium

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state θ_t, the cash-flow level X_t, and the amount of outstanding debt $\{D_{t-}, F_t\}$. The equilibrium requires the following. First, creditors break even; that is, p_t follows equation (2) and y_{t-} follows equation (3). Second, the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t \geq 0$. Finally, an MPE is *smooth* if no jump occurs in long-term debt issuance, in which case we write $dG_t = g_t F_t dt$. In a

[7] One can think of dD_t as the net issuance of short-term debt. Specifically, $dD_t = D_t - D_{t-}$ if there is a jump at t.

smooth equilibrium, the aggregate face value of long-term debt evolves according to

$$dF_t = (g_t - \xi) F_t dt. \tag{6}$$

Let us define J_t as the joint (maximized) continuation value of the borrower and short-term creditors if default does not occur at time t. The following result motivates us to work with J_t for the remainder of this paper.

Proposition 1. *The equity value is $V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_{\theta_t}(X_t, F_t) - D_{t-}, 0\}$, where the joint continuation value J_t is given by the value function $J_{\theta_t}(X_t, F_t)$ of the following problem:*

$$\begin{aligned} J_H(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ & X_s - (r+\xi)F_s + p_s g_s F_s \\ & + (\rho + \lambda - y_{s-})D_{s-} + \lambda \max\{J_L(X_s, F_s) - D_{s-}, 0\} \Big\} ds \Bigg] \end{aligned} \tag{7}$$

$$J_L(X_t, F_t) = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\eta)(s-t)} \Big\{ X_s - (r+\xi)F_s + p_s g_s F_s + (\rho + \eta - y_{s-})D_{s-} \Big\} ds \Bigg],$$

where the maximization is subject to the limited liability constraint $D_{s-} \leq J_{\theta_{s-}}(X_s, F_s)$.

The terms in (7) are related to those in (5). Here, $(\rho + \lambda - y_s)D_{s-}$ reflects the gains from issuing short-term debt. The last term in (7) stands for the event of regime-shifting, upon which the borrower would rather default and renege on the payments if the amount of outstanding short-term debt exceeds the maximum joint value without an immediate default; that is if $D_{s-} > J_L(X_s, F_s)$. The terms in $J_L(X, F)$ can be interpreted similarly.

Proposition 1 generates an interesting economic insight: even though the borrower makes decisions on debt issuance, these decisions are made to maximize the joint value of the borrower and short-term debt. This is because any issuance decisions will be immediately reflected in the credit risk faced by short-term creditors, affecting the price of short-term debt and the proceeds from issuing it. Note that the payoff to existing long-term creditors is ignored in the maximization problem because their debts have been issued in the past, and the borrower ignores how new debt issuance affects their valuation.[8] This result relates to Aguiar et al. (2019) in the context of sovereign debt, where the equilibrium issuance decisions can be characterized by the solution to a planner's problem that ignores the payoff to existing long-term creditors. Meanwhile, the max operator in (7) shows that the borrower and short-term creditors still have conflicts on whether to default immediately.

[8]The payoff to new long-term creditors is at dt order in the smooth equilibrium.

Proposition 1 implies that the state variable D_{t-} only enters the problem by affecting whether the borrower defaults immediately at time t. Following this result, we can suppress the problem's dependence on D_{t-} and treat it as a decision variable. A smooth MPE is characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F,D)$, $D_\theta(X,F)$, and $g_\theta(X,F)$. By exploiting the homogeneity of the problem, we can further reduce the problem's dimension and write $J_\theta(X,F) = Xj_\theta(f)$, $D_\theta(X,F) = Xd_\theta(f)$, where $f = F/X$. The functions g_θ, p_θ, and y_θ are homogeneous of degree zero, so we can write them as $g_\theta(f)$, $p_\theta(f)$, and $y_\theta(f,d)$. For simplicity, we refer to $J_\theta(X,F)$ and $j_\theta(f)$ as the unscaled and scaled *value function* for the rest of this paper.

3 Equilibrium

We solve the model in several steps. Subsection 3.1 studies the model in which the borrower is only allowed to issue short-term debt. This exercise highlights the benefits and costs of short-term debt. Subsection 3.2 builds upon 3.1 but further allows the borrower a one-time opportunity to issue long-term debt at $t = 0$. This exercise serves two purposes. First, it illustrates the tradeoff of long-term debt even without commitment issues. Second, the value functions in this exercise turn out identical to the ones in which the borrower has full flexibility in issuing both types of debt. With results from subsection 3.1 and 3.2, we proceed to solve the model in subsection 3.3 in which the borrower can flexibly issue both types of debt. Subsection 3.4 conducts comparative statics and derives the implied dynamic debt maturity patterns.

3.1 Equilibrium with Only Short-Term Debt

Let us first explore the model where the borrower can only issue short-term debt, which relates to Abel (2018). Heuristically, short-term debt is repaid after each "dt" so that the borrower always renews with zero leverage. The dilution problem therefore no longer exists. The choice of short-term debt follows the standard trade-off theory whereby the equity holder balances cheap debt against costly bankruptcy. Given there is no long-term debt, we work with the unscaled value function, denoted as $J^s_\theta(X_t)$. We also use $D^s_\theta(X_t)$ and $y^s_\theta(X_t)$ for the value and rates of short-term debt.

Note that short-term debt is immune to Brownian shocks and may only default after jump shocks, including regime shifts and disaster shocks. In the low state, default occurs following the disaster, which does not depend on the leverage level. Therefore, the borrower can exhaust her borrowing capacity by fully levering up. In this case, the borrower effectively "sells" the entire firm to creditors whose discount rate is r. The firm value becomes

$$J_L^s(X_t) = \mathbb{E}_t\left[\int_t^\infty e^{-(r+\eta)(s-t)} X_s ds\right] \Rightarrow J_L^s(X_t) = \frac{X_t}{r+\eta-\mu_L}. \tag{8}$$

In the high state, the issuance decision involves a more interesting tradeoff. Default does not occur if $J_t^s \geq D_{t-}^s$; hence, given the potential of a regime switch, the safe level of short-term debt cannot exceed $J_L^s(X_t)$. Meanwhile, the borrower can also exhaust her borrowing capacity by issuing short-term debt $J_H^s(X_t)$, but this becomes risky. The borrower prefers risky short debt $J_H^s(X_t)$ if the benefits from higher leverage exceed the additional bankruptcy cost due to the regime switch, i.e.,

$$\underbrace{(\rho - r)\left(J_H^s(X_t) - J_L^s(X_t)\right)}_{\text{benefit from higher leverage}} > \underbrace{\lambda J_L^s(X_t)}_{\text{additional bankruptcy cost}}. \tag{9}$$

Similar to (8), the associated high-state firm value when the borrower chooses $J_H^s(X_t)$ is $\frac{X_t}{r+\lambda-\mu_H}$. If the borrower chooses riskless short-term debt $J_L^s(X_t)$, the firm value becomes

$$\mathbb{E}_t\left[\int_t^\infty e^{-(\rho+\lambda)(s-t)}\left\{X_s + \underbrace{(\rho-r)J_L^s(X_s)}_{\text{leverage benefits}} + \underbrace{\lambda J_L^s(X_s)}_{\text{firm value in } L}\right\}ds\right] = \frac{X_t}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right).$$

Proposition 1 implies without long-term debt, the borrower chooses short-term debt to maximize the firm value. We have the following results.

Proposition 2 (Equilibrium with only short-term debt)**.** *If only short-term debt is allowed, the unique equilibrium is the following.*

1. *In the low state L, the value function and short-term debt issuance are*

$$D_L^s(X_t) = J_L^s(X_t) = \frac{X_t}{r+\eta-\mu_L}.$$

The borrower only defaults upon the disaster shock, so the short rate is $y_L^s(X_t) = r+\eta$.

2. *In the high state H, the value function is*

$$J_H^s(X_t) = \max\left\{\frac{X_t}{r+\lambda-\mu_H}, \frac{X_t}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right)\right\}.$$

Let

$$\bar{\lambda} \equiv \sqrt{\left(\frac{\rho-\mu_H}{2}\right)^2 + (\rho-r)(\mu_H+\eta-\mu_L)} - \left(\frac{\rho-\mu_H}{2}\right). \tag{10}$$

- *If $\lambda \leq \bar{\lambda}$, short-term debt is*

$$D_H^s(X_t) = \frac{X_t}{r + \lambda - \mu_H}$$

The borrower defaults upon the regime switch, and the short rate is $y_H^s(X_t) = r + \lambda$.

- *If $\lambda > \bar{\lambda}$,*

$$D_H^s(X_t) = \frac{X_t}{r + \eta - \mu_L}.$$

The borrower does not default upon the regime switch, and the short rate is $y_H^s(X_t) = r$.

3.2 Equilibrium with One-Time Long-Term Debt Issuance

Next, we build upon the model in subsection 3.1 and allow the borrower to issue long-term debt *once* at $t = 0$. In other words, the borrower can commit to not issuing any long-term debt after $t = 0$. We solve this model backward.

Short-Term Debt and Value Function After $t = 0$

Let F_0 be the level of long-term debt issued at $t = 0$. Without any new issuance, the outstanding long-term debt evolves according to

$$dF_t = -\xi F_t dt.$$

Now that there is outstanding long-term debt, we will work with the scaled value function after $t = 0$, denoted as $j_\theta^0(f)$, where the superscript 0 highlights that we are imposing $g_\theta(f) = 0$ after $t = 0$. By considering the change in the value function in Proposition 1 over a small interval and substituting $J_\theta^0(X, F) = X j_\theta^0(f)$, we get the following Hamilton-Jacobi-Bellman (HJB) equation:

$$(\rho + \eta - \mu_L)\, j_L^0(f) = \max_{d_L^0 \in [0, j_L^0(f)]} 1 - (r + \xi) f + \left(\rho + \eta - y_L^0\right) d_L^0 - (\mu_L + \xi) f j_L^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_L^{0\prime\prime}(f) \quad (11)$$

Results on the short-term debt are similar to those in Proposition 2. With outstanding long-term debt, the borrower endogenously defaults when the fundamental X_t deteriorates sufficiently compared to the outstanding long-term debt F_t, or equivalently the ratio of long-term debt to earnings $f_t = \frac{F_t}{X_t}$ hits an endogenous boundary f_L^b, where f_L^b satisfies the value matching condition $j_L^0(f_L^b) = 0$ and the smooth pasting condition $j_L^{0\prime}(f_L^b) = 0$.

Following a similar analysis, we arrive at the HJB for the scaled value function $j_H^0(f)$:

$$(\rho+\lambda-\mu_H)\,j_H^0(f)=\max_{d_H^0\in[0,j_H^0(f)]} 1-(r+\xi)\,f+\left(\rho+\lambda-y_H^0\right)d_H^0+\lambda\max\left\{j_L^0(f)-d_H^0,0\right\}$$
$$-(\mu_H+\xi)fj_H^{0\prime}(f)+\frac{1}{2}\sigma^2f^2j_H^{0\prime\prime}(f). \quad (12)$$

As usual, the value function satisfies the value matching and smooth pasting conditions $j_H^0(f_H^b)=0$ and $j_H^{0\prime}(f_H^b)=0$ at the default boundary $f=f_H^b$.

In the upturn, the choice of short-term debt entails a tradeoff similar to that in Proposition 2. The borrower chooses between the risky level $j_H^0(f)$ and the safe level $j_L^0(f)$. The former has the benefits of higher leverage but carries the risk of default and a loss in firm value upon the regime switch from H to L. Expecting so, short-term creditors demand a short rate

$$y_H^0\left(f,d_H^0\right)=\begin{cases} r & \text{if } d_H^0\le j_L^0\left(f\right)\\ r+\lambda & \text{if } d_H^0>j_L^0(f).\end{cases} \quad (13)$$

The condition determining the level of short-term debt is similar to (9), which becomes

$$\underbrace{(\rho-r)\left(j_H^0(f)-j_L^0(f)\right)}_{\text{benefit from higher leverage}}\ge\underbrace{\lambda j_L^0(f)}_{\text{additional bankruptcy cost.}} \quad (14)$$

Note that a comparison between (14) and (9) shows that here, the choice of short-term debt depends additionally on the level of outstanding long-term debt. In Lemma 2 of the appendix, we establish the following single-crossing property. For $\lambda>\bar{\lambda}$, the optimal issuance policy is characterized by a threshold $f_\dagger>0$, such that (14) holds for $f\ge f_\dagger$ but fails for $f<f_\dagger$. For $\lambda\le\bar{\lambda}$, (14) holds for any $f>0$, in which case we define $f_\dagger=0$ without loss of generality.

Figure 1 offers a graphical illustration to (14) under $\lambda>\bar{\lambda}$. Note that the additional bankruptcy cost $\lambda j_L^0(f)$ associated with risky short-term debt is high when f is low but low when f is high. Eventually, it converges to zero as f approaches f_L^b. In contrast, the difference in borrowing benefits declines much slower as f grows and only reaches zero at f_H^b.

Proposition 3 (Short-term debt Issuance)**.** *When the borrower has outstanding long-term debt and can only issue short-term debt, the optimal short-term debt issuance is as follows:*

- *In state $\theta=L$, the borrower issues short-term debt $d_L^0(f)=j_L^0(f)$ and pays a short rate $y_L^0\left(f,d_L^0(f)\right)=r+\eta$.*

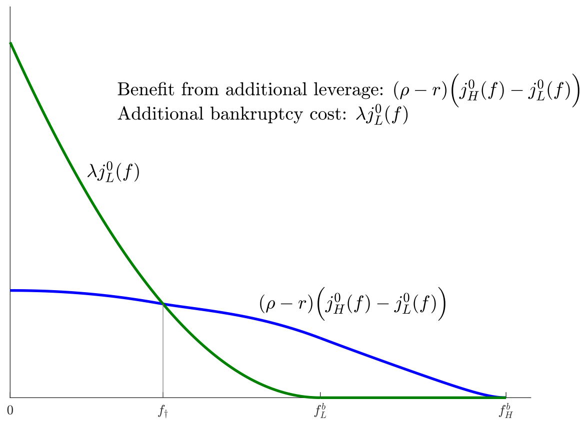


Figure 1: Cost and benefit of riskless short-term debt

- *In state $\theta = H$, the borrower issues short-term debt*

$$d_H^0(f) = \begin{cases} j_L^0(f) & \text{if } f < f_\dagger \\ j_H^0(f) & \text{if } f \geq f_\dagger \end{cases}$$

  and pays a short rate given by (13).

- *If $\lambda > \bar{\lambda}$, there exists a unique $f_\dagger \in (0, f_L^b)$ such that* (14) *holds (fails) for $f \geq f_\dagger$ ($f < f_\dagger$). If $\lambda \leq \bar{\lambda}$,* (14) *holds for any $f \geq 0$.*

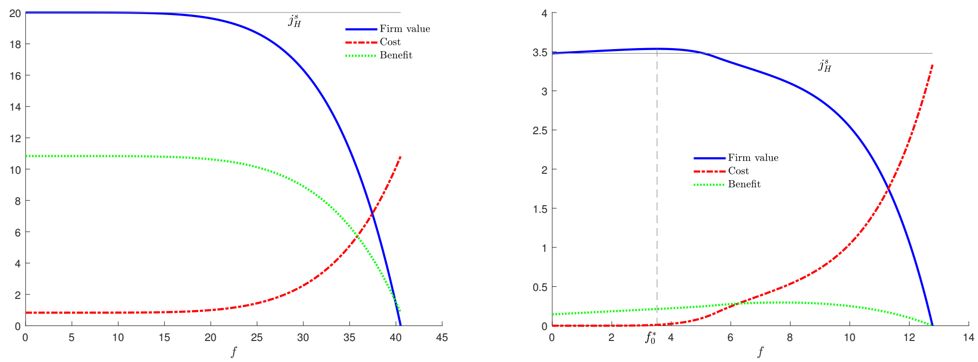
The expressions for the value function $j_\theta^0(f)$ can be found in Proposition 12 in Internet Appendix A.2.

Long-term Debt Issuance at $t = 0$

Now, we turn to the problem of initial issuance at $t = 0$. Let $p_\theta^0(f)$ be the price of long-term debt. The borrower chooses the initial amount of short-term debt D_0 and long-term debt F_0 to maximize the firm value, which includes equity and the proceeds from debt issuance. Given $\theta_0 = H$ and the homogeneity of the borrower's problem, we can write this problem as

$$\max_{f_0} \; j_H^0(f_0) + p_H^0(f_0) f_0. \tag{15}$$

The blue lines in Figure 2 plot (15) for $\lambda \leq \bar{\lambda}$ and $\lambda > \bar{\lambda}$, respectively. The left panel shows that when $\lambda \leq \bar{\lambda}$, the (scaled) firm value $j_H^0(f_0) + p_H^0(f_0) f_0$ decreases with f, so it is never optimal to issue any long-term debt. By contrast, the right panel shows when $\lambda > \bar{\lambda}$, it is optimal for the



(a) $\lambda \leq \bar{\lambda}$ (b) $\lambda > \bar{\lambda}$

Figure 2: Marginal benefits of long-term debt

The baseline parameters in this figure are as follows: $\rho = 0.1$, $r = 0.05$, $\mu_H = 0.01$, $\mu_L = -0.1$, $\xi = 0.1$, $\eta = 1$, $\sigma = 0.3$, In panel (a) $\lambda = 0.1$ and in panel (b) $\lambda = 0.4$. We also plot $j_H^s = \frac{J_H^s(X_t)}{X_t}$ as a reference.

borrower to issue a positive amount of long-term debt. We formalize this observation.

Proposition 4 (Equilibrium with short-term debt and one-time long-term debt issuance)**.** *If the borrower can only issue long-term debt once at $t = 0$, the optimal issuance is as follows.*

- *If $\lambda \leq \bar{\lambda}$,*

$$j_H^0(f_0) + p_H^0(f_0)f_0 < \frac{J_H^s(X_t)}{X_t}, \ \forall f_0 > 0.$$

Therefore, the borrower does not issue any long-term debt at $t = 0$.

- *If $\lambda > \bar{\lambda}$, there exists a $f_0^* > 0$ such that*

$$j_H^0(f_0^*) + p_H^0(f_0^*)f_0^* > \frac{J_H^s(X_t)}{X_t}.$$

Therefore, the borrower optimally issues a positive amount of long-term debt f_0^* at $t = 0$.

Results in Proposition 4 are closely linked to those in Proposition 2. Proposition 2 shows that when a downturn is not very likely ($\lambda \leq \bar{\lambda}$), it is optimal for the borrower to choose the risky amount of short-term debt and use up her borrowing capacity. In this case, if there is a one-time opportunity to issue long-term debt, Proposition 4 shows that the borrower will not issue it. Matters are different when the downturn is more likely to arrive ($\lambda > \bar{\lambda}$). According to Proposition 2, the borrower optimally chooses safe short-term debt and preserves some borrowing capacity. In this case, if there is a one-time opportunity to issue some long-term debt, Proposition 4 shows that the borrower will have strict incentives to issue it.

As in the standard tradeoff theory of capital structure, we can express the firm value as the unlevered value plus the benefits from issuing debt minus the expected cost of financial distress. In our setting, this decomposition becomes

$$\text{Firm Value} = \underbrace{\mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t} X_t dt\right]}_{\text{unlevered value}} + \underbrace{\mathbb{E}_0\left[\int_0^{\tau_b} \left(e^{-\rho t}(\rho - r)D_{t-} + \left(e^{-rt} - e^{-\rho t}\right)(r+\xi)F_t\right) dt\right]}_{\text{benefit of debt}} - \underbrace{\mathbb{E}_0\left[\int_{\tau_b}^{\infty} e^{-\rho t} X_t dt\right]}_{\text{expected bankruptcy cost}}.$$

Figure 2 plots these components (scaled by X_0); the green line shows the gains from debt issuance, while the red line describes the expected bankruptcy cost.[9] If $\lambda \leq \bar{\lambda}$, adding long-term debt reduces the benefit of debt while increasing the expected bankruptcy cost. As a result, the firm's value is reduced, so it is never beneficial to issue it. However, when $\lambda > \bar{\lambda}$, adding some long-term debt can increase the overall benefit of debt, as shown in the right panel. This is because long-term debt increases the firm's borrowing capacity. While it also increases the expected bankruptcy cost, this effect is relatively small for small amounts of long-term debt. Therefore, starting from a situation with no long-term debt ($f = 0$), the marginal benefit of issuing some long-term debt outweighs the marginal cost. This means that issuing long-term debt can increase the value of the firm.

The Hedging Benefits of Long-term Debt

We have shown that the borrower never issues any long-term debt in a model without jump risks. In such a model, short-term debt can immediately adjust following small and negative shocks and completely avoid default. With downward jumps, adjustment in short-term debt is not always possible, and the borrower may default. Long-term debt, however, will share losses from downward jumps. Due to the ex-post (i.e., after the jump) loss-sharing benefits, the borrower wants to issue it ex-ante (i.e., before the jump). Figure 3 illustrates the borrower's balance sheet before and after the transition. Note that a transition from the high to the low state reduces the (scaled) firm value from $j_H^0(f) + p_H^0(f) \cdot f$ to $j_L^0(f) + p_L^0(f) \cdot f$. Specifically, it reduces the equity value by $j_H^0(f) - j_L^0(f)$ and the long-term debt's value by $\left(p_H^0(f) - p_L^0(f)\right) \cdot f$, but leaves the value of short-term debt d intact unless the borrower immediately defaults. In other words, short-term debt does not share

[9]For detailed calculations, please refer to Appendix B.

State $\theta = H$

Assets	Liabilities
Cash flow grows at μ_H	ST debt: d
	LT debt: $p_H^0 \cdot f$
	Equity: $j_H^0 - d$

State $\theta = L$

Assets	Liabilities
Cash flow grows at μ_L	ST debt: d
	LT debt: $p_L^0 \cdot f$
	Equity: $j_L^0 - d$

Figure 3: Balance sheet upon the state transition without immediate default

any loss if default does not immediately occur after the state transition.

The loss-sharing feature of long-term debt reduces the probability of default, preserving the firm's future value. This finding is consistent with the results of Diamond and He (2014), who demonstrate that in certain contexts, long-term debt can lead to less debt overhang compared to short-term debt. To understand why long-term debt may reduce default incentives relative to short-term debt, consider the impact that maturity has on the incentives to exercise an American option, such as the default option. When an American option is far from its maturity date, the holder has a greater incentive to hold the option rather than exercise it, given the possibility that its exercise value may increase over time. Similarly, a borrower is less likely to default following a transition from the low to the high state when the outstanding debt has a longer maturity, as they maintain the option to default in the future. In bad times when default is likely, the longer maturity in the default option has little value, so there is little difference in debt overhang between short- and long-term debt. However, in good times when default is unlikely in the short run, the longer-term default option is valuable. In this case, long-term debt has the potential to generate less debt overhang, so the borrower has incentives to issue it.

An Alternative Hedging Instrument

To further illustrate the role of long-term debt in hedging shocks to θ_t, we depart temporarily by considering an alternative hedging instrument to the regime-shift shock. Specifically, let us assume the borrower can buy a short-term derivative contract written on θ_t, which pays \$1 if the regime shifts from the upturn to the downturn. No arbitrage implies the premium on this derivative contract must be λ. We have the following result.

Proposition 5. *Under the derivative contract, the borrower never issues any long-term debt but issues risky short-term debt $d_\theta(0) = j_\theta^0(0)$ for $\theta \in \{H, L\}$. In addition, the borrower buys $j_H^0(0) - j_L^0(0)$ units of the derivative in the high state.*

Proposition 5 shows that this derivative contract crowds out long-term debt. Compared to the derivative contract, long-term debt is a more costly hedging instrument because of the cost associated with potential bankruptcy. Note that here, the derivative provides a perfect hedge against regime-shift shocks. The interpretation is that the regime-shift shock is an aggregate one. In Internet Appendix B.1, we also explore the case where the hedge is imperfect so that the regime-shift shock can be interpreted as an idiosyncratic one. Results show that when the hedging provided by the derivative becomes less perfect, the borrower has more reasons to issue long-term debt. Clearly, long-term debt and derivative contract are substitutes.

For the remainder of this paper, we will continue to explore the model without such a derivative contract.

3.3 Equilibrium with Continuous Long-Term Debt Issuance

Next, we proceed to solve the model in section 2 where the borrower has the flexibility to issue both long- and short-term debt at all times. Our first result shows that the lack of commitment to future issuance policies fully erodes the value of long-term debt. In equilibrium, the borrower's payoff is identical to the one in which she cannot issue any long-term debt. This result is analogous to Coase (1972) conjecture and has a similar counterpart in DeMarzo and He (2021) (see Proposition 2 there).

Proposition 6. *Suppose the borrower can flexibly issue long-term debt, and there is no commitment to future debt issuance. In any smooth equilibrium, the joint valuation of equity and short-term debt $j_\theta(f)$ equals the one without new issuance of long-term debt. That is, $j_\theta(f) = j_\theta^0(f)$.*

Proposition 6 implies that the evolution of $j_\theta(f)$ is identical to that of $j_\theta^0(f)$ described by (11) and (12), respectively. In equilibrium, long-term debt's price must satisfy

$$p_\theta(X, F) = -\frac{\partial J_\theta(X, F)}{\partial F} \Rightarrow p_\theta(f) = -j_\theta'(f), \tag{16}$$

where $p_\theta(X, F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, and $\frac{\partial J_\theta(X,F)}{\partial F}$ is the associated drop in the continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, the marginal proceeds must be fully offset by the drop in continuation value.[10] Given so, in equilibrium, the borrower is indifferent and gains no marginal benefit from adjusting long-term debt, so her equilibrium payoff is the same as if she were never to issue any debt going forward.

[10] If the price were higher than the marginal cost, the borrower would benefit from accelerating any issuance, bringing the price down immediately. Alternatively, if the price were lower than the marginal cost, the borrower would benefit from accelerating any repurchase, bringing the price up.

Note that by definition $j_\theta^0(0) = j_\theta^s \coloneqq J_\theta^s(X_t)/X_t$. Therefore, Proposition 6 immediately implies $j_\theta^s = j_\theta(0)$, so that if the borrower has no outstanding long-term debt, the ability to flexibly issue it does not affect the total firm value.[11] Intuitively, whereas Proposition 4 shows that long-term debt could have hedging benefits, these benefits are completely dissipated by the lack of commitment to future issuance. The lack of commitment to future issuance means that long-term debt is not valuable in equilibrium.

Even though the borrower is indifferent between issuing long-term debt and not, this result does not imply she never borrows long on the equilibrium path. In fact, the equilibrium long-term debt issuance policy needs to be consistent with the price of long-term debt.[12] Let us now derive the equilibrium issuance policies.

It follows from Itô's lemma that, before the disaster $dN_t = 1$ in the low state, f_t evolves according to[13]

$$df_t = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right) f_t dt - \sigma f_t dB_t. \tag{17}$$

In the downturn $\theta_t = L$, the price satisfies the following HJB equation:

$$(r + \xi + \eta)\, p_L(f) = \underbrace{r + \xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)}_{\text{expected change in bond price}}. \tag{18}$$

To derive the issuance function g_L, we plug $d_L = j_L(f)$ into (11) (replace $j_L^0(f)$ with $j_L(f)$), differentiate the resulting equation once, add (18) on both sides, and apply equation (16). Turning to the upturn $\theta_t = H$, the price $p_H(f)$ satisfies the following HJB equation:

$$(r + \xi + \lambda)\, p_H(f) = r + \xi + \mathbb{1}_{\{f<f_\dagger\}}\lambda p_L(f) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f). \tag{19}$$

Compared with (18), (19) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \leq f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The

[11] Formally, we can establish $j_\theta(0) = \max\ j_\theta(f) + p_\theta(f) \cdot f$. To see this, note that the first-order condition of the right-hand side becomes $j_\theta'(f) + p_\theta'(f) \cdot f + p_\theta(f) = p_\theta'(f) \cdot f$, which is zero at either $f = 0$ or $p_\theta'(f) = 0$. The latter case is ruled out because we prove in the appendix that $p_\theta'(f) = -j_\theta''(f) < 0$ for $f \in [0, f_\theta^b]$. The second-order condition evaluated at $f = 0$ becomes $p_\theta'(0) = -j_\theta''(0) < 0$, so that the objective function is concave at $f = 0$. Note that the objective function differs from the one in subsection 3.2, which is $j_\theta^0(f) + p_\theta^0(f) \cdot f$.

[12] The reason is analogous to the logic behind mixed strategies: a player needs to be indifferent to her choice of action, yet equilibrium strategies are uniquely determined to maintain that indifference. In our context, the equilibrium long-term debt issuance needs to be consistent with the price of long-term debt.

[13] We omit the disaster shock dN_t when $\theta_t = L$. Upon the disaster shock $dN_t = 1$, X_t gets absorbed at 0, so f_t jumps to ∞. The borrower then defaults immediately, and the price of the long-term debt jumps to zero.

derivation of the issuance policy $g_H(f)$ follows the same steps as the one in the low state.

Proposition 7 (Long-term debt issuance)**.** *The equilibrium price of long-term debt is* $p_\theta(f) = -j'_\theta(f)$ *for* $\theta \in \{H, L\}$ *and* $f \in [0, f^b_\theta)$*. The issuance policies are as follows.*

1. *Downturn* $\theta = L$*:* $\forall f \in [0, f^b_L)$*, the issuance policy is*

$$g_L(f) = 0.$$

2. *Upturn* $\theta = H$*:*

   - *For* $f \in [0, f_\dagger)$

$$g_H(f) = \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-f p'_H(f)}. \tag{20}$$

   - *For* $f \in [f_\dagger, f^b_H)$

$$g_H(f) = 0.$$

   - *When* $\lambda \leq \bar{\lambda}$*,* $f_\dagger = 0$ *so* $g_H(f) = 0$ *for all* $f \in [0, f^b_H)$*.*

According to Proposition 7, the borrower does not issue any long-term debt in the low state and in the high state when $f \geq f_\dagger$. By contrast, in the high state when $f < f_\dagger$, the borrower issues long-term debt. The numerator in equation (20) captures the benefit of long-term debt coming from the incremental borrowing capacity provided by long-term debt, whereas the denominator captures the cost of issuance due to price impact. We can provide a heuristic derivation of the equilibrium issuance policy based on a local perturbation approach that illustrates this cost-benefit analysis.

Let's consider the upturn state, $\theta = H$, and an existing long-term debt level $f = f_0 \in [0, f_\dagger)$. Suppose the borrower deviates for "one period" by issuing an extra amount of long-term debt Δ at time t and buying it back at $t + dt$. The proceeds from additional long-term debt issuance is $p_H(f_0 + \Delta) \cdot \Delta$. Meanwhile, this adjustment reduces the proceeds from issuing riskless short-term debt by $j_L(f_0) - j_L(f_0 + \Delta)$. Note that the difference in patience implies that each dollar proceeds from total debt issuance results in a flow benefit of $\rho - r$. Combining the proceeds from long- and short-term debt, the total marginal benefit from this adjustment is

$$(\rho - r)\left[\underbrace{p_H(f_0 + \Delta) \cdot \Delta}_{\approx p_H(f_0)\Delta} - \underbrace{(j_L(f_0) - j_L(f_0 + \Delta))}_{\approx -j'_L(f_0)\Delta = p_L(f_0)\Delta}\right] \approx (\rho - r)\,(p_H(f_0) - p_L(f_0)) \cdot \Delta. \tag{21}$$

The crux of the matter is that the proceeds of long-term debt issuance depend on its price in the *high* state, whereas the impact on short-term borrowing is determined by long-term debt's price in the *low* state.

The cost of this adjustment depends on the price impact of such trade, which is

$$\text{Cost of one period adjustment} \approx -p_H(f_0+\Delta)\Delta + p_H(f_0)\Delta = -p_H'(f_0)\,(\Delta)^2\,.$$

In equilibrium, the marginal benefit is equal to the marginal cost, so

$$(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)\Delta \approx -p_H'(f_0)\,(\Delta)^2 \Longrightarrow \Delta = \frac{(\rho - r)\big(p_H(f_0) - p_L(f_0)\big)}{-p_H'(f_0)}.$$

The issuance function in the other cases of Proposition 7 can be derived using similar heuristic arguments. When the borrower fully exhausts borrowing capacity, the increment in long-term debt's market value is fully offset by the reduction in short-term debt, so no long-term debt is issued. For example, in the high state, when $f > f_\dagger$, the reduction in short-term debt becomes $j_H(f_0) - j_H(f_0+\Delta)$, which is approximately $p_H(f_0)\cdot\Delta$. In this case, the total marginal benefit from this adjustment (the counterpart of (21)) becomes zero.

Long-Term Debt Buyback

An interesting feature in (20) is that the optimal issuance of long-term debt could be negative, implying that the borrower actually buys back long-term debt. This result differs from the literature on the leverage-ratchet effect, which predicts a borrower without commitment to debt issuance would never actively buy back the outstanding debt (DeMarzo and He, 2021; Admati et al., 2018). The reason is as follows. Once $f > f_\dagger$, the borrower will take too much short-term debt, generating a rollover risk whereby default will occur following a regime shift. For f sufficiently close to (but still below) $f_\dagger$, the borrower could find it optimal to buy back some long-term debt and reduce the chances that f rises above $f_\dagger$. Note that (20) implies buyback occurs whenever $p_H(f) < p_L(f)$, suggesting that long-term debt is riskier in the upturn compared to the downturn. Intuitively, for f sufficiently close to (but still below) $f_\dagger$, a few negative Brownian shocks could push f above $f_\dagger$, which leads to an immediate default following a regime switch. By contrast, under the same f in the low state, a default may only occur after a long sequence of negative Brownian shocks that push f to f_L^b. Therefore, the default risk can be higher in the high state than in the low state.

That being said, we would like to point out that ex-post buyback is not why the borrower issues long-term debt ex-ante. Note that the exercise in subsection 3.2 shows that the borrower will benefit from long-term debt issuance at $t = 0$ even if there is no further opportunity afterward to either issue long-term debt to buy back short-term debt or to repurchase outstanding long-term debt issued at $t = 0$.

Moreover, we provide a sufficient condition under which the borrower never buys back long-term debt. Our mechanism still goes through under this condition.

Corollary 1. *The borrower never repurchases its long-term debt if $\eta \geq \lambda$.*

Initial Debt Issuance

So far, our analysis has shown that for a given level $f > 0$, the borrower could issue/repurchase some long-term debt. However, it remains a question whether an initially unlevered borrower would issue any long-term debt. The next proposition provides the necessary conditions under which this would be the case.

Proposition 8. *Suppose that $\lambda > \bar{\lambda}$. If $\eta > 0$ and $p'_H(0) > -\infty$, then there exists an equilibrium in which an initially unlevered borrower will issue long-term debt, that is $\lim_{f \to 0} g_H(f) f > 0$.*

Without the disaster shock, an unlevered borrower would never issue long-term debt. Intuitively, if there is no disaster, a marginal unit of long-term debt is riskless for the unlevered borrower in both the upturn and the downturn; that is, $p_H(0) = p_L(0) = 1$. According to (20), there is no long-term debt issuance. The second condition $p'_H(0) > -\infty$ is needed to ensure that the price impact of issuing an additional unit of long-term debt is not too large to deter the unlevered borrower from issuing it.

Let us conclude this subsection by pointing out that under the conditions in Proposition 8, there exists another Markov Perfect Equilibrium in which the unlevered borrower only issues short-term debt, and the reasons are similar to the zero-leverage equilibrium in (DeMarzo and He, 2021, p. 35).

3.4 Comparative Statics and Debt Dynamics

We start by considering how different primitive variables affect $f_\dagger$, which captures the incentives to issue long-term debt. As shown in Figure 1, this threshold is determined by the trade-off between the additional leverage benefits versus the bankruptcy cost. Thus, any parameter that increases the continuation value after the regime switch (thus, increasing bankruptcy cost) should increase $f_\dagger$, whereas parameters that increase borrowing capacity in the high state should decrease $f_\dagger$. It is immediate that $f_\dagger$ is increasing λ and decreasing in μ_H, because both affect $j_H(f)$ but not $j_L(f)$. Figure 4 presents some further comparative statics. The left panel shows that $f_\dagger$ increases in μ_L. Intuitively, a higher growth rate of cash flows in the low state increases the expected bankruptcy cost upon regime switching and makes risky short-term debt more costly. The right panel confirms the earlier result that $f_\dagger$ increases in λ. A comparison across different curves in both panels shows that $f_\dagger$ decreases in ξ or equivalently increases in the maturity of long-term debt. Intuitively, debt with longer maturity is more sensitive to changes in firm value and, therefore, provides more hedging benefits.

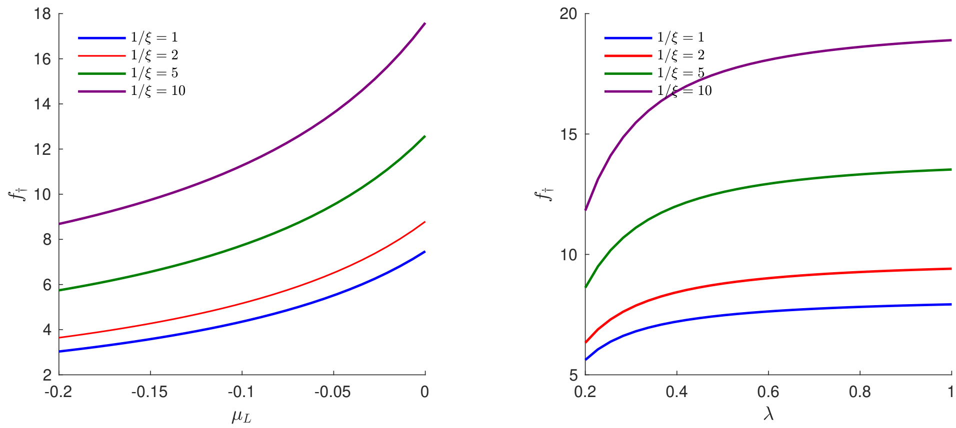


Figure 4: Comparative statics $f_\dagger$.

The baseline parameters in this figure are as follows: $\rho = 0.1$, $r = 0.05$, $\mu_H = 0.1$, $\mu_L = 0$, $\lambda = 0.5$, $\eta = 0.1$, $\sigma = 0.5$, $\pi = 0$.

The left panel Figure 5 describes how the maturity of long-term debt affects firm value, and the results show some interesting non-monotonic patterns. Intuitively, when the maturity of long-term debt gets longer, the dilution problem gets more severe. Meanwhile, the longer-term debt provides better hedging, so the overall effect can be non-monotonic. The right panel shows that in the economy with only long-term debt, the firm value decreases with maturity because longer-maturity debt is subject to more future dilution.[14]

For the rest of this subsection, we focus on a limiting model $\sigma \to 0$. This allows us to conduct more comparative statics and study debt structure dynamics. We impose an additional assumption $\mu_L + \xi < 0 < \mu_H + \xi$ so that in the absence of further long-term debt issuance, f_t decreases in the high state but increases in the low state. In this case, the equilibrium issuance function can be explicitly written in primitive variables. Below, we briefly describe the results, and details are presented in Internet Appendix A.3.

Proposition 9 (Limiting long-term debt issuance policy)**.** *Suppose* $\lambda > \bar{\lambda}$*,* $\mu_L + \xi < 0$*, and*

[14] This model corresponds to DeMarzo and He (2021) adapted with regime-shift shocks. The solution is presented in the Internet Appendix.

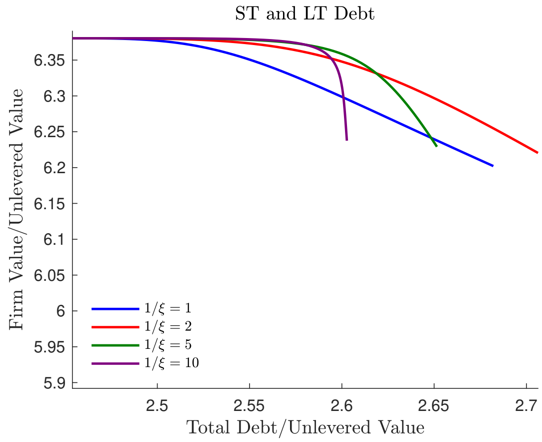


Figure 5: Comparative statics firm value and debt capacity.

The baseline parameters in this figure are as follows: $\rho = 0.075$, $r = 0.05$, $\mu_H = 0.2$, $\mu_L = 0.1$, $\lambda = 0.25$, $\eta = 0.1$, $\sigma = 1$. The figures consider the relation between total debt and firm value. They are constructed by considering the upper branch of the graph $\{(p_H(f)f + d_H(f), p_H(f)f + j_H(f))\}$. In the case of both short- and long-term debt, we restrict attention to the interval $(0, f_\dagger)$. On $(f_\dagger, f_H^b)$, total debt and firm value decrease in f.

$\mu_H + \xi > 0$. Let $\gamma = \frac{r+\eta-\mu_L}{-(\xi+\mu_L)} > 1$. When $\sigma \to 0$, the equilibrium issuance policy converges to:

$$g_\theta(f) = \frac{\rho - r}{\rho + \lambda - r}\left[\frac{\eta(\xi + \mu_H) + (\mu_H - \mu_L)(r + \xi)}{r + \eta + \xi} + \eta\left(\frac{\xi + \mu_H}{\rho + \lambda + \xi} + \frac{-(\xi + \mu_L)}{r + \eta + \xi}\right)\left(\frac{f}{f_L^b}\right)^{-(\gamma - 1)}\right]\mathbb{1}_{\{f < f_\dagger, \theta = H\}}. \quad (22)$$

For any $f \in (0, f_\dagger)$, the issuance function $g_H(f)$ is:

- *Increasing in ρ, η, and μ_H, and decreasing in λ.*
- *If $\eta > 0$, there is $\tilde{f} \in (0, f_\dagger]$ such that $g_H(f)$ is increasing in μ_L for $f < \tilde{f}$ and decreasing in μ_L for $f > \tilde{f}$. If $\eta = 0$, $g_H(f)$ is decreasing in μ_L for all $f \in [0, f_\dagger]$.*

The results in Proposition 9 are straightforward. Higher ρ increases the benefits of leverage; higher η and μ_H both increase the difference between $p_H(f) - p_L(f)$ for any given f. Therefore, the borrower issues more long-term debt. Meanwhile, when λ gets higher, $p_L(f)$ stays unchanged, whereas $p_H(f)$ gets lower. As a result, the borrower issues less long-term debt.

Next, we characterize the dynamics of f_t.

Proposition 10. *Consider the limiting model ($\sigma \to 0$). Under the parametric conditions in Proposition 9, the ratio of long-term debt to earnings f_t follows the piecewise-deterministic process.*

$$\frac{df_t}{dt} = \begin{cases} -(\xi + \mu_L) f_t & \text{if } \theta_t = L \\ -\frac{\nu}{\gamma - 1}\left[1 - \kappa \left(\frac{f_L^b}{f_t}\right)^{\gamma - 1}\right] f_t & \text{if } \theta_t = H \text{ and } f_t \in (0, f_\dagger) \\ -(\xi + \mu_H) f_t & \text{if } \theta_t = H \text{ and } f_t \in (f_\dagger, f_H^b), \end{cases}$$

where $\nu \cdot \kappa \geq 0$ and $\nu > 0$ only if

$$\frac{\xi + \mu_H}{-(\xi + \mu_L)} \frac{\lambda}{\rho - r} > \frac{r + \xi}{r + \eta + \xi}.$$

If $\nu > 0$, let $f^\intercal \equiv \kappa^{\frac{1}{\gamma - 1}} f_L^b$ be the unique solution to $g_H(f^\intercal) = \mu_H + \xi$. In the high state:

- *If $\nu < 0$ or $f^\intercal > f_\dagger$, f_t converges to $f_\dagger$.*
- *If $\nu \geq 0$ and $f^\intercal < f_\dagger$, f_t converges to $f^\intercal$. In this case,*
  - *The speed of adjustment ν is increasing in η, λ, μ_H, and μ_L, and it is decreasing in ρ. It is increasing in ξ if and only if*

    $$\xi > -\mu_L - \sqrt{\frac{\lambda (\mu_H - \mu_L)(r + \eta - \mu_L)}{\rho + \lambda - r}}.$$

  - *The target $f^\intercal$ is increasing in ρ, η, and decreasing in λ and μ_H.*

One can interpret $f^\intercal$ as the target ratio of long-term debt to cash flow. Figure 6 illustrates the dynamics when $f^\intercal \in (0, f_\dagger)$, with the left and right panels, respectively, describe the evolution of long- and short-term debt. Starting in the high state, the path of f_t converges towards the target $f^\intercal$ until the regime switches. The convergence path for $f_0 < f^\intercal$ (red path in the figure) is straightforward. For $f_0 > f_\dagger$ (blue path in the figure), the borrower initially borrows risky short-term debt and retires maturing long-term debt until $f_\dagger$ (this corresponds to time $\tau_\dagger$ in the figure). Once this threshold is reached, the borrower reduces the amount of short-term debt and starts to issue long-term debt. After the regime shift at τ_λ, the borrower stops issuing long-term debt and only borrows short-term, and f_t increases until the firm eventually defaults.

If either $\nu < 0$ or $f^\intercal > f_\dagger$, then f_t converges towards $f_\dagger$, after which it stays there until the state transition.[15]

[15]This last case presents some technical complications because there is a difference between the limit equilibrium

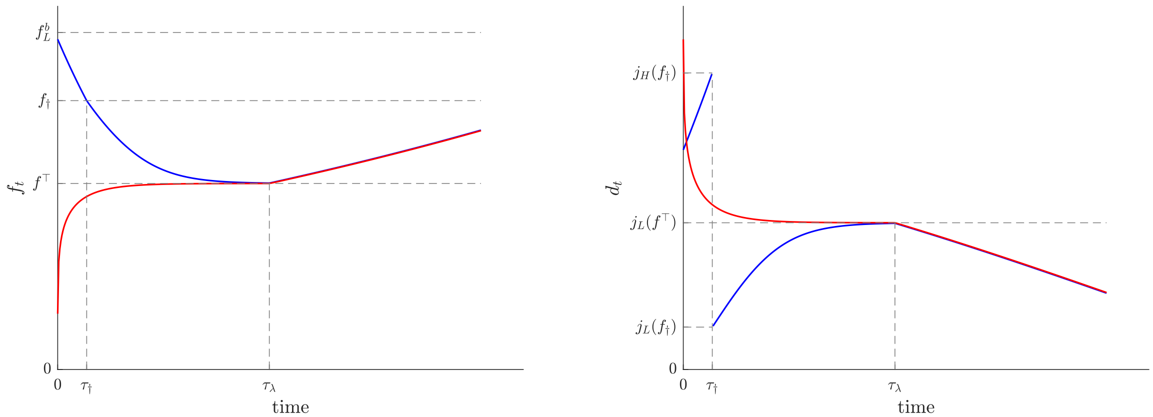


Figure 6: Sample path f_t and d_t for different values of f_0.

The parameters in this figure are as follows: $\rho = 0.2$, $r = 0.1$, $\mu_H = 0.2$, $\mu_L = -0.2$, $\xi = 0.1$, $\lambda = 0.3$, $\eta = 0.1$. The shock arrives at $\tau_\lambda = 5$. The blue corresponds to an initial value $f_0 = 4.90$, while the red line has $f_0 = 0.82$.

4 Extensions, Robustness, and Empirical Implications

This section starts by considering the several extensions to illustrate the main economic mechanisms. In subsection 4.1, we explore the role of market incompleteness by introducing the possibilities of short-term debt renegotiation. Results show that long-term debt is not issued if renegotiation is frictionless. In subsection 4.2, we show that results are similar if there are no regime shifts but cash flows have downside jumps. Finally, subsection 4.3 discusses the model's empirical relevance.

4.1 Restructuring of Short-Term Debt

We have established in subsection 3.2 that long-term debt has hedging benefits because with downward jumps, the adjustment in short-term debt is not always possible, and the borrower may default. Now, we show that if, instead, short-term debt can be frictionlessly restructured, it crowds out reasons to use long-term debt.

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when $\sigma \to 0$ and the equilibrium in a model with $\sigma = 0$. At $f_\dagger$, we have that $g_H(f_\dagger-) > \mu_H + \xi$ and $g_H(f_\dagger+) < \mu_H + \xi$. A classical solution for the path of f_t only exists if we set $g_H(f_\dagger) = \xi + \mu_H$ – so the threshold $f_\dagger$ is absorbing. If $\sigma = 0$, this policy is consistent with the equilibrium price $p_H(f_\dagger) = j_H'(f_\dagger)$ only if the probability of defaulting upon a transition is positive but less than one. We can construct an equilibrium with this property by setting $d_H(f_\dagger) = j_L(f_\dagger)$ and specifying a mixed strategy of default (upon a transition to the low state) so the price of long-term debt satisfies no-arbitrage at $f_\dagger$. Such construction is possible because at $f_\dagger$ the equity holders are indifferent. When $\sigma > 0$, the particular issuance policy at $f_\dagger$ is not a problem because f_t fluctuates around the threshold $f_\dagger$. If we set $g_H(f_\dagger) = g_H(f_\dagger-)$, the existence result in Nakao (1972) implies the existence of a unique strong solution to the SDE for f_t for any $\sigma > 0$. We can interpret the path of f_t in the limit as an approximation for small $\sigma > 0$ where in the high state f_t mean reverts to $f_\dagger$.

To do this, we need to distinguish between default and bankruptcy. Whenever the borrower announces a default, and there is outstanding short-term debt, short-term debt can be restructured with some probability. Notice that in both states $\theta \in \{H, L\}$, when the borrower defaults at f_θ^b, the amount of short-term debt is zero. Therefore, renegotiating short-term debt is only relevant upon a regime shift from H to L. The renegotiation game goes as follows. With probability $1-\alpha$, renegotiating is impossible, and the firm goes bankrupt. With probably α, the firm enters into a renegotiation process. In this case, the equity holder makes the offer with probability β and short-term creditors with probability $1-\beta$. If the short-term creditors make the offer, and this offer is rejected, the firm goes bankrupt. If the equity holder makes the offer and the offer is rejected, she can still choose between repaying the original short-term debt and bankruptcy.

We have the following result.

Proposition 11. *The firm never issues long-term debt if the short-term debt can be renegotiated without friction. That is, if $\alpha = 1$, then $g_\theta(f) = 0$ for all f. If $f_\dagger > 0$, then $f_\dagger$ is decreasing in α.*

The intuition behind this result is similar to the one with derivative contracts in subsection 3.2. The benefit of long-term debt is to provide hedging against state transition. However, hedging is no longer needed if the short-term debt can be restructured.

4.2 Jump Risk

This subsection shows that our mechanism continues to hold if the large negative shock is modeled as downward jumps to the cash flow. Specifically, we assume the cash flow follows a jump-diffusion process:

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \omega^{-1}\right) X_{t-} dN_t, \tag{23}$$

where N_t is a Poisson process with intensity λ and $\omega \in (1, \infty)$ is a constant. We can construct an equilibrium characterized by thresholds $f_\dagger$ and f^b. It is easily established that the scaled value function $j(f)$ satisfies the delay differential equation

$$(\rho + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) \\ + \max\left\{(\rho + \lambda - r)\frac{j(\omega f)}{\omega}, (\rho - r) j(f)\right\},$$

with value matching and smooth pasting conditions $j(f^b) = j'(f^b) = 0$. The optimal short-term debt policy is given by

$$d(f) = \begin{cases} \frac{j(\omega f)}{\omega} & \text{if } f \in [0, f_\dagger) \\ j(f) & \text{if } f \in [f_\dagger, f^b], \end{cases}$$

where the threshold $f_\dagger$ satisfies the condition $(\rho + \lambda - r)\frac{j(\omega f_\dagger)}{\omega} = (\rho - r)j(f_\dagger)$. The issuance of long-term debt satisfies $g(f) = 0$, for $g(f) = 0\ \forall f \in (f_\dagger, f^b]$, where f^b is the endogenous default boundary. The issuance of long-term debt follows

$$g\,(f) = \frac{(\rho - r)\,(p\,(f) - p\,(\omega f))}{-fp'\,(f)}\mathbb{1}_{\{f<f_\dagger\}}. \tag{24}$$

In other words, long-term debt is issued if the amount of outstanding long-term debt is low relative to the operating cash flow. Equation (24) resembles (20): the difference in prices $p\,(f) - p\,(\omega f)$ reflects the drop in the long-term debt's price following the downward jump, and the denominator captures the sensitivity of long-term debt price to issuance.

The issuance of short-term debt is also similar to the full model in section 3. Short-term debt is riskless when $f \leq f_\dagger$ and the amount of issuance is $d(f) = \frac{j(\omega f)}{\omega}$. On the other hand, when $f > f_\dagger$, short-term debt becomes risky, and the amount of issuance becomes $d(f) = j(f)$. We delegate the details to the Internet Appendix B.3.

4.3 Transitory Shocks and Empirical Implications

In the main model, we have assumed that the low state $\theta_t = L$ is absorbing. If we interpret the changes in the regime as business cycles, it is natural to assume that states are transitory. We can extend the model to consider this situation. Such a model is solved in Internet Appendix B.5, and we explore a few patterns.

Time series implications. Our result implies that market leverage is countercyclical, which is consistent with the evidence provided in Halling et al. (2016). Our model implies the borrower's debt maturity is pro-cyclical, consistent with the findings in Chen et al. (2021) at the aggregate level. At the firm level, Mian and Santos (2018) show that firms manage maturity to hedge refinancing risk in good times, and the time variation is driven by demand-side (i.e., firms) factors. Moreover, this pattern is concentrated in investment grade-rated firms.

In Figure 8, we simulate a sample path and plot the time series of the cash-flow rate and debt maturity. Here, debt maturity is defined as the average maturity of total debt outstanding weighted

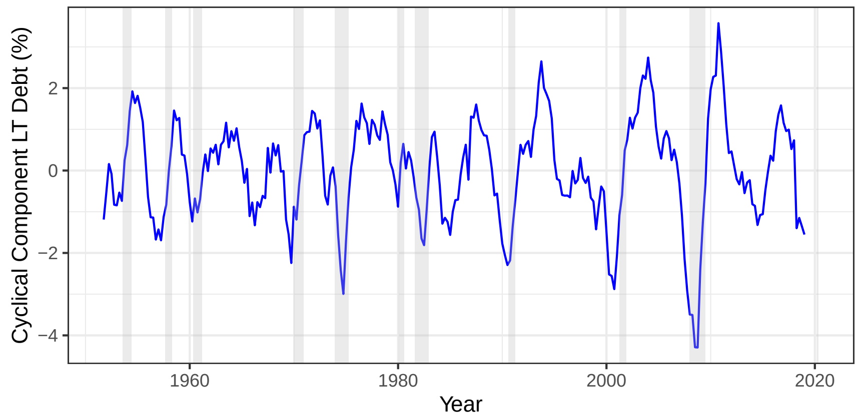

Figure 7: Pro-cyclical long-term debt share in the business cycle

This figure applies the Hodrick-Prescott filter with a multiplier of 1600 to the share of long-term debt of non-financial firms in the United States and extracts the cyclical components. The shaded areas denote NBER-dated recessions. From Chen et al. (2021) using Flow of Funds Accounts data (Table L.103)

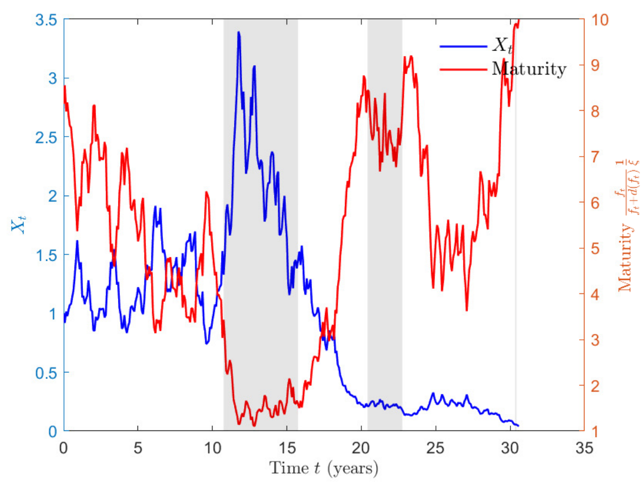

Figure 8: Sample path of leverage and maturity

This figure simulates the sample path of one firm and plots the time series of X_t, maturity, and market leverage, with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\eta = 0.05$.

by their book value:

$$\text{Maturity}_t := \frac{F_t}{F_t + D_t}\frac{1}{\xi}.$$

In the absence of a regime shift, the maturity of debt seems to move in the opposite direction to cash flows. In other words, the borrower expands the average debt maturity following a negative Brownian shock to X_t. Intuitively, this pattern holds because, after a negative Brownian shock to X_t, the borrower immediately rolls over less short-term debt, whereas she only reduces long-term debt outstanding gradually over time. Meanwhile, when the regime shifts and the downturn arrives, the borrower exclusively borrows short-term debt and the average maturity goes down.[16] Therefore, our model implies that within a regime, cash flows and debt maturity negatively comove with each other. However, if we compare across regimes, the regime with higher cash-flow growth rates has on average longer debt maturity.

Stock versus flow Static models of debt maturity tend to make the same predictions regarding the stock (outstanding) and the flow (issuance) of debt. One merit of constructing a dynamic model of debt maturity is to differentiate between the two. Our paper implies that the relationship between credit risk and maturity depends on whether we consider outstanding debt or new issuance as the dependent variable. For example, in the upturn, credit risk is high when the borrower has a significant amount of long-term debt outstanding (i.e., f is very close to f_H^b). Thus, a positive relationship exists between credit risk and the maturity of outstanding long-term debt. However, close to the default boundary, newly issued debt is exclusively short-term, so there is a negative relation between credit risk and the maturity of the newly issued debt.

Gradual and sudden defaults. Our model generates novel empirical implications on debt maturity structure and defaults. In particular, the borrower defaults in two circumstances. First, the ratio of long-term debt to cash flow f_t gets sufficiently high such that the borrower approaches the default boundary gradually from below ($f_t \uparrow f_L^b$ or $f_t \uparrow f_H^b$). In this case, default occurs *gradually* after the deterioration of the fundamental cash flows relative to the outstanding long-term debt. In the second circumstance, default occurs after a transition from the upturn to the downturn, and the borrower has taken too much risky short-term debt before the transition. These can be implications for future empirical tests.

[16] This result depends on the binary-state setup, where no additional downside risk exists in the low state. With more than two states, the borrower may still issue long-term debt in the low state. The broader message is the transition to a worse state, the borrower may only issue short-term debt for a while.

Cross-sectional implications. We have shown that with only small shocks (diffusion risks), the borrower borrows exclusively short-term debt. By contrast, the borrower issues a combination of long- and short-term debt when there are large downside risks (such as the regime switch and the jump risks introduced in subsection 4.2) to hedge. Cross-sectionally, one should observe that firms more exposed to large downside shocks use more long-term debt. One interpretation is that Brownian shocks are small and diversifiable, whereas regime-shift shocks are systematic and non-diversifiable. Under such an interpretation, our paper implies that firms with more non-diversifiable risk use more long-term debt.

Anecdotal examples We can illustrate the predictions of our model by considering real-world examples. Figure 9 plots debt maturity structure from 1990 onward for the Pacific Gas and Electric Company (PG&E) and General Motors (GM). PG&E entered bankruptcy twice in the last two decades. It initially entered Chapter 11 bankruptcy on April 6, 2001, and emerged from bankruptcy in April 2004. In 2019, it filed for bankruptcy on January 29 again and successfully exited on June 20. The left panel plots the maturity of newly issued long-term debt, weighted by the offering amount. The red-shaded areas marked the two bankruptcies, and the gray areas are NBER recessions. Consistent with our model, the newly-issued bonds have shorter maturities in the NBER recessions and shortly before the bankruptcies.[17] The right panel displays similar patterns for GM, which filed for bankruptcy on June 8, 2009.

5 Final Remarks

Our paper offers a theory of debt maturity based on a tradeoff between commitment and hedging. Short-term debt mitigates the lack of commitment problem and incentivizes the borrower to reduce leverage after negative shocks. Long-term debt offers the borrower hedging benefits by delaying default after large and negative shocks.

Modeling Choices: We introduce two types of risks in the model. The Brownian motion captures continuous fluctuations in day-to-day operating cash flows, which are meant to be small and frequent. Meanwhile, a transition across the two states affects the expected growth in cash flow and captures large and infrequent shocks.

Our modeling choice of short- and long-term debt is motivated by the discrete-time microfoundation. There, short-term debt would last for one period and therefore mature simultaneously. In

[17] The maturity of newly-issued debt was also short in 2011, which might be due to the 2010 San Bruno explosion: PG&E was on probation after being found criminally liable in the fire. In the context of our model, the San Bruno explosion can be thought of as a transition from a high to a low state after a Poisson shock.

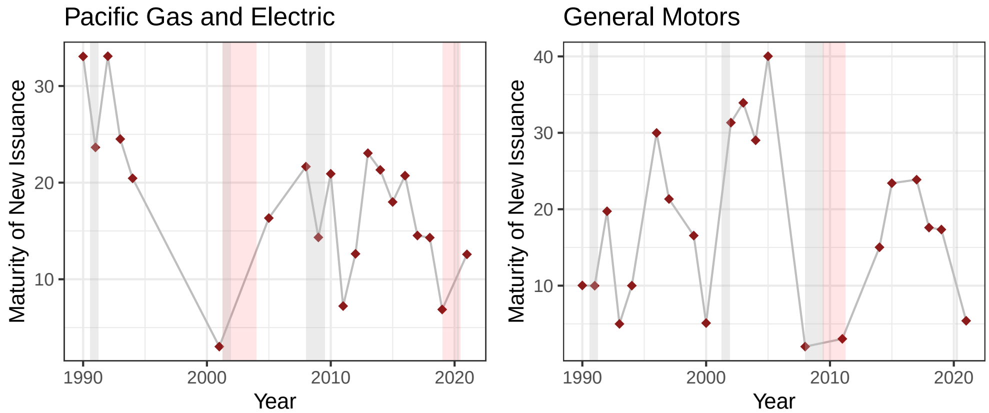

Figure 9: Time series maturity of new issuance for PG&E and GM

This figure shows the average maturity of bonds in a year (weighted by market value at issuance) and the share of long-term debt maturing within one year for Pacific Gas & Electric (PG&E) and General Motors (GM). The gray-shaded area indicates NBER recession, whereas the red-shaded area indicates periods over which these companies were in bankruptcy procedures. Source: Mergent FISD.

the continuous-time setup, this feature is captured by zero-maturity debt that needs to be continuously rolled over. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing it each period would lead to a staggered structure. This feature is well captured by exponentially maturing debt in the continuous-time setup.

Relation to Short-term Debt and Commitment: The role of short-term debt as a commitment device has been discussed by the previous literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001), which emphasizes the demandable feature of debt and the externalities from depositor runs. Calomiris and Kahn (1991) is about stopping a crime in progress through a run, and the prospect of a run creates a reward for information acquisition. In Diamond and Rajan (2001), there is no crime to be stopped. Instead, uninformed depositors must run when held up — they are solving a severe incentive problem. Our paper is similar in that it also solves the incentive problems in commitment. Whereas both Calomiris and Kahn (1991) and Diamond and Rajan (2001) emphasize how run externalities create commitment in banking, our paper shows how the repricing feature of short-term debt (independent of run externalities) resolves the commitment issue in the context of dynamic capital structure.[18]

An alternative commitment device widely used by firms is covenant. Introducing covenants

[18] Also see Hu and Varas (2021) on this feature of short-term debt in the context of financial intermediaries.

would allow the borrower to reap more benefits from long-term debt issuance. For example, a covenant that restricts the issuance of long-term debt to be lower than some threshold can limit the extent of dilution. However, covenants do not eliminate the benefits of short-term debt for two reasons. First, covenants are written on imperfect proxies of the firm's fundamentals, and therefore they don't completely rule out dilution. Second, following small and frequent shocks to cash flows, it is more costly for the borrower to adjust long-term debt. By contrast, short-term debt is more flexible. Therefore, our main mechanism between commitment and hedging continues to work under covenants.

Relation to Hedging in Corporate Finance: In a Leland-type setup, Diamond and He (2014) also explore the hedging benefits of long-term debt. Specifically, their paper studies how debt of different maturities affects debt overhang, broadly defined as any value-enhancing activities taken by equity holders. Recognizing the benefit of long-term debt, they state that "the long-term debt holders–due to less frequent repricing–share more losses with equity holders when asset-in-place deteriorate" (p. 740); while discussing the disadvantage of short-term debt, they state that "not sharing losses in bad times pushes equity holders to default, eliminating future investment opportunities" (p. 741). Although we do not explicitly model investment, injecting funds by equity holders to avoid immediate default enhances the firm value and, therefore, can be considered analogous to investment.

The corporate finance literature has recognized that bankruptcy costs can introduce hedging benefits to risk-neutral borrowers. (Smith and Stulz, 1985, p. 392) define hedging generally as the covariance of the firm value with respect to a state variable and argues that hedging reduces the dependence of firm value on changes in the state variable. In the context of our model, this state variable is the regime θ. More specifically, (Smith and Stulz, 1985, p. 396) state that "By reducing the variability of the future value of the firm, hedging lowers the probability of incurring bankruptcy costs. This decrease in expected bankruptcy costs benefits shareholders."

Relation to the Sovereign Debt: A similar trade-off is studied by Arellano and Ramanarayanan (2012), who calibrate a quantitative model of sovereign borrowing with two maturities. Our paper complements their analysis in several dimensions. First, we develop a tractable model with a transparent characterization of the equilibrium. Unlike Arellano and Ramanarayanan (2012), we can fully characterize the optimal debt policy and highlight the fundamental economic forces underlying the maturity choices. Second, our framework identifies the type of risk – large downside risk – that the borrower wants to hedge using long-term debt. Specifically, we emphasize that by delaying default, long-term debt provides better hedging against downside risk, which is valued in good times. We show that long- and short-term debt offer different kinds of flexibility following shocks. Short-term debt incentivizes the borrower to reduce leverage in response to negative shocks. If there are only small shocks, short-term debt is chosen such that leverage is never excessive. When there are large negative shocks, long-term debt provides a hedge that reduces bankruptcy costs and enhances borrowing capacity. By contrast, Arellano and Ramanarayanan (2012) emphasizes that long-term debt offers a hedge against future fluctuations in spreads. However, because credit spreads are endogenous, it is unclear what underlying risks drive the fluctuations and, hence, where the hedging benefits come from. Finally, The literature in sovereign debt typically assumes a risk-averse borrower and can benefit from hedging. These hedging benefits are absent in our paper, given that we assume a risk-neutral borrower. In our paper (and also DeMarzo et al. (2021)), it is the bankruptcy cost that induces a *risk-neutral* borrower to hedge.

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Appendix

A Proofs

Proof of Proposition 1

Proof. We prove the result for J_H, and the one for J_L follows similar steps. Let $\theta_t = H$ and $\tau_\lambda \geq t$ be the time that the state switches from H to L. By the principle of dynamic programming,

$$\begin{aligned}
V_{t-} &= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b \wedge \tau_\lambda} e^{-\rho(s-t)} \Big((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \Big) \\
&\qquad\qquad + e^{-\rho(\tau_b \wedge \tau_\lambda - t)} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
&= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \Big) \\
&\qquad\qquad + e^{-\rho(\tau_\lambda - t)} V_{\tau_\lambda} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
&= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \Big) \\
&\qquad\qquad + e^{-\rho(\tau_\lambda - t)} \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\} \mathbb{1}_{\{\tau_b \geq \tau_\lambda\}} \Bigg] \\
&= \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s + p_s g_s F_s - y_{s-} D_{s-})\, ds + dD_s \\
&\qquad\qquad + \lambda \max\{J_s - D_{s-}, 0\}\, ds \Big) \Bigg],
\end{aligned}$$

where we have used the definition $V_{\tau_\lambda} = \max\{J_{\tau_\lambda} - D_{\tau_\lambda -}, 0\}$. Using the integration by parts formula for semi-martingales (Corollary 2 in Section 2.6 of Protter (2005)), we get

$$\mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} dD_s \right] = \mathbb{E}_t \left[e^{-(\rho+\lambda)(\tau_b - t)} D_{\tau_b} \right] - D_{t-} + \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} (\rho+\lambda) D_{s-} ds \right].$$

At the time of default, $D_{\tau_b} = 0$. Hence

$$\begin{aligned}
V_{t-} = \sup_{\tau_b, g_s, D_s} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big\{ & X_s - (r+\xi)F_s + p_s g_s F_s \\
& + (\rho + \lambda - y_{s-}) D_{s-} + \lambda \max\{J_L(X_s, F_s) - D_{s-}, 0\} \Big\} ds \Bigg] - D_{t-}.
\end{aligned}$$

Limite liability requires that $V_{t-} \geq 0$, so the amount of short-term borrowing must satisfy the constraint $D_{t-} \leq J_{\theta_t}(X_t, F_t)$. $\square$

Proof of Proposition 2

Proof. In the low state, the borrower chooses short-term debt $D_L = J_L^s$ and only defaults upon the disaster shock. So the short rate is $y_L^s(X) = r + \eta$, which implies the value of the firm is

$$J_L^s(X) = \frac{X}{r + \eta - \mu_L}.$$

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage, in which case

$$J_H^s(X) = \frac{X}{r + \lambda - \mu_H}.$$

On the other hand, if she borrows riskless debt up to $J_L^s(X)$, the firm value is

$$J_H^s(X) = \frac{X}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - r}{r + \eta - \mu_L}\right).$$

From here, we get that the value of the firm is

$$J_H^s(X) = X \max\left\{\frac{1}{r + \lambda - \mu_H}, \frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho + \lambda - r}{r + \eta - \mu_L}\right)\right\}.$$

$\square$

Proof of Proposition 3

First, we derive an equation for the scaled value function. Second, we prove the existence and uniqueness of a solution to the scaled value function and a single-crossing property in the high state.

Low state $\theta_t = L$. Let $g_\theta(f) = 0$, we can derive the following equation:

$$\begin{aligned}(\rho + \eta)J_L^0(X, F) = \max_{D_L^0 \in [0, J_L^0(X,F)]} & X - (r + \xi)F + \left(\rho + \eta - y_L^0\right)D_L^0 \\ & - \xi F \frac{\partial J_L^0(X, F)}{\partial F} + \mu_L X \frac{\partial J_L^0(X, F)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 J_L^0(X, F)}{\partial X^2}. \quad \text{(A.1)}\end{aligned}$$

Note that the choice of short-term debt D_L^0 is capped by the level of value function J_L^0, and the choice of D_L^0 also affects J_L^0. Substituting $J_\theta^0(X,F) = Xj_\theta^0(f)$ in equation (A.1), we get the following HJB for the scaled value function $j_L^0(f)$:

$$(\rho+\eta-\mu_L)\, j_L^0(f) = \max_{d_L^0 \in [0, j_L^0(f)]} 1-(r+\xi)\, f+\left(\rho+\eta-y_L^0\right) d_L-(\mu_L+\xi) f j_L^{0\prime}(f)+\frac{1}{2}\sigma^2 f^2 j_L^{0\prime\prime}(f). \tag{A.2}$$

Given that the coefficient in front of d_L^0 satisfies $\rho+\eta-y_L^0 = \rho - r > 0$, it is always optimal for the borrower to issue as much short-term debt as possible, which leads to $d_L^0(f) = j_L^0(f)$. The rest of the problem becomes standard. The borrower defaults if $f \geq f_L^b$, where f_L^b satisfies the value matching condition $j_L^0(f_L^b) = 0$ and the smooth pasting condition $j_L^{0\prime}(f_L^b) = 0$.

High state $\theta_t = H$. We arrive at the HJB:

$$\begin{aligned}(\rho+\lambda)J_H^0(X,F) = \max_{D_H^0 \in \left[0, J_H^0(X,F)\right]} & X-(r+\xi)\, F+\left(\rho+\lambda-y_H^0\right) D_H^0+\lambda \max\left\{J_L^0(X,F)-D_H^0, 0\right\} \\ & -\xi F \frac{\partial J_H^0(X,F)}{\partial F}+\mu_H X \frac{\partial J_H^0(X,F)}{\partial X}+\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 J_H^0(X,F)}{\partial X^2}. \end{aligned} \tag{A.3}$$

The scaled value function $j_H^0(f)$ therefore satisfies

$$\begin{aligned}(\rho+\lambda-\mu_H)\, j_H^0(f) = \max_{d_H^0 \in [0, j_H^0(f)]} & 1-(r+\xi)\, f+\left(\rho+\lambda-y_H^0\right) d_H^0+\lambda \max\left\{j_L^0(f)-d_H^0, 0\right\} \\ & -(\mu_H+\xi) f j_H^{0\prime}(f)+\frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f). \end{aligned} \tag{A.4}$$

As usual, the value function satisfies the value matching and smooth pasting conditions $j_H^0(f_H^b) = 0$ and $j_H^{0\prime}(f_H^b) = 0$ at the default boundary f_H^b.

The formulas for the value function $j_\theta^0(f)$ can be found in Proposition 12 in Internet Appendix A.2.

The rest of the proof consists of characterizing the optimal short-term debt policy in the high state, which consists of two parts. In the first part, we show the existence and uniqueness of a solution. In the second part, we prove a single-crossing property and therefore show that it is optimal for the borrower to issue riskless short-term debt $d_H^0 = j_L^0(f)$ if $f \leq f_\dagger$. We start by establishing the uniqueness of the equilibrium.

Existence and Uniqueness: For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t} \left(1 - (r+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f \right]$$

$$\text{subject to} \quad dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t,$$

where

$$\nu(z, \tilde{j}) \equiv \max_{d_H^0 \in [0, \tilde{j}]} \left(\rho + \lambda - y_H^0(z, d_H^0)\right) d_H^0 + \lambda \max\left\{ j_L^0(z) - d_H^0, 0 \right\} = \max\{(\rho + \lambda - r)\, j_L^0(z), (\rho - r)\, \tilde{j}\},$$

and $\hat{\rho} \equiv \rho + \lambda - \mu_H$. It follows from the HJB equation that the value function j_H^0 is a fixed point $j_H^0(f) = \Phi(j_H^0)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f, \tilde{j}_1) \geq \nu(f, \tilde{j}_0)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\nu(z, \tilde{j} + a) = \max\{(\rho + \lambda - r)\, j_L^0(z), (\rho - r)\, (\tilde{j} + a)\} \leq (\rho - r) a + \nu(z, \tilde{j}),$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j} + a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t} \left(1 - (r+\xi)z_t + \nu(z_t, \tilde{j}(z_t) + a)\right) dt \Big| z_0 = f \right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t} \left(1 - (r+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f \right] \\
&\quad + \frac{\rho - r}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f \right] a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t} \left(1 - (r+\xi)z_t + \nu(z_t, \tilde{j}(z_t))\right) dt \Big| z_0 = f \right] \\
&\quad + \frac{\rho - r}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f \right] a \\
&= \Phi(\tilde{j})(f) + \frac{\rho - r}{\hat{\rho}} \mathbb{E}\left[1 - e^{-\hat{\rho}\tau^*(\tilde{j}+a)} \Big| z_0 = f \right] a \leq \Phi(\tilde{j})(f) + \frac{\rho - r}{\rho + \lambda - \mu_H} a.
\end{aligned}$$

Thus, the operator Φ is monotone and satisfies discounting, it follows then by Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H^0(f) = \Phi(j_H^0)(f)$.

Optimal Short-Term Debt Policy: We start with the following result, which will be used later on. First, let

$$\bar{\lambda} \equiv -\frac{\rho-\mu_H}{2} + \sqrt{\left(\frac{\rho-\mu_H}{2}\right)^2 + (\rho-r)\left(\mu_H-\mu_L+\eta\right)}. \tag{A.5}$$

Lemma 1. *The condition*

$$(\rho+\lambda-r)\, j_L^0\,(0) > (\rho-r)\, j_H^0\,(0)\,.$$

is satisfied if and only if $\lambda > \bar{\lambda}$.

Proof. See Internet Appendix. □

Next, the following result shows that it is optimal for the borrower to issue $d_H^0 = j_L^0\,(f)$ when $f \leq f_\dagger$ and $d_H^0 = j_H^0\,(f)$ otherwise.

Lemma 2 (Single-crossing)**.** *There exists a unique $f_\dagger \in \left(0, f_L^b\right)$ such that $(\rho+\lambda-r)\, j_L^0\,(f) \geq (\rho-r)\, j_H^0\,(f)$ if and only if $f \leq f_\dagger$.*

Proof. See Internet Appendix. □

Proof of Proposition 4

Proof. When the borrower can commit to not issuing any long-term debt after $t=0$, i.e., $g_H\,(f) = g_L\,(f) \equiv 0$. The debt price in the low state satisfies the following asset pricing equation

$$(r+\xi+\eta)\, p_L^0(f) = (r+\xi) + \left(-\xi-\mu_L+\sigma^2\right) f p_L^{0'}\,(f) + \frac{1}{2}\sigma^2 f^2 p_L^{0''}\,(f)\,, \tag{A.6}$$

where the default boundary f_L^b is determined from Internet Appendix A.2. We guess the solution of the debt price takes the form

$$p_L^0(f) = A_1^0 + A_2^0 f^{\gamma_1 - 1} + A_3^0 f^{\gamma_2 - 1}.$$

Combining with $p_L^0(f_L^b) = 0$ and $\lim_{f\to 0} p_L^0(f) < \infty$, we know

$$p_L^0(f) = \frac{r+\xi}{r+\eta+\xi} - \frac{r+\xi}{r+\eta+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma_1-1}. \tag{A.7}$$

The equation for the price of debt in the high state depends on the value of λ. If $\lambda \leq \bar{\lambda}$. asset pricing equation is

$$(r+\xi+\lambda)\, p_H^0(f) = (r+\xi) + \left(-\xi-\mu_H+\sigma^2\right) f p_H^{0'}\,(f) + \frac{1}{2}\sigma^2 f^2 p_H^{0''}\,(f)\,, \tag{A.8}$$

where the default boundary f_H^b is determined from Internet Appendix A.2. We guess the solution of the debt price takes the form

$$p_H^0(f) = \tilde{B}_1^0 + \tilde{B}_2^0 f^{\beta_1 - 1} + \tilde{B}_3^0 f^{\beta_2 - 1}.$$

Combining with $p_H^0(f_H^b) = 0$ and $\lim_{f \to 0} p_H^0(f) < \infty$, we obtain

$$p_H^0(f) = \frac{r+\xi}{r+\xi+\lambda} - \frac{r+\xi}{r+\xi+\lambda} \left(\frac{f}{f_H^b} \right)^{\beta_1 - 1}, \tag{A.9}$$

where the default boundary f_H^b is determined from Internet Appendix A.2.

On the other hand, if $\lambda > \bar{\lambda}$, the price in the high state satisfies the asset pricing equation

$$(r+\xi+\lambda)\, p_H^0(f) = (r+\xi) + \lambda p_L^0(f) \mathbf{1}_{\{d_H^0(f) \le j_L^0(f)\}} + \left(-\xi - \mu_H + \sigma^2\right) f p_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 p_H^{0\prime\prime}(f), \tag{A.10}$$

where the default boundary f_H^b and the threshold $f_\dagger$ are determined from Internet Appendix A.2.

When $f \in [0, f_\dagger]$, we guess a solution of the form

$$p_H^0(f) = B_1^0 + B_2^0 f^{\gamma_1 - 1} + B_3^0 f^{\beta_1 - 1} + B_4^0 f^{\beta_2 - 1}. \tag{A.11}$$

Plugging into equation (A.10), we can get

$$B_1^0 = \frac{(r+\xi)(r+\eta+\xi+\lambda)}{(r+\xi+\lambda)(r+\eta+\xi)};$$
$$B_2^0 = \frac{-\lambda \frac{r+\xi}{r+\eta+\xi} \left(f_L^b\right)^{1-\gamma_1}}{r+\xi+\lambda - (-\xi - \mu_H + \sigma^2)(\gamma_1 - 1) - \frac{1}{2}\sigma^2 (\gamma_1 - 1)(\gamma_1 - 2)};$$

The condition $\lim_{f \to 0} p_H^0(f) < \infty$ implies that $B_4^0 = 0$, which implies B_3^0 is the only unknown in equation (A.11).

When $f \in \left(f_\dagger, f_H^b\right]$, we guess a solution of the form

$$p_H^0(f) = C_1^0 + C_2^0 f^{\beta_1 - 1} + C_3^0 f^{\beta_2 - 1}, \tag{A.12}$$

Plugging into equation (A.10), we can get $C_1^0 = \frac{r+\xi}{r+\xi+\lambda}$. $\left\{C_2^0, C_3^0\right\}$ are the unknowns in equation (A.12). In the end, we solve the three unknowns $\left\{B_3^0, C_2^0, C_3^0\right\}$ from the following boundary conditions:

$$p_H^0(f_H^b) = 0,$$
$$p_H^0(f_\dagger^-) = p_H^0(f_\dagger^+),$$
$$p_H^{0'}(f_\dagger^-) = p_H^{0'}(f_\dagger^+).$$

Having determined the price of long-term debt, we can analyze the issuance decision at $t = 0$. Let $H^0(f_0) = j_H^0(f_0) + p_H^0(f_0) f_0$ denote the value of the firm given long-term debt over cash flows f_0. When $\lambda \leq \bar{\lambda}$, from the closed-form solutions of $j_H^0(f)$ and $p_H^0(f)$, we know $p_H^0(f) = -j_H^0(f)$ for any $f_0 \in \left[0, f_H^b\right]$. Since $H^{0'}(f_0) = j_H^{0'}(f_0) + p_H^0(f_0) + p_H^{0'}(f_0) f_0 = -j_H^{0''}(f_0) f_0 < 0$, for any $f_0 > 0$, $H^0(f_0) < H^0(0) = \frac{J_H^s(X_t)}{X_t}$. This implies it is optimal for the borrower does not issue any long-term debt at $t = 0$ when $\lambda \leq \bar{\lambda}$.

When $\lambda > \bar{\lambda}$, $H^{0'}(f_0) = j_H^{0'}(f_0) + p_H^0(f_0) + p_H^{0'}(f_0) f_0$. From the closed-form solutions of $j_H^0(f)$ and $p_H^0(f)$, we can get

$$H^{0'}(f_0)\,|_{f_0=0} = \frac{r+\xi}{r+\eta+\xi} \frac{\eta(\rho - r)}{(r+\xi+\lambda)(\rho+\xi+\lambda)} > 0. \tag{A.13}$$

It implies there exists a $f_0^* > 0$ such that

$$H^0(f_0^*) = j_H^0(f_0^*) + p_H^0(f_0^*) f_0^* > H^0(0) = \frac{J_H^s(X_t)}{X_t}.$$

Therefore, the borrower optimally issues a positive amount of long-term debt f_0^* at $t = 0$. □

Proof of Proposition 6

First, we demonstrate that the joint value $j_\theta(f)$ under no commitment is equivalent to the value $j_\theta^0(f)$ without long-term debt issuance. Second, we prove $j_\theta(f)$ is strictly convex so that it is optimal for the borrower to issue long-term debt smoothly.

From the value function (7) and substituting $J_\theta(X, F) = X j_\theta(f)$, we get the following HJB equations for the scaled value function $j_\theta(f)$:

$$\begin{aligned}
(\rho+\eta-\mu_L)\, j_L(f) &= \max_{g_L,\, d_L\in[0,j_L(f)]} 1-(r+\xi)\, f+\Big(p_L(f)+j_L^{'}(f)\Big)\, g_L f+(\rho+\eta-y_L)\, d_L \\
&\quad -(\mu_L+\xi) f j_L^{'}(f)+\frac{1}{2}\sigma^2 j_L^{''}(f) \\
(\rho+\lambda-\mu_H)\, j_H(f) &= \max_{g_H,\, d_H\in[0,j_H(f)]} 1-(r+\xi)\, f+\Big(p_H(f)+j_H^{'}(f)\Big)\, g_H f+(\rho+\lambda-y_H)\, d_H \\
&\quad +\lambda\max\{j_L(f)-d_H,0\}-(\mu_H+\xi) f j_H^{'}(f)+\frac{1}{2}\sigma^2 f^2 j_H^{''}(f).
\end{aligned} \tag{A.14}$$

If the borrower finds it optimal to adjust debt smoothly, then it must be the case that this coefficient equals zero, or equivalently,

$$p_\theta(f)=-j_\theta^{'}(f). \tag{A.15}$$

Given any smooth equilibrium with value function $j_\theta(f)$, (A.14) and (A.15) imply (11) and (12), and as a result setting the issuance policy to $g_\theta=0$ does not change the equity value $j_\theta(f)$. Hence, the value $j_\theta(f)$ under this equilibrium could be obtained under no long-term debt issuance, which corresponds to $j_\theta^0(f)$.

We also need to verify that the equity holders cannot benefit from issuing an atom Δ, in which case they get a payoff $j_\theta(f+\Delta)+p_\theta(f+\Delta)\Delta$. The first order condition with respect to Δ is $j_\theta'(f+\Delta)+p_\theta(f+\Delta)+p_\theta'(f+\Delta)\Delta=p_\theta'(f+\Delta)\Delta=0$. Which means that either $\Delta=0$ or $p_\theta'(f+\Delta)=0$. The second order condition is $p_\theta''(f+\Delta)\Delta+p_\theta'(f+\Delta)\le 0$, which evaluated at $\Delta=0$ yields $p_\theta'(f)\le 0$. It follows from the first and second-order conditions that a smooth equilibrium requires that $p_\theta'(f)=-j_\theta^{''}(f)<0$.

From the closed-form solution for $j_L(f)$ we immediately obtain that $j_L^{''}(f)>0$. Next, we verify that $j_H(f)$ is also a strictly convex function on $\left[0,f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly.

Strict convexity of $j_H(f)$ on $\left[0,f_H^b\right]$. The proof relies on a few auxiliary lemmas.

Lemma 3.

$$j_H^{'}(f)\ge -1,\qquad \forall f\in\left[0,f_H^b\right],$$

Proof. See Internet Appendix. □

Lemma 4.

$$f_H^b>\frac{1}{r+\xi}\qquad \textit{and}\qquad \min\left\{j_H^{''}(0),j_H^{''}(f_H^b)\right\}>0,$$

Proof. See Internet Appendix. □

Lemma 5.

$$j_H'''(f_\dagger^-) > j_H'''(f_\dagger^+).$$

Proof. See Internet Appendix. □

Now we are ready to verify that the solution to the HJB equation is convex. We differentiate the HJB twice and let $\tilde{u} \equiv f j_H''$ to get

$$(\rho+\lambda+\xi)\,\tilde{u} = (\rho+\lambda-r)\,f j_L'' - \left(\mu_H+\xi-\sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in (0, f_\dagger) \tag{A.16}$$

$$(r+\lambda+\xi)\,\tilde{u} = -\left(\mu_H+\xi-\sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in (f_\dagger, f_H^b). \tag{A.17}$$

By the maximum principle in Theorem 1, $\tilde{u}$ cannot have an interior nonpositive local minimum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonpositive minimum is that $\tilde{u}(f_\dagger) < 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are positive, this requires that $j_H''(f_\dagger -) + f_\dagger j_H'''(f_\dagger -) = \tilde{u}'(f_\dagger -) < \tilde{u}'(f_\dagger +) = j_H''(f_\dagger +) + f_\dagger j_H'''(f_\dagger +)$. From the HJB equation it follows that j_H is twice continuously differentiable at $f_\dagger$, so such a kink would require $j_H'''(f_\dagger^-) < j_H'''(f_\dagger^+)$, which is ruled out by Lemma 5. We can conclude that $\tilde{u}$ does not have an interior nonpositive minimum, so it follows that $\tilde{u}(f) = f j_H''(f) > 0$ on $(0, f_H^b)$.

Proof of Proposition 7

Proof. First, we consider the slow state. The debt price satisfies the asset pricing equation

$$(r+\xi+\eta)\,p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f). \tag{A.18}$$

The indifference condition (16) implies that the price of debt is $p_L(f) = -j_L'(f)$. Substituting in (A.18) we get

$$(r+\xi+\eta)\,j_L'(f) = -(r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f). \tag{A.19}$$

Combining (A.18) and (A.19) we get

$$g_L(f) = 0.$$

In the high state, the debt price follows

$$(r+\xi+\lambda)\,p_H(f) = (r+\xi) + \lambda p_L(f) \mathbf{1}_{\{d_H(f) \le j_L(f)\}} + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f).$$

Suppose $\lambda > \bar{\lambda}$. First, consider the case when $f \in (0, f_{\dagger})$. Differentiating the HJB equation for $j_H(f)$ we obtain,

$$(r+\lambda+\xi)\, j_H'(f) + (r+\xi) - \lambda j_L'(f) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f)$$
$$= (\rho - r)\, j_L'(f) - (\rho - r)\, j_H'(f).$$

Combining with the indifference condition (16) – which requires that $p_H(f) = -j_H'(f)$ – we find that

$$g_H(f) = \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-f p_H'(f)}.$$

Finally, for $f \in (f_{\dagger}, f_H^b)$, we differentiate the HJB equation (A.4) to obtain,

$$(r+\lambda+\xi)\, j_H'(f) + (r+\xi) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0,$$

which combined with the optimality condition $p_H(f) = -j_H'(f)$ yields

$$g_H(f) = 0.$$

□

Proof of Corollary 1

Proof. We differentiate the HJB at $\theta = L$, which leads to

$$(r+\eta+\xi)\, j_L'(f) + (r+\xi) + \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f) = 0, \quad \forall f \in \left[0, f_L^b\right].$$

When $f \in [0, f_{\dagger}]$, we differentiate the HJB at $\theta = H$:

$$(\rho+\lambda+\xi)\, j_H'(f) + (r+\xi) - (\rho+\lambda-r)\, j_L'(f) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0.$$

The difference is

$$\frac{1}{2}\sigma^2 f^2 \left[j_L'''(f) - j_H'''(f)\right] - \left(\mu_H + \xi - \sigma^2\right) f \left(j_L''(f) - j_H''(f)\right) - (\rho+\lambda+\xi)\left(j_L'(f) - j_H'(f)\right)$$
$$= \eta j_L'(f) - (\mu_H - \mu_L)\, f j_L''(f) < 0$$

since $j_L'(f) \leq 0$ and $j_L''(f) > 0$ from the strict convexity. By the maximum principle, $\triangle(f) = j_L'(f) - j_H'(f)$ can not have a nonpositive minimum in the region $f \in [0, f_\dagger]$.

When $f \in \left(f_\dagger, f_L^b\right]$, we differentiate the HJB at $\theta = H$:

$$(r+\lambda+\xi)\, j_H^{'}(f) + (r+\xi) + \left(\mu_H + \xi - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = 0.$$

The difference is

$$\begin{aligned} &\frac{1}{2}\sigma^2 f^2 \left(j_L'''(f) - j_H'''(f)\right) - \left(\mu_H + \xi - \sigma^2\right) f \left(j_L''(f) - j_H''(f)\right) \\ &- (r+\lambda+\xi)\left(j_L^{'}(f) - j_H^{'}(f)\right) = (\eta - \lambda)\, j_L^{'}(f) - (\mu_H - \mu_L)\, f j_L''(f) < 0 \end{aligned}$$

where we assume $\eta \geq \lambda$ and $j_L'(f) \leq 0$ and $j_L''(f) > 0$ from the strict convexity. By the maximum principle, $\triangle(f) \equiv j_L'(f) - j_H'(f)$ can not have a nonpositive minimum in the region $f \in \left(f_\dagger, f_L^b\right]$. Since both $j_L'(f)$ and $j_H'(f)$ are continuous for all $f \in \left[0, f_L^b\right]$, $\triangle(f)$ is continuous for all $f \in \left[0, f_L^b\right]$. It implies that $\triangle(f)$ can not have a nonpositive minimum in the region $f \in \left[0, f_L^b\right]$.

In addition, given that

$$\begin{aligned} \triangle(0) &= -\frac{r+\xi}{r+\eta+\xi} + \frac{(r+\xi)(\eta+\rho+\lambda+\xi)}{(\rho+\lambda+\xi)(r+\eta+\xi)} = \frac{r+\xi}{r+\eta+\xi}\frac{\eta}{\rho+\lambda+\xi} > 0, \\ \triangle\left(f_L^b\right) &= -j_H'(f_L^b) > 0, \end{aligned}$$

we know $\triangle(f) > 0$ for any $f \in \left[0, f_L^b\right]$.

From $p_L(f) = -j_L'(f)$, $p_H(f) = -j_H'(f)$, we know $p_H(f) > p_L(f)$ for any $f \in \left[0, f_L^b\right]$. Furthermore, when $f \in \left[0, f_L^b\right]$, the firm never repurchases the long-term debt since

$$g(f) = \frac{(\rho - r)(p_H(f) - p_L(f))}{-f p_H'(f)} = \frac{(\rho - r)(p_H(f) - p_L(f))}{f j_H''(f)} > 0.$$

□

Proof of Proposition 8

Proof. When $\lambda > \bar{\lambda}$ and $\theta = H$, in the region $f < f_\dagger$, the issuance function is given by

$$g_H(f) f = \frac{(\rho - r)(j_L'(f) - j_H'(f))}{-p_H'(f)}.$$

As $\gamma > 1$ and $\phi > 1$, we have that

$$\lim_{f \to 0} j_L'(f) + 1 = \frac{\eta}{r+\eta+\xi},$$
$$\lim_{f \to 0} \left(j_L'(f) - j_H'(f)\right) = \frac{\eta}{\rho+\lambda+\xi} \frac{r+\xi}{r+\eta+\xi},$$

which are strictly positive if $\eta > 0$. Therefore, $\lim_{f \to 0}(g_H(f)f)$ is strictly positive as long as $p_H'(0) > -\infty$, which requires that $\gamma \geq 2$ and $\phi \geq 2$.

□

B Details of Firm Value Decomposition in Subsection 3.2

We derive the equations under $\lambda > \bar{\lambda}$. The case $\lambda \leq \bar{\lambda}$ relates to $f_\dagger = 0$. For any given F_0, we know

$$F_t = F_0 e^{-\xi t}.$$

The joint value and the value of long-term debt are

$$J_H^0(X_0, F_0) = \mathbb{E}_0 \left[\int_0^{\tau_b \wedge \tau_\lambda} e^{-\rho t} \Big\{ X_t - (r+\xi)F_t + (\rho - r)D_{t-} \Big\} dt + e^{-\rho \tau_b \wedge \tau_\lambda} J_L^0(X_{\tau_\lambda}, F_{\tau_\lambda}) \mathbb{1}_{\tau_\lambda < \tau_b} \right]$$
$$F_0 P_H^0(X_0, F_0) = \mathbb{E}_0 \left[\int_0^{\tau_b \wedge \tau_\lambda} e^{-rt}(r+\xi)F_t dt + e^{-r\tau_b \wedge \tau_\lambda} P_L^0(X_{\tau_\lambda}, F_{\tau_\lambda}) F_{\tau_\lambda} \mathbb{1}_{\tau_\lambda < \tau_b} \right]$$
$$J_L^0(X_0, F_t) = \mathbb{E}_0 \left[\int_0^{\tau_b} e^{-\rho t} \Big\{ X_t - (r+\xi)F_t + (\rho - r)D_{t-} \Big\} dt \right]$$
$$F_0 P_L^0(X_0, F_0) = \mathbb{E}_0 \left[\int_0^{\tau_b} e^{-rt}(r+\xi)F_t dt \right].$$

Let us define $W_\theta = J_\theta + FP_\theta^0$ as the firm value, then

$$W_L(X_{\tau_\lambda}, F_{\tau_\lambda}) = \mathbb{E}_{\tau_\lambda} \left[\int_{\tau_\lambda}^{\tau_b} e^{-\rho(t-\tau_\lambda)} X_t dt \right] + \mathbb{E}_{\tau_\lambda} \left[\int_{\tau_\lambda}^{\tau_b} e^{-\rho(t-\tau_\lambda)} (\rho - r) D_{t-} dt \right]$$
$$+ \mathbb{E}_{\tau_\lambda} \left[\int_{\tau_\lambda}^{\tau_b} \left(e^{-r(t-\tau_\lambda)} - e^{-\rho(t-\tau_\lambda)} \right) (r+\xi) F_t dt \right]$$

and

$$
\begin{aligned}
W_H(X_0,F_0) = \mathbb{E}_0&\left[\int_0^{\tau_b\wedge\tau_\lambda} e^{-\rho t}X_t dt + e^{-\rho\tau_b\wedge\tau_\lambda}W_L(X_{\tau_\lambda},F_{\tau_\lambda})\mathbb{1}_{\tau_\lambda<\tau_b}\right] \\
&+\mathbb{E}_0\left[\left(e^{-r\tau_b\wedge\tau_\lambda}-e^{-\rho\tau_b\wedge\tau_\lambda}\right)P_L^0(X_{\tau_\lambda},F_{\tau_\lambda})F_{\tau_\lambda}\mathbb{1}_{\tau_\lambda<\tau_b}\right] \\
&+\mathbb{E}_0\left[\int_0^{\tau_b\wedge\tau_\lambda} e^{-\rho t}(\rho-r)D_{t-}dt\right]+\mathbb{E}_0\left[\int_0^{\tau_b\wedge\tau_\lambda}\left(e^{-rt}-e^{-\rho t}\right)(r+\xi)F_t dt\right].
\end{aligned}
$$

From here, we can rewrite firm value as

$$
\begin{aligned}
W_H(X_0,F_0) = &\underbrace{\mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t}X_t dt\right]}_{\text{unlevered firm value}}+\underbrace{\mathbb{E}_0\left[\int_0^{\tau_b}\left(e^{-rt}-e^{-\rho t}\right)(r+\xi)F_t dt\right]}_{\text{benefit LT debt}}+\underbrace{\mathbb{E}_0\left[\int_0^{\tau_b} e^{-\rho t}(\rho-r)D_{t-}dt\right]}_{\text{benefit ST debt}} \\
&-\underbrace{\mathbb{E}_0\left[\int_{\tau_b}^{\infty} e^{-\rho t}X_t dt\right]}_{\text{bankruptcy cost}}.
\end{aligned}
$$

The remainder of the proof solves term one by one.

- Unlevered firm value

$$
\mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t}X_t dt\right]=\frac{X_0}{\rho+\lambda-\mu_H}+\frac{X_0}{\rho+\lambda-\mu_H}\frac{\lambda}{\rho+\eta-\mu_L}.
$$

- Benefit from long-term debt

$$
\begin{aligned}
\mathbb{E}_0\left[\int_0^{\tau_b}\left(e^{-rt}-e^{-\rho t}\right)(r+\xi)F_t dt\right] &= (r+\xi)F_0\mathbb{E}_0\left[\int_0^{\tau_b}\left(e^{-(r+\xi)t}-e^{-(\rho+\xi)t}\right)dt\right] \\
&= F_0\left(\frac{\rho-r}{\rho+\xi}+\frac{r+\xi}{\rho+\xi}\mathbb{E}_0\left[e^{-(\rho+\xi)\tau_b}\right]-\mathbb{E}_0\left[e^{-(r+\xi)\tau_b}\right]\right).
\end{aligned}
$$

Define $\varphi_\theta(f|\alpha)=\mathbb{E}\left[e^{-(\alpha+\xi)(\tau_b-t)}|f_t=f\right]$, we can derive the following HJB equations

$$
\begin{aligned}
(\alpha+\xi+\eta)\,\varphi_L(f|\alpha) &= \eta-(\mu_L+\xi-\sigma^2)f\varphi_L'(f|\alpha)+\frac{1}{2}\sigma^2 f^2\varphi_L''(f|\alpha) \quad f\in[0,f_L^b) \\
(\alpha+\xi+\lambda)\,\varphi_H(f|\alpha) &= \lambda\varphi_L(f|\alpha)-(\mu_H+\xi-\sigma^2)f\varphi_H'(f|\alpha)+\frac{1}{2}\sigma^2 f^2\varphi_H''(f|\alpha) \quad f\in[0,f_\dagger] \\
(\alpha+\xi+\lambda)\,\varphi_H(f|\alpha) &= \lambda-(\mu_H+\xi-\sigma^2)f\varphi_H'(f|\alpha)+\frac{1}{2}\sigma^2 f^2\varphi_H''(f|\alpha) \quad f\in[f_\dagger,f_H^b).
\end{aligned}
$$

The boundary conditions include value matching and smooth pasting at $f_\dagger$. In addition, we need for $\theta \in \{H, L\}$, $\varphi_\theta(0|\alpha)$ is finite and $\varphi_\theta(f_\theta^b|\alpha) = 1$.

- Benefit from short-term debt. Let

$$s_\theta(f) = \frac{1}{X_0}\mathbb{E}_0\left[\int_0^{\tau_b} e^{-\rho t}(\rho - r)D_{t-}dt\right].$$

We can derive the following HJB equations after taking into account the optimal short-term debt issuance:

$$(\rho + \eta - \mu_L)\, s_L(f) = (\rho - r)\, j_L(f) - (\mu_L + \xi) f s_L^{'}(f) + \frac{1}{2}\sigma^2 f^2 s_L^{''}(f) \quad f \in [0, f_L^b)$$
$$(\rho + \lambda - \mu_H)\, s_H(f) = (\rho - r)\, j_L(f) + \lambda s_L(f) - (\mu_H + \xi) f s_H^{'}(f) + \frac{1}{2}\sigma^2 f^2 s_H^{''}(f) \quad f \in [0, f_\dagger]$$
$$(\rho + \lambda - \mu_H)\, s_H(f) = (\rho - r)\, j_H(f) - (\mu_H + \xi) f s_H^{'}(f) + \frac{1}{2}\sigma^2 f^2 s_H^{''}(f) \quad f \in [f_\dagger, f_H^b).$$

The boundary conditions include value matching and smooth pasting at $f_\dagger$. In addition, we need for $\theta \in \{H, L\}$, $s_\theta(0)$ is finite and $s_\theta(f_\theta^b) = 0$.

- Bankruptcy cost. Define

$$c_\theta(f) = \frac{1}{X_0}\mathbb{E}_0\left[\int_{\tau_b}^{\infty} e^{-\rho t}X_t dt\right].$$

We can derive the following HJB equations

$$(\rho + \eta - \mu_L)\, c_L(f) = 0 - (\mu_L + \xi) f s_L^{'}(f) + \frac{1}{2}\sigma^2 f^2 s_L^{''}(f) \quad f \in [0, f_L^b)$$
$$(\rho + \lambda - \mu_H)\, c_H(f) = \lambda c_L(f) - (\mu_H + \xi) f c_H^{'}(f) + \frac{1}{2}\sigma^2 f^2 c_H^{''}(f) \quad f \in [0, f_\dagger]$$
$$(\rho + \lambda - \mu_H)\, c_H(f) = \frac{\lambda}{\rho + \eta - \mu_L} - (\mu_H + \xi) f c_H^{'}(f) + \frac{1}{2}\sigma^2 f^2 c_H^{''}(f) \quad f \in [f_\dagger, f_H^b).$$

The boundary conditions include value matching and smooth pasting at $f_\dagger$. In addition, we need for $\theta \in \{H, L\}$, $c_\theta(0)$ is finite and

$$c_H(f_H^b) = \frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\lambda}{\rho + \eta - \mu_L}\right)$$
$$c_L(f_L^b) = \frac{1}{\rho + \eta - \mu_L}.$$

Internet Appendix for "Debt Maturity Management"

Yunzhi Hu Felipe Varas Chao Ying

This Internet Appendix contains additional analysis to accompany the manuscript. Section A provides the remaining proofs for the analysis in Section 3, including all technical lemmas. Section B provides the details for Section 4.

A Remaining proofs

Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

Theorem 1 (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{A.20}$$

in an interval $(0, b)$ with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Theorem 2 (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that u is a nonconstant solution of the differential inequality* (A.20) *having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

Corollary 2. *If u satisfies* (A.20) *in an interval (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u(a) \leq 0$, $u(b) \leq 0$, then $u(x) < 0$ in (a, b) unless $u \equiv 0$.*

A.1 Proofs of Auxiliary Lemmas

Proof of Lemma 1

Proof. The proof of Proposition 2 makes it clear that the condition $\lambda > \bar{\lambda}$ guarantees that

$$(\rho + \lambda - r)\, j_L^0\,(0) > (\rho - r)\, j_H^0\,(0)\,.$$

This inequality is satisfied only if

$$\frac{\rho + \lambda - r}{\rho - r} > \frac{\rho + \lambda + \eta - \mu_L}{\rho + \lambda - \mu_H}.$$

Combining terms, we can write this as the following quadratic inequality

$$\lambda^2 + (\rho - \mu_H)\,\lambda - (\rho - r)\,(\mu_H - \mu_L + \eta) > 0.$$

The left hand side is positive if and only if λ is greater than the unique positive root of the quadratic equation for $\bar{\lambda}$

$$\bar{\lambda}^2 + (\rho - \mu_H)\,\bar{\lambda} - (\rho - r)\,(\mu_H - \mu_L + \eta) = 0,$$

which is given by (A.5). □

Proof of Lemma 2

Proof. Define $a \equiv 1 + \frac{\lambda}{\rho - r}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function u let,

$$L^{0\dagger}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\, f u' - (\rho + \lambda - \mu_H)\, u$$
$$L^{\dagger b}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi)\, f u' - (r + \lambda - \mu_H)\, u.$$

The HJB in state $\theta = H$ can be written as

$$L^{0\dagger} j_H^0 + 1 - (r + \xi)\, f + (\rho + \lambda - r)\, j_L^0 = 0,\ \ f \in (0, f_\dagger)$$
$$L^{\dagger b} j_H^0 + 1 - (r + \xi)\, f = 0,\ \ f \in (f_\dagger, f_H^b).$$

Similarly, the HJB in state $\theta = L$ can be written as

$$L^{0\dagger} a j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} + a\left(\rho + \lambda - (r+\eta) + \mu_L - \mu_H\right) j_L^0 + a\left(1 - (r+\xi) f\right) = 0$$
$$L^{\dagger b} a j_L^0 + a(\mu_H - \mu_L) f j_L^{0'} - a\left(\mu_H - \mu_L + \eta - \lambda\right) j_L^0 + a\left(1 - (r+\xi) f\right) = 0.$$

Therefore, we have

$$L^{0\dagger}\left(a j_L^0 - j_H^0\right) + H(f) = 0$$
$$L^{\dagger b}\left(a j_L^0 - j_H^0\right) + H(f) = 0,$$

where the function $H(f)$ defined as

$$H(f) \equiv a(\mu_H - \mu_L) f j_L^{0'} - a\left(\mu_H - \mu_L + \eta - \lambda\right) j_L^0 + (a-1)\left(1 - (r+\xi) f\right),$$

and

$$\begin{aligned} H''(f) &= \left[(\mu_H - \mu_L) a \frac{f j_L^{0'''}}{j_L^{0''}} + (\mu_H - \mu_L) a + a\left(\lambda - \eta\right)\right] j_L^{0''} \\ &= \left[(\mu_H - \mu_L)(\gamma - 1) + \lambda - \eta\right] a j_L^{0''}. \end{aligned} \tag{A.21}$$

We need to distinguish two cases. If $\lambda \geq \eta - (\mu_H - \mu_L)(\gamma - 1)$, $H''(f) \geq 0$, which implies $H(f)$ is convex and the maximum of $H(f)$ on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. Evaluating $H(f)$ at the two boundaries and using the hypothesis $\lambda > \bar{\lambda}$, we have

$$\left(\rho + \lambda - \mu_H\right) a - \left(\rho + \lambda - \eta - \mu_L\right) > 0,$$

from Lemma 1. Then, we get

$$\begin{aligned} H(0) &= -a\left(\mu_H - \mu_L + \eta - \lambda\right) \frac{1}{r + \eta - \mu_L} + (a-1) \\ &= \frac{1}{r + \eta - \mu_L}\left(\left(\rho + \lambda - \mu_H\right) a - \left(\rho + \lambda - \eta - \mu_L\right)\right) > 0, \\ H(f_L^b) &= (a-1)\left(1 - (r+\xi) f_L^b\right) < 0. \end{aligned}$$

Therefore, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.

- On $f \in [0, f_{\dagger}]$, we know $H(f) > 0$ and $L^{0\dagger}\left(aj_L^0 - j_H^0\right) < 0$ on $[0, f_{\dagger}]$. Using Theorem 1, we know that $aj_L^0(f) - j_H^0(f)$ cannot have a negative interior minimum on $[0, f_{\dagger}]$. Given $aj_L^0(0) - j_H^0(0) > 0$, we know that $aj_L^0(f) - j_H^0(f) > 0$, $\forall f \in [0, f_{\dagger})$. Moreover, Theorem 2 and Corollary 2 imply $aj_L^{0'}(f_{\dagger}) - j_H^{0'}(f_{\dagger}) < 0$.
- On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}\left(aj_L^0 - j_H^0\right) \geq 0$. Using Theorem 1, we know that $aj_L^0(f) - j_H^0(f)$ cannot have a nonnegative interior maximum. Given that $aj_L^0\left(f_L^b\right) - j_H^0\left(f_L^b\right) < 0$, $aj_L^0(f) - j_H^0(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
- On $f \in [f_{\dagger}, f']$. Suppose there exists a $f'' \in (f_{\dagger}, f')$ such that $aj_L^0(f'') - j_H^0(f'') > 0$. Given that $aj_L^0(f_{\dagger}) - j_H^0(f_{\dagger}) = 0$ and $aj_L^{0'}(f_{\dagger}) - j_H^{0'}(f_{\dagger}) < 0$, it must be that $aj_L^0(f) - j_H^0(f)$ has a nonpositive interior minimum on $[f_{\dagger}, f'']$. Meanwhile, from $L^{\dagger b}(aj_L^0(f) - j_H^0(f)) \leq 0$ for $f \in (f_{\dagger}, f'')$, we know from Theorem 1 that $aj_L^0(f) - j_H^0(f)$ cannot have a nonpositive interior minimum on $(f_{\dagger}, f'')$, which constitutes a contradiction.

- Case 2: $f' \leq f_{\dagger}$.
  - On $f \in [f_{\dagger}, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}\left(aj_L^0 - j_H^0\right) \leq 0$. From Theorem 1 and 2, we know $aj_L^0(f) - j_H^0(f) \leq 0$ and $aj_L^{0'}(f_{\dagger}) - j_H^{0'}(f_{\dagger}) \leq 0$.
  - On $f \in [f', f_{\dagger}]$, $L^{0\dagger}\left(aj_L^0 - j_H^0\right) \geq 0$ so that $aj_L^0(f) - j_H^0(f)$ cannot have a nonnegative interior maximum. Together with $aj_L^{0'}(f_{\dagger}) - j_H^{0'}(f_{\dagger}) \leq 0$, this shows $aj_L^0(f) - j_H^0(f) \geq 0$.
  - On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}\left(aj_L^0 - j_H^0\right) < 0$ on $[0, f_{\dagger}]$. Using Theorem 1, we know that $aj_L^0(f) - j_H^0(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L^0(0) - j_H^0(0) > 0$, we know that $aj_L^0(f) - j_H^0(f) > 0$, $\forall f \in [0, f')$.

□

Proof of Lemma 3

Proof. Let $\hat{u} = j_H^{'}(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (A.36) that $\hat{u}(0) \geq 0$ and (A.32) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\,\hat{u} = -(\rho + \lambda - r)\,(j_L^{'} + 1) < 0, \qquad f \in [0, f_{\dagger}]$$
$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\,\hat{u} = -\lambda < 0 \qquad f \in [f_{\dagger}, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. □

Proof of Lemma 4

Proof. For any $f \leq \frac{1}{r+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1-(r+\xi)\,f+(\rho+\lambda-y)\,d>0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1}{r+\xi}$. Plugging (A.31) and (A.32) into the HJB equation for $j_H(f)$, we get $j_H^{''}\left(f_H^b\right)$ whenever $f_H^b > \frac{1}{r+\xi}$.

Next, let us turn to prove that $j_H^{''}(0) \geq 0$. Let us define $u \equiv j_H^{'}$ and differentiate the HJB equation once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H+\xi-\sigma^2\right) f u' - (\rho+\lambda+\xi)\,u = (r+\xi) - (\rho+\lambda-r)\,j_L^{'}.$$

Moreover, let z be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H+\xi-\sigma^2\right) f z' - (\rho+\lambda+\xi)\,z = (r+\xi) - (\rho+\lambda-r)\,j_L^{'}(0)$$

with boundary conditions

$$\lim_{f\downarrow 0} z(f) < \infty$$
$$z(f_\dagger) = u(f_\dagger) = j_H^{'}(f_\dagger).$$

The solution is

$$z(f) = -\frac{r+\xi}{\rho+\lambda+\xi} + \frac{(\rho+\lambda-r)\,j_L^{'}(0)}{\rho+\lambda+\xi} + \left(j_H^{'}(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-r)\,j_L^{'}(0)}{\rho+\lambda+\xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1 = \frac{\left(\mu_H+\xi-\frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu_H+\xi-\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho+\lambda+\xi)}}{\sigma^2} > 0.$$

Let $\delta(f) = z - u$. It is easily verified that $\delta(0) = 0$ and $\delta(f_\dagger) = 0$. Moreover, δ satisfies

$$\frac{1}{2}\sigma^2 f^2 \delta'' - \left(\mu_H+\xi-\sigma^2\right) f\delta' - (\rho+\lambda+\xi)\,\delta = (\rho+\lambda-r)\left(j_L^{'}(f) - j_L^{'}(0)\right) \geq 0.$$

By Theorem 1, δ cannot have an interior nonnegative maximum, and the maximum is attained at

$f = 0$. Theorem 2 further implies $\delta'(0) < 0$ so $u'(0) > z'(0)$. Finally, we know that

$$z'(f) = \omega_1 \left(j_H^{'}(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-r)\, j_L^{'}(0)}{\rho+\lambda+\xi} \right) f_\dagger^{-\omega_1} f^{\omega_1 - 1} = \omega_1 \left(j_H^{'}(f_\dagger) + 1 \right) f_\dagger^{-\omega_1} f^{\omega_1 - 1},$$

which implies $z'(f) \geq 0$ given that $j_H^{'}(f_\dagger) \geq -1$. Therefore, $u'(0) = j_H^{''}(0) > 0$. □

Proof of Lemma 5

Proof. We differentiate the HJB (A.4) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2 (j_H^{'''}(f_\dagger+) - j_H^{''''}(f_\dagger-)) = (\rho - r)\left[a j_L^{'}(f_\dagger) - j_H^{'}(f_\dagger) \right],$$

where $a \equiv 1 + \frac{\lambda}{\rho - r}$ The proof of Proposition 2 shows $a j_L^{'}(f_\dagger) - j_H^{'}(f_\dagger) < 0$ so that $j_H^{''''}(f_\dagger-) > j_H^{'''}(f_\dagger+)$. □

A.2 Detailed Solutions of the Joint Value $j_\theta^0(f)$

Proposition 12 (Value function). *In state $\theta = L$, the joint continuation value is*

$$j_L^0(f) = \underbrace{\frac{1}{r+\eta-\mu_L} - \frac{r+\xi}{r+\eta+\xi} f}_{\textit{no default value}} + \underbrace{\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^\gamma}_{\textit{default option}}, \tag{A.22}$$

where $\gamma > 1$ is provided in equation (A.27)*, and the default boundary f_L^b in* (A.28)*.*

In state $\theta = H$:

- *If $\lambda > \bar{\lambda}$, the joint continuation value is*

$$j_H^0(f) = \begin{cases} u_0(f) + \left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^\phi & f \in [0, f_\dagger) \\ u_1(f) + \left(j_H^0(f_\dagger) - u_1(f_\dagger)\right) h_0(f) - u_1(f_H^b) h_1(f) & f \in \left[f_\dagger, f_H^b\right], \end{cases} \tag{A.23}$$

where $\phi > 1$ is provided in (A.35)*, and*

$$u_0(f) = \underbrace{\frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right) - \frac{r+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-r}{r+\eta+\xi}\right)f}_{\text{no default value}} + \underbrace{\delta\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\text{default option in low state}} \tag{A.24}$$

$$u_1(f) = \underbrace{\frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi}f}_{\text{no default value}}. \tag{A.25}$$

The discount factors δ, $h_0(\cdot)$, and $h_1(\cdot)$ are defined in equations (A.37) *and* (A.40)*. The boundaries $f_\dagger$ and f_H^b are determined using the boundary conditions equation* (A.30) *and equation*(A.32)*.*

- *If $\lambda \leq \bar{\lambda}$, the joint continuation value is*

$$j_H^0(f) = \underbrace{\frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi}f}_{\text{no default value}} + \underbrace{\frac{r+\xi}{r+\lambda+\xi}\frac{f_H^b}{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_1}}_{\text{default option}} \tag{A.26}$$

where $\beta_1 > 1$ is provided in equation (A.38)*, and the default boundary f_H^b in* (A.45)*.*

Solution to the HJB equation in the low state. Equation (11) is a second-order ODE, and a standard solution takes the form

$$j_L^0(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$\begin{aligned}
A_0 &= \frac{1}{r+\eta-\mu_L} \\
A_1 &= \frac{r+\xi}{r+\eta+\xi} \\
\gamma_1 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\eta-\mu_L\right)}}{\sigma^2} > 1 \\
\gamma_2 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\eta-\mu_L\right)}}{\sigma^2} < 0.
\end{aligned} \tag{A.27}$$

The condition $\lim_{f\to 0} j_L^0(f) < \infty$ implies $A_3 = 0$. We define $\gamma \equiv \gamma_1$. Combining with value-matching and smooth-pasting condition, we get the default boundary is

$$f_L^b = \frac{\gamma}{\gamma - 1} \frac{r + \eta + \xi}{(r + \eta - \mu_L)(r + \xi)}. \tag{A.28}$$

Therefore, the joint value at $\theta = L$ is

$$j_L^0(f) = \frac{1}{r + \eta - \mu_L} - \frac{r + \xi}{r + \eta + \xi} f + \frac{r + \xi}{r + \eta + \xi} \frac{f_L^b}{\gamma} \left(\frac{f}{f_L^b} \right)^{\gamma}.$$

Solution to the HJB equation in the high state for $f_\dagger > 0$. The value function satisfies equation (A.4) together with the boundary conditions

$$j_H^0(f_\dagger -) = j_H^0(f_\dagger +) \tag{A.29}$$
$$j_H^{0\prime}(f_\dagger -) = j_H^{0\prime}(f_\dagger +) \tag{A.30}$$
$$j_H^0(f_H^b) = 0 \tag{A.31}$$
$$j_H^{0\prime}(f_H^b) = 0 \tag{A.32}$$
$$\lim_{f\to 0} j_H^0(f) < \infty \tag{A.33}$$
$$j_H^0(f_\dagger) = \frac{\rho + \lambda - r}{\rho - r} j_L^0(f_\dagger). \tag{A.34}$$

First, we consider the solution for $f \in [0, f_\dagger]$, in which region the value function satisfies the equation

$$(\rho + \lambda - \mu_H) j_H^0(f) = 1 - (r + \xi) f + (\rho + \lambda - r) j_L^0(f) - (\mu_H + \xi) f j_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f).$$

The unique solution to this ODE satisfying condition (A.33) takes the form

$$j_H^0(f) = u_0(f) + B f^{\phi},$$

where the coefficient ϕ is given by

$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (\rho + \lambda - \mu_H)}}{\sigma^2} > 1, \tag{A.35}$$

and a particular solution u_0 is given by

$$u_0(f) = \underbrace{\frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r+\eta-\mu_L}\right) - \frac{r+\xi}{\rho+\lambda+\xi}\left(1+\frac{\rho+\lambda-r}{r+\eta+\xi}\right) f}_{\text{no default value}}$$

$$+ \underbrace{\delta \frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\text{default option in low state}} \quad \text{(A.36)}$$

where the discount factor δ is

$$\delta \equiv \frac{\rho+\lambda-r}{\rho+\lambda-r-\eta+(\mu_H-\mu_L)(\gamma-1)} \in (0,1). \quad \text{(A.37)}$$

From the Feynman-Kac formula, we know that the solution to the particular solution admits the following stochastic representation:

$$u_0(f) = \mathbb{E}_0\left[\int_0^\infty e^{-(\rho+\lambda-\mu_H)t}\left(1-(r+\xi)\,\tilde{f}_t + (\rho+\lambda-r)\,j_L^0(\tilde{f}_t)\right)dt\right]$$

where $\tilde{f}_t$ corresponds to the process

$$d\tilde{f}_t = -(\mu_H+\xi) f_t dt - \sigma f_t d\tilde{B}_t, \quad \tilde{f}_0 = f$$

for some Brownian motion $\tilde{B}_t$. Equation (A.36) follows by Girsanov's theorem after a change of measure using the Radon-Nikodym derivative $e^{-\mu_H t}(X_t/X_0)$.

The coefficient B is pinned down from the value at $j_H^0\,(f_\dagger)$

$$B = f_\dagger^{-\phi}\left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H^0(f) = u_0(f) + \left(j_H^0(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi}, \qquad \forall f \in [0, f_\dagger],$$

where $j_H^0\,(f_\dagger) = \frac{\rho+\lambda-r}{\rho-r} j_L^0\,(f_\dagger)$. The solution on the interval $\left[f_\dagger, f_H^b\right]$ can be obtained in a similar way. In this interval, the value function satisfies the equation

$$(r+\lambda-\mu_H)\,j_H^0(f) = 1-(r+\xi)\,f + \mathcal{D}^H j_H^0(f).$$

The homogeneous equation

$$(r+\lambda-\mu_H)\,\varphi=\mathcal{D}^H\varphi$$

has two solution f^{β_1} and f^{β_2}, where

$$\begin{aligned}\beta_1&=\frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\lambda-\mu_H\right)}}{\sigma^2}>1\\ \beta_2&=\frac{\mu_H+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\lambda-\mu_H\right)}}{\sigma^2}<0.\end{aligned}\tag{A.38}$$

Hence, the value function takes the form

$$j_H^0\,(f)=u_1(f)+D_1f^{\beta_1}+D_2f^{\beta_2}.$$

As before, the particular solution

$$u_1(f)=\frac{1}{r+\lambda-\mu_H}-\frac{r+\xi}{r+\lambda+\xi}f\tag{A.39}$$

admits the representation

$$u_1(f)=\mathbb{E}_0\left[\int_0^\infty e^{-(r+\lambda-\mu_H)t}\Big(1-(r+\xi)\tilde{f}_t\Big)dt\right],$$

which, after an appropriate change of measure, can be written as equation (A.39). Finally, by combining equations (A.29) and (A.31), we get

$$\begin{aligned}D_1&=\frac{j_H^0(f_\dagger)+u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}-u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}\\ D_2&=(f_H^b)^{-\beta_2}\left(-u_1(f_H^b)-D_1(f_H^b)^{\beta_1}\right).\end{aligned}$$

It follows that the solution to the value function on this interval is given by

$$j_H^0(f)=u_1(f)+\big(j_H^0(f_\dagger)-u_1(f_\dagger)\big)h_0\left(f,f_\dagger,f_H^b\right)-u_1(f_H^b)h_1\left(f,f_\dagger,f_H^b\right),$$

where

$$
\begin{aligned}
h_0\left(f \middle| f_\dagger, f_H^b\right) &= \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \\
h_1\left(f \middle| f_\dagger, f_H^b\right) &= \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}.
\end{aligned}
\tag{A.40}
$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions (A.30) and (A.32). These two conditions lead to the two-variable, non-linear equation system below

$$
u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = u_1'(f_H^b)f_H^b + \left(j_H^0(f_\dagger) - u_1(f_\dagger)\right)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}
\tag{A.41}
$$

$$
\begin{gathered}
\left(u_0'(f_\dagger) - u_1'(f_\dagger)\right)f_\dagger + \phi\left(j_H^0(f_\dagger) - u_0(f_\dagger)\right) = \\
u_1(f_H^b)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2} + \left(j_H^0(f_\dagger) - u_1(f_\dagger)\right)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.
\end{gathered}
\tag{A.42}
$$

Solution to the HJB equation in the high state if $\lambda \leq \bar{\lambda}$. The value function satisfies

$$
\left(r + \lambda - \mu_H\right) j_H^0(f) = 1 - \left(r + \xi\right) f - \left(\mu_H + \xi\right) f j_H^{0\prime}(f) + \frac{1}{2}\sigma^2 f^2 j_H^{0\prime\prime}(f).
\tag{A.43}
$$

We guess the solution of the value function takes the form

$$
j_H^0(f) = D_0^0 - D_1^0 f + D_2^0 f^{\beta_1} + D_3^0 f^{\beta_2}.
$$

Plugging into equation (A.43), we can get

$$
\begin{aligned}
D_0^0 &= \frac{1}{r + \lambda - \mu_H}, \\
D_1^0 &= \frac{r + \xi}{r + \lambda + \xi}.
\end{aligned}
$$

The condition $\lim_{f\to 0} j_H^0(f) < \infty$, we know $D_3^0 = 0$. Since $j_H^0(f_H^b) = 0$ and $j_H^{0\prime}(f_H^b) = 0$, we know

$$j_H^0(f) = \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi} f + \frac{r+\xi}{r+\lambda+\xi} \frac{f_H^b}{\beta_1} \left(\frac{f}{f_H^b}\right)^{\beta_1}, \tag{A.44}$$

where

$$f_H^b = \frac{\beta_1}{\beta_1 - 1} \frac{r+\lambda+\xi}{(r+\xi)(r+\lambda-\mu_H)}. \tag{A.45}$$

A.3 Detailed Analysis of Section 3.4

Proof of Proposition 9

The first step in the analysis is to derive the limit of the value function. This is given in the following result.

Proposition 13 (Limit value function)**.** *Suppose that $\mu_L + \xi < 0$ and $\mu_H + \xi > 0$. Consider the case when $\lambda > \bar{\lambda}$, where $\bar{\lambda}$ is given in equation* (A.5)*. Let*

$$\gamma \equiv \frac{r+\eta-\mu_L}{-(\xi+\mu_L)} > 1$$
$$\psi \equiv \frac{r+\lambda-\mu_H}{\xi+\mu_H} > 0.$$

In the limit when $\sigma^2 \to 0$, the value function converges to

$$j_L(f) = \frac{1}{r+\eta-\mu_L} - \frac{r+\xi}{r+\eta+\xi} f + \frac{1}{\gamma-1} \frac{1}{r+\eta-\mu_L} \left(\frac{f}{f_L^b}\right)^{\gamma}$$
$$j_H(f) = \begin{cases} u_0(f) & f \in [0, f_\dagger] \\ u_1(f) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big) \left(\frac{f}{f_\dagger}\right)^{-\psi} & f \in \left(f_\dagger, f_H^b\right], \end{cases}$$

where $u_0(f)$ and $u_1(f)$ are given in equations (A.36) *and* (A.39)*. The default boundary in the low state is $f_L^b = \frac{1}{r+\xi}$. In the high state, the threshold $f_\dagger$ solves*

$$j_L(f_\dagger) = \frac{\rho - r}{\rho+\lambda-r} u_0(f_\dagger)$$

where the functions $u_0(f)$ and $u_1(f)$ are given in equations (A.36) *and* (A.39)*. The default boundary solves*

$$u_1(f_H^b) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big) \left(\frac{f_\dagger}{f_H^b}\right)^{\psi} = 0.$$

Proof. Under the assumption that $\mu_L + \xi < 0$, the default boundary becomes

$$f_L^b = \frac{1}{r+\xi}.$$

Using L'Hôpital rule, and noticing that $\sqrt{x^2} = \pm|x|$, we get

$$\begin{aligned}
\lim_{\sigma^2 \to 0} \gamma &= \frac{1}{2} + \frac{1}{2}\left[(\mu_L+\xi)^2\right]^{-1/2}\left[(\mu_L+\xi) + 2\,(r+\eta-\mu_L)\right] \\
&= \frac{1}{2} - \frac{1}{2}\,(\mu_L+\xi)^{-1}\left[(\mu_L+\xi) + 2\,(r+\eta-\mu_L)\right] \\
&= -\frac{r+\eta-\mu_L}{\xi+\mu_L}
\end{aligned}$$

Similarly, under the assumption that $\mu_H + \xi > 0$, we get that

$$\begin{aligned}
\lim_{\sigma \to 0} \phi &= \infty \\
\lim_{\sigma \to 0} \beta_1 &= \infty \\
\lim_{\sigma \to 0} \beta_2 &= -\frac{r+\lambda-\mu_H}{\mu_H+\xi} = -\psi.
\end{aligned}$$

The smooth pasting condition for f_H^b can be written as

$$u_1(f_H^b)\left[\frac{\frac{\beta_2}{\beta_1}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = \frac{1}{\beta_1}u_1'(f_H^b)f_H^b + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{1-\frac{\beta_2}{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.$$

In the limit as $\sigma^2 \to 0$, this equation simplifies to

$$u_1(f_H^b) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\left(\frac{f_\dagger}{f_H^b}\right)^{\psi} = 0$$

Similarly, we can write the smooth pasting condition at $f_\dagger$ as

$$\begin{aligned}
&\frac{1}{\beta_1}\big(u_0'(f_\dagger) - u_1'(f_\dagger)\big)f_\dagger + \frac{\phi}{\beta_1}\big(j_H(f_\dagger) - u_0(f_\dagger)\big) = \\
&\qquad u_1(f_H^b)\frac{1-\frac{\beta_2}{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2} + \big(j_H(f_\dagger) - u_1(f_\dagger)\big)\frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \frac{\beta_2}{\beta_1}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}},
\end{aligned}$$

and taking the limit we get

$$j_H(f_\dagger) = u_0(f_\dagger)$$

Substituting in the smooth pasting condition for f_H^b, we get the following equation for f_H^b

$$u_1(f_H^b) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big)\left(\frac{f_\dagger}{f_H^b}\right)^{\psi} = 0,$$

Substituting the solution for $j_H(f_\dagger)$ in indifference condition

$$j_L(f_\dagger) = \frac{\rho - r}{\rho + \lambda - r} j_H(f_\dagger),$$

we obtain the following equation for $f_\dagger$:

$$\frac{1}{r+\eta-\mu_L}\left(1 - \frac{(\rho-r)(\rho+\lambda+\eta-\mu_L)}{(\rho+\lambda-\mu_H)(\rho+\lambda-r)}\right) - \left(1 - \frac{(\rho-r)(\rho+\lambda+\eta+\xi)}{(\rho+\lambda-r)(\rho+\lambda+\xi)}\right)\frac{1}{r+\eta+\xi}\left(\frac{f_\dagger}{f_L^b}\right) + \frac{\lambda+(\mu_H-\mu_L)(\gamma-1)-\eta}{\rho+\lambda-r+(\mu_H-\mu_L)(\gamma-1)-\eta}\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f_\dagger}{f_L^b}\right)^{\gamma} = 0$$

Finally, from the limit coefficients (ϕ, β_1, β_2), we obtain that the value function in the H state converges to

$$j_H(f) = \begin{cases} u_0(f) & f \in [0, f_\dagger] \\ u_1(f) + \big(u_0(f_\dagger) - u_1(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{-\psi} & f \in \left(f_\dagger, f_H^b\right], \end{cases}$$

□

Substituting the previous expressions on the equilibrium conditions determining the price, $p_\theta = -j'_\theta$, we get

Proposition 14 (Limit price of long-term debt)**.** *Under the assumptions in Proposition 13, the limit price of long-term debt when $\sigma^2 \to 0$ is*

$$p_L(f) = \frac{r+\xi}{r+\eta+\xi}\left[1 - \left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] \tag{A.46}$$

$$p_H(f) = \begin{cases} \frac{r+\xi}{r+\eta+\xi}\left[1 + \frac{\eta}{\rho+\lambda+\xi} - \delta\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] & f \in [0, f_\dagger] \\ \frac{r+\xi}{r+\lambda+\xi} + \psi\big(u_0(f_\dagger) - u_1(f_\dagger)\big)\frac{1}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{-(\psi+1)} & f \in \left(f_\dagger, f_H^b\right], \end{cases} \tag{A.47}$$

where, as before, the constant δ is given by

$$\delta = \frac{\rho+\lambda-r}{\rho+\lambda-r+(\mu_H-\mu_L)(\gamma-1)-\eta} \in (0,1).$$

Proof. From the solution for the value function, we can obtain the price of the long-term debt. The price of the long-term bond is

$$p_L(f) = \frac{r+\xi}{r+\eta+\xi}\left[1-\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] \tag{A.48}$$

and

$$p_H(f) = \begin{cases} \frac{r+\xi}{r+\eta+\xi}\left[1+\frac{\eta}{\rho+\lambda+\xi}-\delta\left(\frac{f}{f_L^b}\right)^{\gamma-1}\right] & f\in[0,f_\dagger] \\ \frac{r+\xi}{r+\lambda+\xi}+\psi\big(u_0(f_\dagger)-u_1(f_\dagger)\big)\frac{1}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{-(\psi+1)} & f\in\left(f_\dagger,f_H^b\right], \end{cases}$$

□

Having computed the price, we obtain the issuance function by substituting p_L, p_H, fp_L', fp_H'.

$$g_H(f) = \frac{-(\xi+\mu_L)(\rho-r)}{\delta}\left[\frac{(1-\delta)}{r+\eta+\xi}+\frac{\eta}{(r+\eta+\xi)(\rho+\lambda+\xi)}\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)}\right].$$

We can substitute δ and γ to express $g_H(f)$ exclusively in terms of the primitive parameters

$$g_H(f) = \frac{\rho-r}{\rho+\lambda-r}\left[\frac{\eta(\xi+\mu_H)+(\mu_H-\mu_L)(r+\xi)}{r+\eta+\xi}\right.$$
$$\left.+\eta\left(\frac{(\xi+\mu_H)(r+\eta+\xi)-(\xi+\mu_L)(\rho+\lambda+\xi)}{(r+\eta+\xi)(\rho+\lambda+\xi)}\right)\left(\frac{f}{f_L^b}\right)^{\frac{r+\eta+\xi}{\xi+\mu_L}}\right]$$

The equation for $f_\dagger$ reduces to

$$\frac{\lambda(\rho+\lambda-\mu_H)-(\rho-r)(\mu_H-\mu_L+\eta)}{(r+\eta-\mu_L)(\rho+\lambda-\mu_H)(\rho+\lambda-r)} - \frac{\lambda(\rho+\lambda+\xi)-(\rho-r)\eta}{(\rho+\lambda-r)(\rho+\lambda+\xi)(r+\eta+\xi)}\left(\frac{f_\dagger}{f_L^b}\right)$$
$$+\frac{\lambda+(\mu_H-\mu_L)(\gamma-1)-\eta}{\rho+\lambda-r+(\mu_H-\mu_L)(\gamma-1)-\eta}\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f_\dagger}{f_L^b}\right)^{\gamma} = 0.$$

Comparative Statics g_H: The following comparative statics follow immediately: for any $f \in (0,f_\dagger)$, $g_H(f)$ is decreasing in λ and increasing in ρ, η and μ_H. The effect of μ_L is more difficult to determine. Differentiating the function we get

$$\frac{\partial g_H(f)}{\partial \mu_L} = \frac{\rho - r}{\rho + \lambda - r}\left[-\frac{r+\xi}{r+\eta+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma-1} - \frac{\eta}{r+\eta+\xi} + \frac{\eta}{\xi+\mu_L}\left(\frac{(\gamma-1)(\xi+\mu_H)}{\rho+\lambda+\xi}+1\right)\log\left(\frac{f}{f_L^b}\right)\right]\left(\frac{f}{f_L^b}\right)^{-(\gamma-1)}$$

The sign of the derivative depends on the sign of

$$\Psi(f) \equiv -\frac{r+\xi}{r+\eta+\xi}\left(\frac{f}{f_L^b}\right)^{\gamma-1} - \frac{\eta}{r+\eta+\xi} - \frac{\eta}{-(\xi+\mu_L)}\left(\frac{(\gamma-1)(\xi+\mu_H)}{\rho+\lambda+\xi}+1\right)\log\left(\frac{f}{f_L^b}\right),$$

the function $\Psi(f)$ is decreasing, with $\Psi(f_L^b) < 0$. For any $\eta > 0$, the limit when f goes to zero is $\Psi(f) \to \infty$. Thus, there is $\tilde{f}$ such that $\Psi(f) > 0$ on $[0, \tilde{f})$ and $\Psi(f) < 0$ on $(\tilde{f}, f_L^b]$. If $f_\dagger > \tilde{f}$, then g_H is increasing in μ_L for $f < \tilde{f}$ and decreasing for $f > \tilde{f}$. When $\eta = 0$, the issuance function reduces to $g_H(f) = \frac{(\rho-r)(\mu_H-\mu_L)}{\rho+\lambda-r}$, which is decreasing in μ_L.

Proof of Proposition 10

Sample Path: The ODE describing the evolution of f_t on $(0, f_\dagger)$ can be solved in closed form. Let

$$a_0 = \frac{-(\xi+\mu_L)(\rho-r)(1-\delta)}{(r+\eta+\xi)\delta} - (\xi+\mu_H)$$
$$a_1 = \frac{-(\xi+\mu_L)}{\delta}\frac{(\rho-r)\eta}{(r+\eta+\xi)(\rho+\lambda+\xi)}{f_L^b}^{\gamma-1},$$

so for $f < f_\dagger$, f_t solves

$$\dot{f}_t = a_0 f_t + a_1 f_t^{2-\gamma}.$$

This coefficients can be written as

$$\frac{a_1}{a_0} = -\underbrace{\frac{\eta}{\rho+\lambda+\xi}\left[\left(1+\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{r+\eta+\xi}{\rho-r}\right)\delta - 1\right]^{-1}}_{\equiv\kappa}{f_L^b}^{(\gamma-1)}$$

$$a_0(\gamma-1) = -\underbrace{(r+\eta+\xi)\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)} - \frac{(\rho-r)\Big((\mu_H-\mu_L)(\gamma-1)-\eta\Big)}{(r+\eta+\xi)(\rho+\lambda-r)}\right]}_{\equiv\nu}.$$

Substituting γ and simplifying terms, we get

$$\nu = \frac{(r+\eta+\xi)(\rho-r)}{\rho+\lambda-r}\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r} - \frac{r+\xi}{r+\eta+\xi}\right],$$

which is positive only if

$$\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r} > \frac{r+\xi}{r+\eta+\xi}.$$

In addition, $\kappa \propto \nu$, so it is positive only if ν is positive as well.

Given these definitions, we can can write

$$\frac{\dot{f}_t}{f_t} = -\frac{\nu}{\gamma-1}\left[1-\kappa\left(\frac{f_t}{f_L^b}\right)^{1-\gamma}\right].$$

Moreover, this equation can be solved in closed form. Letting $z_t = \log f_t$, we get the equation

$$\dot{z}_t = -\frac{\nu}{\gamma-1}\left[1-\kappa {f_L^b}^{\gamma-1} e^{(1-\gamma)z_t}\right].$$

The general solution to these equation is given provided in (Zaitsev and Polyanin, 2002, p. 162), and is given by

$$z = \frac{1}{\gamma-1}\log\left(Ce^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\right).$$

From here we get that

$$f = \left[Ce^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\right]^{\frac{1}{\gamma-1}}.$$

The integration constant is determined by the initial condition

$$C = f_0^{\gamma-1} - \kappa {f_L^b}^{\gamma-1},$$

so it follows that

$$f_t = \left[f_0^{\gamma-1}e^{-\nu t} + \kappa {f_L^b}^{\gamma-1}\left(1-e^{-\nu t}\right)\right]^{\frac{1}{\gamma-1}}.$$

Comparative Statics Path Let start considering the speed of adjustment ν

- ξ:

$$\frac{\partial \nu}{\partial \xi} = \frac{\lambda\mu_H\left(\eta-\mu_L+r\right) - \mu_L\left(\eta\lambda + 2\xi(\lambda+\rho) + \mu_L(\rho-r) + r(\lambda-2\xi)\right) + \xi^2(-\lambda-\rho+r)}{\left(\mu_L+\xi\right)^2\left(\rho+\lambda-r\right)}$$

The denominator is positive, and the numerator is positive if and only if

$$\sqrt{\frac{\lambda\,(\mu_H-\mu_L)\,(\eta-\mu_L+r)}{\lambda+\rho-r}}+\mu_L+\xi>0$$

- μ_L:

$$\frac{\partial \nu}{\partial \mu_L}=\frac{\lambda\,(\mu_H+\xi)\,(\eta+\xi+r)}{(-\mu_L-\xi)^{\,2}((\rho+\lambda-r)}>0.$$

- μ_H:

$$\frac{\partial \nu}{\partial \mu_H}=\frac{\lambda\,(\mu_H+\xi)\,(\eta+\xi+r)}{(-\mu_L-\xi)^{\,2}((\rho+\lambda-r)}>0.$$

- λ:

$$\frac{\partial \nu}{\partial \lambda}=\frac{\eta\xi\rho+\mu_H(\rho+r)(\eta+\xi+r)-\mu_L(\xi+r)(\rho-r)+\xi r(\eta+2\xi+2r)}{-\,(\mu_L+\xi)\,((\rho+\lambda-r)^2}>0.$$

- η:

$$\frac{\partial \nu}{\partial \eta}=\frac{\lambda\,(\mu_H+\xi)}{-\,(\mu_L+\xi)\,((\rho+\lambda-r)}>0.$$

- ρ:

$$\begin{aligned}\nu&=\frac{(r+\eta+\xi)(\rho-r)}{(\rho+\lambda-r)}\left[\frac{\xi+\mu_H}{-(\xi+\mu_L)}\frac{\lambda}{\rho-r}-\frac{r+\xi}{r+\eta+\xi}\right]\\&=-\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)(\xi+\mu_L)}-\frac{(\rho-r)(r+\xi)}{(\rho+\lambda-r)}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial \nu}{\partial \rho}&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}-\frac{(r+\xi)(\rho+\lambda-r)-(\rho-r)(r+\xi)}{(\rho+\lambda-r)^2}\\&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}-\frac{(r+\xi)\lambda}{(\rho+\lambda-r)^2}\\&=\frac{(r+\eta+\xi)(\xi+\mu_H)\lambda-(r+\xi)(\xi+\mu_L)\lambda}{(\rho+\lambda-r)^2(\xi+\mu_L)}<0\end{aligned}$$

B Analysis of Extensions

B.1 The derivative contract in Section 3.2

In this section, we extend the analysis of the impact of a derivative contract to consider the case in which the hedging is not perfect. Suppose that the borrower can buy a short-term derivative contract written on a variable $\hat{\theta}_t$ that is correlated with θ_t in the following way. Without a regime switch, $\hat{\theta}_t$ remains a constant. However, if there is a regime switch, $\hat{\theta}_t$ switches with probability $q \in [0,1]$ and remains a constant otherwise. The case of $q = 1$ corresponds to perfect insurance that we discussed in Section 3.2. If so, markets are dynamically complete, and shocks to θ_t can be perfectly insured.

The buyers of the derivative pay a premium $\varsigma \cdot dt$ over the period $[t, t+dt)$, in exchange of a payment of \$1 at time $t + dt$ if there is a change in $\hat{\theta}_t$. The expected payoff of this contract over the period $[t, t+dt)$ is $e^{-rdt}q(1-e^{-\lambda dt})$, so no arbitrage implies

$$\varsigma = \lim_{dt\to 0} \frac{e^{-rdt}(1-e^{-\lambda dt})q}{dt} = \lambda q.$$

Let z_t denote the number of contracts bought by the equity holder at time t. The analysis in the low state is unchanged. In the high state, upon the regime shifting, the borrower receives z_t with probability q and nothing with probability $1-q$. In the first case, default occurs if $j_L(f_{t-}) + z_{t-} \leq d_{t-}$, whereas in the second case, default happens if $j_L(f_{t-}) \leq d_{t-}$. Consistent with the assumption of zero recovery, we assume that in the event of default, the payment from the derivative contract cannot be used to pay long-term creditors.

Given the position z_t, the short rate is given by

$$y_H(f,d,z) = \begin{cases} r & \text{if } d \leq j_L(f) \\ r + \lambda(1-q) & \text{if } j_L(f) < d \leq j_L(f) + z \\ r + \lambda & \text{if } d > j_L(f) + z. \end{cases}$$

In the high state, the HJB equation is

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = \max_{d\in[0,j_H(f)],z\geq 0} \; & 1-(r+\xi)\, f - q\lambda z + (\rho+\lambda-y_H)\, d - (\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ & + \lambda q \max\{j_L(f) + z - d, 0\} + \lambda(1-q)\max\{j_L(f) - d, 0\}\,. \end{aligned} \tag{A.49}$$

The following Lemma characterizes the solution to the maximization problem in equation (A.49)

Lemma 6. *The optimal short-term debt and hedging policy $d_H(f), z(f)$ is*

$$d_H(f) = \begin{cases} j_L(f) & \textit{if } j_L(f) \geq \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) & \textit{Otherwise.} \end{cases}$$

$$z(f) = \begin{cases} 0 & \textit{if } j_L(f) \geq \frac{\rho - r}{\rho + \lambda(1-q) - r} j_H(f) \\ j_H(f) - j_L(f) & \textit{Otherwise.} \end{cases}$$

Proof. In equation (A.49), d_H and z are chosen to maximize

$$-q\lambda z + (\rho + \lambda - y_H)\, d_H + \lambda q \max\{j_L(f) + z - d_H, 0\} + \lambda(1-q) \max\{j_L(f) - d_H, 0\}.$$

There are three situations that we need to consider:

1. If $d_H \leq j_L(f)$, the objective becomes

   $$-q\lambda z + (\rho + \lambda - r)\, d_H + \lambda q\, (j_L(f) + z - d_H) + \lambda(1-q)\, (j_L(f) - d_H) = (\rho - r)\, d_H + \lambda j_L(f),$$

   which is maximized at $d_H = j_L(f)$ with the maximum value

   $$(\rho + \lambda - r)\, j_L(f).$$

2. If $d_H \in \big(j_L(f), j_L(f) + z\big]$, the objective becomes

   $$-q\lambda z + (\rho + \lambda q - r)\, d_H + \lambda q\, (j_L(f) + z - d_H) = (\rho - r)\, d_H + \lambda q j_L(f),$$

   which is maximized at $d_H = j_L(f) + z$ with the maximum value

   $$(\rho - r + \lambda q) j_L(f) + (\rho - r) z.$$

   Given that $d_H = j_L(f) + z \leq j_H(f)$, we know $z \leq j_H(f) - j_L(f)$. The maximized $z = j_H(f) - j_L(f)$, and the maximum value is

   $$(\rho - r) j_H(f) + \lambda q j_L(f).$$

3. If $d_H > j_L(f) + z$, the objective becomes $-q\lambda z + (\rho - r)\, d_H$, which is clearly maximized at

$z = 0$ and $d_H = j_H(f)$, with a maximum value

$$(\rho - r)\, j_H(f).$$

Clearly, the last one is dominated, so the borrower's choice is

- If $(\rho - r)j_H(f) + \lambda q j_L(f) \leq (\rho + \lambda - r)j_L(f)$, then $d_H = j_L(f)$, and z is irrelevant so without loss of generality set as zero.
- Otherwise, then $d_H = j_H(f)$ and $z = j_H(f) - j_L(f)$.

□

When $d_H(f) = j_L(f)$, the firm will survive the regime switch anyway, so insurance is unnecessary. By contrast, when $d_H(f) = j_H(f)$ so that short-term debt is risky, the equity holder buys enough derivative contracts to insure against the regime shift. The equilibrium takes a similar form as the one in section 3. The amount of short term debt is $d_H(f) = j_L(f)$ when $f < f_\dagger$, and $d_H(f) = j_H(f)$ if $f > f_\dagger$, with the threshold $f_\dagger$ given by the indifference condition

$$f_\dagger = \min\big\{f \geq 0 : (\rho + \lambda(1-q) - r)j_L(f) \leq (\rho - r)j_H(f)\big\}.$$

Given the optimal policy in Lemma 6, we can write the HJB equation in simpler form

$$\begin{aligned}
(\rho + \lambda - \mu_H)\, j_H(f) &= 1 - (r+\xi)\, f + (\rho + \lambda - r)\, j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (0, f_\dagger) \\
(r + \lambda - \mu_H)\, j_H(f) &= 1 - (r+\xi)\, f + q\lambda j_L(f) - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f \in (f_\dagger, f_H^b).
\end{aligned} \tag{A.50}$$

The price of debt is now given by the solution to the asset pricing equation.

$$\begin{aligned}
(r + \xi + \lambda)\, p_H(f) &= r + \xi + \lambda p_L(f) + \big(g_H(f) + \sigma^2 - \mu_H - \xi\big)\, f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f \in (0, f_\dagger) \\
(r + \xi + \lambda)\, p_H(f) &= r + \xi + \lambda q p_L(f) + \big(g_H(f) + \sigma^2 - \mu_H - \xi\big)\, f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f), \quad f \in (f_\dagger, f_H^b).
\end{aligned}$$

From, here, together with the indifference condition $j_H'(f) = -p_H(f)$, we can obtain the equilibrium issuance function. We omit the details, but a similar calculations to the ones in the absence of hedging show that the equilibrium issuance policy is given by (20). Notice that although the form of the issuance function does not change the total issuance of long-term debt does change as the price of long-term debt is now different. The main impact of hedging though is on the value of the threshold $f_\dagger$.

It immediately follows from this indifference condition that when $q = 1$, the threshold $f_\dagger$ is equal to zero as $(\rho - r)(j_H(f) - j_L(f)) > 0$. In other words, the borrower does not issue any long-term debt if she can perfectly insure against the regime shift. The solution in Lemma 6 becomes $d_H(f) = j_H(f)$ and $z(f) = j_H(f) - j_L(f)$ for all $f \in [0, f_H^b]$. This provides a proof of Proposition 5.

Solution HJB Equation

The solution to the HJB equation takes the same form as in the baseline model. The function u_0 is still given by (A.36), but the expression for u_1 is different to the one in equation (A.39) because it includes the term $q\lambda j_L(f)$ capturing the continuation value after the regime switch.

For $f \in (0, f_\dagger)$, there is no change in the differential equation, so the solution remains the same. On $f \in (f_\dagger, f_H^b)$ the HJB equation becomes

$$(r + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f + q\lambda j_L(f) + \mathcal{D}^H j_H(f)$$

If $f \geq f_L^b$, the continuation value $j_L(f)$ and the particular solution is

$$u_1(f) = \frac{1}{r + \lambda - \mu_H} - \frac{r + \xi}{r + \lambda + \xi} f.$$

When $f < f_L^b$, the particular solution takes the form

$$u_1(f) = \frac{1}{r + \lambda - \mu_H}\left(1 + q\frac{\lambda}{r + \eta - \mu_L}\right) - \frac{r + \xi}{r + \lambda + \xi}\left(1 + q\frac{\lambda}{r + \eta + \xi}\right) f + C\left(\frac{f}{f_L^b}\right)^{\gamma},$$

Substituting in the previous the ODE, we find that the constant C is given by

$$C = \frac{\lambda q}{\lambda - \eta + (\mu_H - \mu_L)(\gamma - 1)} \frac{1}{\gamma - 1} \frac{1}{r + \eta - \mu_L}.$$

The solution then to the HJB equation is

$$j_H(f) = \begin{cases} u_0(f) + \big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{\phi} & f \in [0, f_\dagger] \\ u_1(f) + \big(j_H(f_\dagger) - u_1(f_\dagger)\big) h_0\left(f \middle| f_\dagger, f_L^b\right) + \big(j_H(f_L^b) - u_1(f_L^b)\big) h_1\left(f \middle| f_\dagger, f_L^b\right) & f \in \left(f_\dagger, f_L^b\right) \\ u_1(f) + \big(j_H(f_L^b) - u_1(f_L^b)\big) h_0\left(f \middle| f_L^b, f_H^b\right) - u_1(f_H^b) h_1\left(f \middle| f_L^b, f_H^b\right) & f \in \left[f_L^b, f_H^b\right] \end{cases}$$

where

$$u_0(f) = \mathcal{A}\frac{1}{\rho+\lambda-\mu_H} - \mathcal{B}\frac{r+\xi}{\rho+\lambda+\xi}f + \delta\frac{1}{\gamma-1}\frac{1}{r+\eta-\mu_L}\left(\frac{f}{f_L^b}\right)^\gamma$$

$$u_1(f) = \begin{cases} \frac{1}{r+\lambda-\mu_H}\left(1+\frac{\lambda q}{r+\eta-\mu_L}\right) - \frac{r+\xi}{r+\lambda+\xi}\left(1+\frac{\lambda q}{r+\eta+\xi}\right)f + C\left(\frac{f}{f_L^b}\right)^\gamma & \text{if } f < f_L^b \\ \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\lambda+\xi}f & \text{if } f \geq f_L^b, \end{cases}$$

and the constants $\mathcal{A}, \mathcal{B}, \delta$ are given by

$$\mathcal{A} = \frac{\rho+\lambda+\eta-\mu_L}{r+\eta-\mu_L} \qquad \mathcal{B} = \frac{\rho+\lambda+\eta+\xi}{r+\eta+\xi} \qquad \delta = \frac{\rho+\lambda-r}{\rho+\lambda-r-\eta+(\mu_H-\mu_L)(\gamma-1)}.$$

The functions $h_0(\cdot)$ and $h_1(\cdot)$ are defined in equation (A.40).

Finally, to show that $f_\dagger$ is decreasing in q when $f_\dagger > 0$, it suffices to show that $j_H(f)$ is increasing in q.

Lemma 7. *If $(\rho+\lambda(1-q)-r)\, j_L(0) > (\rho-r)\, j_H(0)$, then the value function $j_H(f)$ is strictly increasing in q.*

Proof. For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau\geq 0}\mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right]$$
$$\text{subject to} \quad dz_t = -(\xi+\mu_H)z_t dt - \sigma z_t dB_t,$$

where

$$\nu(z,\tilde{j}|q) \equiv \max\{(\rho+\lambda-r)\, j_L(z)\,,\, q\lambda j_L(z) + (\rho-r)\,\tilde{j}\}$$

and $\hat{\rho} \equiv \rho+\lambda-\mu_H$. It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\nu(f,\tilde{j}_1|q) \geq \nu(f,\tilde{j}_0|q)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\begin{aligned}\nu(z,\tilde{j}+a|q) &= \max\{(\rho+\lambda-r)\, j_L(z), q\lambda j_L(z) + (\rho-r)(\tilde{j}+a)\} \\ &\leq \max\{(\rho+\lambda-r)\, j_L(z) + (\rho-r)a, q\lambda j_L(z) + (\rho-r)(\tilde{j}+a)\} = (\rho-r)a + \nu(z,\tilde{j}|q),\end{aligned}$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)+a|q)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\nu(z_t,\tilde{j}(z_t)|q)\right)dt\Big|z_0=f\right] \\
&\quad+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f)+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \leq \Phi(\tilde{j})(f)+\frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}$$

As Φ is monotone and satisfies discounting, it follows from Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f)=\Phi(j_H)(f)$.

For any pair of parameters $q_1 \geq q_0$, the inequality $\nu(f,\tilde{j}|q_1) \geq \nu(f,\tilde{j}|q_0)$ implies that the operator Φ is increasing q. It follows from Theorem 1 in Villas-Boas (1997) that the fixed point $j_H(f)=\Phi(j_H)(f)$ increases in q. $\square$

B.2 Section 4.1 with Restructuring

In this section, we provide the analysis of the equilibrium when short-term debt can be restructured. Whenever the borrower announces a default, and there is outstanding short-term debt, short-term debt can be restructured with some probability. Notice that in both states $\theta \in \{H, L\}$, when the borrower defaults at f_θ^b, the amount of short-term debt is zero. Therefore, renegotiating short-term debt is only relevant upon a regime shift from H to L. The renegotiation game goes as follows. With probability $1-\alpha$, it is impossible to renegotiate, and the firm goes bankrupt. With probably α, the firm enters into a renegotiation process. In this case, the equity holder makes the offer with probability β and short-term creditors with probability $1-\beta$. If the short-term creditors make the offer, and this offer is rejected, the firm goes bankrupt. If the equity holder makes the offer and the offer is rejected, she can still choose between repaying the original short-term debt and bankruptcy. Figure 1 presents the timing of events.

Following the state transition, the borrower receives $j_L(f)-d_H$ if she does not default. If there is a default, with probability $1-\alpha$, there is no renegotiation, and she receives zero. With probability α, there is renegotiation. In this case, if short-term creditors make an offer, they receive

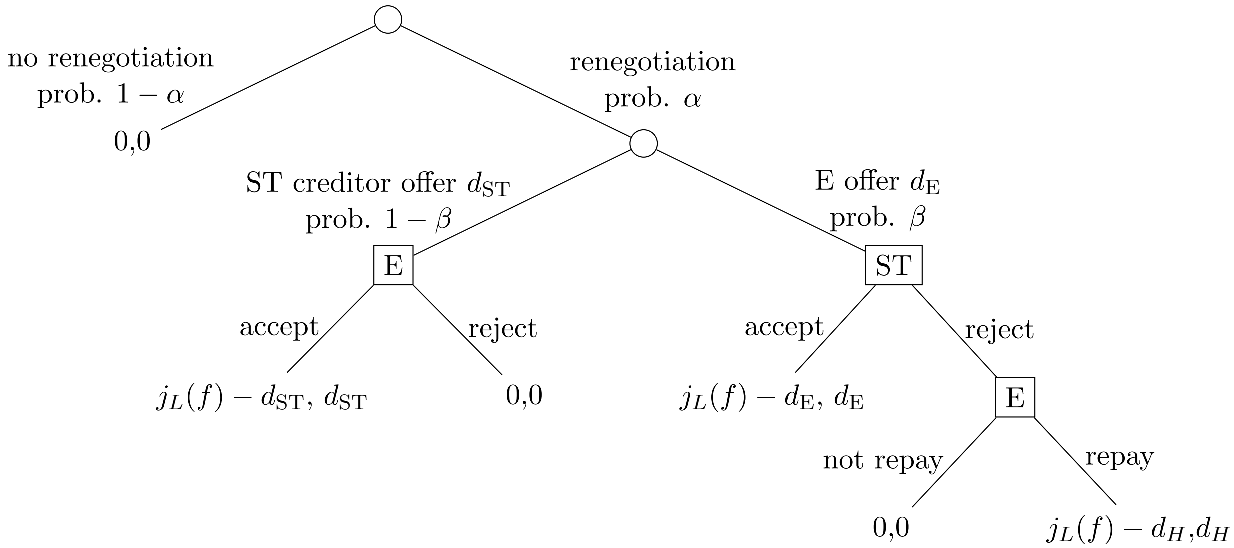

Figure 1: Renegotiation process.

The game tree illustrates the renegotiation process. If the firm defaults, renegotiation is triggered with probability α; otherwise, there is bankruptcy. In the event of renegotiation, the equity holder gets to make an offer with probability β, in which case she offers d_{E}. Otherwise, the offer is made by short-term creditors, in which case they offer d_{ST}. In the tree, E indicates nodes where the equity holder moves and ST nodes where short-term creditors move. At the end of the tree, the first coordinate indicates the payoff to the equity holder, and the second coordinate indicates the payoff to short-term creditors.

$j_L(f)$ while the borrower receives 0. If the equity holder makes an offer, they offer 0 and obtain $j_L(f)$, while the short-term creditors get 0; however, such an offer is credible only if $j_L(f) < d_H$. If $j_L(f) \geq d_H$, then the only credible offer is d_H.[1] It is easy to verify that renegotiation is triggered only after the regime switch and $j_L(f_t) < d_H(f_t)$. The firm goes bankrupt only if the restructuring process fails, which happens with probability $1-\alpha$. To determine the interest rate, we need to analyze the expected recovery. If $d_H(f) \leq j_L(f)$, there is no default, and short-term creditors are paid in full. The equity holder does not have incentives to default because no credible offer would allow paying less than d_H. If $d_H > j_L(f)$, the equity holder default so the expected payoff is $(1-\alpha\beta) \times 0 + \alpha\beta \times j_L(f)$. In the event of default, each creditor gets zero with probability $1-\alpha(1-\beta)$; that is, if either renegotiation is not possible or if it is possible, but equity holder is the one to make the offer. With probability $\alpha(1-\beta)$, the short-term debt recovery per dollar is

[1]When there is indifference, we break ties in favor of the efficient outcome of continuation.

$j_L(f)/d_H$. Hence, the short rate is given by

$$y_H(f,d_H) = \begin{cases} r & \text{if } d_H \leq j_L(f) \\ r + \lambda\left(1-\alpha(1-\beta)\frac{j_L(f)}{d_H}\right) & \text{if } d_H > j_L(f). \end{cases}$$

The analysis in state L is unchanged. In state H, we construct an equilibrium similar to the one without restructuring. The HJB in the high state follows

$$\begin{aligned}(\rho+\lambda-\mu_H)\, j_H(f) = & \max_{d_H\in[0,j_H(f)]} (1-\pi)-(r+\xi)\, f + (\rho+\lambda-y_H)\, d_H \\ & + \lambda\Big(\big(j_L(f)-d_H\big)\mathbb{1}_{d_H\leq j_L(f)} + \alpha\beta j_L(f)\mathbb{1}_{d_H>j_L(f)}\Big) - (\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \quad \text{(A.51)}\end{aligned}$$

The optimal solution for short-term debt is

$$d_H(f) = \begin{cases} j_L(f) & \text{if } j_L(f) \geq \frac{\rho-r}{\rho+\lambda(1-\alpha)-r} j_H(f) \\ j_H(f) & \text{Otherwise.} \end{cases}$$

Note that when $\alpha = 0$, we are back to the benchmark model. Interestingly, α and β serve different purposes: the former leads to efficiency loss, and the latter is only about how to redistribute the surplus across the coalition. The threshold now is determined by the indifference condition

$$f_\dagger = \min\big\{f \geq 0 : (\rho+\lambda(1-\alpha)-r) j_L(f) \leq (\rho-r) j_H(f)\big\}. \quad \text{(A.52)}$$

The HJB equation A.51 can be written as

$$\begin{aligned}(\rho+\lambda-\mu_H)\, j_H(f) &= 1-(r+\xi)\, f + (\rho+\lambda-r) j_L(f) - (\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f\in(0,f_\dagger) \\ (r+\lambda-\mu_H)\, j_H(f) &= 1-(r+\xi)\, f + \lambda\alpha j_L(f) - (\mu_H+\xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f), \quad f\in(f_\dagger, f_H^b).\end{aligned} \quad \text{(A.53)}$$

We see that the only difference with the original equation is that now, when the firm is fully levered, there is a term $\lambda\alpha j_L(f)$ capturing the continuation value after the regime shift. Notice that the HJB equation (A.53) takes the same form as the one with hedging in equation (A.50), so hedging and renegotiation serve a similar economic purpose in the model. The asset pricing equation for bond prices becomes

$$(r+\xi+\lambda)\,p_H\,(f) = r+\xi+\lambda p_L\,(f) + \big(g_H(f)+\sigma^2-\mu_H-\xi\big)\,fp'_H(f) + \frac{1}{2}\sigma^2 f^2 p''_H(f), \quad f\in(0,f_\dagger)$$
$$(r+\xi+\lambda)\,p_H\,(f) = r+\xi+\lambda\alpha p_L + \big(g_H(f)+\sigma^2-\mu_H-\xi\big)\,fp'_H(f) + \frac{1}{2}\sigma^2 f^2 p''_H(f), \quad f\in(f_\dagger,f_H^b).$$

Thus, together with the indifference condition $j'_H(f) = -p_H(f)$, we obtain the equilibrium issuance function in the high state is given by the same expression as the one in equation (20). The HJB and price equations are identical to the ones with hedging, so the proof of Proposition 11 follows the one in Appendix B.1.

B.3 Section 4.2 with Cash Flow Jumps

Suppose that

$$dX_t = \mu X_t dt + \sigma X_t dB_t - (1-\omega^{-1})X_{t-}dN_t,$$

where N_t is a Poisson process with intensity λ and $\omega>1$. Using Ito's Lemma, f_t solves

$$df_t = (g_t-\mu-\xi+\sigma^2)f_t dt - \sigma f_t dB_t + (\omega-1)f_{t-}dN_t$$

Thus, the scaled value function satisfies the delay differential equation

$$(\rho+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f) + \max\left\{(\rho-r)\frac{j(\omega f)}{\omega}+\lambda\frac{j(\omega f)}{\omega},(\rho-r)j(f)\right\}$$

We guess and verify that the optimal short-term debt policy is given by

$$d(f) = \begin{cases} \frac{j(\omega f)}{\omega} & \text{if } f\in[0,f_\dagger] \\ j(f) & \text{if } f\in(f_\dagger,f^b]. \end{cases}$$

The HJB equation can be written as

$$(\rho+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f) + (\rho+\lambda-r)\frac{j(\omega f)}{\omega},\ f\in(0,f_\dagger)$$
$$(r+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f),\ f\in(f_\dagger,f^b).$$

The default boundary solves the value matching and smooth pasting conditions $j(f^b) = j'(f_b) = 0$.

Long-term bonds satisfy the asset pricing equation

$$(r+\xi+\lambda)\,p(f) = 1-(r+\xi)\,f + \left(g(f)-\mu-\xi+\sigma^2\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + \lambda p(\omega f),\ \ f \in (0, f_\dagger)$$
$$(r+\xi+\lambda)\,p(f) = 1-(r+\xi)\,f + \left(g(f)-\mu-\xi+\sigma^2\right) f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f),\ \ f \in (f_\dagger, f^b).$$

Finally, we derive the issuance policy $g(f)$ combining the asset pricing equation with the indifference condition $p(f) = -j'(f)$. This yields

$$g\left(f\right) = \frac{\left(\rho - r\right)\left(p\left(f\right) - p\left(\omega f\right)\right)}{f j''\left(f\right)}.$$

Numerical Computation: For computational purposes, it is easier to work with the the state variable $x = \log(1/f) = -\log f$. Let $\tilde{j}(x) \equiv j(e^{-x})$ and $\delta = \log \omega$. Then, we get

$$\begin{aligned}
\tilde{j}'(x) &= -j'(e^{-x})e^{-x} \\
\tilde{j}''(x) &= j''(e^{-x})e^{-2x} + j'(e^{-x})e^{-x} = j''(e^{-x})e^{-2x} - \tilde{j}'(x)
\end{aligned}$$

Substituting in the HJB equation we get

$$\begin{aligned}
(r+\lambda-\mu)\,\tilde{j}(x) = 1 - (r+\xi)\,e^{-x} + \left(\mu + \xi + \frac{1}{2}\sigma^2\right)\tilde{j}'(x) + \frac{1}{2}\sigma^2 \tilde{j}''(x) \\
- (\rho - r)\min\left\{\tilde{j}(x) - a\frac{\tilde{j}(x-\delta)}{e^\delta}, 0\right\},
\end{aligned}$$

where

$$a \equiv \frac{\rho + \lambda - r}{\rho - r}.$$

We write this as a system of two first order equations. Letting $y_0(x) = \tilde{j}(x)$ and $y_1(x) = \tilde{j}'(x)$, we can reduce the second order equation to the following system of first order equations

$$\begin{aligned}
y_0'(x) &= y_1(x) \\
y_1'(x) &= \frac{2}{\sigma^2}\left[(r+\xi)\,e^{-x} - 1 + (r+\lambda-\mu)\,y_0(x) - \left(\mu+\xi+\frac{1}{2}\sigma^2\right) y_1(x) \right. \\
&\quad \left. + (\rho - r)\min\left\{y_0(x) - a\frac{y_0(x-\delta)}{e^\delta}, 0\right\}\right].
\end{aligned}$$

The previous equation is a system of two first order delay differential equations with constant coefficient that can be solved using standard numerical routines. The value matching and smooth pasting conditions at the default boundary $x^b = -\log f^b$ are $y_0(x^b) = y_1(x^b) = 0$. The only remaining step is to specify the transversality condition. From the HJB equation we get $j(0) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}$, so we have the transversality condition

$$\lim_{x\to\infty} y_0(x) = \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega}.$$

To incorporate this transversality condition, we approximate the value function for $f = \epsilon$. This corresponds to a value of x given by $x_\epsilon = -\log\epsilon$. Differentiating the HJB we get

$$(\rho+\lambda+\xi)\, j'(f) = -(r+\xi) - \left(\mu+\xi-\sigma^2\right) f j''(f) + (\rho+\lambda-r) j'(\omega f),$$

and evaluating at $f = 0$, we get

$$j'(0) = -1.$$

Hence, for ϵ close to zero

$$j(\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - \epsilon,$$

which means that

$$y_0(x_\epsilon) = \tilde{j}(x_\epsilon) \approx \frac{\omega}{(\rho+\lambda)(\omega-1)+r-\mu\omega} - e^{-x_\epsilon}.$$

Finally, we can write the price and issuance function in terms of the functions $y_0(x), y_1(x)$. The price of long-term bonds is given by

$$p(f) = -j'(f) = \tilde{j}'(x)e^x = y_1(x)e^x.$$

Letting $x_\dagger = -\log f_\dagger$, we get that on the $[x_\dagger, \infty)$, the issuance policy is

$$g(x) = \frac{(\rho-r)\big(p\left(e^{-x}\right) - p\left(e^{-(x-\delta)}\right)\big)}{-e^{-x}p'\left(e^{-x}\right)} = \frac{(\rho-r)\big(y_1(x) - y_1(x-\delta)e^{-\delta}\big)}{-e^{-x}(y_1'(x)+y_1(x))}$$

B.4 Equilibrium with Only Long-term Debt

We first describe the equilibrium when the borrower is only allowed to issue long-term debt.

Proposition 15 (Equilibrium with only long-term debt)**.** *If only long-term debt is allowed, the unique equilibrium is the following.*

1. *In state L, the value function is*

$$v_L^\ell(f) = \frac{1}{\rho+\eta-\mu_L} - \frac{r+\xi}{\rho+\eta+\xi} f + \frac{1}{\rho+\eta-\mu_L} \frac{1}{\gamma_\ell - 1} \left(\frac{f}{f_L^{b\ell}} \right)^{\gamma_\ell},$$

where γ_ℓ *is given in* (A.56) *and the default boundary is given in* (A.55).

2. *In state H, when* $f \le f_L^{b\ell}$, *the value function is*

$$v_H^\ell(f) = u_0^\ell(f) + \left(v_H^\ell\left(f_L^{b\ell}\right) - u_0^\ell\left(f_L^{b\ell}\right) \right) \left(\frac{f}{f_L^{b\ell}} \right)^{\phi_1},$$

and

$$u_0^\ell(f) = \frac{1}{\rho+\lambda-\mu_H} \left(1 + \frac{\lambda}{\rho+\eta-\mu_L} \right) - \frac{r+\xi}{\rho+\lambda+\xi} \left(1 + \frac{\lambda}{\rho+\eta+\xi} \right) f + \delta^\ell \frac{1}{\gamma_\ell - 1} \frac{1}{\rho+\eta-\mu_L} \left(\frac{f}{f_L^{b\ell}} \right)^{\gamma_\ell},$$

where

$$\delta^\ell = \frac{\lambda}{\lambda - \eta + (\mu_H - \mu_L)(\gamma_\ell - 1)}.$$

When $f \in \left(f_L^{b\ell}, f_H^{b\ell}\right]$,

$$v_H^\ell(f) = u_1^\ell(f) + \left(v_H^\ell(f_L^{b\ell}) - u_1^\ell(f_L^{b\ell})\right) \tilde{h}_0\left(f \middle| f_L^{b\ell}, f_H^{b\ell}\right) - u_1^\ell(f_H^{b\ell}) \tilde{h}_1\left(f \middle| f_L^{b\ell}, f_H^{b\ell}\right),$$

where $\tilde{h}_0\left(f \middle| f_L^{b\ell}, f_H^{b\ell}\right)$ *and* $\tilde{h}_1\left(f \middle| f_L^{b\ell}, f_H^{b\ell}\right)$ *are given in (*(A.59)*) and*

$$u_1^\ell(f) = \frac{1}{\rho+\lambda-\mu_H} - \frac{r+\xi}{\rho+\lambda+\xi} f.$$

The borrower defaults upon the state transition if $f > f_L^{b\ell}$.

3. *In both states* $\theta \in \{L, H\}$, *the debt price is* $p_\theta^\ell = -v_\theta^{\ell\prime}$, *and the issuance function follows*

$$g_\theta^\ell = \frac{(\rho - r)\, p_\theta^\ell}{-f p_\theta^{\ell\prime}}.$$

Proof of Proposition 15

Proof. Again, let $\tilde{V}_L = X\tilde{v}_L$ so that $\frac{\partial \tilde{V}_L}{\partial F} = \tilde{v}'_L$, $\frac{\partial \tilde{V}_L}{\partial X} = \tilde{v}_L - f\tilde{v}'_L$, and $X\frac{\partial^2 \tilde{V}_L}{\partial X^2} = f^2\tilde{v}''_L$. For notation convenience, we use $\tilde{v}_\theta = v_\theta^\ell$. The scaled HJB becomes

$$(\rho + \eta - \mu_L)\,\tilde{v}_L = 1 - (r + \xi)\,f - (\mu_L + \xi)\,f\tilde{v}'_L + \frac{1}{2}\sigma^2 f^2 \tilde{v}''_L.$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_L(f) < \infty$, $\tilde{v}_L\left(\tilde{f}_L^b\right) = 0$, and $\tilde{v}'_L\left(\tilde{f}_L^b\right) = 0$, we obtain the solution

$$\tilde{v}_L(f) = \frac{1}{\rho + \eta - \mu_L} - \frac{r + \xi}{\rho + \eta + \xi} f + \frac{r + \xi}{\rho + \eta + \xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}} \tag{A.54}$$

$$\tilde{f}_L^b = \frac{1}{\rho + \eta - \mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma} - 1}\frac{\rho + \eta + \xi}{r + \xi} \tag{A.55}$$

where

$$\tilde{\gamma} = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho + \eta - \mu_L)}}{\sigma^2} > 1. \tag{A.56}$$

In a smooth equilibrium, $\tilde{p}_L = -\tilde{v}'_L$, and $\tilde{p}_L$ satisfies

$$(r + \xi + \eta)\,\tilde{p}_L = (r + \xi) + \left(g_L - \xi - \mu_L + \sigma^2\right) f\tilde{p}'_L + \frac{1}{2}\sigma^2 f^2 \tilde{p}''_L.$$

Differentiating once the HJB for $\tilde{v}_L$, we get $\tilde{g}_L = \frac{(\rho - r)\tilde{p}_L}{f\tilde{v}''_L}$.

In the high state, the scaled HJB becomes

$$(\rho - \mu_H)\,\tilde{v}_H = 1 - (r + \xi)\,f - (\mu_H + \xi)\,f\tilde{v}'_H + \frac{1}{2}\sigma^2 f^2 \tilde{v}''_H + \lambda(\tilde{v}_L - \tilde{v}_H).$$

From (A.54), we know

$$\tilde{v}_L(f) = \begin{cases} \frac{1}{\rho+\eta-\mu_L} - \frac{r+\xi}{\rho+\eta+\xi} f + \frac{r+\xi}{\rho+\eta+\xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}} & \text{when } f \le \tilde{f}_L^b \\ 0 & \text{when } f > \tilde{f}_L^b \end{cases}.$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_H(f) < \infty$, we know when $f \le \tilde{f}_L^b$,

$$\tilde{v}_H(f) = \tilde{u}_0(f) + \left(\tilde{v}_H\left(\tilde{f}_L^b\right) - \tilde{u}_0\left(\tilde{f}_L^b\right)\right)\left(\frac{f}{\tilde{f}_L^b}\right)^{\phi_1},$$

where

$$\tilde{u}_0(f) = \frac{(\rho+\eta+\lambda-\mu_L)}{(\rho+\eta-\mu_L)(\rho+\lambda-\mu_H)} - \frac{(r+\xi)(\rho+\eta+\lambda+\xi)}{(\rho+\eta+\xi)(\rho+\lambda+\xi)} f + \frac{\lambda\frac{r+\xi}{\rho+\eta+\xi}}{\lambda-\eta+(\mu_H-\mu_L)(\tilde{\gamma}-1)} \frac{\tilde{f}_L^b}{\tilde{\gamma}} \left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

$$\phi_1 = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} > 1.$$

The solution on the interval $\left[\tilde{f}_L^b, \tilde{f}_H^b\right]$ can be obtained in a similar way. In this interval, the value function satisfies the equation

$$(\rho+\lambda-\mu_H)\tilde{v}_H(f) = 1-(r+\xi)f + \mathcal{D}^H\tilde{v}_H(f).$$

The homogeneous equation

$$(\rho+\lambda-\mu_H)\varphi = \mathcal{D}^H\varphi$$

has two solution f^{ϕ_1} and f^{ϕ_2}, where

$$\begin{aligned}\phi_1 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} > 1 \\ \phi_2 &= \frac{\mu_H+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} < 0.\end{aligned} \tag{A.57}$$

Hence, the value function takes the form

$$\tilde{v}_H(f) = u_1(f) + D_1 f^{\phi_1} + D_2 f^{\phi_2}.$$

As before, the particular solution

$$u_1(f) = \frac{1}{\rho+\lambda-\mu_H} - \frac{r+\xi}{\rho+\lambda+\xi} f \tag{A.58}$$

Finally, by combining equations $\tilde{v}_H\left(\tilde{f}_L^b-\right) = \tilde{v}_H\left(\tilde{f}_L^b+\right)$ and $\tilde{v}_H\left(\tilde{f}_H^b\right) = 0$, we get

$$\begin{aligned}D_1 &= \frac{\tilde{v}_H(\tilde{f}_L^b) + u_1(\tilde{f}_H^b)\left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_2} - u_1(\tilde{f}_L^b)}{(\tilde{f}_H^b)^{\phi_1}\left[\left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_1} - \left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_2}\right]} \\ D_2 &= (\tilde{f}_H^b)^{-\phi_2}\left(-u_1(\tilde{f}_H^b) - D_1(\tilde{f}_H^b)^{\phi_1}\right).\end{aligned}$$

It follows that the solution to the value function on this interval is given by

$$\tilde{v}_H(f) = u_1(f) + \left(\tilde{v}_H(\tilde{f}_L^b) - u_1(\tilde{f}_L^b)\right)\tilde{h}_0\left(f \middle| \tilde{f}_L^b, \tilde{f}_H^b\right) - u_1(\tilde{f}_H^b)\tilde{h}_1\left(f \middle| \tilde{f}_L^b, \tilde{f}_H^b\right),$$

where

$$\begin{aligned}\tilde{h}_0\left(f \middle| \tilde{f}_L^b, \tilde{f}_H^b\right) &= \frac{\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi_1} - \left(\frac{f}{\tilde{f}_H^b}\right)^{\phi_2}}{\left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_1} - \left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_2}} \\ \tilde{h}_1\left(f \middle| \tilde{f}_L^b, \tilde{f}_H^b\right) &= \frac{\left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_2}\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi_1} - \left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_1}\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi_2}}{\left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_2} - \left(\frac{\tilde{f}_L^b}{\tilde{f}_H^b}\right)^{\phi_1}}.\end{aligned} \tag{A.59}$$

It remains to solve $\left\{\tilde{v}_H\left(\tilde{f}_L^b\right), f_H^b\right\}$ from $\tilde{v}_H'\left(\tilde{f}_L^b-\right) = \tilde{v}_H'\left(\tilde{f}_L^b+\right)$ and $\tilde{v}_H'\left(\tilde{f}_H^b\right) = 0$.

In a smooth equilibrium, $\tilde{p}_H = -\tilde{v}_H'$, and $\tilde{p}_H$ satisfies

$$(r+\xi)\,\tilde{p}_H = (r+\xi) + \left(\tilde{g}_H - \xi - \mu_H + \sigma^2\right) f\tilde{p}_H' + \frac{1}{2}\sigma^2 f^2 \tilde{p}_H'' + \lambda\left(\tilde{p}_L - \tilde{p}_H\right).$$

Differentiating once the HJB for $\tilde{v}_H$, we get $\tilde{g}_H = \frac{(\rho - r)\tilde{p}_H}{f\tilde{v}_H''}$. □

B.5 Transitory Shocks

In this section, we extend the model to consider some further empirical implications. In the main model, we have assumed that the state $\theta_t = L$, is absorbing. If we interpret the changes in regime as business-cycles, it is natural to assume that these are transitory. We can extend the model to consider this situation. We denote the transition rate from the high state to the low state by λ_{HL}, and the transition rate from the low state to the high state by λ_{LH}. The stationary distribution of the process θ_t is then given by $\Pr(\theta = H) = \lambda_{LH}/(\lambda_{LH} + \lambda_{HL})$.

The equilibrium has the same qualitative features. The only changes is that in the HJB equation (A.2) for $j_L(f)$ and in the asset pricing equation (18) for $p_L(f)$, we have to add additional terms $\lambda_{LH}\big(j_H(f) - j_L(f)\big)$ and $\lambda_{LH}\big(p_H(f) - p_L(f)\big)$, respectively. The issuance policy takes the general form provided in equation (20). When shocks are transitory, the HJB equation for $j_L(f)$ becomes

$$(\rho + \eta - \mu_L)\, j_L(f) = 1 - (r+\xi)\, f + (\rho - r)\, j_L(f) + \mathcal{D}^L j_L(f) + \lambda_{LH}\big(j_H(f) - j_L(f)\big).$$

The indifference condition for the issuance of short-term debt in high state remains the same and is given by

$$(\rho + \lambda_{HL} - r)\, j_L(f_\dagger) \geq (\rho - r)\, j_H(f_\dagger).$$

We solve the equation in this region, and the combines the solution using smooth pasting and value matching conditions at the threshold $f_\dagger$. The default boundary are determined using the same value matching and smooth pasting conditions as in the main version of the model.

Solution for $f \in (0, f_\dagger)$. The characteristic equation of the associated homogenous equation is now a quartic equation instead of a quadratic one. Hence the solution takes the general form:

$$\begin{aligned} j_L(f) &= A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2} + A_4 f^{\gamma_3} + A_5 f^{\gamma_4} \\ j_H(f) &= B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2} + B_4 f^{\gamma_3} + B_5 f^{\gamma_4}. \end{aligned}$$

Substituting the conjecture in the ODE, we get the linear system

$$\begin{aligned} (r + \lambda_{LH} + \eta - \mu_L) A_0 &= 1 + \lambda_{LH} B_0 \\ (\rho + \lambda_{HL} - \mu_H) B_0 &= 1 + (\rho + \lambda_{HL} - r) A_0 \\ (r + \xi + \lambda_{LH} + \eta) A_1 &= (r + \xi) + \lambda_{LH} B_1 \\ (\rho + \xi + \lambda_{HL}) B_1 &= (r + \xi) + (\rho + \lambda_{HL} - r) A_1. \end{aligned}$$

It follows that

$$\begin{aligned} A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + \lambda_{LH}(r - \mu_H)} \\ A_1 &= \frac{(\rho + \xi + \lambda_{HL} + \lambda_{LH})(r + \xi)}{(\rho + \xi + \lambda_{HL})(r + \xi + \eta) + \lambda_{LH}(r + \xi)} \\ B_0 &= \frac{1 + (\rho + \lambda_{HL} - r) A_0}{\rho + \lambda_{HL} - \mu_H} \\ B_1 &= \frac{(r + \xi) + (\rho + \lambda_{HL} - r) A_1}{\rho + \xi + \lambda_{HL}} \end{aligned}$$

In addition, for any $i = 1, \ldots, 4$

$$\begin{aligned} (r + \lambda_{LH} + \eta - \mu_L) A_{i+1} &= -(\mu_L + \xi) A_{i+1}\gamma_i + \lambda_{LH} B_{i+1} + \frac{1}{2}\sigma^2 A_{i+1}\gamma_i(\gamma_i - 1) \\ (\rho + \lambda_{HL} - \mu_H) B_{i+1} &= (\rho + \lambda_{HL} - r) A_{i+1} - (\mu_H + \xi) B_{i+1}\gamma_i + \frac{1}{2}\sigma^2 B_{i+1}\gamma_i(\gamma_i - 1) \end{aligned}$$

If we multiply the equation for A_2 by γ_1, we get

$$(r + \lambda_{LH} + \eta - \mu_L)\gamma_1 A_2 = -(\mu_L + \xi) A_2\gamma_1^2 + \lambda_{LH} B_2 \gamma_1 + \frac{1}{2}\sigma^2 A_2 \gamma_1^2 (\gamma_1 - 1)$$

when $\lambda_{LH} \neq 0$, from the equation for A_2 we have

$$\lambda_{LH} B_2 = \left[(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1) \right] A_2.$$

Substituting in the equation for B_2 we obtain an expression for $B_2\gamma_1$ that can be then substituted back in the equation for A_2 (multiplied by γ_1). Canceling A_2, we obtain the characteristic equation for the homogenous equation

$$\begin{aligned}
&\frac{1}{4}\sigma^4\gamma_1^4 + \frac{1}{2}\sigma^2\left(\mu_L + \mu_H + 2\xi + \sigma^2\right)\gamma_1^3 \\
&\quad + \left[\frac{1}{4}\sigma^4 - \frac{1}{2}\sigma^2\left(\rho + \lambda_{HL} + r + \lambda_{LH} + \eta - 2\left(\mu_L + \mu_H + \xi\right)\right) + \left(\mu_L + \xi\right)\left(\mu_H + \xi\right)\right]\gamma_1^2 \\
&\quad + \left[\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\left(r + \lambda_{LH} + \eta - \mu_L\right) + \left(\rho + \lambda_{HL} - \mu_H\right)\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\right]\gamma_1 \\
&\qquad + \left(\rho + \lambda_{HL} - \mu_H\right)\left(r + \eta - \mu_L\right) + \left(r - \mu_H\right)\lambda_{LH} = 0
\end{aligned}$$

This equation has four roots.

Solution for $f \in (f_\dagger, f_H^b)$. In this case, we guess a solution of the form

$$\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4} \\
j_H(f) &= D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.
\end{aligned}$$

From the HJB equation for $j_H(f)$, we get that β_1 and β_2 are the roots for the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta + \mu_H - r - \lambda_{HL} = 0,$$

which are given by

$$\begin{aligned}
\beta_1 &= \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2}, \\
\beta_2 &= \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2}.
\end{aligned}$$

From the equation for j_L, we get that β_3 and β_4 are given by the roots to the quadratic equation

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\beta - \left(r + \lambda_{LH} + \eta - \mu_L\right) = 0,$$

which are

$$\beta_3 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} > 1,$$
$$\beta_4 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} + \eta - \mu_L\right)}}{\sigma^2} < 0.$$

Matching coefficients, we get that

$$D_0 = \frac{1}{r + \lambda_{HL} - \mu_H}$$
$$D_1 = \frac{r + \xi}{r + \xi + \lambda_{HL}}$$
$$C_0 = \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L}$$
$$C_1 = \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta}$$
$$C_2 = \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\beta_1 - \frac{1}{2}\sigma^2\left(\beta_1 - 1\right)\beta_1}$$
$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\beta_2 - \frac{1}{2}\sigma^2\left(\beta_2 - 1\right)\beta_2}.$$

Boundary Conditions. We still need to determine the coefficients (A_i, B_i) for $i = 2, \ldots, 5$, the coefficients D_2, D_3, and C_4, C_5, as well as the thresholds $f_\dagger, f_H^b, f_L^b$.

We start considering $f \in (0, f_\dagger)$. Under reasonable parameters, we have found that all four roots of the quartic characteristic equation are real, and that two of them are positive (let the positive roots be γ_1 and γ_2). If this is the case, the transversality conditions

$$\lim_{f \to 0} j_H(f) < \infty,$$
$$\lim_{f \to 0} j_L(f) < \infty,$$

imply that $A_4 = A_5 = B_4 = B_5 = 0$. Thus, we can write the value function as

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}$$
$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2},$$

where the coefficients A_0, A_1, B_0, B_1 have already been determined. Moreover, from the previous analysis we already have that for $i = 2, 3$

$$\left[(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\, \gamma_{i-1} - \frac{1}{2}\sigma^2 \gamma_i\, (\gamma_{i-1} - 1) \right] A_i = \lambda_{LH} B_i,$$

so the coefficients $\{B_2, B_3\}$ are immediately determined by the 2 free coefficients $\{A_2, A_3\}$.

Next, we consider the intervals $(f_\dagger, f_H^b)$ and $(f_\dagger, f_L^b)$. Here we have that $j_\theta(f)$ takes the form

$$\begin{aligned} j_L(f) &= C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4} \\ j_H(f) &= D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}. \end{aligned}$$

where we have 4 free coefficients $\{C_2, C_3, C_4, C_5\}$ since $\{D_2, D_3\}$ are fully determined by $\{C_2, C_3\}$. Thus, we have to determine $(A_2, A_3, C_2, C_3, C_4, C_5)$ in addition to the free boundary $(f_\dagger, f_L^b, f_H^b)$; hence, we need 9 boundary conditions. The first boundary condition is the indifference condition

$$(\rho + \lambda_{HL} - r)\, j_L(f_\dagger) = (\rho - r)\, j_H(f_\dagger).$$

The value function must be continuously differentiable at $f_\dagger$ se we have the value matching and smooth pasting conditions at $f_\dagger$

$$\begin{aligned} j_H(f_\dagger -) &= j_H(f_\dagger +) \\ j_L(f_\dagger -) &= j_L(f_\dagger +) \\ j_H'(f_\dagger -) &= j_H'(f_\dagger +) \\ j_L'(f_\dagger -) &= j_L'(f_\dagger +). \end{aligned}$$

Finally, we have the value matching and smooth pasting conditions at the default boundary

$$\begin{aligned} j_L(f_L^b) &= 0 \\ j_H(f_H^b) &= 0 \\ j_L'(f_L^b) &= 0 \\ j_H'(f_H^b) &= 0. \end{aligned}$$

Substituting the value function in these conditions, we get

$$
\begin{aligned}
A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= \frac{\rho - r}{\rho + \lambda_{HL} - r} \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right) \\
B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} &= D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} \\
A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4} \\
-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} &= -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} \\
-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} &= -C_1 + \beta_1 C_1 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1} \\
&\quad + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}
\end{aligned}
$$

and

$$
\begin{aligned}
C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} &= 0 \\
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0 \\
-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} &= 0 \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0.
\end{aligned}
$$

We can simply the above 9 equations into 3 equations and only solve the three unknowns $(f_\dagger, f_L^b, f_H^b)$: From

$$
\begin{aligned}
D_0 - D_1 f_H^b + D_2 \left(f_H^b\right)^{\beta_1} + D_3 \left(f_H^b\right)^{\beta_2} &= 0, \\
-D_1 + \beta_1 D_2 \left(f_H^b\right)^{\beta_1 - 1} + \beta_2 D_3 \left(f_H^b\right)^{\beta_2 - 1} &= 0,
\end{aligned}
$$

we know

$$
\begin{aligned}
D_2 &= \frac{\frac{\beta_2}{r + \lambda_{HL} - \mu_H} - (\beta_2 - 1) \frac{r + \xi}{r + \xi + \lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_1} (\beta_1 - \beta_2)} \\
D_3 &= \frac{\frac{\beta_1}{r + \lambda_{HL} - \mu_H} - (\beta_1 - 1) \frac{r + \xi}{r + \xi + \lambda_{HL}} f_H^b}{\left(f_H^b\right)^{\beta_2} (\beta_2 - \beta_1)}.
\end{aligned}
$$

Then we know C_2, C_3 from

$$C_2 = \frac{\lambda_{LH} D_2}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\beta_1 - \frac{1}{2}\sigma^2 (\beta_1 - 1)\,\beta_1}$$
$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\,\beta_2 - \frac{1}{2}\sigma^2 (\beta_2 - 1)\,\beta_2}$$

From

$$B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2} = D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2}$$
$$-B_1 + \gamma_1 B_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1} = -D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1},$$

we know

$$B_2 = \frac{-D_1 + \beta_1 D_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 D_3 (f_\dagger)^{\beta_2 - 1} + B_1 - \gamma_2 B_3 (f_\dagger)^{\gamma_2 - 1}}{\gamma_1 (f_\dagger)^{\gamma_1 - 1}},$$
$$B_3 = \frac{D_0 - D_1 f_\dagger + D_2 (f_\dagger)^{\beta_1} + D_3 (f_\dagger)^{\beta_2} - \left(B_0 - B_1 f_\dagger + \frac{1}{\gamma_1}\left(-D_1 f_\dagger + \beta_1 D_2 (f_\dagger)^{\beta_1} + \beta_2 D_3 (f_\dagger)^{\beta_2} + B_1 f_\dagger\right)\right)}{\left(1 - \frac{\gamma_2}{\gamma_1}\right)(f_\dagger)^{\gamma_2}}$$

Then we know A_2 and A_3 where

$$A_2 = \frac{\lambda_{LH} B_2}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\,\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1)}.$$
$$A_3 = \frac{\lambda_{LH} B_3}{(r + \lambda_{LH} + \eta - \mu_L) + (\mu_L + \xi)\,\gamma_2 - \frac{1}{2}\sigma^2 \gamma_2 (\gamma_2 - 1)}$$

From

$$C_0 - C_1 f_L^b + C_2 \left(f_L^b\right)^{\beta_1} + C_3 \left(f_L^b\right)^{\beta_2} + C_4 \left(f_L^b\right)^{\beta_3} + C_5 \left(f_L^b\right)^{\beta_4} = 0$$
$$-C_1 + \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} + \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} + \beta_3 C_4 \left(f_L^b\right)^{\beta_3 - 1} + \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1} = 0$$

we know

$$C_4 = \frac{C_1 - \beta_1 C_2 \left(f_L^b\right)^{\beta_1 - 1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1} - \beta_4 C_5 \left(f_L^b\right)^{\beta_4 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}},$$

$$C_5 = \frac{-C_0 + C_1 f_L^b - C_2 \left(f_L^b\right)^{\beta_1} - C_3 \left(f_L^b\right)^{\beta_2} - \frac{C_1 f_L^b - \beta_1 C_2 \left(f_L^b\right)^{\beta_1} - \beta_2 C_3 \left(f_L^b\right)^{\beta_2}}{\beta_3}}{\left(1 - \frac{\beta_4}{\beta_3}\right)\left(f_L^b\right)^{\beta_4}}$$

Therefore, we only need to solve $(f_\dagger, f_L^b, f_H^b)$ from the following 3 equations:

$$\begin{aligned}
A_0 - A_1 f_\dagger + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= \frac{\rho - r}{\rho + \lambda_{HL} - r} \left(B_0 - B_1 f_\dagger + B_2 (f_\dagger)^{\gamma_1} + B_3 (f_\dagger)^{\gamma_2}\right) \\
A_0 - A_1 (f_\dagger) + A_2 (f_\dagger)^{\gamma_1} + A_3 (f_\dagger)^{\gamma_2} &= C_0 - C_1 f_\dagger + C_2 (f_\dagger)^{\beta_1} + C_3 (f_\dagger)^{\beta_2} + C_4 (f_\dagger)^{\beta_3} + C_5 (f_\dagger)^{\beta_4} \\
-A_1 + \gamma_1 A_2 (f_\dagger)^{\gamma_1 - 1} + \gamma_2 A_3 (f_\dagger)^{\gamma_2 - 1} &= -C_1 + \beta_1 C_2 (f_\dagger)^{\beta_1 - 1} + \beta_2 C_3 (f_\dagger)^{\beta_2 - 1} \\
&\quad + \beta_3 C_4 (f_\dagger)^{\beta_3 - 1} + \beta_4 C_5 (f_\dagger)^{\beta_4 - 1}
\end{aligned}$$

Limit when $\sigma \to 0$

As in the case where the low state is absorbing, we can obtain a more explicit solution for the equilibrium in the limit when $\sigma \to 0$.

Proposition 16 (Limit long-term debt issuance policy)**.** *Suppose that $\mu_L + \xi < 0$, $\mu_H + \xi > 0$, and*

$$(r + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH} \geq 0.$$

In the limit when $\sigma \to 0$, the issuance policy is

$$g_\theta(f) = \frac{\rho - r}{\gamma - 1}\left[g_0 + g_1 \left(\frac{f}{f_\dagger}\right)^{-(\gamma - 1)}\right] \mathbb{1}_{\{f < f_\dagger,\, \theta = H\}}$$

where g_0 and g_1 are positive coefficients and $\gamma > 1$ is the unique positive root of

$$\gamma^2 + \left(\frac{\rho + \lambda_{HL} - \mu_H}{\mu_H + \xi} - \frac{r + \lambda_{LH} + \eta - \mu_L}{-(\mu_L + \xi)}\right)\gamma - \frac{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + (r - \mu_H)\lambda_{LH}}{-(\mu_L + \xi)(\mu_H + \xi)} = 0.$$

Consider the case when $\mu_L + \xi < 0 < \mu_H + \xi$. The characteristic equation for γ_i converges to the quadratic equation

$$\begin{aligned}(\mu_L+\xi)(\mu_H+\xi)\gamma_1^2+[(\mu_H+\xi)(r+\lambda_{LH}+\eta-\mu_L)+(\rho+\lambda_{HL}-\mu_H)(\mu_L+\xi)]\gamma_1 \\ +(\rho+\lambda_{HL}-\mu_H)(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH}=0\end{aligned}$$

The present value of cash flow is finite given the creditors' discount rate only if

$$(r+\lambda_{HL}-\mu_H)(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH}>0,$$

which implies that the quadratic equation has one negative and one positive root. Let γ be the positive root, which can be verified to always be greater than 1. Similarly, the roots β_i converge to

$$\begin{aligned}\beta_1 &= \infty \\ \beta_2 &= -\frac{r+\lambda_{HL}-\mu_H}{\mu_H+\xi} \\ \beta_3 &= \frac{r+\lambda_{LH}+\eta-\mu_L}{-(\mu_L+\xi)} \\ \beta_4 &= -\infty.\end{aligned}$$

Thus the solution to the HJB equation on $(0,f_\dagger)$, becomes

$$\begin{aligned}j_L(f) &= A_0-A_1f+A_2f^\gamma \\ j_H(f) &= B_0-B_1f+B_2f^\gamma,\end{aligned}$$

while the solution for $(f_\dagger,f_H^b)$ and $(f_\dagger,f_L^b)$ becomes

$$\begin{aligned}j_L(f) &= C_0-C_1f+C_3f^{\beta_2}+C_4f^{\beta_3} \\ j_H(f) &= D_0-D_1f+D_3f^{\beta_2}.\end{aligned}$$

The coefficient A_2,B_2 are given by

$$A_2=\frac{\lambda_{LH}}{(r+\lambda_{LH}+\eta-\mu_L)+(\mu_L+\xi)\gamma}\frac{B_1-D_1}{\gamma(f_\dagger)^{\gamma-1}}.$$

From the continuity and smoothness of $j_H(f)$ at $f_\dagger$, we know

$$
\begin{aligned}
B_0 - B_1 f_\dagger + B_2 f_\dagger^{\gamma} &= D_0 - D_1 f_\dagger + D_3 f_\dagger^{\beta_2} \\
-B_1 + \gamma B_2 f_\dagger^{\gamma-1} &= -D_1 + \beta_2 D_3 f_\dagger^{\beta_2-1},
\end{aligned}
$$

which implies that

$$
\begin{aligned}
B_2 &= \frac{\beta_2 \left(D_0 - B_0\right) + \left(1-\beta_2\right)\left(D_1 - B_1\right) f_\dagger}{\left(\beta_2 - \gamma\right) f_\dagger^{\gamma}} \\
D_3 &= \frac{\left(1-\gamma\right)\left(D_1 - B_1\right) f_\dagger + \gamma\left(D_0 - B_0\right)}{\left(\beta_2 - \gamma\right) f_\dagger^{\beta_2}}.
\end{aligned}
$$

The coefficients C_3, C_4 are

$$
\begin{aligned}
C_3 &= \frac{\lambda_{LH} D_3}{r + \lambda_{LH} + \eta - \mu_L + \left(\mu_L + \xi\right)\beta_2} \\
C_4 &= \frac{C_1 - \beta_2 C_3 \left(f_L^b\right)^{\beta_2 - 1}}{\beta_3 \left(f_L^b\right)^{\beta_3 - 1}}.
\end{aligned}
$$

Substituting in the HJB equation, we can write the solution for $f \in [0, f_\dagger]$ as

$$
\begin{aligned}
j_L(f) &= A_0 - A_1 f + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + \left(\mu_L + \xi\right)\gamma} \frac{\beta_2 \left(D_0 - B_0\right) + \left(1-\beta_2\right)\left(D_1 - B_1\right) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\gamma} \\
j_H(f) &= B_0 - B_1 f + \frac{\beta_2 \left(D_0 - B_0\right) + \left(1-\beta_2\right)\left(D_1 - B_1\right) f_\dagger}{\beta_2 - \gamma} \left(\frac{f}{f_\dagger}\right)^{\gamma}.
\end{aligned}
$$

For $f > f_\dagger$, we can write

$$
\begin{aligned}
j_H(f) &= D_0 - D_1 f + \left[\frac{\left(1-\gamma\right)\left(D_1 - B_1\right) f_\dagger + \gamma\left(D_0 - B_0\right)}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2} \\
&= D_0 \left[1 + \frac{\gamma}{\beta_2 - \gamma}\left(\frac{f}{f_\dagger}\right)^{\beta_2}\right] - D_1 f \left[1 - \frac{1-\gamma}{\beta_2 - \gamma}\left(\frac{f}{f_\dagger}\right)^{\beta_2 - 1}\right] - \left[\frac{\left(1-\gamma\right) B_1 f_\dagger + \gamma B_0}{\beta_2 - \gamma}\right] \left(\frac{f}{f_\dagger}\right)^{\beta_2}
\end{aligned}
$$

and

$$
\begin{aligned}
j_L(f) &= C_0 - C_1 f + C_3 f^{\beta_2} + C_4 f^{\beta_3} \\
&= C_0 - C_1 f \left[1 - \frac{1}{\beta_3}\left(\frac{f}{f_L^b}\right)^{\beta_3 - 1}\right] + C_3 f^{\beta_2} \left[1 - \frac{\beta_2}{\beta_3}\left(\frac{f}{f_L^b}\right)^{\beta_3 - \beta_2}\right].
\end{aligned}
$$

where we recollect that the constant $A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1$ are

$$
\begin{aligned}
A_0 &= \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r + \eta - \mu_L) + \lambda_{LH}(r - \mu_H)} \\
A_1 &= \frac{(\rho + \xi + \lambda_{HL} + \lambda_{LH})(r + \xi)}{(\rho + \xi + \lambda_{HL})(r + \xi + \eta) + \lambda_{LH}(r + \xi)} \\
B_0 &= \frac{1 + (\rho + \lambda_{HL} - r) A_0}{\rho + \lambda_{HL} - \mu_H} \\
B_1 &= \frac{(r + \xi) + (\rho + \lambda_{HL} - r) A_1}{\rho + \xi + \lambda_{HL}} \\
D_0 &= \frac{1}{r + \lambda_{HL} - \mu_H} \\
D_1 &= \frac{r + \xi}{r + \xi + \lambda_{HL}} \\
C_0 &= \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} + \eta - \mu_L} \\
C_1 &= \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH} + \eta}.
\end{aligned}
$$

Finally, we get the equations determining the thresholds $f_\dagger, f_H^b, f_L^b$ which are given now by

$$
\begin{aligned}
j_L(f_\dagger -) &= j_L(f_\dagger +) \\
j_L(f_L^b) &= 0 \\
j_H(f_H^b) &= 0
\end{aligned}
$$

which can be written as

$$
\begin{aligned}
A_0 - A_1 f_\dagger + A_2 f_\dagger^\gamma &= C_0 - C_1 f_\dagger + C_3 {f_\dagger}^{\beta_2} + C_4 {f_\dagger}^{\beta_3} \\
C_0 - C_1 f_L^b + C_3 {f_L^b}^{\beta_2} + C_4 {f_L^b}^{\beta_3} &= 0 \\
D_0 - D_1 f_H^b + D_3 {f_H^b}^{\beta_2} &= 0.
\end{aligned}
$$

Issuance Function: Before deriving the issuance function we need to derive the debt price, which is given by $p_\theta(f) = -j_\theta'(f)$. Taking derivatives for $f \in (0, f_\dagger)$ we get

$$
\begin{aligned}
p_L(f) &= A_1 + \frac{\lambda_{LH}}{r + \lambda_{LH} + \eta - \mu_L + (\mu_L + \xi)\gamma} \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1} \\
p_H(f) &= B_1 + \frac{\gamma}{\gamma - \beta_2} \frac{\beta_2 (D_0 - B_0) + (1 - \beta_2)(D_1 - B_1) f_\dagger}{f_\dagger} \left(\frac{f}{f_\dagger}\right)^{\gamma - 1}.
\end{aligned}
$$

Finally, we compute the issuance function. We need to find $p'_H(f)$ for $f \in (0, f_\dagger)$. This expression is given by

$$-fp'_H(f) = -\frac{\gamma(\gamma-1)}{\gamma-\beta_2}\frac{\beta_2\,(D_0-B_0)+(1-\beta_2)\,(D_1-B_1)\,f_\dagger}{f_\dagger}\left(\frac{f}{f_\dagger}\right)^{\gamma-1}.$$

Next, we compote $p_H(f)-p_L(f)$ which is

$$p_H(f)-p_L(f) = B_1-A_1+\frac{r+\eta-\mu_L+(\mu_L+\xi)\,\gamma}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\,\gamma}\frac{\beta_2\,(D_0-B_0)+(1-\beta_2)\,(D_1-B_1)\,f_\dagger}{f_\dagger}\frac{\gamma}{\gamma-\beta_2}\left(\frac{f}{f_\dagger}\right)^{\gamma-1}$$

From here we get that

$$\begin{aligned} g_H(f) &= \frac{(\rho-r)(p_H(f)-p_L(f))}{-fp'_H(f)} \\ &= \frac{\rho-r}{\gamma-1}\left[g_0+g_1\left(\frac{f}{f_\dagger}\right)^{-(\gamma-1)}\right] \end{aligned}$$

where

$$\begin{aligned} g_0 &= -\frac{r+\eta-\mu_L+(\mu_L+\xi)\gamma}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\,\gamma} \\ g_1 &= -\left(1-\frac{\beta_2}{\gamma}\right)\frac{(B_1-A_1)f_\dagger}{\beta_2\,(D_0-B_0)+(1-\beta_2)\,(D_1-B_1)\,f_\dagger} \end{aligned}$$

Finally, noting that $p_L(0)=A_1$, $p_H(0)=B_1$, and $B_0=j_H(0)$, an substituting the relations

$$\begin{aligned} (\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0) &= 1-(r+\lambda_{HL}-\mu_H)\,j_H(0) \\ (\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0) &= (r+\xi)-(r+\xi+\lambda_{HL})\,p_H(0), \end{aligned}$$

we can write

$$g_1 = \left(\frac{1}{\gamma}+\frac{\mu_H+\xi}{r+\lambda_{HL}-\mu_H}\right)\frac{(r+\lambda_{HL}-\mu_H)(p_H(0)-p_L(0))f_\dagger}{(\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0)-\Big[(\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0)\Big]f_\dagger}$$

Letting

$$\varphi(f) \equiv (\rho-r)j_H(f)-(\rho+\lambda_{HL}-r)\,j_L(f)$$

we can write

$$(\rho-r)j_H(0)-(\rho+\lambda_{HL}-r)\,j_L(0)-\Big[(\rho-r)p_H(0)-(\rho+\lambda_{HL}-r)\,p_L(0)\Big]f_\dagger=\varphi(0)+\varphi'(0)f_\dagger.$$

For $f\in(0,f_\dagger)$,

$$\varphi''(f)=(\rho+\lambda_{HL}-r)\left[\frac{\rho-r}{\rho+\lambda_{HL}-r}-\frac{\lambda_{LH}}{r+\lambda_{LH}+\eta-\mu_L+(\mu_L+\xi)\,\gamma}\right]\gamma(\gamma-1)\mathcal{Q}f^{\gamma-2},$$

where

$$\mathcal{Q}\equiv\frac{\gamma}{\gamma-\beta_2}\frac{\beta_2\,(D_0-B_0)+(1-\beta_2)\,(D_1-B_1)\,f_\dagger}{f_\dagger}\left(\frac{1}{f_\dagger}\right)^{\gamma-1}.$$

$j_\theta(f)$ is strictly convex only if $\mathcal{Q}>0$. Hence, the sign of φ'' is determined by the sign of the term within the parenthesis. The coefficient γ is the positive root of the quadratic equation

$$\gamma^2+\left(\frac{\rho+\lambda_{HL}-\mu_H}{\mu_H+\xi}-\frac{r+\lambda_{LH}+\eta-\mu_L}{-(\mu_L+\xi)}\right)\gamma-\frac{(\rho+\lambda_{HL}-\mu_H)\,(r+\eta-\mu_L)+(r-\mu_H)\lambda_{LH}}{-(\mu_L+\xi)(\mu_H+\xi)}=0.$$

This equation can be rewritten more conveniently as

$$\frac{\lambda_{LH}}{r+\lambda_{LH}+\eta+(\gamma-1)\mu_L+\gamma\xi}=\frac{\rho+\lambda_{HL}+(\gamma-1)\mu_H+\xi\gamma}{\rho+\lambda_{HL}-r}>\frac{\rho-r}{\rho+\lambda_{HL}-r},$$

which implies that the term within the parenthesis in $\varphi''(f)$ is negative. Thus, we conclude that $\varphi(f)$ is concave on $[0,f_\dagger]$, so $\varphi(f_\dagger)\le\varphi(0)+\varphi'(0)f_\dagger$. By construction, $\varphi(f_\dagger)=0$, so it follows that $\varphi(0)+\varphi'(0)f_\dagger\ge 0$, which means that $g_1>0$. Moreover, from the previous equation for γ we also get that $r+\lambda_{LH}+\eta+(\gamma-1)\mu_L+\gamma\xi>0$ and

$$r+\eta+(\gamma-1)\mu_L+\gamma\xi=-\lambda_{LH}\frac{r+(\gamma-1)\mu_H+\xi\gamma}{\rho+\lambda_{HL}+(\gamma-1)\mu_H+\xi\gamma}<0,$$

so it follows that $g_0>0$.

B.6 The microfoundation of the disaster shock

Now we show that the disaster shock can be microfounded by a model with three states, high (H), low (L), and disaster (ℓ), where $\mu_H>\mu_L>\mu_\ell$. In other words, the low state can still get worse. As before, let λ be the transition intensity from H to L and η be the one from L to ℓ. We are interested in the condition such that in the low state L, the borrower optimally choose to issue risky short-term debt. In other words, the corresponding $f_\dagger$ is zero in the low state L. To do so, we only need to study the value functions in $\theta = L$ and $\theta = \ell$.

When $\theta = L$, the HJB is

$$\begin{aligned}(\rho+\eta)\, j_L(f) &= \max_{\{0\le d_L\le j_L\}} 1-(r+\xi)\, f+(\rho+\eta-y)\, d_\theta(f)+\eta\,(j_\ell(f)-d_L(f))^+ \\ &\quad +\mu_L\left(j_L(f)-j_L'(f)\, f\right)+\frac{1}{2}\sigma^2 f^2 j_L''(f)-\xi f j_L'(f) \\ \Rightarrow (\rho+\eta-\mu_L)\, j_L(f) &= \max_{\{0\le d_L\le j_L\}} 1-(r+\xi)\, f+(\rho+\eta-y_L)\, d_L(f) \\ &\quad +\eta\,(j_\ell(f)-d_L(f))^+-(\mu_L+\xi)\, f j_L'(f)+\frac{1}{2}\sigma^2 f^2 j_L''(f)\end{aligned}$$

The HJB when $\theta = \ell$ is

$$\begin{aligned}(\rho-\mu_\ell)\, j_\ell(f) &= \max_{\{0\le d_\ell\le j_\ell\}} 1-(r+\xi)\, f+(\rho-y_\ell)\, d_\ell(f)-(\mu_\ell+\xi)\, f j_\ell'(f)+\frac{1}{2}\sigma^2 f^2 j_\ell''(f)\,. \\ &= 1-(r+\xi)\, f+(\rho-r)\, j_\ell(f)-(\mu_\ell+\xi)\, f j_\ell'(f)+\frac{1}{2}\sigma^2 f^2 j_\ell''(f)\,,\end{aligned}$$

which implies that

$$(r-\mu_\ell)\, j_\ell(f) = 1-(r+\xi)\, f-(\mu_\ell+\xi)\, f j_\ell'(f)+\frac{1}{2}\sigma^2 f^2 j_\ell''(f)$$

The short rate is

$$y_L(d,f) = r+\eta \mathbb{1}_{d>j_\ell(f)}$$

and

$$y_\ell(d,f) = r.$$

In state ℓ, we have $d = j_\ell(f)$. In state L we have

1. If $d_L = j_\ell(f)$, the flow benefit of issuing short-term debt is

$$(\rho+\eta-r)\, j_\ell(f)$$

2. If $d_L = j_L(f)$, the flow benefit of issuing short-term debt is

$$(\rho-r)\, j_L(f)$$

3. Hence, $d = j_\ell$ is optimal if

$$(\rho + \eta - r)\, j_\ell(f) \geq (\rho - r) j_L(f)$$

We can conclude that

- If $(\rho + \eta - r)\, j_\ell(f) \geq (\rho - r)\, j_L(f)$ the HJB equation is

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L\,(f) &= 1 - (r + \xi)\, f + (\rho + \eta - r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi)\, f j_L'\,(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

  which can be reduced to

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L(f) &= 1 - (r + \xi)\, f + (\rho + \eta - r)\, j_\ell(f) \\
&\quad - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

- If $(\rho + \eta - r)\, j_\ell(f) < (\rho - r)\, j_H(f)$ the HJB equation is

$$\begin{aligned}
(r - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
(\rho + \eta - \mu_L)\, j_L\,(f) &= 1 - (r + \xi)\, f + (\rho - r)\, j_L(f) \\
&\quad - (\mu_L + \xi)\, f j_L'\,(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

  which can be reduced to

$$\begin{aligned}
((r + \eta) - \mu_\ell)\, j_\ell\,(f) &= 1 - (r + \xi)\, f - (\mu_\ell + \xi)\, f j_\ell'\,(f) + \frac{1}{2}\sigma^2 f^2 j_\ell''\,(f) \\
((r + \eta) - \mu_L)\, j_L(f) &= 1 - (r + \xi)\, f - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''\,(f)
\end{aligned}$$

- In state ℓ, the default boundary is f_ℓ^b, satisfying

$$\begin{aligned}
j_\ell(f_\ell^b) &= 0 \\
j_\ell'(f_\ell^b) &= 0.
\end{aligned}$$

Since $j_L(0) = \frac{1}{\rho+\eta-\mu_L}\frac{\rho+\eta-\mu_\ell}{r-\mu_\ell}$ and $j_\ell(0) = \frac{1}{r-\mu_\ell}$, to let $f_\dagger = 0$ in the low state, it is necessary and sufficient to have

$$(\rho+\eta-r)\, j_\ell(0) \leq (\rho-r)\, j_L(0).$$

That is

$$(\rho+\eta-r)\frac{1}{(r+\eta)-\mu_\ell} \leq (\rho-r)\frac{1}{\rho+\eta-\mu_L}\frac{\rho+\eta-\mu_\ell}{r-\mu_\ell}.$$

From here, we get

$$\eta^2 + (\rho-\mu_L)\,\eta - (\rho-r)(\mu_L-\mu_\ell) \leq 0.$$

This implies that

$$0 < \eta \leq \bar{\eta} = \frac{-(\rho-\mu_L) + \sqrt{(\rho-\mu_L)^2 + 4(\rho-r)(\mu_L-\mu_\ell)}}{2}. \tag{A.60}$$