Debt Maturity Management

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Abstract

This paper studies how a borrower issues long- and short-term debt in response to shocks to the enterprise value. Short-term debt protects creditors from future dilution; long-term debt allows the borrower to share losses with creditors in a downturn by effectively reducing her payments. We develop a theory of debt maturity that highlights the tradeoff between commitment and risk management. Borrowers far from default value risk management and use a combination of long- and short-term debt. By contrast, distressed borrowers exclusively issue short-term debt. Our model predicts pro-cyclical debt maturity and long-term debt issuance, consistent with data.

Keywords: capital structure; debt maturity; risk management; dynamic tradeoff theory.

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1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in a number of aspects, and an important one is its maturity. Borrowing can be short as in the case of repurchase agreement and trade credit, or long as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing, following shocks to their enterprise value?

Yet, the academic literature falls behind in providing a useful framework to study these questions, despite the obvious importance. For example, the Leland model (Leland, 1994) and the vast follow-up literature typically assume 1) all debt has the same (expected) maturity, and 2) the borrower either commits to the total leverage or may increase leverage only after paying some exogenous issuance cost.[1] While these assumptions simplify the analysis, they are not consistent with the ample empirical evidence that borrowers often times issue a mix of long- and short-term debt, and the adjustment of debt maturity structure can be slow and take time to accomplish.

In this paper, we introduce a simple and tractable framework to study these questions. Our theory highlights the tradeoff between commitment and risk management in borrowing long and short. Long-term debt has a staggered structure: it is not due soon and the borrower cannot commit to not issuing more debt before the existing one is due. Due to this lack of commitment, creditors of long-term debt are exposed to the risks of being diluted and therefore charge higher spreads. By contrast, short-term debt does not suffer from dilution, because it matures very soon and needs to be rolled over on a continuing basis. In other words, existing short-term debt must be retired before new one is issued. On the other hand, long-term debt has an important benefit of risk management: if a downturn arrives, losses in the enterprise value are shared between the borrower and long-term creditors, but not short-term creditors. This risk-sharing property of long-term debt increases the enterprise value, benefiting the borrower.

Let us be more specific. A risk-neutral borrower has assets in place, which generate an income flow that follows a geometric Brownian motion (GBM) process. The expected growth rate of the income is high in an upturn but low in a downturn. A transition from the upturn to the downturn can be interpreted as the downside risk. Creditors are competitive, risk-neutral, and have a lower cost of capital compared to the borrower. The difference in the cost of capital offers a reason for the borrower to issue debt, but too much debt could trigger default. Two types of debt are available: short-term debt matures instantaneously (i.e., has zero maturity) and needs to be simultaneously

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss in the subsection on related literature.

rolled over, and long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing either type of debt at any time to adjust the maturity profile of the outstanding debt. This feature differs us from the existing literature.

The flexibility to issue more debt exposes long-term creditors to dilution, due to the leverage-ratchet effect (Fama and Miller, 1972; Black and Scholes, 1973; Admati et al., 2018). Specifically, the borrower always has incentives to borrow more after existing long-term debt has been issued, because the additional borrowing dilutes the existing long-term claims. Note this is the case even though the asset has no value in default, because the additional debt pushes the borrower closer to default and reduces the price of long-term debt. In equilibrium, creditors anticipate the future dilution and the price of long-term debt adjusts downwards to the level that the borrower cannot capture any benefit. In other words, the borrower does not benefit from borrowing long even though creditors have a lower cost of capital.

By contrast, the instantaneous and simultaneous feature of short-term debt protects it from being diluted and resolves the commitment problem. Given that all the short-term debt needs to be rolled over on a continuing basis, existing short-term debt must be retired before any new debt is issued. In other words, for short-term creditors, their debt matures before the borrower can issue again and therefore does not suffer from dilution. As a result, short-term creditors need to be compensated only by the probability of default within a very short period of time, but not by the cost of being diluted in the future.

This distinction between long- and short-term debt echoes the leasing solution to the Coase conjecture on durable-goods monopoly (Coase, 1972). Specifically, the borrower can be thought of as the monopolistic issuer of her debt, and long-term debt can be thought of as the durable goods. While Coase (1972) conjectured that a monopoly without commitment to future prices does not benefit from the monopoly power, our paper shows the borrower does not benefit from issuing long-term debt. Coase (1972) proposes leasing as a solution to the commitment problem, which is later formalized by Bulow (1982). Informally, leasing fulfills commitment because all the goods need to be repriced in each period. In our context, short-term debt, which needs to be continuously rolled over, serves a similar role as leasing.

The advantage of short-term debt in resolving the commitment problem offers the borrower a natural reason to issue it. Indeed, our results show that long-term debt is never issued in the downturn, when there is no additional downside risk. Instead, the borrower fully levers up by borrowing short, which is riskless and does not suffer from dilution. Similar results hold in the upturn if the borrower is very close to default. There, creditors anticipate immediate default if the downturn arrives, so that short-term debt is not riskless. However, long-term debt is exposed not only to the same potential default following downturn but also the dilution and hence more expensive.

Results are different in the upturn if borrower is far from default. Instead, the borrower issues both long- and short-term debt. In this case, creditors do not anticipate an immediate default if the downturn arrives, but the enterprise value still gets reduced. While the enterprise value immediately drops upon the downturn hits, the borrower still needs to fully repay her short-term debt; otherwise she must default. As a result, the reduced enterprise value without immediate default is only shared between the borrower and long-term creditors. This risk-sharing property is reflected in the drop in long-term debt's price, which highlights an important role of long-term debt in risk management: it allows the borrower to effectively make state-contingent payments. The state-contingent payments act as a cushion to reduce the borrower's burden in the downturn, mitigate the borrower's incentives to default, and eventually increase the enterprise value. Compared to long-term debt, short-term debt is a harder claim: the borrower must make non-state-contingent payments; otherwise she has to default. Note that the merit of long-term debt in sharing the downside risk is valued by the borrower, even though she is risk-neutral. The reason is, the cost in default introduces constraints in financing, which makes the borrower behave as if she is risk-averse.

We show the enterprise value gets higher if the borrower is prohibited from borrowing long. The reason is, long-term debt does not benefit the borrower due to the lack of commitment, but, instead, introduces more defaults. Meanwhile, if the borrower is prohibited from borrowing short, the commitment problem becomes more severe: the enterprise value gets lower, so is the price of long-term debt. In other words, the ability to borrow short also increases the value of long-term debt. This result implies that long and short-term debt can be complements.

Our model implies firms far from default, more levered, and endowed with fewer growth options tend to use short-term debt, consistent with the empirical findings in Barclay and Smith Jr (1995). Moreover, the debt maturity structure is pro-cyclical, consistent with findings in Xu (2018) and Chen et al. (2021). We study the borrower's impulse response to different negative shocks to cash flow and find the responses differ by the nature of the shocks. Following frequent and small negative shocks to operating cash flows, the borrower immediately reduces the issuance of short-term debt. The reduction in long-term debt issuance is slowly adjusted over time. By contrast, following infrequent and large negative shocks, the adjustment of both debt is slow. While the issuance of long-term debt is reduced over time, the issuance of short-term debt is actually increased!

Related literature

Our paper builds on the literature of dynamic corporate finance, pioneered by Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019). Two important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the same problem when the borrower can only issue zero-maturity debt. In both papers, the borrower can only issue one type of debt so the tradeoff between borrowing long and short is not explicitly studied.[2] Our paper fills the gap by combining the approaches and insights from both papers. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed and the borrower can choose between two types of exponentially-maturing debt. Our paper differs in two aspects. First, we allow for full flexibility in adjusting leverage. Second, we model short-term debt as one that matures simultaneously. The different approaches in modeling short-term debt render the mechanisms of the two papers drastically different: whereas we emphasize the tradeoff between commitment and risk-sharing, their paper focuses on rollover losses and dilution. Brunnermeier and Yogo (2009) also study debt maturity in the context of liquidity risk, and they show long-term debt is optimal if the firm is close to default (or debt restructuring in their paper). Our results are the opposite: the borrower will issue exclusively short if she is close to default. The difference is driven by the assumption that in our model, the borrower can issue debt at any time without commitment. By contrast, the borrower in Brunnermeier and Yogo (2009) can only issue new debt after the current one is repaid and effectively has commitment.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. The insight that short-term debt resolves lack of commitment is also present in another closely-related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runable feature of short-term debt. In our paper, the reason why short-term debt resolves commitment is fundamentally different: the short rate would increase drastically had the borrower issued more debt.[3] We show short-term debt has the shortcoming of limited risk sharing. Relatedly, Gertner and Scharfstein (1991) show that conditional on financial distress, short-term debt has a higher market value and increases leverage, leading to more ex post debt overhang (also see Diamond and He (2014)).

The insight that long-term debt allows for more state-contingent payments when markets are incomplete is also present in the literature on fiscal policy and sovereign debt. For example, Angeletos (2002) shows how the Arrow-Debreu allocation can be implemented with noncontingent

[2] Malenko and Tsoy (2020) model the role of firm reputation under one type of debt, where maturity is chosen and fixed at the firm's origination. The tradeoff highlighted involves the feature that debt principal is not tax-deductible whereas interest payments are deductible. The channel is clearly different from us.

[3] Also see Hu and Varas (2021) on this feature of short-term debt in the context of repo and shadow banking.

debt of different maturities. According to Aguiar et al. (2019), the borrower never actively issues any long-term debt due to the lack of commitment. By contrast, we show the motives for risk sharing lead the borrowers to issue a combination of long and short (also see Niepelt (2014)). Bigio et al. (2021) study debt maturity management under liquidity cost but without dilution. In their model, the borrower's choice depends on the demand curve for bonds, microfounded via search (Duffie et al., 2005). The mechanisms of the two papers are therefore complementary. Arellano and Ramanarayanan (2012) consider maturity choice in a quantitative model of sovereign default with tradeoffs similar to ours. Our paper has two important differences. First, we fully characterize the optimal policy in debt maturity management. This characterization allows us to study the nature of the shocks that the borrower wants to hedge using long-term debt. Second, the borrower in our model is risk-neutral and therefore does not have a priori reason to value the merit of risk sharing by long-term debt. The cost of default makes the borrower to behave as if she is risk-averse, as emphasized by the corporate finance literature on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To the best of our knowledge, there is no previous work that establishes the link between maturity management and risk management.

2 The Model

2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower, often times interpreted as a firm for the remainder of the paper. The relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate ρ, which exceeds r, the discount rate of creditors.

The borrower's asset generates cash flows at a rate X_t, where X_t follows the regime-switching diffusion

$$\frac{dX_t}{X_t} = \mu_{\theta_t} dt + \sigma dB_t, \tag{1}$$

B_t is a standard Brownian motion, and $\theta_t \in \{H, L\}$ follows a two-state Markov chain, independent of B_t, with transition intensity λ_{LH} and λ_{HL}, respectively. The drift μ_{θ_t} differs across the two states with $\mu_L < \mu_H$, so that the high state H is associated with a higher growth rate in the borrower's expected income. Below, we refer to the high state as the *upturn* and the low state as the *downturn*.

2.2 Debt Maturity Structure

The difference between the discount rates $\rho - r$ offers benefits for the borrower to issue debt.[4] Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the outstanding debt maturity structure. In particular, we do not restrict the borrower to *commit* to a particular issuance path, but, instead, let the issuance decisions to be made at each instant.

All *short-term debt* matures *instantaneously* and *simultaneously* and therefore needs to be rolled over continuously. We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt\downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time t and y_{t-} be the associated short rate. By contrast, *long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing bonds with coupon rate r and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. Let F_t be the aggregate facevalue of long-term debt outstanding at time t and p_t the price per unit of the facevalue.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume that the bankruptcy cost is 100%. In other words, creditors cannot recover any value once the borrower defaults. Subsection 4.2 studies the model under positive recovery.

2.3 Debt Price and Rollover

Let τ_b be the endogenous time at which the borrower defaults. For $t < \tau_b$, the price of the long-term debt per unit of facevalue satisfies

$$p_t = \mathbb{E}_t \left[\int_t^{\tau_\xi \wedge \tau_b} e^{-r(s-t)} r ds + e^{-r(\tau_b - t)} \mathbb{1}_{\{\tau_b > \tau_\xi\}} \right], \tag{2}$$

where the two components in the expression correspond to the coupon and final payments. The short rate y_{t-} depends on the borrower's equilibrium default decisions:

$$y_{t-} = r + \lim_{dt \downarrow 0} \frac{\Pr\left(\tau_b \le t | \tau_b \ge t - dt\right)}{dt}, \tag{3}$$

where the second term on the right-hand side is the hazard rate of default. Clearly, $y_{t-} = r$ whenever the short-term debt is default free. On the other hand, if default is predicted to happen at time t, creditors will refuse to rollover short-term debt at $t-$, and equivalently, $y_{t-} \to \infty$. In

[4] The difference can be related to differences in liquidity, contracting costs or market segmentation. An alternative setup is to introduce tax shields, and the results are similar. In both setup, we take the debt contract as given and acknowledge that it can be the optimal solution under certain agency frictions. Subsection 4.2 studies debt restructuring.

general, y_{t-} compensates the creditors for the probability of default occurring between $t - dt$ and t. For example, if the borrower is anticipated to default following a transition from H to L state, $y_{t-} = r + \lambda_{HL}$.

Over a short time interval $[t, t+dt)$, the net income to the borrower is

$$\Big[X_t - (r+\xi) F_t - y_{t-} D_{t-}\Big] dt + p_t dG_t + dD_t, \tag{4}$$

where $(r+\xi) F_t$ is the interest and principal payments to long-term creditors, $y_{t-} D_{t-}$ the interest payments to short-term creditors. The remaining two terms $p_t dG_t$ and dD_t are the proceeds from issuing long- and short-term debt.[5] Note that the notations dG_t and dD_t allow for both atomistic and flow issuance, and the price of long-term debt p_t could also depend on the issuance amount dG_t.

Define V_t as the continuation value of the borrower, which we sometimes refer to as the equity value at time t. The borrower chooses the endogenous time of default as well as the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate function as given. Once again, let us emphasize that all these decisions, default and issuance, are made without commitment.

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \geq t\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-\rho(s-t)} \Big(\left(X_s - (r+\xi) F_s - y_{s-} D_{s-}\right) ds + p_s dG_s + dD_s \Big) \right]. \tag{5}$$

2.4 Smooth Equilibrium

The heuristic timing within a short time horizon $[t, t+dt)$ goes as follows.

1. The borrowers arrives at time t with outstanding debt $\{D_{t-}, F_t\}$.
2. The exogenous state θ_t is realized, and the borrower decides whether to repay or default on the outstanding debt.
   - If she defaults, the game ends, and nobody receives anything.
   - If she does not default, she repays $y_{t-} D_{t-}$ and $(r+\xi) F_t$ to short- and long-term creditors.
3. In the case of no default, the borrower receives income $X_t dt$. Moreover, she borrows long-term debt dG_t and issues a net amount of dD_t short-term debt.

[5] One can think of dD_t as the net issuance of short-term debt. Specifically, $dD_t = D_t - D_{t-}$ if there is a jump at t.

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state θ_t, the income level X_t, and the amount of outstanding debt $\{D_{t-}, F_t\}$. The equilibrium requires: 1) creditors break even, i.e., p_t follows (2), and y_{t-} follows (3); 2) the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t \geq 0$. Finally, a MPE is *smooth* if there is no jump in long-term debt issuance, in which case we write $dG_t = g_t F_t dt$. In a smooth equilibrium, the aggregate facevalue of long-term debt evolves according to

$$dF_t = (g_t - \xi) F_t dt. \tag{6}$$

Let us define $J_t = V_t + D_{t-}$ as the joint continuation value of the borrower and short-term creditors if default does not happen at time t. The following result motivates us to work with J_t for the remainder of this paper.

Lemma 1. *The equity value equals* $V_t = V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_t - D_{t-}, 0\}$, *where* $J_t = J_{\theta_t}(X_t, F_t)$ *is given by*

$$\begin{aligned} J_\theta(X,F) = \sup_{\tau_b, \{G_s, D_s : D_s \leq J_\theta(X_s,F_s)\}} \mathbb{E}\Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda_{\theta\theta'})(s-t)} \Bigg(\Big(X_s - (r+\xi)F_s + p_s dG_s \\ + (\rho + \lambda_{\theta\theta'} - y_{s-})D_{s-} + \lambda_{\theta\theta'} \max\{J_{\theta'}(X_s,F_s) - D_{s-}, 0\} \Big) ds \Bigg) \Bigg| X_t = X,\ F_t = F,\ \theta_t = \theta \Bigg]. \end{aligned} \tag{7}$$

Note that $V_t = \max\{J_t - D_{t-}, 0\}$ follows, because if the outstanding short-term debt D_{t-} exceeds J_t, the maximized joint continuation value without default, the borrower chooses to default at time t and renege on the payments D_{t-}. Indeed, such default might happen upon the exogenous state θ_t changing, captured by the term $\lambda_{\theta\theta'} \max\{J_{\theta'}(X_s, F_s) - D_{s-}, 0\}$ in (7). Using (3), we can write the two terms in the second line of (7) as $(\rho - r)D_{s-} + \mathbb{1}_{J_{\theta'}(X_s,F_s) \geq D_{s-}} \cdot \lambda_{\theta\theta'} J_{\theta'}(X_s, F_s)$, where $(\rho - r)D_{s-}$ captures the gains from borrowing short-term debt, and $\lambda_{\theta\theta'} J_{\theta'}(X_s, F_s)$ the continuation value upon state transition, which only accrues if $J_{\theta'}(X_s, F_s) \geq D_{s-}$, so that the borrower chooses not to default.

Lemma 1 suppresses the problem's dependence on D_{t-}. A smooth Markov perfect equilibrium is therefore characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F)$, $D_\theta(X,F)$, and $g_\theta(X,F)$, where

$$J_\theta(X,F) = X J_\theta\left(1, \frac{F}{X}\right) = X j_\theta(f), \quad D_\theta(X,F) = X D_\theta\left(1, \frac{F}{X}\right) = X d_\theta(f),$$

are homogeneous of degree one, and the rest are homogeneous of degree zero. Let $f = \frac{F}{X}$ be the long-term debt to income ratio, which is the *endogenous* state variable of the model. The results below show that a higher f is also associated with a shorter distance to default. It follows from Itô's lemma that f_t evolves according to

$$\frac{df_t}{f_t} = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right) dt - \sigma dB_t. \tag{8}$$

Lemma 1 has interesting economic insights. In particular, it implies that issuance and default decisions are made to maximize the joint valuation of the borrower and short-term creditors, while ignoring the payoff to existing long-term creditors.[6] This result relates to Aguiar et al. (2019) in the context of sovereign debt, where the equilibrium issuance decisions can be characterized by the solution to a planner's problem that ignores payoff to existing long-term creditors.

2.5 Modeling Discussion

Risk and binary state. The borrower faces two sources of risks. The Brownian motion captures continuous fluctuations in day-to-day operating cash flows, which is meant to capture small and frequent shocks. On the other hand, a transition across the two states affects the expected growth in cash flow and captures large and infrequent shocks. For the rest of the paper, we label the transition from the upturn H to downturn L as *a downside risk*, which can be interpreted as shocks occurring at either the industry- or the macroeconomy-level. The choice of binary state is made for tractability. Given that we focus on perspective of downside risk sharing, results in the upturn should be interpreted more broadly. Note that one special feature in the upturn is the expected growth rate μ_θ can only fall.

Debt maturity. Our modeling choice of short- and long-term debt is motivated by the discrete-time microfoundation. There, short-term debt would last for one period and therefore mature simultaneously. In the continuous-time setup, this feature is captured by zero-maturity debt that needs to be continuously rolled over. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing them each period would lead to the staggered structure. This feature is well captured by exponentially-maturing debt in the continuous-time setup.

Zero recovery in default. The assumption that creditors do not recover any value once the borrower defaults is made for simplicity and does not affect our mechanism. It implies that debt seniority becomes irrelevant, ruling out the theoretical channel highlighted in Brunnermeier and

[6]The payoff to new long-term creditors is at dt order in the smooth equilibrium.

Oehmke (2013) whereby the equity holder dilutes existing creditors' recovery value in bankruptcy through issuing new debt. In subsection 4.2, we assumes instead that the borrower can restructure her debt after defaulting, in which case creditors still recover some positive value.

Parametric assumptions. To make the problem non-trivial, we impose the following parametric restrictions.

Assumption 1. *The parameters of the model satisfy*

(a) Downturn is absorbing: $\lambda_{LH} = 0$ *and* $\lambda_{HL} = \lambda$.

(b) Unlevered values are finite without Brownian shocks: $r + \lambda > \mu_H$ *and* $r > \mu_L$.

Assumption 1.(a) says once in downturn, the exogenous state will never return to the upturn. Thus, the low state $\theta = L$ is absorbing. By contrast, the state switches from high to low with a Poisson intensity λ. This assumption enables us to obtain a tractable solution. Yet, as shown in section 4, the assumption is innocuous to the main results of the paper. Assumption 1.(b) is a standard one in the literature to guarantee that the valuation remains finite. Specifically, it requires in both states, the creditor's effective discount rate is above the expected growth rate of the cash flow.

3 Equilibrium

Subsection 3.1 derives the value function and the issuance of short-term debt in both states. Subsection 3.2 focuses on the issuance policy of long-term debt and explains the tradeoff behind borrowing long v.s. short. In subsection 3.3, we compare the equilibrium with one where only long- or short-term debt is allowed. Results there highlight the different roles of two types of debt. Finally, we study the debt issuance policies by an unlevered borrower in subsection 3.4.

3.1 Value Function and Short-term Debt Issuance

Low state $\theta_t = L$. Under Assumption 1.(a), the exogenous state θ_t will no longer change once it enters the downturn. Therefore, the remaining risk comes exclusively from the Brownian shock, and default can be anticipated by short-term creditors. As a result, short-term debt is riskless and demands a short rate $y_L(X, F) \equiv r$. By considering the change in the value function in (7) over a small interval, we can derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\underbrace{\rho J_L(X,F)}_{\text{required return}} = \max_{D_L \in [0, J_L(X,F)],\ g_L} \underbrace{X - (r+\xi)F}_{\text{cash flow net long payments}} + \underbrace{(\rho - r) D_L}_{\text{gains from borrowing short}} + \underbrace{p_L(X,F) g_L F}_{\text{proceeds from issuing long}}$$
$$+ \underbrace{\frac{\partial J_L(X,F)}{\partial F}(g_L - \xi) F}_{\text{evolution of } dF} + \underbrace{\frac{\partial J_L(X,F)}{\partial X} X \mu_L + \frac{1}{2}\frac{\partial^2 J_L(X,F)}{\partial X^2} X^2 \sigma^2}_{\text{evolution of } dX}. \quad (9)$$

The net benefits of issuing long-term debt become clear once we examine all the terms that involve g_L on the right-hand side. Whereas $p_L(X,F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, $\frac{\partial J_L(X,F)}{\partial F}$ is the drop in the borrower's continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, then it must be that the marginal proceeds are fully offset by the drop in continuation value, so that the borrower is indifferent, i.e.,

$$p_L(X,F) + \frac{\partial J_L(X,F)}{\partial F} = 0. \quad (10)$$

Under (10), the value function $J_L(X,F)$ can be solved as if $g_L \equiv 0$, which is the case that the borrower will never issue any further long-term debt. We defer the characterization of long-term debt issuance until the next subsection. For now, let us plug (10) into (9) and use the fact that

$$\frac{\partial J_L(X,F)}{\partial F} = j_L'(f), \quad \frac{\partial J_L(X,F)}{\partial X} = j_L(f) - f j_L'(f), \quad X\frac{\partial^2 J_L(X,F)}{\partial X^2} = f^2 j_L''(f),$$

to get the following HJB for the scaled value function $j_L(f)$:

$$(\rho - \mu_L) j_L(f) = \max_{d_L \in [0, j_L(f)]} 1 - (r+\xi) f + (\rho - r) d_L - (\xi + \mu_L) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f). \quad (11)$$

Turning to the issuance of short-term debt, whose net benefits are captured by the term $(\rho - r) d_L$ in (11) (or $(\rho - r) D_L$ in (9)). Intuitively, the creditor is more patient than the borrower, so that financing the enterprise by borrowing d_L in short brings a flow benefit of $(\rho - r) d_L$. Given $\rho > r$, it is always optimal for the borrower to lever up using as much short-term debt as possible, which leads to $d_L = j_L(f)$ (or equivalently $D_L = J_L(X,F)$) so that the limited liability constraint becomes binding.

The rest of the problem is standard. We look for a solution to (11) on $f \in [0, f_L^b]$, where the endogenous default boundary f_L^b satisfies the value-matching condition $j_L(f_L^b) = 0$ and smooth-pasting condition $j_L'(f_L^b) = 0$. Proposition 1 describes the equilibrium outcome.

Proposition 1 (Equilibrium when $\theta_t = L$)**.** *In the unique equilibrium, the value function is*

$$j_L(f) = \underbrace{\frac{1}{r-\mu_L} - f}_{\textit{no default value}} + \underbrace{\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma}_{\textit{default option value}}, \tag{12}$$

where

$$\gamma \equiv \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2} > 1.$$

The default boundary is $f_L^b = \frac{\gamma}{\gamma-1}\frac{1}{r-\mu_L}$. For $\forall f \in [0, f_L^b)$, the borrower issues short-term debt $d_L(f) = j_L(f)$ and pays a short rate $y_L(f) = r$.

High state $\theta_t = H$. The smooth equilibrium leads to an indifference condition in long-term debt issuance that relates to (10):

$$p_H(X,F) + \frac{\partial J_H(X,F)}{\partial F} = 0.$$

Besides the Brownian shock, there is the additional downside risk, whereby the state may transit from high to low. If default does not occur right upon the downturn arrives, the borrower and short-term creditors receive a maximum value of $j_L(f)$, among which d_H must be repaid to short-term creditors. Clearly, the borrower will default right upon the state transition if and only if $d_H > j_L(f)$. Expecting so, short-term creditors demand a short rate

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \le j_L(f) \\ r + \lambda & \text{if } d_H > j_L(f). \end{cases} \tag{13}$$

Following a similar analysis to the one in the low state, we arrive at the HJB for the scaled value function $j_H(f)$:

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = \max_{0 \le d_H \le j_H(f)} \; & 1 - (r + \xi) f + (\rho - r)\, d_H + \mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f) \\ & - (\xi + \mu_H) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).\end{aligned} \tag{14}$$

Compared to (11), there are two differences. First, the borrower's effective discount rate becomes $\rho + \lambda$, due to the possibility of state transition. Second, the term $\mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f)$ captures the scenario that with intensity λ, the downside risk is realized, upon which the borrower and short-term creditors receive a continuation payoff $j_L(f)$ if and only if $d_H \le j_L(f)$. Otherwise, the borrower defaults and both receive nothing. Note that the flow benefit of short-term debt is still captured by $(\rho - r)\, d_H$ and in particular does not include the credit risk premium λ, even in the case that short-term debt is risky (i.e., $d_H > j_L(f)$). The reason is, the risk premium serves as a transfer between the borrower and short-term creditors and therefore does not enter the joint continuation value.

The optimal issuance of short-term debt is straightforward: the borrower borrows either $j_L(f)$ at rate r or $j_H(f)$ at $r + \lambda$. Equation (14) can be therefore written as

$$(\rho + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f - (\mu_H + \xi)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ + \max\Big\{(\rho - r) j_L(f) + \lambda j_L(f), (\rho - r) j_H(f)\Big\}, \quad (15)$$

where the term $\max\Big\{(\rho - r) j_L(f) + \lambda j_L(f), (\rho - r) j_H(f)\Big\}$ captures the tradeoff between borrowing riskless $j_L(f)$ and risky $j_H(f)$. If the equity holder borrows riskless short-term debt, the flow benefit is lower $[(\rho - r) j_L(f) < (\rho - r) j_H(f)]$. However, because there is no immediate default after the state transition, the borrower avoids the expected bankruptcy cost $\lambda j_L(f)$.

If $(\rho - r) j_L(0) + \lambda j_L(0) > (\rho - r) j_H(0)$, a borrower without any outstanding long-term debt will borrow riskless short-term debt so that $d_H(0) = j_L(0)$. Meanwhile, the maximum amount of riskless short-term borrowing decreases as f increases and eventually reduces to zero as f approaches f_L^b. Therefore, the borrower chooses risky short-term borrowing $d_H(f) = j_H(f)$ for f sufficiently high. We show in Lemma 3 of the appendix that there exists a unique threshold $f_\dagger \in \left(0, f_L^b\right)$ such that $(\rho + \lambda - r)\, j_L(f) \leq (\rho - r)\, j_H(f)$ if and only if $f \geq f_\dagger$. Given so, the HJB becomes a second-order ordinary differential equation (ODE) on both $(0, f_\dagger)$ and $\left(f_\dagger, f_H^b\right)$. The solutions and the two free boundaries $\left\{f_\dagger, f_H^b\right\}$ are pinned down by six boundary conditions: 1) value-matching and smooth-pasting at $\left\{f_\dagger, f_H^b\right\}$; 2) Transversality condition at $f = 0$; and 3) the indifference between issuing risky and riskless short-term debt at $f_\dagger$. The detailed expressions are available in (34) to (39) in the appendix. Proposition 2 describes the equilibrium, where the constant ϕ and the expressions for $\left\{h_0\left(f, f_\dagger, f_H^b\right), h_1\left(f, f_\dagger, f_H^b\right)\right\}$ are provided in the appendix.

Proposition 2 (Equilibrium when $\theta_t = H$)**.** *Let*

$$\bar{\lambda} \equiv \sqrt{\left(\frac{\rho - \mu_H}{2}\right)^2 + (\rho - r)(\mu_H - \mu_L)} - \left(\frac{\rho - \mu_H}{2}\right).$$

In the high state, the unique smooth equilibrium is:

1. *If $\lambda \le \bar{\lambda}$, the value function is*

$$j_H(f) = \underbrace{\frac{1}{r+\lambda-\mu_H} - f}_{no\ default\ value} + \underbrace{\frac{f_H^b}{\beta}\left(\frac{f}{f_H^b}\right)^{\beta}}_{default\ option\ value}, \tag{16}$$

where

$$\beta = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r+\lambda-\mu_H\right)}}{\sigma^2} > 1.$$

The default boundary is $f_H^b = \frac{\beta}{\beta-1}\frac{1}{r+\lambda-\mu_H}$. For $\forall f \in [0, f_H^b)$, the borrower issues short-term debt $d_H(f) = j_H(f)$ and pays a short rate $y_H(f) = r + \lambda$.

2. *If $\lambda > \bar{\lambda}$, the value function is*

$$j_H(f) = \begin{cases} u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi} & f \in [0, f_\dagger] \\ u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right) & f \in \left(f_\dagger, f_H^b\right], \end{cases} \tag{17}$$

where

$$u_0(f) \equiv \frac{1}{\rho+\lambda-\mu_H}\left(1 + \frac{\rho - r}{r-\mu_L} + \frac{\lambda}{r-\mu_L}\right) - f + \frac{(\rho+\lambda-r)}{(\rho+\lambda-r)+(\mu_H-\mu_L)(\gamma-1)}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^{\gamma} \tag{18}$$

$$u_1(f) \equiv \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\xi+\lambda}f. \tag{19}$$

The threshold for long term debt issuance $f_\dagger$ and the default boundary f_H^b are determined by conditions (40) and (41) in the appendix. For $\forall f \in [0, f_H^b)$. The borrower issues short-term debt

$$d_H(f) = \begin{cases} j_L(f) & \text{if } f \le f_\dagger \\ j_H(f) & \text{if } f > f_\dagger \end{cases}$$

and pays a short rate given by equation (13).

The first case in Proposition 2 is isomorphic to the results in Proposition 1 on the equilibrium in state L. Intuitively, if λ is low so that the downside risk is small, even a borrower without any outstanding long-term debt will borrow riskless short-term debt ($(\rho - r)j_L(0) + \lambda j_L(0) > (\rho - r)j_H(0)$). In this case, the borrower borrows short-term debt $j_H(f)$, which is expected to default upon the state transition and therefore commands a short rate $r+\lambda$.

Results are more interesting for $\lambda > \bar{\lambda}$. In fact, the expressions in (17) have clear and intuitive interpretations. To see this, let us define $\tau_\dagger$ and τ_H^b as the first hitting time of $f_\dagger$ and f_H^b. We supplement the details to (17) as follows:

$$\begin{aligned}
\left(\frac{f}{f_\dagger}\right)^\phi &= \mathbb{E}\left[e^{-(\rho+\lambda)\tau_\dagger}\left(\frac{X_{\tau_\dagger}}{X_0}\right)\Big| f_0 = f\right] \\
h_0\left(f, f_\dagger, f_H^b\right) &= \mathbb{E}\left[e^{-(r+\lambda)\tau_\dagger}\left(\frac{X_{\tau_\dagger}}{X_0}\right)\mathbb{1}_{\{\tau_\dagger < \tau_b\}}\Big| f_0 = f\right] \\
h_1\left(f, f_\dagger, f_H^b\right) &= \mathbb{E}\left[e^{-(r+\lambda)\tau_b}\left(\frac{X_{\tau_b}}{X_0}\right)\mathbb{1}_{\{\tau_\dagger > \tau_b\}}\Big| f_0 = f\right].
\end{aligned}$$

The first expression is the present value of investing \$1 at $t=0$ into a hypothetical claim which pays out the unlevered return of the asset at the first time that f_t reaches $f_\dagger$. However, if the downturn arrives before $\tau_\dagger$, this claim pays out nothing. The function h_0 has a similar interpretation for $f_0 > f_\dagger$, but now the claim only pays if the firm has not defaulted. The last function h_1 is the present value of a claim that pays the unlevered return on assets at the time of default if default occurs before f reaches $f_\dagger$.[7]

For $f \in [0, f_\dagger)$, $u_0(f)$ in (17) captures the joint value of equity holder and short-term creditors if the issuance of short-term debt follows $d_H(f) = j_L(f)$. In particular, the second term $\big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^\phi$ is the option value of f reaching $f_\dagger$, in which case the borrower exercises the option of changing the short-term debt issuance to $d_H(f) = j_H(f)$. For the first term $u_0(f)$, we can rewrite it as the sum of the payoff without state transition and the one following the transition:

$$u_0(f) = \mathbb{E}\left[\int_0^\infty e^{-(\rho+\lambda)t}\Big(\underbrace{1-(r+\xi)f_t+(\rho-r)d_H(f_t)}_{\text{no state transition}} + \underbrace{\lambda j_L(f_t)}_{\text{state transition}}\Big)\left(\frac{X_t}{X_0}\right)dt\Big| f_0 = f\right],$$

Note that $d_H(f_t) = j_L(f_t)$ holds for $f \leq f_\dagger$, so that the terms in the first line of (18) capture the discounted value of the cash flow and the benefits from borrowing short-term debt net the payments to long-term creditors. The term in the second line of (18) represents the value of defaulting after the downturn arrives and the value of borrowing against this default option prior to the state

[7]The factor (X_t/X_0) accounts for the adjustment σ^2 in the drift of f_t in (8) due to the stochastic growth of X_t. Alternatively, we can omit the factor (X_t/X_0) and compute the expectations under a new measure $\mathbf{Q}$ such

$$df_t = -(\mu_H + \xi)f_t dt - \sigma f_t d\bar{B}_t,$$

and $d\bar{B}_t = dB_t - \sigma dt$ is a Brownian motion under $\mathbf{Q}$, adjusting the discount factors to $e^{-(\rho+\lambda-\mu_H)t}$ and $e^{-(r+\lambda-\mu_H)t}$. The change of measure adjusts for the stochastic growth of X_t, and the change of discount factors adjust for the expected growth of X_t.

transition. Note that these valuations are calculated *as if* $g_\theta(f) = 0$, that is, as if the borrower would not issue any further long-term debt, due to the indifference condition in long-term debt issuance.

The value function for $f > f_\dagger$ can be interpreted in a similar vein. Specifically in (17), $u_1(f)$ is the value if the issuance of short-term debt follows $d_H(f) = j_H(f)$, whereas the remaining two terms $\big(j_H(f_\dagger) - u_1(f_\dagger)\big)h_0\left(f, f_\dagger, f_H^b\right)$ and $u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right)$ are the option value of switching short-term debt issuance to $d_H(f) = j_L(f)$ and default, respectively. Similarly, we can rewrite $u_1(f)$ as

$$u_1(f) = \mathbb{E}_0\left[\int_0^\infty e^{-(r+\lambda)t}\Big(1-(r+\xi)f_t\Big)\left(\frac{X_t}{X_0}\right)dt\Big|f_0 = f\right],$$

where the discount rate is r because the borrower is fully levered and creditors become the effective owners. The two terms in (19) correspond to the value of cash flow and the expected payments to long-term creditors, with the latter being discounted by $\frac{r+\xi}{r+\xi+\lambda}$, because the borrower immediately defaults upon the downturn arrives.

3.2 The Issuance of Long-term Debt

We have shown that in the smooth equilibrium, the borrower is indifferent between issuing long-term debt or not. However, the result doesn't imply that she won't borrow long on the equilibrium path. In this subsection, we solve for the issuance policy of long-term debt.

Let us start with the downturn $\theta_t = L$, where Equation (10) (or equivalently $p_L(f) = -j_L'(f)$) is the necessary condition for the borrower to be indifferent between issuing long-term debt or not. Meanwhile, the price satisfies the following HJB equation

$$(r+\xi)\,p_L\left(f\right) = \underbrace{r+\xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)}_{\text{expected change in bond price}}. \tag{20}$$

To derive the issuance function g_L, we plug $d_L = j_L(f)$ into (11), differentiate the resulting equation once, and add (20) on both sides. Turning to the upturn $\theta_t = H$. The equity holder's indifference in long-term debt issuance becomes $p_H(f) = -j_H'(f)$, and $p_H(f)$ satisfies the following HJB equation

$$(r+\xi+\lambda)\,p_H\left(f\right) = r+\xi+\mathbb{1}_{\{f\le f_\dagger\}}\cdot\lambda p_L\left(f\right) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f). \tag{21}$$

Compared to (20), (21) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \le f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The derivation of the issuance policy $g_H(f)$ follows the same steps as the one in the low state.

Proposition 3 (Long-term debt issuance)**.** *The equilibrium price and issuance of long-term debt are :*

- *Downturn $\theta = L$: for $\forall f \in [0, f_L^b)$, the price of long-term debt is $p_L(f) = -j'_L(f)$, and firm does not issue long-term debt, $g_L(f) = 0$.*
- *Upturn $\theta = H$: for $\forall f \in [0, f_H^b)$, the price of long-term debt is $p_H(f) = -j'_H(f)$. The long-term debt issuance policy is as follows: Let $\lambda \le \bar{\lambda}$ be threshold in Proposition 2.*
  - *If $\lambda \le \bar{\lambda}$, then the firm never issues long-term debt, $g_H(f) = 0 \ \forall f \in [0, f_H^b)$;*
  - *If $\lambda > \bar{\lambda}$, the long-term debt issuance policy is*

$$g_H(f) = \begin{cases} \frac{(\rho - r)\left(p_H(f) - p_L(f)\right)}{-f p'_H(f)} & f \le f_\dagger \\ 0 & f > f_\dagger. \end{cases} \tag{22}$$

Proposition 3 shows that in the low state, the equity holder never issues any long-term debt, but, instead, borrows the maximum amount of short-term debt. Similar results hold in the high state if $\lambda \le \bar{\lambda}$, so that the downside risk is relatively small. By contrast, long-term debt is issued in the high state if the downside risk is prominent and the amount of short-term borrowing is riskless, i.e., $\lambda > \bar{\lambda}$ and $f \le f_\dagger$. Why might the equity holder borrow long in the high state but not in low? Why would the equity holder borrow long in the high state only if the amount of short-term borrowing is riskless? What are the differential roles of short- and long-term debt?

Due to the leverage-ratchet effect, the equity holder faces a time-inconsistency problem when borrowing long: she is unable to commit to a path of issuance, but, instead, always has incentives to issue more and dilute legacy long-term creditors. As shown by (10), this lack of commitment implies the borrower is unable to capture any benefits from borrowing long, even though the creditors are more patient. By contrast, short-term debt, in particular its combined nature of instantaneous maturity and simultaneity, resolves the commitment problem, because all outstanding debt must be rolled over continuously, i.e., the existing short-term debt must be retired before issuing any new one. Given that short-term debt is riskless and cheap ($y_L = r$) in the low state, the equity holder only borrows short.[8] Similar results hold in the high state when short-term debt is risky, i.e., when $\lambda \le \bar{\lambda}$ or $\lambda > \bar{\lambda}$ but $f > f_\dagger$. Given that the borrower is expected to default upon the state

[8]Note that the result of no long-term debt issuance in the low state stays unchanged even if $\lambda_{LH} > 0$, because the borrower will not default upon a state transition from low to high.

transition, short-term creditors demand a short rate $r + \lambda$. Meanwhile, long-term debt is subject not only to the same downside risk of state transition but also the dilution effect. Therefore, the equity holder, again, only borrows short.

As shown by DeMarzo (2019), the borrower's problem when she only issues long-term debt is related to the Coase conjecture on durable-goods monopoly (Coase, 1972). Specifically, the borrower can be thought of as the monopolistic issuer of her debt, and long-term debt can be thought of as the durable goods. Short-term debt in our context echoes the leasing solution to the Coase conjecture, which was originally proposed by Coase (1972) and later formalized by Bulow (1982). Effectively, short-term debt achieves commitment it needs to be continuously rolled over and repriced, similar to leasing.

Matters are different in the high state when short-term debt is riskless, i.e., when $\lambda > \bar{\lambda}$ and $f \leq f_{\dagger}$. In this case, default does not occur after the downside risk is realized, but the enterprise value experiences a discontinuous jump. Whereas a transition from the high to the low state reduces the equity value by $j_H(f) - j_L(f)$ and the long-term debt price by $p_H(f) - p_L(f)$, it leaves the value of short-term debt intact. In other words, the loss in enterprise value is shared between the borrower and long-term creditors. Short-term creditors, on the other hand, do not share any loss. This result highlights an important role of long-term debt in risk sharing: it allows the borrower to make state-contingent payment (effectively $p_H(f)$ in H but $p_L(f)$ in L) without default. The state-contingent payments act as a cushion which reduces the borrower's burden in the downturn and mitigates the incentives to default, thereby increasing the enterprise value. By contrast, short-term debt is a *harder* claim: the borrower must make non-state-contingent payments; otherwise she has to default. Figure 1 provides a graphical illustration upon the state transition. Clearly, both equity value and long-term debt price get reduced, whereas the value of short-term debt stays unchanged (unless the borrower default).

Assets	Liabilities
	ST debt: j_L
EBIT grows at μ_H	LT debt: $p_H \cdot f$
	Equity: $j_H - j_L$

Assets	Liabilities
	ST debt: j_L
EBIT grows at μ_L	LT debt: $p_L \cdot f$
	Equity: 0

Figure 1: Balance Sheet upon the State Transition without immediate default

A careful examination of the issuance function (22) shows that the amount of long term debt issued, fg_H, equals the ratio of the flow benefits from risk-sharing $(\rho - r)\big(p_H(f) - p_L(f)\big)$ to the price sensitivity of new issuance $-p'_H(f)$.Let us offer a heuristic derivation based on a local perturbation argument. Consider a policy that the borrower smoothly issues long-term debt Δdt at time t and buys it back at time $t+dt$. As a result, the amount of short-term debt outstanding drops from $j_L(f)$ to $j_L(f+\Delta dt)$ at time t, and resumes $j_L(f)$ at $t+dt$. The change in total leverage at t is approximately $(p_H(f)+j'_L(f))\,\Delta dt$ for dt sufficiently small. During $[t,t+dt)$, this increase in leverage brings a marginal benefit of $(\rho-r)\,dt\cdot(p_H(f)+j'_L(f))\,\Delta dt$. Meanwhile, this operation brings issuance proceeds $p_H(f+\Delta dt)\Delta dt$ at time t and $-p_H(f)\Delta dt$ at $t+dt$, resulting in a marginal cost of $(p_H(f)-p_H(f+\Delta dt))\,\Delta dt$.[9] For Δdt sufficiently small, this marginal cost becomes $-p'_H(f)\,(\Delta dt)^2$. The optimal issuance Δdt must equalize marginal benefit and marginal cost:

$$(\rho-r)\cdot(p_H(f)+j'_L(f))\Delta(dt) = -p'_H(f)(\Delta dt)^2 \Longrightarrow \Delta = gf = \frac{(\rho-r)(p_H(f)-p_L(f))}{-p'_H(f)}$$

For f sufficiently close to $f^0_\dagger$, the issuance of long-term debt can be negative. The equation above implies that $p_H(f) < p_L(f)$ in this case. In these states, the probability of the borrower default in the near future could be higher in the high state than in the low state, why? Note that in the high state, the Brownian shock may increase the state variable above $f_\dagger$, in which case default will happen following a regime-shift. By contrast, for the same level of f in the low state, default cannot happen unless f increases up to f^b_L, and this probability can be lower.

To summarize, the choice of maturity is determined by the trade-off between commitment and risk sharing. Short-term debt resolves the issue of lack of commitment; long-term debt shares the downside risk. This insight relates to the previous work in sovereign debt literature that emphasizes how long-term debt allows for more state-contingency (Angeletos, 2002). A key difference is, we cast the model in the context of a risk-neutral borrower, as it is typically the case in capital structure studies. An immediate question, then, is why would a risk-neutral borrower value the long-term debt's merit in sharing the downside risk? The reason is, the bankruptcy cost introduces concavity into her objective function, so that she behaves as if risk-averse. As emphasized by previous work on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010), a risk-neutral corporation has incentives to insure or hedge against negative shocks when the cost of external financing is costly and can fluctuate.

Our result shows a pecking order among the choice of the riskiness and maturity of debt. In particular, short-term riskless debt is the most favored, as illustrated by the borrower's choice in the low state. If short-term debt may not be riskless, as the case in the high state, the borrower starts to borrow long-term debt that is subject to the potential of borrower default and tries to

[9]Note the proceeds in short-term debt issuance t and $t+dt$ cancel out on each other.

maintain the riskless feature of short-term debt. This is the situation in the high state when $f < f_\dagger$ and $\lambda > \bar{\lambda}$. Furthermore, if the amount of outstanding long-term debt gets too high or the downside risk is small ($\lambda \leq \bar{\lambda}$), the borrower turns to borrowing exclusively short-term risky debt and in the meantime reduces the outstanding long-term debt by not issuing any additional amount.

3.3 Benefits and Costs of Long- and Short-term Debt

In this subsection, we explore the benefits and costs of both types of debt by studying the equilibrium if only long or short-term debt is allowed.

Proposition 4 (Equilibrium with only short-term debt)**.** *There is a unique equilibrium if the borrower can only issue short-term debt.*

1. *In the low state L, the borrower never defaults. The value function is*

$$\tilde{J}_L(X) = \frac{X}{r - \mu_L}.$$

Short-term debt is $\tilde{D}_L(X) = \tilde{J}_L(X)$ and the short rate is $y_L = r$.

2. *In the high state H, the value function is*

$$\tilde{J}_H(X) = X \max\left\{\frac{1}{r + \lambda - \mu_H}, \frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho - r + \lambda}{r - \mu_L}\right)\right\}.$$

Let $\bar{\lambda}$ be the threshold in Proposition 2.

- *If $\lambda \leq \bar{\lambda}$, then short term debt is $\tilde{D}_H(X) = \tilde{J}_H(X)$. The borrower defaults as soon as θ switches from H to L so the interest rate is $y_H = r + \lambda$.*
- *If $\lambda > \bar{\lambda}$, then $\tilde{D}_H(X) = \tilde{J}_L(X)$, and the borrower never defaults and $y_H = r$.*

3. *The total enterprise value is higher than in the case if the borrower can issue both types of debt, i.e., $\tilde{J}_\theta(X) \geq J_\theta(X, F) + p_\theta(X, F)\,F$, $\forall \theta \in \{L, H\}$.*

If only short-term debt is allowed, the commitment problem in debt issuance no longer exists. Instead, the choice of capital structure is a static problem and follows the standard trade-off theory whereby equity holders balance cheap debt against costly bankruptcy. Given that the problem is scalable with respect to X_t, the solution is one with a constant leverage level. Interestingly, the total enterprise value is higher if the borrower is prohibited from issuing long-term debt. Intuitively, long-term debt does not benefit the borrower at all due to the lack of commitment, but, instead, introduces the possibility of default and the associated bankruptcy cost if the cash flow gets sufficiently low (or equivalently f gets very high).

Proposition 5 (Equilibrium with only long-term debt)**.** *There is a unique equilibrium. Define*

$$\tilde{\gamma} \equiv \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 \left(\rho - \mu_L\right)}}{\sigma^2} > 1.$$

1. *In state L, the value function is*

$$\tilde{v}_L\left(f\right) = \frac{1}{\rho - \mu_L} - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi} \frac{\tilde{f}_L^b}{\tilde{\gamma}} \left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

where the default boundary is $\tilde{f}_L^b = \frac{1}{\rho-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\xi}{r+\xi}$.

2. *In state H, the value function is*

$$\tilde{v}_H\left(f\right) = \tilde{u}_0\left(f\right) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi},$$

where

$$\tilde{u}_0\left(f\right) \equiv \frac{1}{\rho+\lambda-\mu_H}\left(1 + \frac{\lambda}{\rho-\mu_L}\right) - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi}\frac{\lambda}{\lambda + (\mu_H - \mu_L)(\tilde{\gamma}-1)}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}}.$$

The borrower defaults upon the state transition if and only if $f > \tilde{f}_L^b$.

3. *In both states $\theta \in \{L, H\}$, the debt price is $\tilde{p}_\theta = -\tilde{v}_\theta'$, and the issuance function follows*

$$\tilde{g}_\theta = \frac{\left(\rho - r\right)\tilde{p}_\theta}{-f\tilde{p}_\theta'}.$$

When the borrower is only allowed to issue long-term debt, the setup resembles the one in DeMarzo and He (2021). Without commitment to the issuance policy, equity holders do not reap the benefits from issuing cheaper debt, as the debt price will adjust for the future issuance policy. In equilibrium, long-term debt is issued smoothly. The next proposition compares the equilibrium with only long-term debt with the one where the borrower can issue both types of debt.

Proposition 6 (Comparison of equilibrium)**.**

1. *The total enterprise value is lower than the case if the borrower can issue both types of debt, i.e.,* $\tilde{v}_\theta(f) + \tilde{p}_\theta(f) f \le j_\theta(f) + p_\theta(f) f$, $\forall f$.

2. *The default boundary is higher in the presence of short term debt. That is* $f_\theta^b > \tilde{f}_\theta^b$.

3. *In the low state, the price of debt is higher in the presence of short-term debt. That is,* $\tilde{p}_L < p_L$, $\forall f \in [0, \tilde{f}_L^b]$. *In the high state, if* $\rho > r + \lambda$, *there are thresholds* $\underline{f} \in [0, f_\dagger]$ *and* $\overline{f} \in [f_\dagger, \tilde{f}_H^b]$ *such that* $\tilde{p}_H(f) \le p_H(f)$ *on* $[0, \underline{f}] \cup [\overline{f}, \tilde{f}_H^b]$.

In both states, the enterprise value is higher when the borrower can issue both types of debt. Intuitively, borrowing short not only increases the leverage but also allows the equity holder to reap some benefits from issuing debt. This higher enterprise value is reflected with higher default boundaries in both states. Turning to the comparison of long-term debt's price, which can be either higher or lower when the borrower can issue both types of debt. On one hand, whereas short-term debt increases the enterprise value, it also pushes up the default boundary, so that under the same level of long-term debt, the borrower is further away from the default boundary. On the other hand, in the high state when f is close to $f_\dagger$, the price of long-term debt can be lower if she can issue both debt. Intuitively, taking short-term debt leads to the borrower default following the realization of the downside risk when f is above $f_\dagger$. Without short-term debt, the borrower won't default following the same transition unless f rises above $\tilde{f}_L^b$. Therefore, the price of long-term debt is relatively low when f exceeds $f_\dagger$ but is still far from $\tilde{f}_L^b$ yet. This result suggests that long-term debt and short-term debt could be either complements or substitutes, depending on the borrower's distance to default.

3.4 Initial Debt Issuance

Does an initially unlevered borrower issue any long-term debt? From the issuance function (22), it is easily established that

$$g_H(0) = \frac{(\rho - r)(\mu_H - \mu_L)}{\rho + \lambda - r}, \qquad \lim_{f \to 0} g_H(f) f = 0,$$

so that an unlevered borrower does not issue any long-term debt. The intuition is straightforward. For the unlevered borrower, both $p_H(f) \to 1$ and $p_L(f) \to 1$ hold as $f \to 0$ so that a marginal unit of long-term debt is riskless and does not share any downside risk. Given so, the unlevered borrower has no reason to issue it. Indeed, the issuance function (22) makes it clear that, for an unlevered borrower to issue long-term debt, that is, $\lim_{f\to 0} g_H(f) \cdot f > 0$, it must be $p_H(0) > p_L(0)$ so that a marginal unit of long-term debt shares some losses following the transition to the downturn.

There are several approaches to motivate an unlevered borrower to issue long-term debt. One is to introduce an exogenous disaster in the low state, modeled as a Poisson event with intensity ζ, upon which X_t permanently drops to zero. In this case, both $p_H(0)$ and $p_L(0)$ are less than 1 (and therefore not default free) and $p_H(0) > p_L(0)$.

Proposition 7. *In the model with a disaster event, the expressions of issuance policy are identical to those in Proposition 3. Let*

$$\underline{\zeta} \equiv \sigma^2 - (r + \mu_L + 2\xi), \quad \overline{\zeta} \equiv \frac{\lambda(\rho + \lambda) - (\rho + \lambda - r)\mu_H + (\rho - r)\mu_L}{\rho - r}.$$

If $\underline{\zeta} < \overline{\zeta}$, $\lambda \geq \sigma^2 - (\rho + \mu_H + 2\xi)$, and $\zeta \in [\underline{\zeta}, \overline{\zeta}]$, an unlevered borrower issue some long-term debt in the high state. That is, $\lim_{f \to 0} g_H(f)f > 0$. Otherwise, $\lim_{f \to 0} g_H(f)f = 0$.

Note that an unlevered borrower issues some long-term debt if the downside risk of state transition is significantly high, and the disaster intensity is neither too high nor too low. The requirement on downside risk is intuitive, due to the role of long-term debt in risk sharing. Let us explain the intuitions behind the conditions on disaster risk. If $\zeta > \overline{\zeta}$, the disaster risk is very high in the downturn, so that the continuation value $j_L(0)$ is very low, even if the borrower is unlevered. As a result, the amount of riskless short-term debt that she can borrow is also low, so much so that she would rather issue risky short-term debt. Given so, the borrower is anticipated to default immediately upon the downturn arrives, and there is no role of long-term debt in sharing the downside risk. On the other hand, if $\zeta < \underline{\zeta}$, disaster risk is so low such that the difference between $p_H(f)$ and $p_L(f)$ is small and dominated by the price impact of issuing long-term debt. In this case, the unlevered borrower would again, refrain herself from borrowing long to begin with.

The analysis above further highlights the earlier mechanism that long-term debt helps with risk sharing. If the downside risk only includes the state transition from high to low, the marginal unit of long-term debt is riskless for an unlevered borrower. Now that there is the additional downside risk from the disaster, a marginal unit of long-term debt is more exposed to the disaster risk in the low state than in the high state, even if the borrower is unlevered. This exercise confirms that an unlevered borrower has incentives to issue long-term debt as long as the marginal unit of long-term debt shares either some downside risk or some disaster risk, and the price impact of a marginal dollar of long-term debt is limited.

4 Empirical Implications, Extensions, and Robustness

Subsection 4.1 explores the model's empirical implications in both the cross-section and time-series. In subsection 4.2, we study debt issuance and restructuring when the recovery value in default is positive. We show in subsection 4.3 that the modeling of downside risk is not restricted to regime shifting. In particular, a downward jump to the cash-flow process also motivates the issuance of long-term debt. Finally, whereas the benchmark model has assumed competitive creditors, subsection 4.4 introduces investors clienteles for specific maturity segments so that the discount rates differ across creditors with different investment horizons. We show how the motives to manage interest-rate risk affect the borrower's decision to issue long and short.

4.1 Empirical Implications and Impulse Responses

This subsection numerically solves the model with the disaster event introduced in subsection 3.4. Moreover, we allow for $\lambda_{LH} > 0$ so that shocks to the cash flow's expected growth rate are transitory. We look for an equilibrium similar to the one in section 3 under reasonable parameters. The model's cross-sectional and time-series implications on debt maturity structure will be linked to empirical studies.

Our central object of interest is a firm's debt maturity structure, defined as the average maturity of total debt outstanding weighted by their book value:[10]

$$\text{Maturity}_t := \frac{F_t}{F_t + D_t}\frac{1}{\xi} = \frac{f_t}{f_t + d_t}\frac{1}{\xi}. \tag{23}$$

Cross-sectional implications. Figure 2 shows how the average maturity varies within a cross-section of borrowers with different characteristics. The left panel plots how the maturity changes with f in the high state, under different levels of μ_L.[11] One interpretation of f is the borrower's distance to default (DD), and a higher f is associated with the borrower being closer to default. Three patterns are prominent. First, a borrower closer to default has more long-term debt. In our model, this pattern holds because in the absence of regime shift, default is only triggered by a large amount of outstanding long-term debt, whereas short-term debt can be easily adjusted and default can be anticipated by its creditors. Second, the average maturity could has a discontinuous

[10]We focus on maturity measured using book values rather than market values because this is the one most commonly used in empirical studies due to data limitations. That said, results stay qualitative unchanged when debt maturity are weighted by market values.

[11]We argue the results in the high state are a more precise description for the following reason. In the binary state setup, there is no additional downside risk once the borrower enters the low state. In practice, it is likely that the borrower always faces some downside risk, which motivates a reason to use long-term debt. In other words, the high state in our model is meant to capture any real-world scenario as long as the borrower still faces some downside risk.

downwards jump when the borrower gets closer to default. In our model, this pattern holds because when f increases above $f_{\dagger}$, the borrower stops borrowing long and exclusively issues short-term debt. In practice, firms get downgraded are found to mostly rely on short-term borrowing. The third pattern comes from the comparison across the two lines. Specifically, for a given f, the borrower whose cash flows grow at a lower rate in the low state (captured by a lower μ_L) has more long-term debt outstanding. Intuitively, this borrower has more incentives to hedge against the downside risk if its size gets bigger.[12]

The middle panel of Figure 2 shows how maturity differs across firms with different leverage, where leverage is defined as $\frac{d+f}{j+f}$, the book value of total debt divided by the sum of market value of equity and book value of debt.[13] This measurement corresponds to the market leverage ratio in most empirical papers.[14] Results show that more levered borrowers use more long-term debt. In our model, this result happens because a borrower's equity value is $j_H(f) - j_L(f)$, which decreases with f in the high state when $f < f_{\dagger}$. Therefore, an increase in f leads to both a higher leverage and a longer average maturity. Moreover, a comparison across the two lines confirms the earlier result that controlling for market leverage, a borrower with higher downside risk uses more long-term debt.

Finally, the right panel plots average maturity across firms with different asset market to book ratio, defined as $\frac{p(f)f+j}{f+j(f)}$. A borrower with higher market-to-book ratio has relatively more short-term debt. In our model, this pattern holds because the price of long-term debt declines with f. This result is consistent with previous findings on firms use more growth options have more short-term debt outstanding (Stohs and Mauer, 1996; Barclay and Smith Jr, 1995). Short-term debt can be easily adjusted once these firms exercise the growth options.

The evidence above points out the importance of differentiating stock versus flow in studying debt maturities. For example, the left-panel of Figure 2 implies that default is triggered by too much long-term debt in stock, but, in this situation, the borrower only issues short-term debt. This result is consistent with Friewald et al. (2021), who show that when a firm has a large amount of long-term debt due in the next three years, it is more exposed to systematic risk and commands a

[12] Similar patterns hold for higher λ_{HL} if we keep μ_L the same, due to the same intuition reason.

[13] The leverage is 100% in the low state and in the high state when $f > f_{\dagger}$, implying that borrowers with high levels of leverage (100% in this case) could have different maturity structures. In this sense, our model implies that debt maturity structure has additional predictable power in defaulting after controlling for a borrower's leverage.

[14] For example, empirical papers using Compustat data typically define market leverage of firm i in year t as

$$\text{Lev}_{it} = \frac{\text{DLTT}_{it} + \text{DLC}_{it}}{\text{DLTT}_{it} + \text{DLC}_{it} + \text{CSHO}_{it} \times \text{PRCC\_F}_{it}},$$

where DLTT_{it} and DLC_{it} are the amount of long-term debt and debt in current liabilities. PRCC_F is the fiscal year-end common share price and CSHO is the fiscal year-end number of shares outstanding.

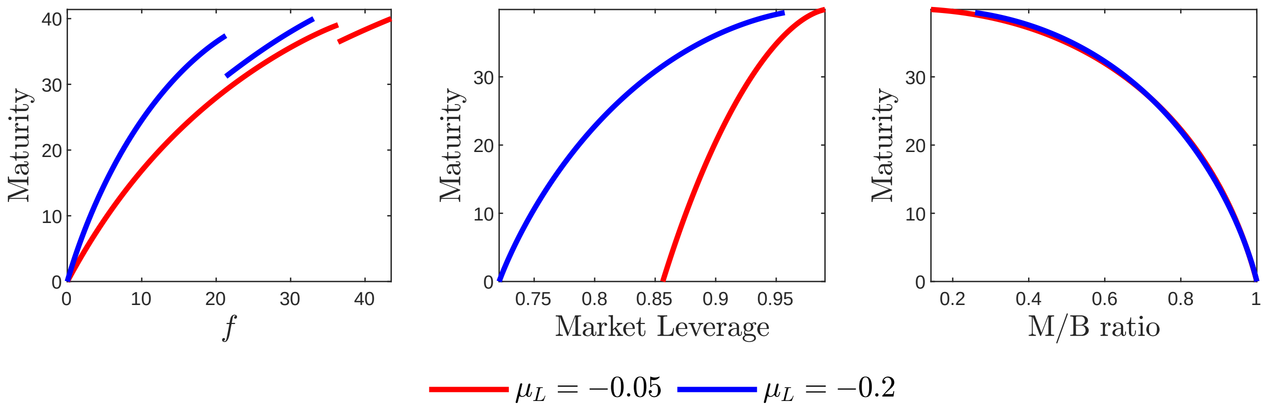


Figure 2: Cross-sectional Debt Maturity Structure

This figure plots maturity as a function of f, market leverage and market-to-book ratio in the high state. The parameters are as follows: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.2$, $\sigma = 0.3$, $\xi = 0.025$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$. The first figure plots maturity as a function of f on $[0, f_H^b]$. The second and third figure plot maturity as a function of leverage and market to book ratio for f on $[0, f_\dagger]$.

higher equity return.[15]

Time-series implications. Now we turn to the time series implications. Our results that long-term debt is only issued in the high state $\theta_t = H$ immediately implies that the borrower's debt maturity is pro-cyclical, if one interprets these two states as business-cycle frequency boom and bust. This prediction is consistent with the findings in Chen et al. (2021), which we replicate in Figure 3 below.

We simulate a sample path and plots the time-series of cash-flow rate and debt maturity in Figure 4.[16] Without the regime shift, it seems that average maturity and cash flow rate comove negatively with each other. In other words, the borrower expands the average debt maturity following a negative Brownian shock to X_t. Intuitively, this pattern holds because after a negative Brownian shock to X_t, the borrower rolls over less short-term debt which is easier to adjust. On the other hand, when the regime shifts and the downturn arrives, the borrower borrows exclusively short-term debt and the average maturity goes down.[17] This result suggests that the borrower's

[15] In Friewald et al. (2021), this is referred to as short-term leverage, though.

[16] Note that our result implies market leverage is on average counter-cyclical, if we interpret the state transition as business-cycle shocks. Adrian and Shin (2014) offer consistent evidence.

[17] This result depends on the binary state setup, where there is no additional downside risk in the low state. With more than two states, the borrower may still issue long-term debt in the low state. The broader message is the transition to a worse state, the borrower may only issue short-term debt for a while.

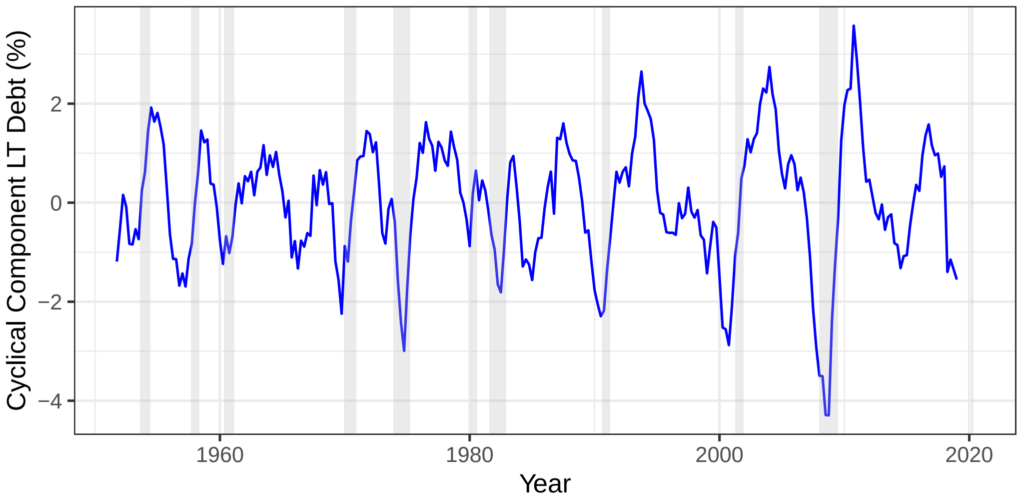


Figure 3: Pro-cyclical Long-term Debt Share in the Business Cycle

This figure applies the Hodrick-Prescott filter with multiplier 1600 to the share of long-term debt of non-financial firms in the U.S. and extracts the cyclical components. The shaded areas denote NBER-dated recessions. From Chen et al. (2021) using Flow of Funds Accounts data (Table L.103)

debt issuance policy has a different response to small, frequent Brownian shocks versus large, infrequent regime-shift shocks. Below, we formalize these results by studying the impulse response functions of debt issuance to both Brownian and regime-shifting shocks.

Impulse responses functions and shock elasticity. How does a borrower's long- and short-term debt issuance respond following a negative cash-flow (Brownian) shock and/or a regime-shift shock whereby the downturn arrives? We answer these questions by studying the impulse response of time-t outstanding debt F_t and D_t to the two different shocks occurring at time 0. Given the model is non-linear, we cannot follow the traditional approach in macroeconomic studies by assuming a one-time shock at time 0 and no further shocks afterwards. Instead, we need to simulate entire sample path during $[0, t]$ by taking into account subsequent shocks after time 0 and study how an average borrower responds to the shock at time 0. We follow Borovička et al. (2014) by defining the shock elasticity of F_t and D_t with respect to the cash flow shock and regime-shift

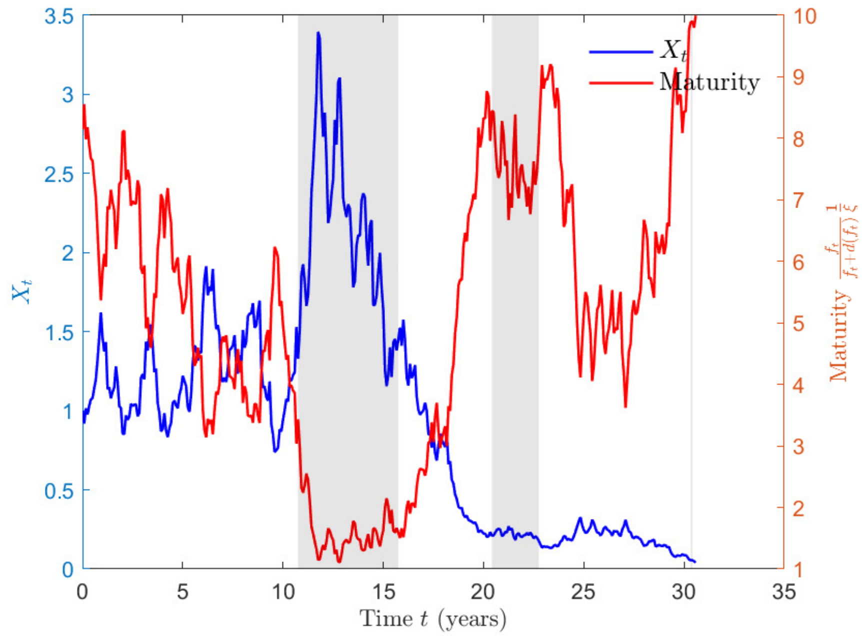


Figure 4: Sample Path of Leverage and Maturity

This figure simulates the sample path of one firm and plots the time series of X_t, maturity, and market leverage, with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$.

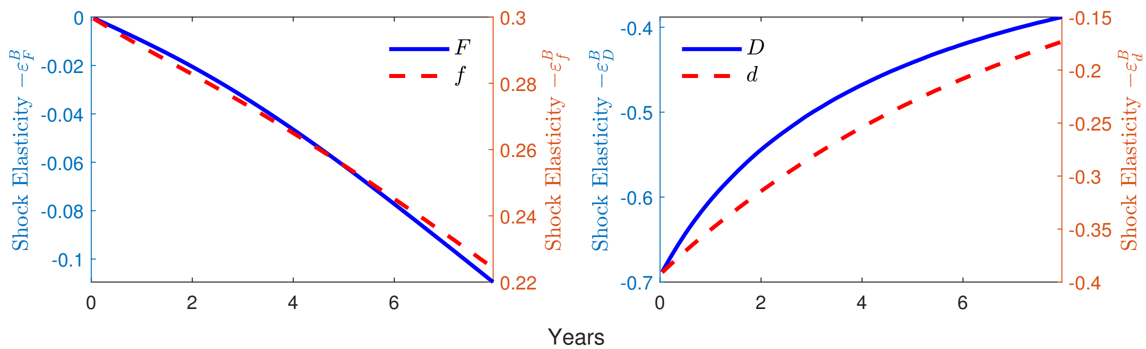
shock. Specifically, for a process $Y_t \in \{F_t, D_t\}$, let us define

$$\varepsilon_Y^B(t, f, \theta) \equiv \frac{\mathbb{E}\left[\mathcal{D}_0 Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = \theta\right]}{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = \theta\right]}$$

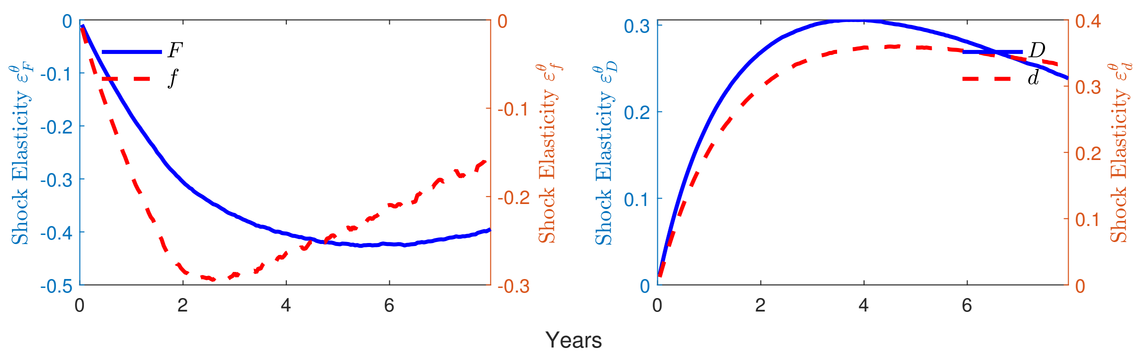
$$\varepsilon_Y^\theta(t, f) \equiv \frac{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0, \theta_0 = L\right] - \mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = H\right]}{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = H\right]}$$

where $\mathcal{D}_0 Y_t$ corresponds to the Malliavin derivative of the process Y_t, and the indicator function $\mathbb{1}_{\{t<\tau_b\}}$ accounts for the possibility of the borrower defaulting before time t. Intuitively, ε_F^B (ε_D^B) captures the proportional change of an average borrower's outstanding long-term (short-term) debt at time t as a response to a cash flow shock $dB_0 = 1$, whereas ε_F^θ (ε_D^θ) is the same response to the regime shift shock. Details are supplemented in Proposition 10 in the appendix.

Figure 5 plots the shock elasticity. The top two panels show that after a negative Brownian shock $dB_0 = -1$, an average borrower slowly changes the outstanding long-term debt but immediately reduces the amount of short-term debt. Over time, the borrower changes the composition of debt by reducing long and increasing short (relative to time 0). The bottom two panels describe the shock elasticity with respect to the regime shift shock, i.e., θ_0 shifts from the upturn H to the downturn L. Interestingly, whereas the borrower also slowly reduces the outstanding long-term debt, she actually increases the amount of short-term borrowing, because downside risk is mitigated after the downturn has already arrived. Note that in this case, the adjustment of both long and short are slow.



(a) Shock elasticity with respect to Brownian shock



(b) Shock elasticity with respect to θ

Figure 5: Shock Elasticity of Short- and Long-term Debt.

This figure plots the impulse response function to cash flow shocks for $\theta_0 = H$, $X_0 = 1$, and $f_0 = 8.5$. The parameter values are the following: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$. With these parameters, $f_\dagger = 17.10$, $f_H^b = 23.00$, and $f_L^b = 19.91$. The scales on the left axes stand for the variables F and D, and the scales on the right axes stand for the variables f and d.

Whereas the majority of existing empirical research on debt maturity has focused on the cross-sectional comparisons, our results highlight the importance of studying the borrower's dynamic adjustments in the outstanding maturity structure. Specifically, in a Difference-in-Difference (DiD) estimation, the dynamic treatment effects would be considerably different for long- and short-term debt. In the situation that borrowers in the treatment group experience a negative cash-flow shock (Brownian shock), our model predicts no immediate reaction in the outstanding long-term debt. Over time, the amount of long-term debt gets gradually reduced. By contrast, borrowers in the treatment group immediately and abruptly reduces the issuance of short-term debt. Overtime, this reduction gets mean-reverted.

The dynamic treatment effects could also differ by the nature of the shock. In the situation that borrowers in the treatment group experience a regime-shift shock, our model predicts no immediate reaction in either long-term debt or short-term debt. In the near future, the amount of outstanding long-term debt gets gradually reduced, whereas the amount of short-term debt gets gradually increased. All these implications can be testable hypotheses for future empirical studies.

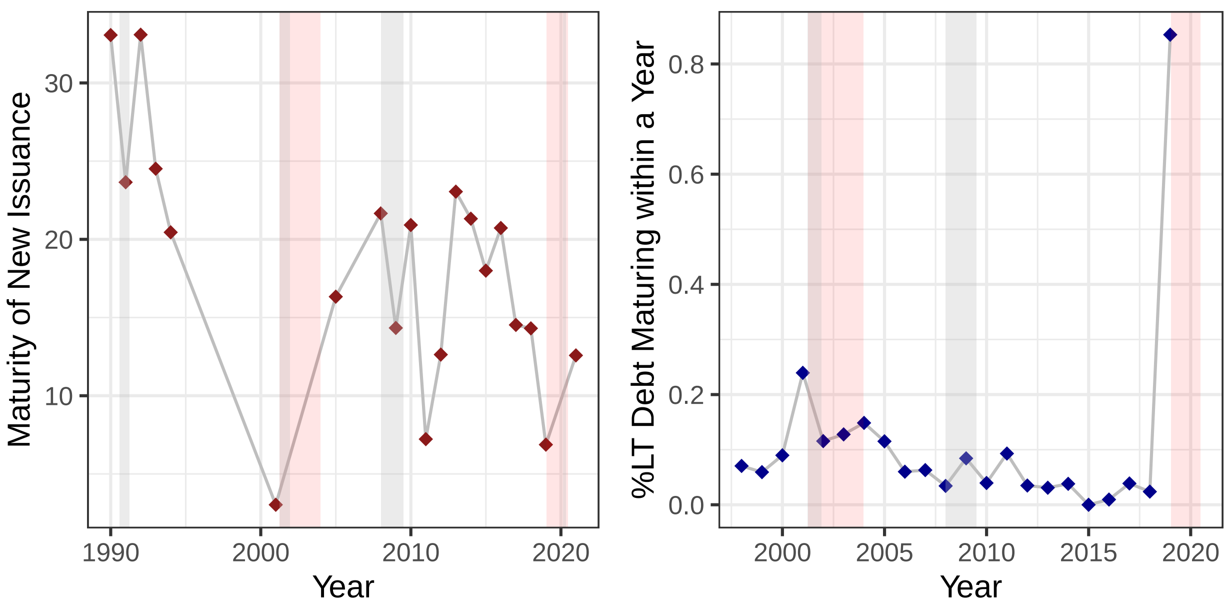


Figure 6: Maturity Structure of Pacific Gas & Electric Company

This figure shows the average maturity of bonds in a year (weighted by issuance amount) and the share of long-term debt maturing within one year for Pacific Gas & Electric (PG&E). The gray shaded area indicates NBER recession while the red shaded area indicates periods over which PG&E was in bankruptcy procedures. Source: Compustat and Mergent FISD.

An example: Pacific Gas and Electric Company. Let us connect our model's empirical predictions to a real-world example. The Pacific Gas and Electric Company (PG&E) is an American-based public company that entered bankruptcy twice in the last two decades. It initially entered Chapter 11 bankruptcy on April 6, 2001 and emerged from bankruptcy in April 2004. In 2019, it filed bankruptcy on January 29 again and successfully exited on June 20. Figure 6 plots the PG&E's debt maturity structure from 1990 onward. The left panel plots the maturity of newly issued long-term debt, weighted by the offering amount. The red-shaded areas marked the two bankruptcies, and the gray areas are NBER recessions. Consistent with our model, the newly-issued bonds had shorter maturities in the NBER recessions and shortly prior to the bankruptcies.[18] The right panel plots the ratio of long-term debt due within a year to the sum of current and long-term liabilities. Clearly, this ratio spiked the year prior to both bankruptcies. The patterns in both panels suggest that bankruptcy was associated with large amount of long-term debt maturing soon and prior to bankruptcy, PG&E issued more short-term debt.

4.2 Positive Recovery and Debt Restructuring

Thus far we have assumed the recovery value is zero if the borrower defaults. If instead, the recovery value is positive, a borrower without any restriction in issuance can effectively steal the entire recovery value from creditors by issuing a large amount of new debt just prior to default and using the proceeds to pay a dividend. In practice, this transaction is referred to as a preference action and can be voided using the clawback provision.

In this subsection, we consider a possibility that in the upturn $\theta_t = H$, the borrower can restructure the outstanding debt under the assumption that the recovery value is $\alpha X j_H(0)$, a fraction α of the unlevered value.[19] We assume short-term debt is junior to outstanding long-term debt; otherwise the borrower can fully dilute long-term creditors by issuing a large amount of short-term debt prior to default. Once the borrower defaults, all relevant parties enter a restructuring process whereby long-term creditors can recover $R(f)$ per unit of face value.[20] Finally, we assume that there is a restructuring cost $Xb(f)$ that captures frictions in the renegotiation process.

In principle, equity holders may still issue more long-term debt prior to default to dilute existing creditors. Let this amount be Δ. Creditors – anticipating default and restructuring shortly after – expect to receive $R(f + \Delta)$ per unit of face value, and this is the maximum price they are willing

[18] The maturity was also short in 2011, which might be due to the 2010 San Bruno fire: PG&E was on probation after being found criminally liable in the fire.

[19] It is straightforward to extend to analysis to both states.

[20] One implementation of the restructuring is to convert long-term debt to equity share $fR(f)/\alpha j(0)$ and long-term debt to equity share $d(f)/\alpha j(0)$. The original borrower retains a fraction $1 - (d(f) + fR(f))/\alpha j(0)$ of the equity of the restructured enterprise.

to pay before bankruptcy. The net proceeds from issuance are therefore $fR(f+\Delta)$. From here we get the value that equity holders obtain from restructuring is $v_H^R(f,d) = \max\{j_H^R(f) - d, 0\}$,[21] where

$$j_H^R(f) \equiv \alpha j_H(0) - \min_{\Delta} \{fR(f+\Delta) + b(f+\Delta)\}.$$

We make the following assumptions:

Assumption 2. *The functions $R(f)$ and $b(f)$ satisfy*

a. $R(f) \in [0,1]$.

b. $R(f)$ *is non-increasing, and* $b(f)$ *is non-decreasing.*

c. The total recovery by long-term creditors is lower than firm value.

$$\alpha j_H(0) \geq \min_{\Delta \geq -f} \{fR(f+\Delta) + b(f+\Delta)\}.$$

d. For all $f \in \mathbb{R}_+$, *the function* $fR(f+\Delta) + b(f+\Delta)$ *is continuously differentiable and quasi-convex in* Δ.

Assumption 2.a. states that long-term creditors are protected by limited liability and never recover more than the face value. Assumption 2.b. captures in reduced-form that a higher leverage makes it more difficult to restructure debt and decreases the expected recovery. Assumption 2.c. restricts the total recovery value received by long-term creditors to be less than the enterprise value. Under these assumption, the optimal issuance at the time of default $\Delta^b(f)$ is given by the first order condition

$$b'\left(f + \Delta^b(f)\right) = -fR'\left(f + \Delta^b(f)\right).$$

The right-hand side $-fR'(f+\Delta)$ captures the marginal benefit from diluting existing long-term creditors, whereas the left-hand side is the marginal cost of renegotiation. Note that the adjustment $\Delta^b(f)$ could be negative, which is necessarily the case if $b'(f) > -fR'(f)$. A negative issuance captures a situation in which the borrower injects cash to buy back some long-term debt at a discount. By doing so, the borrower facilitates the restructuring process by reducing the renegotiation cost. Indeed, debt repurchase is common across financially-distressed firms.

The rest of the model is solved similarly to the zero recovery case. Indeed, the HJB equations stay unchanged, and the boundary conditions are replaced by $j_H(f_H^b) = j_H^R(f)$, and $j'_H(f_H^b) =$

[21] In equilibrium, $j_H^R(f) - d \geq 0$, because otherwise, short-term creditors anticipate default.

$j_H^{R'}(f_H^b) = -R(f_H^b + \Delta^b(f_H^b))$.[22] The linear recovery case in DeMarzo and He (2021) and DeMarzo et al. (2021) corresponds to the case where $R(f)$ and $b(f)$ are constants, in which case we can take $\Delta^b(f) = 0$. The following proposition provide some implications of corporate restructuring on the use of long-term debt.

Proposition 8 (Impact of Restructuring on Long Term Debt)**.** *Both a higher cost of restructuring and a higher recovery value expand the region of long-term debt issuance. That is,*

- *Consider two restructuring environments $\{R_1(f), b_1(f)\}$ and $\{R_2(f), b_2(f)\}$, such that $R_2(f) \geq R_1(f)$ and $b_2(f) \geq b_1(f)$. Let $f_\dagger^1$ and $f_\dagger^2$ be the threshold for the issuance of long-term debt, respectively. Then $f_\dagger^2 > f_\dagger^1$.*
- *The threshold $f_\dagger$ is increasing in bankruptcy cost $1 - \alpha$. That is, if $\alpha_2 > \alpha_1$ then $f_\dagger^2 < f_\dagger^1$.*

4.3 Jump Risk

In the benchmark model, the borrower is subject to two types of risks. The Brownian motion captures small frequent shocks to the cash flow, which has a continuous effect on the enterprise value. By contrast, a transition from the high to the low state, i.e., the regime shift, captures large infrequent shocks that reduce the enterprise value discontinuously. In this subsection, we show the modeling choice of regime shift is unimportant. In particular, our mechanism continues to hold if large infrequent shocks are modeled as downward jump risks to the cash flow. Specifically, we assume the cash flow follows a jump-diffusion process:

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \eta^{-1}\right) X_{t-} dN_t, \tag{24}$$

where N_t is a Poisson process with intensity λ and $\eta \in (1, \infty)$ is a constant. We can construct an equilibrium characterized by thresholds $f_\dagger$ and f^b. The issuance of long-term debt satisfies $g(f) = 0$, for $g(f) = 0 \; \forall f \in (f_\dagger, f^b]$, where f^b is the endogenous default boundary. For $f \in [0, f_\dagger]$, the issuance of long-term debt follows

$$g\left(f\right) = \frac{\left(\rho - r\right)\left(p\left(f\right) - p\left(\eta f\right)\right)}{f j''\left(f\right)}. \tag{25}$$

In other words, long-term debt is issued if and only the amount of outstanding long-term debt is low relative to the operating cash flow. Examining (25), it becomes clear that the intuitive reason

[22] By the envelope theorem we have

$$j_H^{R'}(f) = -fR'(f + \Delta^b(f)) - R(f + \Delta^b(f)) - b'(f + \Delta^b(f)).$$

Substituting the first order condition for $\Delta^b(f)$ we get $j_H^{R'}(f) = -R(f + \Delta^b(f))$.

again fall into the benefits of long-term debt in sharing downside risks, modeled as downward jumps in this case. The difference in prices $p(f) - p(\eta f)$ reflects the drop in the long-term debt's price following the downward jump, and the expression thus can be similarly interpreted as (22).

The issuance of short-term debt is also similar to that in section 3. Short-term debt is riskless when $f \leq f_\dagger$ and the amount of issuance is $d(f) = j(\eta f)$. On the other hand, when $f > f_\dagger$, short-term debt becomes risky, and the amount of issuance becomes $d(f) = j(f)$. The scaled-value function $j(f)$ satisfies a second order delay differential equation, which cannot be solved in closed form. Detailed analysis is available in appendix C.1.

4.4 Managing Interest-rate risk

According to Graham and Harvey (2001), corporate firms' choice between long- and short-term debt can depend on the concurrent and anticipated interest rates.[23] In this subsection, we modify the model to study how a borrower issues long- and short-term debt to hedge against fluctuations in interest-rate risk.

The benchmark model presented in section 2 can be extended to consider variations in interest rate, so the discount rate of creditors can differ across the upturn and the downturn. Moreover, we assume there are investor clienteles for specific maturity segments (Vayanos and Vila, 2021), so that the discount rate for short-term creditors can differ from the one for long-term creditors. The short-term creditors discount rate in state θ is r^0_θ, while the long term creditors discount rate is r^ξ_θ. For simplicity, we assume that the equity holders discount rate ρ is constant across state (it is straightforward to incorporate variations in ρ). We assume r^0_θ and r^ξ_θ are both strictly lower than ρ in both states. Moreover, we assume $\mu_H - \mu_L$ is sufficiently large so that in equilibrium, the value function always satisfies $j_H(f) > j_L(f)$, $\forall f$. We can derive the equilibrium following similar steps to the ones in the benchmark model (the details can be found in the appendix). The next proposition presents a characterization of the debt issuance policy.

Proposition 9 (Debt issuance with interest-rate risk)**.** *The debt issuance policies are as follows.*

1. *In the low state $\theta_t = L$,*

$$d_L = j_L(f) \tag{26}$$

$$g_L(f) = \frac{\left(r^0_L - r^\xi_L\right) p_L(f)}{-f p'_L(f)}. \tag{27}$$

[23] For example, 28.70% of the CFOs claim they issue short-term when they are waiting for long-term market interest rates to decline.

2. *In the high state $\theta_t = H$:*

(a) *If* $\left(\rho + \lambda_{HL} - r_H^0\right) j_L(f) \geq \left(\rho - r_H^0\right) j_H(f)$,

$$d_H(f) = j_L(f) \tag{28}$$

$$g_H(f) = \frac{\left(\rho - r_H^{\xi}\right)\left(p_H(f) - p_L(f)\right) + \left(r_H^0 - r_H^{\xi}\right) p_L(f)}{-f p_H'(f)}. \tag{29}$$

(b) *If* $\left(\rho + \lambda_{HL} - r_H^0\right) j_L(f) < \left(\rho - r_H^0\right) j_H(f)$,

$$d_H(f) = j_H(f) \tag{30}$$

$$g_H(f) = \frac{\left(r_H^0 - r_H^{\xi}\right) p_H(f)}{-f p_H'(f)}. \tag{31}$$

Whereas the short-term debt policy resembles that in the benchmark model, the issuance of long-term debt is different because now it considers both hedging and interest rate management. Equation (27) shows in the low state, when long-term creditors have lower cost of capital, i.e., $r_L^{\xi} < r_L^0$, the borrower will issue long-term debt. Given the lower interest rate for long-term bonds (due to an inverted yield curve), the firm replaces short-term debt for long-term debt. However, due to price impact, the adjustment is gradual. The rate of issuance, $g_L(f) \cdot f$, is the marginal flow benefits of replacing short- with long-term debt $(r_L^0 - r_L^{\xi}) p_L(f)$, scaled by the price impact $-p_L'(f)$. In the high state, the issuance policy depends on whether short-term debt is risky or not. When short-term debt is risky, the issuance policy in equation (31) resembles the one in the low state. The firm actively trades long-term debt only to adjust due to differences in discount rate. If $r_H^{\xi} > r_H^0$, the firm will buy back long-term debt to replace it with short-term debt. When short-term debt is riskless, the issuance policy in equation (29) is driven by two factors. As in the model without interest rate risk, the numerator term includes the benefits of sharing the downside risk $(\rho - r_H^{\xi})\left(p_H(f) - p_L(f)\right)$. However, there is an additional motive to issue long-term debt when long-term creditors have lower cost of capital, i.e., $r_H^0 > r_H^{\xi}$, which is capture by the second term in the numerator of equation (29). In the presence of interest rate risk, the borrower may optimally buy-back long-term debt when long-term creditors have a higher discount rate than short-term ones. It is important to notice thought that variation in interest rates alone is not sufficient to generate this maturity adjustment. In fact, issuance policy reduces to the one in the benchmark model if $r_\theta^{\xi} = r_\theta^0$. Hence, market segmentation is crucial to generate buy-backs of long-term debt driven by an upward slopping yield curve.

Empirical evidence has shown that the real interest rate is counter-cyclical (Winberry, 2021), and the yield curve is upward (downward) sloping in upturns (downturns). Following these observations, we expect $r_H^0 < r_H^\xi$ and $r_L^0 > r_L^\xi$ to hold. Given so, $g_L(f) > 0$ so that in downturns, the borrower would like to issue long-term debt for interest-rate risk management purpose. In the upturn, the borrower would buy back long-term debt and issue short-term debt when she is close to default. When she is far from default, the downward risk-sharing factor continues to motivate her to borrow long. But the upward-sloping yield curve reduces the rate of long-term debt issuance.

Figure 7 shows the issuance function of long-term debt. The left panel plots the issuance function in the low and high state. In the low state, the firm issues long-term debt due to the inverted yield curve. In the high state the interaction between hedging and interest-rate risk creates a nonlinear affect. In the right panel, we decompose the issuance function in the high state into two components. Whereas the first component captures hedging downside risks as in the benchmark model, the second component captures the interest-rate risk. The hedging component is positive, leading to a positive issuance of long-term debt. By contrast, the interest-rate risk component is negative (because $r_H^\xi > r_H^0$), leading to the buy back of long-term debt. For very low and large values of f, the interest-rate risk component dominates and the firm actively buys back long-term debt. For intermediate values of f, the hedging component could dominate, and the firm issues long-term debt. Overall, the long-term debt issuance policy is non-monotone.

5 Conclusion

Our paper offers a theory of debt maturity, which is fundamentally a tradeoff between commitment and risk sharing. Short-term debt mitigates the lack of commitment problem but does not share any downside risk. Long-term debt suffers from dilution but shares the downside risk. In a model with binary state, risk sharing is not valued in the downturn or in the upturn if the borrower is close to default. If the borrower is far from default in the upturn, she borrows both debt.

We have not modeled callable bonds and exchange offers, which are common tools for corporate firms to manage debt maturity in practice. In our model, if long-term debt can be frictionlessly called back, then there is no essential difference between long- and short-term debt, and debt maturity becomes irrelevant. The assumption of frictionless callback is not an innocuous simplification, and further exploration towards this direction goes beyond the scope of this paper. In practice, another motive behind debt maturity management is to match assets and liabilities, which we intend to study in follow-up work. Another extension is to introduce collateral and secured debt. In our model, short-term debt is essentially senior to long-term debt because it matures earlier. Fully-collateralized long-term debt is similar to short-term debt, and payments can not be diluted. The interaction between maturity and collateral in establishing priority is understudied and deserves

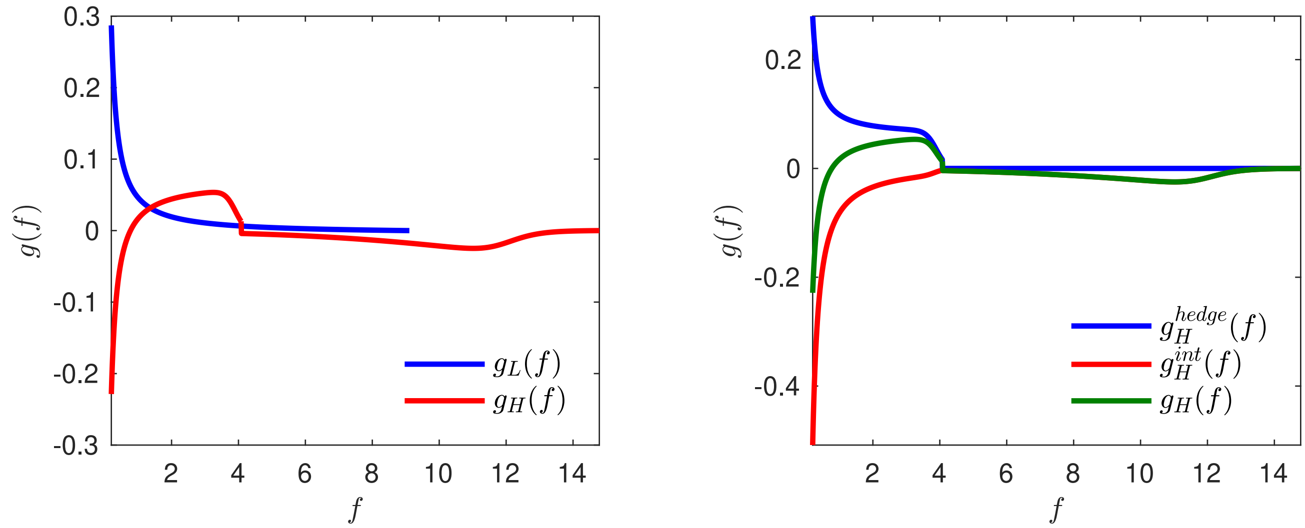


Figure 7: Equilibrium Issuance Rate

This figure plots the issuance function in the model with interest-rate risk. The right hand panel presents a decomposition of the issuance rate in the hight state. $g^{hedge}(f) = (\rho - r_H^{\xi})\,(p_H(f) - p_L(f))\,/\,(-fp'_H\,(f))$ corresponds to the first term in (29) while $g^{int}(f) = (r_H^0 - r_H^{\xi})p_L(f)/\,(-fp'_H\,(f))$ correspond to the second term. The parameter values are the following: $\rho = 0.1$, $r_H^0 = 0.01$, $r_H^{\xi} = 0.015$, $r_L^0 = 0.02$, $r_L^{\xi} = 0.015$, $\mu_H = 0.015$, $\mu_L = -0.2$, $\sigma = 0.1$, $\xi = 0.1$, $\lambda_{HL} = 0.2$. With these parameters, $f_{\dagger} = 4.07$, $f_H^b = 14.78$, and $f_L^b = 9.10$.

more careful analysis in the future.

References

Abel, A. B. (2018). Optimal debt and profitability in the trade-off theory. *Journal of Finance 73*(1), 95–143.

Admati, A. R., P. M. DeMarzo, M. F. Hellwig, and P. Pfleiderer (2018). The leverage ratchet effect. *Journal of Finance 73*(1), 145–198.

Adrian, T. and H. S. Shin (2014). Procyclical leverage and value-at-risk. *The Review of Financial Studies 27*(2), 373–403.

Aguiar, M., M. Amador, H. Hopenhayn, and I. Werning (2019). Take the short route: Equilibrium default and debt maturity. *Econometrica 87*(2), 423–462.

Angeletos, G.-M. (2002). Fiscal policy with noncontingent debt and the optimal maturity structure. *Quarterly Journal of Economics 117*(3), 1105–1131.

Arellano, C. and A. Ramanarayanan (2012). Default and the maturity structure in sovereign bonds. *Journal of Political Economy 120*(2), 187–232.

Barclay, M. J. and C. W. Smith Jr (1995). The maturity structure of corporate debt. *the Journal of Finance 50*(2), 609–631.

Benzoni, L., L. Garlappi, R. S. Goldstein, J. Hugonnier, and C. Ying (2019). Optimal debt dynamics, issuance costs, and commitment. *Issuance Costs, and Commitment (September 16, 2019)*.

Bigio, S., G. Nuño, and J. Passadore (2021). Debt-maturity management with liquidity costs.

Black, F. and M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy 81*(3), 637–654.

Borovička, J., L. P. Hansen, and J. A. Scheinkman (2014). Shock elasticities and impulse responses. *Mathematics and Financial Economics 8*(4), 333–354.

Brunnermeier, M. K. and M. Oehmke (2013). The maturity rat race. *Journal of Finance 68*(2), 483–521.

Brunnermeier, M. K. and M. Yogo (2009). A note on liquidity risk management. *American Economic Review 99*(2), 578–83.

Bulow, J. I. (1982). Durable-goods monopolists. *Journal of Political Economy 90*(2), 314–332.

Calomiris, C. W. and C. M. Kahn (1991). The role of demandable debt in structuring optimal banking arrangements. *American Economic Review*, 497–513.

Chen, H., Y. Xu, and J. Yang (2021). Systematic risk, debt maturity, and the term structure of credit spreads. *Journal of Financial Economics 139*(3), 770–799.

Coase, R. H. (1972). Durability and monopoly. *Journal of Law and Economics 15*(1), 143–149.

Dangl, T. and J. Zechner (2020). Debt maturity and the dynamics of leverage. *Available at SSRN 890228*.

DeMarzo, P. and Z. He (2021). Leverage dynamics without commitment. *Journal of Finance 73*(3), 1195–1250.

DeMarzo, P. M. (2019). Presidential address: Collateral and commitment. *Journal of Finance 74*(4), 1587–1619.

DeMarzo, P. M., Z. He, and F. Tourre (2021). Sovereign debt ratchets and welfare destruction. Technical report, National Bureau of Economic Research.

Diamond, D. W. (1991). Debt maturity structure and liquidity risk. *Quarterly Journal of Economics 106*(3), 709–737.

Diamond, D. W. and Z. He (2014). A theory of debt maturity: the long and short of debt overhang. *The Journal of Finance 69*(2), 719–762.

Diamond, D. W. and R. G. Rajan (2001). Liquidity risk, liquidity creation, and financial fragility: A theory of banking. *Journal of Political Economy 109*(2), 287–327.

Duffie, D., N. Gârleanu, and L. H. Pedersen (2005). Over-the-counter markets. *Econometrica 73*(6), 1815–1847.

Fama, E. F. and M. Miller (1972). *The theory of finance.* Holt Rinehart & Winston.

Flannery, M. J. (1986). Asymmetric information and risky debt maturity choice. *The Journal of Finance 41*(1), 19–37.

Friewald, N., F. Nagler, and C. Wagner (2021). Debt refinancing and equity returns. *Journal of Finance, Forthcoming*.

Froot, K. A., D. S. Scharfstein, and J. C. Stein (1993). Risk management: Coordinating corporate investment and financing policies. *Journal of Finance 48*(5), 1629–1658.

Gertner, R. and D. Scharfstein (1991). A theory of workouts and the effects of reorganization law. *Journal of Finance 46*(4), 1189–1222.

Goldstein, R., N. Ju, and H. Leland (2001). An ebit-based model of dynamic capital structure. *The Journal of Business 74*(4), 483–512.

Graham, J. R. and C. R. Harvey (2001). The theory and practice of corporate finance: Evidence from the field. *Journal of Financial Economics 60*(2-3), 187–243.

He, Z. and K. Milbradt (2016). Dynamic debt maturity. *Review of Financial Studies 29*(10), 2677–2736.

Hu, Y., D. Nualart, X. Sun, and Y. Xie (2019). Smoothness of density for stochastic differential equations with markovian switching. *Discrete & Continuous Dynamical Systems – Series B 24*(8), 3615–3631.

Hu, Y. and F. Varas (2021). Intermediary financing without commitment. Technical report, Working paper, Duke University.

Leland, H. E. (1994). Corporate debt value, bond covenants, and optimal capital structure. *The journal of finance 49*(4), 1213–1252.

Leland, H. E. (1998). Agency costs, risk management, and capital structure. *Journal of Finance 53*(4), 1213–1243.

Malenko, A. and A. Tsoy (2020). Optimal time-consistent debt policies. *Available at SSRN 3588163*.

Niepelt, D. (2014). Debt maturity without commitment. *Journal of Monetary Economics 68*, S37–S54.

Panageas, S. (2010). Bailouts, the incentive to manage risk, and financial crises. *Journal of Financial Economics 95*(3), 296–311.

Protter, M. H. and H. F. Weinberger (1967). *Maximum Principles in Differential equations.* Prentice-Hall.

Protter, P. E. (2005). Stochastic integration and differential equations. In *Stochastic integration and differential equations*, pp. 249–361. Springer.

Rampini, A. A. and S. Viswanathan (2010). Collateral, risk management, and the distribution of debt capacity. *Journal of Finance 65*(6), 2293–2322.

Stohs, M. H. and D. C. Mauer (1996). The determinants of corporate debt maturity structure. *Journal of business*, 279–312.

Vayanos, D. and J.-L. Vila (2021). A preferred-habitat model of the term structure of interest rates. *Econometrica 89*(1), 77–112.

Winberry, T. (2021). Lumpy investment, business cycles, and stimulus policy. *American Economic Review 111*(1), 364–96.

Xu, Q. (2018). Kicking maturity down the road: early refinancing and maturity management in the corporate bond market. *Review of Financial Studies 31*(8), 3061–3097.

Appendix

A Proofs of Section 2

Proof of Lemma 1

Proof. Let $\tau \geq t$ be the time that the state switches from θ to θ'. By the principle of dynamic programming,

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t,\tau)\}} \mathbb{E}_t \left[\int_t^{\tau_b \wedge \tau} e^{-\rho(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-}D_{s-})\, ds + p_s dG_s + dD_s \Big) + e^{-\rho \tau_b \wedge \tau} V_\tau \mathbb{1}_{\{\tau_b \geq \tau\}} \right]$$
$$= \sup_{\tau_b, \{G_s, D_s : s \in [t,\tau)\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-}D_{s-})\, ds + p_s dG_s + dD_s \Big) + e^{-\rho\tau} V_\tau \mathbb{1}_{\{\tau_b \geq \tau\}} \right].$$

Given the definition of J_t, the equity value can be written as $V_t = \max\{J_t - D_{t-}, 0\}$, where the max operator takes into account the borrower's limited liability constraint. In particular, the borrower defaults at time τ if $J_\tau < D_{\tau-}$, so that $V_\tau = \max\{J_\tau - D_{\tau-}, 0\}$. Hence,

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t,\tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-}D_{s-})\, ds + p_s dG_s + dD_s \Big)$$
$$+ e^{-\rho\tau} \max\{J_\tau - D_{\tau-}, 0\} \mathbb{1}_{\{\tau_b \geq \tau\}} \Bigg]$$
$$= \sup_{\tau_b, \{G_s, D_s : s \in [t,\tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(\big(X_s - (r+\xi)F_s - y_{s-}D_{s-} + \lambda_{\theta_s \theta_s'} \max\{J_s - D_{s-}, 0\}\big)\, ds$$
$$+ p_s dG_s + dD_s \Big) \Bigg]$$

Using the integration by parts formula for semi-martingales in Corollary 2 in Section 2.6 of Protter (2005), we get

$$\mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} dD_s \right] = \mathbb{E}_t \left[e^{-(\rho+\lambda)(\tau_b - t)} D_{\tau_b} \right] - D_{t-} + \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} (\rho+\lambda) D_{s-} ds \right].$$

At the time of default, $D_T = 0$. Hence

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t,\tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(\big(X_s - (r+\xi)F_s + (\rho+\lambda-y_{s-})D_{s-} + \lambda_{\theta_s \theta_s'} (J_s - D_{s-})^+\big)\, ds$$
$$+ p_s dG_s + dD_s \Big) \Bigg] - D_{t-}.$$

□

B Proofs of Section 3

B.1 Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

Theorem 1 (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{32}$$

in an interval $(0, b)$ with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Theorem 2 (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that u is a nonconstant solution of the differential inequality (32) having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

Corollary 1. *If u satisfies (32) in an interval (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u(a) \leq 0$, $u(b) \leq 0$, then $u(x) < 0$ in (a, b) unless $u \equiv 0$.*

B.2 Equilibrium

Proof of Proposition 1

Equation (11) is a second-order ODE, and a standard solution takes the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$\begin{aligned}
(r-\mu_L)\,A_0 &= 1\\
(r+\xi)\,A_1 &= (r+\xi)\\
(r-\mu_L)\,A_2 &= -\left(\mu_L+\xi\right)A_2\gamma_1+\frac{1}{2}\sigma^2A_2\gamma_1\left(\gamma_1-1\right)\\
(r-\mu_L)\,A_3 &= -\left(\mu_L+\xi\right)A_3\gamma_2+\frac{1}{2}\sigma^2A_3\gamma_2\left(\gamma_2-1\right),
\end{aligned}$$

which implies

$$A_0=\frac{1}{r-\mu_L},\ A_1=1$$

and $\{\gamma_1,\gamma_2\}$ are the two roots of

$$\frac{1}{2}\sigma^2\gamma_1^2-\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\gamma_1-\left(r-\mu_L\right)=0.$$

In particular,

$$\begin{aligned}
\gamma_1 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r-\mu_L\right)}}{\sigma^2}>1,\\
\gamma_2 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r-\mu_L\right)}}{\sigma^2}<0.
\end{aligned}$$

The condition

$$\lim_{f\to 0} j_L(f)<\infty$$

implies $A_3=0$. Therefore, we define

$$\gamma\equiv\gamma_1=\frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r-\mu_L\right)}}{\sigma^2}.$$

Combining with value-matching and smooth-pasting condition, we get the solution to $j_L(f)$ and f_L^b.

To derive $g_L(f)$, let us first write down the HJB for $p_L(f)$:

$$(r+\xi)\,p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)\,, \tag{33}$$

where we have used the condition (8). The result of $g_L(f) \equiv 0$ follows from differentiating (11), applying in condition (10), and subtracting (33).

Proof of Proposition 2

We start establishing the uniqueness of the equilibrium.

Uniqueness: For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\begin{aligned}
\Phi(\tilde{j})(f) \equiv \sup_{\tau \geq 0} \mathbb{E}&\left[\int_0^\tau e^{-\hat{\rho}t}\left(1-(r+\xi)z_t + \pi(z_t,\tilde{j}(z_t))\right)dt \Big| z_0 = f\right] \\
&\text{subject to} \\
dz_t &= -(\xi+\mu_H)z_t dt - \sigma z_t dB_t,
\end{aligned}$$

where

$$\begin{aligned}
\pi(z,\tilde{j}) &\equiv \max_{d\in[0,\tilde{j}]} (\rho - r)\,d + \mathbb{1}_{\{d\leq j_L(z)\}}\cdot \lambda j_L(z) \\
&= \max\{(\rho-r)\,j_L(z) + \lambda j_L(z), (\rho - r)\,\tilde{j}\}.
\end{aligned}$$

It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\pi(f,\tilde{j}_1) \geq \pi(f,\tilde{j}_0)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\begin{aligned}
\pi(z,\tilde{j}+a) &= \max\{(\rho-r)\,j_L(z) + \lambda j_L(z), (\rho-r)\,(\tilde{j}+a)\} \\
&\leq \max\{(\rho-r)\,j_L(z) + \lambda j_L(z) + (\rho - r)a, (\rho-r)\,(\tilde{j}+a)\} \\
&= (\rho-r)a + \pi(z,\tilde{j}),
\end{aligned}$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t)+a)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right]+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right]+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f)+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \Phi(\tilde{j})(f)+\frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}$$

Thus, the operator Φ is monotone and satisfies discounting, it follows then by Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f)=\Phi(j_H)(f)$.

Next, we provide a the solution to the HJB equation when both long- and short-term debt are issued. That is, when $\lambda>\bar{\lambda}$.

Solution HJB Equation $\lambda>\bar{\lambda}$: Let us first write down the HJBs in different regions as well as the boundary conditions. Specifically, the value function satisfies

$$\begin{aligned}
&(\rho+\lambda-\mu_H)\,j_H(f)=1-(r+\xi)\,f+(\rho+\lambda-r)\,j_L(f)-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\ \ f\in[0,f^\dagger] \\
&(r+\lambda-\mu_H)\,j_H(f)=1-(r+\xi)\,f-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\ \ f\in[f^\dagger,f_H^b].
\end{aligned}$$

The boundary conditions are

$$j_H(f\dagger-)=j_H(f\dagger+) \tag{34}$$
$$j_H'(f_\dagger-)=j_H'(f_\dagger+) \tag{35}$$
$$j_H(f_H^b)=0 \tag{36}$$
$$j_H'(f_H^b)=0 \tag{37}$$
$$\lim_{f\to 0}\ j_H(f)<\infty \tag{38}$$
$$j_H\left(f_\dagger\right)=\left(1+\frac{\lambda}{\rho-r}\right)j_L\left(f_\dagger\right). \tag{39}$$

Next, let us supplement the expressions of the auxiliary functions. The solutions to the constants

$\{\phi, \beta_1, \beta_2\}$ and the boundaries $\left\{f_\dagger, f_H^b\right\}$ are provided as we solve the ODE system.

$$
\begin{aligned}
u_0(f) &\equiv \frac{1}{r-\mu_L}\frac{\rho+\lambda-\mu_L}{\rho+\lambda-\mu_H} - f + \frac{(\rho+\lambda-r)}{(\rho+\lambda-r)+(\mu_H-\mu_L)(\gamma-1)}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma \\
u_1(f) &\equiv \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\xi+\lambda}f \\
h_0\left(f, f_\dagger, f_H^b\right) &= \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \\
h_1\left(f, f_\dagger, f_H^b\right) &= \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.
\end{aligned}
$$

$$
j_L(f) = \frac{1}{r-\mu_L} - f + \frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma
$$

The rest of the proof includes three parts. In the first part, we detail the solutions to the ODE system (15) combined with the boundary conditions (34)-(39). In the second part, we prove a single-crossing property and therefore shows that it is optimal for the borrower to issue riskless short-term debt $d_H = j_L(f)$ if and only if $f \le f_\dagger$. Finally, we verify that $j_H(f)$ is a convex function on $\left[0, f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly.

Part 1: the solution to the ODE system. On $[0, f_\dagger]$, the solution to the ODE taking into condition (38) shows that

$$
j_H(f) = u_0(f) + Bf^\phi,
$$

where

$$
\phi = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} > 1
$$

solves

$$
\frac{1}{2}\sigma^2\phi^2 - \left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)\phi - (\rho+\lambda-\mu_H) = 0.
$$

The coefficient B is pinned down from the value at $j_H\left(f_\dagger\right)$

$$B = f_\dagger^{-\phi}\left(j_H(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H(f) = u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi}, \qquad \forall f \in [0, f_\dagger],$$

where $j_H(f_\dagger) = \left(1 + \frac{\lambda}{\rho - r}\right) j_L(f_\dagger)$.

On $\left[f_\dagger, f_H^b\right]$, the solution to the ODE is

$$j_H\left(f\right) = u_1(f) + D_1 f^{\beta_1} + D_2 f^{\beta_2},$$

where

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} > 1$$
$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} < 0$$

solve

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta - \left(r + \lambda - \mu_H\right) = 0.$$

Using (36) and (34), we get

$$D_1 = \frac{j_H(f_\dagger) + u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}$$
$$D_2 = (f_H^b)^{-\beta_2}\left(-u_1(f_H^b) - D_1(f_H^b)^{\beta_1}\right).$$

so that

$$j_H(f) = u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right).$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions

(35) and (37). These two conditions lead to the two-variable, non-linear equation system below

$$
\begin{aligned}
u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}-\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] &= \\
u_1'(f_H^b)f_H^b+\big(j_H(f_\dagger)-u_1(f_\dagger)\big)\frac{\beta_1-\beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} & \qquad (40)
\end{aligned}
$$

$$
\begin{aligned}
\big(u_0'(f_\dagger)-u_1'(f_\dagger)\big)f_\dagger+\phi\big(j_H(f_\dagger)-u_0(f_\dagger)\big) &= \\
u_1(f_H^b)\frac{\beta_1-\beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2}+\big(j_H(f_\dagger)-u_1(f_\dagger)\big)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}-\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}. & \qquad (41)
\end{aligned}
$$

Part 2: Optimality Short-term Debt Policy We start with the following result, which will be used later on.

Lemma 2. *If $\lambda > \bar{\lambda}$, then*

$$\left(1+\frac{\lambda}{\rho-r}\right)(\rho+\lambda-\mu_H) > (\rho+\lambda-\mu_L).$$

Proof. The proof of Proposition 4 makes it clear that the condition $\lambda > \bar{\lambda}$ guarantees

$$\frac{1}{r+\lambda-\mu_H} < \frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r-\mu_L}\right).$$

From here, we get

$$
\begin{aligned}
\frac{\rho+\lambda-\mu_H}{r+\lambda-\mu_H} &< \frac{\rho+\lambda-\mu_L}{r-\mu_L} \\
\Rightarrow \frac{r+\lambda-\mu_H}{\rho+\lambda-\mu_H} &> \frac{r-\mu_L}{\rho+\lambda-\mu_L} \\
\Rightarrow \frac{r-\rho}{\rho+\lambda-\mu_H} &> \frac{r-\rho-\lambda}{\rho+\lambda-\mu_L} \\
\Rightarrow \frac{\rho-r}{\rho+\lambda-\mu_H} &< \frac{\rho+\lambda-r}{\rho+\lambda-\mu_L} \\
\Rightarrow \left(1+\frac{\lambda}{\rho-r}\right)(\rho+\lambda-\mu_H) &> (\rho+\lambda-\mu_L).
\end{aligned}
$$

□

Next, the following result shows that it is optimal for the borrower to issue $d_H = j_L(f)$ when $f \leq f_\dagger$ and $d_H = j_H(f)$ otherwise.

Lemma 3 (Single-crossing)**.** *There exists a unique* $f_\dagger \in \left(0, f_L^b\right)$ *such that* $(\rho+\lambda-r)j_L(f) \geq (\rho-r)j_H(f)$ *if and only if* $f \leq f_\dagger$*.*

Proof. Define $a \equiv 1 + \frac{\lambda}{\rho-r}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function u let,

$$
\begin{aligned}
L^{0\dagger}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi) f u' - (\rho + \lambda - \mu_H) u \\
L^{\dagger b}u &\equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi) f u' - (r + \lambda - \mu_H) u.
\end{aligned}
$$

The HJB in state $\theta = H$ (15), can be therefore written as

$$
\begin{aligned}
L^{0\dagger}j_H + 1 - (r+\xi) f + (\rho+\lambda-r) j_L = 0,\ & f \in (0, f^\dagger) \\
L^{\dagger b}j_H + 1 - (r+\xi) f = 0,\ & f \in (f^\dagger, f_H^b).
\end{aligned}
$$

Similarly, the HJB in state $\theta = L$, (11) can be written as

$$
\begin{aligned}
L^{0\dagger}aj_L + a(\mu_H - \mu_L) f j_L' + a\left(\rho + \lambda - r + \mu_L - \mu_H\right) j_L + a\left(1 - (r+\xi) f\right) = 0 \\
L^{\dagger b}aj_L + a(\mu_H - \mu_L) f j_L' - a(\mu_H - \mu_L - \lambda) j_L + a\left(1 - (r+\xi) f\right) = 0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
L^{0\dagger}\left(aj_L - j_H\right) + H(f) = 0 \\
L^{\dagger b}\left(aj_L - j_H\right) + H(f) = 0,
\end{aligned}
$$

where the function $H(f)$ defined as

$$
H(f) \equiv a(\mu_H - \mu_L) f j_L' - a(\mu_H - \mu_L - \lambda) j_L(f) + (a-1)\left(1 - (r+\xi) f\right),
$$

is convex as

$$
\begin{aligned}
H''(f) &= \left[(\mu_H - \mu_L) a \frac{f j_L'''}{j_L''} + (\mu_H - \mu_L) a + (\rho + \lambda - r)(a-1)\right] j_L'' \\
&= \left[(\mu_H - \mu_L) a (\gamma - 1) + (\rho + \lambda - r)(a-1)\right] j_L'' > 0.
\end{aligned}
$$

Therefore, $H(f)$ attains its maximum on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. Evaluating

$H(f)$ at the two boundaries and using the hypothesis $\lambda > \bar{\lambda}$, we have

$$\left(1 + \frac{\lambda}{\rho - r}\right)(\rho + \lambda - \mu_H) > (\rho + \lambda - \mu_L),$$

from Lemma 2. Then, we get

$$H(0) = \frac{a(\rho + \lambda - \mu_H) - (\rho + \lambda - \mu_L)}{r - \mu_L} > 0$$
$$H(f_L^b) = (a-1)\left(1 - (r + \xi)\, f_L^b\right) < 0.$$

Therefore, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.
  - On $f \in [0, f_\dagger]$, we know $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f_\dagger]$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f_\dagger)$. Moreover, Theorem 2 and Corollary 1 imply $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$.
  - On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}(aj_L - j_H) \geq 0$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Given that $aj_L\left(f_L^b\right) - j_H\left(f_L^b\right) < 0$, $aj_L(f) - j_H(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
  - On $f \in [f_\dagger, f']$. Suppose there exists a $f'' \in (f_\dagger, f')$ such that $aj_L(f'') - j_H(f'') > 0$. Given that $aj_L(f_\dagger) - j_H(f_\dagger) = 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$, it must be that $aj_L(f) - j_H(f)$ has a nonpositive interior minimum on $[f_\dagger, f'']$. Meanwhile, from $L^{\dagger b}(aj_L(f) - j_H(f)) \leq 0$ for $f \in (f_\dagger, f'')$, we know from Theorem 1 that $aj_L(f) - j_H(f)$ cannot have a nonpositive interior minimum on $(f_\dagger, f'')$, which constitutes a contradiction.

- Case 2: $f' \leq f_\dagger$.
  - On $f \in [f_\dagger, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}(aj_L - j_H) \leq 0$. From Theorem 1 and 2, we know $aj_L(f) - j_H(f) \leq 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$.
  - On $f \in [f', f_\dagger]$, $L^{0\dagger}(aj_L - j_H) \geq 0$ so that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Together with $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$, this shows $aj_L(f) - j_H(f) \geq 0$.
  - On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f')$.

$\square$

Part 3: Strict convexity of $j_H(f)$ on $\left[0, f_H^b\right]$. The proof relies on a few auxiliary lemmas.

Lemma 4.

$$j_H'(f) \geq -1, \qquad \forall f \in \left[0, f_H^b\right],$$

Proof. Let $\hat{u} = j_H'(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (17) that $\hat{u}(0) = 0$ and (37) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\,\hat{u} = -(\rho + \lambda - r)\left(j_L' + 1\right) < 0, \qquad f \in [0, f_\dagger]$$
$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\,\hat{u} = -\lambda < 0 \qquad f \in [f_\dagger, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. $\square$

Lemma 5.

$$f_H^b > \frac{1}{r+\xi} \qquad \textit{and} \qquad \min\left\{j_H''(0), j_H''\left(f_H^b\right)\right\} > 0,$$

Proof. For any $f \leq \frac{1}{r+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1 - (r+\xi) f + (\rho + \lambda - y) d > 0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1}{r+\xi}$. Plugging (36) and (37) into (15), we get $j_H''\left(f_H^b\right)$ whenever $f_H^b > \frac{1}{r+\xi}$.

Next, let us turn to prove that $j_H''(0) \geq 0$. Let us define $u \equiv j_H'$ and differentiate the HJB once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H + \xi - \sigma^2\right) f u' - (\rho + \lambda + \xi)\, u = (r + \xi) - (\rho + \lambda - r)\, j_L'.$$

Moreover, let z be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H + \xi - \sigma^2\right) f z' - (\rho + \lambda + \xi)\, z = (r + \xi) - (\rho + \lambda - r)\, j_L'(0)$$

with boundary conditions

$$\lim_{f \downarrow 0} z(f) < \infty$$
$$z(f_\dagger) = u(f_\dagger) = j_H'(f_\dagger).$$

The solution is

$$z(f)=-\frac{r+\xi}{\rho+\lambda+\xi}+\frac{(\rho+\lambda-r)\,j_L'(0)}{\rho+\lambda+\xi}+\left(j_H'(f_\dagger)+\frac{r+\xi}{\rho+\lambda+\xi}-\frac{(\rho+\lambda-r)\,j_L'(0)}{\rho+\lambda+\xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1=\frac{\left(\mu_H+\xi-\frac{1}{2}\sigma^2\right)+\sqrt{\left(\mu_H+\xi-\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(\rho+\lambda+\xi\right)}}{\sigma^2}>0.$$

Let $\delta(f)=z-u$. It is easily verified that $\delta(0)=0$ and $\delta(f_\dagger)=0$. Moreover, δ satisfies

$$\frac{1}{2}\sigma^2 f^2\delta''-\left(\mu_H+\xi-\sigma^2\right)f\delta'-\left(\rho+\lambda+\xi\right)\delta=\left(\rho+\lambda-r\right)\left(j_L'(f)-j_L'(0)\right)\geq 0.$$

By Theorem 1, δ cannot have an interior nonnegative maximum, and the maximum is attained at $f=0$. Theorem 2 further implies $\delta'(0)<0$ so $u'(0)>z'(0)$. Finally, we know that

$$z'(f)=\omega_1\left(j_H'(f_\dagger)+\frac{r+\xi}{\rho+\lambda+\xi}-\frac{(\rho+\lambda-r)\,j_L'(0)}{\rho+\lambda+\xi}\right)f_\dagger^{-\omega_1}f^{\omega_1-1}=\omega_1\left(j_H'(f_\dagger)+1\right)f_\dagger^{-\omega_1}f^{\omega_1-1},$$

which implies $z'(f)\geq 0$ given that $j_H'(f_\dagger)\geq -1$. Therefore, $u'(0)=j_H''(0)>0$. $\square$

Lemma 6.

$$j_H'''(f_\dagger^-)>j_H'''(f_\dagger^+).$$

Proof. We differentiate the HJB (15) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2(j_H'''(f_\dagger+)-j_H''''(f_\dagger-))=(\rho-r)\left[aj_L'(f_\dagger)-j_H'(f_\dagger)\right],$$

where $a\equiv 1+\frac{\lambda}{\rho-r}$ The proof of Proposition 3 shows $aj_L'(f_\dagger)-j_H'(f_\dagger)<0$ so that $j_H''''(f_\dagger-)>j_H'''(f_\dagger+)$. $\square$

Now we are ready to verify that the solution to the HJB equation is convex. We differentiate the HJB twice and let $\tilde{u}\equiv fj_H''$

$$\left(\rho+\lambda+\xi\right)\tilde{u}=\left(\rho+\lambda-r\right)fj_L''-\left(\mu_H+\xi-\sigma^2\right)f\tilde{u}'+\frac{1}{2}\sigma^2 f^2\tilde{u}''\qquad f\in[0,f_\dagger]\tag{42}$$

$$\left(r+\lambda+\xi\right)\tilde{u}=-\left(\mu_H+\xi-\sigma^2\right)f\tilde{u}'+\frac{1}{2}\sigma^2 f^2\tilde{u}''\qquad f\in\left[f_\dagger,f_H^b\right].\tag{43}$$

By the maximum principle in Theorem 1, $\tilde{u}$ cannot have an interior nonpositive local minimum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonpositive minimum is that $\tilde{u}(f_\dagger) < 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are positive, this requires that $j_H''(f_\dagger-) + f_\dagger j_H'''(f_\dagger-) = \tilde{u}'(f_\dagger-) < \tilde{u}'(f_\dagger+) = j_H''(f_\dagger+) + f_\dagger j_H'''(f_\dagger+)$. From the HJB equation it follows that j_H is twice continuously differentiable at $f_\dagger$, so such a kink would require $j_H'''(f_\dagger^-) < j_H'''(f_\dagger^+)$, which is ruled out by Lemma 6. We can conclude that $\tilde{u}$ does not have an interior nonpositive minimum, so it follows that $\tilde{u}(f) = f j_H''(f) > 0$ on $(0, f_H^b)$.

Solution HJB Equation $\lambda \le \bar{\lambda}$: In the case that $\lambda \le \bar{\lambda}$, the firm never issues long term debt, so the analysis reduces to the one in Case 1 for $f' > f_\dagger$.

B.3 Extensions and Robustness

Proof of Proposition 7

Proof. The proof proceeds in a few steps.

State $\theta_t = L$. The HJB in the low state is

$$\begin{aligned}
(\rho + \zeta - \mu_L)\, j_L(f) &= \max_{d \le j_L(f)} 1 - (r+\xi)\, f + (\rho + \zeta - y)\, d - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
&= 1 - (r+\xi)\, f + (\rho - r)\, j_L(f) - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
\Rightarrow (r + \zeta - \mu_L)\, j_L(f) &= 1 - (r+\xi)\, f - (\mu_L + \xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f),
\end{aligned}$$

where we have used the condition $y = r + \zeta$ that compensates the disaster risk. Take derivative of the above equation

$$(r + \zeta + \xi)\, j_L'(f) = -(r+\xi) - \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f).$$

The debt price follows

$$(r + \xi + \zeta)\, p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)$$

From here we get,

$$g_L(f) = \frac{(r+\xi+\zeta)\, j_L'(f) + (r+\xi) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)} = 0.$$

Next, we solve for $j_L(f)$, which follows

$$(r+\zeta-\mu_L)\, j_L(f) = 1-(r+\xi)\, f-(\mu_L+\xi)\, f j_L'(f)+\frac{1}{2}\sigma^2 f^2 j_L''(f)$$

with boundary condition

$$\begin{aligned} j_L(f_L^b) &= 0 \\ j_L'(f_L^b) &= 0. \end{aligned}$$

The solution is

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1},$$

where

$$\begin{aligned} A_0 &= \frac{1}{r+\zeta-\mu_L} \\ A_1 &= \frac{r+\xi}{r+\zeta+\xi} \\ A_2 &= \frac{1}{\gamma_1}\frac{r+\xi}{r+\zeta+\xi}\left(f_L^b\right)^{1-\gamma_1} \end{aligned}$$

and

$$f_L^b = \frac{\gamma_1}{\gamma_1-1}\frac{1}{r+\zeta-\mu_L}\frac{r+\zeta+\xi}{r+\xi}$$

is the default boundary. Moreover,

$$\gamma_1 = \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\zeta-\mu_L\right)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\gamma^2-\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\gamma-(r+\zeta-\mu_L) = 0.$$

State H. The function of HJB and debt price are the same as benchmark so the function of long-term debt issuance is also the same, i.e, when $f < f_\dagger$,

$$g_H(f) = \frac{(\rho-r)\left(j_L'(f)-j_H'(f)\right)}{f j_H''(f)}$$

and $g_H(f) = 0$ when $f > f_\dagger$.

Region $(\rho+\lambda-r)\, j_L(f) \geq (\rho-r)\, j_H(f)$. The HJB is

$$(\rho+\lambda-\mu_H)\, j_H(f) = 1-(r+\xi)\, f-(\mu_H+\xi)\, f j_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f)+(\rho-r+\lambda) j_L(f).$$

The solution is

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1}$$

where

$$\begin{aligned}
B_0 &= \frac{\rho+\lambda+\zeta-\mu_L}{(\rho+\lambda-\mu_H)\,(r+\zeta-\mu_L)} \\
B_1 &= \frac{(r+\xi)\,(\rho+\lambda+\zeta+\xi)}{(\rho+\xi+\lambda)\,(r+\zeta+\xi)} \\
B_2 &= \frac{\rho+\lambda-r}{(\rho+\lambda-\mu_H)+(\mu_H+\xi)\,\gamma_1-\frac{1}{2}\sigma^2\gamma_1\,(\gamma_1-1)} A_{2.}
\end{aligned}$$

and

$$\phi_1 = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\,(\rho+\lambda-\mu_H)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\phi^2-\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)\phi-(\rho+\lambda-\mu_H) = 0.$$

The coefficient B_3 will be determined by the boundary conditions.

Region $(\rho+\lambda-r)\, j_L(f) < (\rho-r)\, j_H(f)$. The HJB is

$$(r+\lambda-\mu_H)\, j_H(f) = 1-(r+\xi)\, f-(\mu_H+\xi)\, f j_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f).$$

The solution is

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2},$$

where

$$\begin{aligned}
D_0 &= \frac{1}{r+\lambda-\mu_H} \\
D_1 &= \frac{r+\xi}{r+\xi+\lambda}
\end{aligned}$$

and

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 \left(r + \lambda - \mu_H\right)}}{\sigma^2} > 1,$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 \left(r + \lambda - \mu_H\right)}}{\sigma^2} < 0$$

solve

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta + \mu_H - r - \lambda = 0.$$

The coefficients (D_2, D_3) will be determined by the boundary conditions.

Long-term debt issuance at $f = 0$. In the region $f < f_\dagger$, we know

$$\begin{aligned}
j'_H(f) &= -B_1 + \gamma_1 B_2 f^{\gamma_1 - 1} + \phi_1 B_3 f^{\phi_1 - 1} \\
j''_H(f) &= \gamma_1 \left(\gamma_1 - 1\right) B_2 f^{\gamma_1 - 2} + \phi_1 \left(\phi_1 - 1\right) B_3 f^{\phi_1 - 2} \\
j'_L(f) &= -A_1 + \gamma_1 A_2 f^{\gamma_1 - 1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
g_H(f) f &= \frac{(\rho - r)\left(j'_L(f) - j'_H(f)\right)}{j''_H(f)} \\
&= \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1\left(A_2 - B_2\right) f^{\gamma_1 - 1} - \phi_1 B_3 f^{\phi_1 - 1}\right)}{B_2\gamma_1(\gamma_1 - 1) f^{\gamma_1 - 2} + B_3\phi_1\left(\phi_1 - 1\right) f^{\phi_1 - 2}}.
\end{aligned}$$

Given that $\min\{\phi_1, \gamma_1\} > 1$ it becomes immediately clear that as long as $\min\{\phi_1, \gamma_1\} \geq 2$, $\lim_{f\to 0} g_H(f) f > 0$. To see this, assume $\phi_1 > \gamma_1$. If $\gamma_1 > 2$,

$$\lim_{f\to 0} g_H(f) f = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2\left(A_2 - B_2\right) f^{\gamma_1 - 1} - \phi_1^2 B_3 f^{\phi_1 - 1}\right)}{B_2\gamma_1(\gamma_1 - 1)^2 f^{\gamma_1 - 2} + B_3\phi_1\left(\phi_1 - 1\right)^2 f^{\phi_1 - 2}} = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{0} = \infty.$$

If $\gamma_1 = 2$,

$$
\begin{aligned}
\lim_{f\to 0} g_H(f)f &= \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2 (A_2 - B_2) f^{\gamma_1-1} - \phi_1^2 B_3 f^{\phi_1-1}\right)}{B_2\gamma_1(\gamma_1-1)^2 f^{\gamma_1-2} + B_3\phi_1(\phi_1-1)^2 f^{\phi_1-2}} \\
&= \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{B_2\gamma_1(\gamma_1-1)^2} = \frac{(\rho-r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{2B_2} \\
&= \frac{2(\rho+\lambda-r-\zeta+\mu_H-\mu_L)(\rho-r)\zeta(r+\zeta+\xi)}{(\rho+\lambda-r)(r+\zeta-\mu_L)(r+\xi)(\rho+\xi+\lambda)}.
\end{aligned}
$$

The results are similar if $\phi_1 < \gamma_1$. The condition $\zeta > \underline{\zeta}$ is required for $\gamma_1 \geq 2$.

The condition $(\rho+\lambda-r)\, j_L(0) > (\rho-r)\, j_H(0)$. Note that the assumption $f_\dagger > 0$ requires

$$
\begin{aligned}
& (\rho+\lambda-r)\, j_L(0) > (\rho-r)\, j_H(0) \\
\Rightarrow \quad & (\rho+\lambda-r)\, A_0 - (\rho-r)\, B_0 > 0 \\
\Rightarrow \quad & \zeta < \frac{\lambda}{\rho-r}(\rho+\lambda-\mu_H) - (\mu_H-\mu_L).
\end{aligned}
$$

□

Proof of Proposition 4

Proof. In the low state, short-term debt is riskless and the borrower never defaults. The borrower chooses short-term debt $D_t = X j_L$, so the value of the firm is

$$\tilde{J}_L(X) = \frac{X}{r-\mu_L}.$$

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage, in which case

$$\hat{J}_H(X) = \frac{X}{r+\lambda-\mu_H}.$$

On the other hand, if she borrows riskless debt up to $X_t j_L$, the firm value is

$$\widehat{J}_H(X) = \frac{X}{\rho+\lambda-\mu_H}\left(1+\frac{\rho-r+\lambda}{r-\mu_L}\right).$$

From here we get that the value of the firm is

$$\tilde{J}_H(X) = X \max\left\{\frac{1}{r+\lambda-\mu_H}, \frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho-r+\lambda}{r-\mu_L}\right)\right\}$$

Finally, $\tilde{J}_L(X) \geq J_L(X,F) + p_L(X,F)\,F$ is straightforward given the former is the first-best firm value. In the high state, this is equivalent to proving $\tilde{j}_H \geq j_H(f) + p_H(f)\,f$. It is easily verified that $\tilde{j}_H \geq j_H(0)$ (and the equality holds for both cases no matter the value of λ). The result follows from

$$\frac{d\left[j_H(f) + p_H(f)\,f\right]}{df} = p_H'(f)f < 0.$$

$\square$

Proof of Proposition 5

Proof. In the low state, the HJB becomes

$$\rho\tilde{V}_L = X - (r+\xi)\,F - \frac{\partial\tilde{V}_L}{\partial F}\xi F + \frac{\partial\tilde{V}_L}{\partial X}\mu_L X + \frac{1}{2}\sigma^2 X^2\frac{\partial^2\tilde{V}_L}{\partial X^2}.$$

Again, let $\tilde{V}_L = X\tilde{v}_L$ so that

$$\begin{aligned}\frac{\partial\tilde{V}_L}{\partial F} &= \tilde{v}_L' \\ \frac{\partial\tilde{V}_L}{\partial X} &= \tilde{v}_L - f\tilde{v}_L' \\ X\frac{\partial^2\tilde{V}_L}{\partial X^2} &= f^2\tilde{v}_L''.\end{aligned}$$

The scaled HJB becomes

$$(\rho-\mu_L)\,\tilde{v}_L = 1 - (r+\xi)\,f - (\mu_L+\xi)\,f\tilde{v}_L' + \frac{1}{2}\sigma^2 f^2\tilde{v}_L''.$$

Using the conditions

$$\begin{aligned}\lim_{f\to 0}\tilde{v}_L(f) &< \infty \\ \tilde{v}_L\left(\tilde{f}_L^b\right) &= 0 \\ \tilde{v}_L'\left(\tilde{f}_L^b\right) &= 0,\end{aligned}$$

we obtain the solution

$$\tilde{v}_L(f) = \frac{1}{\rho - \mu_L} - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi} \frac{\tilde{f}_L^b}{\tilde{\gamma}} \left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}}, \qquad \tilde{f}_L^b = \frac{1}{\rho-\mu_L} \frac{\tilde{\gamma}}{\tilde{\gamma}-1} \frac{\rho+\xi}{r+\xi}$$

where

$$\tilde{\gamma} = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho - \mu_L)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2 \tilde{\gamma}^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\tilde{\gamma} - (\rho - \mu_L) = 0.$$

In a smooth equilibrium, $\tilde{p}_L = -\tilde{v}_L'$, and $\tilde{p}_L$ satisfies

$$(r+\xi)\,\tilde{p}_L = (r+\xi) + \left(g_L - \xi - \mu_L + \sigma^2\right) f \tilde{p}_L' + \frac{1}{2}\sigma^2 f^2 \tilde{p}_L''.$$

Differentiating once the HJB for $\tilde{v}_L$, we get

$$\tilde{g}_L = \frac{(\rho - r)\,\tilde{p}_L}{f \tilde{v}_L''}.$$

In the high state , the scaled HJB becomes

$$(\rho - \mu_H)\,\tilde{v}_H = 1 - (r+\xi) f - (\mu_H + \xi) f \tilde{v}_H' + \frac{1}{2}\sigma^2 f^2 \tilde{v}_H'' + \lambda(\tilde{v}_L - \tilde{v}_H).$$

Using conditions

$$\begin{aligned} \lim_{f\to 0} \tilde{v}_H(f) &< \infty \\ \tilde{v}_H\left(\tilde{f}_H^b\right) &= 0 \\ \tilde{v}_H'\left(\tilde{f}_H^b\right) &= 0, \end{aligned}$$

we obtain the solution

$$\tilde{v}_H(f) = \tilde{u}_0(f) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi},$$

where

$$\tilde{u}_0(f) = \frac{1}{\rho-\mu_L}\frac{\rho+\lambda-\mu_L}{\rho+\lambda-\mu_H} - \frac{r+\xi}{\rho+\xi}f + \frac{\lambda\frac{r+\xi}{\rho+\xi}}{(\rho+\lambda-\mu_H)+\tilde{\gamma}(\mu_H+\xi)-\frac{1}{2}\sigma^2\tilde{\gamma}(\tilde{\gamma}-1)}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

and

$$\phi = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} > 1$$

solves

$$\frac{1}{2}\sigma^2\phi^2 - \left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)\phi - (\rho+\lambda-\mu_H) = 0.$$

Finally, the boundary $\tilde{f}_H^b$ is pinned down by the smooth-pasting condition

$$\tilde{f}_H^b\tilde{u}_0'\left(\tilde{f}_H^b\right) - \phi\tilde{u}_0\left(\tilde{f}_H^b\right) = 0.$$

In a smooth equilibrium, $\tilde{p}_H = -\tilde{v}_H'$, and $\tilde{p}_H$ satisfies

$$(r+\xi)\tilde{p}_H = (r+\xi) + \left(\tilde{g}_H-\xi-\mu_H+\sigma^2\right)f\tilde{p}_H' + \frac{1}{2}\sigma^2 f^2\tilde{p}_H'' + \lambda(\tilde{p}_L-\tilde{p}_H).$$

Differentiating once the HJB for $\tilde{v}_H$, we get

$$\tilde{g}_H = \frac{(\rho-r)\tilde{p}_H}{f\tilde{v}_H''}.$$

Lemma 7. *$j_L(f) \geq \tilde{v}_L(f)$ and $f_L^b \geq \tilde{f}_L^b$*

Proof.

$$\begin{aligned}
J_L(X,F) &= \sup_{\tau_b,\{G_s,D_s\leq J_L(X_s,F_s)\}}\mathbb{E}\left[\int_t^{\tau_b}e^{-\rho(s-t)}\Big(X_s-(r+\xi)F_s+(\rho-y_{s-})D_{s-}\Big)ds+p_s dG_s\Big)\Big|X_t=X,\ F_t=F\right]\\
&> \sup_{\tau_b,\{G_s\}}\mathbb{E}\left[\int_t^{\tau_b}e^{-\rho(s-t)}\Big(X_s-(r+\xi)F_s\Big)ds+p_s dG_s\Big)\Big|X_t=X,\ F_t=F\right]\\
&= \sup_{\tau_b}\mathbb{E}\left[\int_t^{\tau_b}e^{-\rho(s-t)}\Big(X_s-(r+\xi)F_s\Big)ds\Big)\Big|X_t=X,\ F_t=F\right] = \tilde{V}_L(X,F) \qquad (44)
\end{aligned}$$

where the inequality comes from that $(\rho-y_{s-})D_{s-}$ is positive.It implies that the value function in the low state in the benchmark is higher than that in the economy with long-term debt only for any (X, F).

Let T_L and $\tilde{\tau}_{bL}$ be the default time in both economies in the low state. We have that $\tilde{v}_L(f_{\tilde{\tau}_{bL}}) = 0 \Rightarrow j_L(f_{\tilde{\tau}_{bL}}) > 0$, which means that $T_L > \tilde{\tau}_{bL}$. It follows then that $f_L^b > \tilde{f}_L^b$. □

Lemma 8. *$\tilde{p}_L(f) \leq p_L(f)$, and the inequality is strict for $\forall f > 0$.*

Proof. Given $f_L^b > \tilde{f}_L^b$ and $\tilde{g}_L(f) > g_L(f) = 0$ and $T_L > \tilde{T}_L$, the result follows from the definition

$$
\begin{aligned}
p_L &= \mathbb{E}\left[\int_t^{\tau_\xi \wedge T_L} e^{-r(s-t)} r ds + e^{-r(T_L - t)} \mathbb{1}_{T_L > \tau_\xi}\right] \\
\tilde{p}_L &= \mathbb{E}\left[\int_t^{\tau_\xi \wedge \tilde{\tau}_{bL}} e^{-r(s-t)} r ds + e^{-r(\tilde{\tau}_{bL} - t)} \mathbb{1}_{\tilde{\tau}_{bL} > \tau_\xi}\right].
\end{aligned}
$$

□

Lemma 9. *$j_H(f) \geq \tilde{v}_H(f)$ and $f_H^b \geq \tilde{f}_H^b$*

Proof. We define an auxiliary process z_t that satisfies $z_0 = f_0$ and

$$
dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t.
$$

The value functions are equivalently

$$
\begin{aligned}
j_H(z) &\equiv \sup_{\tau_b, d_t \in [0, j_H(z_t)]} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) z_t + (\rho - r) d_t + \mathbb{1}_{\{d_t \leq j_L(f_t)\}} \cdot \lambda j_L(z_t)\right) dt\right] \\
\tilde{v}_H(z) &\equiv \sup_{\tau_b} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) z_t + \lambda \tilde{v}_L(z_t)\right) dt\right],
\end{aligned}
$$

where $\hat{\rho} \equiv \rho + \lambda - \mu_H$. Note that

$$
\begin{aligned}
j_H(f) &\geq \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + (\rho - r) j_L(f_t) + \lambda j_L(f_t)\right) dt\right] \\
&> \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + \lambda j_L(f_t)\right) dt\right] \\
&\geq \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + \lambda \tilde{v}_L(f_t)\right) dt\right] \\
&= \tilde{v}_H(f).
\end{aligned}
$$

It follows that $f_H^b > \tilde{f}_H^b$. $\square$

Lemma 10. *There is* $0 \leq \underline{f} \leq f_\dagger \leq \overline{f} \leq \tilde{f}_H^b$ *such that* $\tilde{p}_H(f) \leq p_H(f)$ *on* $[0, \underline{f}] \cup [\overline{f}, \tilde{f}_H^b]$

Proof. Define $\Delta_H(f) = p_H(f) - \tilde{p}_H(f)$. We get

- On $f \in (0, f_\dagger)$ It should be

$$(\rho + \lambda + \xi)\,\Delta_H(f) = (\rho - r + \lambda)\,\Delta_L(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta'_H(f) + \frac{1}{2}\sigma^2 f^2 \Delta''_H(f),$$

  where we have used the condition

$$g_H f p'_H - \tilde{g}_H f \tilde{p}'_H = -(\rho - r)(p_H - p_L) + (\rho - r)(\tilde{p}_H - \tilde{p}_L) = -(\rho - r)\,\Delta_H + (\rho - r)\,\Delta_L.$$

  From here we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta''_H(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta'_H(f) - (\rho + \lambda + \xi)\,\Delta_H(f) \leq 0.$$

  By the maximum principle, $\Delta_H(f)$ cannot have a nonpositive minimum. In addition, $\Delta_H(0) = 1 - \frac{r+\xi}{\rho+\xi} > 0$. Hence, $\Delta_H(f)$ single crosses 0 from above when f starts from $f = 0$.

- On $f \in (f_\dagger, \tilde{f}_H^b)$

$$\frac{1}{2}\sigma^2 f^2 \Delta''_H(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta'_H(f) - (\rho + \xi)\,\Delta_H(f) + (\rho - r - \lambda)\, p_H(f) + \lambda\left(\tilde{p}_H(f) - \tilde{p}_L(f)\right) = 0.$$

  Given $\rho > r + \lambda$, we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta''_H(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta'_H(f) - (\rho + \xi)\,\Delta_H(f) \leq 0.$$

  It follows that $\Delta_H(f)$ cannot have a nonpositive minimum. In addition, $\Delta_H(\tilde{f}_H^b) \geq 0$ since $f_H^b \geq \tilde{f}_H^b$. Hence, $\Delta_H(f)$ single crosses 0 from below when f goes to $f = \tilde{f}_H^b$.

$\square$

$\square$

C Proofs of Section 4

C.1 Jump Risk

The value function satisfies the equation

$$(\rho + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + \max\Big\{(\rho - r) j(\eta f) + \lambda j(\eta f), (\rho - r) j(f)\Big\}$$

In the region $(f_\dagger, f^b)$, the equation reduces to

$$(r + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f),$$

while in the region $(f_\dagger, f^b)$, the equation reduces to

$$(\rho + \lambda - \mu)\, j(f) = 1 - (r + \xi)\, f - (\mu + \xi)\, f j'(f) + \frac{1}{2}\sigma^2 f^2 j''(f) + (\rho + \lambda - r) j(\eta f).$$

The threshold $f_\dagger$ is determined by the condition

$$(\rho + \lambda - r) j(\eta f_\dagger) = (\rho - r) j(f_\dagger)$$

C.2 Model solution under $\lambda_{LH} > 0$

C.2.1 Region $(\rho + \lambda_{HL} - r)\, j_L(f) \geq (\rho - r)\, j_H(f)$:

Guess a solution of the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2} + A_4 f^{\gamma_3} + A_5 f^{\gamma_4}$$
$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2} + B_4 f^{\gamma_3} + B_5 f^{\gamma_4}.$$

$$
\begin{aligned}
&(r+\lambda_{LH}-\mu_L)\left(A_0-A_1 f+A_2 f^{\gamma_1}+A_3 f^{\gamma_2}+A_4 f^{\gamma_3}+A_5 f^{\gamma_4}\right)\\
&=1-(r+\xi) f-(\mu_L+\xi)\left(-A_1 f+A_2\gamma_1 f^{\gamma_1}+A_3\gamma_2 f^{\gamma_2}+A_4\gamma_3 f^{\gamma_3}+A_5\gamma_4 f^{\gamma_4}\right)\\
&+\lambda_{LH}(B_0-B_1 f+B_2 f^{\gamma_1}+B_3 f^{\gamma_2}+B_4 f^{\gamma_3}+B_5 f^{\gamma_4})\\
&+\frac{1}{2}\sigma^2\left(A_2\gamma_1(\gamma_1-1) f^{\gamma_1}+A_3\gamma_2(\gamma_2-1) f^{\gamma_2}+A_4\gamma_3(\gamma_3-1) f^{\gamma_3}+A_5\gamma_4(\gamma_4-1) f^{\gamma_4}\right)\\
&(\rho+\lambda_{HL}-\mu_H)\left(B_0-B_1 f+B_2 f^{\gamma_1}+B_3 f^{\gamma_2}+B_4 f^{\gamma_3}+B_5 f^{\gamma_4}\right)\\
&=1-(r+\xi) f+(\rho+\lambda_{HL}-r)\left(A_0-A_1 f+A_2 f^{\gamma_1}+A_3 f^{\gamma_2}+A_4 f^{\gamma_3}+A_5 f^{\gamma_4}\right)\\
&-(\mu_H+\xi)\left(-B_1 f+B_2\gamma_1 f^{\gamma_1}+B_3\gamma_2 f^{\gamma_2}+B_4\gamma_3 f^{\gamma_3}+B_5\gamma_4 f^{\gamma_4}\right)\\
&+\frac{1}{2}\sigma^2\left(B_2\gamma_1(\gamma_1-1) f^{\gamma_1}+B_3\gamma_2(\gamma_2-1) f^{\gamma_2}+B_4\gamma_3(\gamma_3-1) f^{\gamma_3}+B_5\gamma_4(\gamma_4-1) f^{\gamma_4}\right)
\end{aligned}
$$

From here we get

$$
\begin{aligned}
(r+\lambda_{LH}-\mu_L) A_0 &= 1+\lambda_{LH} B_0\\
(\rho+\lambda_{HL}-\mu_H) B_0 &= 1+(\rho+\lambda_{HL}-r) A_0\\
(r+\xi+\lambda_{LH}) A_1 &= (r+\xi)+\lambda_{LH} B_1\\
(\rho+\xi+\lambda_{HL}) B_1 &= (r+\xi)+(\rho+\lambda_{HL}-r) A_1
\end{aligned}
$$

where we can solve for $\{A_0,\, B_0\}$ analytically where

$$
\begin{aligned}
A_0 &= \frac{\rho+\lambda_{HL}+\lambda_{LH}-\mu_H}{(\rho+\lambda_{HL}-\mu_H)(r-\mu_L)+\lambda_{LH}(r-\mu_H)},\\
B_0 &= \frac{1+(\rho+\lambda_{HL}-r) A_0}{\rho+\lambda_{HL}-\mu_H}.
\end{aligned}
$$

and $A_1 = B_1 = 1$.

and

$$\begin{aligned}
(r+\lambda_{LH}-\mu_L)\,A_2 &= -(\mu_L+\xi)\,A_2\gamma_1+\lambda_{LH}B_2+\frac{1}{2}\sigma^2A_2\gamma_1\,(\gamma_1-1)\\
(r+\lambda_{LH}-\mu_L)\,A_3 &= -(\mu_L+\xi)\,A_3\gamma_2+\lambda_{LH}B_3+\frac{1}{2}\sigma^2A_3\gamma_2\,(\gamma_2-1)\\
(r+\lambda_{LH}-\mu_L)\,A_4 &= -(\mu_L+\xi)\,A_4\gamma_3+\lambda_{LH}B_4+\frac{1}{2}\sigma^2A_4\gamma_3\,(\gamma_3-1)\\
(r+\lambda_{LH}-\mu_L)\,A_5 &= -(\mu_L+\xi)\,A_5\gamma_4+\lambda_{LH}B_5+\frac{1}{2}\sigma^2A_5\gamma_4\,(\gamma_4-1)\\
(\rho+\lambda_{HL}-\mu_H)\,B_2 &= (\rho+\lambda_{HL}-r)\,A_2-(\mu_H+\xi)\,B_2\gamma_1+\frac{1}{2}\sigma^2B_2\gamma_1\,(\gamma_1-1)\\
(\rho+\lambda_{HL}-\mu_H)\,B_3 &= (\rho+\lambda_{HL}-r)\,A_3-(\mu_H+\xi)\,B_3\gamma_2+\frac{1}{2}\sigma^2B_3\gamma_2\,(\gamma_2-1)\\
(\rho+\lambda_{HL}-\mu_H)\,B_4 &= (\rho+\lambda_{HL}-r)\,A_4-(\mu_H+\xi)\,B_4\gamma_3+\frac{1}{2}\sigma^2B_4\gamma_3\,(\gamma_3-1)\\
(\rho+\lambda_{HL}-\mu_H)\,B_5 &= (\rho+\lambda_{HL}-r)\,A_5-(\mu_H+\xi)\,B_5\gamma_4+\frac{1}{2}\sigma^2B_5\gamma_4\,(\gamma_4-1)
\end{aligned}$$

If we multiply the equation for A_2 by γ_1, we get

$$(r+\lambda_{LH}-\mu_L)\,\gamma_1A_2=-(\mu_L+\xi)\,A_2\gamma_1^2+\lambda_{LH}B_2\gamma_1+\frac{1}{2}\sigma^2A_2\gamma_1^2\,(\gamma_1-1)$$

when $\lambda_{LH}\neq 0$, from the equation for A_2 we have

$$\lambda_{LH}B_2=\left[(r+\lambda_{LH}-\mu_L)+(\mu_L+\xi)\,\gamma_1-\frac{1}{2}\sigma^2\gamma_1\,(\gamma_1-1)\right]A_2.$$

Substituting in the equation for B_2, we get

$$B_2\gamma_1=\frac{(\rho+\lambda_{HL}-r)}{(\mu_H+\xi)}A_2-\frac{\left(\rho+\lambda_{HL}-\mu_H-\frac{1}{2}\sigma^2\gamma_1\,(\gamma_1-1)\right)}{\lambda_{LH}\,(\mu_H+\xi)}\left[(r+\lambda_{LH}-\mu_L)+(\mu_L+\xi)\,\gamma_1-\frac{1}{2}\sigma^2\gamma_1\,(\gamma_1-1)\right]A_2.$$

Substituting back in the equation for A_2 (multiplied by γ_1), and canceling A_2, we get

$$\begin{gathered}
0=-\frac{1}{4}\sigma^4\gamma_1^4+\left(\frac{1}{2}\sigma^2\left(\mu_L+\mu_H+2\xi+\sigma^2\right)\right)\gamma_1^3+\\
+\left[-(\mu_L+\xi)\,(\mu_H+\xi)+\frac{1}{2}\sigma^2\,(\rho+\lambda_{HL}+r+\lambda_{LH}-2\,(\mu_L+\mu_H+\xi))-\frac{1}{4}\sigma^4\right]\gamma_1^2+\\
+\left[-\left(\frac{1}{2}\sigma^2+\mu_H+\xi\right)(r+\lambda_{LH}-\mu_L)-(\rho+\lambda_{HL}-\mu_H)\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\right]\gamma_1\\
+(\mu_H-r)\,\lambda_{LH}-(\rho+\lambda_{HL}-\mu_H)\,(r-\mu_L)
\end{gathered}$$

This equation has four roots. Numerically I find under reasonable parameters, all the four roots are real and two of them are positive. Therefore, the transversality conditions

$$\lim_{f\to 0} j_H(f) < \infty,$$
$$\lim_{f\to 0} j_L(f) < \infty,$$

implies that $A_4 = A_5 = B_4 = B_5 = 0$.

C.2.2 Region $(\rho + \lambda_{HL} - r)\, j_L(f) < (\rho - r)\, j_H(f)$:

We can also look for a solution of the form

$$\begin{aligned} j_L(f) &= C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4} \\ j_H(f) &= D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}. \end{aligned}$$

From the equation for $j_H(f)$ we get

$$\begin{aligned} (\rho + \lambda_{HL} - \mu_H)\left(D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}\right) &= 1 - (r+\xi)\, f + (\rho - r)\left(D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}\right) \\ &- (\mu_H + \xi)\left(-D_1 f + \beta_1 D_2 f^{\beta_1} + \beta_2 D_3 f^{\beta_2}\right) + \frac{1}{2}\sigma^2 \left(\left(\beta_1 - 1\right)\beta_1 D_2 f^{\beta_1} + \left(\beta_2 - 1\right)\beta_2 D_3 f^{\beta_2}\right) \end{aligned}$$

so

$$\begin{aligned} D_0 &= \frac{1}{r + \lambda_{HL} - \mu_H} \\ D_1 &= \frac{r+\xi}{r + \xi + \lambda_{HL}} \\ (\rho + \lambda_{HL} - \mu_H) D_2 &= (\rho - r)\, D_2 - (\mu_H + \xi)\, \beta_1 D_2 + \frac{1}{2}\sigma^2 \left(\beta_1 - 1\right)\beta_1 D_2 \\ (\rho + \lambda_{HL} - \mu_H) D_3 &= (\rho - r)\, D_3 - (\mu_H + \xi)\, \beta_2 D_3 + \frac{1}{2}\sigma^2 \left(\beta_2 - 1\right)\beta_2 D_3 \end{aligned}$$

It can be rewritten as

$$\frac{1}{2}\sigma^2 \beta_1^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta_1 + \mu_H - r - \lambda_{HL} = 0$$

This equation has two roots and let

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2},$$
$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2}.$$

Next, we look at the solution for j_L.

$$\begin{gathered}(r + \lambda_{LH} - \mu_L)\left(C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4}\right) = 1 - (r + \xi) f \\ - (\mu_L + \xi)\left(-C_1 f + C_2\beta_1 f^{\beta_1} + C_3\beta_2 f^{\beta_2} + C_4\beta_3 f^{\beta_3} + C_5\beta_4 f^{\beta_4}\right) + \lambda_{LH}(D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}) \\ + \frac{1}{2}\sigma^2\left((\beta_1 - 1)\beta_1 C_2 f^{\beta_1} + (\beta_2 - 1)\beta_2 C_3 f^{\beta_2} + (\beta_3 - 1)\beta_3 C_4 f^{\beta_3} + (\beta_4 - 1)\beta_4 C_5 f^{\beta_4}\right)\end{gathered}$$

From here we get

$$\begin{gathered}C_0 = \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} - \mu_L} \\ C_1 = \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH}} \\ (r + \lambda_{LH} - \mu_L) C_2 = -(\mu_L + \xi) C_2\beta_1 + \lambda_{LH} D_2 + \frac{1}{2}\sigma^2(\beta_1 - 1)\beta_1 C_2 \\ (r + \lambda_{LH} - \mu_L) C_3 = -(\mu_L + \xi) C_3\beta_2 + \lambda_{LH} D_3 + \frac{1}{2}\sigma^2(\beta_2 - 1)\beta_2 C_3 \\ (r + \lambda_{LH} - \mu_L) C_4 = -(\mu_L + \xi) C_4\beta_3 + \frac{1}{2}\sigma^2(\beta_3 - 1)\beta_3 C_4 \\ (r + \lambda_{LH} - \mu_L) C_5 = -(\mu_L + \xi) C_5\beta_4 + \frac{1}{2}\sigma^2(\beta_4 - 1)\beta_4 C_5\end{gathered}$$

which can simplified to

$$
\begin{aligned}
C_0 &= \frac{1+\lambda_{LH} D_0}{r+\lambda_{LH}-\mu_L} \\
C_1 &= \frac{r+\xi+\lambda_{LH} D_1}{r+\xi+\lambda_{LH}} \\
C_2 &= \frac{\lambda_{LH} D_2}{r+\lambda_{LH}-\mu_L+(\mu_L+\xi)\,\beta_1-\frac{1}{2}\sigma^2\,(\beta_1-1)\,\beta_1} \\
C_3 &= \frac{\lambda_{LH} D_3}{r+\lambda_{LH}-\mu_L+(\mu_L+\xi)\,\beta_2-\frac{1}{2}\sigma^2\,(\beta_2-1)\,\beta_2} \\
&\frac{1}{2}\sigma^2\beta_3^2-\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)\beta_3-(r+\lambda_{LH}-\mu_L)=0
\end{aligned}
$$

The last equation has two roots and let

$$
\begin{aligned}
\beta_3 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\,(r+\lambda_{LH}-\mu_L)}}{\sigma^2}>1, \\
\beta_4 &= \frac{\mu_L+\xi+\frac{1}{2}\sigma^2-\sqrt{\left(\mu_L+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\,(r+\lambda_{LH}-\mu_L)}}{\sigma^2}<0.
\end{aligned}
$$

C.2.3 Equilibrium Issuance

In the region $f<f^{\dagger}$ we have (where $A_1=B_1=1$)

$$
\begin{aligned}
j_L(f) &= A_0-f+A_2 f^{\gamma_1}+A_3 f^{\gamma_2} \\
j_H(f) &= B_0-f+B_2 f^{\gamma_1}+B_3 f^{\gamma_2}.
\end{aligned}
$$

so

$$
g_H(f)=\frac{(\rho-r)\left(j_L'(f)-j_H'(f)\right)}{f j_H''(f)}=\frac{(\rho-r)\left(\gamma_1\,(A_2-B_2)\,f^{\gamma_1-1}+\gamma_2\,(A_3-B_3)\,f^{\gamma_2-1}\right)}{B_2\gamma_1(\gamma_1-1)f^{\gamma_1-1}+B_3\gamma_2\,(\gamma_2-1)\,f^{\gamma_2-1}}
$$

C.3 Shock elasticity

The next Proposition provides a formula for computing the shock elasticity.

Proposition 10. *Consider the process $Y_t=\varphi(f_t,\theta_t)$. The shock elasticity of Y_t is*

$$
\varepsilon_Y^B(t,f,\theta)\equiv\frac{\mathbb{E}[\varphi'(f_t,\theta_t)\mathcal{D}_0 f_t \mathbb{1}_{\{t<\tau_b\}}|f_0=f,\theta_0=\theta]}{\mathbb{E}[\varphi(f_t,\theta_t)\mathbb{1}_{\{t<\tau_b\}}|f_0=f,\theta_0=\theta]},
$$

where

$$\mathcal{D}_0 f_t = -\sigma f_t \exp\left\{\int_0^t g'_{\theta_s}(f_s) f_s ds\right\},$$

is the Malliavin derivative of f_t. In particular,

$$\varepsilon_f^B(t,f,\theta) = -\sigma \frac{\mathbb{E}\left[f_t \exp\left\{\int_0^t g'_{\theta_s}(f_s) f_s ds\right\} \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, \theta_0 = \theta\right]}{\mathbb{E}\left[f_t \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, \theta_0 = \theta\right]}$$

$$\varepsilon_d^B(t,f,\theta) = -\sigma \frac{\mathbb{E}\left[d'_{\theta_t}(f_t) f_t \exp\left\{\int_0^t g'_{\theta_s}(f_s) f_s ds\right\} \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, \theta_0 = \theta\right]}{\mathbb{E}\left[d_{\theta_t}(f_t) \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, \theta_0 = \theta\right]}.$$

The shock elasticities of F_t and D_t are given by

$$\varepsilon_F^B(t,f,\theta) = \frac{\mathbb{E}\left[X_t\left(\sigma f_t + \mathcal{D}_0 f_t\right) \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}{\mathbb{E}\left[X_t f_t \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}$$

$$\varepsilon_D^B(t,f,\theta) = \frac{\mathbb{E}\left[X_t\left(\sigma d_t + d'_{\theta_t}(f_t)\mathcal{D}_0 f_t\right) \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}{\mathbb{E}\left[X_t d_t \mathbb{1}_{\{t<\tau_b\}} \middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}.$$

Proof. We start deriving the Malliavin derivative of f_t. Using Ito's Lemma we get a stochastic differential equation for $\log(f_t)$

$$d\log(f_t) = \left(g_{\theta_t}\left(e^{\log(f_t)}\right) - \xi - \mu_{\theta_t} + \frac{1}{2}\sigma^2\right) dt - \sigma dB_t.$$

The Malliavin derivative of $\log(f_t)$ is given by the solution to the differential equation

$$d\mathcal{D}_0 \log f_t = \left(g'_{\theta_t}(f_t) f_t\right) \mathcal{D}_0 \log f_t dt,$$

with initial condition $\mathcal{D}_0 \log f_0 = -\sigma$ (Hu et al., 2019). It follows from the previous equation that

$$\mathcal{D}_0 \log f_t = -\sigma \exp\left\{\int_0^t g'_{\theta_s}(f_s) f_s ds\right\}.$$

From here, we can find the derivative of f_t using the chain rule

$$\mathcal{D}_0 f_t = f_t \mathcal{D}_0 \log f_t = -\sigma f_t \exp\left\{\int_0^t g'_{\theta_s}(f_s) f_s ds\right\}.$$

It follows that $\mathcal{D}_0 Z_t = \mathcal{D}_0\varphi(f_t,\theta_t) = \varphi'(f_t,\theta_t)\mathcal{D}_0 f_t$. Next, we look at the Malliavin derivatives of F_t and D_t. These derivatives can be obtained using the product rule, which yield $\mathcal{D}_0 F_t = X_t\mathcal{D}_0 f_t + f_t\mathcal{D}_0 X_t$. Noticing that

$$d\log X_t = \left(\mu_{\theta_t} - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t,$$

we get $\mathcal{D}_0 \log X_t = \sigma$. Thus, using the chain rule we get $\mathcal{D}_0 X_t = \sigma X_t$. It follows that $\mathcal{D}_0 F_t = X_t\left(\mathcal{D}_0 f_t + \sigma f_t\right)$. Similarly, for short-term debt we have that $\mathcal{D}_0 D_t = X_t\mathcal{D}_0 d_t + d_t\mathcal{D}_0 X_t$, so $\mathcal{D}_0 D_t = \sigma\left(X_t d_t + d'_{\theta_t}(f_t)\mathcal{D}_0 f_t\right)$. □

C.4 Interest-rate risk

We offer detailed solution to the model with interest-rate risk introduced in subsection 4.4.

C.4.1 Model and HJB.

Let creditors' discount rates be $\left\{r_L^0, r_H^0, r_L^\xi, r_H^\xi\right\}$, where the subscripts stand for the state $\theta_t \in \{H, L\}$ and the superscripts represent the discount rate of short- and long-term creditors . We assume $\max\left\{r_L^0, r_H^0, r_L^\xi, r_H^\xi\right\} < \rho$. Moreover, we assume $\mu_H - \mu_L$ is sufficiently large so that in equilibrium, the value function always satisfies $j_H(f) > j_L(f)$, $\forall f$. Given so, the short rate is

$$y_H\left(d,f\right) = r_H^0 + \lambda_{HL}\mathbb{1}_{d>j_L(f)} \tag{45}$$

$$y_L\left(d,f\right) = r_L^0 \tag{46}$$

The HJBs are

$$\begin{aligned}
\left(\rho + \lambda_{LH} - \mu_L\right) j_L(f) = & \max_{d_L \le j_L(f)} 1 - \left(r_L^\xi + \xi\right) f + \left(\rho + \lambda_{LH} - y_L\right) d_L - \left(\mu_L + \xi\right) f j_L'(f) + \lambda_{LH}(j_H(f) - d_L)^+ \\
& + \frac{1}{2}\sigma^2 f^2 j_L''\left(f\right) \\
\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = & \max_{d_H \le j_H(f)} 1 - \left(r_H^\xi + \xi\right) f + \left(\rho + \lambda_{HL} - y_H\right) d_H - \left(\mu_H + \xi\right) f j_H'(f) + \lambda_{HL}(j_L(f) - d_H)^+ \\
& + \frac{1}{2}\sigma^2 f^2 j_H''\left(f\right)
\end{aligned}$$

Given $j_H(f) > j_L(f)$, $\forall f$, it continues to hold in state L that $d_L = j_L(f)$. In state H, there are still two candidates, $d_H \in \{j_L(f), j_H\left(f\right)\}$, and $d_H = j_L(f)$ iff

$$\left(\rho + \lambda_{HL} - r_H^0\right) j_L \ge \left(\rho - r_H^0\right) j_H.$$

The HJB in the L state becomes,

$$\left(r_L^0 + \lambda_{LH} - \mu_L\right) j_L = 1 - \left(r_L^\xi + \xi\right) f - \left(\mu_L + \xi\right) f j_L' + \lambda_{LH} j_H + \frac{1}{2}\sigma^2 f^2 j_L''. \tag{47}$$

In the H state, the HJB becomes

$$\begin{aligned}\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - \left(\mu_H + \xi\right) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ + \max\left\{(\rho + \lambda_{HL} - r_H^0) j_L, (\rho - r_H^0) j_H\right\}.\end{aligned}$$

Finally, the state variable evolves according to

$$df_t = \left(g_{\theta_t}(f_t) - \mu_{\theta_t} - \xi + \sigma^2\right) f_t dt - \sigma f_t dB_t.$$

C.4.2 Debt Price and Issuance.

The prices of debt satisfy the following HJB:

$$\begin{aligned}\left(r_L^\xi + \xi + \lambda_{LH}\right) p_L(f) &= \left(r_L^\xi + \xi\right) + \lambda_{LH} p_H(f) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f) \\ \left(r_H^\xi + \xi + \lambda_{HL}\right) p_H(f) &= \left(r_H^\xi + \xi\right) + \lambda_{HL} p_L(f) \mathbb{1}_{d_H \le j_L(f)} + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)\end{aligned}$$

From here we get,

$$\begin{aligned} g_L(f) &= \frac{\left(r_L^\xi + \xi + \lambda_{LH}\right) p_L(f) - \left(r_L^\xi + \xi\right) - \lambda_{LH} p_H(f) + \left(\xi + \mu_L - \sigma^2\right) f p_L'(f) - \frac{1}{2}\sigma^2 f^2 p_L''(f)}{f p_L'(f)} \\ &= \frac{\left(r_L^\xi + \xi + \lambda_{LH}\right) j_L'(f) + \left(r_L^\xi + \xi\right) - \lambda_{LH} j_H'(f) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)}. \\ g_H(f) &= \frac{\left(r_H^\xi + \xi + \lambda_{HL}\right) p_H(f) - \left(r_H^\xi + \xi\right) - \lambda_{HL} p_L(f) \mathbb{1}_{d_H \le j_L(f)} + \left(\xi + \mu_H - \sigma^2\right) f p_H'(f) - \frac{1}{2}\sigma^2 f^2 p_H''(f)}{f p_H'(f)} \\ &= \frac{\left(r_H^\xi + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^\xi + \xi\right) - \lambda_{HL} j_L'(f) \mathbb{1}_{d_H \le j_L(f)} + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f)}{f j_H''(f)}. \end{aligned}$$

We differentiate the HJB in the low state

$$\left(r_L^\xi + \xi + \lambda_{LH}\right) j_L'(f) + \left(r_L^\xi + \xi\right) - \lambda_{LH} j_H'(f) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f) = \left(r_L^\xi - r_L^0\right) j_L'(f).$$

Hence, we get

$$g_L(f) = \frac{\left(r_L^{\xi} - r_L^0\right) j_L'(f)}{f j_L''(f)} = \frac{\left(r_L^{\xi} - r_L^0\right) j_L'(f)}{-f p_L'(f)}.$$

Case 1: $\left(\rho + \lambda_{HL} - r_H^0\right) j_L \geq \left(\rho - r_H^0\right) j_H$. In this case, $d_H(f) = j_L(f)$, and we differentiate the HJB in the high state:

$$\begin{aligned}\left(r_H^{\xi} + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^{\xi} + \xi\right) - \lambda_{HL} j_L'(f) + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) \\ = \left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^{\xi}\right) j_H'(f)\end{aligned}$$

Hence,

$$g_H(f) = \frac{\left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^{\xi}\right) j_H'(f)}{f j_H''(f)} = \frac{\left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^{\xi}\right) j_H'(f)}{-f p_H'(f)}$$

Case 2: $\left(\rho + \lambda_{HL} - r_H^0\right) j_L < \left(\rho - r_H^0\right) j_H$. In this case, $d_H(f) = j_H(f)$, and we differentiate the HJB in the high state:

$$\left(r_H^{\xi} + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^{\xi} + \xi\right) + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = \left(r_H^{\xi} - r_H^0\right) j_H'(f)$$

so

$$g_H(f) = \frac{\left(r_H^{\xi} - r_H^0\right) j_H'(f)}{f j_H''(f)} = \frac{\left(r_H^{\xi} - r_H^0\right) j_H'(f)}{-f p_H'(f)}.$$

C.4.3 Solve the value function under $\lambda_{LH} = 0$

State $\theta = L$. Under $\lambda_{LH} = 0$,

$$\left(r_L^0 - \mu_L\right) j_L = 1 - \left(r_L^{\xi} + \xi\right) f - \left(\mu_L + \xi\right) f j_L' + \frac{1}{2}\sigma^2 f^2 j_L''. \tag{48}$$

Let

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Then

$$\begin{aligned}\left(r_L^0 - \mu_L\right)\left(A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}\right) = 1 - \left(r_L^{\xi} + \xi\right) f - \left(\mu_L + \xi\right)\left(-A_1 f + A_2 \gamma_1 f^{\gamma_1} + A_3 \gamma_2 f^{\gamma_2}\right) \\ + \frac{1}{2}\sigma^2\left(A_2 \gamma_1\left(\gamma_1 - 1\right) f^{\gamma_1} + A_3 \gamma_2\left(\gamma_2 - 1\right) f^{\gamma_2}\right)\end{aligned}$$

From here we get

$$\left(r_L^0 - \mu_L\right) A_0 = 1$$
$$\left(r_L^0 + \xi\right) A_1 = (r_L^\xi + \xi)$$

where we can solve for

$$A_0 = \frac{1}{r_L^0 - \mu_L},\ A_1 = \frac{r_L^\xi + \xi}{r_L^0 + \xi}$$

and

$$\left(r_L^0 - \mu_L\right) A_2 = -\left(\mu_L + \xi\right) A_2 \gamma_1 + \frac{1}{2}\sigma^2 A_2 \gamma_1 \left(\gamma_1 - 1\right)$$
$$\left(r_L^0 - \mu_L\right) A_3 = -\left(\mu_L + \xi\right) A_3 \gamma_2 + \frac{1}{2}\sigma^2 A_3 \gamma_2 \left(\gamma_2 - 1\right)$$

which implies

$$\frac{1}{2}\sigma^2 \gamma_1^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\gamma_1 - \left(r_L^0 - \mu_L\right) = 0$$

which has two roots and

$$\gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r_L^0 - \mu_L\right)}}{\sigma^2} > 1,$$
$$\gamma_2 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r_L^0 - \mu_L\right)}}{\sigma^2} < 0,$$

Since $\lim_{f\to 0} j_L(f) < \infty$, $A_3 = 0$ and

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1}$$

State $\theta = H$ and $(\rho + \lambda_{HL} - r_H^0) j_L \geq (\rho - r_H^0) j_H$. The HJB becomes

$$\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - \left(\mu_H + \xi\right) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) + (\rho + \lambda_{HL} - r_H^0) j_L \tag{49}$$

Guess a solution of the form

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1} + B_4 f^{\phi_2}.$$

$$
\begin{aligned}
&\left(\rho+\lambda_{HL}-\mu_H\right)\left(B_0-B_1 f+B_2 f^{\gamma_1}+B_3 f^{\phi_1}+B_4 f^{\phi_2}\right)\\
&=1-\left(r_H^{\xi}+\xi\right) f+\left(\rho+\lambda_{HL}-r_H^0\right)\left(A_0-A_1 f+A_2 f^{\gamma_1}\right)\\
&-\left(\mu_H+\xi\right)\left(-B_1 f+B_2\gamma_1 f^{\gamma_1}+\phi_1 B_3 f^{\phi_1}+\phi_2 B_4 f^{\phi_2}\right)\\
&+\frac{1}{2}\sigma^2\left(B_2\gamma_1\left(\gamma_1-1\right) f^{\gamma_1}+B_3\phi_1\left(\phi_1-1\right) f^{\phi_1}+B_4\phi_2\left(\phi_2-1\right) f^{\phi_2}\right)
\end{aligned}
$$

From here we get

$$
\begin{aligned}
\left(\rho+\lambda_{HL}-\mu_H\right) B_0&=1+\left(\rho+\lambda_{HL}-r_H^0\right) A_0\\
\left(\rho+\xi+\lambda_{HL}\right) B_1&=\left(r_H^{\xi}+\xi\right)+\left(\rho+\lambda_{HL}-r_H^0\right) A_1
\end{aligned}
$$

where we can solve for

$$
B_0=\frac{1+\left(\rho+\lambda_{HL}-r_H^0\right) A_0}{\rho+\lambda_{HL}-\mu_H}=\frac{1+\left(\rho+\lambda_{HL}-r_H^0\right)\frac{1}{r_L^0-\mu_L}}{\rho+\lambda_{HL}-\mu_H},
$$

$$
B_1=\frac{\left(r_H^{\xi}+\xi\right)+\left(\rho+\lambda_{HL}-r_H^0\right) A_1}{\rho+\xi+\lambda_{HL}}=\frac{\left(r_H^{\xi}+\xi\right)+\left(\rho+\lambda_{HL}-r_H^0\right)\frac{r_L^{\xi}+\xi}{r_L^0+\xi}}{\rho+\xi+\lambda_{HL}}
$$

In addition, we have

$$
\begin{aligned}
\left(\rho+\lambda_{HL}-\mu_H\right) B_2&=\left(\rho+\lambda_{HL}-r_H^0\right) A_2-\left(\mu_H+\xi\right) B_2\gamma_1+\frac{1}{2}\sigma^2 B_2\gamma_1\left(\gamma_1-1\right)\\
\left(\rho+\lambda_{HL}-\mu_H\right) B_3&=-\left(\mu_H+\xi\right)\phi_1 B_3+\frac{1}{2}\sigma^2 B_3\phi_1\left(\phi_1-1\right)\\
\left(\rho+\lambda_{HL}-\mu_H\right) B_4&=-\left(\mu_H+\xi\right)\phi_2 B_4+\frac{1}{2}\sigma^2 B_4\phi_2\left(\phi_2-1\right)
\end{aligned}
$$

which implies

$$
B_2=\frac{\rho+\lambda_{HL}-r_H^0}{\left(\rho+\lambda_{HL}-\mu_H\right)+\left(\mu_H+\xi\right)\gamma_1-\frac{1}{2}\sigma^2\gamma_1\left(\gamma_1-1\right)} A_2
$$

The function

$$
\frac{1}{2}\sigma^2\phi_1^2-\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)\phi_1-\left(\rho+\lambda_{HL}-\mu_H\right)=0
$$

has two roots where

$$\phi_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda_{HL} - \mu_H\right)}}{\sigma^2} > 1,$$
$$\phi_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda_{HL} - \mu_H\right)}}{\sigma^2} < 0.$$

The transversality conditions

$$\lim_{f\to 0} j_H(f) < \infty$$

imply that $B_4 = 0$. Hence,

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1}.$$

State $\theta = H$ and $(\rho + \lambda_{HL} - r_H^0) j_L < (\rho - r_H^0) j_H$. The HJB becomes

$$\left(r_H^0 + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - \left(\mu_H + \xi\right) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).$$

We can also look for a solution of the form

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.$$

From the equation for $j_H(f)$, we get

$$\left(r_H^0 + \lambda_{HL} - \mu_H\right)\left(D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}\right) = 1 - \left(r_H^\xi + \xi\right) f$$
$$- \left(\mu_H + \xi\right)\left(-D_1 f + \beta_1 D_2 f^{\beta_1} + \beta_2 D_3 f^{\beta_2}\right) + \frac{1}{2}\sigma^2\left(\left(\beta_1 - 1\right)\beta_1 D_2 f^{\beta_1} + \left(\beta_2 - 1\right)\beta_2 D_3 f^{\beta_2}\right)$$

so

$$D_0 = \frac{1}{r_H^0 + \lambda_{HL} - \mu_H}$$
$$D_1 = \frac{r_H^\xi + \xi}{r_H^0 + \xi + \lambda_{HL}}$$
$$\left(r_H^0 + \lambda_{HL} - \mu_H\right) D_2 = -\left(\mu_H + \xi\right)\beta_1 D_2 + \frac{1}{2}\sigma^2\left(\beta_1 - 1\right)\beta_1 D_2$$
$$\left(r_H^0 + \lambda_{HL} - \mu_H\right) D_3 = -\left(\mu_H + \xi\right)\beta_2 D_3 + \frac{1}{2}\sigma^2\left(\beta_2 - 1\right)\beta_2 D_3.$$

It can be rewritten as

$$\frac{1}{2}\sigma^2\beta_1^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta_1 + \mu_H - r_H^0 - \lambda_{HL} = 0.$$

This equation has two roots and let

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r_H^0 - \lambda_{HL}\right)}}{\sigma^2} > 1,$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r_H^0 - \lambda_{HL}\right)}}{\sigma^2} < 0.$$

C.4.4 Numerical Example

Parameters are as follows:

$$\rho = 0.1,\ r_H^0 = 0.015,\ r_H^\xi = 0.017,\ r_L^0 = 0.02,\ r_L^\xi = 0.018,\ \mu_H = 0.015,\ \mu_L = -0.2,\ \sigma = 0.1,\ \lambda_{HL} = 0.2.$$

Under these parameters, we get

$$f_L^b = 8.88$$
$$f_H^b = 14.41$$
$$f_\dagger = 4.16.$$

Numerically, the condition

$$(\rho + \lambda_{HL} - r_H^0)j_L \geq (\rho - r_H^0)j_H$$

holds in state H if and only if $f \leq f_\dagger$. Moreover, all the value functions are convex, confirming the equilibrium.

C.5 Upward Regime Shift

Suppose $\theta \in \{L, H, G\}$, where $\mu_G > \mu_H > \mu_L$. We use the notation G so that we do not need to change the notation in the benchmark model. The transitional intensity is $\lambda_{HG} = \lambda_G$ and $\lambda_{HL} = \lambda_L$, and the other intensities are zero.

In both $\theta = G$ and $\theta = L$, there is no long-term debt issuance. The value functions are

$$j_L(f) = \frac{1}{r-\mu_L} - f + \frac{f_L^b}{\gamma_L}\left(\frac{f}{f_L^b}\right)^{\gamma_L}$$
$$j_G(f) = \frac{1}{r-\mu_G} - f + \frac{f_G^b}{\gamma_G}\left(\frac{f}{f_G^b}\right)^{\gamma_G},$$

where

$$\gamma_L = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(r-\mu_L)}}{\sigma^2} > 1$$
$$\gamma_G = \frac{\mu_G + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_G + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(r-\mu_G)}}{\sigma^2} > 1$$
$$f_L^b = \frac{\gamma_L}{\gamma_L - 1}\frac{1}{r-\mu_L}$$
$$f_G^b = \frac{\gamma_G}{\gamma_G - 1}\frac{1}{r-\mu_G}.$$

The HJB in the high state is

$$(\rho + \lambda_G + \lambda_L - \mu_H)\, j_H(f) = \max_{0\le d_H \le j_H(f)} 1 - (r+\xi)\, f + (\rho - r)\, d_H + \mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda_L j_L(f) + \lambda_G j_G(f)$$
$$- (\xi + \mu_H)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).$$

Equivalently,

$$(\rho + \lambda_G + \lambda_L - \mu_H)\, j_H(f) = \max_{0\le d_H \le j_H(f)} 1 - (r+\xi)\, f - (\xi + \mu_H)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f)$$
$$+ \max\left\{(\rho - r)\, j_L(f) + \lambda j_L(f), (\rho - r) j_H(f)\right\}.$$

The price satisfies

$$(r + \xi + \lambda_G + \lambda_L)\, p_H(f) = r + \xi + \lambda_G p_G(f) + \mathbb{1}_{\{f \le f_\dagger\}} \lambda_L p_L(f) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)$$

Again,

$$g_H(f) = \begin{cases} \frac{(\rho - r)(p_H(f) - p_L(f))}{-f p_H'(f)} & f \le f_\dagger \\ 0 & f > f_\dagger \end{cases}.$$

Finally, we need the condition that

$$
\begin{aligned}
&(\rho - r + \lambda_G)\, j_L(0) > (\rho - r) j_H(0) \\
\Rightarrow\ & \frac{\rho - r + \lambda_L}{r - \mu_L} > (\rho - r) \frac{1 + \frac{\lambda_G}{r - \mu_G} + \frac{(\rho - r + \lambda_L)}{r - \mu_L}}{\rho + \lambda_G + \lambda_L - \mu_H} \\
& \frac{\rho - r + \lambda_L}{\rho - r} > \frac{\rho - \mu_L + \lambda_L + \lambda_G \frac{\rho - r + \lambda_L}{r - \mu_G}}{\rho + \lambda_G + \lambda_L - \mu_H}.
\end{aligned}
$$

If $\lambda_G = 0$, the condition goes back to the one we had in the paper. With the upside, the functional form of $g_H(\cdot)$ stays unchanged. However, $p_H(\cdot)$ is different, $f_\dagger$ is different, and the condition that $(\rho - r + \lambda_G)\, j_L(0) > (\rho - r) j_H(0)$ is also different.