Bargaining in Securities*

Isaías N. Chaves† and Felipe Varas‡

September 1, 2020

Abstract

A privately informed buyer and a seller negotiate over the terms of a joint project. The buyer has private information that affects both his standalone value and the net returns from the project. The seller makes offers in a one-dimensional family of *securities* (debt's face value, share of equity, etc), so that the value of an accepted offer depends on the buyer's private information. We characterize Markovian bargaining dynamics in continuous time. We show that equilibria either have instant trade, or delay of a particular form: trade begins smoothly in a gradual concessions phase; reaches an impasse of random length during which no offers are accepted; and then ends suddenly with an atom of types trading in an instant. Whenever there is delay, steeper security families (those that are more informationally sensitive) lead buyer types above a cutoff to pay strictly less, and types below to pay weakly more. We provide conditions under which the buyer prefers bargaining in a flatter family of securities, and we show that he may prefer flatter securities even though these may cause higher expected delay in equilibrium.

*We are grateful to Brett Green, Harry Pei, Andy Skrzypacz, and Vish Viswanathan for insightful comments.

†Kellogg School of Management, Northwestern University, Evanston, IL 60208, U.S.A. Email: isaias.chaves@kellogg.northwestern.edu

‡Fuqua School of Business, Duke University, Durham, NC 27708, U.S.A. Email: felipe.varas@duke.edu

1 Introduction

Many real world negotiations involve contract terms that are complex, with payments that are contingent on future outcomes: the value of offers depends on outcomes that are unknown at the time of bargaining, and the parties often have different information about the likelihood of favorable outcomes.

For example, in merger and acquisitions (M&A), acquirers often pay the target using shares of their own companies (Malmendier et al., 2016). Unlike the case of a pure cash payment, when the payment includes equity, the final transfer to the seller ultimately depends on the realized value of the joint entity. Contingent payments are also the norm in oil and gas leases agreements as well as in procurement contracts. Lease agreements, which are typically negotiated between an individual land owner and a local oil and gas producer, tend to specify an upfront cash payment and a pre-specified royalty over future revenues.[1] Likewise, procurement contracts specify some cost sharing rule, and many such contracts are arrived at via negotiation with suppliers. The previous examples feature bargaining over an equity-like payment, but the phenomenon is more general and can involve other kinds of contingent payments. In Chapter 11 bankruptcy procedures, for instance, claim holders bargain over the terms of the restructuring plan such as the face value, maturity and seniority of new debt (White, 1989).

In short, negotiating parties frequently make offers in *securities*. Despite the widespread use of contingent payments in negotiation, its impact on bargaining remains largely unexplored. Although there is an extensive literature in finance looking at the use of non-cash means of payment and its attendant effects on signaling and screening, all these models are either static or, if dynamic, assume full commitment by the party making the offers. We therefore lack an understanding of how the use of securities as payment can affect the outcomes of actual *bargaining*—that is, how do they affect the back and forth of offers in which neither party has the power to commit?

In the foregoing, we abstract somewhat from the institutional detail in order to isolate the impact of security payments on bargaining without commitment. In the model, a privately informed buyer and a seller negotiate over the terms of a joint project. The buyer has private information that affects both his standalone value and the net return of the project. The offers are being made in a *fixed* family of contingent payments (i.e., an indexed security class such as debt, indexed by the face value, or equity, indexed by the share being traded), so that the value of an accepted offer depends on the buyer's private information. Time is continuous, so the seller can revise her offers infinitely frequently. We focus on a tractable class of Markovian "skimming" equilibria, in which buyer types accept gradually in a given order.

We derive two sets of results. First, we completely characterize the bargaining dynamics in continuous time (Theorem 1). Depending on both the primitives and the security being used, the environment can be downward-skimming (high types dislike delay relative more, and accept first) or upward-skimming (high types dislike delay relatively less, and

[1] Government oil lease agreements are usually auctioned but individual lease agreements are commonly settled by negotiation.

accept later). If the environment is downward-skimming, or if it is upward-skimming but adverse selection is not too severe (efficiency is attainable in a one shot model), trade happens instantly. If upward skimming, and adverse selection is sufficiently severe, then there is delay of a very particular form: (i) trade begins smoothly in a phase of gradual concessions; (ii) reaches an impasse of random length during which no offers are accepted; and (iii) suddenly ends in an atom of trade with all remaining types.

Second, we analyze how the negotiation changes as one increases the informational sensitivity of the security being used. For example, equity is very informationally sensitive, since its value depends tightly on the private information on the company that is offering that equity. Cash, on the other hand, is informationally *in*sensitive, since the value of a cash payment does not depend on the private information of the party offering the cash.

We measure the informational sensitivity of securities by their *steepness*, as defined by DeMarzo et al. (2005). We show that whenever there is delay, steeper securities (more informationally sensitive ones) lead high types to pay strictly less, and low types to pay weakly more (Proposition 2). Steeper securities also change the negotiation dynamics: they slow down the concessions phase, but lead to less pooling at the final atom (Proposition 1). Types under a cutoff always prefer flatter securities (Proposition 3), and there always exists a positive-measure range of types who prefer the flatter security even though it increases their expected payments.

We provide tighter results for security families whose steepness can be ranked in terms of a one-dimensional parameter. For example, the equity of a levered company is more informationally sensitive than the equity of an all-equity company; in other words, levered equity is steeper than unlevered equity, and the steepness of equity payments can be parameterized by the amount of underlying leverage. For such parametrized steepness comparisons, steeper securities increase discounting costs for types outside an intermediate region. In particular, small increases in steepness raise discounting costs for all but a vanishing portion of types (Proposition 4). Moreover, we provide conditions under which all buyer types prefer flatter securities (Proposition 5). Since delay is random, one must distinguish between higher discounting costs—which depend on certainty-equivalent delay—and higher expected delay. We show through examples that, even though a flatter security lowers discounting costs for sufficiently low and sufficiently high types, and even though all types prefer this security, using it may impose strictly higher expected delay on all but a vanishing fraction of types.

Finally, we apply our general results to study the impact of financial constrains in M&A negotiations. The effect of financial constraints is subtle; we show that loosening them significantly may in fact increase bargaining frictions and lead to longer expected delay for some types. Because the impact of financial constraints varies across the distribution of deals, aggregate measures may be misleading. For example, additional financial slack lowers the probability of deal failure for high types, but it may raise it for lower types. More broadly, our results point out two potential problems with how to interpret empirical evidence on M&A negotiations. First, particular care is needed when using evidence on realized delay to make inferences about inefficiency. As financial constraints are loosened, expected delay may increase while certainty-equivalent delay decreases. The former is observable, while the latter, which is the actual quantity of interest for efficiency comparisons, is unobservable. Second, evidence on ex post payoffs like abnormal returns may be misleading regarding the optimality of acquirer decisions. We show that even if a higher cash component increases an acquirer's ex-ante payoffs, the impact in ex-post (observed) payoffs is ambiguous, as high types always end up paying more while low types end up paying weakly less. Thus, even if we observe empirically that the use of (more) cash reduces realized returns, firms may still have strict incentives to increase the cash component of their offers.

Related Literature This work relates to a literature on bargaining with asymmetric information and frequent offers, e.g., (Fudenberg et al., 1985; Gul et al., 1986; Fuchs and Skrzypacz, 2010). Deneckere and Liang (2006)'s key contribution characterized frequent offers dynamics in a bargaining model with interdependent values and a lemons problem. The bargaining in that model is in cash, time is discrete, and the type space is discrete. A main result in that paper is that, in the gap case, if efficiency is unattainable in a static model, there must be delay in the frequent offers limit; surprisingly, in equilibrium bursts of trade are followed by long quiet periods in which no serious offers are made. In contrast, in our paper the level of interdependence in values is endogenous to the security and the offers made. Our type space is continuous, and we formulate the bargaining problem directly in continuous time, which dramatically streamlines the analysis. Moreover, the explicit expressions for the speed of trade allow for clean comparative statics on the space of securities and a study of how the model primitives affect delay. Notably, the dynamics (even the direction of screening) in our model depend not only on the information primitives, but also on the security being used.

The continuous-time formulation that we use originated in Ortner (2017) and Daley and Green (2020), which are models with discrete types and driving Brownian process (changing costs in the former, news about the informed party's type in the latter). These were adapted to standard continuous-type Coasean bargaining (without a driving Brownian process) in Chaves (2019). The bargaining dynamics we find in the upward skimming case (smooth trade, followed by an atom of trade) are reminiscent of those in Daley and Green (2020), though the forces leading to delay, and to gradual trade in particular, are different: without news, their model reduces to the frequent-offers limit of Deneckere and Liang (2006), and the equilibrium features two bursts of trade with a quiet period in between.

We also contribute to a nascent bargaining literature that considers bargaining over richer objects than cash. Strulovici (2017) considers a two-type Coasean bargaining model where parties negotiate over the terms of contracts, including, for instance, the quantity or quality of goods traded. He shows that agreement is efficient in the continuous-time limit for a broad class of contracting spaces. Hanazono and Watanabe (2018) consider the splitting of a stochastic pie in a common value setting: both players receive private noisy signals about the size of the pie, so their offers about how to divide the pie are a form of equity claims on a variable surplus. They characterize the conditions under which parties can efficiently aggregate their dispersed information in equilibrium.

Finally, there is a large literature in finance studying security design in static settings or settings with full commitment power. We briefly describe some points of connection. Since Myers and Majluf (1984), a key concern in corporate finance, and in particular in the security design literature, is to understand the impact of asymmetric information on financing. One of the key insights of this literature is that using informationally sensitive securities (steep securities) is costly due to adverse selection. For example, the pecking order theory developed by Myers and Majluf (1984) states that companies choose informationally insensitive securities, such as debt, as their main source of financing, and only rely on informationally sensitive securities, such as equity, when other sources of financing are unavailable. While Myers and Majluf (1984) only consider the choice between debt an equity, their insight has been extended to consider more general securities. DeMarzo and Duffie (1999) consider the implications that adverse selection has on security design: more informationally sensitive securities generate a more severe lemons problem and a less liquid market, while less informationally sensitive securities reduce the amount of capital that can be raised.

The literature on mergers and acquisitions has also studied the effects of different security choices. Fishman (1989) considers securities bidding in take-over competitions; he shows that flatter securities are more effective in equilibrium at preempting competing bids. Hansen (1987) shows how an uninformed acquirer can use stock offers to screen out low quality targets. Rhodes-Kropf and Viswanathan (2004) consider equity auctions to explain the existence of merger waves driven by aggregate changes in valuation.

Our work also speaks to a growing literature on auctions with contingent payments that emphasizes the effects of informational sensitivity on equilibrium outcomes (Hansen, 1985; DeMarzo et al., 2005; Che and Kim, 2010). Our definition of informational sensitivity ("steepness") is taken from DeMarzo et al. (2005). Using Linkage-Principle techniques (Milgrom and Weber, 1982), they show, under a condition related to our "downward-skimming" case in Lemma 1, that steeper securities increase revenue. We follow Che and Kim (2010) in allowing bidders' private information to affect their standalone value. As in our upward-skimming case, Che and Kim (2010) show that this can cause severe adverse selection and equilibria in decreasing strategies. The condition determining upward- vs downward-skimming with pure-equity bids was introduced by them. We provide a comparison between Che and Kim (2010)'s results and ours in Section 6. Outside of the security auction literature, Tung (2020) studies the impact of steepness in a directed search model. Workers with privately known productivity match with owners of heterogeneous assets of known quality. Tung (2020) characterizes the inefficiencies that arise as the market moves (exogenously) from cash transfers to output share (equity) payments; when asset owners are free to choose among securities, competition drives them to offer only cash payments.

We contribute to this prior literature by characterizing the interaction between the means of payment and the lack of commitment. In particular, we relate the informational sensitivity of securities to the amount of bargaining delay, and we show how changes in inefficient delay can overwhelm the Linkage Principle forces that are at play in the security auctions literature.

2 Setup

Players and Information A buyer (he) and a seller (she) negotiate over the terms of a joint project, the rights to which initially rest with the seller. (To fix ideas, one can think of the buyer as an acquirer and the seller as a target). The buyer has a standalone value $\tilde{X}$, and the project generates a net return $\tilde{b}$. The value of the project is therefore $\tilde{Z} := \tilde{b} + \tilde{X}$.

The buyer observes a private signal $v \sim U[0,1]$ that is informative about both his standalone value and the net return of the project. In particular, we assume that $\tilde{Z}|v$, $\tilde{b}|v$, and $\tilde{X}|v$ each have conditional densities $g_Z(z|v)$, $g_b(b|v)$, and $g_X(x|v)$ that have full support on $[0, \bar{z}]$ for each v, are smooth in both arguments, and satisfy the monotone likelihood ratio property (MLRP), with $g_Z(z|v)$ satisfying strict MLRP.

Let $b(v)$ denote $\mathbb{E}[\tilde{b}|v]$ and similarly for $X(v)$ and $Z(v)$. Throughout we assume that $b'(\cdot)$ and $X'(\cdot)$ are non-negative and bounded above (which follows by MLRP), and that $Z'(\cdot)$ is strictly positive.

Securities and Bargaining Protocol The seller makes offers to the buyer, who at each point in time chooses whether to accept or reject. This is as in standard models of Coasean bargaining. Unlike those models, the seller offers *contingent payments* belonging to a particular *security class*. With minor modifications to the defintions in DeMarzo et al. (2005), a class of securities is a function $S : [\underline{\alpha}, \bar{\alpha}] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that (i) $S(\alpha, \tilde{Z})$ is weakly increasing in both arguments, strictly so in α, and for every $\alpha > \underline{\alpha}$, strictly so in $\tilde{Z}$ on some open interval;[2] (ii) $Z - S(\alpha, Z)$ is weakly increasing in Z for all α.

The class S is fixed throughout the bargaining interaction; different offers by the seller therefore correspond to different indeces α and α'. Below, when we write "the seller makes an offer of α," and the security class is S, we mean that the seller asks for a contingent payment $S(\alpha, \tilde{Z})$ in order to agree to the merger.

We repeatedly use $\bar{S}(\alpha, v)$ to denote $\mathbb{E}[S(\alpha, \tilde{Z})|v]$, the expected payment to the seller from an offer α when the buyer's information is v and the security class is S.

Below we use the following non-degeneracy assumptions.

Assumption 1 (Non-degeneracy)**.**

1. No deal at lowest offer: $c > \bar{S}(\underline{\alpha}, v)\ \forall v$.
2. No deal at highest offer: $b(1) < \bar{S}(\bar{\alpha}, 1)$.
3. High v's strictly good news the seller:
   For all $\alpha > \underline{\alpha}$, $\mathbb{E}[S(\alpha, \tilde{Z})|v \in [k, k']]$ is strictly increasing in k, k'.

[2] Hence, we focus on securities other than pure cash, which is a (trivially) contingent payment.

In words, (i) the seller prefers not trading to trading at the lowest offer, (ii) the highest buyer type prefers not trading to trading at the highest feasible offer, and (iii), for all non-trivial offers, the seller's revenue *strictly* increases as the selection of types who accept the offer improves.

Payoffs The seller enjoys a flow payoff of rc before agreement is reached. For example, in the mergers and acquisitions example, c can represent either the target's cashflows or management's flow benefits of control. Hence, if the buyer with gross realized returns $\tilde{Z}$ and type v accepts a security with index α at time t, the seller receives ex post profits of

$$(1 - e^{-rt})c + e^{-rt} S(\alpha, \tilde{Z}).$$

while the buyer earns (in expectation over $\tilde{Z}$, conditional on knowing v),

$$(1 - e^{-rt})X(v) + e^{-rt} \left(Z(v) - \bar{S}(\alpha, v) \right)$$

Direction of Skimming With quasilinear payoffs (i.e., bargaining in cash), a now standard argument by Fudenberg et al. (1985) shows that equilibria satisfy a "skimming" property: if a type v is indifferent between accepting and rejecting an offer p after history H_t, then all types $v' > v$ strictly prefer to accept p at H_t; beliefs after every history are therefore right-truncations of the prior. Intuitively, high types like cash just as much as low types, but they dislike delay relatively more. Analysis usually focuses on Markovian equilibria with the truncation point as a state variable.

When bargaining in non-cash securities, the buyer's true type affects his expected payment, so the standard argument will not apply. When bargaining in equity, for example, high buyer types dislike delay more, but they also dislike giving up their equity more. High types may therefore be *more* willing than low types to wait for better equity offers.

For tractability, we will also focus on Markovian equilibria with a skimming structure—buyer types accept in a predetermined "order"—but what that order is, and whether it matches the natural order of types, depends on additional conditions. We describe these in Lemma 1 and Definition 1.

Lemma 1 (Direction of skimming)**.** *Fix a security class S. Let ι^S and V be given by*

$$\iota^S(v, \alpha) := -\frac{b(v) - \bar{S}(\alpha, v)}{\bar{S}_\alpha(\alpha, v)}$$
$$V(t, \alpha, v) := (1 - e^{-rt})X(v) + e^{-rt}(Z(v) - \bar{S}(\alpha, v)),$$

and let $\alpha^f(\cdot)$ denote the solution to

$$Z(v) - \bar{S}(\alpha^f(v), v) = X(v). \tag{1}$$

Fix an arbitrary (deterministic) sequence of offers $\{\tilde{\alpha}_t\}_{t \geq 0}$, and let $T(v) := \arg\max_{t \in \mathbb{R}_+ \cup \{+\infty\}} V(t, \tilde{\alpha}_t, v)$.

1. *If $\iota^S(\cdot, \alpha)$ is strictly increasing for all α, every selection from $T(v)$ is non-decreasing,*[3] *and $\alpha^f(\cdot)$ is strictly decreasing.*

2. *If $\iota^S(\cdot, \alpha)$ is strictly decreasing for all α, every selection from $T(v)$ is non-increasing, and $\alpha^f(\cdot)$ is strictly increasing.*

The superscript on α^f stands for "final": it is the highest take-it-or-leave-it offer that v would consider accepting. Hence, when $\iota_v^S > 0$ for every α, then (i) if two types $v' < v''$ accept offers from $\{\tilde{\alpha}_t\}_{t\geq 0}$ at different times, then v'' accepts strictly later, and (ii), if v' is indifferent between accepting and rejecting a final offer, then $v'' > v'$ strictly prefers to reject it.

Lemma 1 thus motivates the following definition:

Definition 1 (Upward vs Downward Skimming)**.** Say the environment satisfies **upward skimming** if $\iota^S(\cdot, \alpha)$ is strictly increasing for every α. The environment satisfies **downward skimming** if $\iota^S(\cdot, \alpha)$ is strictly decreasing for every α. The environment satisfies the **skimming property** if it is either upward skimming or downward skimming.

Example 1 (Bargaining in Equity; Relation to Che and Kim (2010))**.** Suppose the buyer is an acquirer, the seller is a target, and both parties are negotiating over a merger. The seller makes offers in terms of equity in the merged entity, i.e., $S(\alpha, \tilde{Z}) = \alpha\tilde{Z}$. Then $\iota^S(v, \alpha) = -\left(b(v)Z(v)^{-1} - \alpha\right)$. The environment is upward skimming iff

$$\frac{b'(v)}{b(v)} < \frac{X'(v)}{X(v)} \text{ for all } v \tag{2}$$

and downward skimming if the inequality is everywhere reversed. For a quick intuition, note that, if b/X is decreasing, then, as the buyer's type grows, his disagreement motive (i.e., the standalone value) grows proportionally faster than his agreement motive (i.e., the net surplus); the reverse is true when b/X is increasing.

In their (static) analysis of security auctions, Che and Kim (2010) identify Condition (2) as determining whether second-price sealed bid equity auctions have equilibria in decreasing strategies. We postpone a fuller discussion of Che and Kim (2010) to section 6.

Remark 1 (Relation to Usual Skimming Notions)**.** Lemma 1 is weaker than the usual skimming result invoked in the literature on cash bargaining, so our focus on Markovian skimming equilibria is a stronger restriction than the analogous restriction in models with cash bargaining. To highlight the differences, focus on the downward-skimming case. In the literature on Coasean bargaining with cash, if a type v is indifferent between accepting and rejecting an offer p after a history H_t, then all types $v' > v$ strictly prefer to accept p at H_t regardless of continuation play after the rejection. In contrast, the present lemma covers only deterministic offer paths, and it allows for offer histories where both v and

[3] In the usual order on the extended real line.

v' are indifferent between accepting and rejecting. Hence, the Lemma does not entirely rule out histories in which v' accepts strictly earlier—for example, if v and v' are both indifferent and randomize over their acceptance decisions.

Rather, we use Lemma 1 in two ways. First, the result motivates the search for equilibria in particular tractable classes, depending on the shape of ι^S. From the Lemma, if the environment is upward skimming, it will be fruitless to search for skimming equilibria where higher types accept first. Instead, in such a case we will look for upward skimming Markov equilibria, where (i) the seller's beliefs about the buyer are *left*-truncations of the prior (*lower* types accept first), and (ii) the truncation point is the relevant state variable for continuation play. (Vice versa for downward skimming environments).

Second, the offer paths in the equilibria we construct, while stochastic, are such that Lemma 1 will suffice to verify incentive compatibility for the buyer. We provide further details on the equilibrium in the subsection after next.

Static Benchmarks From Deneckere and Liang (2006), one expects that bargaining dynamics will depend crucially on whether (i) there are strict gains from trade for every type of buyer in a one-shot, static game, and (ii) whether efficiency is achievable in that static game, i.e., whether there exists on offer that all buyer types accept on which the seller can break even. If such an offer does not exist, a "Static Lemons Condition" holds.

Regarding (i), we assume throughout that $b(0) \geq c$, i.e. there are gains from trade with every type of buyer. As usual in the literature (Fudenberg et al., 1985; Gul et al., 1986), we distinguish between the *gap* ($b(0) > c$) and *no gap* ($b(0) = c$) cases. Regarding (ii), motivated by Lemma 1, we define a Static Lemons Condition separately for upward and downward skimming environments, since the relevant "lowest" type differes in the two cases.

Definition 2 (Static Lemons Condition)**.**

1. In an *upward skimming environment*, say the **Static Lemons Condition** (SLC) holds iff
$$\mathbb{E}[S(\alpha^f(1), \tilde{Z})] < c.$$
For such an environment, let k^{SLC} be defined by
$$k^{SLC} = \inf\{k \leq 1 : \mathbb{E}\left[S(\alpha^f(1), \tilde{Z})|v \in [k, 1]\right] \geq c\}.$$

2. In a *downward skimming environment*, say the Static Lemons Condition holds iff
$$\mathbb{E}[S(\alpha^f(0), \tilde{Z})] < c.$$

Below we refer to k^{SLC} as the "critical type."

Note that in downward skimming environments, the SLC will necessarily fail. Indeed,
$$\mathbb{E}[S(\alpha^f(0), \tilde{Z})] > \mathbb{E}[S(\alpha^f(0), \tilde{Z})|v = 0] = b(0) \geq c,$$

where the strict inequality uses the non-degeneracy conditions 1 and 3 in Assumption 1.

In words, under downward skimming, pooling "favors the seller:" higher types accept any final offer that a lower type accepts, so if the seller can makes money by trading with the lower type at a given offer, she makes money at that offer with any type that pools with that lowest one.

Equilibrium Notion We focus on a class of Markov equilibria that allow for rich dynamics. For environments that satisfy the skimming property, by Lemma 1 the game has a natural state variable: the truncation of the seller's prior beliefs that yields her current posterior. For upward skimming environments, this is a *left* truncation: if the state at time t is $K_t = k$, then, given the history of offers and rejections, the seller believes $v > k$. For downward skimming environments, k is a *right* truncation, i.e., if $K_t = k$, the seller believes $v < k$.

The equilibrium objects have minor differences depending on whether the environment has upward or downward skimming. To simplify the exposition, we describe the equilibrium notion for upward skimming environments, and later explain how the notion must be adapted for downward skimming.

The equilibria that we study are Markovian in the relevant truncation K_t (henceforth, "the cutoff"). Following recent formulations of Coasean bargaining in continuous time (see Ortner (2017), Daley and Green (2020), and, most relevant for the current setup, Chaves (2019)), we model the seller as solving a (Markovian) optimal stopping-control problem, and the buyer as solving a (Markovian) optimal stopping problem. Roughly, that machinery writes the seller's problem in *quantity space.* The buyer's chooses a reservation offer strategy $\alpha(\cdot)$. On path, the seller chooses how fast to screen through buyer types, taking as given that to screen through types $v < k$, she must offer $\alpha(k)$. Formally, the seller chooses paths of belief cutoffs $t \mapsto K_t$, which result in paths of offers $t \mapsto \alpha(K_t)$, i.e., the seller is quoting prices from the reservation offer schedule. We also give the seller an option to "give up on screening": she can make a pooling offer $\alpha(1)$ that would be accepted by all remaining types, thereby ending the game. Given a law of motion for cutoffs, and that future offers are given by $\alpha(K_t)$, the buyer solves an optimal stopping problem with state K_t.

The description just given is insufficient in two regards. First, the seller is potentially choosing over all non-decreasing paths, which is an unmanageably large space. Second, if the reservation offer curve $\alpha(\cdot)$ is discontinuous in equilibrium, the vignette given above does not specify what happens after an off-path offer $\alpha' \notin \alpha([0, 1])$.

We describe our solution to these two issues in the next definition. First, using the approach in Chaves (2019), we impose some restrictions on seller strategies; these make the analysis tractable while still allowing a rich set of dynamics.

Definition 3 (Seller Strategy Space)**.**

1. A *plan* of on-path offers by the seller consists of a non-decreasing cutoff path $t \mapsto K_t$ and a stopping time T at which to make a pooling offer $\alpha(1)$. We denote an entire cutoff path $(K_t)_{t \geq 0}$ by K. K is *admissible* if it has no singular-continuous parts.

We allow for mixed strategies in the stopping time T, which are represented by a CDF $F = (F_t)_{t\geq 0}$. Thus a plan for the seller is given by a pair (K, F), and we denote by $\mathcal{A}_k^U$ the set of admissible plans (K, F) satisfying $K_{0^-} = k$, i.e., with initial value k, and generic element K^k. We say that the stopping time T is Markov if its hazard measure $dF(t)/(1 - F(t-))$ is a function of K_{t-}. At any point where F_t is absolutely continuous, we denote its hazard rate by the arrival rate λ_t. In this case, the Markov assumption amounts to $\lambda_t = \lambda(K_t)$, for some non-negative function $\lambda(\cdot)$.

2. Time intervals $[\underline{t}, \bar{t})$ where $dF_t = 0, \Delta K_{t^-} = 0$ are *smooth trade regions*, and $\dot{K}_t$ is the *trading speed*. A special case of a smooth trade region is a *quiet period*, i.e., an interval $[\underline{t}, \bar{t})$ with $\dot{K}_t = dF_t = \Delta K_{t^-} = 0$.

3. A plan for on-path offers is supplemented by a plan following off-path offers. For any off-equilibrium offer $\alpha' \notin \alpha([0,1])$ made at time t, we let $\sigma_t(\alpha') \in \Delta([0,1])$ be the randomized offer that "immediately" follows the rejection of α'.

We briefly explain the rationale for the third item. As we show below, sometimes $\alpha(\cdot)$ must be discontinuous in equilibrium, so to complete the specification of the strategies, we need to specify the seller's strategy following an off-equilibrium offer $\alpha' \notin \alpha([0,1])$. As in Fudenberg et al. (1985) and Gul et al. (1986), to sustain Markovian behavior on path, we need to allow for randomization following the rejection of an off-path offer α'. The usual complexities in continuous time games arise when we try to capture the randomization that follows an off-equilibrium offer in a discrete time model. To overcome this difficulty, we stop the clock whenever an off-equilibrium offer is made, allowing the seller to immediately make a new offer if such an offer is rejected.[4] We allow the new offer to depend on the off-path offer α'; in the lingo of the discrete time literature, our equilibrium will be "weak Markov."

Definition 4 (Buyer and Seller Problems)**.** At state k, a buyer type v takes $\alpha(\cdot)$ and K as given, and solves

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[(1-e^{-r(\tau\wedge T)})X(v) + e^{-r(\tau\wedge T)}\left(Z(v) - \bar{S}(\alpha(K_{\tau\wedge T}), v)\right)\right] \tag{3}$$

where by definition $K_T = 1$, and $\mathcal{T}$ is the set of stopping times adapted to the filtration generated by T. Meanwhile, the seller S takes $\alpha(\cdot)$ as given. Given any path Q_t and realization of the stopping time T, the seller payoff is

$$\Pi(Q,T) \equiv \int_0^T e^{-rt}\mathbb{E}\left[\bar{S}\left(\alpha(Q_t), v\right)\middle| v \in [Q_{t-}, Q_t]\right] dQ_t + e^{-rT}\mathbb{E}\left[\bar{S}\left(\alpha^f(1), v\right)\middle| v \in [Q_{T-}, 1]\right] + \left(1 - (1-Q_T)e^{-rT} - \int_0^T e^{-rt} dQ_t\right) c,$$

[4]The idea of stopping the clock to allow for multiple sequential moves in a continuous time game has been used in bargaining models by Smith and Stacchetti (2002) and Fanning (2016). An alternative approach is to follow the formalization in Fudenberg and Tirole (1985) and to consider "intervals of consecutive atoms."

and, at each k, the seller strategy (Q, F) solves

$$\sup_{(Q,F)\in\mathcal{A}_k^U} \int_0^\infty \Pi(Q,T)dF(T). \tag{4}$$

We can now fully define a weak Markov equilibrium.

Definition 5 (Equilibrium)**.** A weak *Markov Equilibrium* of an upward-skimming game consists of a tuple

$$(\{K^k\}_{k\in[0,1]}, F, \alpha(\cdot), \sigma(\cdot|\cdot,\cdot))$$

together with a value $J(\cdot)$ for the the seller and a value $B(\cdot,\cdot)$ for the buyer such that

1. For all $v \in [0,1]$, $k \in [0,1]$, accepting at $\tau^* = \inf\{t : \alpha(K_t^k) \leq \alpha(v)\}$ solves the buyer's problem (3) and delivers value $B(v,k)$.

2. $\alpha(1) = \alpha^f(1)$, where α^f is defined in (1).

3. For all $k \in [0,1]$ and T in the support of F, K^k is an admissible path and T is a Markov stopping time that together solve (4) and deliver value $J(k)$.

4. For any point of discontinuity of $\alpha(\cdot)$, k', and any off-equilibrium offer $\alpha' \in (\alpha(k'+), \alpha(k'-))$, $\sigma(\cdot|k',\alpha')$ maximizes[5]

$$\int_0^1 \Big\{(\alpha^{-1}(\tilde{\alpha}) - k')^+ \mathbb{E}\left[\bar{S}(\tilde{\alpha}, v)\,\Big|\, v \in [k', \alpha^{-1}(\tilde{\alpha}) \wedge k']\right] \\ + (1-\alpha^{-1}(\tilde{\alpha}))J\left(\alpha^{-1}(\tilde{\alpha})\right)\Big\}\, d\sigma(\tilde{\alpha}|k',\alpha')$$

5. For any point of discontinuity of $\alpha(\cdot)$, k', and any off-equilibrium offer $\alpha' \in (\alpha(k'+), \alpha(k'-))$, $\sigma(\cdot|k',\alpha')$ satisfies

$$Z(k') - \bar{S}(\alpha', k') \leq B(k',k') \int_0^{\alpha(k')} d\sigma(\tilde{\alpha}|k',\alpha') \\ + \int_{\alpha(k')}^1 \left(Z(k') - \bar{S}(\tilde{\alpha}, k')\right) d\sigma(\tilde{\alpha}|k',\alpha')$$

Condition 2 is a refinement inspired by the corresponding discrete time game. In a stationary equilibrium of the discrete time game, for any positive period length, the seller would never offer more than $\alpha^f(1)$ when her beliefs are concentrated at $v = 1$. (And $\alpha(1)$ can never be above $\alpha^f(1)$, since $v = 1$ would strictly prefer to reject, i.e. $\alpha(1)$ cannot be a reservation offer for $v = 1$.) Given how we have written the seller's problem, the

[5]Here, $\alpha^{-1}(\cdot)$ represents the generalized inverse defined as $\alpha^{-1}(y) \equiv \sup\{x > 0 : \alpha(x) \geq y\}$.

same kind of analysis is not quite well defined in our continuous time game, so we impose Condition 2 directly.[6]

Conditions 4 and 5, inspired by the discrete-time analyses in Fudenberg et al. (1985) and Gul et al. (1986), will be necessary to tackle off-path behavior. We will see that, in the class of equilibria we consider, $\alpha(\cdot)$ must be discontinuous. The usual continuous-time technique of reducing the seller to controlling a path of cutoffs will not be faithful to the idea that the seller is choosing offers. Her strategy space must be augmented in some way, which these conditions provide. Condition 4 says that, when the seller makes an off-path offer, the buyer still accepts according the reservation offer curve $\alpha(\cdot)$. Condition 5 specifies that, after making a "mistake" and charging an off-path offer, the seller randomizes in way that justifies the buyer's choice to accept according to $\alpha(\cdot)$.

Finally, to streamline the derivation of necessary conditions, we restrict our search to equilibria in an amenable subclass:

Definition 6 (Regularity)**.** A weak Markov Equilibrium is *regular* if

1. J is continuous and C^1 in the interior of smooth regions;
2. $\dot{K}_t$ is continuous in the interior of smooth trade regions.
3. Jump discontinuities in cutoff paths are *isolated*.

Below, we refer to regular weak Markov Equilibria as simply "equilibria."

Remark 2 (Modifications for Downward Skimming)**.** In downward skimming environments, regular weak Markov Equilibria are defined almost identically, with the following changes:

1. Admissible paths $t \mapsto K_t^k$ are non-*increasing* and satisfy $K_0^1 = 0$. The admissible set at state k is denoted $\mathcal{A}_k^D$.
2. Condition 2 in Definition 5 becomes $\alpha(0) = \alpha^f(0)$.
3. The seller's objective is now written as

$$\Pi(Q) = \int_0^\infty e^{-rt}\mathbb{E}\left[\bar{S}\left(\alpha(Q_t), v\right) \middle| v \in [Q_t, Q_{t-}]\right] d(1 - Q_t) \\ + \left(1 - \int_0^\infty e^{-rt} d(1 - Q_t)\right) c \quad (5)$$

[6] See, for example, the discussions in Ortner (2017) and Daley and Green (2020), who impose conditions similar to our Condition 1; Ortner (2017) shows that, absent this kind of refinement, continuous time equilibria can violate this natural discrete-time property.

3 Dynamics for General Securities

Within our class of equilibria, we can fully characterize equilibrium dynamics. We provide an informal derivation of the equilibrium in an upward skimming case where the SLC holds, relegating the full proof of necessary conditions and equilibrium verification to the appendix.

We construct an equilibrium where the game starts with smooth trade. By the usual Coasean logic, whenever the seller is trading smoothly, her payoff is pinned down at c: otherwise, she would have strict incentives to speed up trade. To wit, the HJB equation in the smooth trading region is given by

$$rJ(k) = \sup_{\dot{k} \geq 0} \left(\bar{S}(\alpha(k), k) - J(k) \right) \frac{\dot{k}}{1-k} + J'(k)\dot{k} + rc. \tag{6}$$

The optimization problem in the HJB equation (6) is well-defined only if

$$J(k) \geq \bar{S}(\alpha(k), k) + (1-k)J'(k).$$

If trade is happening at a positive speed ($\dot{k} > 0$ is optimal), this condition must hold with equality. Substituting $\bar{S}(\alpha(k), k)$ into the right hand side of (6), it follows that $J(k) = c$, with $\alpha(k)$ implicitly defined by $\bar{S}(\alpha(k), k) = c$.

The speed of trade at state k depends on the marginal buyer $v = k$ and his incentives to delay. The value function for buyer v at state a k with smooth trade must satisfy the HJB equation

$$rB(v, k) = rX(v) + \dot{k}B_k(v, k).$$

Evaluating at $v = k = K_t$, so that $B(K_t, K_t) = Z(K_t) - \bar{S}(\alpha(K_t), K_t)$, we get

$$r\left(Z(K_t) - \bar{S}(\alpha(K_t), K_t)\right) = rX(K_t) - \dot{K}_t\alpha'(K_t)\bar{S}_\alpha(\alpha(K_t), K_t).$$

Finally, substituting, $b(k) = Z(k) - X(k)$, $c = \bar{S}(\alpha(k), k)$, and

$$0 = \alpha'(k)\bar{S}_\alpha(\alpha(k), k) + \bar{S}_v(\alpha(k), k),$$

we obtain a differential equation for K_t: starting at a state k with smooth trade, K_t evolves according to

$$\dot{K}_t = r\frac{b(K_t) - c}{\bar{S}_v(\alpha(K_t), K_t)}, K_0 = k. \tag{7}$$

A natural guess is that, using (7), one can construct an equilibrium in smooth trade. However, smooth trading cannot persist indefinitely. If the seller were to continue screening more and more types starting from the bottom, eventually the state would reach *and cross* k^{SLC}. At that point, trading instantly with all remaining types at an offer of $\alpha^f(1)$ would become strictly more profitable than trading smoothly with the marginal type: $\mathbb{E}[\bar{S}(\alpha^f(1), v)|v \in [k, 1]] > c$ for $k > k^{SLC}$.

Our equilibrium construction therefore specifies smooth trade at $k < k^{SLC}$, and an atom of trade at $k > k^{SLC}$. It remains only to specify what happens at state k^{SLC}. The key fact is that, to make types in $(k^{SLC}, 1]$ all willing to wait and trade at the offer $\alpha^f(1)$, when they could have traded at the offer $\alpha(k^{SLC})$, there must be an impasse when state k^{SLC} is reached, i.e., trade must cease for while.

To see this, note that, for $k^{SLC} < 1$, the offer curve $\alpha(\cdot)$ must be drop discontinuously at k^{SLC}: on the one hand, $\alpha(k^{SLC})$ must satisfy $\bar{S}(\alpha(k^{SLC}), k^{SLC}) = c$, since the seller trades smoothly for $k < k^{SLC}$. On the other, given that all types in $(k^{SLC}, 1]$ trade at the final offer $\alpha^f(1)$, $\mathbb{E}[\bar{S}(\alpha(k^{SLC}+), v) | v \in (k^{SLC}, 1]] = c$. It follows that $\alpha(k^{SLC}+) < \alpha(k^{SLC})$.

The seller, then, must delay offering $\alpha^f(1)$ just long enough to make $v = k^{SLC}$ indifferent between accepting $\alpha(k^{SLC})$ "now" and rejecting it in in hopes of receiving $\alpha^f(1)$ "later." The expected discount until $\alpha^f(1)$ is offered, denoted by D, must solve

$$Z\left(k^{SLC}\right) - \bar{S}\left(\alpha(k^{SLC}), k^{SLC}\right) = (1 - D) X\left(k^{SLC}\right) + D\left(Z\left(k^{SLC}\right) - \bar{S}\left(\alpha^f(1), k^{SLC}\right)\right), \tag{8}$$

which simplifies[7] to

$$c = (1 - D) b(k^{SLC}) + D\bar{S}(\alpha^f(1), k^{SLC}). \tag{9}$$

Since the seller must use a Markov stopping time at k^{SLC}, she can implement this delay by postponing the final offer until the first tick of a Poisson clock with a rate λ given by $\lambda/(r + \lambda) = D$.

For these equilibrium dynamics, Lemma 1 and standard mechanism design arguments imply that it is globally incentive-compatible for buyers to accept from lowest to highest according to $\alpha(\cdot)$, and a verification approach shows that these screening dynamics are optimal for the seller, given $\alpha(\cdot)$.

We have outlined the construction of an equilibrium, but in fact, in Theorem 1 we prove that these are the *only* possible equilibrium dynamics. Formally, we prove the following theorem, which also covers the remaining cases:

Theorem 1. *In skimming environments, there exists a (regular weak Markov) equilibrium.*

1. *In a downward-skimming environment, all equilibria have instant trade at an offer* $\alpha^f(0)$.

2. *In an upward-skimming environment, if SLC fails, all equilibria have instant trade at an offer of* $\alpha^f(1)$.

3. *In an upward-skimming environment, if SLC holds, and there is no gap, there is no trade in any equilibrium.*

[7]The simplification uses $\bar{S}(\alpha(k^{SLC}), k^{SLC}) = c$ and $b(k^{SLC}) - \bar{S}(\alpha^f(k^{SLC}), k^{SLC}) = X(k^{SLC})$. Such a $D \in (0, 1)$ always exists: since there are gains of trade and $b(\cdot)$ is strictly increasing, we always have that

$$\bar{S}(\alpha^f(k^{SLC}), k^{SLC}) = b(k^{SLC}) > c = \bar{S}(\alpha(k^{SLC}), k^{SLC}).$$

4. *In an upward-skimming environment, if SLC holds and there is a gap, there is a unique on-path equilibrium triple* $\left(\{K^k\}_{k\in[0,1]}, F, \alpha(\cdot)\right)$*:*

 - *The buyer's acceptance strategy is*

$$\alpha(k) = \begin{cases} \alpha^f(1) & \text{if } k \in (k^{SLC}, 1] \\ \bar{S}^{-1}(c,k) & \text{if } k \in [0, k^{SLC}], \end{cases}$$

 - *There is smooth trade for* $k \in [0, k^{SCL})$*. In the smooth trade region, the cutoff* K_t *is the unique solution to equation* (7)*.*
 - *At state* k^{SCL}*, there is a temporary (random) breakdown in trade. The seller makes the final offer* $\alpha^f(1)$ *with a Poisson arrival intensity* $\lambda = rD/(1-D)$*, where* D *is defined by equation* (9)*.*
 - *For* $k > k^{SLC}$*, the seller immediately offers* $\alpha^f(1)$*.*

Figure 1 illustrates typical realized paths of outcomes for the case with non-trivial delay dynamics (Theorem 1.4).[8] The cutoff rises gradually from 0 until it reaches k^{SLC}, with the seller gradually dropping her offers from $\alpha(0)$ to $\alpha(k^{SLC})$. When the state arrives at k^{SLC}, the game reaches an impasse, with the cutoff frozen at k^{SLC} for a random amount of time $T - \tau(k^{SLC})$. During the impasse, the seller "stubbornly" refuses to move her offer from $\alpha(k^{SLC})$, until finally, at a random time, she concedes, dropping her offer to $\alpha^f(1)$. At that point, all remaining types $v \in (k^{SLC}, 1]$ accept suddenly, and the cutoff jumps to $k = 1$.

It is instructive to compare the path of offers $\alpha(K_t)$ to the expected equilibrium payments by the different types $\bar{S}(\alpha(v), v)$. While the offer $\alpha(K_t)$ is dropping gradually, the payments made by types in $[0, k^{SLC}]$ are *constant* in type and equal to c. In contrast, while all types in $(k^{SLC}, 1]$ accept the same offer $\alpha^f(1)$, they all pay different amounts according to $\bar{S}(\alpha^f(1), v)$, which is strictly increasing in v.

4 Means of Payment and Bargaining Dynamics

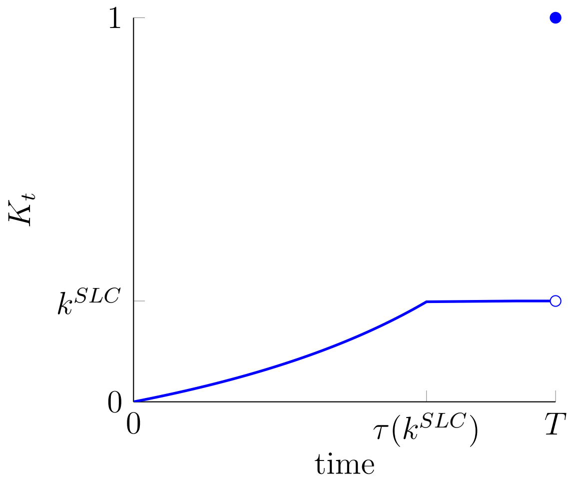
Using Theorem 1, we can characterize how the means of payment used in the negotiation affect the bargaining dynamics. A key feature of a security is its *steepness*.

DeMarzo et al. (2005) define steepness as folows:

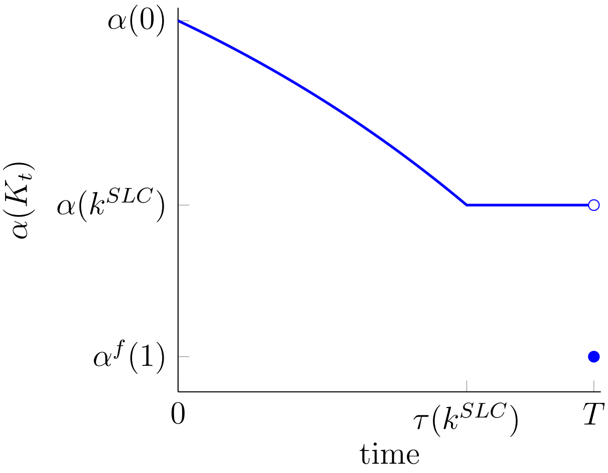
Definition 7 (Steepness)**.** Take two securities $S^1 : [\underline{\alpha}_1, \bar{\alpha}_1] \times \mathbb{R}_+ \to \mathbb{R}_+$,$S^2 : [\underline{\alpha}_2, \bar{\alpha}_2] \times \mathbb{R}_+ \to \mathbb{R}_+$. S^1 is steeper than S^2 if, for any feasible indices $\alpha_1 \in [\underline{\alpha}_1, \bar{\alpha}_1]$ and $\alpha_2 \in [\underline{\alpha}_2, \bar{\alpha}_2]$,

$$\bar{S}^1(\alpha_1, v) = \bar{S}^1(\alpha_2, v) \Rightarrow \bar{S}^1_v(\alpha_1, v) > \bar{S}^2_v(\alpha_2, v)$$

[8]The graphs are actual model output, using the setup in Section 5.1 below.



(a) Equilibrium path of cutoff K_t



(b) Equilibrium path of offers $\alpha(K_t)$

Figure 1: Illustration of a realized equilibrium path in an upward skimming environment satisfying the SLC. $\tau(v)$ denotes the realized time at which state v is reached on-path.

For this comparative exercise below, we focus on pairs of securities that both generate non-trivial delay dynamics.

Definition 8 (Delayed trade)**.** Given c and a joint distribution for $\tilde{b}$ and $\tilde{X}$, say S *generates delayed trade*, denoted $S \in \mathcal{D}_{\tilde{b},\tilde{X},c}$, if

1. There is upward-skimming under S: $\iota^S(\cdot, \alpha)$ is strictly increasing for every α.
2. The Static Lemons Condition holds: $\mathbb{E}[S(\alpha^f(1), \tilde{Z})] < c$.
3. There are strict gains from trade: $b(0) > c$.

By Theorem 1, whenever $S \in \mathcal{D}_{\tilde{b},\tilde{X},c}$, trade does not break down completely and does not happen instantaneously. Focusing on that non-trivial case, our next result characterizes the effect on trading dynamics of increasing steepness.

Proposition 1. *Take S^1 and S^2 in $\mathcal{D}_{\tilde{b},\tilde{X},c}$, with S^1 steeper. Let $k^{SLC}(S^i)$ denote the critical type under security $S^i, i = 1, 2$. Under S^1 there is less pooling, and a slower gradual concessions phase:*

1. $k^{SLC}(S^1) > k^{SLC}(S^2)$.

2. Buyer types $[0, k^{SLC}(S^2)]$ trade strictly later under S^2.

Proof. First, we show that types that trade smoothly in both securities trade later under S^1. Let α^1 and α_2 be the reservation offer curves for the bargaining games with securities

S^1 and S^2. Suppose S^1 is steeper than S^2, Recall that, when trade is smooth, the seller's indifference condition across speeds,

$$\bar{S}^1(\alpha^1(k), k) = \bar{S}^2(\alpha^2(k), k) = c.$$

Since S_1 is steeper, this implies $\bar{S}^1_v(\alpha^1(k), k) > \bar{S}^2_v(\alpha^2(k), k)$. Let K^1 and K^2 be any equilibrium cutoff paths under the S^1 and S^2 environments, respectively. By the inverse function theorem, for any $k \in (0, k^{SLC}(S^i))$, we have that

$$\tau^{i'}(k) = \frac{1}{\dot{K}^i_{\tau^i(k)}} = \frac{\bar{S}^i_v(\alpha^i(k), k)}{r(b(k) - c)}$$

Hence, we have that

$$\tau^i(k) = \int_0^k \frac{\bar{S}^i_v(\alpha^i(s), s)}{r(b(s) - c)} ds.$$

It immediately follows that $\tau^1(k) > \tau^2(k)$ for any $k \in [0, \min\{k^{SLC}(S^1), k^{SLC}(S^2)\}]$.

Next, we show that $k^{SLC}(S^1) > k^{SLC}(S^2)$. Let $\alpha^{f,i}$ denote the final offer locus (1) under security S^i, and consider the functions $h^i(k) = \mathbb{E}[S^i(\alpha^{f,i}(1), \tilde{Z})|v \in [k, 1]]$. We claim $h^1(k) < h^2(k)$, which implies, since both are strictly increasing, that $k^{SLC}(S^1) > k^{SLC}(S^2)$. Security S^1 is steeper and $\bar{S}^1(\alpha^{f,1}(1), 1) = \bar{S}^2(\alpha^{f,2}(1), 1) = b(1)$, so

$$\bar{S}^1(\alpha^{f,1}(1), v) < \bar{S}^2(\alpha^{f,2}(1), v) \text{ for all } v < 1. \tag{10}$$

Therefore,

$$h^1(k) = \frac{1}{1-k} \int_k^1 \bar{S}^1(\alpha^{f,1}(1), v) dv < \frac{1}{1-k} \int_k^1 \bar{S}^2(\alpha^{f,2}(1), v) dv < h^2(k),$$

as required.

□

The intuition is as follows. First, consider the gradual concessions phase. The buyer's incentive to reject an offer is that, by rejecting, he can affect the seller's beliefs about his type and can obtain a better price in the future. When using S^i, if a type k rejects an offer and pretends to be $k + dk$, he expects price to drop by $(\alpha^i)'(k)\bar{S}^i_\alpha(\alpha(k), k)dk < 0$. However, because of the Coasean force the seller's expected payment is constant in the state–pinned down to $\bar{S}^i(\alpha^i(k), k) = c$. Taking total derivatives we have

$$\bar{S}^i(\alpha(k), k) = c \Rightarrow \underbrace{\overbrace{(\alpha^i)'(k)\bar{S}^i_\alpha(\alpha(k), k)}^{<0}}_{\text{Price improvement from changing seller's belief}} = -\underbrace{\overbrace{\frac{\partial}{\partial v}\bar{S}^i(\alpha(k), v)\Big|_{v=k}}^{>0}}_{\text{steepness of } S^i \text{ at } k}$$

Hence, the more sensitive price is to the buyer's private information, the greater the price improvement that he expects from rejecting an offer, and the greater his incentives to reject.

Second, consider the size of the pooling region. The pooling region makes the seller on average break even with the final offer $\alpha^{f,i}(1)$. Since type-by-type expected payments $\bar{S}^i(\alpha^{f,i}(1), v)$ are higher under S^2—the payments of the flatter security cannot cross those of the steeper one from below—the seller must average over a strictly worse pool (include even lower types) to break even under S^2.

Increasing steepness has subtle effects on payoffs. Our next few results decompose these effects. First, increased steepness has opposite effects on the equilibrium payments of high types and low types:

Proposition 2 (Expected payments are "single-crossing")**.** *Take S^1 and S^2 in $\mathcal{D}_{\tilde{b},\tilde{X},c}$, with S^1 steeper. Let $\pi_i(v) := \bar{S}^i(\alpha^i(v), v)$, v's equilibrium expected payment under S^i.*

1. *Under S^1, high types pay strictly less, low types pay more: there exists a unique $k^{cross} \in (k^{SLC}(S^2), 1)$ such that*

$$\begin{aligned} \pi_1(v) &= \pi_2(v), \quad v \in [0, k^{SLC}(S^2)]. \\ \pi_1(v) &> \pi_2(v), \quad v \in (k^{SLC}(S^2), k^{cross}). \\ \pi_1(v) &< \pi_2(v), \quad v \in (k^{cross}, 1). \end{aligned}$$

2. *Let k^* solve $\bar{S}^2(\alpha^{f,2}(1), k^*) = c$. Then $k^{cross} = \min\{k^*, k^{SLC}(S^1)\}$.*

Proof. The first line of (11) follows from smooth trading: $\bar{S}^i(\alpha^i(v), v)\pi_i(v) = c$ on $[0, k^{SLC}(S^2))$. To show the latter two lines, use (10) to conclude

$$\pi_1(v) = \bar{S}^1(\alpha^{f,1}(1), v) < \bar{S}^2(\alpha^{f,2}(1), v) = \pi_2(v),\ v \in (k^{SLC}(S^1, 1).$$

Meanwhile, by definition, $\mathbb{E}[\bar{S}^2(\alpha^{f,2}(1), v)|v \in (k^{SLC}(S^2), 1]] = c$, so there exists a $k^* \in (k^{SLC}(S^2), 1)$ such that

$$\bar{S}(\alpha^{f,2}(1), v) = \pi_2(v) \begin{cases} < c, v \in (k^{SLC}(S^2), k^*), \\ = c, v = k^* \\ > c, v \in (k^*, 1] \end{cases} \tag{11}$$

Since $\pi_1(v) = c$ for $v \leq k^{SLC}(S^1)$, if $k^* < k^{SLC}(S^1)$, then (10) and (11) yield $k^{cross} = k^*$. On the other hand, if $k^* \geq k^{SLC}(S^1)$, (10) and (11) yield $k^{cross} = k^{SLC}(S^1)$. □

Since the expected payment of the highest type is the same under both securities ($\bar{S}^i(\alpha^{f,i}(1), 1) = b(1)$), and high enough types accept $\alpha^{f,i}(1)$ in either case, they must pay less under the steeper security. At the same time, since flatter security has a larger pooling region, there are types at the bottom of the interval that would be separated under a steep security (and pay c), but get cross-subsidized by very high types when they face the steep security (and therefore pay less than c). Finally, types at the very bottom of the distribution are separated and pay c in either case.

In general, it is not possible to rank the payoffs of two securities directly, for two reasons. First, due to pooling at the final atom, it is not necessarily the case that all types suffer more delay under a steeper security. Proposition 1 provides a delay ranking only for types who trade smoothly with both securities—and since $k^{SLC}(S^2)$ can equal 0, the ranking can become vacuous. Second, Proposition 2 shows that high types (other than $v = 1$) pay strictly more with a flatter security. Even if a flatter security were to lead to lower delay, it seems possible that high types would not prefer it.

To rank payoffs more generally, we introduce a convenient integral representation of the equilibrium payoffs. Equilibrium payoffs are

$$U_i(v) = X(v) + e^{-r\tau_i^{CE}(v)} \left(b(v) - \bar{S}^i(\alpha^i(v), v)\right), \tag{12}$$

and, by the envelope theorem, its derivative is

$$U_i'(v) = X'(v) + e^{-r\tau_i^{CE}(v)} \left(b'(v) - \bar{S}_v^i(\alpha^i(v), v)\right) \tag{13}$$

almost everywhere. Using equation (12) to substitute $e^{-r\tau_i^{CE}(v)}$ into the envelope condition (13) we get[9]

$$\frac{U_i'(v) - X'(v)}{U_i(v) - X(v)} = \frac{\partial}{\partial v} \log\left(U_i(v) - X(v)\right) = \frac{b'(v) - \bar{S}_v^i(\alpha^i(v), v)}{b(v) - \bar{S}^i(\alpha^i(v), v)}$$

almost everywhere. Integrating with respect to v, we obtain

Lemma 2. *Let $U_i(v)$ be the equilibrium indirect utility under an upward-skimming security S^i, and let α^i be the associated equilibrium offer. Define*

$$\eta^i(v, \alpha) := \frac{b'(v) - \bar{S}_v^i(\alpha, v)}{b(v) - \bar{S}^i(\alpha, v)}$$

Then for $v < 1$,

$$U_i(v) = X(v) + (b(0) - c) \exp\left\{\int_0^v \eta^i(y, \alpha^i(y)) dy\right\}. \tag{14}$$

This representation allows us to rank preferences over securities by comparing $\eta^i(v, \alpha^i(v))$. We repeatedly rely on two key facts:

- for two securities S^1 and S^2, if $\eta^1(v, \alpha^1(v)) < \eta^2(v, \alpha^2(v))$ for all $v \leq \bar{v}$, then types below $\bar{v}$ must prefer S^2.[10]
- If for some v, $U_1(v) = U_2(v)$, but $\eta^1(v, \alpha^1(v)) < \eta^2(v, \alpha^2(v))$, U_2 must cross U_1 from below.

[9] We know from Theorem 1 that $b(v) > \bar{S}^i(\alpha^i(v), v)$ for all $v < 1$.

[10] Note that the ranking is not immediate from the shape of the η^i's, since it depends on $\alpha^i(v)$, which is an endogenous object.

Using this overall approach, we can rank preferences over steepness for types who pay less than c under the flatter security:

Proposition 3. *Take two securities S^1 and S^2 in $\mathcal{D}_{c,\tilde{b},\tilde{X}}$, with S^1 steeper. Let k^* be given by equation* (11)*. Then there exists $k' > k^*$ such all types $v \in [0, k')$ prefer the flatter security S^2, and strictly so for $v > 0$.*

Proof. We rank the utilities by considering the two cases $v \in [0, k^{SLC}(S^2)]$ and $v \in (k^{SLC}(S^2), k^*)$ separately, The ranking on $[0, k^{SLC}(S^2)]$ follows directly from (14), smooth trading, and the definition of steepness. Let U_1 and U_2 denote the equilibrium indirect utilities under securities S^1 and S^2. Since $\bar{S}^1(\alpha^1(v), v) = \bar{S}^2(\alpha^2(v), v) = c$ for $v \leq k^{SLC}(S^2)$, $\eta^1(v, \alpha^1(v)) < \eta^2(v, \alpha^1(v))$.

The ranking then extends to $(k^{SLC}(S^2), k^*]$ by using the upward-skimming property. Let $\tilde{\alpha}^2 : [0,1] \to [\underline{\alpha}_2, \bar{\alpha}_2]$ be defined by $\bar{S}^2(\tilde{\alpha}^2(v), v) = c$. This is the offer that would have been taken by each type under security S^2 if the S^2 equilibrium had featured smooth trade until the end. Now consider an artificial utility function

$$\tilde{U}_2(v) = X(v) + (b(0) - c) \exp\left\{ \int_0^v \eta^2(y, \tilde{\alpha}^2(y)) dy \right\}.$$

Clearly $U_2(v) = \tilde{U}_2(v)$ for $v \leq k^{SLC}(S^2)$, and by the same steepness argument used to rank utilities on $[0, k^{SLC}(S^2)]$, $\tilde{U}_2(v) > U_1(v)$ for all $v \in (k^{SLC}(S^2), k^*]$.

We claim that, if $U_2(v) = \tilde{U}_2(v)$ for some $v \in (k^{SLC}(S^2), k^*]$, $U_2'(v) > \tilde{U}_2'(v)$. Given $U_2(k^{SLC}(S^2)) = \tilde{U}_2(k^{SLC}(S^2))$ and $\tilde{U}_2(v) > U_1(v)$ for $v \in (k^{SLC}(S^2), k^*]$, this will imply $U_2(v) > U_1(v)$ in that range. Taking partial derivatives on η^2,

$$\begin{aligned} \eta^2_\alpha(v, \alpha) \propto \frac{\partial}{\partial \alpha} \left(b'(v) - \bar{S}^2_v(\alpha, v) \right) \left[b(v) - \bar{S}^2(\alpha, v) \right] + \bar{S}^2_\alpha(\alpha, v) \left[b'(v) - \bar{S}^2_v(\alpha, v) \right] \\ = -\frac{\partial}{\partial v} \iota^{S^2}(v, \alpha) < 0 \end{aligned}$$

where the inequality uses the definition of upward skimming. Since $\alpha^2(v) = \alpha^{f,2}(1) < \tilde{\alpha}^2(v)$ for $v \in (k^{SLC}(S^2), k^*]$, the claim follows from our integral representation in (14). $\square$

Proposition 3 provides a partial payoff ranking for an arbitrary pair of securities S^1 and S^2 in $\mathcal{D}_{c,\tilde{b},\tilde{X}}$. Partly because of the atom of trade at the end, steepness-based arguments alone do not suffice for a payoff ranking that covers all types. To move forward, we introduce additional structure on the paired comparison between S^1 and S^2.

Definition 9. An ordered security class with parametrized steepness is a function $S : [\underline{\alpha}, \overline{\alpha}] \times \mathbb{R}_+ \times [\underline{\gamma}, \overline{\gamma}] \to \mathbb{R}_+$ such that

1. for any γ, $S(\cdot, \cdot; \gamma)$ is an ordered security class satisfying Assumption 1.

2. for any $(\alpha, v) \in [\underline{\alpha}, \overline{\alpha}] \times [0, 1]$, $\bar{S}(\alpha, v; \gamma) := \mathbb{E}[S(\alpha, \tilde{Z}; \gamma)|v]$ is a continuous function of γ; and

3. for any pair $\gamma', \gamma'' \in [\underline{\gamma}, \overline{\gamma}]$ with $\gamma' < \gamma''$, $S(\cdot,\cdot;\gamma'')$ is steeper than $S(\cdot,\cdot;\gamma')$.

Some examples of ordered security classes with parametrized steepness include[11]

- Equity plus a fixed cash component γ: $S(\alpha, Z; \gamma) = (\bar{L} - \gamma) + \alpha Z$.
- Levered equity, with face value of debt γ: $S(\alpha, Z; \gamma) = \alpha \max\{Z - \gamma, 0\}$.
- Cash plus royalty rate γ: $S(\alpha, Z; \gamma) = \alpha + \gamma Z$.

For steepness comparisons along such parametrized paths, we show that, with flatter securities, bargaining inefficiencies (in the sense of expected discounting costs until the time of trade) drop for types outside an intermediate region.

Proposition 4 (Steepness raises discounting costs outside intermediate region)**.** *Let S be an ordered security class of parametrized steepness. Take $\gamma' < \gamma''$ such that $S(\cdot,\cdot;\gamma) \in \mathcal{D}_{c,\tilde{b},\tilde{X}}$ for all $\gamma \in [\gamma', \gamma'']$. For security $S(\cdot,\cdot;\gamma)$, let $\tau^{CE}(v;\gamma)$ denote the certainty-equivalent delay for v, and let $k^{SLC}(\gamma)$ denote the critical type. Then*

1. *Discounting costs decrease for types who (i) reach the impasse under both securities, or (ii) do not reach it under either security:*
$$\tau^{CE}(v;\gamma'') > \tau^{CE}(v;\gamma'), \quad \textit{for all } v \in (0,1] \setminus (k^{SLC}(\gamma'), k^{SLC}(\gamma'')]. \tag{15}$$
2. *Small steepness changes increase discounting costs for most types: for any $\delta > 0$, $\gamma \in [\gamma', \gamma'')$, there exists $\varepsilon > 0, [\gamma, \gamma+\varepsilon] \subset [\gamma', \gamma'']$, such that a measure $1-\delta$ of types endures strictly higher discounting costs under $S(\cdot,\cdot;\gamma+\varepsilon)$ than under $S(\cdot,\cdot;\gamma)$.*

Proof. We prove that, for all $\gamma \in [\gamma', \gamma'']$, $\tau^{CE}(1;\gamma)$ is strictly increasing in γ. Point 1 then follows from $\tau^{CE}(v;\gamma) = \tau^{CE}(1;\gamma)$ for all $v > k^{SLC}(\gamma)$ and Proposition 1.

Let $k^*(\gamma)$ be given by equation (11), evaluated at security $S(\cdot,\cdot;\gamma)$. By Proposition 2, for any $\gamma \in [\gamma', \gamma'']$ we have $k^*(\gamma) > k^{SLC}(\gamma)$.

Assume for the moment that $k^{SLC}(\cdot)$ is continuous—we prove this at the end. Then for all $\hat{\gamma} > \gamma$ sufficiently close to γ, $k^*(\gamma) > k^{SLC}(\hat{\gamma})$. By Proposition 3, for small enough $\xi > 0$, buyer $k^{SLC}(\hat{\gamma}) + \xi$ must then strictly prefer $S(\cdot,\cdot;\gamma)$ to $S(\cdot,\cdot;\hat{\gamma})$. And yet, by Proposition 2, in expectation $k^{SLC}(\hat{\gamma}) + \xi$ pays strictly less under $S(\cdot,\cdot;\gamma)$ than under $S(\cdot,\cdot;\hat{\gamma})$. It follows that $k^{SLC}(\hat{\gamma}) + \xi$ must suffer strictly higher discounting costs under $S(\cdot,\cdot;\hat{\gamma})$. Since $k^{SLC}(\hat{\gamma}) + \xi$ is in the final trading atom under both securities, $\tau^{CE}(1;\hat{\gamma}) > \tau^{CE}(1;\gamma)$, as required.

Finally, we verify the continuity of $k^{SLC}(\cdot)$. By the continuity of $\bar{S}(\alpha, v; \cdot)$, $\alpha^f(1;\cdot)$ is continuous, using the inverse function theorem. Now write $k^{SLC}(\gamma)$ as the solution to

$$\int_k^1 \bar{S}(\alpha^f(1;\gamma), v, \gamma)dv - (1-k)c = 0.$$

By the continuity of $\bar{S}$ and $\alpha^f(1;\cdot)$, $k^{SLC}(\cdot)$ is continuous, too. Given Point 1, this proves Point 2. □

[11]It follows from Lemma 5 in DeMarzo et al. (2005) that higher γ's correspond to steeper securities in these examples.

Even though types in the pooling region trade faster under a flatter security in a parametrized steepness class, a payoff ranking is not immediate. For one thing, high types pay strictly more in expectation under the flatter security; for another, there is an intermediate region of types who pool under a flat security but are separated under a steep security, and for those types no ranking of discounting costs is possible.

Nonetheless, our next result introduces a sufficient condition, easy to check in applications, that guarantees that all types prefer steeper securities within a family of parametrized steepness. With a slight abuse of notation, let $\alpha^f(1;\gamma)$ be the pooling offer under $S(\cdot,\cdot;\gamma)$ and let $\eta(v,\alpha;\gamma) := \frac{b'(v)-\bar{S}_v(\alpha,v;\gamma)}{b(v)-\bar{S}(\alpha,v;\gamma)}$.

Proposition 5. *Let S be an ordered security class of parametrized steepness, and let $U(v;\gamma)$ denote v's equilibrium indirect utility under $S(\cdot,\cdot;\gamma)$. Take $\gamma' < \gamma''$ such that $S(\cdot,\cdot;\gamma) \in \mathcal{D}_{c,\tilde{b},\tilde{X}}$ for all $\gamma \in [\gamma',\gamma'']$. If for any $v \in [0,1]$, $\gamma \in [\gamma',\gamma'']$, $\eta(v,\alpha^f(1;\gamma);\gamma)$ is decreasing in γ, then $U(v;\gamma)$ is also decreasing in γ, and strictly decreasing for $v \in (0,1)$.*

Proof. Following the argument in Proposition 4, for all $\hat{\gamma} > \gamma$ sufficiently close to γ, $k^*(\gamma) > k^{SLC}(\hat{\gamma})$. By Proposition 3, buyer types in $[0,k^*(\gamma)]$ must then prefer $S(\cdot,\cdot;\gamma)$ to $S(\cdot,\cdot;\hat{\gamma})$, strictly so for $v > 0$.

Using the payoff representation in Lemma 2, we can write the indirect utility for type $v \in (k^*(\gamma),1)$ under security $S(\cdot,\cdot;\gamma^\dagger)$, $\gamma^\dagger \in \{\gamma,\hat{\gamma}\}$ as

$$U(v;\gamma^\dagger) = X(v) + U(k^*(\gamma);\gamma^\dagger)\exp\left\{\int_{k^*(\gamma)}^{v} \eta(y,\alpha^f(1;\gamma^\dagger);\gamma^\dagger)dy\right\}. \tag{16}$$

It follows that from the assumption that $\eta(v,\alpha^f(1;\gamma);\gamma)$ is decreasing in γ, and from $U(k^*(\gamma);\gamma) > U(k^*(\gamma);\hat{\gamma})$, that

$$U(k^*(\gamma);\gamma)\exp\left\{\int_{k^*(\gamma)}^{v} \eta(y,\alpha^\dagger(1;\gamma);\gamma)dy\right\} \geq \\ U(k^*(\gamma);\hat{\gamma})\exp\left\{\int_{k^*(\gamma)}^{v} \eta(y,\alpha^\dagger(1;\hat{\gamma});\hat{\gamma})dy\right\}$$

Plugging this into the representation (16) above, $U(v,\gamma) \geq U(v,\hat{\gamma})$ for all $v \in [0,1]$, and strictly so for interior v's. Since $\gamma \in [\gamma',\gamma'']$ was arbitrary, we conclude that for any $U(v,\gamma)$ is a weakly decreasing function of γ, and strictly decreasing for interior v's. □

Our sufficient condition on $\eta(v,\alpha^f(1;\gamma);\gamma)$ is not stated in terms of the primitive securities, and it does not follows directly from steepness. However, the condition is easily verified when we study particular classes of securities, as we do in section 5.

5 Applications

In this section, we apply the previous results to study applications in which securities are a crucial part of the negotiation.

5.1 Mergers and Acquisitions with Financial Constrains

In this section, we apply our framework to a common situation in mergers and acquisitions: an acquirer and a target bargain over the terms of the merger using a combination of cash and equity in the merged entity. Even though using cash reduces bargaining frictions, companies in practice rely on other securities.

On reason why use shares as a mean of payment in acquisitions is financial constraints. The acquirer might not have enough cash in hand to complete the transaction, and external financing might be prohibitively expensive. Even if the company has sufficient cash, the opportunity cost of depleting its cash reserves may outweigh the efficiency benefits from negotiating in a less informationally sensitive security. For example, Alshwer et al. (2011) finds that financially constrained acquirers rely more on stock as a method of payment than financially unconstrained one ones. Other empirical studies have found that, even when acquirers have enough cash to complete a transaction, they tend to use stock as a means of payment if they are financially constrained.

We capture this situation by considering the case when a liquidity-constrained acquirer, who takes on the role of the privately informed buyer in the model, has cash holdings of $L > 0$.[12] The offers made by the seller consist of a fixed cash component L, "sweetened" by some negotiable amount of equity α in the merged firm. The security used for the negotiation is therefore $S(\alpha, Z; L) = L + \alpha Z$, where $L = 0$ corresponds to bargaining only in equity. The parties to the negotiation take the amount of cash used in the transaction as given, and we consider comparative statics with respect to the amount of cash.

As a shorthand, let S^L denote the function $S(\cdot, \cdot; L)$. We assume below that $S^0 \in \mathcal{D}_{\tilde{b},\tilde{X},c}$—there are non-trivial dynamics in the absence of cash—and $S^L \in \mathcal{D}_{\tilde{b},\tilde{X},c}$—there are non-trivial dynamics at the original liquidity level. We study the effect of increasing liquidity from L to $L' > L$ in this environment, assuming that $S^{L'}$ is also in $\mathcal{D}_{\tilde{b},\tilde{X},c}$. This exercise is well defined for L and L' sufficiently small:

Lemma 3. *If $S^0 \in \mathcal{D}_{\tilde{b},\tilde{X},c}$, then there exists $L^* < c$ such that $S^L \in \mathcal{D}_{\tilde{b},\tilde{X},c}$ for all $L \leq L^*$.*

It is easy to verify that, for any $L_1 < L_2$, $S^1 := S^{L_1}$ is steeper than $S^2 := S^{L_2}$. Hence, we can apply Propositions 1 through 5 to look at the effect that higher liquidity has on the outcome of the acquisition. These propositions imply:

Corollary 1. *Assume $S^0 \in \mathcal{D}_{\tilde{b},\tilde{X},c}$, and take two liquidity levels $L_1 < L_2 < L^*$ as in Lemma 3. Let $k^*(L_2) := Z^{-1}\left(Z(1)\frac{c-L_2}{b(1)-L_2}\right)$. Then*

1. *More buyer types reach the impasse when liquidity constraints are looser: $k^{SLC}(L_1) > k^{SLC}(L_2)$.*

[12] For the present discussion, L is not part of the standalone value, but needs to be raised for the purpose of bidding.

2. *Types outside* $\left(k^{SLC}(L_2), k^{SLC}(L_1)\right]$ *—those who do not reach the impasse under either security, and those who do reach the impasse under* both *securities—endure higher discounting costs when liquidity constraints are tighter.*

3. *When liquidity constraints are tighter, types above* $\min\{k^*(L_2), k^{SLC}(L_1)\}$ *pay more, and those below pay less.*

4. *Types below* $k^*(L_2)$ *prefer the looser liquidity constraint, strictly so for* $v > 0$.

5. *If* $\varphi(v) := \frac{b(1)-b(v)}{X(1)-X(v)}$ *is decreasing, then all types prefer the looser liquidity constraint, strictly so for interior types.*

To unpack the assumptions in the corollary, note that $S^0 \in \mathcal{D}_{\tilde{b},\tilde{X},c}$ requires that the standalone value is more sensitive to the private information than the joint return (Remark 1). Assumption 1 applied to S^L implies that $L < c$: the buyer's liquidity constraints are tight enough that the two parties cannot agree on a pure cash offer.

Next, we study the interaction between cash constraints and equilibrium delay. Since in a Markov equilibrium delay is random and discounting is a convex function of delay, it is essential to distinguish between the certainty-equivalent delay, which determines discounting costs and payoffs, from the expected delay that an outside observer is able to measure. Let $\tau^{sm}(v; L)$ ("sm" for "smooth") be the time at which $v \leq k^{SLC}(L)$ trades smoothly in equilibrium, and let $D(L)$ be the expected discount in the impasse phase for S^L, given by (8). Then we can distinguish between the *expected delay*

$$\tau^{Exp}(v; L) = \begin{cases} \tau^{sm}(v; L), & v \leq k^{SLC}(L), \\ \tau^{sm}(v; L) + \frac{1-D(L)}{rD(L)}, & v > k^{SLC}(L) \end{cases}$$

and the *certainty-equivalent delay*

$$\tau^{CE}(v; L) = \begin{cases} \tau^{sm}(v; L), & v \leq k^{SLC}(L), \\ \tau^{sm}(v; L) - \frac{\log D(L)}{r}, & v > k^{SLC}(L) \end{cases}$$

Specializing linear-uniform primitives, we show through numerical examples that

- Tightening up-front liquidity constraints can *uniformly raise* expected delay all types $v > 0$, or it can *uniformly lower* expected delay for all but a vanishing fraction $[0, \varepsilon)$

- Even in cases where expected delay is (almost) uniformly lower under tighter liquidity constraints, all types can uniformly prefer loosening that constraint.

We illustrate these points in the following parametric class:

Example 2 (Linear-Uniform Example)**.** Take $\tilde{b}|v \sim U[c + \Delta, c + \Delta + 2v]$ for some fixed $\Delta > 0$, and $\tilde{X}|v \sim U[0, 2av]$ for some fixed $a > 0$.

For these primitives, $X(v) = av$ and $b(v) = c + \Delta + v$ are linear functions of v, and we can solve for the equilibrium in closed form:

Corollary 2. *For the primitives in Example 2, the environment under S^L is upward-skimming and satisfies SLC*[13] *so long as*

$$L < \frac{2ac - (c + \Delta + 1)(2\Delta + (1 + a))}{(1 + a)} := L_{max}.$$

1. *For $L < L_{max}$, trade dynamics are given by*

$$\tau^{sm}(k; L) = \frac{(c - L)(1 + a)}{r(c - a\Delta)} \log\left(\frac{1}{\Delta}\frac{(c + \Delta)(\Delta + k)}{c + \Delta + (1 + a)k}\right)$$
$$k^{SLC}(L) = \frac{(1 + a)(1 + L) + c(2\Delta + 1 - a) + (3 + a)\Delta + 2\Delta^2}{(1 + a)(c + \Delta + 1 - L)}$$
$$D(L) = \frac{\Delta + k^{SLC}}{c + \Delta + k^{SLC} - L - \alpha^f(1)(c + \Delta + (1 + a)k^{SLC})}$$

2. *Take $L_1 < L_2 < L_{max}$. Types in $(k^{SLC}(L_1), 1]$ suffer less discounting* (trade sooner in certainty equivalent terms) *under L_2, and all buyer types prefer L_2, strictly so for $v \in (0, 1)$.*

In spite of Corollary 2.2, increasing liquidity below L_{max} can increase or decrease the expected delay suffered by any given buyer type. We present two extreme examples in Figure 2. The two rows correspond to two different parameter configurations; the left panel on a row depicts expected delay for two different liquidity levels, while the right panel on a row depicts certainty-equivalent delay. In the top row, corresponding to $c = 5, \Delta = 1/5, a = 15, r = 4, L_1 = 0, L_2 = 1.18$, all types suffer higher expected delay when liquidity is lower. For all types, expected delay and certainty-equivalent delay move in the same direction as L increases. By contrast, in the bottom row ($c = 5, \Delta = 1/5, a = 5, r = 1, L_1 = 1.79, L_2 \approx L_{max} - 10^{-3}$), all but an arbitrarily small fraction of types ($v < 10^{-3}$ in the simulation) suffer *lower* expected delay when liquidity is lower. This holds even though types in the pooling region of both securities suffer *higher* certainty equivalent delay when liquidity is lower, and even though all types prefer having higher liquidity. Altogether, the model suggests that a measure of caution is needed when interpreting empirical data on delay.

Real delay may be hard to measure in applications, since the researcher may not have a consistent way of identifying the point in time that corresponds to $t = 0$ in any given negotiation. However, by reinterpreting the discount rate r as the arrival rate of

[13] Under S^0, the environment is always upward-skimming, and it satisfies SLC when

$$2ac > (c + \Delta + 1)(2\Delta + (1 + a)).$$

an exogenous breakdown in negotiations, we can connect discounting costs in our model to observed probabilities of deal failure. If negotiations break down at a Poisson rate r, then $e^{-r\tau^{CE}(v;L)}$ is the probability of a negotiation failure for an acquirer of type v. With this interpretation, Corollary 1 indicates that negotiations with sufficiently low and sufficiently high types are more likely to fail if they have a larger equity component. This is broadly consistent with the evidence provided by Malmendier et al. (2016), if one focuses on small differences in the equity component of deals.[14] However, as Panel (d) in Figure 2 suggests, aggregate measures on the probability of deal failure may mask heterogeneous effects of large changes in equity components.

Another implication of our results is that the acquirer's ex-ante payoffs increase in the cash component in the acquisition. However, our analysis also shows that we need to be careful when we evaluate the return that acquirers obtain from acquisitions. Even when a larger cash component increases ex-antes payoffs (by reducing delay or increasing the probability of a deal), Proposition 2 shows that it may decrease ex post (observed) payoffs, as high types end paying more. Thus, even if empirically we observe that more cash-heavy offers reduce realized returns, those offers may still be in the (ex ante) best interests of the acquiring firm's shareholders.

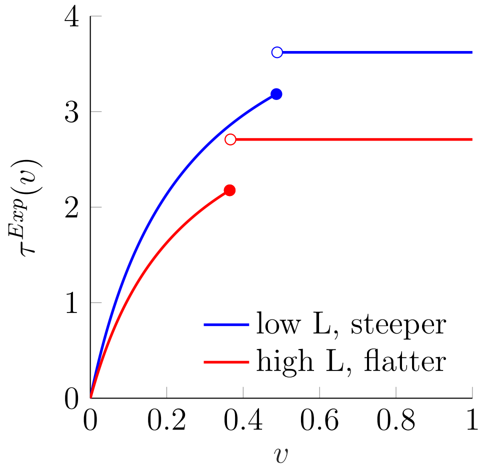
One drawback of our analysis is that we fix the cash level and only allow the parties to negotiate on the equity component (we study the dual problem of fixing the equity component and negotiating over cash in the next subsection). In reality, companies negotiate over the cash and equity components simultaneously. Analyzing this formally would require solving a multidimensional screening problem, which is beyond the scope of our model.[15]

5.2 Cash and Fixed Royalty Rate

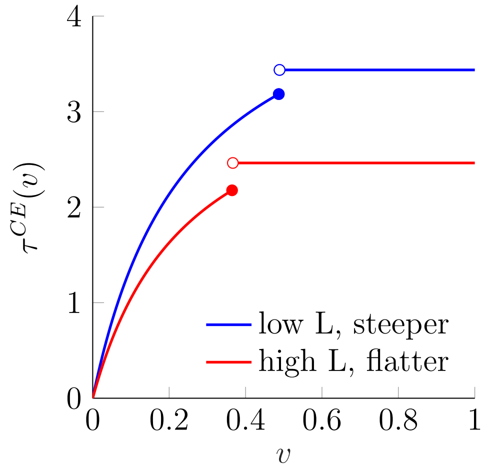
Payment schemes with royalties are another common example of contingent payments. Negotiations over natural resources often involve security offers, in the form of royalties. While many mineral leases are assigned through auctions, others, especially those for smaller, privately owned properties, are assigned through private negotiations. The terms of the lease usually consider cash payments as well as royalties over the proceeds. When the land is owned by the government, these royalties are often determined by law and not part of the negotiation. We model these interactions by considering a security family $\beta \in (0,1)$: $S(\alpha, \tilde{Z}; \beta) = \alpha + \beta\tilde{Z}$. That is, parties bargain over a cash transfer, given a

[14] In their sample, successful acquisitions have a smaller equity component than unsuccessful acquisitions (46% and 55%, respectively). Interestingly, on average, both tend to have similar cash component (45 % and 44%, respectively). However, the difference between successful and unsuccessful acquisition is significant if we ignore outliers, and we consider the median transaction. The median equity component among successful acquisitions is 43%, while the median equity component among unsuccessful acquisitions is 75%. Similarly, while the median cash component among successful acquisitions is 37%, the median cash component among failed acquisitions is 19%.

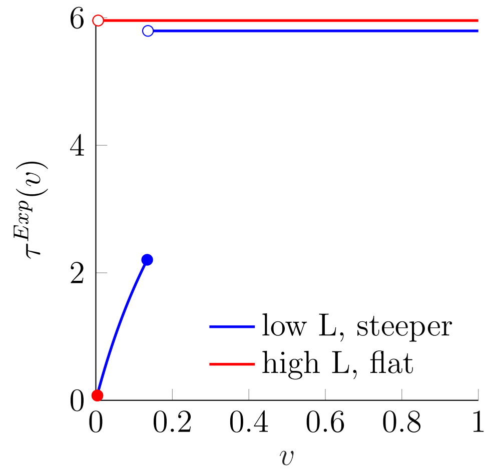
[15] See, for instance, the analysis in Strulovici (2017), who studies two parties negotiating over a multidimensional contract space. However, extending his results from a purely private value, finite-type environment to a common value, continuous-type environment would be a highly non-trivial task.



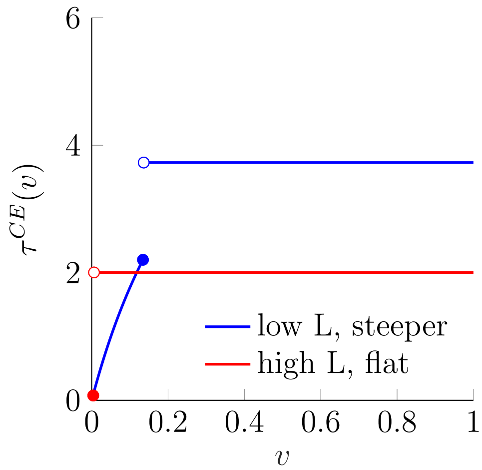
(a) Expected Delay



(b) Certainty-Equivalent Delay



(c) Expected Delay



(d) Certainty-Equivalent Delay

Figure 2: Discounting and observable delay are distinct. *Panels (a) and (b)*: parameters are $c = 5, \Delta = 1/5, a = 15, r = 4, L_1 = 0, L_2 = 1.18$. All types suffer higher expected delay when liquidity is lower. *Panels (c) and (d)*: parameters are $c = 5, \Delta = 1/5, a = 5, r = 1, L_1 = 1.79, L_2 = 2.98$. Types in the pooling region suffer *higher* certainty equivalent delay when liquidity is lower, and all types prefer having higher liquidity. However, all but a vanishing fraction of types suffer *lower* expected delay when liquidity is lower.

fixed royalty rate.

We focus again on the primitives in Example 2, and we illustrate two facts:

- Minor changes in the royalty contract (that is, in the security) can dramatically change the outcome of the negotiation and the payoffs to both parties.
- For parts of the parameter space in which trade happens instantly, both parties' payoffs are non-monotone in steepness. When the environment is downward-skimming, steeper securities benefit the seller at the expense of every type of buyers; when the environment is upward-skimming, steeper securities benefit every type of buyer at the expense of the seller.[16]

First, we show how equilibrium delay changes discontinuously in the security. Minor algebra yields $\alpha^f(1;\beta) = b(1) - \beta Z(1)$. The environment is upward skimming if

$$\beta > \sup_{v\in[0,1]} \frac{b'(v)}{b'(v) + X'(v)},$$

while the SLC holds so long as

$$1 > \beta > \frac{b(1) - c}{Z(1) - \mathbb{E}[\tilde{Z}]}.$$

In Example 2, these conditions amount to, respectively,

$$\beta > \frac{1}{1+a} \quad \text{and} \quad 1 > \beta > \frac{2(\Delta+1)}{(1+a)}$$

Let $\bar{\beta} := 2(\Delta + 1)/(1 + a)$. Then it follows that, when $\bar{\beta} < 1$, $\bar{\beta}$ is the maximum royalty that leads to instant trade in equilibrium, and for all $\beta \in ((1 + a)^{-1}, \bar{\beta})$, there is instant trade at an offer of $\alpha^f(1;\beta)$. On the other hand, as β tends to $\bar{\beta}$ from above, $k^{SLC}(\beta) \downarrow 0$, and for an arbitrarily small $\varepsilon > 0$, all types in $[\varepsilon, 1]$ have an expected discount until trading close to

$$\lim_{\beta\downarrow\bar{\beta}} \frac{b(k^{SLC}(\beta)) - c}{b(k^{SLC}(\beta)) - \bar{S}(\alpha^f(1;\beta), k^{SLC}(\beta);\beta)} = \frac{\Delta}{\bar{\beta}(1+a) - 1} = \frac{\Delta}{2\Delta + 1} \tag{17}$$

An arbitrarily small change in the royalty rate around $\bar{\beta}$ therefore has dramatic effects on the equilibrium delay. Our result here echoes Deneckere and Liang (2006).[17] The contribution is to illustrate (i) the additional tractability offered by a continuous time formulation, and (ii) how the discontinuity can be triggered by minor changes in the security.

[16] This echoes Che and Kim (2010)'s findings in a static security auction setting. See Section 6 for more details.

[17] One can show from their Theorem 3 that the limiting delay changes discontinuously as the static inefficiency condition ceases to hold.

Using the primitives in Example 2, we can also look at the impact of the royalty rate on the interim payoffs of the buyer. The environment is downward-skimming if $\beta \leq 1/(1+a)$, which leads to instant trade at an offer of $\alpha^f(0;\beta) = b(0) - \beta Z(0)$. If instead $\beta \in \big((1+a)^{-1}, \bar{\beta}\big)$, as seen above, the environment is upward-skimming, with instant trade at an offer of $\alpha^f(1;\beta) = b(1) - \beta Z(1)$. Altogether, for $\beta < \bar{\beta}$, equilibrium payoffs for type v are

$$U(v;\beta) = \begin{cases} Z(v) - (b(0) - \beta Z(0)) - \beta Z(v) = av + (1 - \beta(1+a))v, & \beta \in \left(0, \frac{1}{1+a}\right), \\ Z(v) - (b(1) - \beta Z(1)) - \beta Z(v) = av + (\beta(1+a) - 1)(1-v), & \beta \in \left(\frac{1}{1+a}, \bar{\beta}\right). \end{cases} \tag{18}$$

For all interior types, payoffs are strictly decreasing in β for $\beta < 1/(1+a)$, and strictly increasing for $\beta \in (1/(1+a), \bar{\beta})$.

In addition, payoffs are discontinuous at $\beta = \bar{\beta}$. The function $\eta(v, \alpha^f(1;\beta);\beta)$ is non-decreasing in β for all v if $\varphi(v) = \frac{b(v)-b(1)}{X(v)-X(1)}$ is non-increasing in v. For linear-uniform primitives, $\varphi(v)$ is constant, so Proposition 5 implies that $U(v;\beta)$ is *decreasing* in β for $\beta > \bar{\beta}$. Using the limiting expected discount in (17),

$$\lim_{\beta \downarrow \bar{\beta}} U(v;\beta) = \left(\frac{\Delta+1}{2\Delta+1}\right) av + \left(\frac{\Delta}{2\Delta+1}\right)(\bar{\beta}(1+a) - 1)(1-v) < \lim_{\beta \uparrow \bar{\beta}} U(v;\beta).$$

6 Discussion

Connection to Security Auction Literature In the security auctions literature, Che and Kim (2010) first pointed out that having the standalone value $X(v)$ be increasing in v could lead to decreasing bidding strategies, i.e., higher types bid less and are less likely to win the auction. They study the effect on revenue of moving from a flat security S^2 to a steeper one S^1, when both have equilibria in decreasing strategies.[18]

With the usual mapping between higher probability of winning and lower expected discounting costs, their comparative statics correspond to our comparison between S^1, $S^2 \in \mathcal{D}_{c,\tilde{b},\tilde{X}}$. We now contrast our results to theirs.

First, Che and Kim (2010) prove, both for first and second price auctions, that expected payments are higher type-by-type under S^2, the flatter security.[19] The difference

[18] Their results are more general, but we emphasize this aspect of them to streamline the comparison to ours. In fact, they show that, whenever S^2 has decreasing bidding strategies ("upward-skimming," in our terminology), S^1 also will. Moreover, they show that upward-skimming steeper securities yield lower revenue than flatter securities, regardless of whether the flatter ones are upward- or downward-skimming. In our setup, it is easy to show that if SLC holds for S^2, then it must hold for S^1, but it does not follow that if S^2 is upward-skimming, S^1 will also be.

[19] We refer to the working paper version, which contains results on both auction formats (available on the authors' website here: `https://emu-perch-bjgm.squarespace.com/s/security-comment-1.pdf`). Proposition 3 in that version shows that payments are strictly higher *ex-post* in the second price auction. The proof of their Proposition 5 shows that interim utilities are strictly higher (except for $v = 1$) under S^1; since the allocations are the same in both cases (both security auctions have decreasing strategies), interim expected payments must be higher under S^2.

between our results and theirs stems from the seller's extreme commitment problem in our model. Because of her extreme lack of commitment, *the seller makes exactly c on every trade.* Therefore,

- Low types ($v \in [0, k^{SLC}(S^2)]$) face an expected payment that is *constant* across securities, since they are separated under either security.
- Intermediate types ($v \in (k^{SLC}(S^2), k^{cross})$) pay strictly less under S^1, since types in that range are separated in the S^1 equilibrium, but they get cross-subsidized (by types above k^*) in the S^2 equilibrium.

A consequence of the revenue rankings in Che and Kim (2010) is that in their model, among two securities that induce decreasing strategies bidders weakly prefer the steeper one (strictly so for types below the highest). In our model, that preference can be reversed uniformly for all types—even for types who pay more under the flatter security—and in fact is always reversed for types under k^*. The reason is as follows: the allocation in Che and Kim (2010)'s comparison is the same across both securities, so payment rankings translate into utility rankings. Meanwhile, in our bargaining game, changing the security changes the amount of delay and the expected allocation; the impact of this allocation change on payoffs can overwhelm the impact of the higher expected payments under S^2.[20]

The role of increasing net surplus We assumed throughout that $b' \geq 0$. Here we describe the role of this assumption and how the equilibrium changes when it is relaxed. Note that there is nothing pathological about a strictly decreasing b.[21] For example, in the M&A setting, $b(v)$ is a measure of the synergies in a merger, which can be higher or lower for high types. Suppose that the buyer is acquiring the seller for access to a proprietary technology. v measures how close the buyer is to the technological frontier. A higher v would then increase the expected standalone value X and may even increase the total value Z, but the marginal value of the seller's technology $Z - X = b$ can be lower the closer the buyer is to the technological frontier.

So long as the non-degeneracy conditions in Assumption 1 hold, and the environment is upward-skimming, a b with decreasing portions does not change the equilibrium analysis, with one key exception. Even if $b(0) > c$—i.e., there is a gap at the bottom—one could have $b(1) = c$—i.e., there is no gap at the *top*. In such a case, per Theorem 1, we avoid a complete breakdown of trade, but there will be no final atom. Consider the SLC. Since $c = b(1) := \bar{S}(\alpha^f(1), 1)$, $\mathbb{E}[\bar{S}(\alpha^f(1), v) | v \in [k, 1]] < c$ for all k. In other words, no matter how many types the seller has screened, adverse selection is always severe enough that she prefers trading with the marginal type to trading with the remaining types.

[20]To emphasize this point—that our departure from the results in Che and Kim (2010) and DeMarzo et al. (2005) is "allocation-driven," consider two upward-skimming securities that both fail SLC. Since all types trade instantly in either case, the allocation, meaning the expected discount until trade, is constant across securities. Then, given that all types pay $\bar{S}^i(\alpha^{f,i}(1), v)$ under S^i, it follows from (10) that flatter securities lead to higher revenue and make buyers worse off, as in Che and Kim (2010). Taking two downward-skimming securities, we get the opposite result, as in DeMarzo et al. (2005).

[21]We are grateful to Brett Green for suggesting this possibility.

The ranking results in Section 4 also simplify considerably: if $b(0) > b(1) = c$, for any two upward-skimming securities ranked by steepness, it follows that all types trade faster with the flatter security, and they are better off for it.

We can contrast this $b(1) = c$ case to the examples in Section 5.1. There we showed that flatter securities could increase bargaining frictions, raising discounting costs for some types and even raising expected delay for practically all types. The comparison suggests that whether or not $b(1) > c$ is a crucial condition for determining the equilibrium effects of changing securities.

References

Alshwer, Abdullah A, Valeriy Sibilkov, and Nataliya S Zaiats (2011), "Financial constraints and the method of payment in mergers and acquisitions." *Available at SSRN 1364455*.

Chaves, Isaías N. (2019), "Privacy in Bargaining: The Case of Endogenous Entry." Working paper, Northwestern University, Kellogg School of Management.

Che, Yeon-Koo and Jinwoo Kim (2010), "Bidding with Securities: Comment." *American Economic Review*, 100, 1929–35.

Coddington, Earl A. and Norman Levinson (1955), *Theory of Ordinary Differential Equations*. McGraw-Hill Book Company, Inc.

Daley, Brendan and Brett Green (2020), "Bargaining and News." *American Economic Review*, 110, 428–74.

DeMarzo, Peter and Darrell Duffie (1999), "A Liquidity-based Model of Security Design." *Econometrica*, 67, 65–99.

DeMarzo, Peter M, Ilan Kremer, and Andrzej Skrzypacz (2005), "Bidding with Securities: Auctions and Security Design." *American Economic Review*, 95, 936–959.

Deneckere, Raymond and Meng-Yu Liang (2006), "Bargaining With Interdependent Values." *Econometrica*, 74, 1309–1364.

Edlin, Aaron S and Chris Shannon (1998), "Strict Monotonicity in Comparative Statics." *Journal of Economic Theory*, 81, 201–219.

Fanning, Jack (2016), "Mediation in reputational bargaining." Working Paper, Brown University.

Fishman, Michael J (1989), "Preemptive Bidding and the Role of the Medium of Exchange in Acquisitions." *Journal of Finance*, 44, 41–57.

Fuchs, William and Andrzej Skrzypacz (2010), "Bargaining With Arrival Of New Traders." *American Economic Review*, 100, 802–836.

Fudenberg, Drew, David K. Levine, and Jean Tirole (1985), "Infinite-Horizon Models of Bargaining With One-Sided Incomplete Information." In *Game-theoretic Models of Bargaining* (Alvin E. Roth, ed.), 73–98, Cambridge University Press, Cambridge, UK.

Fudenberg, Drew and Jean Tirole (1985), "Preemption and rent equalization in the adoption of new technology." *Review of Economic Studies*, 52, 383–401.

Gul, Faruk, Hugo Sonnenschein, and Robert Wilson (1986), "Foundations of dynamic Monopoly and the Coase Conjecture." *Journal of Economic Theory*, 39, 155–190.

Hanazono, Makoto and Yasutora Watanabe (2018), "Equity bargaining with common value." *Economic Theory*, 65, 251–292.

Hansen, Robert G (1985), "Auctions With Contingent Payments." *American Economic Review*, 75, 862–865.

Hansen, Robert G (1987), "A theory for the choice of exchange medium in mergers and acquisitions." *Journal of Business*, 75–95.

Malmendier, Ulrike, Marcus M Opp, and Farzad Saidi (2016), "Target revaluation after failed takeover attempts: Cash versus stock." *Journal of Financial Economics*, 119, 92–106.

Milgrom, Paul (2004), *Putting Auction Theory to Work*. Cambridge University Press, Cambridge.

Milgrom, Paul and Chris Shannon (1994), "Monotone Comparative Statics." *Econometrica*, 62, 157–157.

Milgrom, Paul R. and Robert J. Weber (1982), "A Theory of Auctions and Competitive Bidding." *Econometrica*, 1089–1122.

Myers, Stewart C. and Nicholas S. Majluf (1984), "Corporate Financing and Investment Decisions When Firms Have Information That Investors do not Have." *Journal of Financial Economics*, 13, 187–221.

Ortner, Juan (2017), "Durable Goods Monopoly with Stochastic Costs." *Theoretical Economics*, 12, 817–861.

Rhodes-Kropf, Matthew and S. Viswanathan (2004), "Market valuation and merger waves." *Journal of Finance*, 59, 2685–2718.

Smith, Lones and Ennio Stacchetti (2002), "Aspirational bargaining." Working Paper, NYU.

Strulovici, Bruno (2017), "Contract Negotiation and the Coase Conjecture: A Strategic Foundation for Renegotiation-Proof Contracts." *Econometrica*, 85, 585–616.

Teschl, Gerald (2012), *Ordinary Differential Equations and Dynamical Systems*, volume 140. American Mathematical Society.

Tung, Lam Wing (2020), "Inefficient sorting under output sharing." *Journal of Economic Theory*, 105031.

White, Michelle J (1989), "The corporate bankruptcy decision." *Journal of Economic Perspectives*, 3, 129–151.

Omitted Proofs

Proof of Lemma 1. We prove the case of strictly increasing $\iota^S(\cdot, \alpha)$ (the argument for an decreasing $\iota^S(\cdot, \alpha)$ is symmetric). The statements on α^f follow by implicit differentiation: since α^f solves $b(v) = \bar{S}(\alpha^f(v), v)$,

$$\frac{\partial}{\partial v}\alpha^f(v) = \frac{b'(v) - \bar{S}_v(\alpha^f(v), v)}{\bar{S}_\alpha(\alpha^f(v), v)} - \overbrace{\left[b(v) - \bar{S}(\alpha^f(v), v)\right]}^{=0} \frac{\bar{S}^{\alpha v}(\alpha^f(v), v)}{\bar{S}_\alpha(\alpha^f(v), v)^2}$$
$$= -\frac{\partial}{\partial v}\iota^S(v, \alpha^f(v)) < 0 \quad (19)$$

We separate the statement on $T(v)$ into two claims:

1. First, we show that selections from $\arg\max_{t\in\mathbb{R}_+} V(t, \tilde{\alpha}_t, v)$ are non-decreasing.
2. Then we show that if $t = +\infty \in T(\underline{v})$, then for any $\bar{v} > \underline{v}$, $T(\bar{v}) = \{+\infty\}$. Formally, if $\sup_{t\in R_+} V(t, \tilde{\alpha}, \underline{v}) = X(\underline{v})$, then $\sup_{t\in R_+} V(t, \tilde{\alpha}, \bar{v}) = X(\bar{v})$ and $V(t, \tilde{\alpha}_t, \bar{v}) < X(\bar{v})$ for all $t \in \mathbb{R}_+$.

Claim 1: The key step is an argument by Milgrom and Shannon (1994) and Edlin and Shannon (1998). Edlin and Shannon (1998)'s Theorem 2 has additional conditions that are violated in our setting, but which are only necessary to derive their conclusions on strict comparative statics. For completeness, we reproduce here the part of the argument that suffices for our purposes:

Definition 10. For $V : \mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}] \times [0, 1] \to \mathbb{R}$, V satisfies the strict Spence-Mirrlees condition in $((t, \alpha), v)$ if if V is C^1, $V_t/|V_\alpha|$ is strictly increasing in t, and $V_\alpha \neq 0$ and has a constant sign.

Lemma 4 (Adapted from Theorem 2 in Edlin and Shannon (1998))**.** *Assume V : $\mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}] \times [0, 1]$ satisfies the strict Spence-Mirrlees condition and has path-connected indifference sets. Then every selection from $\arg\max_{t\in\mathbb{R}_+} V(t, \tilde{\alpha}_t, v)$ is non-decreasing.*

Proof. By Theorem 3 in Milgrom and Shannon (1994), V is strictly single crossing in $((t, \alpha); v)$, where $\mathbb{R}_+ \times [0, 1]$ is endowed with the lexicographic order. With that order on $\mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}]$, V is quasisupermodular in (t, α) and the set $\{(t, \alpha) : \alpha = \hat{\alpha}_t\}$ is a sublattice of $\mathbb{R}_+ \times [0, 1]$. The result then follows by Theorem 4' in Milgrom and Shannon (1994). $\square$

The Spence-Mirrlees condition follows from simple calculus: using $V_\alpha = -e^{-rt}\bar{S}_\alpha < 0$,

$$\frac{V_t}{|V_\alpha|} = -\frac{r(b(v) - \bar{S}(\alpha, v))}{\bar{S}_\alpha(\alpha, v)} = r\iota^S(\alpha, v). \quad (20)$$

so V satisfies the strict Spence-Mirrlees condition whenever $\iota^S(\cdot, v)$ is strictly increasing.

To show complete regularity of V, fix v and $\underline{t} < \bar{t}$ and $\underline{\alpha}, \bar{\alpha}$ such that $V(\bar{t}, \bar{\alpha}, v) = V(\underline{t}, \underline{\alpha}, v) = \bar{u}$. We construct a continuous function $\tilde{\alpha} : [\underline{t}, \bar{t}] \to [0, 1]$ satisfying $\tilde{\alpha}(\underline{t}) = \underline{\alpha}$, $\tilde{\alpha}(\bar{t}) = \bar{\alpha}$; then $\{(t, \tilde{\alpha}(t), t \in [\underline{t}, \bar{t}]\}$ is a path connecting $(\underline{t}, \underline{\alpha})$ and $(\bar{t}, \bar{\alpha})$.

By the Implicit Function Theorem, since $V_\alpha < 0$, for any $t_0 \in [\underline{t}, \bar{t}]$, there exists some open neighborhood $\mathcal{O} \subset [\underline{t}, \bar{t}]$ with $t_0 \in \mathcal{O}$ and a $C^1(\mathcal{O})$ function $\tilde{\alpha} : \mathcal{O} \to [0, 1]$ satisfying

$$\tilde{\alpha}'(t) = -\frac{V_t(t, \tilde{\alpha}(t), v)}{V_\alpha(t, \tilde{\alpha}(t), v)}, t \in \mathcal{O}, \quad V(t_0, \tilde{\alpha}(t_0), v) = \bar{u},$$

i.e., $\tilde{\alpha}$ is in fact a local solution to an initial value problem.[22]

We extend the solution to the IVP to yield the desired function. Take the open domain $\mathcal{D} = (\underline{t}, \bar{t}) \times (\underline{\alpha}, \bar{\alpha})$. We only show how to extend $\tilde{\alpha}$ continuously rightward up to $\bar{t}$, since extending it leftward to $\underline{t}$ is done symmetrically. Since V is C^1, $V_t = -(b(v) - \bar{S}(\alpha, v))$ is bounded below by $-X(v)$ and above by $b(v)$, and $V_\alpha < 0$, $g(t, \alpha) := -V_t(t, \alpha, v)/V_\alpha(t, \alpha, v)$ is continuous and bounded on $\mathcal{D}$. By standard extension theorems (Lemma 2.14 in Teschl (2012) and Theorem 4.1 in Coddington and Levinson (1955)), either $\tilde{\alpha}$ can be extended rightwards inside $\mathcal{D}$ to all of $[t_0, \bar{t})$, or there is is some $t' \in (t_0, \bar{t}]$ such that $\tilde{\alpha}$ extends rightwards up to $[t, t')$ with $\tilde{\alpha}(t') = \bar{\alpha}$.[23] In the latter case, if $t' = \bar{t}$, we are done, so suppose $t' < \bar{t}$. Since $V(t', \bar{\alpha}, v) = \bar{u} = V(\bar{t}, \bar{\alpha}, v)$, by Rolle's theorem there exists some $t'' \in (t', \bar{t})$ such that $V_t(t'', \bar{\alpha}, v) = 0$. However, this is only possible if $b(v) = \bar{S}(\bar{\alpha}, v))$, in which case $V_t(t, \bar{\alpha}, v) = 0$ for all $t \in [t', \bar{t}]$, and we can trivially extend $\tilde{\alpha}$ rightwards to all of $[t_0, \bar{t}]$ while satisfying the right boundary condition. In the former case, $V(\bar{t}-, \tilde{\alpha}(\bar{t}-), v) = \bar{u} = V(\bar{t}, \bar{\alpha}, v)$; since V is continuous and $V_\alpha < 0$, $\tilde{\alpha}(\bar{t}-)$ must in fact equal $\bar{\alpha}$.

Claim 2: If $\sup_{t \in R_+} V(t, \tilde{\alpha}, \underline{v}) = X(\underline{v})$, so that $t = +\infty$ achieves that supremum, it must be that, for all $t \in \mathbb{R}_+$, $b(\underline{v}) \leq \bar{S}(\tilde{\alpha}_t, \underline{v})$. But then, using $\bar{S}_\alpha < 0$, it follows that, for all $t \in \mathbb{R}_+$,

$$\tilde{\alpha}_t \geq \alpha^f(\underline{v}) > \alpha^f(\bar{v}).$$

where the strict inequality was shown in (19). Therefore, using $\bar{S}_\alpha < 0$ and $b(\bar{v}) = \bar{S}(\alpha^f(\bar{v}), v)$, we have that, for all $t \in \mathbb{R}_+$,

$$b(\bar{v}) < \bar{S}(\tilde{\alpha}_t, \bar{v}) \Rightarrow V(t, \tilde{\alpha}_t, \bar{v}) < X(\bar{v}) \text{ and } \sup_{t \in R_+} V(t, \tilde{\alpha}, \bar{v}) = X(\bar{v})$$

We conclude that $\arg\max_{t \in \mathbb{R}_+ \cup \{+\infty\}} V(t, \tilde{\alpha}_t, \bar{v}) = \{+\infty\}$, as required.

□

Proof of 3. A minor calculation yields $\iota^{S^L}(v, \alpha) = -[b(v) - L]Z(v)^{-1} + \alpha$. $\iota^{S^0}(v, \alpha)$ is increasing in v for every α if and only iff b/X is strictly decreasing. Hence, by continuity, there exists some L_a small enough that, for all $L \leq L_a$, $\iota^{S^L}(v, \alpha)$ remains strictly increasing in v for all α.

Let $\alpha^f(1; L) = \frac{b(1)-L}{Z(1)}$ denote the final offer that makes $v = 1$ just indifferent under S^L. Using $\alpha^f(1; 0) = \frac{b(1)}{Z(1)}$, SLC holds under S^0 (equity bargaining) iff $\mathbb{E}[\tilde{Z}]\frac{b(1)}{Z(1)} < c$, while

[22] Since $V_\alpha < 0$, one can solve for $\tilde{\alpha}(t_0)$ in $V(t_0, \tilde{\alpha}(t_0), v) = \bar{u}$.

[23] To be precise, $\tilde{\alpha}$ extends up to $[t, t')$ and $\tilde{\alpha}(t'-) = \bar{\alpha}$.

SLC holds under S^L iff

$$L \leq \frac{c\left[1 - \frac{b(1)}{c}\frac{\mathbb{E}[\tilde{Z}]}{Z(1)}\right]}{1 - \frac{\mathbb{E}[\tilde{Z}]}{Z(1)}}$$

Therefore, whenever SLC holds under S^0, there exists $L_b \in (0, c)$ such that SLC holds under S^L for all $L \leq L_b$. Taking $L^* = \min\{L_a, L_b\}$ concludes the proof. $\square$

A Necessary Conditions

Proof of Theorem 1, Necessary Conditions.

Downward Skimming The proof for the downward skimming case proceeds by contradiction. We assume the existence of an equilibrium with smooth screening and show that the seller has incentive to accelerate trade as much as possible, which contradicts the optimality of smooth screening.

Since higher types trade first, the seller's beliefs are right-truncations of the prior, and the truncation cutoff k is the Markov state controlled by the seller. Let k' be a state in the interior of a smooth trade region. Then, for some $\varepsilon > 0$, the seller's HJB at all $k \in (k' - \varepsilon, k' + \varepsilon)$ is

$$rJ(k) = \sup_{\dot{k} \geq 0} \big(\underbrace{\bar{S}(\alpha(k), k)}_{\mathbb{E}[S(\alpha(k), Z)|k]} - J(k)\big) \frac{|\dot{k}|}{k} - J'(k)|\dot{k}| + rc.$$

For $|\dot{k}| < \infty$ to be indeed optimal for the seller, it must be that for all $k \in (k' - \varepsilon, k + \varepsilon)$,

$$\mathbb{E}[S(\alpha(k), Z)|k] \leq J(k) + J'(k)k \Rightarrow J(k) = c. \tag{21}$$

Consider first the case in which $|\dot{k}| \in (0, \infty)$ is optimal. Then the inequality above is an equality, and $\mathbb{E}[S(\alpha(k'), Z)|k'] = J(k') = c$. In particular, $\alpha(\cdot)$ is C^1 in the region $(k' - \varepsilon, k' + \varepsilon)$. Differentiating both sides with respect to k',

$$\alpha'(k)\bar{S}_\alpha(\alpha(k), k) = -\bar{S}_v(\alpha(k), k) \tag{22}$$

Meanwhile, for buyer K_t in the interior of a smooth trade region to buy at t and not mimic any type in that region, the following local incentive constraint is necessary:

$$r(Z(K_t) - \bar{S}(\alpha(K_t), K_t)) = rX(K_t) - \dot{K}_t\alpha'(K_t)\bar{S}_\alpha(\alpha(K_t), K_t).$$

Plugging in (22) and the seller's indifference condition (21), we obtain

$$\dot{K}_t = \frac{r(b(K_t) - c)}{\bar{S}_v(\alpha(K_t), K_t)}. \tag{23}$$

In particular, $\dot{K}_t \geq 0$, contradicting either forward skimming or the optimality of $|\dot{K}_t| \in (0, \infty)$ at time t for the seller.

If $\dot{k} = 0$ is (strictly) optimal for the seller on an interval of time with positive measure, by a simple dynamic programming argument, indefinitely ceasing trade is also optimal and $J(k) = c$. For no trade to be optimal, the reservation offer $\hat{\alpha}(k)$ of buyer k must satisfy $Z(k) - \bar{S}(\hat{\alpha}(k), k) = X(k)$. In other words, $\hat{\alpha} = \alpha^f$, where the latter is defined in Definition 2. By the arguments in Lemma 1, $\hat{\alpha}(\cdot)$ must be weakly increasing. The seller can therefore offer $\alpha^f(0)$ to trade with all remaining types $[0, k]$ at prices of $\hat{\alpha}(0)$ (create an atom of trade of size k to obtain

$$\mathbb{E}[\bar{S}(\alpha^f(0), v)|v \le k] > \bar{S}(\alpha^f(0), 0) = b(0) \ge c,$$

where the strict inequality follows from the nondegeneracy conditions in Assumption 1 and $\alpha^f(0) \neq \underline{\alpha}$.[24]

Altogether, in a downward skimming environment, there can be no smooth trade and no quiet periods. Given that jumps are well-separated in an regular equilibrium, this implies that the equilibrium must consist of a single atom of measure 1, i.e., instant trade.

Upward Skimming Higher types trade *later*, so the seller now controls the *left* truncation of her posterior beliefs as a Markov state.

As before, we first identify implications of smooth trading, which then guide the analysis of all possible dynamics. The HJB for k in the interior of a smooth trade region (in cutoff space) is now

$$rJ(k) = \sup_{\dot{k} \ge 0} \left(\bar{S}(\alpha(k), k) - J(k)\right)\frac{\dot{k}}{1-k} + J'(k)\dot{k} + rc. \tag{24}$$

Therefore $J(k) = c$. If in addition, $\dot{k} \neq 0$ at such a state, $\bar{S}(\alpha(k), k) = c$.

Second, we show that if there is smooth trade, it happens only on a set of states $[0, k^{smooth})$, i.e., the games starts with smooth trading and ends with a jump. Suppose that, on some continuation game, there were a jump from k to $k' > k$. Since there are countably many jumps, and jumps are isolated,[25] smooth trade must recommence at k'. In particular, $J(k') = c$ and $\mathbb{E}[S(\alpha(k'), \tilde{Z})|v = k'] \le c$. The seller's payoff from jumping to k' is therefore

$$\left(\frac{k'-k}{1-k}\right)\mathbb{E}[\bar{S}(\alpha(k'), v)|v \in [k, k')] + \left(\frac{1-k'}{1-k}\right)c.$$

The seller can always freeze trade and ensure a payoff of c, so for such a jump to be optimal,

$$c \le \mathbb{E}[\bar{S}(\alpha(k'), v)|v \in [k, k')] \le \mathbb{E}[\bar{S}(\alpha(k'), v)|v = k'] \le c.$$

[24] $\alpha^f(0)$ satisfies $\bar{S}(\alpha^f(0), 0) = b(0) \ge c$, so it must be greater than $\underline{\alpha}$.

[25] If jumps discontinuities were not isolated, then a jump from k to k^1 would be followed almost immediately by a jump from k^1 to k^2. However, this gives the types in $(k^1, k^2]$ no incentive to reject the offer that led to state k^1. Buyer optimality would therefore require all of $[k, k^2]$ to accept instantly. Hence, the distance between two jumps must be bounded away from zero.

If the second inequality were an equality, then $\alpha(k') = \underline{\alpha}$, by the nondegeneracy condition in Assumption 1. But that would contradict the last inequality, since $\bar{S}(\underline{\alpha}, k') < c$. The second inequality must therefore be strict, which is a contradiction. Therefore, the set of smooth trade states must be an interval $[0, k^{smooth})$.

Third, we show that $k^{smooth} = k^{SLC}$. By Condition 2 in the equilibrium definition, buyer $v = 1$'s equilibrium reservation price $\alpha(1)$ must be *equal* to $\alpha^f(1)$, the highest take-it-or-leave-it offer that he would acccept. Hence, if $k^{smooth} < k^{SLC}$, so that continuation play prescribes a jump before the state reaches k^{SLC}, then it is weakly optimal for the seller to jump directly to $k = 1$ at k^{smooth}. Her payoffs at k^{smooth} are therefore $\mathbb{E}[S(\alpha(1), \tilde{Z})|v \in [k^{smooth}, 1]] < c$, by definition of k^{SLC}. This violates the seller's individual rationality, so we must have $k^{smooth} \geq k^{SLC}$. However, if $k^{smooth} > k^{SLC}$, then for any state $k \in (k^{SLC}, k^{smooth})$, the seller can jump the state to $v = 1$ with an offer of $\alpha^f(1)$; this gives her a payoff of $\mathbb{E}[S(\alpha^f(1), \tilde{Z})|v \in [k, 1]] > c$, a profitable deviation. In particular, $k^{smooth} = k^{SLC}$ implies that, if the SLC fails, all equilibria have instant trade.

Fourth, we show that, for $k \in (0, k^{SLC})$, the optimal $\dot{k}$ in (24) must be strictly positive, i.e., there are no quiet periods. Suppose otherwise. By a typical dynamic programming argument, if starting at state $k \in (0, k^{SLC})$ we have $\dot{k} = 0$, then the continuation value of the marginal buyer equals $X(k)$, and $\alpha(k) = \alpha^f(k)$ as defined in Lemma 1. But then $\bar{S}(\alpha(k), k) = b(k) > c$, so (24) fails, a contradiction.

Finally, we show that if the SLC holds, but there is no gap, trade breaks down. Since the SLC holds, $k^{SLC} > 0$. By the previous point, The states $[k^{SLC}, 1]$ must be reached via smooth trade. By identical arguments to those in the downward skimming case, if trade is smooth at a positive speed, the speed of trade must satisfy (23). In particular, for a small enough but positive $\varepsilon > 0$, the unique smooth trade cutoff path that is locally incentive compatible for the buyer and the seller on $t \in [0, \varepsilon)$ solves the initial value problem

$$\dot{K}_t = \frac{r(b(K_t) - c)}{\bar{S}_v(\alpha(K_t), K_t)}, \quad K_t = 0,$$

i.e, $K_t = 0$ on $[0, \varepsilon)$.[26]

At $t = \varepsilon$, the situation replicates itself, so $k^{SLC} > 0$ is never reached. Again by previous arguments, the only remaining non-trivial trading dynamic involves instant trade. However, since the SLC holds, this is strictly suboptimal for the seller. □

B Equilibrium Verification

Proof of Theorem 1: Equilibrium Verification. We present details for the case with non-trivial delayed trade, $S \in \mathcal{D}_{c,\tilde{b},\tilde{X}}$; the remaining cases are similar, but much simpler.

[26] The right hand side is C^1, given the assumptions on primitives and the expression for $\alpha(\cdot)$.

Verification: Seller's On-Path Strategies Now, we verify that the seller's choice of $\{K^k\}_{k\in[0,1]}$ and F are optimal, given the buyer's strategy

$$\alpha(k) = \begin{cases} \alpha^f(1) & \text{if } k \in (k^{SLC}, 1] \\ \bar{S}^{-1}(c,k) & \text{if } k \in [0, k^{SLC}), \end{cases} \tag{25}$$

where the inverse $\bar{S}^{-1}(c,k)$ is defined by $\bar{S}\big(\bar{S}^{-1}(c,k),k\big) = c$. From the previous, given $\alpha(k)$ in equation (25), seller's the continuation value is

$$J(k) = \begin{cases} c & \text{if } k \in [0, k^{SLC}] \\ \mathbb{E}[S(\alpha^f(1), \tilde{Z})|v \in [k,1]] & \text{if } k \in (k^{SLC}, 1] \end{cases} \tag{26}$$

Notice that $J(\cdot)$ has a kink at k^{SLC} as

$$J'(k^{SLC}-) = 0 < J'(k^{SLC}+) = \frac{\partial}{\partial k}\mathbb{E}[S(\alpha^f(1), \tilde{Z})|v \in [k,1]]\big|_{k=k^{SLC}}$$

The value function fails to be differentiable at at k^{SLC} due to the discontinuity in $\alpha(\cdot)$, Moreover, this implies that the HJB equation is discontinuous at this point. To avoid the technical complications associated to working with discontinuous HJB equations, and the theory of viscosity solutions, we take advantage that admissible cutoff polices are non-decreasing, and we split the verification of the optimal policies in two steps: First starting at $k_0 \in (k^{SLC}, 1]$, and then starting $k_0 \in [0, k^{SLC}]$.

Verification for $k_0 \in [k^{SLC}, 1]$. Let's ignore the fact that for $\alpha(K_t) = \alpha^f(1)$, all types accept the offer, and consider a relaxed formulation in which the seller is allowed to smoothly screen on $k_0 \in [k^{SLC}, 1]$. To simplify notation, we consider F which are absolutely continuous and let $\Lambda_t = \int_0^t \lambda(s)ds$. The seller's value function is

$$\begin{aligned} J(k_0) = \sup_{Q,\lambda} \int_0^\infty e^{-rt-\Lambda_t}\Big(\mathbb{E}\left[\bar{S}\left(\alpha(Q_t), v\right)\Big| v \in [Q_{t-}, Q_t]\right] \frac{dQ_t}{1-k_0} \\ + \lambda_t \mathbb{E}\left[\bar{S}\left(\alpha^f(1), v\right)\Big| v \in [Q_{t-}, 1]\right]\Big) + \left(1 - \int_0^\infty e^{-rt-\Lambda_t}\left(\frac{1-Q_t}{1-k_0}\lambda_t dt + \frac{dQ_t}{1-k_0}\right)\right)c. \end{aligned}$$

Rather than working with the value function $J(\cdot)$, it is convenient to work with the equivalent value function $\bar{J}(k) \equiv (1-k)J(k)$, so

$$\begin{aligned} \bar{J}(k_0) = \sup_{Q,\lambda} \int_0^\infty e^{-rt-\Lambda_t}\Big(\mathbb{E}\left[\bar{S}\left(\alpha(Q_t), v\right)\Big| v \in [Q_{t-}, Q_t]\right] dQ_t \\ + \lambda_t(1-k_0)\mathbb{E}\left[\bar{S}\left(\alpha^f(1), v\right)\Big| v \in [Q_{t-}, 1]\right]\Big) + \left(1 - k_0 - \int_0^\infty e^{-rt-\Lambda_t}\big(\lambda_t(1-Q_t)dt + dQ_t\big)\right)c. \end{aligned}$$

We conjecture, and then verify, that the value function $\bar{J}(\cdot)$ satisfies the quasi-variational inequality

$$0 = \max\Bigg\{ \sup_{\dot{k}\geq 0, \lambda\geq 0} \big(\bar{S}(\alpha(k),k) + \bar{J}'(k)\big)\dot{k} + \lambda\left(\int_k^1 \bar{S}\left(\alpha^f(1),v\right)dv - \bar{J}(k)\right) + r(1-k)c - r\bar{J}(k)\,,\, \mathcal{M}\bar{J}(k) - \bar{J}(k)\Bigg\}, \quad (27)$$

where the operator $\mathcal{M}$ is defined by

$$\mathcal{M}\bar{J}(k) = \max_{k'\in[k,1]} \left\{(k'-k)\,\mathbb{E}[\bar{S}(\alpha(k'),v)|v\in[k,k']] + \bar{J}(k')\right\}$$

First we verify that $\bar{J}(k) = (1-k)J(k)$, where $J(k)$ is as in (26) satisfies this quasi-variational inequality. First, it is immediate to verify that $\bar{J}(k) = \mathcal{M}\bar{J}(k)$, so the second term of (27) is satisfied. For the first term, notice that

$$\big(\bar{S}(\alpha(k),k) + \bar{J}'(k)\big)\dot{k} + \lambda\left(\int_k^1 \bar{S}\left(\alpha^f(1),v\right)dv - \bar{J}(k)\right) + r(1-k)c - r\bar{J}(k) \leq \big(\bar{S}(\alpha(k),k) + \bar{J}'(k)\big)\dot{k} = 0,$$

where we have used that $\bar{J}'(k) = -\bar{S}(\alpha^f(1),k)$. From here on, the verification is standard. Consider an arbitrary admissible policy Q_t. Using the change of value formula, we get that

$$e^{-rt-\Lambda_t}\bar{J}(Q_t) = J(k_0) + \int_0^t e^{-rs-\Lambda_s}\left(\dot{q}_s\bar{J}'(Q_{s-}) + \lambda_s\left(\int_k^1 \bar{S}\left(\alpha^f(1),v\right)dv - \bar{J}(Q_{s-})\right) - r\bar{J}(Q_{s-})\right)ds + \sum_{s<t} e^{-rs-\Lambda_s}\left(\bar{J}(Q_{s-} + \Delta Q^d_{s-}) - \bar{J}(Q_{s-})\right)$$

From the quasi-variational inequality (27) we get that

$$\bar{J}(Q_s) - \bar{J}(Q_{s-}) \leq (Q_s - Q_{s-})\,\mathbb{E}[\bar{S}(\alpha(Q_s),v)|v\in[Q_{s-},Q_s]]$$

and the term in the integral is less than

$$-r(1-Q_s)c - \dot{q}_s\bar{S}(\alpha(Q_{s-}),Q_{s-}) - \lambda_s\int_k^1 \bar{S}\left(\alpha^f(1),v\right)dv.$$

It follows that

$$
\begin{aligned}
\bar{J}(k_0) \geq & \int_0^t e^{-rs-\Lambda_s} \left(r(1-Q_{s-})c + \dot{q}_s \bar{S}(\alpha(Q_{s-}), Q_{s-}) \right. \\
& \left. + \lambda_s \int_k^1 \bar{S}\left(\alpha^f(1), v\right) dv \right) ds \\
& \sum_{s<t} e^{-rs-\Lambda_s} \left(Q_s^d - Q_{s-}^d\right) \mathbb{E}[\bar{S}(\alpha(Q_s), v) | v \in [Q_{s-}, Q_s]] + e^{-rt-\Lambda_t} \bar{J}(Q_t) \\
= & \int_0^t e^{-rt-\Lambda_t} \mathbb{E}\left[S\left(\alpha(Q_s), \tilde{Z}\right) \middle| v \in [Q_{s-}, Q_s] \right] dQ_s \\
& + \left(1 - k_0 - e^{-rt-\Lambda_t}(1-Q_t) - \int_0^t e^{-rs-\Lambda_s} \big((1-Q_s)\lambda_s + dQ_s \big) \right) c \\
& + e^{-rt-\Lambda_t} \left(\bar{J}(Q_t) - (1-Q_t)c \right),
\end{aligned}
$$

where the equality

$$
1 - k_0 - e^{-rt-\Lambda_t}(1-Q_t) - \int_0^t e^{-rs-\Lambda_s} \big((1-Q_s)\lambda_s + dQ_s \big) = \int_0^t e^{-rs-\Lambda_s} r(1-Q_{s-}) ds,
$$

follows by integration by parts. Taking the limit when $t \to \infty$, we get that $\bar{J}(k_0)$ is an upper bound on the payoff that the seller can attain starting at any $k_0 \geq k^{SLC}$. Finally, because all the inequalities hold with equality in the case of equation for our conjecture policy K, it follows that K is optimal starting at $k_0 \in [k^{SLC}, 1]$.

Verification for $k_0 \in [0, k^{SLC})$. Using the previous characterization of the value function $\bar{J}(\cdot)$ on $[k^{SLC}, 1]$, by the principle of dynamic programming, we can state the optimization problem on $[0, k^{SLC})$, as

$$
\begin{aligned}
\bar{J}(k_0) = & \sup_Q \int_0^{\tau(Q)} e^{-rt-\Lambda_t} \mathbb{E}\left[\bar{S}\left(\alpha(Q_t), v\right) \middle| v \in [Q_{t-}, Q_t] \right] dQ_t \\
+ & \left(1 - k_0 - \int_0^{\tau(Q)} e^{-rt-\Lambda_t} \big(\lambda_t (1-Q_t) + dQ_t \big) \right) c + e^{-r\tau(Q)} \left(\bar{J}(Q_{\tau(Q)}) - (1-Q_{\tau(Q)})c \right).
\end{aligned}
$$

where $\tau(Q) = \inf\{t > 0 : Q_t \geq k^{SLC}\}$. Notice that the factor $(1-Q_{\tau(Q)})c$ is added to account for the constant $(1-k)c$ in the expected payoff. Once again, we conjecture that the value function $\bar{J}(\cdot)$ satisfies the quasi-variational inequality (27).

First, we can verify that $\bar{J}(\cdot)$ defined by (26) (multiplied by $1-k$) satisfies equation (27) on $[0, k^{SLC})$. By construction, $\bar{S}(\alpha(k), k) = \bar{J}'(k) = -c$. Also,

$$
\mathcal{M}\bar{J}(k) - \bar{J}(k) = \max_{k' \in [k,1]} \left\{ (k'-k)\, \mathbb{E}[S(\alpha(k'), \tilde{Z}) | v \in [k, k']] + \bar{J}(k') \right\} - (1-k)c < 0,
$$

so $\max_{\lambda\geq 0}\{\lambda(\int_k^1 \bar{S}\left(\alpha^f(1),v\right)dv - \bar{J}(k))\} = 0$. Thus, the first term of the variational inequality is equal to zero, and because $\mathcal{M}\bar{J}(k) - \bar{J}(k)) \leq 0$, the second term also satisfies the required inequality. It follows then that $\bar{J}(k) = (1-k)c$ is a solution of (27) on $[0, k^{SLC})$. Consider an arbitrary policy Q, so, once again, using the change of value formula we get that

$$\begin{aligned}\mathbb{E}^Q\left[e^{-rt\wedge\tau(Q)}\bar{J}(Q_{t\wedge\tau(Q)})\right] = J(k_0) + \int_0^{t\wedge\tau(Q)} e^{-rs}\left(\dot{q}_s\bar{J}'(Q_{s-}) + \lambda_s\left(\bar{J}(Q_{s-}+\Delta Q^s_{s-})\right.\right.\\ \left.\left.-\bar{J}(Q_{s-})\right) - r\bar{J}(Q_{s-})\right)ds + \sum_{s<t\wedge\tau(Q)} e^{-rs}\left(\bar{J}(Q_{s-}+\Delta Q^d_{s-}) - \bar{J}(Q_{s-})\right)\end{aligned}$$

Following the same steps that we did before, we get

$$\begin{aligned}\bar{J}(k_0) \geq &\int_0^{t\wedge\tau(Q)} e^{-rs-\Lambda_s}\left(r(1-Q_{s-})c + \dot{q}_s\bar{S}(\alpha(Q_{s-}),Q_{s-})\right.\\ &\left.+\lambda_s\left(Q^s_s - Q^s_{s-}\right)\mathbb{E}[\bar{S}(\alpha(Q_s),v)|v\in[Q_{s-},Q_s]]\right)ds\\ &\sum_{s<t\wedge\tau(Q)} e^{-rs-\Lambda_s}\left(Q^d_s - Q^d_{s-}\right)\mathbb{E}[\bar{S}(\alpha(Q_s),v)|v\in[Q_{s-},Q_s]] + e^{-rt\wedge\tau(Q)-\Lambda_{t\wedge\tau(Q)}}\bar{J}(Q_{t\wedge\tau(Q)})\\ = &\int_0^{t\wedge\tau(Q)} e^{-rs-\Lambda_s}\mathbb{E}\left[\bar{S}\left(\alpha(Q_s),v\right)\middle|v\in[Q_{s-},Q_s]\right]dQ_s + (1-k_0)c\\ &-\int_0^{t\wedge\tau(Q)} e^{-rs-\Lambda_s}\left((1-Q_s)\lambda_s + dQ_s\right)c + e^{-rt\wedge\tau(Q)-\Lambda_{t\wedge\tau(Q)}}\left(\bar{J}(Q_{t\wedge\tau(Q)}) - (1-Q_{t\wedge\tau(Q)})c\right).\end{aligned}$$

Taking the limit as $t\to\infty$ we get that $t\wedge\tau(Q)\to\tau(Q)$. It follows that $\bar{J}(k_0)$ is and upper bound on the seller's expected payoff. Finally, because in the case of the policy K all the inequalities hold with equality, we get that the value of the policy K is given by $\bar{J}(k_0)$, so K is optimal on $[0,k^{SLC})$.

Verification: Seller's Off-Path Strategy Finally, we characterize the off-equilibrium seller's offer $\sigma(\cdot|k',\alpha')$, where $\sigma(\cdot|k',\alpha')$ has to maximize

$$\int_0^1 \left\{(\alpha^{-1}(\tilde{\alpha})-k')^+\mathbb{E}\left[S\left(\tilde{\alpha},\tilde{Z}\right)\middle|v\in[k',\alpha^{-1}(\tilde{\alpha})\wedge k']\right]\right.\\ \left.+(1-\alpha^{-1}(\tilde{\alpha}))J\left(\alpha^{-1}(\tilde{\alpha})\right)\right\}d\sigma(\tilde{\alpha}|k',\alpha').$$

We consider an off-equilibrium offer with two mass points, given by

$$\sigma(\alpha|k',\alpha') = \begin{cases}\alpha(k') & \text{w.p. } p(k',\alpha')\\ \alpha^f(1) & \text{w.p. } 1-p(k',\alpha'),\end{cases}$$

If $k' < k^{SLC}$, then, conditional on rejection of α', the cut-off is $\alpha^{-1}(\alpha') = k^{SLC}$. In this case, $\bar{S}(\alpha(k^{SLC}),k^{SLC}) = \mathbb{E}\left[S(\alpha^f(1),\tilde{Z})|v\in[k^{SLC},1]\right] = c = J(k^{SLC})$, and this payoff is higher than any other serious offer. Thus, any probability $p(k', \alpha') \in [0,1]$ is optimal, and in particular $p(k', \alpha')$ solving

$$\bar{S}(\alpha', k^{SLC}) = p(k^{SLC}, \alpha')\bar{S}(\alpha^f(1), k^{SLC}) + (1 - p(k^{SLC}, \alpha'))\bar{S}(\alpha(k^{SLC}), k^{SLC}).$$

If $k > k^{SLC}$, then the optimal offer is $p(k', \alpha') = 1$, as any other offer that is accepted with positive probability yields $\mathbb{E}\left[S(\alpha^f(1), \tilde{Z})|v \in [k', k]\right] < \mathbb{E}\left[S(\alpha^f(1), \tilde{Z})|v \in [k', 1]\right] = J(k')$.

Verification: Buyer's On-Path Strategy The proof use a direct mechanism representation of the continuation play together with the characterization in Lemma 1. We cannot apply Lemma 1 directly because the characterization only applies to a deterministic path of cut-offs, and the path cut-off is stochastic in our equilibrium construction (it jumps to $K_T = 1$ at time T). The first step then is to establish that, given the seller strategy, the buyer acceptance strategy is incentive compatible only if it incentive compatible for a deterministic path with the same delay for the pooling offer $\alpha^f(1)$. Let $\tau(k) = \inf\{t > 0 : K_t \geq k\}$, let $\alpha(k) \equiv \alpha(K_{\tau(k)})$, and $y(k) = 1 - \mathbb{E}[e^{-r\tau(k)}]$. Notice that, given the seller strategy K we have that $\alpha(k)$ is a deterministic function of k, so the only random variable is $\tau(k)$. Thus, we can write the buyer's problem as

$$\begin{aligned} B(v,k) &= \max_{k' \in [k,1]} \mathbb{E}^{K^k}[(1 - e^{-r\tau(k')})X(v) + e^{-r\tau(k')}\left(Z(v) - \bar{S}(\alpha(K_{\tau(k')}), v)\right)] \\ &= \max_{k' \in [k,1]} y(k')X(v) + (1 - y(k'))\left(Z(v) - \bar{S}(\alpha(k'), v)\right) \\ &= \max_{k' \in [k,1]} U(y(k'), \alpha(k'), v). \end{aligned}$$

It follows that it is without loss of generality to consider the incentive compatibility condition for a deterministic mechanism inducing the same $y(k)$ as K^k. By the arguments in Lemma 1, we know, for increasing $\iota^S(\alpha, v)$, $U(y, \alpha, v)$ satisfies strict single crossing differences in $((y, \alpha), v)$, where (y, α) is ordered lexicographically. Hence, for any $y \mapsto \tilde{\alpha}(y)$, $U(y, \tilde{\alpha}(y), v)$ has strict single-crossing differences in (y, v).

We have shown that $y(v)$ is non-decreasing. If we prove that $U(y, \tilde{\alpha}(y), v)$ satisfies *smooth* single crossing differences, taking $\tilde{\alpha}(y)$ to be the candidate equilibrium mapping between choice of (1 minus) expected delay and equilibrium offer, and if the following envelope condition is satisfied

$$U(y(v), \alpha(v), v) = U(y(0), \alpha(0), 0) + \int_0^v U_v(y(s), \alpha(s), s)ds, \tag{28}$$

then by Theorem 4.2 in Milgrom (2004), the buyer acceptance strategy $\alpha(v)$ will incentive compatible To check smooth single-crossing differences, take (y, v) such that $\frac{d}{dy}U(y, \tilde{\alpha}(y), v) = 0$. Taking the derivative, we have

$$\bar{S}_\alpha(\tilde{\alpha}(y), v)\left[\iota^S(\tilde{\alpha}(y), v) - \tilde{\alpha}'(y)\right] = 0. \tag{29}$$

By assumption, $\tilde{S}_\alpha > 0$, so if the above display is 0, $\iota^S(\tilde{\alpha}(y), v) = \tilde{\alpha}'(y)$. Then whenever the derivative exists,

$$\frac{\partial}{\partial v}\frac{d}{dy}U(y, \tilde{\alpha}(y), v) = \bar{S}_\alpha(\tilde{\alpha}(y), v)\left[\frac{\partial}{\partial v}\iota^S(\tilde{\alpha}(y), v)\right] > 0,$$

since the environment is upward skimming.

Now we show the relevant envelope condition. By definition, we have that for any v and any (y, α)

$$U_v(y, \alpha, v) = yX'(v) + (1-y)\left(Z'(v) - \bar{S}_v(\alpha, v)\right)$$

For any $v \in [0, k^{SLC}]$ we have

$$\begin{aligned} U(y(v), \alpha(v), v) = U(y(0), \alpha(0), 0) + \int_0^v & (U_v(y(s), \alpha(s), s) \\ & + U_y(y(s), \alpha(s), s)y'(s) + U_\alpha(y(s), \alpha(s), s)\alpha'(s))\, ds, \end{aligned}$$

where

$$\begin{aligned} U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s) &= \left(X(s) - Z(s) + \bar{S}(\alpha(s), s)\right)y'(s) - (1-y(s))\bar{S}_\alpha(\alpha(s), s)\alpha'(s) \\ &= -\left(b(s) - \bar{S}(\alpha(s), s)\right)y'(s) - (1-y(s))\bar{S}_\alpha(\alpha(s), s)\alpha'(s). \end{aligned}$$

From the local IC constraint way have that

$$r\left(b(K_t) - \bar{S}(\alpha(K_t), K_t)\right) = -\dot{K}_t\alpha'(K_t)\bar{S}_\alpha(\alpha(K_t), K_t).$$

By definition, on $[0, k^{SLC})$, $y'(k) = re^{-r\tau(k)}\tau'(k)$ and $\alpha'(k) = \alpha'(K_{\tau(k)})\dot{K}_{\tau(k)}\tau'(k)$. Hence, multiplying both sides of the local incentive compatibility constraint by $e^{-r\tau(k)}\tau'(k)$, and using the definition $K_{\tau(k)} = k$, we get

$$\left(b(k) - \bar{S}(\alpha(k), k)\right)y'(k) = -(1-y(k))\alpha'(k)\bar{S}_\alpha(\alpha(k), k),$$

so $U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s)$, and we obtain equation (28). Next, we verify he envelope representation (28) for $k \in (k^{SLC}, 1]$. Because $\alpha(k)$ and $y(k)$ are constant on $(k^{SLC}, 1]$ and $U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s) = 0$ on $v \in [0, k^{SLC}]$ we have that

$$\begin{aligned} U(y(v), \alpha(v), v) = U(y(0), \alpha(0), 0) + \int_0^v & U_v(y(s), \alpha(s), s)ds \\ & + U(y(k^{SLC}+), \alpha(k^{SLC}+), k^{SLC}) - U(y(k^{SLC}), \alpha(k^{SLC}), k^{SLC}). \end{aligned}$$

By construction, the delay D in equation (9) is such

$$U(y(k^{SLC}+), \alpha(k^{SLC}+), k^{SLC}) = U(y(k^{SLC}), \alpha(k^{SLC}), k^{SLC}),$$

so the expected payoff $U(y(v), \alpha(v), v)$ satisfies the envelope condition (28).

Verification: Buyer's Off-Path Strategy The only step left is to verify the optimality of the reservation price strategy $\alpha(k)$ following an off-equilibrium offer $\alpha' \notin \alpha([0,1])$. By construction, the $\sigma(\alpha|k', \alpha')$ is such the type k^{SLC} buyer is indifferent between accepting α' and reject it. Thus, we only need to verify that types above k^{SLC} are better off rejecting it. By construction

$$\bar{S}(\alpha', k^{SLC}) = p(k^{SLC}, \alpha')\bar{S}(\alpha^f(1), k^{SLC}) + (1 - p(k^{SLC}, \alpha'))\bar{S}(\alpha(k^{SLC}), k^{SLC}).$$

Let $p' \equiv p(k^{SLC}, \alpha')$, because $\bar{S}(\alpha', v)$ is increasing in v, we have that

$$\begin{aligned}
Z(v) - \bar{S}(\alpha', v) &< Z(v) - \bar{S}(\alpha', k^{SLC}) \\
&= p' \left(Z(v) - \bar{S}(\alpha^f(1), k^{SLC})\right) + (1 - p') \left(Z(v) - \bar{S}(\alpha(k^{SLC}), k^{SLC})\right) \\
&< p' \left(Z(v) - \bar{S}(\alpha^f(1), k^{SLC})\right) + (1 - p')B(v, k^{SLC}),
\end{aligned}$$

which means that types $v > k^{SLC}$ are strictly better off rejecting α'. A similar calculation shows that types $v < k^{SLC}$ are strictly better off accepting α'.

$\square$