

Bargaining in Securities*

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Abstract

Many corporate negotiations involve contingent payments or securities, yet the bargaining literature overwhelmingly focuses on pure cash transactions. We characterize equilibria in a continuous-time model of bargaining in securities. A privately informed buyer and a seller negotiate the terms of a joint project. The buyer's private information affects both his standalone value and the net returns from the project. The seller makes offers in a one-dimensional family of securities (e.g., equity splits). We show how outcomes change as the underlying security becomes more sensitive to the buyer's information, and we apply the framework to mergers and acquisitions under financial constraints.

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1 Introduction

Many corporate negotiations involve payments other than cash. In merger and acquisitions (M&A), acquirers often pay the target using shares of their own companies ([Malmendier et al., 2016](#)). When an individual land owner and a local oil and gas producer negotiate a lease agreement, they tend to specify an upfront cash payment and a pre-specified royalty over future revenues.¹ Likewise, procurement contracts, many of which are arrived at via negotiation with suppliers, specify some cost sharing rule. And in Chapter 11 bankruptcy procedures, claim holders bargain not only over cash payments, but also over the terms of the restructuring plan such as the face value, maturity, and seniority of new debt ([White, 1989](#)).

In short, negotiating parties frequently make offers in contingent payments or securities, and yet, the bargaining literature overwhelmingly focuses on pure cash transactions. We therefore lack an understanding of how negotiations change when the value of offers depends on the private information of parties, as is the case with contingent payments.

This paper characterizes equilibria in a continuous-time model of bargaining in securities. We abstract from some institutional details in order to isolate the impact of security payments on bargaining without commitment. In the model, a privately informed buyer and a seller negotiate over the terms of a joint project. The buyer has private information that affects both his assets in place (standalone value) and the net return of the project; the seller has a commonly known cost. The seller makes offers in a given one-dimensional family of securities (e.g., makes debt offers with different face values, or makes equity offers with different ownership shares) so that the value of an accepted offer depends on the buyer’s private information. The seller can revise her offers infinitely frequently, and both players discount the future at the same exponential rate. We focus on a tractable class of Markovian “skimming” equilibria in which buyer types accept gradually in a given order.

¹Government oil lease agreements are usually auctioned but individual lease agreements are commonly settled by negotiation.

The model can be interpreted in a few different ways. If one interprets the seller’s cost as an investment required for the joint project, then the model is a monopolistic, dynamic version of [Myers and Majluf \(1984\)](#), with the seller as a financier and the buyer as an entrepreneur with private information about returns and assets in place. One can also interpret the model as an M&A negotiation between an acquirer and a target, in which one of the two parties has private information about the value of its assets in place and the value of the potential synergies from the acquisition. It would be standard to think of the acquirer as the buyer and the target as the seller, but the reverse mapping is also possible: we stress that the labels are somewhat arbitrary when bargaining in securities, and the choice of labels will mostly depend on which party one thinks has the relevant private information.² Our model could also capture negotiations in corporate restructuring – either out-of-court or as part of Chapter 11 procedures – which restructure the terms of the firm’s liabilities. The seller would represent the debt holders, and the buyer would represent either existing equity holders or the manager acting on their behalf.

We provide two sets of results: we characterize bargaining dynamics, and we show how outcomes depend on the security’s sensitivity to the buyer’s information.

The bargaining dynamics depend on the buyer’s marginal rate of substitution between delay (or time of trade) and offers.³ Depending on both the primitives and the security family, that marginal rate of substitution may be increasing or decreasing in the buyer’s type. When it is decreasing in type—so that high types are less willing to endure additional delays in exchange for receiving a better offer—trade is fully efficient, and there is no delay. In contrast, when that rate is increasing, there will be delay provided adverse selection is sufficiently severe. Delay takes on a very particular form. The

²For example, if the parties are negotiating over an equity split, one can just as well think of the target (in layman’s terms a “seller”) as “buying” shares in a merged entity using its own equity as payment.

³Note that, unlike cash bargaining models, “offer” here is not synonymous with “payment.” For example, when bargaining over equity splits, the “offer” is the share of the gross returns being proposed; the “payment” is the value of that share.

game begins with a phase of gradual concessions, in which the seller’s offer becomes more generous smoothly, and buyer types accept gradually in ascending order. Eventually, the negotiation reaches an impasse of random length, during which the seller intransigently refuses to improve his offers, even though they are continuously rejected. And finally, the impasse ends in a flash: the seller “submits,” drops his ask discontinuously and trades immediately with all remaining types. Whenever there is delay, the seller breaks even on every offer that is accepted in equilibrium. The inefficiencies therefore involve not only delay, but also cross subsidization: high types who trade at the final offer “subsidize” low types who trade at that offer, allowing them to receive more generous terms than they would have had they traded individually.

Second, focusing on scenarios with non-trivial delay, we show how equilibrium delay, ex-post payments, and ex-ante payoffs are affected by the sensitivity of the security to the buyer’s private information. We rely on [DeMarzo et al. \(2005\)](#)’s notion of steepness to partially order security families according to their informational sensitivity. For example, if there is a fixed royalty rate that must be paid to the seller, but the parties negotiate over an additional cash payment, then the higher the royalty rate, the steeper (more informationally sensitive) the overall “cash + royalty” security family will be.

For concreteness, here we explain our steepness results for the case in which the informational sensitivity of the security is indexed by a one-dimensional parameter, and we defer the more general comparisons to the main text. The parameter can be, e.g., the royalty rate in the “cash + royalty” example above, or, as we explain below, it can be the tightness of the buyer’s financial constraints in an M&A application. We note that the bulk of the results generalize to steepness comparisons that are not “parametrized.”⁴

A common thread throughout our results is that steeper securities—those with a higher informational sensitivity parameter—tend to raise bargaining

⁴Equity offers, for instance, are more informationally sensitive than cash offers—the value of a cash offer to the seller does not depend on what the buyer knows about the project, but the value of an equity offer does—but there is no overarching (one-dimensional) parametrized security family that includes pure equity offers and pure cash offers as special cases.

frictions and lower equilibrium payments. However, the cross-subsidization that takes place in equilibrium often generates non-monotonic effects for types in the middle of the distribution. We measure bargaining frictions according to the (type-specific) certainty-equivalent delay: the deterministic delay that causes the same expected discounting cost as the random equilibrium delay. Increasing the steepness of the security will raise certainty-equivalent delay for sufficiently low and sufficiently high types; for intermediate types, the effect may be reversed. For our leading case, a local increase in steepness raises bargaining frictions for “most” types but must lower them for a small but positive measure of types. Likewise, in that leading case, steeper securities lead to weakly lower expected payments for high types and low types but strictly raise payments for some intermediate types.

Since steeper securities lead payments to fall and bargaining frictions to rise, the net effect on the buyer’s utility could be ambiguous. We prove that low types (those below a threshold) are always harmed by bargaining in steeper securities, even though their payments decrease. And we provide an easy-to-check sufficient condition under which using a steeper security harms all types.

Our main application is to M&A negotiations under financial constraints. We present two formulations of equity bargaining under financial constraints for which the tightness of the buyer’s pre-existing financial constraints indexes the steepness of the equity being offers. In the first formulation, the parties negotiate in equity, but the buyer has a fixed amount of cash that is added to the offer. Insofar as up-front cash is costly to procure, the amount of cash indexes both the tightness of the buyer’s liquidity constraints and the informational sensitivity of the security offer: the lower the cash amount, the steeper the overall security is with respect to the buyer’s information. In the second formulation, the parties negotiate in equity, but the buyer has pre-existing debt and maximizes the total value of debt and equity holders. The higher the leverage, the more financially constrained the buyer is, and the more informationally sensitive levered equity will be.

Our general results then imply that tighter financial constraints tend to increase bargaining frictions and reduce equilibrium payments. These findings

are mostly in line with empirical studies, but they point to some avenues for additional research and some challenges to inference. If the discount rate reflects an underlying exogenous probability of deal failure, then our measure of bargaining frictions maps one-to-one to deal failure probabilities. Our results then say that tightening financial constraints on the margin will raise deal failure probabilities for most of the distribution (and large increases in tightness will often raise those probabilities for all types). Previous studies ([Malmendier et al., 2016](#); [Uysal, 2011](#)) indeed have shown that tighter financial constraints are associated with higher probabilities of deal failure and lower M&A activity by firms. However, the typical regression specification in these studies will capture only a composite effect that averages both across constraint changes of various sizes and across unobservable types. As such, these studies may be conflating the heterogeneous effects that we identify.

We also show that data on delay—rather than deal failure—can be a misleading guide to the underlying bargaining frictions. Detailed delay data would at best identify an expected delay curve across the distribution of types. Due to the randomness in equilibrium delay and the convexity of exponential discounting, expected delay will differ from certainty-equivalent delay, which is the true welfare-relevant measure of bargaining frictions and not directly observable. We provide numerical examples in which tighter financial constraints lower expected delay for almost all types, even as—in line with our theoretical results—they increase bargaining frictions for most types. In other examples, expected and certainty-equivalent delay move in the same direction for all types. This suggests a challenge for empirical studies of bargaining that try to use data on delay.

Finally, using a normal-linear model parametrization of the levered-equity model, we show how the negotiation depends on the nature of the synergies or net returns generated by the project. In an M&A setting, merger synergies related to cost savings may be easier to estimate than merger synergies from market expansion or product market fit. A mean preserving-spread of future synergies raises the value of levered equity, but it also may dilute how tightly the buyer’s signal covaries with the expected value of the project; what is the

net effect on equilibrium play?

We vary the precision of the buyer’s signal about net returns, and we prove that the net effect of raising that precision is equivalent to negotiating in a steeper security. Hence, as the buyer’s information becomes more precise, high and low types endure worse bargaining frictions. Our theoretical results imply that types below a threshold are always harmed by the increased precision, but in the numerical examples we have computed, the harm seems to be uniform across all types. In other words, we show that more precise information can be value-destroying.

The remainder of this section briefly outlines related work. Section 2 presents the model setup and our continuous-time equilibrium notion, and Section 3 presents our equilibrium construction and our results on equilibrium uniqueness for general securities. Section 4 presents our comparative statics results for general steepness comparisons, both parametrized and not, focusing on the case with non-trivial delay, Section 5 applies the theoretical results to equity bargaining under financial constraints. Section 6 discusses some connections to entrepreneurial finance and bankruptcy negotiations, and it elaborates on some connections the literature. All omitted proofs are in the appendix.

Related Literature: The use of contingent payments makes the seller and buyer values interdependent. Bargaining with interdependent values was first studied by [Deneckere and Liang \(2006\)](#), who consider a discrete-time cash bargaining model with finite types. We discuss the relationship to our paper in Section 6 in more detail. In the interim, we note that, aside from allowing general securities as means of payment, our model also differs in that the type space is continuous, and we formulate the bargaining problem directly in continuous time.

More generally, we contribute to a recent literature that adds tractability and richness to discrete-time models of bargaining with asymmetric information (e.g., [Fudenberg et al., 1985](#); [Gul et al., 1986](#); [Fuchs and Skrzypacz, 2010](#)) by reformulating them directly in continuous time. The original contributions by [Ortner \(2017\)](#) and [Daley and Green \(2020\)](#) used models with discrete

types and driving Brownian process (changing costs in the former, news about the informed party’s type in the latter). These formulations were adapted to continuous-type Coasean bargaining (without driving Brownian processes) in [Chaves \(2019\)](#). While some of the dynamics in [Daley and Green \(2020\)](#) resemble ours (smooth trade, followed by an atom of trade), their model does not generate an impasse phase. The forces leading to gradual trade are also different.

We also contribute to a nascent bargaining literature that considers bargaining over richer objects than cash. [Strulovici \(2017\)](#) considers a two-type Coasean bargaining model where parties negotiate over the terms of contracts, including, for instance, the quantity or quality of goods traded. [Hanazono and Watanabe \(2018\)](#) study the splitting of a stochastic pie in a setting where both parties have noisy signals about the size of the pie. [de Clippel et al. \(2019\)](#) consider a two-stage Nash demand model where the utility possibility sets depend on privately known types, and they provide a non-cooperative foundation to Myerson’s axiomatic solution for these problems.

Finally, we contribute to the corporate finance literature on the interaction between asymmetric information and contingent payments.⁵ The literature has studied security payments in financing decisions ([Myers and Majluf, 1984](#); [DeMarzo and Duffie, 1999](#)), bidding wars for mergers and acquisitions ([Fishman, 1989](#); [Hansen, 1987](#); [Rhodes-Kropf and Viswanathan, 2004](#)), and auctions ([DeMarzo et al., 2005](#); [Che and Kim, 2010](#)), but overwhelmingly these models are static/one-shot interactions, or feature full commitment. A theme from those papers, originating in [Myers and Majluf \(1984\)](#) is that informationally sensitive securities (steep securities) are costly due to adverse selection.

A competing force, noted by [DeMarzo et al. \(2005\)](#) work on security auctions, is that steeper securities create a tighter link between the informed

⁵Outside of the corporate finance literature, [Lam \(2020\)](#) studies the impact of steepness in a directed search environment with owners with heterogeneous assets and workers of privately known productivity. [Lam \(2020\)](#) characterizes the inefficiencies that arise as the market moves (exogenously) from cash transfers to output share (equity) payments; when asset owners are free to choose among securities, competition drives them to offer only cash payments.

party’s payment and his type, and can therefore help surplus extraction. We use their definition of informational sensitivity (“steepness”). We also adopt [Che and Kim \(2010\)](#)’s extension of [DeMarzo et al. \(2005\)](#) that allows bidders’ private information to affect their standalone value; part of our contribution is showing how dynamic considerations interact with, and may overwhelm, these linkage-principle forces at work in the security auction literature. [Section 6](#) elaborates on this connection.

2 Setup

Players, Protocol, Payoffs: A buyer (he) and a seller (she) negotiate over the terms of a joint project, the rights to which initially rest with the seller, but which only the buyer can undertake. Both players discount the future at rate r . The buyer has a privately observed type $\theta \sim U[0, 1]$ that affects both his disagreement payoff and his payoff from agreeing. Before agreement is reached, the seller enjoys a flow payoff of rc , $c > 0$ and the buyer enjoys a flow of $rA(\theta)$. A is mnemonic for “assets in place,” which are the source of the buyer’s disagreement flow payoffs. The project, once undertaken, generates random cash flows with present value \tilde{V} that is affiliated with θ . Hence, once the buyer owns the project, he enjoys a present value $V(\theta) := \mathbb{E}[\tilde{V}|\theta]$ in expectation, gross of payments.

The seller makes offers of payment terms to the buyer, who at each point in time chooses whether to accept or reject. This is as in standard models of Coasean bargaining. Unlike those models, the seller makes offers in a one-dimensional family of contingent payments, which we call the security family (sometimes shortened to “the security.”). Concretely, when the buyer accepts an offer $\alpha \in [\underline{\alpha}, \bar{\alpha}] \subset \mathbb{R}$, he commits to a contingent payment $S(\alpha, \tilde{V})$ as a function of the project’s value. Conditional on his type, he then expects to pay $\bar{S}(\alpha, \theta) := \mathbb{E}[S(\alpha, \tilde{V})|\theta]$. The function S is fixed throughout the negotiation; different offers by the seller therefore correspond to different indices in $[\underline{\alpha}, \bar{\alpha}]$.

Altogether, if the buyer with type θ accepts an offer α at time t , the seller’s expected payoffs are

$$(1 - e^{-rt})c + e^{-rt}\bar{S}(\alpha, \theta), \quad (1)$$

while the buyer's are

$$(1 - e^{-rt})A(\theta) + e^{-rt}(V(\theta) - \bar{S}(\alpha, \theta)) \quad (2)$$

Let $R(\theta) := V(\theta) - A(\theta)$ denote buyer θ 's net return on the project. We impose the following restrictions:

Assumption 1.

1. $R(\theta) \geq c \forall \theta$, strictly so for $\theta > 0$.
2. $\bar{S}(\bar{\alpha}, \theta) \geq R(\theta) \geq c > \bar{S}(\underline{\alpha}, \theta) \forall \theta$.
3. $\bar{S}_\alpha := \frac{\partial \bar{S}}{\partial \alpha}$ exists, is strictly positive, and smooth in α, θ .
4. $\bar{S}_\theta(\alpha, \theta) := \frac{\partial \bar{S}(\alpha, \theta)}{\partial \theta}$ exists, is strictly positive for $\alpha > \underline{\alpha}$, and is smooth in α, θ .
5. $V(\theta)$ and $A(\theta)$ are smooth.

Condition 1 says that there are gains from trade with every type of buyer. Below we distinguish between the gap ($R(0) > c$) and no gap ($R(0) = c$) cases (Fudenberg et al., 1985; Gul et al., 1986). Condition 2 is a non-degeneracy assumption. It ensures that the expected payment is sufficiently variable as a function of the offer: in a one-shot game, there exist sufficiently unfavorable offers that any player would definitely want to reject, and sufficiently favorable ones that any player would definitely accept. The assumption would be trivially satisfied if the parties were bargaining in an unrestricted amount of cash. Conditions 3 and 4 are DeMarzo et al. (2005)'s notion of ordered securities with minor modifications. Condition 3 is essentially just an ordering assumption on offers, such that higher offers correspond to strictly higher expected payments. Condition 4 says that higher types are strictly good news for the seller.⁶

An important object for the analysis is $\alpha^f(\theta)$, the solution to

⁶Conditions 2-4 are satisfied, for example, if $S(\alpha, \tilde{V})$ is weakly increasing in both arguments and $\tilde{V}|\theta$ admits a conditional density $g_V(v|\theta)$ that satisfies strict MLRP, is twice-differentiable in both arguments, with $vg_V(v|\theta)$, $v \left| \frac{\partial}{\partial \theta} g_V(v|\theta) \right|$, and $v \left| \frac{\partial^2}{\partial \theta^2} g_V(v|\theta) \right|$ integrable on $v > 0$. This is Lemma 1 in DeMarzo et al. (2005). Under those assumptions, V is also strictly increasing and smooth.

$$V(\theta) - \bar{S}(\alpha^f(\theta), \theta) = A(\theta). \quad (3)$$

This is the highest take-it-or-leave-it offer that type θ would consider accepting. As a mnemonic, the superscript on α^f stands for “final.”

Discussion of the Model:

Our setting can accommodate different applications of security bargaining.

Entrepreneurial Finance: The seller is a financier, the cost c corresponds to the investment required for the project, $A(\theta)$ corresponds to the value of the entrepreneur’s assets in place, and $R(\theta)$ corresponds to the present value of the cash flows generated by the new project. [Myers and Majluf \(1984\)](#) considers the static case with a competitive market of financiers.

Mergers and Acquisitions: In one version, the “buyer” is the acquirer and the “seller” is the target. $A(\theta)$ then corresponds to the current value of the acquirer, $R(\theta)$ corresponds to the synergies between the two firms, and c measures the current value of the target. In the above we took the acquirer as the privately informed “buyer” and the target as the uninformed “seller”. However, these labels are somewhat arbitrary with security payments. When equity is used, the value of the transaction is shared in a linear way. Thus, if the relevant private information belongs to the target, one can alternatively interpret the “buyer” in the model as the target firm and the “seller” in the model as the acquiring firm.

Corporate Restructuring: The seller is the debt holder and the buyer is the equity holder. $R(\theta)$ corresponds to the net benefit of continuing operation versus liquidating the firm. $A(\theta)$ and c corresponds to the value that the parties expect if the firm is liquidated. If the restructuring involves a debt exchange, the negotiation is over the face value of the new debt. ⁷

Direction of Skimming: With cash bargaining, a now standard argument ([Fudenberg et al., 1985](#)) shows that discrete-time equilibria satisfy a “skimming”

⁷In this case it is natural to allow c to be a function of θ . We can map this case to our current setting by redefining $R(\theta)$ and $\bar{S}(\alpha, \theta)$. In particular, if we let $\hat{R}(\theta) := R(\theta) - c(\theta)$ and $\hat{S}(\alpha, \theta) := \bar{S}(\alpha, \theta) - c(\theta)$, then we can write the debt holder payoff as $c(\theta) + e^{-rt}(\bar{S}(\alpha, \theta) - c(\theta)) = c(\theta) + e^{-rt}\hat{S}(\alpha, \theta)$, and the equity holder payoff as $A(\theta) + e^{-rt}(R(\theta) - \bar{S}(\alpha, \theta)) = A(\theta) + e^{-rt}(\hat{R}(\theta) - \hat{S}(\alpha, \theta))$. See Section 6 for details.

property: if a type θ is indifferent between accepting and rejecting an offer p after history H_t , then all types $\theta' > \theta$ strictly prefer to accept p at H_t ; beliefs after every history are therefore right-truncations of the prior. Intuitively, high types like cash just as much as low types, but they dislike delay relatively more. Analysis typically focuses on Markovian equilibria with the truncation point as a state variable.

The standard argument breaks down when bargaining in non-cash securities because the buyer's true type affects his expected payment. When bargaining in equity, for example, high buyer types dislike delay more, but they also dislike giving up their equity more. High types may therefore be more willing than low types to wait for better equity offers. For tractability, we will focus nonetheless on Markovian equilibria with a skimming structure—buyer types accept in a predetermined “order”—but what that order is, and whether it matches the natural order of types, depends on both the primitives and the security family.

Definition 1 (Upward vs Downward Skimming). Let ι^S given by

$$\iota^S(\theta, \alpha) := -\frac{R(\theta) - \bar{S}(\alpha, \theta)}{\bar{S}_\alpha(\alpha, \theta)} \quad (4)$$

Say the environment satisfies upward skimming if $\iota^S(\cdot, \alpha)$ is strictly increasing for every α . The environment satisfies downward skimming if $\iota^S(\cdot, \alpha)$ is strictly decreasing for every α . The environment satisfies the skimming property if it is either upward skimming or downward skimming.

To unpack the definition, notice that $\iota^S(\theta, \alpha)$ is the marginal rate of substitution between delay (t) and offer (α) in the buyer's utility in (2). When ι^S is increasing in θ , the indifference curves of low types in (α, t) space cross the indifference curves of high types from below, and high types are more willing to trade off additional delays in order to get an improvement in the offer (and vice-versa when ι^S is decreasing in θ).

We use the classification in Definition 1 to guide our search for equilibria in tractable classes. When the environment is upward skimming, it will be fruitless to search for skimming equilibria where higher types accept first, so we

in those environments we look for upward skimming Markov equilibria, where (i) the seller’s beliefs about the buyer are left-truncations of the prior (lower types accept first), and (ii) the truncation point is the relevant state variable for continuation play. (Vice versa for downward skimming environments).⁸

For a quick example, adapted from [Che and Kim \(2010\)](#)’s work on security auctions, suppose that the seller makes offers in equity shares of the gross value of the project, i.e., $S(\alpha, \tilde{V}) = \alpha \tilde{V}$. Then $\iota^S(\theta, \alpha) = -(R(\theta)V(\theta)^{-1} - \alpha)$, and the environment is upward skimming iff R/V is everywhere decreasing, and downward-skimming if its everywhere increasing. Intuitively, if R/V is decreasing, then, as the buyer’s type grows, his disagreement motive (i.e., the assets in place) grows proportionally faster than his agreement motive (i.e., the net surplus), so higher types will tend to trade later than low types.

Severity of Adverse Selection: Bargaining dynamics will depend crucially on whether or not efficiency is achievable in that static game, i.e., whether there exists an offer that all buyer types accept on which the seller can break even ([Deneckere and Liang, 2006](#)):

Definition 2 (Static Lemons Condition).

1. In an upward skimming environment, say the Static Lemons Condition (SL) holds iff

$$\mathbb{E}[\bar{S}(\alpha^f(1), \theta)] < c.$$

If SL holds, let k^{SL} be defined by

$$k^{SL} = \inf\{k \leq 1 : \mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in [k, 1]] \geq c\}.$$

2. In a downward skimming environment, say the Static Lemons Condition holds iff

$$\mathbb{E}[\bar{S}(\alpha^f(0), \theta)] < c.$$

Below we refer to k^{SL} as the “critical type.”

⁸Lemma 4 in the Online Appendix shows that under upward-skimming, for any deterministic offer path, every selection of maximizers is non-decreasing in type. Even though the paths in our case are stochastic, Lemma 4 still suffices to verify buyer’s incentive compatibility for the offer paths in the equilibria we construct.

Equilibrium Notion: For environments that satisfy the skimming property, the game has a natural state variable: the truncation of the seller’s prior beliefs that yields her current posterior. For upward skimming environments, this is a left truncation: if the state at time t is $K_t = k$, then, given the history of offers and rejections, the seller believes $\theta > k$. For downward skimming environments, k is a right truncation, i.e., if $K_t = k$, the seller believes $\theta < k$. We focus on a tractable class of equilibria that are Markovian in the truncation (henceforth, “the cutoff”). To simplify the exposition, we describe the equilibrium notion for upward skimming environments, and later explain how the notion must be adapted for downward skimming.

We first give a brief verbal description of the equilibrium notion. Following recent formulations of Coasean bargaining in continuous time (see [Ortner \(2017\)](#), [Daley and Green \(2020\)](#), and, most relevant for the current setup, [Chaves \(2019\)](#)), the seller solves a Markovian optimal stopping-control problem, and the buyer solves a Markovian optimal stopping problem. Formally, the buyer’s chooses a reservation offer strategy $\alpha(\cdot)$. On path, the seller chooses how fast to screen through buyer types, taking as given that to screen through types $\theta < k$, she must offer $\alpha(k)$. That is, the seller chooses paths of belief cutoffs $t \mapsto K_t$, which result in paths of offers $t \mapsto \alpha(K_t)$. We also give the seller an option to “give up on screening”: she can make a pooling offer $\alpha(1)$ that would be accepted by all remaining types, thereby ending the game. Given a law of motion for cutoffs, and that future offers are given by $\alpha(K_t)$, the buyer then solves an optimal stopping problem with state K_t . Finally (unlike [Ortner \(2017\)](#), [Daley and Green \(2020\)](#), and [Chaves \(2019\)](#)), the buyer’s strategy $\alpha(\cdot)$ must sometimes be discontinuous in equilibrium. We therefore augment the seller’s strategy space off-path: after the rejection of an off-path offer $\alpha' \notin \alpha([0, 1])$, we give the seller the ability to randomize over offers in a way that depends on α' . In the lingo of the discrete time literature, our equilibrium will be “weak Markov” (see [Fudenberg et al. \(1985\)](#) and [Gul et al. \(1986\)](#) for the origins of this “Weak Markov” approach).

The technical details are as follows. First, the set of all measurable non-decreasing paths is an unmanageably large strategy space for the seller. Using

the approach in [Chaves \(2019\)](#), we impose some restrictions on seller strategies that make the analysis tractable while still allowing a rich set of dynamics.

Definition 3 (Seller Strategy Space).

1. A plan of on-path offers by the seller consists of a non-decreasing cutoff path $t \mapsto K_t$ and a stopping time T at which to make a pooling offer $\alpha(1)$. We denote an entire cutoff path $(K_t)_{t \geq 0}$ by K . K is admissible if it has no singular-continuous parts. We allow for mixed strategies in the stopping time T , which are represented by a CDF $F = (F_t)_{t \geq 0}$. Thus a plan for the seller is given by a pair (K, F) , and we denote by \mathcal{A}_k^U the set of admissible plans (K, F) satisfying $K_{0-} = k$, i.e., with initial value k , and generic element K^k . The stopping time T is Markov if its hazard measure $dF_t/(1 - F_{t-})$ is a function of K_{t-} .
2. Time intervals $[t, \bar{t})$ where K and F are absolutely continuous are smooth trade regions. For such regions, \dot{K}_t is the (a.e.) trading speed and γ_t denotes the (a.e.) hazard rate $\frac{dF_t}{dt}(1 - F_t)^{-1}$. A special case of a smooth trade region is a quiet period, i.e., an interval $[t, \bar{t})$ with $\dot{K}_t = \Delta K_{t-} = 0$.
3. An on-path plan is supplemented by an off-path plan. For any off-equilibrium offer $\alpha' \notin \alpha([0, 1])$ made at time t , we let $\sigma_t(\alpha') \in \Delta([0, 1])$ be the randomized offer that “immediately” follows the rejection of α' .

The discrete-time “Weak Markov” equilibria in [Fudenberg et al. \(1985\)](#) and [Gul et al. \(1986\)](#) sustain Markovian behavior on path by prescribing randomization immediately following the rejection of an off-path offer. In continuous time, there is no “next” period immediately after a seller deviation. Hence, to capture this off-path randomization, in the third item we “stop the clock” after an off-equilibrium offer is made, and we allow the seller to immediately make a new offer when the off-path offer is rejected.⁹

Definition 4 (Buyer and Seller Problems). At state k , a buyer type θ takes $\alpha(\cdot)$ and K as given, and solves

⁹See [Smith and Stacchetti \(2002\)](#) and [Fanning \(2016\)](#), who use the technique of “stopping the clock” to allow for multiple sequential moves in a continuous time bargaining games. An alternative approach is to follow the formalization in [Fudenberg and Tirole \(1985\)](#) and to consider “intervals of consecutive atoms.”

$$\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[(1 - e^{-r(\tau \wedge T)}) A(\theta) + e^{-r(\tau \wedge T)} (V(\theta) - \bar{S}(\alpha(K_{\tau \wedge T}), \theta)) \right] \quad (5)$$

where by definition $K_T = 1$, and \mathcal{T} is the set of stopping times adapted to the filtration generated by T . Meanwhile, the seller S takes $\alpha(\cdot)$ as given. Given any path Q_t and realization of the stopping time T , the seller payoff is

$$\begin{aligned} \Pi(Q, T) := & \int_0^T e^{-rt} \mathbb{E} \left[\bar{S}(\alpha(Q_t), \theta) \mid \theta \in [Q_{t-}, Q_t] \right] dQ_t \\ & + e^{-rT} \mathbb{E} \left[\bar{S}(\alpha^f(1), \theta) \mid \theta \in [Q_{T-}, 1] \right] + \left(1 - (1 - Q_T)e^{-rT} - \int_0^T e^{-rt} dQ_t \right) c, \end{aligned}$$

and, at each k , the seller strategy (Q, F) solves

$$\sup_{(Q, F) \in \mathcal{A}_k^U} \int_0^\infty \Pi(Q, T) dF(T). \quad (6)$$

We can now fully define a weak Markov equilibrium.

Definition 5 (Equilibrium). A weak Markov Equilibrium of an upward-skimming game consists of a tuple

$$(\{K^k\}_{k \in [0,1]}, F, \alpha(\cdot), \sigma(\cdot|\cdot, \cdot))$$

together with a value $J(\cdot)$ for the the seller and a value $B(\cdot, \cdot)$ for the buyer such that

1. For all $\theta \in [0, 1]$, $k \in [0, 1]$, accepting at $\tau^* = \inf\{t : \alpha(K_t^k) \leq \alpha(\theta)\}$ solves the buyer's problem (5) and delivers value $B(\theta, k)$.
2. $\alpha(1) = \alpha^f(1)$, where α^f is defined in (3).
3. For all $k \in [0, 1]$ and T in the support of F , K^k is an admissible path and T is a Markov stopping time that together solve (6) and deliver value $J(k)$.
4. For any point of discontinuity of $\alpha(\cdot)$, k' , and any off-equilibrium offer $\alpha' \in (\alpha(k'+), \alpha(k'-))$, $\sigma(\cdot|k', \alpha')$ maximizes¹⁰

¹⁰Here, $\alpha^{-1}(\cdot)$ represents the generalized inverse defined as $\alpha^{-1}(y) \equiv \sup\{x > 0 : \alpha(x) \geq y\}$.

$$\int_0^1 \left\{ (\alpha^{-1}(\tilde{\alpha}) - k')^+ \mathbb{E} \left[\bar{S}(\tilde{\alpha}, \theta) \mid \theta \in [k', \alpha^{-1}(\tilde{\alpha}) \wedge k'] \right] \right. \\ \left. + (1 - \alpha^{-1}(\tilde{\alpha})) J(\alpha^{-1}(\tilde{\alpha})) \right\} d\sigma(\tilde{\alpha} | k', \alpha')$$

5. For any point of discontinuity of $\alpha(\cdot)$, k' , and any off-equilibrium offer $\alpha' \in (\alpha(k'+), \alpha(k'-))$, $\sigma(\cdot | k', \alpha')$ satisfies

$$V(k') - \bar{S}(\alpha', k') \leq B(k', k') \int_0^{\alpha(k')} d\sigma(\tilde{\alpha} | k', \alpha') \\ + \int_{\alpha(k')}^1 (V(k') - \bar{S}(\tilde{\alpha}, k')) d\sigma(\tilde{\alpha} | k', \alpha')$$

Condition 2 is a natural refinement inspired by the corresponding discrete time game.¹¹ In a stationary equilibrium of the discrete time game, for any positive period length, the seller would never offer more than $\alpha^f(1)$ when her beliefs are concentrated at $\theta = 1$. (And $\alpha(1)$ can never be above $\alpha^f(1)$, since $\theta = 1$ would strictly prefer to reject, i.e. $\alpha(1)$ cannot be a reservation offer for $\theta = 1$.) Condition 4 and 5 say that, when the seller makes an off-path offer “by mistake”, the buyer still accepts according the reservation offer curve $\alpha(\cdot)$, and after making the mistake, the seller randomizes in way that justifies the buyer’s choice to accept according to $\alpha(\cdot)$ (Fudenberg et al., 1985; Gul et al., 1986).

Finally, to streamline the derivation of necessary conditions, we restrict our search for equilibria to an amenable subclass, along the lines in Chaves (2019):

Definition 6 (Regularity). A weak Markov Equilibrium is regular if

1. J is continuous.
2. In any interval (k', k'') for which smooth trade is prescribed at all $k \in (k', k'')$, J is C^1 and α is continuous.
3. \dot{K}_t is continuous in the interior of smooth trade regions.
4. Jump discontinuities in cutoff paths are isolated.

¹¹See the discussions in Ortner (2017) and Daley and Green (2020), who impose conditions similar to our Condition 1; Ortner (2017) shows that, absent this kind of refinement, continuous time equilibria can violate this natural discrete-time property.

Below, we refer to regular weak Markov Equilibria as simply “equilibria.”

Remark 1 (Modifications for Downward Skimming). In downward skimming environments, regular weak Markov Equilibria are defined almost identically, with the following changes:

1. Admissible paths $t \mapsto K_t^k$ are non-increasing and satisfy $K_0^1 = 0$. The admissible set at state k is denoted \mathcal{A}_k^D .
2. Condition 2 in Definition 5 becomes $\alpha(0) = \alpha^f(0)$.
3. The seller’s objective for given Q, T is now written

$$\begin{aligned} \Pi(Q, T) = & \int_0^T e^{-rt} \mathbb{E} \left[\bar{S}(\alpha(Q_t), \theta) \mid \theta \in [Q_t, Q_{t-}] \right] d(1 - Q_t) \\ & + e^{-rT} \mathbb{E} \left[\bar{S}(\alpha^f(0), \theta) \mid \theta \in [0, Q_{T-}] \right] \\ & + \left(1 - Q_T e^{-rT} - \int_0^T e^{-rt} d(1 - Q_t) \right) c, \quad (7) \end{aligned}$$

3 Dynamics for General Securities

Within our class of equilibria, we can fully characterize equilibrium dynamics. Here we provide an informal derivation of the equilibrium in an upward skimming case when adverse selection is sufficiently severe, relegating the full proof of necessary conditions and equilibrium verification to the appendix.

We construct an equilibrium where the game starts with smooth trade. By the usual Coasean logic, whenever the seller is trading smoothly, her payoff is pinned down at c : otherwise, she would have strict incentives to speed up trade. To wit, the HJB equation in the smooth trading region is given by

$$rJ(k) = \sup_{\dot{k} \geq 0} \left(\bar{S}(\alpha(k), k) - J(k) \right) \frac{\dot{k}}{1 - k} + J'(k) \dot{k} + rc. \quad (8)$$

The choice variable \dot{k} enters the HJB in an affine way. Hence, if trade is happening at a positive speed ($\dot{k} > 0$ is optimal), the coefficients on \dot{k} must cancel. It follows that $J(k) = c$, and $\bar{S}(\alpha(k), k) = c$, i.e., the seller exactly breaks even, conditional on trading with type k .

The seller is therefore indifferent among speeds of trade in a smooth trading region, and the speed of trade at state k is pinned down instead by the marginal buyer $\theta = k$ and his incentives to delay. Ignoring second-order effects, if the buyer $\theta = K_t$ waits an additional dt units of time before accepting, he suffers discounting costs

$$r dt \left(V(K_t) - \bar{S}(\alpha(K_t), K_t) \right).$$

However, while waiting he receives flow utility $r dt A(K_t)$, and the price he faces improves by

$$\alpha'(K_t) \bar{S}_\alpha(\alpha(K_t), K_t) \dot{K}_t dt.$$

Setting marginal costs equal to marginal benefits, the speed \dot{K}_t must satisfy

$$\dot{K}_t = r \frac{R(K_t) - c}{-\alpha'(K_t) \bar{S}_\alpha(\alpha(K_t), K_t)}.$$

The numerator represents the gains from trade with type K_t , while the denominator represents the (absolute value of) the slope of expected payments with respect to the state. Trade is therefore faster (i.e., types are skimmed more quickly) when the gains of trade are larger, and it is slower when the equilibrium expected payment changes more quickly with respect to the state. Intuitively, when expected payments are more sensitive to the state, the buyer has a stronger incentive to “move the state along” by rejecting offers and misrepresenting his type. Incentive compatibility then requires that trade is slower.

We obtain a useful re-interpretation of the speed of trade by leveraging the seller’s breaks-even condition $S(\alpha(K_t), K_t) = c$, which holds when trade is smooth. Totally differentiating on both sides with respect to K_t ,

$$0 = \alpha'(K_t) \bar{S}_\alpha(\alpha(K_t), K_t) + \bar{S}_\theta(\alpha(K_t), K_t),$$

Plugging into our expression for the speed of trade, we have that, starting at a state k with smooth trade, K_t evolves according to the ODE

$$\dot{K}_t = r \frac{R(K_t) - c}{\bar{S}_\theta(\alpha(K_t), K_t)}, K_0 = k. \tag{9}$$

Hence—foreshadowing our steepness results—the speed of trade depends on

the sensitivity of expected payments to the true type.

A natural guess is that, using (9), one can construct an equilibrium in smooth trade. However, smooth trading cannot persist indefinitely. If the seller were to continue screening more and more types, eventually the state would reach and cross k^{SL} . At that point, trading instantly with all remaining types at an offer they would all accept would become strictly more profitable than trading smoothly with the marginal type: for $k > k^{SL}$, $\mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in [k, 1]] > c$.

Our equilibrium construction therefore specifies smooth trade at $k < k^{SL}$, with each type trading separately at different offers, and an atom of trade at $k > k^{SL}$, with all remaining types $[k, 1]$ trading simultaneously at the pooling offer $\alpha^f(1)$. However, this implies that the offer must drop discontinuously (become discontinuously more favorable for the buyer) at state k^{SL} . Indeed, for a type slightly below k^{SL} , the seller just breaks even conditional on trading only with that type. At the same time, the seller also breaks even when trading simultaneously with all types strictly above k^{SL} , i.e., $\mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in (k^{SL}, 1]] = c$.¹² In order to make types just below k^{SL} willing to trade at the higher offer, when types just above face a discontinuously lower one, the seller must delay offering $\alpha^f(1)$. In equilibrium, she delays just long enough to make $\theta = k^{SL}$ indifferent between accepting $\alpha(k^{SL})$ “now” and rejecting it in hopes of receiving $\alpha^f(1)$ “later.” The expected discount until $\alpha^f(1)$ is offered, denoted by D , must solve

$$\underbrace{V(k^{SL}) - \bar{S}(\alpha(k^{SL}), k^{SL})}_{\text{payoff from accepting } \alpha(k^{SL})} = \underbrace{(1 - D)A(k^{SL}) + D(V(k^{SL}) - \bar{S}(\alpha^f(1), k^{SL}))}_{\text{payoff from waiting for } \alpha^f(1)}, \quad (10)$$

where $\bar{S}(\alpha(k^{SL}), k^{SL}) = c$.¹³

Since the seller must use a Markov stopping time at k^{SL} , she can implement this delay by postponing the final offer until the first tick of a Poisson clock

¹²On the one hand, since the seller trades smoothly for $k < k^{SL}$, $\alpha(k^{SL}-)$ must satisfy $\bar{S}(\alpha(k^{SL}-), k^{SL}) = c$. On the other, given that all types in $(k^{SL}, 1]$ trade at the final offer $\alpha^f(1)$, $\mathbb{E}[\bar{S}(\alpha(k^{SL}+), \theta) | \theta \in (k^{SL}, 1]] = c$. It follows that $\alpha(k^{SL}+) < \alpha(k^{SL})$.

¹³Such a $D \in (0, 1)$ always exists: since there are strict gains from trade, we have $\bar{S}(\alpha^f(1), k^{SL}) < c = \bar{S}(\alpha(k^{SL}), k^{SL}) < R(k^{SL}) = V(k^{SL}) - A(k^{SL})$.

with a rate λ given by $\lambda/(r + \lambda) = D$.

For these equilibrium dynamics, one can use standard mechanism design arguments (together with Lemma 4 in the appendix) to show that it is globally incentive-compatible for buyers to accept from lowest to highest according to $\alpha(\cdot)$. A verification approach shows that these screening dynamics are also optimal for the seller, given $\alpha(\cdot)$.

We have outlined the construction of an equilibrium, but in fact, in Theorem 1 we prove that these are the only possible equilibrium dynamics. Formally, we prove the following theorem, which also covers the remaining cases:

Theorem 1. In skimming environments, there exists a (regular weak Markov) equilibrium.

1. In a downward-skimming environment, all equilibria have instant trade at an offer $\alpha^f(0)$.
2. In an upward-skimming environment, if SL fails, all equilibria have instant trade at an offer of $\alpha^f(1)$.
3. In an upward-skimming environment, if SL holds, and there is no gap, there is no trade in any equilibrium.
4. In an upward-skimming environment, if SL holds and there is a gap, there is a unique on-path equilibrium triple $(\{K^k\}_{k \in [0,1]}, F, \alpha(\cdot))$:
 - The buyer's acceptance strategy is given by $\bar{S}(\alpha(k), k) = c$ for $k \leq k^{SL}$, and $\alpha(k) = \alpha^f(1)$ for $k > k^{SL}$.
 - There is smooth trade for $k \in [0, k^{SL})$. In the smooth trade region, the cutoff K_t is the unique solution to equation (9).
 - At state k^{SL} , there is a temporary (random) breakdown in trade. The seller makes the final offer $\alpha^f(1)$ with a Poisson arrival intensity $\lambda = rD/(1 - D)$, where D is defined by equation (10).
 - For $k > k^{SL}$, the seller immediately offers $\alpha^f(1)$.

Figure 1 illustrates typical realized paths of outcomes for the case with non-trivial delay dynamics (Theorem 1.4). The cutoff rises gradually from 0 until it reaches k^{SL} , with the seller gradually dropping her offers from $\alpha(0)$ to $\alpha(k^{SL})$. When the state arrives at k^{SL} , the game reaches an impasse, with

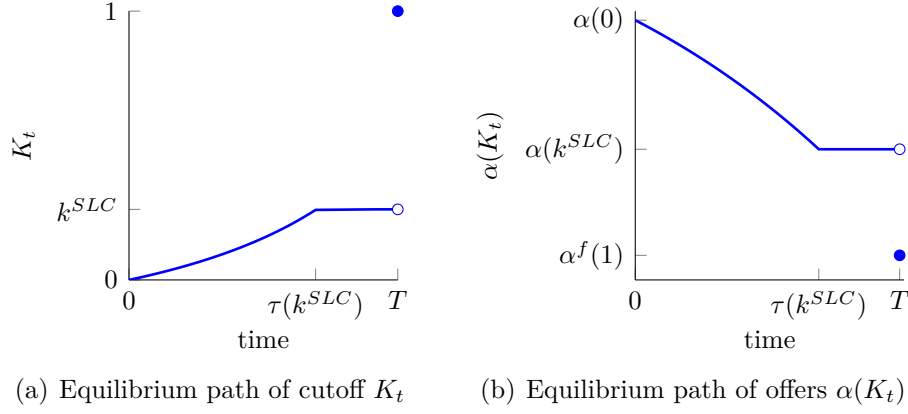


Figure 1: Illustration of a realized equilibrium path in an upward skimming environment satisfying the SL. $\tau(\theta)$ denotes the realized time at which state θ is reached on-path.

the cutoff frozen at k^{SL} for a random amount of time $T - \tau(k^{SL})$. During the impasse, the seller “stubbornly” refuses to move her offer from $\alpha(k^{SL})$, until finally, at a random time, she concedes, dropping her offer to $\alpha^f(1)$. At that point, all remaining types $\theta \in (k^{SL}, 1]$ accept suddenly, and the cutoff jumps to $k = 1$.

We remark in two features of the equilibrium.

First, unlike cash bargaining, with contingent payments one must distinguish between the equilibrium offer $\alpha(\theta)$ that type θ accepts, and the expected equilibrium payment $\bar{S}(\alpha(\theta), \theta)$ that he faces. For types below k^{SL} their offers and expected payments are linked by $\bar{S}(\alpha(\theta), \theta) = c$; while they all accept different offers $\alpha(\theta)$, they make the exact same expected payment c in equilibrium. Meanwhile, all types in $(k^{SL}, 1]$ accept the same exact same offer $\alpha^f(1)$, but they all make different expected payments according to $\bar{S}(\alpha^f(1), \theta)$, which is strictly increasing in θ .

Second, since the equilibrium time at which any θ trades could be random, it is useful to have a one-dimensional summary of how much delay is experienced by type θ . Let $\tau(\theta)$ denote the (possibly random) time at which time θ trades. Then type θ ’s certainty-equivalent delay, $\tau^{CE}(\theta)$, is the (deterministic) delay that solves

$$\mathbb{E}[e^{-r\tau(\theta)}] = e^{-r\tau^{CE}(\theta)}.$$

Since $\tau^{CE}(\theta)$ varies one-to-one with expected discounting costs, it provides a welfare-relevant measure of bargaining frictions. Using Theorem 1, we obtain explicit expressions for τ^{CE} that come in handy below:

$$\tau^{CE}(\theta; L) = \begin{cases} \int_0^\theta \frac{\bar{S}_\theta(\alpha(s), s)}{r(R(s) - c)} ds, & \theta \leq k^{SL}, \\ \int_0^{k^{SL}} \frac{\bar{S}_\theta(\alpha(s), s)}{r(R(s) - c)} ds - \frac{\log D}{r}, & \theta > k^{SL} \end{cases} \quad (11)$$

where D is given by (10).

Remark 2 (Downward Skimming Equilibria). To understand why delay cannot be sustained in downward-skimming environments, notice that in any such case, the SL will necessarily fail. Indeed, for any $k > 0$,

$$\mathbb{E}[\bar{S}(\alpha^f(0), \theta) | \theta \in [0, k]] > \bar{S}(\alpha^f(0), 0) = R(0) \geq c,$$

where the strict inequality uses Assumption 1. This rules out the possibility of any smooth trade or quiet periods, since the seller's payoff in such a case would be exactly c . In words, under downward skimming, pooling “favors the seller:” higher types accept any final offer that a lower type accepts. It therefore becomes overwhelmingly tempting to speed up trade. (Formal details in the appendix).

4 Means of Payment and Bargaining Dynamics

In this section, we study how bargaining dynamics change as a function of the underlying security family used for bargaining. We focus on changes in the “informational sensitivity” or steepness of the security family. DeMarzo et al. (2005) define steepness as follows:

Definition 7 (Steepness). Take two security families $S^1 : [\underline{\alpha}_1, \bar{\alpha}_1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $S^2 : [\underline{\alpha}_2, \bar{\alpha}_2] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. S^1 is steeper than S^2 if, for any feasible offers $\alpha_1 \in [\underline{\alpha}_1, \bar{\alpha}_1]$ and $\alpha_2 \in [\underline{\alpha}_2, \bar{\alpha}_2]$,

$$\bar{S}^1(\alpha_1, \theta) = \bar{S}^2(\alpha_2, \theta) \Rightarrow \bar{S}_\theta^1(\alpha_1, \theta) > \bar{S}_\theta^2(\alpha_2, \theta)$$

Steepness is a partial order on the space of securities. As a simple example, the security family $S^1(\alpha, \tilde{V}) = \alpha \tilde{V}$, which describes equity indexed by the share of gross return, is a steeper security family than $S^2(\alpha, \tilde{V}) = \min\{\alpha, \tilde{V}\}$, which describes debt indexed by its face value. Below, we often use the shorthand “steeper (flatter) security” to mean “steeper (flatter) security family.”

The bulk of our comparative statics results apply to general steepness comparisons as the above example. However, stronger comparative statics results are possible for more structured comparisons where the two security families S^1 and S^2 are part of the same parametrized class:

Definition 8. An security class of parametrized steepness is a function $S : [\underline{\alpha}, \bar{\alpha}] \times \mathbb{R}_+ \times [\underline{\gamma}, \bar{\gamma}] \rightarrow \mathbb{R}_+$ such that

1. $\bar{S}(\alpha, \theta; \gamma) := \mathbb{E}[S(\alpha, \theta; \gamma)]$ is continuous, and for any γ , $\bar{S}(\cdot, \cdot; \gamma)$ satisfies Assumption 1.
2. for any pair $\gamma', \gamma'' \in [\underline{\gamma}, \bar{\gamma}]$ with $\gamma' < \gamma''$, $S(\cdot, \cdot; \gamma'')$ is steeper than $S(\cdot, \cdot; \gamma')$.

Some examples of parametrized steepness classes include¹⁴

- Equity plus a fixed cash component γ : $S(\alpha, \tilde{V}; \gamma) = (\bar{L} - \gamma) + \alpha \tilde{V}$.
- Levered equity, with face value of debt γ : $S(\alpha, \tilde{V}; \gamma) = \alpha \max\{\tilde{V} - \gamma, 0\}$.
- Cash plus royalty rate γ : $S(\alpha, \tilde{V}; \gamma) = \alpha + \gamma \tilde{V}$.

We are interested in comparing pairs of security families that both generate non-trivial delay dynamics.

Definition 9 (Delayed trade). Given c , A , and a conditional distribution $\tilde{V}|\theta$, a security family S generates delayed trade, denoted $S \in \mathcal{D}_{A, \tilde{V}, c}$, if

1. Assumption 1 holds, and the environment is upward-skimming.
2. The Static Lemons Condition holds: $\mathbb{E}[S(\alpha^f(1), \tilde{V})] < c$.
3. There are strict gains from trade: $R(0) > c$.

For two security families S^1 and S^2 in a parametrized steepness class $S(\cdot, \cdot; \cdot)$, with S^1 steeper, we write $[S^2, S^1] \in \mathcal{D}_{A, \tilde{V}, c}$ if all securities “in between” S^1 and S^2 generate delay:

¹⁴It follows from Lemma 5 in [DeMarzo et al. \(2005\)](#) that higher γ ’s correspond to steeper securities in these examples.

1. there exist steepness parameters $\gamma_1 > \gamma_2$ such that $S^1(\cdot, \cdot) = S(\cdot, \cdot; \gamma_1)$, $S^2(\cdot, \cdot) = S(\cdot, \cdot; \gamma_2)$; and
2. $S(\cdot, \cdot; \gamma) \in \mathcal{D}_{A, \tilde{V}, c}$ for all $\gamma \in [\gamma_2, \gamma_1]$.

For pairs of securities that generate delay, we characterize the effects of increasing steepness on trading dynamics, payments, and equilibrium utilities. First, regarding trading dynamics and delay, we show:

Proposition 1. Take S^1 and S^2 in $\mathcal{D}_{A, \tilde{V}, c}$, with S^1 steeper. Let $k^{SL}(S^i)$ and $\tau^{CE}(\theta; S^i)$ denote, respectively, the critical type and the (equilibrium) certainty-equivalent delay suffered by type θ under security S^i .

1. Under S^1 there is less pooling, and a slower gradual concessions phase:
 - (a) $k^{SL}(S^1) > k^{SL}(S^2)$.
 - (b) $\tau^{CE}(\theta; S^1) > \tau^{CE}(\theta; S^2)$ for $\theta \in (0, k^{SL}(S^2)]$.
2. If, in addition, S^1 and S^2 belong to the same parametrized steepness class, with $[S_2, S_1] \in \mathcal{D}_{A, \tilde{V}, c}$ then types above $k^{SL}(S^1)$ also suffer strictly higher certainty-equivalent delay under S^1 :

$$\tau^{CE}(\theta; S^1) > \tau^{CE}(\theta; S^2), \quad \text{for all } \theta \in (0, 1] \setminus (k^{SL}(S^2), k^{SL}(S^1)].$$

We postpone the proof of item 2, which depends on some technical lemmas, to the appendix. Meanwhile, items 1(a) and 1(b) follow directly from Theorem 1 and the definition of steepness. Consider the size of the pooling region. The pooling region makes the seller on average break even with the final offer $\alpha^{f,i}(1)$. Since type-by-type expected payments $\bar{S}^i(\alpha^{f,i}(1), \theta)$ are higher under S^2 —the payments of the flatter security cannot cross those of the steeper one from below—the seller must average over a strictly worse pool (include even lower types) to break even under S^2 . Formally, let $\alpha^{f,i}(1)$ denote the final equilibrium offer under security S^i . By definition, the $\alpha^{f,i}$'s satisfy $\bar{S}^1(\alpha^{f,1}(1), 1) = \bar{S}^2(\alpha^{f,2}(1), 1) = R(1)$, which, by the greater steepness of S^1 , implies that

$$\bar{S}^1(\alpha^{f,1}(1), \theta) < \bar{S}^2(\alpha^{f,2}(1), \theta) \text{ for all } \theta < 1. \quad (12)$$

Therefore, for all $k < 1$,

$$\overbrace{\frac{1}{1-k} \int_k^1 \bar{S}^1(\alpha^{f,1}(1), \theta) d\theta}^{\mathbb{E}[\bar{S}^1(\alpha^{f,1}(1), \theta) | \theta \in [k, 1]]} < \overbrace{\frac{1}{1-k} \int_k^1 \bar{S}^2(\alpha^{f,2}(1), \theta) d\theta}^{\mathbb{E}[\bar{S}^2(\alpha^{f,2}(1), \theta) | \theta \in [k, 1]]}, \quad (13)$$

and item 1(a) follows immediately.

Likewise, consider the gradual concessions phase. The buyer's incentive to reject an offer is that, by rejecting, he can affect the seller's beliefs about his type and can obtain a better price in the future. However, because of the Coasean force the seller's expected payment is constant in the state. Letting α^1 and α^2 denote the reservation offer curves for the bargaining games with securities S^1 and S^2 , we have $\bar{S}^i(\alpha^i(k), k) = c$. The change in price from a change in the seller's beliefs therefore exactly equals the change in price from a change in the buyer's type: for $i = 1, 2$ and $\theta \leq k^{SL}(S^2)$,

$$\bar{S}^i(\alpha(k), k) = c \Rightarrow \underbrace{-\overbrace{(\alpha^i)'(k) \bar{S}_\alpha^i(\alpha(k), k)}^{<0}}_{\text{Price improvement from changing seller's belief}} = \underbrace{\frac{\partial}{\partial \theta} \bar{S}^i(\alpha(k), \theta) \Big|_{\theta=k}}_{\text{sensitivity to } \theta \text{ at } \theta = k}$$

Hence, the more sensitive price is to the buyer's private information, the greater the price improvement that he expects from rejecting an offer, the greater his incentives to reject, and the slower trade must be. Formally, $\bar{S}^i(\alpha^i(k), k) = c$ implies $\bar{S}_\theta^1(\alpha^1(k), k) > \bar{S}_\theta^2(\alpha^2(k), k)$, since S^1 is steeper. Plugging this inequality into (11) yields $\tau^{CE}(\theta; S^1) > \tau^{CE}(\theta; S^2)$ for all $\theta \in (0, k^{SL}(S^2))$.

We also characterize the effect of information sensitivity on expected payments. The effect is always heterogeneous across types, raising equilibrium payments for some and lowering them for others:

Proposition 2 (Expected payments are “single-crossing”). Take S^1 and S^2 in $\mathcal{D}_{A, \tilde{V}, c}$, with S^1 steeper. Let $\pi_i(\theta) := \bar{S}^i(\alpha^i(\theta), \theta)$, θ 's equilibrium expected payment under S^i .

1. Under S^1 , high types pay strictly less, low types pay more: there exists a unique $k^{cross} \in (k^{SL}(S^2), 1)$ such that

$$\begin{aligned}
\pi_1(\theta) &= \pi_2(\theta), & \theta &\in [0, k^{SL}(S^2)]. \\
\pi_1(\theta) &> \pi_2(\theta), & \theta &\in (k^{SL}(S^2), k^{cross}). \\
\pi_1(\theta) &< \pi_2(\theta), & \theta &\in (k^{cross}, 1).
\end{aligned} \tag{14}$$

2. Let k^* solve $\bar{S}^2(\alpha^{f,2}(1), k^*) = c$. Then $k^{cross} = \min\{k^*, k^{SL}(S^1)\}$.

Since the expected payment of the highest type is the same under both securities ($\bar{S}^i(\alpha^{f,i}(1), 1) = R(1)$), and high enough types accept $\alpha^{f,i}(1)$ in either case, they must pay less under the steeper security. At the same time, since flatter security has a larger pooling region, there are types at the bottom of the interval that would be separated under a steep security (and pay c), but get cross-subsidized by very high types when they face the flat security. They must therefore pay less strictly than c under the flat security. Finally, types at the bottom of the distribution are separated and pay c in either case.

The previous propositions deal with arbitrary increases in steepness. It is useful for empirical applications to study the effect of “small” or local changes in steepness. Doing so uncovers important subtleties. For small increases in steepness, expected payments drop and bargaining frictions rise for “almost” all types, but payments must rise and frictions must drop for “some” small measure of types.

Corollary 1 (Local Increases in Steepness). Consider a parametrized steepness class $S(\cdot, \cdot; \cdot), \gamma \in [\underline{\gamma}, \bar{\gamma}]$ For any $\gamma \in [\underline{\gamma}, \bar{\gamma}]$ with $S(\cdot, \cdot; \gamma) \in \mathcal{D}_{A, \bar{V}, c}$:

1. For any $\delta \in (0, 1)$, there exists $\varepsilon > 0$ small enough that, for a measure $1 - \delta$ of types
 - (a) certainty-equivalent delay $\tau^{CE}(\cdot)$ is strictly higher under $S(\cdot, \cdot; \gamma + \varepsilon)$ than under $S(\cdot, \cdot; \gamma)$.
 - (b) expected payments $\pi(\cdot)$ are weakly lower under $S(\cdot, \cdot; \gamma + \varepsilon)$ than under $S(\cdot, \cdot; \gamma)$.
2. There exists $\varepsilon > 0$ small enough that, for a positive measure of types,
 - (a) $\pi(\cdot)$ is strictly higher under $S(\cdot, \cdot; \gamma + \varepsilon)$ than under $S(\cdot, \cdot; \gamma)$.
 - (b) τ^{CE} is strictly lower under $S(\cdot, \cdot; \gamma + \varepsilon)$ than under $S(\cdot, \cdot; \gamma)$.

Point 1 and Point 2(a) follow immediately from Propositions 1 and 2 using the continuity of the critical cutoff k^{SL} with respect to the steepness parameter

(Lemma 3 in the appendix). In contrast, Point 2(b) follows from discontinuity of τ^{CE} . As we increase γ slightly by $d\gamma$, types inside $[k^{SL}(\gamma), k^{SL}(\gamma + d\gamma))$ shift from trading in the final atom—after the impasse—to trading in the initial phase of smooth trading. This has two effects on the bargaining frictions they endure. First, τ^{CE} for these types drops discontinuously so as to lie along a smooth trading locus; second, the smooth trading locus itself rises—continuously—because of the slightly higher steepness. For a small enough change in steepness, the discontinuous drop must dominate, so bargaining frictions must drop for some intermediate types.¹⁵

The next section discusses the empirical implications of the corollary and Propositions 1 and 2 in detail. For the moment, we note that studies summarize the effect of higher steepness with a single parameter may conceal important heterogeneities effects across the distribution of firms.

Having ranked equilibrium delay and payments by type, we now rank, where possible, equilibrium utilities by type.

Proposition 3. Take two securities S^1 and S^2 in $\mathcal{D}_{A, \tilde{V}, c}$, with S^1 steeper.

1. Let k^{cross} be as in Proposition 2. Then there exists $k' > k^{cross}$ such all types $\theta \in [0, k')$ prefer the flatter security S^2 , and strictly so for $\theta > 0$.
2. If, in addition, S^1 and S^2 belong to the same parametrized steepness class $S(\cdot, \cdot; \cdot)$, with $[S_2, S_1] \in \mathcal{D}_{A, \tilde{V}, c}$. If $\varphi(\theta, \gamma) := R(\theta) - \bar{S}(\alpha^f(1; \gamma), v; \gamma)$ is log-supermodular in $(\theta, -\gamma)$, then all types prefer S^2 , strictly so for $\theta \in (0, 1)$.

Said differently, types who pay less under the flatter security always prefer to bargain with it, no matter whether they suffer higher or lower delay under that security. For parametrized steepness comparisons, an easy-to-check sufficient condition ensures that all types prefer bargaining in the flatter security, even when equilibrium requires them to pay more or suffer more equilibrium delay.¹⁶ Our sufficient condition is not stated in terms of the primitive security

¹⁵Hence, the results in Proposition 1 are “tight”: there exist pairs of securities $S^1, S^2 \in \mathcal{D}_{A, \tilde{V}, c}$, with S^1 steeper, for which some intermediate types suffer lower bargaining frictions with S^1 .

¹⁶Note that, under the conditions of Proposition 3, there are no types for whom the steeper

family, and it does not follow directly from steepness. However, the condition is easily verified for particular families of securities, as we do in section 5.

Gains from trade for the highest type—the source of cross-subsidization: We have not directly imposed sign restrictions on the slope of net return R beyond those implied by the upward-skimming/downward-skimming assumption, but it is natural to think of it as strictly increasing. (Our Assumption 1.1, which requires strict gains from trade for types above 0, would then follow).

However, in many applications, there is nothing pathological about a strictly decreasing net return.¹⁷ In the case of M&A, $R(\theta)$ measures the synergies from the merger, which can be higher or lower for high types. Suppose, for instance, that the buyer is acquiring a seller in order to gain access to the seller’s proprietary technology, and θ measures how close the buyer is to the technological frontier. A higher θ would raise the expected value of assets in place $A(\theta)$ and may even raise the total value $V(\theta)$, but the marginal value of the seller’s technology $R(\theta) = V(\theta) - A(\theta)$ can be lower the closer the buyer is to the technological frontier.

With a decreasing R it may happen that there is no gap at the top, i.e. $R(1) = c$, even though there is a gap at the bottom ($R(0) > c$). Our results go through almost identically, with one key difference: cross-subsidization across types may vanish, so all the non-monotonicities caused by cross-subsidization vanish.

Corollary 2. Take two securities S^1, S^2 that both satisfy Assumption 1, with the first item replaced by

$$R(\theta) > c \text{ for } \theta < 1, \text{ and } R(1) = c.$$

If S^1 and S^2 both cause upward-skimming given R and $\tilde{V}|\theta$, and S^1 is steeper,

- $k^{SL}(S^1) = k^{SL}(S^2) = 1$, and under both securities there is smooth trade at all states. Hence $\pi_1(\theta) = \pi_2(\theta) = c$ for all θ and $\tau^{CE}(\theta; S^1) > \tau^{CE}(\theta; S^2)$ for $\theta > 0$.

security leads to strictly more certainty-equivalent delay and strictly higher payments. In that sense, there is a “delay-payment trade-off.”

¹⁷We are grateful to Brett Green for suggesting this possibility.

- Types above $\theta = 0$ strictly prefer S^2 .

The proof follows directly from previous arguments, and is omitted. For a brief intuition, notice that when $c = R(1)$, $\bar{S}(\alpha^f(1), 1) = c$, so it must be that $\mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in [k, 1]] < c$ for all $k < 1$. Therefore, no matter how many types the seller has screened, adverse selection is always severe enough that she prefers trading with the marginal type to trading with the remaining types.

5 Mergers and Acquisitions with Financial Constrains

Here we apply our general results to the issue of mergers and acquisitions under financial constraints. We study this through two different models in which financially constrained acquirers negotiate over the equity split in the merged entity. First, we look at the case in which the acquirer has pre-existing debt that the merged company will have to assume; for any given equity split, the acquirer's leverage lowers the value of the equity, and it makes that equity more sensitive to the acquirer's private information. Second, we look at the case in which the acquirer has a limited amount of cash, fixed at the outset of the negotiation, that is added to the equity payment to the target. For any equity split, lowering the amount of cash also makes the total payment more sensitive, in relative terms, to the acquirer's private information. Insofar as cash is costly for firms—and costlier for the financially constrained ones—the amount of cash added to the equity payment parametrizes the acquirer's liquidity constraints. (For example, the acquirer might not have enough cash in hand to complete the transaction, or external financing might be prohibitively expensive).¹⁸

¹⁸Even if the company has sufficient cash, the opportunity cost of depleting its cash reserves may outweigh the efficiency benefits from negotiating in a less informationally sensitive security. Indeed, there is empirical evidence that financial constraints limit the use of cash. For example, [Alshwer et al. \(2011\)](#) finds that financially constrained acquirers rely more on stock as a method of payment than financially unconstrained ones. Other empirical studies have found that, even when acquirers have enough cash to complete a transaction, they tend to use stock as a means of payment if they are financially constrained.

Using these examples, we first derive empirical implications for impact of financial constraints on bargaining frictions and acquisition returns. Through numerical examples, we also show how a reduced form analyses of the effects that financial constraints have on those quantities of interest can be very misleading. Second, we show how negotiation patterns depend on the nature of the synergies that the merger would create.

We structure our discussion around two parametrized steepness classes that capture these financial constraints:

Equity bids with limited up-front cash $S_{liq}(\alpha, \tilde{V}; L) = \alpha\tilde{V} + L, \alpha \in [0, 1]$.

That is, the target and acquirer negotiation over equity in the merged entity. This equity is “sweetened” by a fixed amount of cash $L < c$, which improves the offer but on its own is insufficient to persuade the target. We write S_{liq}^L as a shorthand for $S_{liq}(\cdot, \cdot; L)$.

Levered equity offers $S_{lev}(\alpha, \tilde{V}; d) = \alpha(\tilde{V} - d)_+, \alpha \in [0, 1], d \geq 0$. That is, the target and acquirer negotiate over equity in the merged entity, which takes on the acquirer’s debt. To match the utility specification in (2), we assume that the acquirer maximizes total firm value and not just the value of equity holders. This would be the case if there are covenants that require approval from debt holders. We write S_{lev}^d as a shorthand for $S_{lev}(\cdot, \cdot; d)$.

For both specifications, tighter financial constraints (lower L and higher d) correspond to steeper security families. Moreover, these examples have the useful property that, if a security causes delay, then every steeper security in the same class also causes delay. We can therefore apply Propositions 1 through 3 to study the effects of tightening financial constraints:

Lemma 1.

- If $S_{liq}^L \in \mathcal{D}_{A, \tilde{V}, c}$, then $S_{liq}^{L'} \in \mathcal{D}_{A, \tilde{V}, c}$ for all $L' \leq L$. Moreover, if $S_{liq}^0 \in \mathcal{D}_{A, \tilde{V}, c}$, then there exists $L^* < c$ such that $S_{liq}^L \in \mathcal{D}_{A, \tilde{V}, c}$ for all $L \leq L^*$.
- If $S_{lev}^d \in \mathcal{D}_{A, \tilde{V}, c}$, then $S_{lev}^{d'} \in \mathcal{D}_{A, \tilde{V}, c}$ for all $d' > d$. Thus, if $S_{lev}^0 \in \mathcal{D}_{A, \tilde{V}, c}$, then $S_{lev}^d \in \mathcal{D}_{A, \tilde{V}, c}$ for all $d \geq 0$.

For concreteness, we often focus on the following convenient parametriza-

tion:

Example 1 (Normal-Linear Primitives). The stand alone value is $A(\theta) = \chi\theta$, and the synergy value, conditional on θ , is distributed $\tilde{R}|\theta \sim \mathcal{N}(c + \Delta + \beta\theta, \eta^{-2})$, where $\Delta, \chi, \beta > 0$. η is therefore the precision of the buyer's signal about synergies.

Empirical implications for deal failure and M&A activity: With a slight reinterpretation of the discounting cost r , our model generates predictions for deal failures and M&A activity. If negotiations break down at a Poisson rate r , then $e^{-r\tau^{CE}(\theta)}$ is the probability of a negotiation failure for an acquirer of type θ . Proposition 1 then predicts that a marginal tightening of financial constraints (a small increase in d or a small decrease in L) increases the probability of deal failure for all types outside a small intermediate region. This is broadly consistent with empirical studies that have looked at financial constraints and M&A activity, e.g. [Malmendier et al. \(2016\)](#) and [Uysal \(2011\)](#). The former study shows that successful acquisitions have a larger cash component, while the latter shows that over-levered firms (firms that are more levered than predicted by other covariates) are less likely to make an acquisition in the observation period.

However, for larger changes in financial constraints (e.g., a large increase in d), there exists a sizeable range of intermediate types for Proposition 1 does not provide a concrete prediction. As we show in the bottom right panel of Figure 2, the deal failure probability may even decrease for those intermediate types. The regressions in [Malmendier et al. \(2016\)](#) and [Uysal \(2011\)](#), which estimate an average effect (averaged over the unobservable type θ), may therefore conflate heterogeneous effects that pull in opposite directions. Identifying those heterogeneous impacts empirically is an interesting possibility for future research.¹⁹

At the same time, our model raises new challenges for reduced-form inference on bargaining frictions. Since the time at which a negotiation starts

¹⁹Similar points apply for the empirical analysis of ex post returns. These are given by $1 - \frac{\tilde{S}_{liq}(\alpha(\theta), \theta; L)}{V(\theta)}$ in the liquidity constraints model and $1 - \frac{\tilde{S}_{lev}(\alpha(\theta), \theta; d)}{V(\theta)}$ in the leverage model.

is seldom observable by the researcher, but longer delay implies higher inefficiencies, a dataset with observations on realized delay may be thought to provide rare direct evidence on bargaining frictions. We explain how this line of reasoning—longer expected delay, measurable in the reduced form, implies greater inefficiency— can be incorrect.

Reduced-form estimation from realized delay could at most recover— even with a rich dataset on M&A negotiation delay that controls for the target value and industry characteristics— the expected delay curve:

$$\tau^{Exp}(\theta) = \mathbb{E}[\tau(\theta)] = \begin{cases} \tau^{CE}(\theta), & \theta < k^{SL}, \\ \tau^{CE}(k^{SL}) + \frac{1-D}{\tau D}, & \theta \geq k^{SL}, \end{cases} \quad (15)$$

with D given by (10). It would seem that a pointwise increase in $\tau^{Exp}(\theta)$ is strong reduced-form evidence in favor of rising bargaining inefficiencies. And yet, not only does τ^{Exp} necessarily differ from the (unobservable) certainty-equivalent delay curve τ^{CE} , which is the true welfare-relevant measure of bargaining frictions—this is immediate from the convexity of exponential discounting and the randomness in equilibrium delay—but the two curves can have completely different behaviors as financial constraints tighten.

Consider the levered equity scenario with normal-linear primitives. Figure 2 shows changes to τ^{Exp} and τ^{CE} as we move from the low leverage (loose financial constraints) case of $d_1 = 0$ to the high leverage case of $d_2 = 5$. The top row displays these changes for the parameter set $(c = 5, \Delta = 1/2, r = 1, \chi = 10, \beta = 1, \eta = 1)$. For these parameters, as when the buyer’s financial constraints tighten, both certainty-equivalent delay (welfare-relevant bargaining frictions) and expected (i.e., observable) delay rise uniformly for all buyer types.

The bottom row displays the effects of the same leverage change for a new parameter set $(c = 5, \Delta = 1/2, r = 1, \chi = 5, \beta = 1, \eta = 1/7)$. Here, tightening the buyer’s financial constraints raises bargaining frictions for buyer types in the top 90% percentile, but lowers them for types below that threshold. In contrast to the top row, tighter financial constraints now uniformly lower expected (i.e., observable) delay for almost all types. Put differently, a change

that worsens bargaining frictions for the top 90% percentile of acquiring firms would show up in the data as “improving” observable delay for nearly all firms!

Remarkably, for either parameter sets used in Figure 2, all buyer types are strictly better off when they are less financially constrained, so that observable delay can vary independently of both bargaining frictions and buyer preferences.

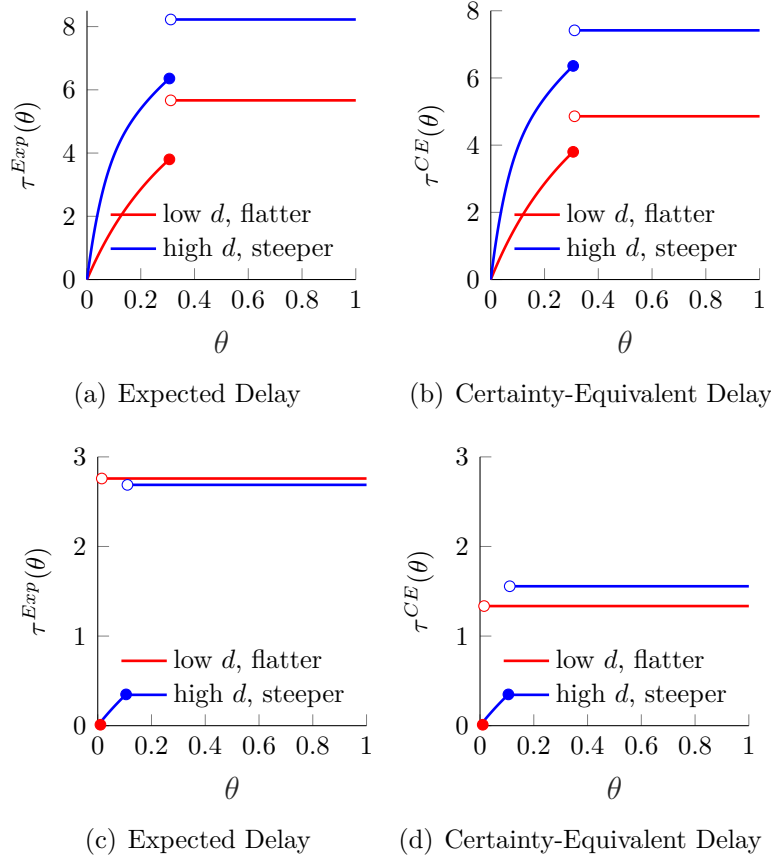


Figure 2: Discounting and observable delay are distinct. Panels (a) and (b): parameters are $c = 5, \Delta = 1/2, \chi = 10, \beta = 1, r = 1, d_1 = 0, d_2 = 5, \eta = 1$. All types suffer higher expected delay when leverage is higher. Panels (c) and (d): parameters are $c = 5, \Delta = 1/2, \chi = 5, \beta = 1, r = 1, d_1 = 0, d_2 = 5, \eta = 1/7$. Types in the pooling region suffer higher certainty equivalent delay when leverage is higher, and all types prefer having lower leverage. However, all but a vanishing fraction of types suffer lower expected delay when leverage is higher.

The impact of rising uncertainty about synergies: Using the leverage model

with normal-linear primitives, we show how bargaining outcomes change as the acquirer's signal about potential synergies becomes more precise. It is typically easier for a merged firm to achieve cost reductions, say by economizing on fixed costs, than it is to raise revenue by expanding into new markets (Berk and DeMarzo, 2013). An acquirer who hopes to achieve cost efficiencies through a merger will therefore tend to have a more precise estimate of the potential synergies than one who hopes to exploit a particular kind of product market fit.

Rising precision about synergies affects the negotiation through two different channels. On the one hand, for any offer α , it lowers the value of the levered equity: $\max\{\tilde{V} - d, 0\}$ is convex in \tilde{V} , and lowering η causes a mean-preserving spread of $\tilde{V}|\theta$. On the other, changes to η affect the slope of levered equity with respect to the buyer's private information; an initial intuition would suggest that, by raising the signal-to-noise ratio for the buyer's signal θ , a higher η would make levered more sensitive to θ . A priori it is not clear how what the equilibrium repercussions of these effects may be. We show that, even though the slope of levered equity with respect to the buyer's type may go up or down in absolute terms, the net effect of increases precision is indeed a "heightened sensitivity": for a given debt level, raising the precision of the buyer's signal is equivalent to bargaining in a steeper security:

Proposition 4. Consider a levered equity model with normal-linear primitives $\tilde{V}|\theta \sim \mathcal{N}(c + \Delta + \zeta\theta, \eta^{-2})$, so that η is the precision of $\tilde{V}|\theta$. Let $\bar{S}_{lev}^d(\alpha, \theta; \eta) := \mathbb{E}[\alpha \max\{\tilde{V} - d, 0\}|\theta]$ denote the expected payment as a function of offer, type, and precision.

1. There exist $\bar{d} > \underline{d}$ such that

$$\frac{\partial^2}{\partial \theta \partial \eta} \bar{S}_{lev}^d(\alpha, \theta; \eta) = \begin{cases} < 0, & d > \bar{d} \\ > 0, & d < \underline{d} \end{cases}$$

2. For any $\eta_1 > \eta_2$, and α_1, α_2 ,

$$\bar{S}_{lev}^d(\alpha_1, \theta; \eta_1) = \bar{S}_{lev}^d(\alpha_2, \theta; \eta_2) \Rightarrow \frac{\partial}{\partial \theta} \bar{S}_{lev}^d(\alpha_1, \theta; \eta_1) > \frac{\partial}{\partial \theta} \bar{S}_{lev}^d(\alpha_2, \theta; \eta_2) \quad (16)$$

3. If there is non-trivial delay for some η (i.e., the environment is upward-skimming and SL holds), there is non-trivial delay for every $\eta' > \eta$.
4. Let $h(\cdot)$ be the hazard rate of the standard normal distribution. A sufficient condition for the environment to be upward-skimming for all d is

$$\frac{\beta}{\beta + \chi} < \frac{\eta(c + \Delta)}{\eta(c + \Delta) + \eta(\beta + \chi) + h(-\eta(c + \Delta))}.$$

Points 2 and 3 imply that a higher precision is equivalent to bargaining in a steeper security, and all the results from Section 4 apply unchanged.

Figure 3 shows the changes in certainty equivalent delay, expected payments $\bar{S}_{lev}(\alpha(\theta; d), v; d)$, and indirect buyer utilities as η increases from $\eta = 1/10$ to $\eta = 1$. The other parameters are held fixed at $c = 5, \Delta = 1/2, \beta = 1, \chi = 10, r = 1, d = 4$. With a higher precision about synergies, bargaining frictions—measured by τ^{CE} —rise for all types, and all types are made better off. As predicted by Propositions 1-3, (i) expected payments rise for types above a threshold and (weakly) drop for types under that threshold, (ii) there are fewer types are in the final pooling region, (iii) bargaining frictions rise for low enough and high enough types, and (iv) types below a threshold are harmed by the increase in precision. For this large rise in the precision, we see moreover that bargaining frictions rise for all types, and all types other than the corners are made strictly worse off.

Put differently, increasing the precision of the buyer's signal destroys value. The seller's equilibrium payoff is unaffected (since the game starts with smooth trade, it equals c regardless of the precision level), so this destruction is all at the buyer's expense. The buyer, in particular, would not want to invest in a technology that improves its prediction about synergies. If, as in the cost reduction vs market examples above, we see different η 's as modeling different possible mergers *ceteris paribus*, this suggests that, net of negotiation inefficiencies, the buyer has an incentive to focus mergers with more uncertain synergies.

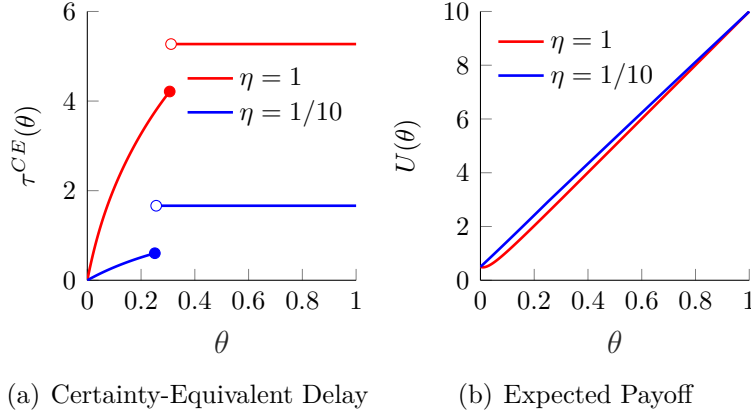


Figure 3: Discounting costs and buyer payoffs as a function of signal precision. Parameters are $c = 5, \Delta = 1/2, \chi = 10, \beta = 1, r = 1, d = 4$.

6 Discussion

Relation to Entrepreneurial Financing Models: If we interpret the seller in our setting as the financier, and the cost c as the required funding, our model corresponds to a non-competitive, no commitment version of [Myers and Majluf \(1984\)](#). To provide some contrast, let us consider a very simple version of their model where there is a competitive market and the firm has to finance the project with equity. Letting \mathcal{I} be the set of types who issue equity in equilibrium, the equilibrium α^* is determined by the zero profit condition

$$(1 - \alpha^*)\bar{V} = c, \quad (17)$$

where $\bar{V} := \mathbb{E}[V(\theta) \mid \theta \in \mathcal{I}]$. Given the equilibrium price of the security, the manager (who acts on behalf of old shareholders and knows θ) invests in the project only if

$$R(\theta) - \alpha^*V(\theta) \geq 0 \Rightarrow \frac{R(\theta)}{A(\theta)} \geq \frac{c}{\bar{V}} - 1 \quad (18)$$

Similar to our upward/downward-skimming distinction, the type of inefficiency depends R/A is increasing or decreasing. If R/A is decreasing—the upward-skimming case, where assets in place are the dominant source of private information—the firm invests if and only if its type is sufficiently low,

whereas if R/A is increasing—the downward-skimming case, it invests only if its type is sufficiently high. For a fixed probability of acceptance, inefficiencies will be worse when the source of private information is assets in place, since the types who do not invest are precisely the high types who generate the greatest net returns. In our setting, this efficiency “wedge” between the upward- and downward-skimming cases becomes especially extreme: when the dominant source of asymmetric information is the return of the project, inefficiencies are not only reduced but vanish.

Implications for Corporate Restructuring: With minor modifications, our model applies to corporate restructuring negotiations of firms in financial distress. These negotiations can occur in a formal bankruptcy procedure (a Chapter 11 process), or outside of one (an out-of-court restructuring). Even though there is significant evidence of inefficiencies in corporate restructuring negotiations, most of the literature consider bargaining models that lead to efficient outcomes.²⁰ In contrast, our model allows us to study the impact that the securities have on the efficiency of the negotiation. Our model also allows to study the impact that other terms of the securities, such as debt maturity, have on the negotiation.

We briefly elaborate on the institutional setting, and we describe how the model maps to the restructuring setting. The identity of the parties (that is, seller and buyer) during a corporate restructuring depends on the circumstances. In Chapter 11 restructuring, negotiation is typically between junior and senior creditors, while in out-of-court restructuring, negotiations are typically between current equity holders and debt holders. The main advantage of an out-of-court restructuring is to avoid the costs (direct and indirect) of going through a formal bankruptcy procedure. There are multiple restructuring options during an out-of-court restructuring depending on the severity of financial distress. The first option, when the firm is not severely distressed, is for equity holders to infuse new capital or repurchase debt, in exchange for some reduction in the value of current debt. The second option, when

²⁰Most models consider either Nash Bargaining or alternating offer with symmetric information (Bebchuk and Chang, 1992; Bernardo et al., 2016; Antill and Grenadier, 2019).

firms are severely distressed, is to exchange the current debt for fresh new debt and equity. The new debt usually has a lower face value and longer maturity (Altman et al., 2019)

Focusing on out-of-court restructuring, let the seller be the debt-holder, and the buyer be the current equity-holder, and θ represent the equity-holder's private information about current and future cash-flows. The security is debt, and the negotiation is over the face value α , in which case $\bar{S}(\alpha, \theta) = \mathbb{E}[(\tilde{V} - \alpha)^+ | \theta]$. $R(\theta)$ then captures the equity-holder's benefits from restructuring relative to liquidation. A and c capture the liquidation value that the parties expect to obtain if the negotiation fails, which depend on their pre-existent claims to the assets. In this scenario, one would expect that c , the debt-holder's liquidation value, can depend on the equity-holder's type θ , i.e., it is a function $c(\theta)$; our setting can accommodate that with minor tweaks.²¹ The discount rate r can be interpreted in multiple ways. One interpretation is that r captures the probability that negotiations break down and trigger a Chapter 11 bankruptcy procedure (in the case of a Chapter 11 negotiation a breakdown in the negotiation can lead to a Chapter 7 liquidation). A second viable interpretation is that f captures in reduced form the deterioration in the value of the firm due to ongoing financial distress.

Using this mapping, our model uncovers an unintended consequence of raising the maturity of debt. Using the valuation model of corporate risky debt with log-normal cash flows developed by Merton (1974), Diamond and He (2014) show that debt with longer maturity is steeper.²² Proposition 1 then

²¹Payoffs are as in footnote 7. The equilibrium offer in the smooth trading region is given by $\bar{S}(\alpha(k), k) - c(k) = \hat{S}(\alpha(k), k) = 0$ and the definition for k^{SL} becomes

$$k^{SL} = \inf \left\{ k \leq 1 : \mathbb{E} \left[\hat{S}(\alpha^f(1), \theta) | \theta \in [k, 1] \right] \geq 0 \right\}$$

A additional requirement is that $\bar{S}_\theta(\alpha(k), k) > c'(k)$, so that $\alpha(k)$ is decreasing.

²²Diamond and He (2014) do not state their results directly in terms of steepness. However, the fact that debt with longer maturity is steeper follows from their Proposition 1, which states the following: If we let $D(V; F, m)$ be the value of debt with face value F and maturity m when the current value of assets is V , then, for any $m_2 > m_1$, $D_V(V; F_1, m_1) < D_V(V; F_2, m_2)$ whenever $D(V; F_1, m_1) = D(V; F_2, m_2)$. This corresponds exactly to our definition of steepness.

implies that increasing the maturity increases bargaining frictions. Raising the maturity of debt is typically seen as a way to spread the firm's payments more thinly over time, giving the firm a greater opportunity to improve performance and meet its debt obligations. However, our analysis shows that increasing maturity has an indirect cost: it increases the frictions in the negotiation, reduces the probability of reaching an agreement, and therefore increases the probability of entering formal bankruptcy.

Connection to Security Auction Literature: [Che and Kim \(2010\)](#) first pointed out that having the value of assets-in-place $A(\theta)$ be increasing in θ could lead to decreasing bidding strategies in security auctions. They study the effect on auction revenue of moving from a flatter security S^2 to a steeper one S^1 , when both have equilibria in decreasing strategies (are “upward-skimming,” in our terminology)²³ We now contrast our results to theirs.

First, [Che and Kim \(2010\)](#) prove, both for first and second price auctions, that expected payments are higher type-by-type under S^2 , the flatter security.²⁴ The difference between our results and theirs stems from the seller's extreme commitment problem. In our model, due to her extreme lack of commitment, the seller makes exactly c on every trade. In particular, intermediate types ($\theta \in (k^{SL}(S^2), k^{cross})$) pay strictly less under S^1 , since types in that range are separated in the S^1 equilibrium, but they get cross-subsidized by high types in the S^2 equilibrium.

Second, a consequence of the revenue rankings in [Che and Kim \(2010\)](#) is that, in their model, among those securities that induce decreasing strategies, bidders prefer the steeper ones. In our model, that preference can be reversed uniformly for all types and in fact is always reversed for types under k^* . The

²³Their results are more general, but we emphasize this aspect of them to streamline the comparison to ours. In fact, they show that, whenever S^2 has decreasing bidding strategies, S^1 also will. In our setup, it is easy to show that if SL holds for S^2 , then it must hold for S^1 , but it does not follow that if S^2 is upward-skimming, S^1 will also be.

²⁴We refer to the working paper version, which contains results on both auction formats (available on the authors' website here: <https://emu-perch-bjgm.squarespace.com/s/security-comment-1.pdf>). Proposition 3 in that version shows that payments are higher ex-post in the second price auction. The proof of their Proposition 5 shows that interim utilities are higher under S^1 in a first price auction; since the allocations are the same under either security, interim expected payments must be higher under S^2 .

reason for the contrast is that the allocation in [Che and Kim \(2010\)](#) remains the same as the security becomes steeper, while in our bargaining game, changing the security changes the amount of delay and the expected allocation. Payment rankings therefore translate into utility rankings in their work, while in ours, the allocation effect on payoffs often overwhelms the effect of higher expected payments under S^2 .²⁵

Connection to [Deneckere and Liang \(2006\)](#):

In their classic analysis of interdependent values cash bargaining, [Deneckere and Liang \(2006\)](#) posit a privately informed seller with type $\theta \sim U[0, 1]$ and preferences

$$(1 - e^{-rt})c(\theta) + e^{-rt}\alpha$$

bargaining with an uninformed buyer with preferences

$$e^{-rt}(v(\theta) - \alpha).$$

where α is the cash offer. To highlight the tractability of our continuous-time analysis, we revisit their setup.

As in our model, their leading case assumes the uninformed party makes all the offers, there is a gap ($v(\theta) - c(\theta) > 0$) and first-best efficiency is unattainable ($\mathbb{E}[v(\theta) - c(1)] < 0$). Unlike our setup, t is on a discrete grid $\{0, \Delta, 2\Delta, 3\Delta, \dots\}$, and they assume v and c are step functions, i.e., there are finitely many payoff types. c is weakly increasing. They study the limit of Markovian equilibria as $\Delta \rightarrow 0$. A subtle proof shows that, in the limit, the sequence of offers is a step function: there are quiet periods of deterministic length, punctuated by discontinuous drops in price.

If one assumes instead that v and c are both smooth and strictly increasing, a direct consequence of Theorem 1 is that there is a unique (regular, weak

²⁵To emphasize this point, consider two upward-skimming securities that both fail SL. All types trade instantly in either case, so the allocation—i.e., the expected discount until trade—is constant across securities. Then, given that all types pay $\bar{S}^i(\alpha^{f,i}(1), \theta)$ under S^i , we know from (12) that flatter securities lead to higher revenue and make buyers worse off, as in [Che and Kim \(2010\)](#). Taking two downward-skimming securities, we get the opposite result, as in [DeMarzo et al. \(2005\)](#).

Markov) outcome path.²⁶ Let $k^* = \inf \{k : \mathbb{E}[v(\theta) - c(1)|\theta \geq k] = 0\}$. Then, for $k < k^*$, trade is smooth according

$$\dot{K}_t = \frac{r(v(K_t) - c(K_t))}{v'(K_t)}, \quad K_0 = k.$$

with offers $\alpha(k) = v(k)$. At k^* , there is a (stochastic) quiet period that causes expected discounting costs δ given by

$$v(k^*) = (1 - \delta)c(k^*) + \delta c(1) \Rightarrow \delta = \frac{v(k^*) - c(k^*)}{c(1) - c(k^*)},$$

after which the pooling offer $c(1)$ is made. Comparing this to their Theorem 3 and equation (15) in their paper, we see three effects of a continuous-type, continuous-time formulation. First, there is a large tractability gain, with the dynamics essentially pinned down by “first order conditions” (8) and (9). Second, the equilibrium acquires a smooth region with gradual concessions. Third, the certainty equivalent delay after reaching k^* is half the amount implied by their discrete-type model— mutatis mutandi, our expression for δ exactly matches their expression for ρ_1 .

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²⁶The MRS between α and t for the privately informed seller is $\iota(\theta, \alpha) = -r(\alpha - c(\theta))$, which is strictly increasing in θ .

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Appendix: Proofs for Section 4

Throughout this section, S^1 and S^2 refer to two order securities in $\mathcal{D}_{A,\tilde{V},c}$, with S^1 steeper. An outline for this section is as follows. The ranking of critical types $k^{SL}(S^i)$ was proved in the text (Proposition 1.1). With that in hand, we prove Proposition 2, which ranks equilibrium expected payments by type. Using an envelope-like representation of equilibrium payoffs (Lemma 2), we prove Proposition 3, our ranking on equilibrium utilities by type. Finally, with the ranking on equilibrium utilities, we return to the issue of certainty-equivalent delay and prove Proposition 1.2, which gives ranks delay for types outside an intermediate region. Auxiliary lemmas are proved in Online Appendix B.

Proof of Proposition 2. The first line of (14) follows from smooth trading: $\pi_i(\theta) = \bar{S}^i(\alpha^i(\theta), \theta) = c$ on $[0, k^{SL}(S^2))$. To show the latter two lines, for $\theta \in (k^{SL}(S^1), 1)$, the equilibrium payment under S^i satisfies $\pi_i(\theta) = \bar{S}^i(\alpha^{f,i}(1), \theta)$; the inequality in (12) then becomes

$$\pi_1(\theta) < \pi_2(\theta), \theta \in (k^{SL}(S^1), 1).$$

Meanwhile, by definition $k^{SL}(S^2)$ must satisfy

$$\mathbb{E}[\bar{S}^2(\alpha^{f,2}(1), \theta) | \theta \in (k^{SL}(S^2), 1]] = c,$$

so there exists a $k^* \in (k^{SL}(S^2), 1)$ such that

$$\bar{S}^2(\alpha^{f,2}(1), \theta) \begin{cases} < c, \theta \in (k^{SL}(S^2), k^*), \\ = c, \theta = k^* \\ > c, \theta \in (k^*, 1] \end{cases} \quad (19)$$

Since $\pi_1(\theta) = c$ for $\theta \leq k^{SL}(S^1)$, if $k^* < k^{SL}(S^1)$, then (12) and (19) yield $k^{cross} = k^*$. On the other hand, if $k^* \geq k^{SL}(S^1)$, (12) and (19) yield $k^{cross} = k^{SL}(S^1)$. \square

We use the following convenient representation of equilibrium payoffs:

Lemma 2. Let $U_i(\theta)$ be the equilibrium indirect utility under an upward-skimming security S^i , and let α^i be the associated equilibrium offer. Define

$$\nu^i(\theta, \alpha) := \frac{R'(\theta) - \bar{S}_\theta^i(\alpha, \theta)}{R(\theta) - \bar{S}^i(\alpha, \theta)}$$

Then for $\theta < 1$,

$$U_i(\theta) = A(\theta) + (R(0) - c) \exp \left\{ \int_0^\theta \nu^i(y, \alpha^i(y)) dy \right\}. \quad (20)$$

With (27), we can prove Proposition 3. We split the proof in two parts.

Proof of Proposition 3.1. Let $\tilde{\alpha}^i(\theta)$ denote the locus $\bar{S}^i(\tilde{\alpha}^i(\theta), \theta) = c$, and $\alpha^i(\theta)$ denote the equilibrium offer accepted by type θ under security i . Using Propositions 1 and 2, we have

$$\begin{aligned} \alpha^1(\theta) &= \tilde{\alpha}^1(\theta), \theta \in [0, k^{cross}] \\ \alpha^2(\theta) &\begin{cases} = \tilde{\alpha}^2(\theta), \theta \in [0, k^{SL}(S^2)] \\ < \tilde{\alpha}^2(\theta), \theta \in (k^{SL}(S^2), k^{cross}) \end{cases} \end{aligned} \quad (21)$$

In addition, let $U_i(\theta)$ denote θ 's equilibrium expected utility when bargaining with security S^i .

Since S^1 is steeper,

$$\nu^1(\theta, \tilde{\alpha}^1(\theta)) = \frac{R'(\theta) - \bar{S}_\theta^1(\tilde{\alpha}^1(\theta), \theta)}{R(\theta) - \bar{S}^1(\tilde{\alpha}^1(\theta), \theta)} < \frac{R'(\theta) - \bar{S}_\theta^2(\tilde{\alpha}^2(\theta), \theta)}{R(\theta) - \bar{S}^2(\tilde{\alpha}^2(\theta), \theta)} = \nu^2(\theta, \tilde{\alpha}^2(\theta)).$$

Integrating with respect to θ for $\theta < k^{cross}$, and using (21), one obtains

$$\begin{aligned} U_1(\theta) &= A(\theta) + (R(0) - c) e^{\int_0^\theta \nu^1(y, \alpha^1(y)) dy} = A(\theta) + (R(0) - c) e^{\int_0^\theta \nu^1(y, \tilde{\alpha}^1(y)) dy} \\ &< A(\theta) + (R(0) - c) e^{\int_0^\theta \nu^2(y, \tilde{\alpha}^2(y)) dy} \\ &< A(\theta) + (R(0) - c) e^{\int_0^\theta \nu^2(y, \alpha^2(y)) dy} = U_2(\theta), \end{aligned}$$

where the second inequality uses

$$\begin{aligned} \frac{\partial}{\partial \alpha} \nu^2(\theta, \alpha) &\propto \frac{\partial}{\partial \alpha} (R'(\theta) - \bar{S}_\theta^2(\alpha, \theta)) [R(\theta) - \bar{S}^2(\alpha, \theta)] \\ &\quad + \bar{S}_\alpha^2(\alpha, \theta) [R'(\theta) - \bar{S}_\theta^2(\alpha, \theta)] \propto -\frac{\partial}{\partial \theta} \iota^{S^2}(\theta, \alpha) < 0 \end{aligned} \quad (22)$$

□

To prove the second part of Proposition 3, we need a quick technical lemma:

Lemma 3. Let $S(\cdot, \cdot, \gamma)$ be a security belonging to a parametrized steepness class. Let $k^{SL}(\gamma)$ be the critical type under security $S(\cdot, \cdot; \gamma)$, and let $k^*(\gamma)$ solve $\bar{S}(\alpha^f(1; \gamma), k^*(\gamma); \gamma) = c$, where $\alpha^f(1; \gamma)$ is final offer under $S(\cdot, \cdot; \gamma)$. For $\hat{\gamma} > \gamma$ sufficiently close to γ , $k^*(\gamma) > k^{SL}(\hat{\gamma})$.

Proof of Proposition 3.2. By Lemma 3, $k^*(\gamma) > k^{SL}(\hat{\gamma})$ for all $\hat{\gamma} > \gamma$ sufficiently close to γ , $k^*(\gamma) > k^{SL}(\hat{\gamma})$. By Proposition 3.1, buyer types in $[0, k^*(\gamma)]$ must then prefer $S(\cdot, \cdot; \gamma)$ to $S(\cdot, \cdot; \hat{\gamma})$, strictly so for $\theta > 0$.

Using the payoff representation in Lemma 2, we can write the indirect utility for type $\theta \in (k^*(\gamma), 1)$ under security $S(\cdot, \cdot; \gamma^\dagger)$, $\gamma^\dagger \in \{\gamma, \hat{\gamma}\}$ as

$$U(\theta; \gamma^\dagger) = A(\theta) + U(k^*(\gamma); \gamma^\dagger) \exp \left\{ \int_{k^*(\gamma)}^v \nu(y, \alpha^f(1; \gamma^\dagger); \gamma^\dagger) dy \right\}. \quad (23)$$

It follows that from the assumption that $\nu(\theta, \alpha^f(1; \gamma); \gamma)$ is decreasing in γ , and from $U(k^*(\gamma); \gamma) > U(k^*(\gamma); \hat{\gamma})$, that

$$U(k^*(\gamma); \gamma) \exp \left\{ \int_{k^*(\gamma)}^v \nu(y, \alpha^\dagger(1; \gamma); \gamma) dy \right\} \geq U(k^*(\gamma); \hat{\gamma}) \exp \left\{ \int_{k^*(\gamma)}^v \nu(y, \alpha^\dagger(1; \hat{\gamma}); \hat{\gamma}) dy \right\}$$

Plugging this into the representation (23) above, $U(\theta, \gamma) \geq U(\theta, \hat{\gamma})$ for all $\theta \in [0, 1]$, and strictly so for interior θ 's. Since $\gamma \in [\gamma', \gamma'']$ was arbitrary, we conclude that for any $U(\theta, \gamma)$ is a weakly decreasing function of γ , and strictly decreasing for interior θ 's. \square

Proof of Proposition 1.2. Let $\tau^{CE}(1; \gamma)$ denote the certainty-equivalent delay for $\theta = 1$ under security $S(\cdot, \cdot; \gamma)$; we show that, for all $\gamma \in [\gamma_2, \gamma_1]$, $\tau^{CE}(1; \gamma)$ is strictly increasing in γ , which implies the result.

By Lemma 3, for $\hat{\gamma} > \gamma$ sufficiently close to γ , $k^*(\gamma) > k^{SL}(\hat{\gamma})$. By Proposition 3.1, for small enough $\xi > 0$, buyer $k^{SL}(\hat{\gamma}) + \xi$ must then strictly prefer $S(\cdot, \cdot; \gamma)$ to $S(\cdot, \cdot; \hat{\gamma})$. And yet, by Proposition 2, in expectation $k^{SL}(\hat{\gamma}) + \xi$ pays strictly less under $S(\cdot, \cdot; \gamma)$ than under $S(\cdot, \cdot; \hat{\gamma})$. It follows that $k^{SL}(\hat{\gamma}) + \xi$ must suffer strictly higher discounting costs under $S(\cdot, \cdot; \hat{\gamma})$. Since $k^{SL}(\hat{\gamma}) + \xi$ is in the final trading atom under $S(\cdot, \cdot; \gamma)$ and under $S(\cdot, \cdot; \hat{\gamma})$, $\tau^{CE}(1; \hat{\gamma}) > \tau^{CE}(1; \gamma)$, as required. \square

Online Appendix

A Necessary Conditions

We begin with a key technical lemma characterizing the direction of skimming as a function of the primitives and the security family. The proof, which adapts classic arguments in [Milgrom and Shannon \(1994\)](#) and [Edlin and Shannon \(1998\)](#), is in Online Appendix B.

Lemma 4 (Skimming). Fix a security family S . Let ι^S be as in (4), U be given by

$$U(t, \alpha, \theta) := (1 - e^{-rt})A(\theta) + e^{-rt}(V(\theta) - \bar{S}(\alpha, \theta)),$$

and let $\alpha^f(\cdot)$ be as in (3).

Fix an arbitrary (deterministic) sequence of offers $\{\tilde{\alpha}_t\}_{t \geq 0}$, and let $T(\theta) := \arg \max_{t \in \mathbb{R}_+ \cup \{+\infty\}} U(t, \tilde{\alpha}_t, \theta)$.

1. If $\iota^S(\cdot, \alpha)$ is strictly increasing for all α , every selection from $T(\theta)$ is non-decreasing,²⁷ and $\alpha^f(\cdot)$ is strictly decreasing.
2. If $\iota^S(\cdot, \alpha)$ is strictly decreasing for all α , every selection from $T(\theta)$ is non-increasing, and $\alpha^f(\cdot)$ is strictly increasing.

Remark 3 (Relation to Usual Skimming Notions). Lemma 4 is weaker than the usual skimming result invoked in the literature on cash bargaining, so our focus on Markovian skimming equilibria is a stronger restriction than the analogous restriction in models with cash bargaining. To highlight the differences, focus on the downward-skimming case. In the literature on Coasean bargaining with cash, if a type θ is indifferent between accepting and rejecting an offer p after a history H_t , then all types $\theta' > \theta$ strictly prefer to accept p at H_t regardless of continuation play after the rejection. In contrast, the present lemma covers only deterministic offer paths, and it allows for offer histories where both θ and θ' are indifferent between accepting and rejecting. Hence, the Lemma does not

²⁷In the usual order on the extended real line.

entirely rule out histories in which θ' accepts strictly earlier—for example, if θ and θ' are both indifferent and randomize over their acceptance decisions.

However, Lemma 4 guides the search for tractable equilibria, since it shows, for example, that when $\iota_\theta^S(\theta, \alpha) > 0$ it is be fruitless to search for skimming equilibria where higher types accept first with certainty. And even though Lemma 4 only covers deterministic offer paths, the stochasticity of equilibrium offers is such that, with some additional arguments, the lemma suffices to verify incentive compatibility for the buyer.

Given the lemma, and the assumption of a skimming environment, we now derive the necessary conditions behind the uniqueness claims in Theorem 1.

Proof of Theorem 1, Necessary Conditions.

Downward Skimming: We assume the existence of an equilibrium with non-trivial smooth screening and derive a contradiction.

Since higher types trade first, the seller's beliefs are right-truncations of the prior, and the truncation cutoff k is the Markov state controlled by the seller. Let k be a state such that smooth trade is prescribed for all $k' \in (k - \varepsilon, k + \varepsilon)$, $\varepsilon > 0$. Then, by regularity of the equilibrium, the seller's value function at k must satisfy the HJB

$$rJ(k) = \sup_{(|\dot{k}|, \gamma) \in \mathbb{R}_+^2} \left\{ \left(\overbrace{\bar{S}(\alpha(k), k)}^{\mathbb{E}[S(\alpha(k), \tilde{V})|k]} - J(k) \right) \frac{|\dot{k}|}{k} - J'(k)|\dot{k}| + \gamma \left(\mathbb{E}[\bar{S}(\alpha^f(0), \theta)|\theta \leq k] - J(k) \right) + rc \right\}. \quad (24)$$

with the right hand side attained by some $|\dot{k}| < \infty, \gamma < \infty$. If $|\dot{k}| < \infty, \gamma < \infty$ is indeed optimal, by the linearity of the objective with respect to $|\dot{k}|, \gamma$, the coefficients on those choice variables must be weakly negative, and (24) must simplify to $J(k) = c$. From the coefficient on γ , we conclude

$$\mathbb{E}[\bar{S}(\alpha^f(0), \theta)|\theta < k] \leq c. \quad (25)$$

Now we have reached a contradiction: $k > 0$, since it is in the interior $(k - \varepsilon, k + \varepsilon)$, so

$$\mathbb{E}[\bar{S}(\alpha^f(0), \theta) | \theta \leq k] > \bar{S}(\alpha^f(0), 0) = R(0) \geq c,$$

where the strict inequality follows Assumption 1 and $\alpha^f(0) \neq \underline{\alpha}$.²⁸

Hence, the states at which smooth trade is prescribed have zero measure, and the equilibrium path K can have no smooth trade regions of positive duration where it is strictly increasing—if there were such an interval (\underline{t}, \bar{t}) , then $(K_{\underline{t}}, K_{\bar{t}})$ would be a positive measure interval of states with smooth trade.

Consider now a smooth trade region of positive duration that is quiet, that is, K is constant at some level k . The seller's payoffs would be

$$(1 - \mathbb{E}[e^{-rT}])c + \mathbb{E}[e^{-rT}]\mathbb{E}[\bar{S}(\alpha^f(0), \theta) | \theta \geq k] < \mathbb{E}[\bar{S}(\alpha^f(0), \theta) | \theta \geq k].$$

The right hand side is achieved with $T = 0$. We again have a contradiction, since the positive-length quiet period is strictly suboptimal.

Hence, the only possible dynamics involve quiet periods at which the pooling offer is made immediately, or jumps in K . The game must therefore end in the first instant, which can only happen at the pooling offer $\alpha^f(0)$.

Upward Skimming: The seller now controls the left truncation of her posterior beliefs as a Markov state. Notice, for $k < k^{SL}$, the seller will never use the pooling offer with positive probability, since she would earn less than c by doing so. With that in mind, we first identify implications of smooth trading. The derivation in the main text then implies that, for any k in the interior of a smooth trade region, $J(k) = c$ and $\bar{S}(\alpha(k), k) \leq c$; if, in addition, $\dot{k} \neq 0$ at such a state, $\bar{S}(\alpha(k), k) = c$.

Second, we show that if there is smooth trade, it happens only on a set of states $[0, k^{smooth})$, i.e., the game starts with smooth trading and ends with a jump. Suppose that, on some continuation game, there were a jump from k to $k' > k$. Since there are countably many jumps, and jumps are isolated, smooth trade must recommence at k' . In particular, $J(k') = c$ and $\mathbb{E}[\bar{S}(\alpha(k'), \theta) | \theta = k'] \leq c$. The seller's payoff from jumping to k' is therefore

$$\left(\frac{k' - k}{1 - k} \right) \mathbb{E}[\bar{S}(\alpha(k'), \theta) | \theta \in [k, k']] + \left(\frac{1 - k'}{1 - k} \right) c.$$

²⁸ $\alpha^f(0)$ satisfies $\bar{S}(\alpha^f(0), 0) = R(0) \geq c$, so it must be greater than $\underline{\alpha}$.

The seller can always freeze trade and ensure a payoff of c , so for such a jump to be optimal,

$$c \leq \mathbb{E}[\bar{S}(\alpha(k'), \theta) | \theta \in [k, k']] \leq \mathbb{E}[\bar{S}(\alpha(k'), \theta) | \theta = k'] \leq c.$$

If the second inequality were an equality, then $\alpha(k') = \alpha$, by the nondegeneracy condition in Assumption 1. But that would contradict the last inequality, since $\bar{S}(\alpha, k') < c$. The second inequality must therefore be strict, which is a contradiction. Therefore, the set of smooth trade states must be an interval $[0, k^{smooth})$.

Third, we show that $k^{smooth} = k^{SL}$. By Condition 2 in the equilibrium definition, buyer $\theta = 1$'s equilibrium reservation price $\alpha(1)$ must be equal to $\alpha^f(1)$, the highest take-it-or-leave-it offer that he would accept. Hence, if $k^{smooth} < k^{SL}$, so that continuation play prescribes a jump before the state reaches k^{SL} , then it is weakly optimal for the seller to jump directly to $k = 1$ at k^{smooth} . Her payoffs at k^{smooth} are therefore $\mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in [k^{smooth}, 1]] < c$, by definition of k^{SL} . This violates the seller's individual rationality, so we must have $k^{smooth} \geq k^{SL}$. However, if $k^{smooth} > k^{SL}$, then for any state $k \in (k^{SL}, k^{smooth})$, the seller can jump the state to $\theta = 1$ with an offer of $\alpha^f(1)$; this gives her a payoff of $\mathbb{E}[\bar{S}(\alpha^f(1), \theta) | \theta \in [k, 1]] > c$, a profitable deviation. In particular, $k^{smooth} = k^{SL}$ implies that, if the SL fails, all equilibria have instant trade.

Fourth, we show that, for $k \in (0, k^{SL})$, the optimal \dot{k} for the seller must be strictly positive, i.e., there are no quiet periods. Suppose otherwise. By a typical dynamic programming argument, if starting at state $k \in (0, k^{SL})$ we have $\dot{k} = 0$, then the continuation value of the marginal buyer equals $A(k)$, and $\alpha(k) = \alpha^f(k)$ as defined in Lemma 4. But then $\bar{S}(\alpha(k), k) = R(k) > c$, a contradiction.

Finally, we show that if the SL holds, but there is no gap, trade breaks down. Since the SL holds, $k^{SL} > 0$. By the previous point, the states $[k^{SL}, 1]$ must be reached via smooth trade. As argued in the main text, local incentive compatibility for the marginal buyer, coupled with the fact that K is C^1 under it reaches k^{SL} , implies that K_t must satisfy (9):

$$\dot{K}_t = \frac{r(R(K_t) - c)}{\bar{S}_\theta(\alpha(K_t), K_t)}, \quad K_0 = 0,$$

The right hand side of this IVP is C^1 , given the assumptions on primitives and the expression for $\alpha(\cdot)$, so the IVP has a unique local solution $K_t = 0$ on a maximum interval of existence $[0, \varepsilon)$, with $\lim_{t \rightarrow \varepsilon} K_t = 0$. (Note that $\alpha(0) > \underline{\alpha}$, since $\bar{s}(\alpha(0), 0) = c$). At $t = \varepsilon$, the situation repeats itself, so K_t never rises above 0. \square

B Auxiliary Lemmas for Sections 3 and 4

Proof of Lemma 2. Equilibrium payoffs are

$$U_i(\theta) = A(\theta) + e^{-r\tau_i^{CE}(\theta)} (R(\theta) - \bar{S}^i(\alpha^i(\theta), \theta)), \quad (26)$$

and, by the envelope theorem, their derivative is

$$U'_i(\theta) = A'(\theta) + e^{-r\tau_i^{CE}(\theta)} (R'(\theta) - \bar{S}_\theta^i(\alpha^i(\theta), \theta)) \quad (27)$$

almost everywhere. Using equation (26) to substitute $e^{-r\tau_i^{CE}(\theta)}$ into the envelope condition (27) we get²⁹

$$\frac{U'_i(\theta) - A'(\theta)}{U_i(\theta) - A(\theta)} = \frac{\partial}{\partial v} \log(U_i(\theta) - A(\theta)) = \frac{R'(\theta) - \bar{S}_\theta^i(\alpha^i(\theta), \theta)}{R(\theta) - \bar{S}^i(\alpha^i(\theta), \theta)}$$

almost everywhere. Integrating with respect to θ yields (20). \square

Proof of Lemma 3. By definition, $\mathbb{E}[\bar{S}(\alpha^f(1; \gamma), \theta; \gamma) | \theta \in [k^{SL}(\gamma), 1]] = c = \bar{S}(\alpha^f(1; \gamma), k^*(\gamma))$, so we must have $k^*(\gamma) > k^{SL}(\gamma)$. The lemma then follows by the continuity of $k^{SL}(\gamma)$, which we now show. Let $\alpha^f(1; \gamma)$ denote the final (pooling) offer under security $S(\cdot, \cdot; \gamma)$. By the continuity of $\bar{S}(\cdot, \theta; \cdot)$, $\alpha^f(1; \cdot)$ is continuous, using the inverse function theorem. The critical type $k^{SL}(\gamma)$ solves

$$\int_k^1 \bar{S}(\alpha^f(1; \gamma), v, \gamma) d\theta - (1 - k)c = 0.$$

with respect to k . By the continuity of \bar{S} and $\alpha^f(1; \cdot)$, $k^{SL}(\cdot)$ is also continuous. \square

²⁹We know from Theorem 1 that $R(\theta) > \bar{S}^i(\alpha^i(\theta), \theta)$ for all $\theta < 1$.

Proof of Lemma 4. We prove the case of strictly increasing $\iota^S(\cdot, \alpha)$ (the argument for an decreasing $\iota^S(\cdot, \alpha)$ is symmetric). The statements on α^f follow by implicit differentiation: since α^f solves $R(\theta) = \bar{S}(\alpha^f(\theta), \theta)$,

$$\begin{aligned} \frac{\partial}{\partial \theta} \alpha^f(\theta) &= \frac{R'(\theta) - \bar{S}_\theta(\alpha^f(\theta), \theta)}{\bar{S}_\alpha(\alpha^f(\theta), \theta)} - \overbrace{[R(\theta) - \bar{S}(\alpha^f(\theta), \theta)]}^{=0} \frac{\bar{S}_{\alpha\theta}(\alpha^f(\theta), \theta)}{\bar{S}_\alpha(\alpha^f(\theta), \theta)^2} \\ &= -\frac{\partial}{\partial \theta} \iota^S(\theta, \alpha^f(\theta)) < 0 \quad (28) \end{aligned}$$

We separate the statement on $T(\theta)$ into two claims:

1. First, we show that selections from $\arg \max_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}_t, \theta)$ are non-decreasing.
2. Then we show that if $t = +\infty \in T(\underline{\theta})$, then for any $\bar{\theta} > \underline{\theta}$, $T(\bar{\theta}) = \{+\infty\}$.
Formally, if $\sup_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}, \underline{\theta}) = A(\underline{\theta})$, then $\sup_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}, \bar{\theta}) = A(\bar{\theta})$ and $U(t, \tilde{\alpha}_t, \bar{\theta}) < A(\bar{\theta})$ for all $t \in \mathbb{R}_+$.

Claim 1: The key step is an argument by [Milgrom and Shannon \(1994\)](#) and [Edlin and Shannon \(1998\)](#). [Edlin and Shannon \(1998\)](#)'s Theorem 2 has additional conditions that are violated in our setting, but which are only necessary to derive their conclusions on strict comparative statics. For completeness, we reproduce here the part of the argument that suffices for our purposes:

Definition 10. For $U : \mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}] \times [0, 1] \rightarrow \mathbb{R}$, U satisfies the strict Spence-Mirrlees condition in $((t, \alpha), \theta)$ if if U is C^1 , $U_t/|U_\alpha|$ is strictly increasing in t , and $U_\alpha \neq 0$ and has a constant sign.

Lemma 5 (Adapted from Theorem 2 in [Edlin and Shannon \(1998\)](#)). Assume $U : \mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}] \times [0, 1]$ satisfies the strict Spence-Mirrlees condition and has path-connected indifference sets. Then every selection from $\arg \max_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}_t, \theta)$ is non-decreasing.

Proof. By Theorem 3 in [Milgrom and Shannon \(1994\)](#), U is strictly single crossing in $((t, \alpha); \theta)$, where $\mathbb{R}_+ \times [0, 1]$ is endowed with the lexicographic order. With that order on $\mathbb{R}_+ \times [\underline{\alpha}, \bar{\alpha}]$, U is quasisupermodular in (t, α) and the set $\{(t, \alpha) : \alpha = \hat{\alpha}_t\}$ is a sublattice of $\mathbb{R}_+ \times [0, 1]$. The result then follows

by Theorem 4' in [Milgrom and Shannon \(1994\)](#). \square

The Spence-Mirrlees condition follows from simple calculus: using $U_\alpha = -e^{-rt}\bar{S}_\alpha < 0$,

$$\frac{U_t}{|U_\alpha|} = -\frac{r(R(\theta) - \bar{S}(\alpha, \theta))}{\bar{S}_\alpha(\alpha, \theta)} = r\iota^S(\alpha, \theta). \quad (29)$$

so U satisfies the strict Spence-Mirrlees condition whenever $\iota^S(\cdot, \theta)$ is strictly increasing.

To show complete regularity of U , fix θ and $\underline{t} < \bar{t}$ and $\underline{\alpha}, \bar{\alpha}$ such that $U(\bar{t}, \bar{\alpha}, \theta) = U(\underline{t}, \underline{\alpha}, \theta) = \bar{u}$. We construct a continuous function $\tilde{\alpha} : [\underline{t}, \bar{t}] \rightarrow [0, 1]$ satisfying $\tilde{\alpha}(\underline{t}) = \underline{\alpha}$, $\tilde{\alpha}(\bar{t}) = \bar{\alpha}$, and $U(t, \tilde{\alpha}(t), \theta) = \bar{u}$.

If $\bar{u} = A(\theta)$, then $\bar{\alpha} = \underline{\alpha} = \alpha^f(\theta)$ and $U(t, \alpha^f(\theta), \theta)$ is constant in t ; setting a constant $\tilde{\alpha}(t) = \alpha^f(\theta)$ trivially suffices. Focus then on $\bar{u} > A(\theta)$; the proof for $\bar{u} < A(\theta)$ is symmetric. For all (t', α') such that $U(t', \alpha', \theta) = \bar{u}$, $R(\theta) - \bar{S}(\alpha', \theta) > 0$ and therefore $U_t(t', \alpha', \theta) = re^{-rt'}(R(\theta) - \bar{S}(\alpha', \theta)) > 0$. By the Implicit Function Theorem, since $U_\alpha < 0$, for any $t_0 \in [\underline{t}, \bar{t}]$, there exists some open neighborhood $\mathcal{O} \subset [\underline{t}, \bar{t}]$ with $t_0 \in \mathcal{O}$ and a $C^1(\mathcal{O})$ function $\tilde{\alpha} : \mathcal{O} \rightarrow [0, 1]$ satisfying

$$\tilde{\alpha}'(t) = -\frac{U_t(t, \tilde{\alpha}(t), \theta)}{U_\alpha(t, \tilde{\alpha}(t), \theta)}, t \in \mathcal{O}, \quad U(t_0, \tilde{\alpha}(t_0), \theta) = \bar{u},$$

i.e., $\tilde{\alpha}$ is in fact a local solution to an initial value problem.³⁰

We extend the solution to the IVP above to yield the desired function $\tilde{\alpha}$. Take the open domain $\mathcal{D} = (\underline{t}, \bar{t}) \times (\underline{\alpha}, \bar{\alpha})$. We only show how to extend $\tilde{\alpha}$ continuously rightward up to \bar{t} , since extending it leftward to \underline{t} is done symmetrically. Since U is C^1 , U_t is bounded above and below on \mathcal{D} , and $U_\alpha < 0$, $g(t, \alpha) := -U_t(t, \alpha, \theta)/U_\alpha(t, \alpha, \theta)$ is continuous and bounded on \mathcal{D} . By standard extension theorems (Lemma 2.14 in [Teschl \(2012\)](#) and Theorem 4.1 in [Coddington and Levinson \(1955\)](#)), either $\tilde{\alpha}$ can be extended rightwards inside \mathcal{D} to all of $[t_0, \bar{t})$, or there exists some $t' \in (t_0, \bar{t})$ such that $\tilde{\alpha}$ extends rightwards up to $[t, t')$ with $\tilde{\alpha}(t') = \bar{\alpha}$.³¹ If $\tilde{\alpha}$ can be extended rightwards to all of $[t_0, \bar{t})$, then by the continuity of U , $\tilde{\alpha}(\bar{t}-) = \bar{\alpha}$, and we are done.

³⁰Since $U_\alpha < 0$, one can solve for $\tilde{\alpha}(t_0)$ in $U(t_0, \tilde{\alpha}(t_0), \theta) = \bar{u}$.

³¹To be precise, $\tilde{\alpha}$ extends up to $[t, t')$ and $\tilde{\alpha}(t'-) = \bar{\alpha}$.

Suppose, then, that $\tilde{\alpha}$ cannot be extended rightwards to all of $[t_0, \bar{t})$, so there exists some $t' \leq \bar{t}$ with $\tilde{\alpha}(t') = \bar{\alpha}$ as above. If $t' = \bar{t}$, we are done, so focus on the remaining case $t' < \bar{t}$. Since $U(t', \bar{\alpha}, \theta) = \bar{u} = U(\bar{t}, \bar{\alpha}, \theta)$, by Rolle's theorem there exists some $t'' \in (t', \bar{t})$ such that $U_{t''}(\bar{\alpha}, \theta) = 0$. That would require $\bar{\alpha} = \alpha^f(\theta)$, a contradiction to $\bar{u} > A(\theta)$.

Claim 2: If $\sup_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}, \underline{\theta}) = A(\underline{\theta})$, so that $t = +\infty$ achieves that supremum, it must be that, for all $t \in \mathbb{R}_+$, $R(\underline{\theta}) \leq \bar{S}(\tilde{\alpha}_t, \underline{\theta})$. But then, using $\bar{S}_\alpha < 0$, it follows that, for all $t \in \mathbb{R}_+$,

$$\tilde{\alpha}_t \geq \alpha^f(\underline{\theta}) > \alpha^f(\bar{\theta}).$$

where the strict inequality was shown in (28). Therefore, using $\bar{S}_\alpha < 0$ and $R(\bar{\theta}) = \bar{S}(\alpha^f(\bar{\theta}), \theta)$, we have that, for all $t \in \mathbb{R}_+$,

$$R(\bar{\theta}) < \bar{S}(\tilde{\alpha}_t, \bar{\theta}) \Rightarrow U(t, \tilde{\alpha}_t, \bar{\theta}) < A(\bar{\theta}) \text{ and } \sup_{t \in \mathbb{R}_+} U(t, \tilde{\alpha}, \bar{\theta}) = A(\bar{\theta})$$

We conclude that $\arg \max_{t \in \mathbb{R}_+ \cup \{+\infty\}} U(t, \tilde{\alpha}_t, \bar{\theta}) = \{+\infty\}$, as required. \square

C Proofs for Section 5

Proof of Lemma 1.

Cash plus Equity: A minor calculation yields $\iota^{S^L}(\theta, \alpha) = -[R(\theta) - L]V(\theta)^{-1} + \alpha$. $\iota^{S^0}(\theta, \alpha)$ is increasing in θ for every α if and only iff b/A is strictly decreasing. Hence, by continuity, there exists some L' such that, for all $L \leq L'$, $\iota^{S^L}(\theta, \alpha)$ remains strictly increasing in θ for all α . Moreover, $\iota_{\theta L}^{S^L}(\theta, \alpha) = -V(\theta)^{-2}V'(\theta) < 0$; hence, if $\iota^{S^L}(\theta, \alpha)$ is increasing in θ for L , then it is increasing for any $L' < L$. Let $\alpha^f(1; L) = \frac{R(1)-L}{V(1)}$ denote the final offer that makes $\theta = 1$ just indifferent under S^L . Using $\alpha^f(1; 0) = \frac{R(1)}{V(1)}$, SL holds under S^0 (equity bargaining) iff $\mathbb{E}[\tilde{V}] \frac{R(1)}{V(1)} < c$, while SL holds under S^L iff

$$L \leq \frac{c \left[1 - \frac{R(1)}{c} \frac{\mathbb{E}[\tilde{V}]}{V(1)} \right]}{1 - \frac{\mathbb{E}[\tilde{V}]}{V(1)}}$$

Therefore, whenever SL holds under S^0 , there exists $L'' \in (0, c)$ such that SL holds under S^L for all $L \leq L''$, we can take $L^* = \min\{L'', L'\}$. Finally, if SL condition holds for L , it also holds for any $L' \leq L$. So, if $S_{liq}^L \in \mathcal{D}_{A, \tilde{V}, c}$, then $S_{liq}^{L'} \in \mathcal{D}_{A, \tilde{V}, c}$ for all $L' \leq L$

Levered Equity: Next, we consider the situation in which the acquirer means of payment is equity over the merged entity, and the acquirer has outstanding debt. The question then is how the acquirers leverage impact the negotiation. We can derive the offer in the smooth trading region $\alpha(\theta)$ and the final offer $\alpha^f(1)$, which are given by

$$\alpha(\theta) = \frac{c}{\mathbb{E}[(Z - d)^+ | \theta]}$$

$$\alpha^f(1) = \frac{R(1)}{\mathbb{E}[(Z - d)^+ | 1]}.$$

Using integration by parts we can write

$$\mathbb{E}[(Z - d)^+ | \theta] = \int_d^\infty (1 - G(x | \theta)) dx.$$

With some abuse of notation we drop the subscript in $g_V(z | \theta)$. Denoting the cumulative density function of V conditional on $\theta \in [k, 1]$ by $\bar{G}(z | k)$, which is given

$$\bar{G}(z | k) := \frac{1}{1 - k} \int_k^1 G(z | \theta) d\theta.$$

We can directly characterize the conditions in terms of the cumulative density function $G(z | \theta)$. The threshold k^{SL} is given by

$$k^{SL} = \inf \left\{ k \leq 1 : \frac{\int_d^\infty (1 - \bar{G}(z | k^{SL})) dz}{\int_d^\infty (1 - G(z | 1)) dz} \geq \frac{c}{R(1)} \right\}.$$

The environment is upward skimming if

$$-\frac{\int_d^\infty G_\theta(z | \theta) dz}{\int_d^\infty (1 - G(z | \theta)) dz} > \frac{R'(\theta)}{R(\theta)} \quad (30)$$

First, we verify that if the environment is upward skimming for d , it is also upward skimming for any $d' > d$. The left hand side of the upward skimming

condition (30) can be written as

$$-\frac{\int_d^\infty G_\theta(x|\theta)dx}{\int_d^\infty (1-G(x|\theta))dx} = \frac{\partial}{\partial v} \log \int_d^\infty (1-G(x|\theta))dx.$$

After changing the order of differentiation, we get

$$-\frac{\partial}{\partial d} \frac{\int_d^\infty G_\theta(x|\theta)dx}{\int_d^\infty (1-G(x|\theta))dx} = -\frac{\partial}{\partial v} \frac{1-G(d|\theta)}{\int_d^\infty (1-G(x|\theta))dx} > 0, \quad (31)$$

where the inequality follows as, for all $z > \theta$, the ratio $\frac{1-G(z|\theta)}{1-G(d|\theta)}$ is increasing in θ by the monotone likelihood ratio property.

Next, we need to verify that if the lemons condition is satisfied for d , then it is also satisfied for $d' > d$. Next, we verify that if the lemon's condition is satisfied for d , then it is also satisfied for $d > d'$. The lemons condition can be written as

$$\begin{aligned} \frac{\int_d^\infty (1-\bar{G}(z|0))dz}{\int_d^\infty (1-G(z|1))dz} &< \frac{c}{R(1)} && \Longleftrightarrow \\ 0 &< R(1) \int_d^\infty (1-\bar{G}(z|0))dz - c \int_d^\infty (1-G(z|1))dz. \end{aligned}$$

Letting

$$\psi(d) := R(1) \int_d^\infty (1-\bar{G}(z|0))dz - c \int_d^\infty (1-G(z|1))dz$$

we get

$$\psi'(d) = (1-G(d|1)) \left[R(1) \left(\frac{1-\bar{G}(d|0)}{1-G(d|1)} \right) - c \right]$$

Notice that

$$R(1) \left(\frac{1-\bar{G}(\underline{z}|0)}{1-G(\underline{z}|1)} \right) - c = R(1) - c > 0;$$

thus, it is enough to show that $\frac{1-\bar{G}(d|0)}{1-G(d|1)}$ is increasing in d . Differentiating with respect to d we get

$$\frac{\partial}{\partial d} \frac{1-\bar{G}(d|0)}{1-G(d|1)} = \frac{1-\bar{G}(d|0)}{1-G(d|1)} \left[\frac{g(d|1)}{1-G(d|1)} - \frac{\bar{g}(d|0)}{1-\bar{G}(d|0)} \right],$$

which is positive as $G(z|1)$ dominates $\bar{G}(z|0)$ in the hazard rate order.

□

Proof of Proposition 4. The offer in the smooth trading region and the final offer are $\alpha(\theta) = c/\mathbb{E}[(\tilde{V} - d)^+|\theta]$ and $\alpha^f(1) = \mathbb{E}[(\tilde{V} - d)^+|1]$, respectively. The expected value of equity is

$$\hat{V}^d(\theta; \eta) := \mathbb{E}[(\tilde{V} - d)^+|\theta] = \int_d^\infty \Phi(\eta(c + \Delta + \zeta\theta - z)) dz,$$

where Φ is the CDF of the standard normal distribution and $\zeta := \chi + \beta$. With this notation, $\bar{S}_{lev}^d(\alpha, \theta; \eta) = \alpha \hat{V}^d(\theta; \eta)$. Using the identity $\phi(x)x = -\phi'(x)$, we have

$$\hat{V}_\eta^d(\theta; \eta) = -\frac{1}{\eta} \int_d^\infty \phi'(\eta(c + \Delta + \zeta\theta - z)) dz = -\frac{1}{\eta^2} \phi(\eta(c + \Delta + \zeta\theta - d))$$

where the second equality uses a change of variables and $\lim_{x \rightarrow -\infty} \phi(x) = 0$. Therefore,

$$\hat{V}_{\eta\theta}^d = -\frac{\zeta}{\eta} \phi'(\eta(c + \Delta + \zeta\theta - d)).$$

The density ϕ is strictly single-peaked around 0, so for sufficiently large d , $\frac{\partial^2}{\partial\theta\partial\eta} \bar{S}_{lev}^d(\alpha, \theta) = \alpha \hat{V}_{\eta\theta}^d < 0$ for all θ , and the reverse holds for sufficiently small d . This proves the first statement.

To prove the second statement, we prove the stronger claim

$$\frac{\hat{V}_\theta^d}{\hat{V}^d} \text{ is strictly increasing in } \eta. \quad (\star)$$

which implies the result. Indeed, if the claim is true, then whenever $\bar{S}_{lev}^d(\alpha_1, \theta; \eta_1) = \bar{S}_{lev}^d(\alpha_2, \theta; \eta_2)$,

$$\frac{\hat{V}_\theta^d(\theta; \eta_1)}{\hat{V}^d(\theta; \eta_1)} \bar{S}_{lev}^d(\alpha_1, \theta; \eta_1) > \frac{\hat{V}_\theta^d(\theta; \eta_2)}{\hat{V}^d(\theta; \eta_2)} \bar{S}_{lev}^d(\alpha_1, \theta; \eta_1) = \frac{\hat{V}_\theta^d(\theta; \eta_2)}{\hat{V}^d(\theta; \eta_2)} \bar{S}_{lev}^d(\alpha_2, \theta; \eta_2)$$

The left hand side equals $\frac{\partial}{\partial\theta} \bar{S}_{lev}^d(\alpha_1, \theta; \eta_1)$ and the right hand side equals $\frac{\partial}{\partial\theta} \bar{S}_{lev}^d(\alpha_2, \theta; \eta_2)$, so (16) follows.

To prove (\star) , we sign

$$\hat{V}_{\theta\eta}^d \hat{V}^d - \hat{V}_\theta^d \hat{V}_\eta^d. \quad (32)$$

which is proportional to $\frac{\partial}{\partial\eta} \frac{\hat{V}_\theta^d}{\hat{V}^d}$ by the quotient rule. First, simplify notation by labeling $\Gamma := \eta(c + \Delta + \zeta\theta - d)$ and $\mu := (c + \Delta + \zeta\theta)$. Note that μ is the

mean of $\tilde{V}|\theta$. With this notation, $\hat{V}_\eta^d = -\phi(\Gamma)\eta^{-2}$, $\hat{V}_{\eta\theta}^d = \zeta\phi'(\Gamma)\eta^{-1}$. We can directly calculate that

$$\hat{V}_\theta^d(\theta; \eta) = \eta\zeta \int_d^\infty \phi(\eta(c + \Delta + \zeta\theta - z))dz = \zeta\Phi(\Gamma)$$

Next, using standard formulas for the moments of censored random variables and letting $h(\cdot) = \phi(\cdot)/(1 - \Phi(\cdot))$ denote the inverse Mills ratio, we calculate \hat{V}^d :

$$\begin{aligned} \hat{V}^d &= -d + \mathbb{E}[\max\{\tilde{V}, d\}|\theta] \\ &= -d + [1 - \Phi(\eta(d - \mu))] \left[\mu + \frac{1}{\eta}h(\eta(d - \mu)) \right] + \Phi(\eta(d - \mu))d \\ &= \frac{1}{\eta} [1 - \Phi(-\Gamma)] [\Gamma + h(-\Gamma)] \end{aligned} \quad (33)$$

Plugging the expressions for $\hat{V}_{\eta\theta}^d$, \hat{V}^d , \hat{V}_η^d and \hat{V}_θ^d into (32), we obtain a quantity proportional to

$$-\phi'(\Gamma)\Phi(\Gamma) [\Gamma + h(-\Gamma)] + \Phi(\Gamma)\phi(\Gamma) \propto -\phi'(\Gamma) [\Gamma + h(-\Gamma)] + \phi(\Gamma) \quad (34)$$

$$\propto \Gamma [\Gamma + h(-\Gamma)] + 1, \quad (35)$$

where the last line uses the identity $\phi'(x) = -x\phi(x)$. If $\Gamma \geq 0$, we are done. Otherwise, if $\Gamma < 0$, we apply the following classic bound on the inverse Mills ratio (see [Gordon \(1941\)](#)):

$$h(x) < x + \frac{1}{x}, x > 0$$

to obtain

$$\Gamma [\Gamma + h(-\Gamma)] + 1 > \Gamma \left[\Gamma + \left(-\Gamma + \frac{1}{-\Gamma} \right) \right] + 1 = 0,$$

which proves that (32) is strictly positive.

To show that upward-skimming is preserved as η rises, note that, similar to an environment with unlevered equity, the environment is upward-skimming if

$$\frac{\hat{V}_\theta^d(\theta; \eta)}{\hat{V}^d(\theta; \eta)} > \frac{R'(\theta)}{R(\theta)}.$$

By (\star) , it follows that if the environment is upward-skimming for η , it is upward-skimming for any $\eta' > \eta$.

The preservation of SL as η rises also follows from (\star) . Indeed, the SL holds if

$$R(1) \int_0^1 \frac{\hat{V}^d(\theta; \eta)}{\hat{V}^d(1; \eta)} d\theta < c \quad (36)$$

where we have used the fact that the final offer is $R(1)/\hat{V}^d(1; \eta)$. From (\star) one has

$$0 < \frac{\partial}{\partial \eta} \frac{\hat{V}_\theta^d(\theta; \eta)}{\hat{V}^d(\theta; \eta)} = \frac{\partial^2}{\partial \eta \partial \theta} \log \hat{V}^d(\theta; \eta) = \frac{\partial}{\partial \theta} \frac{\hat{V}_\eta^d(\theta; \eta)}{\hat{V}^d(\theta; \eta)}.$$

Hence, for all $\theta \in [0, 1]$,

$$\frac{\hat{V}_\eta^d(\theta; \eta)}{\hat{V}^d(\theta; \eta)} \leq \frac{\hat{V}_\eta^d(1; \eta)}{\hat{V}^d(1; \eta)} \Rightarrow \frac{\partial}{\partial \eta} \frac{\hat{V}^d(\theta; \eta)}{\hat{V}^d(1; \eta)} \leq 0.$$

The integrand in (36) therefore decreases as η increases, which concludes the proof.

Finally, the environment is upward-skimming if (omitting some arguments to reduce clutter)

$$\frac{\beta}{c + \Delta + \beta\theta} < \frac{\hat{V}_\theta^d(\theta)}{\hat{V}^d(\theta)} = \frac{\zeta\eta}{\eta(c + \Delta + \zeta\theta - d) + h(-\eta(c + \Delta + \zeta\theta - d))} \quad \forall \theta. \quad (37)$$

where we have substituted our previous expressions for \hat{V}_θ^d and \hat{V}^d . From Lemma 1, it suffices to satisfy (37) for $d = 0$, so we will have upward skimming if

$$\frac{\beta}{\beta + \chi} < \frac{\eta(c + \Delta + \beta\theta)}{\eta(c + \Delta + \zeta\theta) + h(-\eta(c + \Delta + \zeta\theta))}$$

Using the monotone hazard rate property of the normal distribution, a crude lower bound for the right hand side is

$$\frac{\eta(c + \Delta)}{\eta(c + \Delta) + \eta(\beta + \chi) + h(-\eta(c + \Delta))}.$$

Using, $h(0) = 2/\sqrt{2\pi}$, an even cruder bound is

$$\frac{\eta(c + \Delta)}{\eta(c + \Delta) + \eta(\beta + \chi) + 2/\sqrt{2\pi}}.$$

□

D Equilibrium Verification

Proof of Theorem 1: Equilibrium Verification. We present details for the case with non-trivial delayed trade, $S \in \mathcal{D}_{A, \tilde{V}, c}$; the remaining cases are similar, but much simpler.

Verification: Seller's On-Path Strategies: Now, we verify that the seller's choice of $\{K^k\}_{k \in [0, 1]}$ and F are optimal, given the buyer's strategy

$$\alpha(k) = \begin{cases} \alpha^f(1) & \text{if } k \in (k^{SL}, 1] \\ \bar{S}^{-1}(c, k) & \text{if } k \in [0, k^{SL}), \end{cases} \quad (38)$$

where the inverse $\bar{S}^{-1}(c, k)$ is defined by $\bar{S}(\bar{S}^{-1}(c, k), k) = c$. From the previous, given $\alpha(k)$ in equation (38), seller's the continuation value is

$$J(k) = \begin{cases} c & \text{if } k \in [0, k^{SL}] \\ \mathbb{E}[S(\alpha^f(1), \tilde{V}) | \theta \in [k, 1]] & \text{if } k \in (k^{SL}, 1] \end{cases} \quad (39)$$

Notice that $J(\cdot)$ has a kink at k^{SL} as

$$J'(k^{SL}-) = 0 < J'(k^{SL}+) = \frac{\partial}{\partial k} \mathbb{E}[S(\alpha^f(1), \tilde{V}) | \theta \in [k, 1]] \big|_{k=k^{SL}}$$

The value function fails to be differentiable at k^{SL} due to the discontinuity in $\alpha(\cdot)$. Moreover, this implies that the HJB equation is discontinuous at this point. To avoid the technical complications associated to working with discontinuous HJB equations, and the theory of viscosity solutions, we take advantage that admissible cutoff policies are non-decreasing, and we split the verification of the optimal policies in two steps: First starting at $k_0 \in (k^{SL}, 1]$, and then starting $k_0 \in [0, k^{SL}]$.

Verification for $k_0 \in [k^{SL}, 1]$: Let's ignore the fact that for $\alpha(K_t) = \alpha^f(1)$, all types accept the offer, and consider a relaxed formulation in which the seller is

allowed to smoothly screen on $k_0 \in [k^{SL}, 1]$. To simplify notation, we consider F which are absolutely continuous and let $\Lambda_t = \int_0^t \lambda_s ds$, where λ_s denotes the hazard rate of F at s . This is without loss of generality as the payoff from a distribution F with atoms can be approximated by a sequence of absolutely continuous functions (in other words, we can take a sequence of absolutely continuous distributions F^n that converges to F in the weak* topology). The seller's value function is

$$J(k_0) = \sup_{Q, \lambda} \int_0^\infty e^{-rt - \Lambda_t} \left(\mathbb{E} \left[\bar{S}(\alpha(Q_t), v) \mid \theta \in [Q_{t-}, Q_t] \right] \frac{dQ_t}{1 - k_0} + \lambda_t \mathbb{E} \left[\bar{S}(\alpha^f(1), v) \mid \theta \in [Q_{t-}, 1] \right] \right) + \left(1 - \int_0^\infty e^{-rt - \Lambda_t} \left(\frac{1 - Q_t}{1 - k_0} \lambda_t dt + \frac{dQ_t}{1 - k_0} \right) \right) c.$$

Rather than working with the value function $J(\cdot)$, it is convenient to work with the equivalent value function $\bar{J}(k) \equiv (1 - k)J(k)$, so

$$\bar{J}(k_0) = \sup_{Q, \lambda} \int_0^\infty e^{-rt - \Lambda_t} \left(\mathbb{E} \left[\bar{S}(\alpha(Q_t), v) \mid \theta \in [Q_{t-}, Q_t] \right] dQ_t + \lambda_t (1 - k_0) \mathbb{E} \left[\bar{S}(\alpha^f(1), v) \mid \theta \in [Q_{t-}, 1] \right] \right) + \left(1 - k_0 - \int_0^\infty e^{-rt - \Lambda_t} (\lambda_t (1 - Q_t) dt + dQ_t) \right) c.$$

We conjecture, and then verify, that the value function $\bar{J}(\cdot)$ satisfies the quasi-variational inequality

$$0 = \max \left\{ \sup_{\dot{k} \geq 0, \lambda \geq 0} (\bar{S}(\alpha(k), k) + \bar{J}'(k)) \dot{k} + \lambda \left(\int_k^1 \bar{S}(\alpha^f(1), v) d\theta - \bar{J}(k) \right) + r(1 - k)c - r\bar{J}(k), \mathcal{M}\bar{J}(k) - \bar{J}(k) \right\}, \quad (40)$$

where the operator \mathcal{M} is defined by

$$\mathcal{M}\bar{J}(k) = \max_{k' \in [k, 1]} \{ (k' - k) \mathbb{E}[\bar{S}(\alpha(k'), \theta) \mid \theta \in [k, k']] + \bar{J}(k') \}$$

First we verify that $\bar{J}(k) = (1 - k)J(k)$, where $J(k)$ is as in (39) satisfies this quasi-variational inequality. First, it is immediate to verify that $\bar{J}(k) = \mathcal{M}\bar{J}(k)$, so the second term of (40) is satisfied. For the first term, notice that

$$(\bar{S}(\alpha(k), k) + \bar{J}'(k))\dot{k} + \lambda \left(\int_k^1 \bar{S}(\alpha^f(1), v) d\theta - \bar{J}(k) \right) + r(1-k)c - r\bar{J}(k) \leq$$

$$(\bar{S}(\alpha(k), k) + \bar{J}'(k))\dot{k} = 0,$$

where we have used that $\bar{J}'(k) = -\bar{S}(\alpha^f(1), k)$. From here on, the verification is standard. Consider an arbitrary admissible policy Q_t . Using the change of value formula, we get that

$$e^{-rt-\Lambda t} \bar{J}(Q_t) = J(k_0) + \int_0^t e^{-rs-\Lambda s} \left(\dot{q}_s \bar{J}'(Q_{s-}) + \lambda_s \left(\int_k^1 \bar{S}(\alpha^f(1), v) d\theta - \bar{J}(Q_{s-}) \right) \right. \\ \left. - r\bar{J}(Q_{s-}) \right) ds + \sum_{s < t} e^{-rs-\Lambda s} (\bar{J}(Q_{s-} + \Delta Q_{s-}^d) - \bar{J}(Q_{s-}))$$

From the quasi-variational inequality (40) we get that

$$\bar{J}(Q_s) - \bar{J}(Q_{s-}) \leq (Q_s - Q_{s-}) \mathbb{E}[\bar{S}(\alpha(Q_s), \theta) | \theta \in [Q_{s-}, Q_s]]$$

and the term in the integral is less than

$$-r(1 - Q_s)c - \dot{q}_s \bar{S}(\alpha(Q_{s-}), Q_{s-}) - \lambda_s \int_k^1 \bar{S}(\alpha^f(1), v) d\theta.$$

It follows that

$$\bar{J}(k_0) \geq \int_0^t e^{-rs-\Lambda s} \left(r(1 - Q_{s-})c + \dot{q}_s \bar{S}(\alpha(Q_{s-}), Q_{s-}) \right. \\ \left. + \lambda_s \int_k^1 \bar{S}(\alpha^f(1), v) d\theta \right) ds \\ + \sum_{s < t} e^{-rs-\Lambda s} (Q_s^d - Q_{s-}^d) \mathbb{E}[\bar{S}(\alpha(Q_s), \theta) | \theta \in [Q_{s-}, Q_s]] + e^{-rt-\Lambda t} \bar{J}(Q_t) \\ = \int_0^t e^{-rt-\Lambda t} \mathbb{E} \left[S(\alpha(Q_s), \tilde{V}) \mid \theta \in [Q_{s-}, Q_s] \right] dQ_s \\ + \left(1 - k_0 - e^{-rt-\Lambda t} (1 - Q_t) - \int_0^t e^{-rs-\Lambda s} ((1 - Q_s)\lambda_s ds + dQ_s) \right) c \\ + e^{-rt-\Lambda t} (\bar{J}(Q_t) - (1 - Q_t)c),$$

where the equality

$$1 - k_0 - e^{-rt-\Lambda t} (1 - Q_t) - \int_0^t e^{-rs-\Lambda s} ((1 - Q_s)\lambda_s ds + dQ_s) = \int_0^t e^{-rs-\Lambda s} r(1 - Q_{s-}) ds,$$

follows by integration by parts. Taking the limit when $t \rightarrow \infty$, we get that

$\bar{J}(k_0)$ is an upper bound on the payoff that the seller can attain starting at any $k_0 \geq k^{SL}$. Finally, because all the inequalities hold with equality in the case of equation for our conjecture policy K , it follows that K is optimal starting at $k_0 \in [k^{SL}, 1]$.

Verification for $k_0 \in [0, k^{SL}]$. Using the previous characterization of the value function $\bar{J}(\cdot)$ on $[k^{SL}, 1]$, by the principle of dynamic programming, we can state the optimization problem on $[0, k^{SL}]$, as

$$\begin{aligned} \bar{J}(k_0) = & \sup_Q \int_0^{\tau(Q)} e^{-rt-\Lambda_t} \mathbb{E} \left[\bar{S}(\alpha(Q_t), v) \mid \theta \in [Q_{t-}, Q_t] \right] dQ_t \\ & + \left(1 - k_0 - \int_0^{\tau(Q)} e^{-rt-\Lambda_t} (\lambda_t(1 - Q_t)dt + dQ_t) \right) c + e^{-r\tau(Q)} (\bar{J}(Q_{\tau(Q)}) - (1 - Q_{\tau(Q)})c). \end{aligned}$$

where $\tau(Q) = \inf\{t > 0 : Q_t \geq k^{SL}\}$. Notice that the factor $(1 - Q_{\tau(Q)})c$ is added to account for the constant $(1 - k)c$ in the expected payoff. Once again, we conjecture that the value function $\bar{J}(\cdot)$ satisfies the quasi-variational inequality (40).

First, we can verify that $\bar{J}(\cdot)$ defined by (39) (multiplied by $1 - k$) satisfies equation (40) on $[0, k^{SL}]$. By construction, $\bar{S}(\alpha(k), k) = \bar{J}'(k) = -c$. Also,

$$\mathcal{M}\bar{J}(k) - \bar{J}(k) = \max_{k' \in [k, 1]} \left\{ (k' - k) \mathbb{E}[S(\alpha(k'), \tilde{V}) \mid \theta \in [k, k']] + \bar{J}(k') \right\} - (1 - k)c < 0,$$

so $\max_{\lambda \geq 0} \{ \lambda (\int_k^1 \bar{S}(\alpha^f(1), v) d\theta) - \bar{J}(k) \} = 0$. Thus, the first term of the variational inequality is equal to zero, and because $\mathcal{M}\bar{J}(k) - \bar{J}(k) \leq 0$, the second term also satisfies the required inequality. It follows then that $\bar{J}(k) = (1 - k)c$ is a solution of (40) on $[0, k^{SL}]$. Consider an arbitrary policy Q , so, once again, using the change of value formula we get that

$$\begin{aligned} \mathbb{E}^Q \left[e^{-rt \wedge \tau(Q)} \bar{J}(Q_{t \wedge \tau(Q)}) \right] = & J(k_0) + \int_0^{t \wedge \tau(Q)} e^{-rs} (\dot{q}_s \bar{J}'(Q_{s-}) + \lambda_s (\bar{J}(Q_{s-} + \Delta Q_{s-}^s) \\ & - \bar{J}(Q_{s-})) - r \bar{J}(Q_{s-})) ds + \sum_{s < t \wedge \tau(Q)} e^{-rs} (\bar{J}(Q_{s-} + \Delta Q_{s-}^d) - \bar{J}(Q_{s-})) \end{aligned}$$

Following the same steps that we did before, we get

$$\begin{aligned}
\bar{J}(k_0) &\geq \int_0^{t \wedge \tau(Q)} e^{-rs - \Lambda_s} \left(r(1 - Q_{s-})c + \dot{q}_s \bar{S}(\alpha(Q_{s-}), Q_{s-}) \right. \\
&\quad \left. + \lambda_s (Q_s^s - Q_{s-}^s) \mathbb{E}[\bar{S}(\alpha(Q_s), \theta) | \theta \in [Q_{s-}, Q_s]] \right) ds \\
&\quad \sum_{s < t \wedge \tau(Q)} e^{-rs - \Lambda_s} (Q_s^d - Q_{s-}^d) \mathbb{E}[\bar{S}(\alpha(Q_s), \theta) | \theta \in [Q_{s-}, Q_s]] + e^{-rt \wedge \tau(Q) - \Lambda_{t \wedge \tau(Q)}} \bar{J}(Q_{t \wedge \tau(Q)}) \\
&= \int_0^{t \wedge \tau(Q)} e^{-rs - \Lambda_s} \mathbb{E} \left[\bar{S}(\alpha(Q_s), v) \mid \theta \in [Q_{s-}, Q_s] \right] dQ_s + (1 - k_0)c \\
&\quad - \int_0^{t \wedge \tau(Q)} e^{-rs - \Lambda_s} ((1 - Q_s)\lambda_s ds + dQ_s)c + e^{-rt \wedge \tau(Q) - \Lambda_{t \wedge \tau(Q)}} (\bar{J}(Q_{t \wedge \tau(Q)}) - (1 - Q_{t \wedge \tau(Q)})c).
\end{aligned}$$

Taking the limit as $t \rightarrow \infty$ we get that $t \wedge \tau(Q) \rightarrow \tau(Q)$. It follows that $\bar{J}(k_0)$ is an upper bound on the seller's expected payoff. Finally, because in the case of the policy K all the inequalities hold with equality, we get that the value of the policy K is given by $\bar{J}(k_0)$, so K is optimal on $[0, k^{SL})$.

Verification: Seller's Off-Path Strategy: Finally, we characterize the off-equilibrium seller's offer $\sigma(\cdot | k', \alpha')$, where $\sigma(\cdot | k', \alpha')$ has to maximize

$$\begin{aligned}
&\int_0^1 \left\{ (\alpha^{-1}(\tilde{\alpha}) - k')^+ \mathbb{E} \left[S(\tilde{\alpha}, \tilde{V}) \mid \theta \in [k', \alpha^{-1}(\tilde{\alpha}) \wedge k'] \right] \right. \\
&\quad \left. + (1 - \alpha^{-1}(\tilde{\alpha})) J(\alpha^{-1}(\tilde{\alpha})) \right\} d\sigma(\tilde{\alpha} | k', \alpha').
\end{aligned}$$

We consider an off-equilibrium offer with two mass points, given by

$$\sigma(\alpha | k', \alpha') = \begin{cases} \alpha(k') & \text{w.p. } p(k', \alpha') \\ \alpha^f(1) & \text{w.p. } 1 - p(k', \alpha'), \end{cases}$$

If $k' < k^{SL}$, then, conditional on rejection of α' , the cut-off is $\alpha^{-1}(\alpha') = k^{SL}$. In this case, $\bar{S}(\alpha(k^{SL}), k^{SL}) = \mathbb{E} \left[S(\alpha^f(1), \tilde{V}) | \theta \in [k^{SL}, 1] \right] = c = J(k^{SL})$, and this payoff is higher than any other serious offer. Thus, any probability $p(k', \alpha') \in [0, 1]$ is optimal, and in particular $p(k', \alpha')$ solving

$$\bar{S}(\alpha', k^{SL}) = p(k^{SL}, \alpha') \bar{S}(\alpha^f(1), k^{SL}) + (1 - p(k^{SL}, \alpha')) \bar{S}(\alpha(k^{SL}), k^{SL}).$$

If $k > k^{SL}$, then the optimal offer is $p(k', \alpha') = 1$, as any other offer that is accepted with positive probability yields $\mathbb{E} \left[S(\alpha^f(1), \tilde{V}) | \theta \in [k', k] \right] < \mathbb{E} \left[S(\alpha^f(1), \tilde{V}) | \theta \in [k', 1] \right] = J(k')$.

Verification: Buyer's On-Path Strategy: The proof use a direct mechanism representation of the continuation play together with the characterization in Lemma 4. We cannot apply Lemma 4 directly because the characterization only applies to a deterministic path of cut-offs, and the path cut-off is stochastic in our equilibrium construction (it jumps to $K_T = 1$ at time T). The first step then is to establish that, given the seller strategy, the buyer acceptance strategy is incentive compatible only if it incentive compatible for a deterministic path with the same delay for the pooling offer $\alpha^f(1)$. Let $\tau(k) = \inf\{t > 0 : K_t \geq k\}$, let $\alpha(k) \equiv \alpha(K_{\tau(k)})$, and $y(k) = 1 - \mathbb{E}[e^{-r\tau(k)}]$. Notice that, given the seller strategy K we have that $\alpha(k)$ is a deterministic function of k , so the only random variable is $\tau(k)$. Thus, we can write the buyer's problem as

$$\begin{aligned} B(\theta, k) &= \max_{k' \in [k, 1]} \mathbb{E}^{K^k} [(1 - e^{-r\tau(k')})A(\theta) + e^{-r\tau(k')} (V(\theta) - \bar{S}(\alpha(K_{\tau(k')}), \theta))] \\ &= \max_{k' \in [k, 1]} y(k')A(\theta) + (1 - y(k')) (V(\theta) - \bar{S}(\alpha(k'), \theta)) \\ &= \max_{k' \in [k, 1]} U(y(k'), \alpha(k'), \theta). \end{aligned}$$

It follows that it is without loss of generality to consider the incentive compatibility condition for a deterministic mechanism inducing the same $y(k)$ as K^k . By the arguments in Lemma 4, we know, for increasing $\nu^S(\alpha, \theta)$, $U(y, \alpha, \theta)$ satisfies strict single crossing differences in $((y, \alpha), \theta)$, where (y, α) is ordered lexicographically. Hence, for any $y \mapsto \tilde{\alpha}(y)$, $U(y, \tilde{\alpha}(y), \theta)$ has strict single-crossing differences in (y, θ) .

We have shown that $y(\theta)$ is non-decreasing. If we prove that $U(y, \tilde{\alpha}(y), \theta)$ satisfies smooth single crossing differences, taking $\tilde{\alpha}(y)$ to be the candidate equilibrium mapping between choice of (1 minus) expected delay and equilibrium offer, and if the following envelope condition is satisfied

$$U(y(\theta), \alpha(\theta), \theta) = U(y(0), \alpha(0), 0) + \int_0^\theta U_\theta(y(s), \alpha(s), s) ds, \quad (41)$$

then by Theorem 4.2 in [Milgrom \(2004\)](#), the buyer acceptance strategy $\alpha(\theta)$ will incentive compatible. To check smooth single-crossing differences, take (y, θ) such that $\frac{d}{dy}U(y, \tilde{\alpha}(y), \theta) = 0$. Taking the derivative, we have

$$\bar{S}_\alpha(\tilde{\alpha}(y), \theta) [\iota^S(\tilde{\alpha}(y), \theta) - \tilde{\alpha}'(y)] = 0. \quad (42)$$

By assumption, $\tilde{S}_\alpha > 0$, so if the above display is 0, $\iota^S(\tilde{\alpha}(y), \theta) = \tilde{\alpha}'(y)$. Then whenever the derivative exists,

$$\frac{\partial}{\partial v} \frac{d}{dy} U(y, \tilde{\alpha}(y), \theta) = \bar{S}_\alpha(\tilde{\alpha}(y), \theta) \left[\frac{\partial}{\partial v} \iota^S(\tilde{\alpha}(y), \theta) \right] > 0,$$

since the environment is upward skimming.

Now we show the relevant envelope condition. By definition, we have that for any θ and any (y, α)

$$U_\theta(y, \alpha, \theta) = yX'(\theta) + (1 - y) (V'(\theta) - \bar{S}_\theta(\alpha, \theta))$$

For any $\theta \in [0, k^{SL}]$ we have

$$\begin{aligned} U(y(\theta), \alpha(\theta), \theta) &= U(y(0), \alpha(0), 0) + \int_0^\theta (U_\theta(y(s), \alpha(s), s) \\ &\quad + U_y(y(s), \alpha(s), s)y'(s) + U_\alpha(y(s), \alpha(s), s)\alpha'(s)) ds, \end{aligned}$$

where

$$\begin{aligned} U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s) &= (A(s) - V(s) + \bar{S}(\alpha(s), s)) y'(s) - (1 - y(s))\bar{S}_\alpha(\alpha(s), s)\alpha'(s) \\ &= -(R(s) - \bar{S}(\alpha(s), s)) y'(s) - (1 - y(s))\bar{S}_\alpha(\alpha(s), s)\alpha'(s). \end{aligned}$$

From the local IC constraint we have that

$$r (R(K_t) - \bar{S}(\alpha(K_t), K_t)) = -\dot{K}_t \alpha'(K_t) \bar{S}_\alpha(\alpha(K_t), K_t).$$

By definition, on $[0, k^{SL})$, $y'(k) = re^{-r\tau(k)}\tau'(k)$ and $\alpha'(k) = \alpha'(K_{\tau(k)})\dot{K}_{\tau(k)}\tau'(k)$. Hence, multiplying both sides of the local incentive compatibility constraint by $e^{-r\tau(k)}\tau'(k)$, and using the definition $K_{\tau(k)} = k$, we get

$$(R(k) - \bar{S}(\alpha(k), k)) y'(k) = -(1 - y(k))\alpha'(k)\bar{S}_\alpha(\alpha(k), k),$$

so $U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s)$, and we obtain equation (41). Next, we verify the envelope representation (41) for $k \in (k^{SL}, 1]$. Because $\alpha(k)$ and $y(k)$ are constant on $(k^{SL}, 1]$ and $U_y(\cdot)y'(s) + U_\alpha(\cdot)\alpha'(s) = 0$ on $\theta \in [0, k^{SL}]$ we have that

$$\begin{aligned}
U(y(\theta), \alpha(\theta), \theta) &= U(y(0), \alpha(0), 0) + \int_0^\theta U_\theta(y(s), \alpha(s), s) ds \\
&\quad + U(y(k^{SL}+), \alpha(k^{SL}+), k^{SL}) - U(y(k^{SL}), \alpha(k^{SL}), k^{SL}).
\end{aligned}$$

By construction, the delay D in equation (10) is such

$$U(y(k^{SL}+), \alpha(k^{SL}+), k^{SL}) = U(y(k^{SL}), \alpha(k^{SL}), k^{SL}),$$

so the expected payoff $U(y(\theta), \alpha(\theta), \theta)$ satisfies the envelope condition (41).

Verification: Buyer's Off-Path Strategy: The only step left is to verify the optimality of the reservation price strategy $\alpha(k)$ following an off-equilibrium offer $\alpha' \notin \alpha([0, 1])$. By construction, the $\sigma(\alpha|k', \alpha')$ is such the type k^{SL} buyer is indifferent between accepting α' and reject it. Thus, we only need to verify that types above k^{SL} are better off rejecting it. By construction

$$\bar{S}(\alpha', k^{SL}) = p(k^{SL}, \alpha') \bar{S}(\alpha^f(1), k^{SL}) + (1 - p(k^{SL}, \alpha')) \bar{S}(\alpha(k^{SL}), k^{SL}).$$

Let $p' \equiv p(k^{SL}, \alpha')$, because $\bar{S}(\alpha', \theta)$ is increasing in θ , we have that

$$\begin{aligned}
V(\theta) - \bar{S}(\alpha', \theta) &< V(\theta) - \bar{S}(\alpha', k^{SL}) \\
&= p' (V(\theta) - \bar{S}(\alpha^f(1), k^{SL})) + (1 - p') (V(\theta) - \bar{S}(\alpha(k^{SL}), k^{SL})) \\
&< p' (V(\theta) - \bar{S}(\alpha^f(1), k^{SL})) + (1 - p') B(\theta, k^{SL}),
\end{aligned}$$

which means that types $\theta > k^{SL}$ are strictly better off rejecting α' . A similar calculation shows that types $\theta < k^{SL}$ are strictly better off accepting α' .

□