Debt Maturity Management

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Abstract

This paper studies how a borrower issues long- and short-term debt in response to shocks to the enterprise value. We develop a theory of debt maturity that highlights the tradeoff between commitment and risk management. Short-term debt protects creditors from future dilution; long-term debt allows the borrower to share losses with creditors in a downturn. Borrowers far from default value risk management and use a combination of long- and short-term debt. By contrast, distressed borrowers exclusively issue short-term debt. Our model generates novel cross-sectional and time-series implications on the adjustment of debt maturity following different types of shocks.

Keywords: capital structure; debt maturity; risk management; dynamic tradeoff theory.

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1 Introduction

The optimal management of debt obligations is a central problem faced by indebted entities, including households, firms, and sovereign governments. In practice, debt can differ in a number of aspects, and an important one is its maturity. Borrowing can be short as in the case of repurchase agreement and trade credit, or long as in the case of 30-year corporate bonds. How do borrowers choose the maturity profile of their outstanding debt? How do they adjust the mix between long- and short-term borrowing, following shocks to their enterprise value?

Yet, the academic literature falls behind in providing a useful framework to study these questions, despite the obvious importance. For example, the Leland model (Leland, 1994) and the vast follow-up literature typically assume that (1) all debt has the same (expected) maturity and (2) the borrower either commits to the total leverage or may only increase leverage after retiring all existing debt and paying some exogenous issuance cost.[1] Although these assumptions simplify the analysis, they are not consistent with the ample empirical evidence that borrowers often issue a mix of long- and short-term debt, and the adjustment of debt maturity structure can be slow and take time to be accomplished.

In this paper, we introduce a simple and tractable framework to study these questions. Our theory highlights the tradeoff between commitment and risk management in borrowing long and short. Long-term debt has a staggered structure: it is not due soon, and the borrower cannot commit to not issuing more debt before the existing debt is due. Due to this lack of commitment, creditors of long-term debt are exposed to the risk of being diluted and therefore charge higher spreads. By contrast, short-term debt does not suffer from dilution, because it matures before the borrower gets the chance to borrow again. In other words, existing short-term debt must be retired before new debt is issued. However, long-term debt has an important benefit of risk management: if a downturn arrives, the borrower and long-term creditors – but not short-term creditors – share losses in the enterprise value. This risk-sharing property of long-term debt offers hedging benefits to the borrower.

More specifically, a risk-neutral borrower has assets in place, which generate an income flow that follows a geometric Brownian motion (GBM) process. The expected growth rate of the income is high in an upturn but low in a downturn. A transition from the upturn to the downturn can be interpreted as the downside risk. Creditors are competitive, risk-neutral, and have a lower cost of capital than the borrower.[2] The difference in the cost of capital offers a reason for the borrower to

[1] Notable exceptions include He and Milbradt (2016) and DeMarzo and He (2021), which we discuss in the subsection on related literature.

[2] We also explore the model under tax shields, which naturally favor long-term debt issuance due to the differential tax treatments to coupon and principal payments.

issue debt, but too much debt could trigger default. Two types of debt are available: short-term debt matures instantaneously (i.e., has zero maturity) and needs to be simultaneously rolled over, and long-term debt matures exponentially with a constant amortization rate. The key innovation of our model is to allow the borrower to have full flexibility in issuing either type of debt at any time to adjust the maturity profile of the outstanding debt. This feature distinguishes us from the existing literature.

The flexibility to issue more debt exposes long-term creditors to dilution, due to the leverage-ratchet effect (Fama and Miller, 1972; Black and Scholes, 1973; Admati et al., 2018). Specifically, the borrower always has incentives to borrow more after existing long-term debt has been issued, because the additional borrowing dilutes the existing long-term claims. Notice these incentives exist even though the asset has no recovery value in default, because the additional debt pushes the borrower closer to default and reduces the price of long-term debt. In equilibrium, creditors anticipate the future dilution, and the price of long-term debt adjusts downwards to the level at which the borrower cannot capture any benefit. In other words, the borrower does not benefit from borrowing long even though creditors have a lower cost of capital.

By contrast, the instantaneous and simultaneous feature of short-term debt protects it from being diluted and resolves the commitment problem. Given that all the short-term debt needs to be rolled over on a continuing basis, existing short-term debt must be retired before any new debt is issued. In other words, short-term creditors' debt matures before the borrower can issue again, and therefore it does not suffer from dilution. As a result, short-term creditors need to be compensated only by the probability of default within a short period of time, but not by the cost of being diluted in the future.

This distinction between long- and short-term debt echoes the leasing solution to the Coase conjecture on the durable-goods monopoly (Coase, 1972). Specifically, the borrower can be thought of as the monopolistic issuer of her debt, and long-term debt can be thought of as the durable goods. Similar to the Coase (1972) conjecture that a monopoly without commitment to future prices does not benefit from the monopoly power, DeMarzo and He (2021) show the borrower does not benefit from issuing long-term debt. Coase (1972) proposes leasing as a solution to the commitment problem, which Bulow (1982) later formalizes. Informally, leasing fulfills commitment because all the goods need to be repriced in each period. In our context, short-term debt, which needs to be continuously rolled over, serves a role similar to leasing.

The advantage of short-term debt in resolving the commitment problem offers the borrower a natural reason to issue it. Indeed, our results show long-term debt is never issued in the downturn, when no additional downside risk exists. Instead, the borrower fully levers up by borrowing short, which is riskless and does not suffer from dilution. In the upturn, the potential arrival of the downturn renders short-term debt no longer riskless. The borrower chooses between issuing a lower amount of riskless short-term debt and a higher amount of risky short-term debt, with the latter leading to immediate default once the downturn arrives. We show the borrower issues risky (riskless) short-term debt if she is very close (far) to default. The reason is that when the borrower is close to default, both the amount of riskless short-term borrowing and the cost of default – the option value of continuing to operate in the downturn – become very low. In this case, risky short-term debt offers more benefits. When immediate default is anticipated if the downturn arrives, long-term debt is not only exposed to the same potential default following downturn, but is also exposed to dilution. Hence, it is more expensive and the borrower only issues short-term debt.

Results are different in the upturn if the borrower is far from default. Here, the borrower issues both long- and short-term debt. In this case, creditors do not anticipate an immediate default if the downturn arrives, but the enterprise value still gets reduced following the downturn. Although the enterprise value immediately drops upon the downturn hits, the borrower still needs to fully repay her short-term debt; otherwise, she must default. As a result, the reduced enterprise value without immediate default is only shared between the borrower and long-term creditors. This risk-sharing property is reflected in the drop in long-term debt's price, which highlights an important role of long-term debt in risk management: it allows the borrower to effectively make state-contingent payments. The state-contingent payments act as a cushion to reduce the borrower's burden in the downturn and mitigate the borrower's incentives to default. Short-term debt is a harder claim than long-term debt: the borrower must make non-state-contingent payments; otherwise, she has to default. Note the borrower values the merit of long-term debt in sharing the downside risk, even though she is risk-neutral. The reason is that the cost in default introduces constraints in financing, which makes the borrower behave as if she were risk-averse.

Even though long-term debt shares the downside risk, we show the enterprise value becomes higher if the borrower is prohibited from borrowing long. The reason is that due to the lack of commitment, long-term debt does not benefit the borrower but instead introduces more defaults. In other words, even though long-term debt offers hedging benefits against the downside risk, the borrower is unable to capture any hedging benefits in equilibrium, due to her lack of commitment. Meanwhile, if the borrower is prohibited from borrowing short, the commitment problem becomes more severe: the enterprise value becomes lower, as does the price of long-term debt. In other words, the ability to borrow short also increases the value of long-term debt. This result implies long- and short-term debt can be complements.

Our model implies firms far from default, more levered, and endowed with fewer growth options tend to use short-term debt, consistent with the empirical findings in Barclay and Smith Jr (1995). Moreover, the debt maturity structure is pro-cyclical, consistent with findings in Xu (2018) and

Chen et al. (2021). We study the borrower's impulse response to different negative shocks to cash flow and find the responses differ by the nature of the shocks. Following frequent and small negative shocks to operating cash flows, the borrower immediately reduces the issuance of short-term debt. The reduction in long-term debt issuance is slowly adjusted over time. By contrast, following infrequent and large negative shocks, the adjustment of both types of debt is slow. Whereas the issuance of long-term debt is reduced over time, the issuance of short-term debt is actually increased. These patterns are in line with anecdotal examples. Moreover, this theoretical prediction on heterogeneous impulse responses to different negative shocks can be useful hypotheses for future empirical tests on capital structure and debt maturity.

For simplicity, our benchmark model has assumed zero recovery value if the borrower defaults and no debt restructuring occurs. In robustness analysis, we show the main results continue to hold under positive recovery value and if debt restructuring is allowed. Moreover, our main mechanism does not depend on the modeling approach of the downside risk. The results continues to hold if the downside risk is modeled as a downward jump in the realized cash flows as opposed to a regime-shift shock. In practice, financial managers report managing interest-rate risk as a main factor behind the choices of debt maturity. We further modify the model by introducing fluctuations to the interest rates required by short- and long-term creditors. Results show the motivations for managing interest-rate risk leads to the firm issuing additional long-term debt during downturn but buying back long-term debt during upturn.

Related literature

Our paper builds on the literature of dynamic corporate finance, pioneered by Leland (1994). Most of this literature either fixes book leverage (Leland, 1998) or allows for adjustment with some issuance costs (Goldstein et al., 2001; Dangl and Zechner, 2020; Benzoni et al., 2019). Important exceptions are DeMarzo and He (2021) and Abel (2018). Whereas the former studies leverage dynamics when the borrower has full flexibility in issuing exponentially-maturing debt, the latter addresses the same problem when the borrower can only issue zero-maturity debt (see also Bolton et al. (2021), who further model costly equity issuance). In these papers, the borrower can only issue one type of debt, so the tradeoff between borrowing long and short is not explicitly studied. Malenko and Tsoy (2020) model the role of firm reputation under one type of debt, where maturity is chosen and fixed at the initial date. Their tradeoff involves the differential tax treatments on principal and coupon payments, which differs from us. He and Milbradt (2016) also study the problem of dynamic debt maturity management, where the total leverage is fixed and the borrower can choose between two types of exponentially-maturing debt. Our paper differs in two aspects. First, we allow for flexibility in adjusting total leverage. Second, we model short-term debt as debt that matures simultaneously. The different approaches in modeling short-term debt render the mechanisms of the two papers drastically different: whereas we emphasize the tradeoff between commitment and risk-sharing, their paper focuses on rollover losses and dilution. Brunnermeier and Yogo (2009) also study debt maturity in the context of liquidity risk, and they show long-term debt is optimal if the firm is close to default (or close to debt restructuring as in their paper). Our results are the opposite: the borrower will issue exclusively short if she is close to default. The difference is driven by the assumption that in our model, the borrower can issue debt at any time without commitment. By contrast, the borrower in Brunnermeier and Yogo (2009) can only issue new debt after current debt is repaid and effectively has commitment.

More broadly, our paper is related to the literature in corporate finance on debt maturity, starting from Flannery (1986) and Diamond (1991). This literature emphasizes the role of asymmetric information and the signaling role of short-term debt. One advantage of a fully-dynamic setup is that it allows us to make empirical predictions regarding the stock (existing debt) and the flow (new issuance) of debt maturity. The insight that short-term debt resolves the lack of commitment is also present in another related literature (Calomiris and Kahn, 1991; Diamond and Rajan, 2001) that emphasizes the runable feature of short-term debt. In our paper, the reason that short-term debt resolves commitment is fundamentally different: the short rate would increase drastically if the borrower issued more debt.[3] This feature resembles the leasing solution (Bulow, 1982) to the durable-goods monopoly problem.[4] We show that short-term debt has the shortcoming of limited risk-sharing. Relatedly, Gertner and Scharfstein (1991) show that, conditional on financial distress, short-term debt has a higher market value and increases leverage, leading to more ex-post debt overhang (also see Diamond and He (2014)).

The insight that long-term debt can be diluted has been recognized by Fama and Miller (1972) and Black and Scholes (1973), and has been more recently formalized by Admati et al. (2018). Brunnermeier and Oehmke (2013) show equity and short-term debt can dilute long-term debt's recovery value in bankruptcy. Our paper, by assuming zero recovery value in the benchmark model, rules out this mechanism. Instead, we focus on dilution outside the bankruptcy, which comes exclusively from the borrower's lack of commitment on issuance and default.

The mechanism whereby long-term debt allows for more state-contingent payments when markets are incomplete is also present in the literature on fiscal policy and sovereign debt. For example, Angeletos (2002) shows how the Arrow-Debreu allocation can be implemented with noncontingent debt of different maturities. Aguiar et al. (2019) show that in the absence of hedging motives, the

[3] Also see Hu and Varas (2021) on this feature of short-term debt in the context of repo and shadow banking.

[4] As emphasized by DeMarzo (2019), the problem of a borrower without commitment to future debt issuance has many similarities to one on a monopolistic producer selling durable goods without commitment.

borrower never actively issues any long-term debt, due to the lack of commitment. By contrast, we show the motives for risk-sharing lead the borrowers to issue a combination of long- and short-term debt (also see Niepelt (2014)). Bigio et al. (2021) study debt maturity management under liquidity cost but without dilution. In their model, the borrower's choice depends on the demand curve for bonds, microfounded via search (Duffie et al., 2005). The mechanisms of the two papers are therefore complementary. Arellano and Ramanarayanan (2012) consider maturity choice in a quantitative model of sovereign default with tradeoffs similar to ours. Our paper has two important differences. First, we fully characterize the optimal policy in debt maturity management. This characterization allows us to study the nature of the shocks that the borrower wants to hedge using long-term debt. Second, the borrower in our model is risk-neutral and therefore does not have a reason *a priori* to value the merit of risk-sharing by long-term debt. The cost of default makes the borrower behave as if she is risk-averse, as emphasized by the corporate finance literature on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010; Panageas, 2010). To the best of our knowledge, no previous work establishes the link between maturity management and risk management in a corporate setting.

2 The Model

2.1 Agents and the Asset

Time is continuous and goes to infinity: $t \in [0, \infty)$. We study a borrower, often interpreted as a firm for the remainder of the paper. The relevant parties include the borrower as an equity holder and competitive creditors. Throughout the paper, we assume all agents are risk neutral, deep-pocketed, and protected by limited liability. Moreover, the borrower discounts the future at a rate ρ, which exceeds r, the discount rate of creditors.

The borrower's asset generates cash flows at a rate X_t, where X_t follows the regime-switching diffusion:

$$\frac{dX_t}{X_t} = \mu_{\theta_t} dt + \sigma dB_t, \tag{1}$$

where B_t is a standard Brownian motion, and $\theta_t \in \{H, L\}$ follows a two-state Markov chain, independent of B_t, with transition intensity λ_{LH} and λ_{HL}, respectively. The drift μ_{θ_t} differs across the two states with $\mu_L < \mu_H$, so that the high state H is associated with a higher growth rate in the borrower's expected cash flow. Below, we refer to the high state as the *upturn* and the low state as the *downturn*.

2.2 Debt Maturity Structure

The difference between the discount rates $\rho - r$ offers benefits for the borrower to issue debt.[5] Throughout the paper, we allow the borrower to issue two types of debt, short and long, to adjust the outstanding debt maturity structure. In particular, we do not restrict the borrower to commit to a particular issuance path, but instead let the issuance decisions be made at each instant.

All *short-term debt* matures instantaneously and simultaneously and therefore needs to be rolled over continuously. We model short-term debt as one with zero maturity. Let $D_{t-} = \lim_{dt\downarrow 0} D_{t-dt}$ be the amount of short-term debt outstanding (and due) at time t and let y_{t-} be the associated short rate. By contrast, *long-term debt* matures in a staggered manner. We follow the literature and model long-term debt as exponentially maturing bonds with coupon rate r and a constant amortization rate $\xi > 0$. Therefore, $1/\xi$ can be interpreted as the expected maturity. Let F_t be the aggregate face value of long-term debt outstanding at time t and let p_t be the price per unit of the face value.

The borrower may default, in which case the bankruptcy is triggered. To isolate issues related to debt seniority and direct dilution in bankruptcy, we assume the bankruptcy cost is 100%. In other words, creditors cannot recover any value once the borrower defaults. Subsection 4.2 studies the model under positive recovery.

2.3 Valuation

Let τ_b be the endogenous time at which the borrower defaults. For $t < \tau_b$, the price of the long-term debt per unit of face value satisfies

$$p_t = \mathbb{E}_t \left[\int_t^{\tau_\xi \wedge \tau_b} e^{-r(s-t)} r ds + e^{-r(\tau_b - t)} \mathbb{1}_{\{\tau_b > \tau_\xi\}} \right], \tag{2}$$

where the two components in the expression correspond to the coupon and final payments. The short rate y_{t-} depends on the borrower's equilibrium default decisions:

$$y_{t-} = r + \lim_{dt\downarrow 0} \frac{\Pr\left(\tau_b \leq t | \tau_b \geq t - dt\right)}{dt}, \tag{3}$$

where the second term on the right-hand side is the hazard rate of default. Clearly, $y_{t-} = r$ whenever the short-term debt is default free. On the other hand, if default is predicted to happen

[5] The difference can be related to differences in liquidity, contracting costs, or market segmentation. An alternative setup is to introduce tax shields, and the results are similar with some nuanced differences driven by differential tax treatments on coupon vs. principal payments. In both setup, we take the debt contract as given and acknowledge that it can be the optimal solution under certain agency frictions. Subsection 4.2 studies debt restructuring.

at time t, creditors will refuse to roll over short-term debt at $t-$, and equivalently, $y_{t-} \to \infty$. In general, y_{t-} compensates the creditors for the probability of default occurring between $t - dt$ and t. For example, if the borrower is anticipated to default following a transition from state H to state L, $y_{t-} = r + \lambda_{HL}$.

Over a short time interval $[t, t + dt)$, the net cash flow to the borrower is

$$\Big[X_t - (r + \xi)\, F_t - y_{t-} D_{t-}\Big] dt + p_t dG_t + dD_t, \tag{4}$$

where $(r + \xi)\, F_t$ is the interest and principal payments to long-term creditors, and $y_{t-} D_{t-}$ is the interest payments to short-term creditors. The remaining two terms $p_t dG_t$ and dD_t are the proceeds from issuing long- and short-term debt.[6] Note the notations dG_t and dD_t allow for both atomistic and flow issuance, and the price of long-term debt p_t could also depend on the issuance amount dG_t.

Define V_t as the continuation value of the borrower, which we sometimes refer to as the equity value at time t. The borrower chooses the endogenous time of default as well as the issuance of two types of debt to maximize the equity value, taking the price of long-term debt and the short-rate function as given. Once again, let us emphasize that all these decisions, default and issuance, are made without commitment:

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \geq t\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-\rho(s-t)} \Big((X_s - (r + \xi) F_s - y_{s-} D_{s-})\, ds + p_s dG_s + dD_s \Big) \right]. \tag{5}$$

2.4 Smooth Equilibrium

The heuristic timing within a short time horizon $[t, t + dt)$ is as follows:

1. The borrower arrives at time t with outstanding debt $\{D_{t-}, F_t\}$.

2. The exogenous state θ_t is realized, and the borrower decides whether to repay or default on the outstanding debt.

    - If she defaults, the game ends and nobody receives anything.
    - If she does not default, she repays $y_{t-} D_{t-}$ and $(r + \xi)\, F_t$ to short- and long-term creditors.

3. In the case of no default, the borrower receives cash flow $X_t dt$. Moreover, she borrows long-term debt dG_t and issues a net amount of dD_t short-term debt.

[6] One can think of dD_t as the net issuance of short-term debt. Specifically, $dD_t = D_t - D_{t-}$ if a jump occurs at t.

We focus on the Markov perfect equilibrium (MPE) in which the payoff-relevant state variables include the exogenous state θ_t, the cash-flow level X_t, and the amount of outstanding debt $\{D_{t-}, F_t\}$. The equilibrium requires the following: (1) creditors break even; that is, p_t follows equation (2) and y_{t-} follows equation (3); and (2) the borrower chooses optimal default and issuance (i.e., equation (5)), subject to the limited liability constraint $V_t \geq 0$. Finally, an MPE is *smooth* if no jump occurs in long-term debt issuance, in which case we write $dG_t = g_t F_t dt$. In a smooth equilibrium, the aggregate face value of long-term debt evolves according to

$$dF_t = (g_t - \xi) F_t dt. \tag{6}$$

Let us define $J_t = V_t + D_{t-}$ as the joint continuation value of the borrower and short-term creditors if default does not happen at time t. The following result motivates us to work with J_t for the remainder of this paper.

Lemma 1. *The equity value equals* $V_t = V_{\theta_t}(X_t, F_t, D_{t-}) = \max\{J_t - D_{t-}, 0\}$*, where* $J_t = J_{\theta_t}(X_t, F_t)$ *is given by*

$$J_\theta(X,F) = \sup_{\tau_b, \{G_s, D_s : D_s \leq J_\theta(X_s,F_s)\}} \mathbb{E}\Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda_{\theta\theta'})(s-t)}\Bigg(\Big(X_s - (r+\xi)F_s + p_s dG_s$$
$$+ (\rho + \lambda_{\theta\theta'} - y_{s-})D_{s-} + \lambda_{\theta\theta'} \max\{J_{\theta'}(X_s,F_s) - D_{s-}, 0\}\Big)ds\Bigg)\Bigg| X_t = X,\ F_t = F,\ \theta_t = \theta\Bigg]. \tag{7}$$

Note the max operator in $V_t = \max\{J_t - D_{t-}, 0\}$ follows, because if the outstanding short-term debt D_{t-} exceeds J_t, the maximized joint continuation value without default, the borrower chooses to default at time t and renege on the payments D_{t-}. Indeed, such default might happen upon the exogenous state θ_t changing, captured by the term $\lambda_{\theta\theta'} \max\{J_{\theta'}(X_s, F_s) - D_{s-}, 0\}$ in (7). Using (3), we can write the two terms in the second line of (7) as $(\rho - r)D_{s-} + \mathbb{1}_{J_{\theta'}(X_s,F_s) \geq D_{s-}} \cdot \lambda_{\theta\theta'} J_{\theta'}(X_s, F_s)$, where $(\rho - r)D_{s-}$ captures the gains from borrowing short-term debt, and $\lambda_{\theta\theta'} J_{\theta'}(X_s, F_s)$ is the continuation value upon state transition, which only accrues if $J_{\theta'}(X_s, F_s) \geq D_{s-}$, so that the borrower chooses not to default.

Lemma 1 suppresses the problem's dependence on D_{t-}. A smooth MPE is therefore characterized by functions $J_\theta(X,F)$, $p_\theta(X,F)$, $y_\theta(X,F)$, $D_\theta(X,F)$, and $g_\theta(X,F)$, where

$$J_\theta(X,F) = XJ_\theta\left(1, \frac{F}{X}\right) = Xj_\theta(f), \quad D_\theta(X,F) = XD_\theta\left(1, \frac{F}{X}\right) = Xd_\theta(f)$$

are homogeneous of degree one, and the rest are homogeneous of degree zero.[7] Let $f = \frac{F}{X}$ be the long-term debt to cash-flow ratio, which is the endogenous state variable of the model. The results below show a higher f is also associated with a shorter distance to default. It follows from Itô's lemma that f_t evolves according to

$$\frac{df_t}{f_t} = \left(g_{\theta_t}(f_t) - \xi - \mu_{\theta_t} + \sigma^2\right) dt - \sigma dB_t. \tag{8}$$

Lemma 1 has interesting economic insights. In particular, it implies issuance and default decisions are made to maximize the joint valuation of the borrower and short-term creditors, while ignoring the payoff to existing long-term creditors.[8] This result relates to Aguiar et al. (2019) in the context of sovereign debt, where the equilibrium issuance decisions can be characterized by the solution to a planner's problem that ignores payoff to existing long-term creditors.

2.5 Modeling Discussion

Risk and binary state. The borrower faces two sources of risks. The Brownian motion captures continuous fluctuations in day-to-day operating cash flows, which are meant to be small and frequent. On the other hand, a transition across the two states affects the expected growth in cash flow and captures large and infrequent shocks. For the rest of the paper, we label the transition from the upturn H to downturn L as *a downside risk*, which can be interpreted as shocks occurring at either the industry or the macroeconomy level. Given that we focus on the perspective of downside risk-sharing, results in the upturn H should be interpreted more broadly. Note one special feature of the binary state is that, once in the upturn, the expected growth rate μ_θ can only fall. We show in Internet Appendix B.5 that the potential of an upward jump in the state θ (i.e., μ_H can jump to a further higher level) does not change the qualitative results.

Debt maturity. Our modeling choice of short- and long-term debt is motivated by the discrete-time microfoundation. There, short-term debt would last for one period and therefore mature simultaneously. In the continuous-time setup, this feature is captured by zero-maturity debt that needs to be continuously rolled over. In the discrete-time setup, long-term debt would last for multiple periods, and the flexibility in issuing it each period would lead to the staggered structure. This feature is well captured by exponentially-maturing debt in the continuous-time setup.

[7] For any D_{t-} different from $D_\theta(X, F)$, the borrower would immediately default if $D_{t-} > J_\theta(X, F)$. If $D_{t-} \leq J_\theta(X, F)$, the borrower would immediately adjust to $D_\theta(X, F)$.

[8] The payoff to new long-term creditors is at dt order in the smooth equilibrium.

Zero recovery in default. The assumption that creditors do not recover any value once the borrower defaults is made for simplicity and does not affect our mechanism. It implies debt seniority becomes irrelevant, ruling out the theoretical channel highlighted in Brunnermeier and Oehmke (2013) whereby the equity holder dilutes existing creditors' recovery value in bankruptcy through issuing new debt. In subsection 4.2, we assume instead that the borrower can restructure her debt after defaulting, in which case creditors still recover some positive value.

Parametric assumptions. To make the problem non-trivial, we impose the following parametric restrictions.

Assumption 1. *The parameters of the model satisfy*

(a) Downturn is absorbing: $\lambda_{LH} = 0$ *and* $\lambda_{HL} = \lambda$.

(b) Unlevered values are finite without Brownian shocks: $r + \lambda > \mu_H$ *and* $r > \mu_L$.

Assumption 1.(a) says that once in downturn, the exogenous state will never return to the upturn. Thus, the low state $\theta = L$ is absorbing. By contrast, the state switches from high to low with a Poisson intensity λ. This assumption enables us to obtain a tractable solution. Yet, as shown in subsection 4.1, the assumption is innocuous to the main results of the paper. Assumption 1.(b) is a standard one in the literature to guarantee the valuation remains finite. Specifically, it requires in both states that the creditor's effective discount rate is above the expected growth rate of the cash flow.

3 Equilibrium

Subsection 3.1 derives the value function and the issuance of short-term debt in both states. Subsection 3.2 focuses on the issuance policy of long-term debt and explains the tradeoff behind borrowing long versus short. In subsection 3.3, we compare the equilibrium with one in which only one type of debt (either long or short) is allowed. Results there highlight the different roles of two types of debt. Finally, we study the debt-issuance policies by an unlevered borrower in subsection 3.4.

3.1 Value Function and Short-Term Debt Issuance

Low state $\theta_t = L$. Under Assumption 1.(a), the exogenous state θ_t will no longer change once it enters the downturn. Therefore, the remaining risk comes exclusively from the Brownian shock, and default can be anticipated by short-term creditors. As a result, short-term debt is riskless and demands a short rate $y_L(X,F) \equiv r$. By considering the change in the value function in (7) over a small interval, we can derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\underbrace{\rho J_L(X,F)}_{\text{required return}} = \max_{D_L \in [0, J_L(X,F)],\, g_L} \underbrace{X - (r+\xi) F}_{\text{cash flow net long payments}} + \underbrace{(\rho - r) D_L}_{\text{gains from borrowing short}} + \underbrace{p_L(X,F) g_L F}_{\text{proceeds from issuing long}}$$
$$+ \underbrace{\frac{\partial J_L(X,F)}{\partial F}(g_L - \xi) F}_{\text{evolution of } dF} + \underbrace{\frac{\partial J_L(X,F)}{\partial X} X\mu_L + \frac{1}{2}\frac{\partial^2 J_L(X,F)}{\partial X^2} X^2 \sigma^2}_{\text{evolution of } dX}. \quad (9)$$

The net benefits of issuing long-term debt become clear once we examine all the terms that involve g_L on the right-hand side. Whereas $p_L(X,F)$ captures the marginal proceeds from issuing an additional unit of long-term debt, $\frac{\partial J_L(X,F)}{\partial F}$ is the drop in the borrower's continuation value. If the borrower finds it optimal to adjust long-term debt smoothly, the marginal proceeds must be fully offset by the drop in continuation value, so that the borrower is indifferent; that is,

$$p_L(X,F) + \frac{\partial J_L(X,F)}{\partial F} = 0. \quad (10)$$

Under (10), the value function $J_L(X,F)$ can be solved as if $g_L \equiv 0$, which is the case in which the borrower will never issue any further long-term debt. We defer the characterization of long-term debt issuance until the next subsection. For now, let us plug (10) into (9) and use the fact that

$$\frac{\partial J_L(X,F)}{\partial F} = j_L'(f), \quad \frac{\partial J_L(X,F)}{\partial X} = j_L(f) - f j_L'(f), \quad X\frac{\partial^2 J_L(X,F)}{\partial X^2} = f^2 j_L''(f)$$

to get the following HJB for the scaled value function $j_L(f)$:

$$(\rho - \mu_L) j_L(f) = \max_{d_L \in [0, j_L(f)]} 1 - (r+\xi) f + (\rho - r) d_L - (\xi + \mu_L) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f). \quad (11)$$

We turn to the issuance of short-term debt, whose net benefits are captured by the term $(\rho - r) d_L$ in (11) (or $(\rho - r) D_L$ in (9)). Intuitively, the creditor is more patient than the borrower, so that financing the enterprise by borrowing d_L in short brings a flow benefit of $(\rho - r) d_L$. Given $\rho > r$, it is always optimal for the borrower to lever up using as much short-term debt as possible, which leads to $d_L = j_L(f)$ (or equivalently $D_L = J_L(X,F)$) so that the limited liability constraint becomes binding.

The rest of the problem is standard. We look for a solution to (11) on $f \in [0, f_L^b]$, where the endogenous default boundary f_L^b satisfies the value-matching condition $j_L(f_L^b) = 0$ and smooth-pasting condition $j_L'(f_L^b) = 0$. Proposition 1 describes the equilibrium outcome.

Proposition 1 (Equilibrium when $\theta_t = L$)**.** *In the unique equilibrium, the value function is*

$$j_L(f) = \underbrace{\frac{1}{r-\mu_L}}_{\textit{no default value}} - f + \underbrace{\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^{\gamma}}_{\textit{default option value}}, \tag{12}$$

where

$$\gamma \equiv \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2} > 1.$$

The default boundary is $f_L^b = \frac{\gamma}{\gamma-1}\frac{1}{r-\mu_L}$. For $\forall f \in [0, f_L^b)$, the borrower issues short-term debt $d_L(f) = j_L(f)$ and pays a short rate $y_L(f) = r$.

High state $\theta_t = H$. The smooth equilibrium leads to an indifference condition in long-term debt issuance that relates to (10):

$$p_H(X,F) + \frac{\partial J_H(X,F)}{\partial F} = 0.$$

Besides the Brownian shock, an additional downside risk exists whereby the state may transit from high to low. If default does not occur immediately upon the downturn arrival, the borrower and short-term creditors receive a maximum value of $j_L(f)$, among which d_H must be repaid to short-term creditors. Clearly, the borrower will default immediately upon the state transition if and only if $d_H > j_L(f)$. Expecting so, short-term creditors demand a short rate

$$y_H(f, d_H) = \begin{cases} r & \text{if } d_H \le j_L(f) \\ r + \lambda & \text{if } d_H > j_L(f). \end{cases} \tag{13}$$

Following a similar analysis to the one in the low state, we arrive at the HJB for the scaled value function $j_H(f)$:

$$\begin{aligned}(\rho + \lambda - \mu_H)\, j_H(f) = \max_{0 \le d_H \le j_H(f)} \; & 1 - (r+\xi) f + (\rho - r) d_H + \mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f) \\ & - (\xi + \mu_H) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f). \end{aligned} \tag{14}$$

Compared with (11), there are two differences. First, the borrower's effective discount rate becomes $\rho + \lambda$, due to the possibility of state transition. Second, the term $\mathbb{1}_{\{d_H \le j_L(f)\}} \cdot \lambda j_L(f)$ captures the scenario in which with intensity λ, the downside risk is realized, upon which the borrower and short-term creditors receive a continuation payoff $j_L(f)$ if and only if $d_H \le j_L(f)$. Otherwise, the borrower defaults and both receive nothing. Note the flow benefit of short-term debt is still

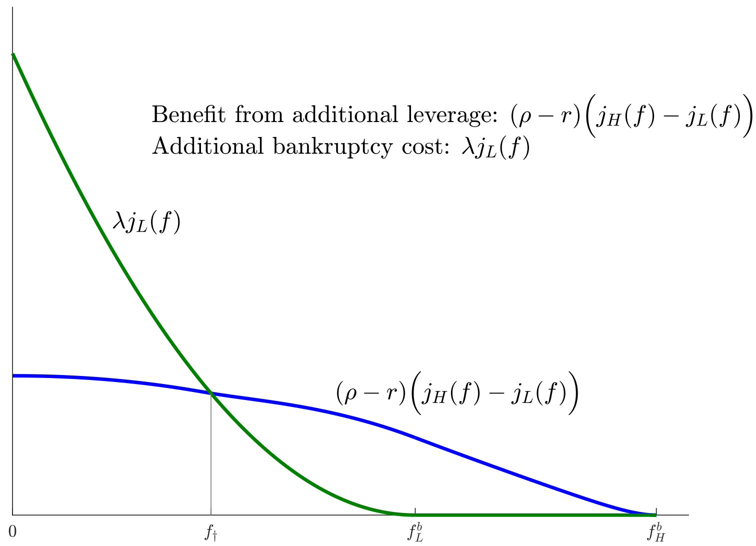


Figure 1: Cost and benefit of riskless short-term debt

captured by $(\rho - r)\, d_H$ and in particular does not include the credit risk premium λ, even in the case in which short-term debt is risky (i.e., $d_H > j_L(f)$). The reason is that the risk premium serves as a transfer between the borrower and short-term creditors and therefore does not enter the joint continuation value.

The optimal issuance of short-term debt is straightforward: the borrower borrows either $j_L(f)$ at rate r or $j_H(f)$ at $r + \lambda$. Equation (14) can therefore be written as

$$(\rho + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f - (\mu_H + \xi)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) + \max\Big\{(\rho - r)j_L(f) + \lambda j_L(f), (\rho - r)j_H(f)\Big\}, \quad (15)$$

where the term $\max\Big\{(\rho - r)j_L(f) + \lambda j_L(f), (\rho - r)j_H(f)\Big\}$ captures the tradeoff between borrowing riskless $j_L(f)$ and risky $j_H(f)$ short-term debt. Figure 1 illustrates the cost and benefit of riskless short-term debt. If the equity holder borrows riskless short-term debt, the flow benefit is lower $[(\rho - r)j_L(f) < (\rho - r)j_H(f)]$. However, because no immediate default occurs after the state transition, the borrower avoids the expected bankruptcy cost $\lambda j_L(f)$.

If $(\rho - r)j_L(0) + \lambda j_L(0) > (\rho - r)j_H(0)$, a borrower without any outstanding long-term debt will borrow riskless short-term debt so that $d_H(0) = j_L(0)$. Meanwhile, the maximum amount of riskless short-term borrowing decreases as f increases and eventually reduces to zero as f approaches f_L^b. Therefore, the borrower chooses risky short-term borrowing $d_H(f) = j_H(f)$ for f sufficiently high. We show in Lemma 3 of the appendix that a unique threshold $f_\dagger \in \left(0, f_L^b\right)$ exists such that $(\rho+\lambda-r)\, j_L(f) \leq (\rho-r)\, j_H(f)$ if and only if $f \geq f_\dagger$. The reason is that the additional bankruptcy cost $\lambda j_L(f)$ associated with risky short-term borrowing is high when f is low but becomes very low when f is high, whereas the difference between the amount of risky and riskless short-term borrowing declines much slower as f grows (Figure 1 offers a graphical illustration). Given the threshold $f_\dagger$, the HJB becomes a second-order ordinary differential equation (ODE) on both $(0, f_\dagger)$ and $\left(f_\dagger, f_H^b\right)$. The solutions and the two free boundaries $\left\{f_\dagger, f_H^b\right\}$ are pinned down by six boundary conditions: (1) value-matching and smooth-pasting at $\left\{f_\dagger, f_H^b\right\}$, (2) the transversality condition at $f=0$, and (3) the indifference between issuing risky and riskless short-term debt at $f_\dagger$. The detailed expressions are available in (35) to (40) in the appendix. Proposition 2 describes the equilibrium, where the constant ϕ and the expressions for $\left\{h_0\left(f, f_\dagger, f_H^b\right), h_1\left(f, f_\dagger, f_H^b\right)\right\}$ are provided in the appendix.

Proposition 2 (Equilibrium when $\theta_t = H$)**.** *Let*

$$\bar{\lambda} \equiv \sqrt{\left(\frac{\rho-\mu_H}{2}\right)^2 + (\rho-r)(\mu_H-\mu_L)} - \left(\frac{\rho-\mu_H}{2}\right).$$

In the high state, the unique smooth equilibrium follows:

1. *If $\lambda \leq \bar{\lambda}$, the value function is*

$$j_H(f) = \underbrace{\frac{1}{r+\lambda-\mu_H}}_{\textit{no default value}} - f + \underbrace{\frac{f_H^b}{\beta}\left(\frac{f}{f_H^b}\right)^{\beta}}_{\textit{default option value}}, \tag{16}$$

   where

$$\beta = \frac{\mu_H+\xi+\frac{1}{2}\sigma^2+\sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2+2\sigma^2\left(r+\lambda-\mu_H\right)}}{\sigma^2} > 1.$$

   The default boundary is $f_H^b = \frac{\beta}{\beta-1}\frac{1}{r+\lambda-\mu_H}$. For $\forall f \in [0, f_H^b)$, the borrower issues short-term debt $d_H(f) = j_H(f)$ and pays a short rate $y_H(f) = r+\lambda$.

2. *If $\lambda > \bar{\lambda}$, the value function is*

$$j_H(f) = \begin{cases} u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^{\phi} & f \in [0, f_\dagger] \\ u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right) h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b) h_1\left(f, f_\dagger, f_H^b\right) & f \in \left(f_\dagger, f_H^b\right], \end{cases} \tag{17}$$

where

$$u_0(f) \equiv \frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho-r}{r-\mu_L}+\frac{\lambda}{r-\mu_L}\right)-f$$
$$+\frac{(\rho+\lambda-r)}{(\rho+\lambda-r)+(\mu_H-\mu_L)(\gamma-1)}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^{\gamma} \tag{18}$$
$$u_1(f) \equiv \frac{1}{r+\lambda-\mu_H}-\frac{r+\xi}{r+\xi+\lambda}f. \tag{19}$$

The threshold for long-term debt issuance $f_\dagger$ and the default boundary f_H^b are determined by conditions (41) *and* (42) *in the appendix. For $\forall f \in [0, f_H^b)$, the borrower issues short-term debt*

$$d_H(f) = \begin{cases} j_L(f) & \text{if } f \le f_\dagger \\ j_H(f) & \text{if } f > f_\dagger \end{cases}$$

and pays a short rate given by equation (13)*.*

The first case in Proposition 2 is isomorphic to the results in Proposition 1 on the equilibrium in state L. Intuitively, if λ is low so that the downside risk is small, even a borrower without any outstanding long-term debt will borrow risky short-term debt ($(\rho-r)j_L(0)+\lambda j_L(0) > (\rho-r)j_H(0)$). In this case, the borrower always issues short-term debt $j_H(f)$, which is expected to default upon the state transition and therefore commands a short rate $r+\lambda$.

Results are more interesting for $\lambda > \bar{\lambda}$. In fact, the expressions in (17) have clear and intuitive interpretations. To see this, let us define $\tau_\dagger$ and τ_H^b as the first hitting time of $f_\dagger$ and f_H^b. We supplement the details to (17) as follows:

$$\left(\frac{f}{f_\dagger}\right)^{\phi} = \mathbb{E}\left[e^{-(\rho+\lambda)\tau_\dagger}\left(\frac{X_{\tau_\dagger}}{X_0}\right)\Big| f_0 = f\right]$$
$$h_0\left(f, f_\dagger, f_H^b\right) = \mathbb{E}\left[e^{-(r+\lambda)\tau_\dagger}\left(\frac{X_{\tau_\dagger}}{X_0}\right)\mathbb{1}_{\{\tau_\dagger<\tau_b\}}\Big| f_0 = f\right]$$
$$h_1\left(f, f_\dagger, f_H^b\right) = \mathbb{E}\left[e^{-(r+\lambda)\tau_b}\left(\frac{X_{\tau_b}}{X_0}\right)\mathbb{1}_{\{\tau_\dagger>\tau_b\}}\Big| f_0 = f\right].$$

The first expression is the present value of investing \$1 at $t=0$ into a hypothetical claim that pays out the unlevered return of the asset the first time f_t reaches $f_\dagger$. However, if the downturn arrives before $\tau_\dagger$, this claim pays out nothing. The function h_0 has a similar interpretation for $f_0 > f_\dagger$, but now the claim only pays if the firm has not defaulted. The last function h_1 is the present value of a claim that pays the unlevered return on assets at the time of default if default occurs before f

reaches $f_\dagger$.[9]

For $f \in [0, f_\dagger)$, $u_0(f)$ in (17) captures the joint value of the equity holder and short-term creditors if the issuance of short-term debt follows $d_H(f) = j_L(f)$. In particular, the second term $\big(j_H(f_\dagger) - u_0(f_\dagger)\big)\left(\frac{f}{f_\dagger}\right)^{\phi}$ is the option value of f reaching $f_\dagger$, in which case the borrower exercises the option of changing the short-term debt issuance to $d_H(f) = j_H(f)$. For the first term $u_0(f)$, we can rewrite it as the sum of the payoff without state transition and the one following the transition:

$$u_0(f) = \mathbb{E}\left[\int_0^\infty e^{-(\rho+\lambda)t}\Big(\underbrace{1-(r+\xi)f_t+(\rho-r)d_H(f_t)}_{\text{no state transition}} + \underbrace{\lambda j_L(f_t)}_{\text{state transition}}\Big)\left(\frac{X_t}{X_0}\right)dt\Big|f_0=f\right].$$

Note $d_H(f_t) = j_L(f_t)$ holds for $f \le f_\dagger$, so that the terms in the first line of (18) capture the discounted value of the cash flow and the benefits from borrowing short-term debt net the payments to long-term creditors. The term in the second line of (18) represents the value of defaulting after the downturn arrives and the value of borrowing against this default option prior to the state transition. Note these valuations are calculated as if $g_\theta(f) = 0$, that is, as if the borrower would not issue any further long-term debt, due to the indifference condition in long-term debt issuance.

The value function for $f > f_\dagger$ can be interpreted in a similar vein. Specifically, in (17), $u_1(f)$ is the value if the issuance of short-term debt follows $d_H(f) = j_H(f)$, whereas the remaining two terms $\big(j_H(f_\dagger) - u_1(f_\dagger)\big)h_0\left(f, f_\dagger, f_H^b\right)$ and $u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right)$ are the option value of switching short-term debt issuance to $d_H(f) = j_L(f)$ and default, respectively. Similarly, we can rewrite $u_1(f)$ as

$$u_1(f) = \mathbb{E}_0\left[\int_0^\infty e^{-(r+\lambda)t}\Big(1-(r+\xi)f_t\Big)\left(\frac{X_t}{X_0}\right)dt\Big|f_0=f\right],$$

where the discount rate is r because the borrower is fully levered and creditors become the effective owners of the asset. The two terms in (19) correspond to the value of cash flow and the expected payments to long-term creditors, with the latter being discounted by $\frac{r+\xi}{r+\xi+\lambda}$, because the borrower immediately defaults upon the downturn arrival.

[9]The factor (X_t/X_0) accounts for the adjustment σ^2 in the drift of f_t in (8) due to the stochastic growth of X_t. Alternatively, we can omit the factor (X_t/X_0) and compute the expectations under a new measure $\mathbf{Q}$ such that

$$df_t = -(\mu_H+\xi)f_t dt - \sigma f_t d\bar{B}_t,$$

and $d\bar{B}_t = dB_t - \sigma dt$ is a Brownian motion under $\mathbf{Q}$, adjusting the discount factors to $e^{-(\rho+\lambda-\mu_H)t}$ and $e^{-(r+\lambda-\mu_H)t}$. The change of measure adjusts for the stochastic growth of X_t, and the change of discount factors adjusts for the expected growth of X_t.

3.2 The Issuance of Long-term Debt

We have shown that in the smooth equilibrium, the borrower is indifferent between issuing long-term debt or not. However, the result doesn't imply she won't borrow long on the equilibrium path. In this subsection, we solve for the issuance policy of long-term debt.

Let us start with the downturn $\theta_t = L$, where equation (10) (or equivalently, $p_L(f) = -j_L'(f)$) is the necessary condition for the borrower to be indifferent between issuing long-term debt or not. Meanwhile, the price satisfies the following HJB equation:

$$(r+\xi)\, p_L(f) = \underbrace{r+\xi}_{\text{coupon and principal}} + \underbrace{\left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)}_{\text{expected change in bond price}}. \tag{20}$$

To derive the issuance function g_L, we plug $d_L = j_L(f)$ into (11), differentiate the resulting equation once, and add (20) on both sides. Turning to the upturn $\theta_t = H$, the equity holder's indifference in long-term debt issuance becomes $p_H(f) = -j_H'(f)$, and $p_H(f)$ satisfies the following HJB equation:

$$(r+\xi+\lambda)\, p_H(f) = r+\xi + \mathbb{1}_{\{f \le f_\dagger\}} \cdot \lambda p_L(f) + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f). \tag{21}$$

Compared with (20), (21) includes the additional event of state transition, upon which the price drops to $p_L(f)$ if $f \le f_\dagger$; otherwise, the borrower defaults and the price drops to zero. The derivation of the issuance policy $g_H(f)$ follows the same steps as the one in the low state.

Proposition 3 (Long-term debt issuance). *The equilibrium price and issuance of long-term debt follow:*

- *Downturn $\theta = L$: for $\forall f \in [0, f_L^b)$, the price of long-term debt is $p_L(f) = -j_L'(f)$, and the firm does not issue long-term debt, $g_L(f) = 0$.*
- *Upturn $\theta = H$: for $\forall f \in [0, f_H^b)$, the price of long-term debt is $p_H(f) = -j_H'(f)$. The long-term debt issuance policy is as follows: let $\lambda \le \bar{\lambda}$ be the threshold in Proposition 2.*
  - *If $\lambda \le \bar{\lambda}$, the borrower never issues long-term debt, $g_H(f) = 0\ \forall f \in [0, f_H^b)$;*
  - *If $\lambda > \bar{\lambda}$, the long-term debt issuance policy is*

$$g_H(f) = \begin{cases} \frac{(\rho - r)\left(p_H(f) - p_L(f)\right)}{-f p_H'(f)} & f \le f_\dagger \\ 0 & f > f_\dagger. \end{cases} \tag{22}$$

Proposition 3 shows that in the low state, the equity holder never issues any long-term debt, but, instead, borrows the maximum amount of short-term debt. Similar results hold in the high state if $\lambda \leq \bar{\lambda}$, so that the downside risk is relatively small. By contrast, long-term debt is issued in the high state if the downside risk is prominent and the amount of short-term borrowing is riskless, that is, $\lambda > \bar{\lambda}$ and $f \leq f_{\dagger}$. Why might the equity holder borrow long in the high state but not in low? Why would the equity holder borrow long in the high state only if the amount of short-term borrowing is riskless? What are the differential roles of short- and long-term debt?

Due to the leverage-ratchet effect, the equity holder faces a time-inconsistency problem when borrowing long: she is unable to commit to a path of issuance, but, instead, always has incentives to issue more and dilute legacy long-term creditors. As (10) shows, this lack of commitment implies the borrower is unable to capture any benefits from borrowing long, even though the creditors are more patient. By contrast, short-term debt, in particular its combined nature of instantaneous maturity and simultaneity, resolves the commitment problem, because all outstanding debt must be rolled over continuously; that is, the existing short-term debt must be retired before issuing any new debt. Given that short-term debt is riskless and cheap ($y_L = r$) in the low state, the equity holder only borrows short.[10] Similar results hold in the high state when short-term debt is risky, that is, when $\lambda \leq \bar{\lambda}$ or $\lambda > \bar{\lambda}$ but $f > f_{\dagger}$. Given that the borrower is expected to default upon the state transition, short-term creditors demand a short rate $r + \lambda$. Meanwhile, long-term debt is subject not only to the same downside risk of state transition but also the dilution effect. Therefore, the equity holder again only borrows short.

As DeMarzo (2019) shows, the borrower's problem when she only issues long-term debt is related to the Coase conjecture on the durable-goods monopoly (Coase, 1972). Specifically, the borrower can be thought of as the monopolistic issuer of her debt, and long-term debt can be thought of as the durable goods. Short-term debt in our context echoes the leasing solution to the Coase conjecture, which was originally proposed by Coase (1972) and later formalized by Bulow (1982). Effectively, short-term debt achieves commitment because it needs to be continuously rolled over and repriced, similar to leasing.

Matters are different in the high state when short-term debt is riskless, that is, when $\lambda > \bar{\lambda}$ and $f \leq f_{\dagger}$. In this case, default does not occur after the downside risk is realized, but the enterprise value experiences a discontinuous jump. Whereas a transition from the high to the low state reduces the equity value by $j_H(f) - j_L(f)$ and the long-term debt price by $p_H(f) - p_L(f)$, it leaves the value of short-term debt intact. In other words, the loss in enterprise value is shared between the borrower and long-term creditors. Short-term creditors, on the other hand, do not share any loss.

[10] Note the result of no long-term debt issuance in the low state stays unchanged even if $\lambda_{LH} > 0$, because the borrower will not default upon a state transition from low to high.

This result highlights an important role of long-term debt in risk sharing: it allows the borrower to make state-contingent payment (effectively $p_H(f)$ in H but $p_L(f)$ in L) without default. The state-contingent payments act as a cushion that reduces the borrower's burden in the downturn and mitigates the incentives to default, thereby increasing the enterprise value. By contrast, short-term debt is a harder claim: the borrower must make non-state-contingent payments; otherwise, she has to default. Figure 2 provides a graphical illustration upon the state transition. Clearly, both equity value and long-term debt price become reduced, whereas the value of short-term debt stays unchanged (unless the borrower defaults).

State $\theta = H$

Assets	Liabilities
Cash flow grows at μ_H	ST debt: j_L
	LT debt: $p_H \cdot f$
	Equity: $j_H - j_L$

State $\theta = L$

Assets	Liabilities
Cash flow grows at μ_L	ST debt: j_L
	LT debt: $p_L \cdot f$

Equity: 0

Figure 2: Balance sheet upon the state transition without immediate default

A careful examination of the issuance function (22) shows the amount of long-term debt issued, fg_H, equals the ratio of the flow benefits from risk-sharing $(\rho - r)\big(p_H(f) - p_L(f)\big)$ to the price sensitivity of new issuance $-p'_H(f)$. Let us offer a heuristic derivation based on a local perturbation argument. Consider a policy whereby the borrower issues a small amount of long-term debt Δf at time t and buys it back at time $t + dt$. As a result, the amount of riskless short-term debt that can be borrowed drops from $j_L(f)$ to $j_L(f + \Delta f)$ at time t, and resumes to be $j_L(f)$ at $t + dt$. At time t, long-term debt increases by $p_H(f+\Delta f)(f+\Delta f) - p_H(f)f \approx p_H(f)\Delta f$, whereas short-term debt decreases by $j_L(f) - j_L(f + \Delta f) \approx j'_L(f)\Delta f$, for Δf sufficiently small. The total change in leverage at t is then approximately $(p_H(f) + j'_L(f))\,\Delta f$. During $[t, t + dt)$, this increase in leverage brings a marginal benefit of $(\rho - r)\,dt \cdot (p_H(f) + j'_L(f))\,\Delta f$. Meanwhile, this operation brings issuance proceeds $p_H(f + \Delta f)\Delta f$ at time t and $-p_H(f)\Delta f$ at $t + dt$, resulting in a marginal cost of $(p_H(f) - p_H(f + \Delta f))\,\Delta f$.[11] For Δf sufficiently small, this marginal cost becomes $-p'_H(f)\,(\Delta f)^2$. The optimal issuance Δdt must equalize the marginal benefit and marginal cost:

$$(\rho - r)\,dt \cdot \big(p_H(f) + j'_L(f)\big)\,\Delta f = -p'_H(f)\,(\Delta f)^2 \Rightarrow \Delta f = g \cdot f dt = \frac{(\rho - r)\,(p_H(f) - p_L(f))}{-p'_H(f)} dt.$$

[11]Note the proceeds in short-term debt issuance t and $t + dt$ cancel out on each other.

Remark 1 (Long-term debt buybacks)**.** *For f sufficiently close to $f_{\dagger}$, the issuance of long-term debt can be negative. In other words, the borrower actively buys back long-term debt. This result contrasts with the literature on the leverage-ratchet effect (DeMarzo and He, 2021; Admati et al., 2018), which proves no buybacks exist in equilibrium. The issuance equation above implies buybacks occur in our model if $p_H(f) < p_L(f)$, that is, if the price of the long-term debt is lower in the high state than in the low state. In these states, the probability of the borrower defaulting in the near future could be higher in the high state than in the low state. To understand why, note that in the high state, the Brownian shock may increase the state variable above $f_{\dagger}$, in which case the borrower will borrow risky short-term debt and default will happen following the regime shift. By contrast, for the same level of f in the low state, default cannot happen unless f increases up to f_L^b, and this probability can be relatively low for f just slightly below $f_{\dagger}$.*

To summarize, the choice of maturity is determined by the trade-off between commitment and risk-sharing. Short-term debt resolves the issue of lack of commitment; long-term debt shares the downside risk. This insight relates to the previous work in the fiscal-policy literature that emphasizes how long-term debt allows for more state contingency (Angeletos, 2002). A key difference is that we cast the model in the context of a risk-neutral borrower, as is typically the case in capital structure studies. An immediate question, then, is why would a risk-neutral borrower value the long-term debt's merit in sharing the downside risk? The reason is that the bankruptcy cost introduces concavity into her objective function, so that she behaves as if risk-averse. As emphasized by previous work on risk management (Froot et al., 1993; Rampini and Viswanathan, 2010), a risk-neutral corporation has incentives to insure or hedge against negative shocks when the cost of external financing is costly and can fluctuate.

3.3 Benefits and Costs of Long- and Short-Term Debt

In this subsection, we explore the benefits and costs of both types of debt by studying the equilibrium if only long- or short-term debt is allowed.

Proposition 4 (Equilibrium with only short-term debt)**.** *A unique equilibrium exists if the borrower can only issue short-term debt.*

1. *In the low state L, the borrower never defaults. The value function is*

$$\tilde{J}_L(X) = \frac{X}{r - \mu_L}.$$

Short-term debt is $\tilde{D}_L(X) = \tilde{J}_L(X)$ and the short rate is $y_L = r$.

2. *In the high state H, the value function is*

$$\tilde{J}_H(X) = X \max\left\{\frac{1}{r+\lambda-\mu_H}, \frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho-r+\lambda}{r-\mu_L}\right)\right\}.$$

Let $\bar{\lambda}$ be the threshold in Proposition 2.

- *If $\lambda \leq \bar{\lambda}$, short-term debt is $\tilde{D}_H(X) = \tilde{J}_H(X)$. The borrower defaults as soon as θ switches from H to L, so the short rate is $y_H = r + \lambda$.*
- *If $\lambda > \bar{\lambda}$, $\tilde{D}_H(X) = \tilde{J}_L(X)$, and the borrower never defaults and $y_H = r$.*

3. *The total enterprise value is higher than in the case where the borrower can issue both types of debt, that is, $\tilde{J}_\theta(X) \geq J_\theta(X,F) + p_\theta(X,F)\,F$, $\forall \theta \in \{L,H\}$.*

If only short-term debt is allowed, the commitment problem in debt issuance no longer exists. Instead, the choice of capital structure is a static problem and follows the standard trade-off theory whereby equity holders balance cheap debt against costly bankruptcy. Given that the problem is scalable with respect to X_t, the solution is one with a constant leverage level. Interestingly, the total enterprise value is higher if the borrower is prohibited from issuing long-term debt. This result may appear paradoxical, given the earlier discussion on how long-term debt benefits the borrower for creating more state contingency. As we have shown in subsection 3.2, the issuance function of long-term debt clearly captures the risk-sharing benefits. However, due to the borrower's lack of commitment to future issuance policies, the price of long-term debt drops to the level at which the benefits from risk-sharing are completely depleted. In other words, given the pricing function of long-term debt $\{p_H(f), p_L(f)\}$, the borrower always has incentives to borrow long to hedge against the downside risk. However, due to lack of commitment, the hedging benefits from the flexibility to issue long-term debt are completely dissipated. In fact, the borrower's equilibrium payoff is strictly lower with the flexibility to issue long-term debt, because outstanding long-term debt introduces the additional defaults (and the associated bankruptcy cost) following a sequence of negative Brownian shocks. To summarize, whereas the borrower values risk-sharing, the lack of commitment to future issuance totally dissipates the benefits from risk-sharing and additionally leads to more default on the equilibrium path, thereby reducing the borrower's payoff.

The equilibrium outcomes under $\lambda \leq \bar{\lambda}$ and $\lambda > \bar{\lambda}$ are worth further discussion, particularly when they are compared with the ones in Proposition 3. In the equilibrium with only short-term debt, if λ is low, downside risk is unlikely to materialize, so that the short rate $y_H = r + \lambda$ is relatively low. In this case, the borrower finds it optimal to borrow risky short-term debt. When λ rises above $\bar{\lambda}$, the risky short-term debt becomes too expensive, in which case she switches to borrowing riskless short-term debt. The results are related when the borrower can borrow both long- and short-term debt. For low λ, the downside risk is low, and the borrower never issues any long-term debt to hedge against the downside risk, which is unlikely to materialized. Instead, she issues risky short-term debt and rationally takes the downside risk. For high λ, this downside risk gets higher, but the flexibility to issue long-term debt to hedge against it. As a result, the borrower could continue to issue riskless short-term debt for f being low.

Proposition 5 (Equilibrium with only long-term debt). *A unique equilibrium exists. Define*

$$\tilde{\gamma} \equiv \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho - \mu_L\right)}}{\sigma^2} > 1.$$

1. *In state L, the value function is*

$$\tilde{v}_L\left(f\right) = \frac{1}{\rho - \mu_L} - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}},$$

   where the default boundary is $\tilde{f}_L^b = \frac{1}{\rho-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\xi}{r+\xi}$.

2. *In state H, the value function is*

$$\tilde{v}_H\left(f\right) = \tilde{u}_0\left(f\right) - \tilde{u}_0\left(\tilde{f}_H^b\right)\left(\frac{f}{\tilde{f}_H^b}\right)^{\phi},$$

   where

$$\tilde{u}_0\left(f\right) \equiv \frac{1}{\rho+\lambda-\mu_H}\left(1 + \frac{\lambda}{\rho-\mu_L}\right) - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi}\frac{\lambda}{\lambda + (\mu_H - \mu_L)(\tilde{\gamma}-1)}\frac{\tilde{f}_L^b}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}_L^b}\right)^{\tilde{\gamma}}.$$

   The borrower defaults upon the state transition if and only if $f > \tilde{f}_L^b$.

3. *In both states $\theta \in \{L, H\}$, the debt price is $\tilde{p}_\theta = -\tilde{v}_\theta'$, and the issuance function follows*

$$\tilde{g}_\theta = \frac{\left(\rho - r\right)\tilde{p}_\theta}{-f\tilde{p}_\theta'}.$$

When the borrower is only allowed to issue long-term debt, the setup resembles the one in DeMarzo and He (2021). Without commitment to the issuance policy, equity holders do not reap the benefits from issuing cheaper debt, because the debt price will adjust for the future issuance policy. In equilibrium, long-term debt is issued smoothly. The next proposition compares the equilibrium with only long-term debt with the one in which the borrower can issue both types of debt.

Proposition 6 (Comparison of equilibrium)**.**

1. *The total enterprise value is lower than the case in which the borrower can issue both types of debt, that is,* $\tilde{v}_\theta(f) + \tilde{p}_\theta(f) f \leq j_\theta(f) + p_\theta(f) f$, $\forall f$.

2. *The default boundary is higher in the presence of short-term debt. That is,* $f_\theta^b > \tilde{f}_\theta^b$.

3. *In the low state, the price of debt is higher in the presence of short-term debt. That is,* $\tilde{p}_L < p_L$, $\forall f \in [0, \tilde{f}_L^b]$. *In the high state, if* $\rho > r + \lambda$, *thresholds* $\underline{f} \in [0, f_\dagger]$ *and* $\overline{f} \in [f_\dagger, \tilde{f}_H^b]$ *exist such that* $\tilde{p}_H(f) \leq p_H(f)$ *on* $[0, \underline{f}] \cup [\overline{f}, \tilde{f}_H^b]$.

In both states, the enterprise value is higher when the borrower can issue both types of debt. Intuitively, borrowing short not only increases the leverage but also allows the equity holder to reap some benefits from issuing debt. This higher enterprise value is reflected in higher default boundaries in both states. We turn to the comparison of long-term debt's price, which can be either higher or lower when the borrower can issue both types of debt. On one hand, whereas short-term debt increases the enterprise value, it also pushes up the default boundary, so that under the same level of long-term debt, the borrower is further away from the default boundary. On the other hand, in the high state when f is close to $f_\dagger$, the price of long-term debt can be lower if she can issue both kinds of debt. Intuitively, taking short-term debt leads to the borrower defaulting following the realization of the downside risk when f is above $f_\dagger$. Without short-term debt, the borrower won't default following the same transition unless f rises above $\tilde{f}_L^b$. Therefore, the price of long-term debt is relatively low when f exceeds $f_\dagger$ but is still far from $\tilde{f}_L^b$ yet. This result suggests long-term debt and short-term debt could be either complements or substitutes, depending on the borrower's distance to default.

3.4 Initial Debt Issuance

Does an initially unlevered borrower issue any long-term debt? From the issuance function (22), it is easily established that

$$g_H(0) = \frac{(\rho - r)(\mu_H - \mu_L)}{\rho + \lambda - r}, \qquad \lim_{f \to 0} g_H(f) f = 0,$$

so that an unlevered borrower does not issue any long-term debt. The intuition is straightforward. For the unlevered borrower, both $p_H(f) \to 1$ and $p_L(f) \to 1$ hold as $f \to 0$ so that a marginal unit of long-term debt is riskless and does not share any downside risk. Therefore, the unlevered borrower has no reason to issue it. Indeed, the issuance function (22) shows that for an unlevered borrower to issue long-term debt, that is, $\lim_{f\to 0}\ g_H(f)\cdot f>0$, it must be that $p_H(0)>p_L(0)$ so that a marginal unit of long-term debt shares some losses following the transition to the downturn.

Several approaches can be taken to motivate an unlevered borrower to issue long-term debt. One is to introduce an exogenous disaster in the low state, modeled as a Poisson event with intensity ζ, upon which X_t permanently drops to zero. In this case, both $p_H(0)$ and $p_L(0)$ are less than 1 (and therefore not default free) and $p_H(0)>p_L(0)$.

Proposition 7. *In the model with a disaster event, the expressions of issuance policy are identical to those in Proposition 3. Let*

$$\underline{\zeta}\equiv\sigma^2-(r+\mu_L+2\xi),\quad \overline{\zeta}\equiv\frac{\lambda(\rho+\lambda)-(\rho+\lambda-r)\mu_H+(\rho-r)\mu_L}{\rho-r}.$$

If $\underline{\zeta}<\overline{\zeta}$, $\lambda\geq\sigma^2-(\rho+\mu_H+2\xi)$, *and* $\zeta\in[\underline{\zeta},\overline{\zeta}]$, *an unlevered borrower issues some long-term debt in the high state. That is,* $\lim\limits_{f\to 0} g_H(f)f>0$. *Otherwise,* $\lim\limits_{f\to 0} g_H(f)f=0$.

Note that an unlevered borrower issues some long-term debt if the downside risk of state transition is significantly high and the disaster intensity is neither too high nor too low. The requirement on downside risk is intuitive, due to the role of long-term debt in risk-sharing. Let us explain the intuitions behind the conditions on disaster risk. If $\zeta>\overline{\zeta}$, the disaster risk is very high in the downturn, so that the continuation value $j_L(0)$ is very low, even if the borrower is unlevered. As a result, the amount of riskless short-term debt that she can borrow is also low, so much so that she would rather issue risky short-term debt. Therefore, the borrower is anticipated to default immediately upon the downturn arrival, and long-term debt has no role in sharing the downside risk. Meanwhile, if $\zeta<\underline{\zeta}$, disaster risk is so low that the difference between $p_H(f)$ and $p_L(f)$ is small and dominated by the price impact of issuing long-term debt. In this case, the unlevered borrower would again refrain from borrowing long to begin with.

The analysis above further highlights the earlier mechanism whereby long-term debt helps with risk-sharing. If the downside risk only includes the state transition from high to low, the marginal unit of long-term debt is riskless for an unlevered borrower. In the presence of the additional downside risk from the disaster, a marginal unit of long-term debt is more exposed to the disaster risk in the low state than in the high state, even if the borrower is unlevered. This exercise confirms that an unlevered borrower has incentives to issue long-term debt as long as the marginal unit of long-term debt shares either some downside risk or some disaster risk, and the price impact of a marginal dollar of long-term debt is limited.

4 Empirical Implications, Extensions, and Robustness

Subsection 4.1 explores the model's empirical implications in both the cross section and time series. In subsection 4.2, we study debt issuance and restructuring when the recovery value in default is positive. We show in subsection 4.3 that the modeling of downside risk is not restricted to regime shifting. In particular, a downward jump to the cash-flow process also motivates the issuance of long-term debt. Whereas the benchmark model has assumed competitive creditors, subsection 4.4 introduces investors‘ clienteles for specific maturity segments so that the discount rates differ across creditors with different investment horizons. We show how the motives to manage interest-rate risk affect the borrower's decision to issue long- and short-term debt. Finally, in subsection 4.5, we introduce tax shields as the reasons for debt issuance. Our main mechanism between commitment and risk-sharing continues to operate, but the results are slighly different from the benchmark model in which debt issuance is motivated by differences in discount rates. The reason comes from differential tax treatments toward coupon and principal payments, which affect long- and short-term debt differently.

4.1 Empirical Implications and Impulse Responses

This subsection numerically solves the model with the disaster event introduced in subsection 3.4. Moreover, we allow for $\lambda_{LH} > 0$ so that shocks to the cash flow's expected growth rate are transitory. We look for an equilibrium similar to the one in section 3 under reasonable parameters. The model's cross-sectional and time-series implications on debt maturity structure will be linked to empirical studies.

Our central object of interest is a firm's debt maturity structure, defined as the average maturity of total debt outstanding weighted by their book value[12]:

$$\text{Maturity}_t := \frac{F_t}{F_t + D_t}\frac{1}{\xi} = \frac{f_t}{f_t + d_t}\frac{1}{\xi}. \tag{23}$$

Cross-sectional implications. Figure 3 shows how the average maturity varies within a cross-section of borrowers with different characteristics. The left panel plots how the maturity changes with f in the high state, under different levels of μ_L.[13] One interpretation of f is the borrower's

[12]We focus on maturity measured using book values rather than market values, because this approach is the one most commonly used in empirical studies, due to data limitations. That said, results stay qualitatively unchanged when debt maturity is weighted by market values.

[13]We argue the results in the high state are a more precise description for the following reason. In the binary-state setup, no additional downside risk is present once the borrower enters the low state. In practice, the borrower is will likely to always face some downside risk, which motivates the use of long-term debt. In other words, the high state in our model is meant to capture any real-world scenario as long as the borrower still faces some downside risk.

distance to default (DD), and a higher f is associated with the borrower being closer to default. Three patterns are prominent. First, a borrower closer to default has more long-term debt. In our model, this pattern holds because in the absence of a regime shift, default is only triggered by a large amount of outstanding long-term debt, whereas short-term debt can be easily adjusted and default can be anticipated by the borrower's creditors. Second, the average maturity could have a discontinuous downwards jump when the borrower gets closer to default. In our model, this pattern holds because when f increases above $f_\dagger$, the borrower stops borrowing long and exclusively issues short-term debt. In practice, downgraded firms are found to mostly rely on short-term borrowing. The third pattern comes from the comparison across the two lines. Specifically, for a given f, the borrower whose cash flows grow at a slower rate in the low state (captured by a lower μ_L) has more long-term debt outstanding. Intuitively, this borrower has more incentives to hedge against the downside risk if its size increases.[14]

The middle panel of Figure 3 shows how maturity differs across firms with different leverage, where leverage is defined as $\frac{d+f}{j+f}$, the book value of total debt divided by the sum of market value of equity and book value of debt.[15] This measurement corresponds to the market leverage ratio in most empirical papers.[16] Results show that more levered borrowers use more long-term debt. In our model, this result happens because a borrower's equity value is $j_H(f) - j_L(f)$, which decreases with f in the high state when $f < f_\dagger$. Therefore, an increase in f leads to both a higher leverage and a longer average maturity. Moreover, a comparison across the two lines confirms the earlier result that, controlling for market leverage, a borrower with higher downside risk uses more long-term debt.

Finally, the right panel plots average maturity across firms with different asset market-to-book ratios, defined as $\frac{p(f)f+j}{f+j(f)}$. A borrower with a higher market-to-book ratio has relatively more short-term debt. In our model, this pattern holds because the price of long-term debt declines with f. This result is consistent with previous findings on firms with more growth options having more short-term debt outstanding (Stohs and Mauer, 1996; Barclay and Smith Jr, 1995). Short-term debt can be easily adjusted once these firms exercise the growth options.

The evidence above points out the importance of differentiating stock versus flow in studying

[14] Similar patterns hold for higher λ_{HL} if we keep μ_L the same, due to the same intuition.

[15] The leverage is 100% in the low state and in the high state when $f > f_\dagger$, implying borrowers with high levels of leverage (100% in this case) could have different maturity structures. In this sense, our model implies debt maturity structure has additional predictive power for defaults after controlling for a borrower's leverage.

[16] For example, empirical papers using Compustat data typically define market leverage of firm i in year t as

$$\text{Lev}_{it} = \frac{\text{DLTT}_{it} + \text{DLC}_{it}}{\text{DLTT}_{it} + \text{DLC}_{it} + \text{CSHO}_{it} \times \text{PRCC\_F}_{it}},$$

where DLTT_{it} and DLC_{it} are the amount of long-term debt and debt in current liabilities. PRCC_F is the fiscal year-end common share price and CSHO is the fiscal year-end number of shares outstanding.

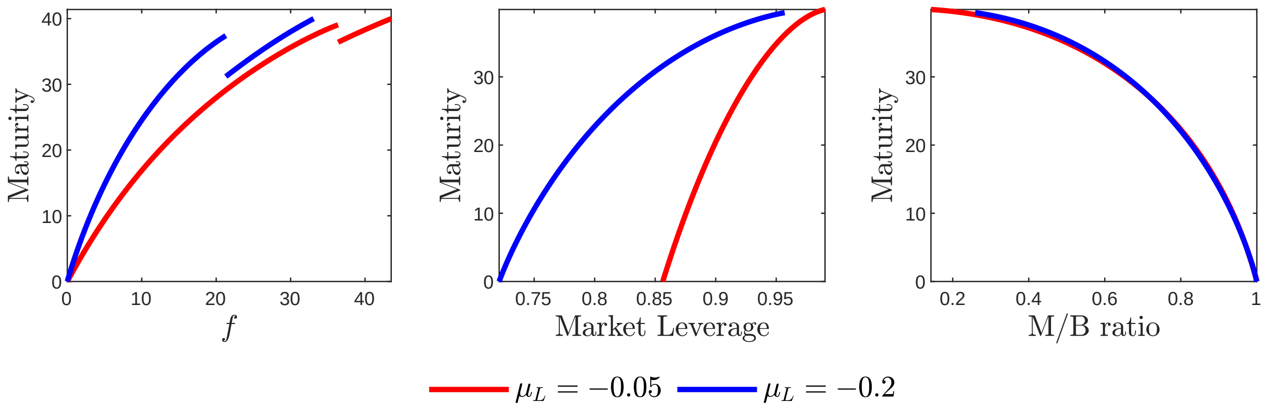


Figure 3: Cross-sectional debt maturity structure

This figure plots maturity as a function of f, market leverage, and market-to-book ratio in the high state. The parameters are as follows: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.2$, $\sigma = 0.3$, $\xi = 0.025$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$. The first figure plots maturity as a function of f on $[0, f_H^b]$. The second and third figures plot maturity as a function of leverage and market-to-book ratio for f on $[0, f_\dagger]$.

debt maturities. For example, the left panel of Figure 3 implies default is triggered by too much long-term debt in stock, but in this situation, the borrower only issues short-term debt. This result is consistent with Friewald et al. (2021), who show that when a firm has a large amount of long-term debt due in the next three years, it is more exposed to systematic risk and commands a higher equity return.[17]

Time-series implications. Now we turn to the time-series implications. Our result that long-term debt is only issued in the high state $\theta_t = H$ immediately implies the borrower's debt maturity is pro-cyclical, if one interprets these two states as business-cycle frequency boom and bust. This prediction is consistent with the findings in Chen et al. (2021), which we replicate in Figure 4 below.

We simulate a sample path and plot the time-series of the cash-flow rate and debt maturity in Figure 5.[18] Without the regime shift, average maturity and the cash-flow rate seem to comove negatively with each other. In other words, the borrower expands the average debt maturity following a negative Brownian shock to X_t. Intuitively, this pattern holds because after a negative Brownian shock to X_t, the borrower rolls over less short-term debt, which is easier to adjust. On

[17]In Friewald et al. (2021), this is referred to as short-term leverage, though. This result in Friewald et al. (2021) is also consistent with Chaderina et al. (2021) on short-maturity financed firms being risky over short holding horizons.

[18]Note our result implies market leverage is on average countercyclical, if we interpret the state transition as business-cycle shocks. Adrian and Shin (2014) offer consistent evidence.

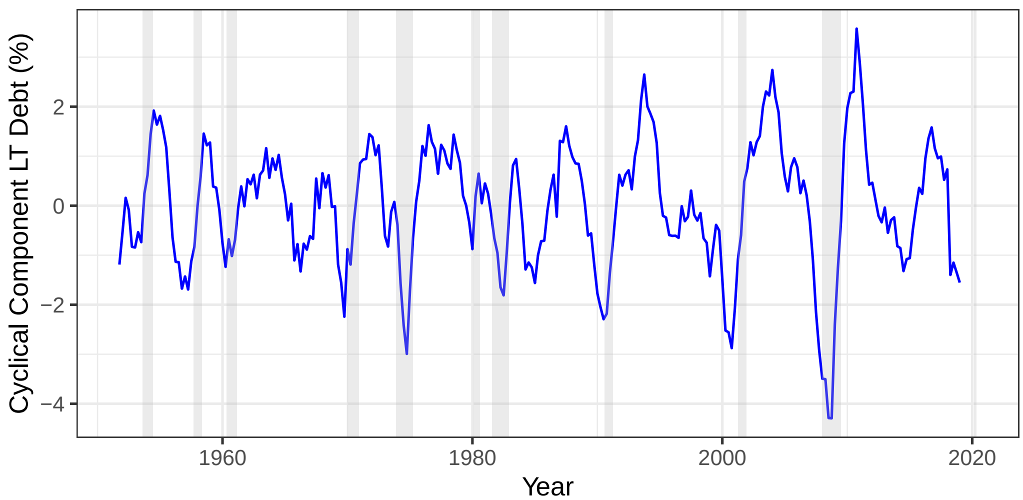


Figure 4: Pro-cyclical long-term debt share in the business cycle

This figure applies the Hodrick-Prescott filter with multiplier 1600 to the share of long-term debt of non-financial firms in the United States and extracts the cyclical components. The shaded areas denote NBER-dated recessions. From Chen et al. (2021) using Flow of Funds Accounts data (Table L.103)

the other hand, when the regime shifts and the downturn arrives, the borrower exclusively borrows short-term debt and the average maturity goes down.[19] This result suggests the borrower's debt-issuance policy has a different response to small, frequent Brownian shocks versus large, infrequent regime-shift shocks. Below, we formalize these results by studying the impulse-response functions of debt issuance to both Brownian and regime-shifting shocks.

Impulse-response functions and shock elasticity. How does a borrower's long- and short-term debt issuance respond to a negative cash-flow (Brownian) shock and/or a regime-shift shock whereby the downturn arrives? We answer this question by studying the impulse response of time-t outstanding debt F_t and D_t to the two different shocks occurring at time 0. Given the model is non-linear, we cannot follow the traditional approach in macroeconomic studies by assuming a one-time shock at time 0 and no further shocks afterwards. Instead, we need to simulate the entire sample path during $[0, t]$ by taking into account subsequent shocks after time 0 and study how an average borrower responds to the shock at time 0. We follow Borovička et al. (2014) by defining the shock

[19]This result depends on the binary-state setup, where no additional downside risk exists in the low state. With more than two states, the borrower may still issue long-term debt in the low state. The broader message is the transition to a worse state, the borrower may only issue short-term debt for a while.

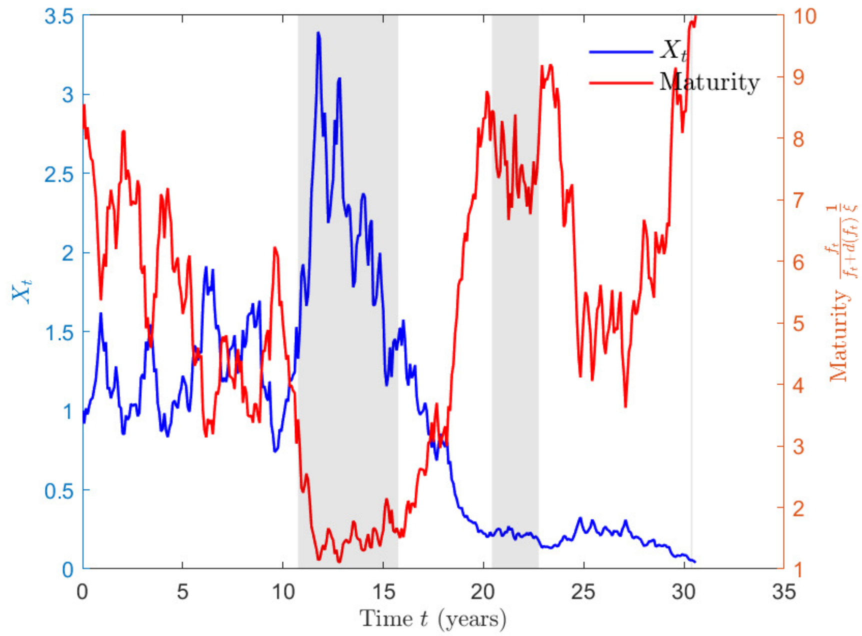


Figure 5: Sample Path of Leverage and Maturity

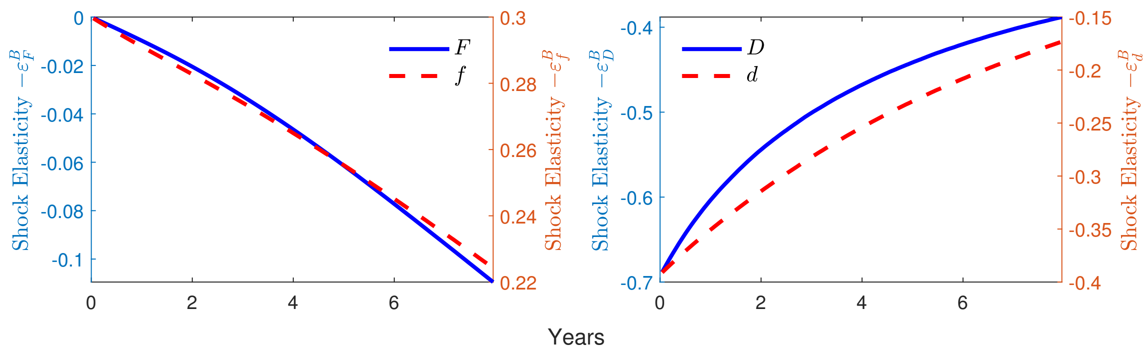
This figure simulates the sample path of one firm and plots the time series of X_t, maturity, and market leverage, with the following parameter values: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$.

elasticity of F_t and D_t with respect to the cash-flow shock and regime-shift shock. Specifically, for a process $Y_t \in \{F_t, D_t\}$, let us define

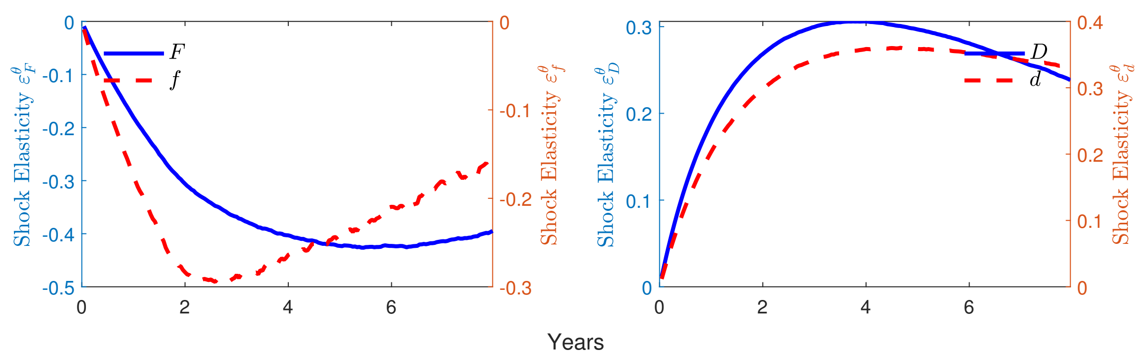
$$\varepsilon_Y^B(t, f, \theta) \equiv \frac{\mathbb{E}\left[\mathcal{D}_0 Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = \theta\right]}{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = \theta\right]}$$
$$\varepsilon_Y^\theta(t, f) \equiv \frac{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0, \theta_0 = L\right] - \mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = H\right]}{\mathbb{E}\left[Y_t \mathbb{1}_{\{t<\tau_b\}} | f_0 = f, \theta_0 = H\right]},$$

where $\mathcal{D}_0 Y_t$ corresponds to the Malliavin derivative of the process Y_t, and the indicator function $\mathbb{1}_{\{t<\tau_b\}}$ accounts for the possibility of the borrower defaulting before time t. Intuitively, ε_F^B (ε_D^B) captures the proportional change of an average borrower's outstanding long-term (short-term) debt at time t as a response to a positive cash-flow shock, whereas ε_F^θ (ε_D^θ) is the same response to the regime-shift shock. Details are provided in Proposition 10 in the appendix.

Figure 6 plots the shock elasticities. The top two panels show that after a negative Brownian shock $dB_0 = -1$ (that is, the plots are about $-\varepsilon_F^B$ and $-\varepsilon_D^B$), an average borrower slowly changes the outstanding long-term debt but immediately reduces the amount of short-term debt. Over time, the borrower changes the composition of debt by reducing long- and increasing short-term debt (relative to time 0). The bottom two panels describe the shock elasticity with respect to the regime shift shock, i.e., θ_0 shifts from the upturn H to the downturn L. Interestingly, whereas the borrower also slowly reduces the outstanding long-term debt, she actually increases the amount of short-term borrowing, because downside risk is mitigated after the downturn has already arrived. Note that in this case, the adjustment of both long- and short-term debt is slow.



(a) Shock elasticity with respect to a negative Brownian shock



(b) Shock elasticity with respect to θ from H to L

Figure 6: Shock elasticity of short- and long-term debt.

The top panels of figure plot the impulse-response functions to a negative cash-flow shock $dB_0 = -1$ for $\theta_0 = H$, $X_0 = 1$, and $f_0 = 8.5$. The bottom panels plot the impulse-response functions to the regime-shifting shock such that θ transits from H to L at $t = 0$. The parameter values in both panels are the following: $\rho = 0.1$, $r = 0.035$, $\mu_H = 0.015$, $\mu_L = -0.1$, $\sigma = 0.3$, $\xi = 0.1$, $\lambda_{HL} = 0.2$, $\lambda_{LH} = 0.4$, $\zeta = 0.05$. With these parameters, $f_\dagger = 17.10$, $f_H^b = 23.00$, and $f_L^b = 19.91$. The scales on the left axes stand for the variables F and D, and the scales on the right axes stand for the variables f and d.

Whereas the majority of existing empirical research on debt maturity has focused on the cross-sectional comparisons, our results highlight the importance of studying the borrower's dynamic adjustments in the outstanding maturity structure. Specifically, in a difference-in-difference (DiD) estimation, the dynamic treatment effects would be considerably different for long- and short-term debt. When borrowers in the treatment group experience a negative cash-flow shock (Brownian shock), our model predicts no immediate reaction in the outstanding long-term debt. Over time, the amount of long-term debt gets gradually reduced. By contrast, borrowers in the treatment group immediately and abruptly reduce the issuance of short-term debt. Over time, this reduction mean-reverts.

The dynamic treatment effects could also differ by the nature of the shock. In the situation in which borrowers in the treatment group experience a regime-shift shock, our model predicts no immediate reaction in either long- or short-term debt. In the near future, the amount of outstanding long-term debt gets gradually reduced, whereas the amount of short-term debt gets gradually increased. All these implications can be testable hypotheses for future empirical studies.

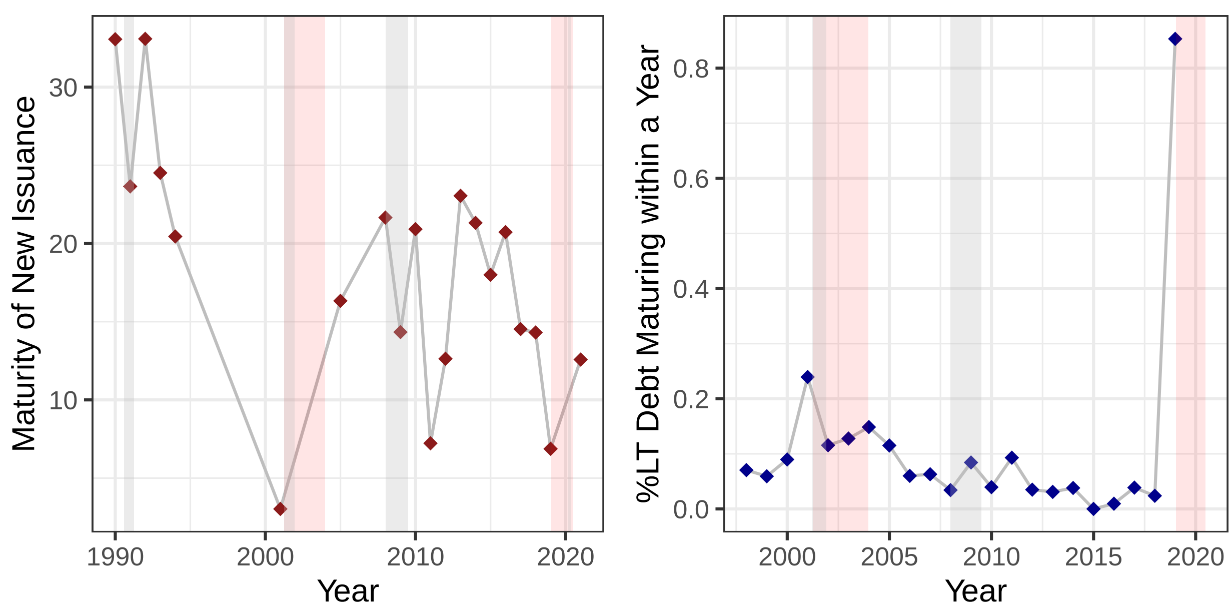


Figure 7: Maturity structure of Pacific Gas & Electric Company

This figure shows the average maturity of bonds in a year (weighted by market value at issuance) and the share of long-term debt maturing within one year for Pacific Gas & Electric (PG&E). The gray-shaded area indicates NBER recession, whereas the red-shaded area indicates periods over which PG&E was in bankruptcy procedures. Source: Compustat and Mergent FISD.

An example: Pacific Gas and Electric Company. Let us connect our model's empirical predictions to a real-world example. The Pacific Gas and Electric Company (PG&E) is a US-based public company that has entered bankruptcy twice in the last two decades. It initially entered Chapter 11 bankruptcy on April 6, 2001, and emerged from bankruptcy in April 2004. In 2019, it filed for bankruptcy on January 29 again and successfully exited on June 20. Figure 7 plots PG&E's debt maturity structure from 1990 onward. The left panel plots the maturity of newly issued long-term debt, weighted by the offering amount. The red-shaded areas marked the two bankruptcies, and the gray areas are NBER recessions. Consistent with our model, the newly-issued bonds have shorter maturities in the NBER recessions and shortly prior to the bankruptcies.[20] The right panel plots the ratio of long-term debt due within a year to the sum of current and long-term liabilities. Clearly, this ratio spiked the year prior to both bankruptcies. The patterns in both panels suggest bankruptcy was associated with a large amount of long-term debt maturing soon and prior to bankruptcy, PG&E issued more short-term debt. In online appendix section C, we also plot similar patterns for General Motors Company and American Airlines, Inc., with both showing similar patterns.

4.2 Positive Recovery and Debt Restructuring

Thus far, we have assumed the recovery value is zero if the borrower defaults. If instead, the recovery value is positive, a borrower without any restriction in issuance can effectively steal the entire recovery value from creditors by issuing a large amount of new debt just prior to default and using the proceeds to pay a dividend. In practice, this transaction is referred to as a preference action and can be voided using the clawback provision.

In this subsection, we consider a possibility that in the upturn $\theta_t = H$, the borrower can restructure the outstanding debt under the assumption that the recovery value is $\alpha X j_H(0)$, a fraction α of the unlevered value.[21] We assume short-term debt is junior to outstanding long-term debt; otherwise, the borrower can fully dilute long-term creditors by issuing a large amount of short-term debt prior to default. Once the borrower defaults, all relevant parties enter a restructuring process whereby long-term creditors can recover $R(f)$ per unit of face value.[22] Finally, we assume a restructuring cost $Xb(f)$ that captures frictions in the renegotiation process.

[20] The maturity of newly-issued debt was also short in 2011, which might be due to the 2010 San Bruno explosion: PG&E was on probation after being found criminally liable in the fire. In the context of our model, the San Bruno explosion can be thought of as a transition from a high to low state after a Poisson shock.

[21] It is straightforward to extend the analysis to both states.

[22] One implementation of the restructuring is to convert long-term debt to equity share $fR(f)/\alpha j(0)$ and long-term debt to equity share $d(f)/\alpha j(0)$. The original borrower retains a fraction $1 - (d(f) + fR(f))/\alpha j(0)$ of the equity of the restructured enterprise.

In principle, equity holders may still issue more long-term debt prior to default to dilute existing creditors. Let this amount be Δ. Creditors – anticipating default and restructuring shortly after – expect to receive $R(f+\Delta)$ per unit of face value, and this price is the maximum they are willing to pay before bankruptcy. The net proceeds from issuance are therefore $fR(f+\Delta)$. From here, the value that equity holders obtain from restructuring is $v_H^R(f,d) = \max\{j_H^R(f) - d, 0\}$,[23] where

$$j_H^R(f) \equiv \alpha j_H(0) - \min_{\Delta} \{fR(f+\Delta) + b(f+\Delta)\}.$$

We make the following assumptions:

Assumption 2. *The functions $R(f)$ and $b(f)$ satisfy*

a. $R(f) \in [0,1]$.

b. *$R(f)$ is non-increasing, and $b(f)$ is non-decreasing.*

c. *The total recovery by long-term creditors is lower than firm value:*

$$\alpha j_H(0) \geq \min_{\Delta \geq -f} \{fR(f+\Delta) + b(f+\Delta)\}.$$

d. *For all $f \in \mathbb{R}_+$, the function $fR(f+\Delta) + b(f+\Delta)$ is continuously differentiable and quasi-convex in Δ.*

Assumption 2.a. states that long-term creditors are protected by limited liability and never recover more than the face value. Assumption 2.b. captures in reduced form that higher leverage makes restructuring debt more difficult and decreases the expected recovery. Assumption 2.c. restricts the total recovery value received by long-term creditors to be less than the enterprise value. Under these assumptions, the optimal issuance at the time of default $\Delta^b(f)$ is given by the first-order condition:

$$b'\left(f + \Delta^b(f)\right) = -fR'\left(f + \Delta^b(f)\right).$$

The right-hand side $-fR'(f+\Delta)$ captures the marginal benefit from diluting existing long-term creditors, whereas the left-hand side is the marginal cost of renegotiation. Note the adjustment $\Delta^b(f)$ could be negative, which is necessarily the case if $b'(f) > -fR'(f)$. A negative issuance captures a situation in which the borrower injects cash to buy back some long-term debt at a discount.

[23] In equilibrium, $j_H^R(f) - d \geq 0$, because otherwise, short-term creditors anticipate default.

By doing so, the borrower facilitates the restructuring process by reducing the renegotiation cost. Indeed, debt repurchase is common across financially distressed firms.

The rest of the model is solved similarly to the zero recovery case. Indeed, the HJB equations remain unchanged, and the boundary conditions are replaced by $j_H(f_H^b) = j_H^R(f)$, and $j_H'(f_H^b) = j_H^{R'}(f_H^b) = -R(f_H^b + \Delta^b(f_H^b))$.[24] The linear recovery case in DeMarzo and He (2021) and DeMarzo et al. (2021) corresponds to the case in which $R(f)$ and $b(f)$ are constants, in which case we can take $\Delta^b(f) = 0$. The following proposition provides some implications of corporate restructuring on the use of long-term debt.

Proposition 8 (Impact of Restructuring on Long-Term Debt)**.** *Both a higher cost of restructuring and a higher recovery value expand the region of long-term debt issuance. That is,*

- *Consider two restructuring environments $\{R_1(f), b_1(f)\}$ and $\{R_2(f), b_2(f)\}$, such that $R_1(f) \geq R_2(f)$ and $b_1(f) \geq b_2(f)$. Let $f_{1\dagger}$ and $f_{2\dagger}$ be the threshold for the issuance of long-term debt, respectively. Then, $f_{2\dagger} > f_{1\dagger}$.*
- *The threshold $f_\dagger$ is increasing in bankruptcy cost $1 - \alpha$. That is, if $\alpha_1 > \alpha_2$, $f_{2\dagger} > f_{1\dagger}$.*

The previous proposition implies that a more costly renegotiation process and a higher bankruptcy cost increases the incentives to use long-term debt.

4.3 Jump Risk

In the benchmark model, the borrower is subject to two types of risks. The Brownian motion captures small frequent shocks to the cash flow, which has a continuous effect on the enterprise value. By contrast, a transition from the high to the low state, that is, the regime shift, captures large infrequent shocks that reduce the enterprise value discontinuously. In this subsection, we show the modeling choice of a regime shift is unimportant. In particular, our mechanism continues to hold if large infrequent shocks are modeled as downward jump risks to the cash flow. Specifically, we assume the cash flow follows a jump-diffusion process:

$$dX_t = \mu X_{t-} dt + \sigma X_{t-} dB_t - \left(1 - \eta^{-1}\right) X_{t-} dN_t, \tag{24}$$

where N_t is a Poisson process with intensity λ and $\eta \in (1, \infty)$ is a constant. We can construct an equilibrium characterized by thresholds $f_\dagger$ and f^b. The issuance of long-term debt satisfies

[24] By the envelope theorem, we have

$$j_H^{R'}(f) = -fR'(f + \Delta^b(f)) - R(f + \Delta^b(f)) - b'(f + \Delta^b(f)).$$

Substituting the first-order condition for $\Delta^b(f)$, we get $j_H^{R'}(f) = -R(f + \Delta^b(f))$.

$g(f) = 0$, for $g(f) = 0 \; \forall f \in (f_\dagger, f^b]$, where f^b is the endogenous default boundary. For $f \in [0, f_\dagger]$, the issuance of long-term debt follows

$$g\left(f\right) = \frac{\left(\rho - r\right)\left(p\left(f\right) - p\left(\eta f\right)\right)}{f j''\left(f\right)}. \tag{25}$$

In other words, long-term debt is issued if and only if the amount of outstanding long-term debt is low relative to the operating cash flow. Examining (25) shows that the intuitive reason again falls into the benefits of long-term debt in sharing downside risks, modeled as downward jumps in this case. The difference in prices $p\left(f\right) - p\left(\eta f\right)$ reflects the drop in the long-term debt's price following the downward jump, and the expression thus can be similarly interpreted as (22).

The issuance of short-term debt is also similar to that in section 3. Short-term debt is riskless when $f \leq f_\dagger$ and the amount of issuance is $d(f) = j(\eta f)$. On the other hand, when $f > f_\dagger$, short-term debt becomes risky and the amount of issuance becomes $d(f) = j(f)$. The scaled-value function $j(f)$ satisfies a second-order delay differential equation, which cannot be solved in closed form. Detailed analysis is available in Internet Appendix B.1.

4.4 Managing Interest-Rate Risk

According to Graham and Harvey (2001), corporate firms' choice between long- and short-term debt can depend on the concurrent and anticipated interest rates.[25] In this subsection, we modify the model to study how a borrower issues long- and short-term debt to hedge against fluctuations in interest-rate risk.

The benchmark model presented in section 2 can be extended to consider variations in interest rate, so the discount rate of creditors can differ across the upturn and the downturn. Moreover, we assume investor clienteles exist for specific maturity segments (Vayanos and Vila, 2021), so that the discount rate for short-term creditors can differ from the one for long-term creditors. The short-term creditors' discount rate in state θ is r_θ^0, whereas the long-term creditors discount rate is r_θ^ξ. For simplicity, we assume the equity holders' discount rate ρ is constant across states (incorporating variations in ρ is straightforward). We assume r_θ^0 and r_θ^ξ are both strictly lower than ρ in both states. Moreover, we assume $\mu_H - \mu_L$ is sufficiently large so that in equilibrium, the value function always satisfies $j_H(f) > j_L(f)$, $\forall f$. We can derive the equilibrium following steps similar to the ones in the benchmark model (the details can be found in Internet Appendix B.4). The next proposition presents a characterization of the debt-issuance policy.

Proposition 9 (Debt issuance with interest-rate risk). *The debt-issuance policies are as follows.*

[25] For example, 28.70% of the CFOs claim they issue short-term debt when they are waiting for long-term market interest rates to decline.

1. In the low state $\theta_t = L$,

$$d_L = j_L(f) \tag{26}$$

$$g_L(f) = \frac{\left(r_L^0 - r_L^\xi\right) p_L(f)}{-f p_L'(f)}. \tag{27}$$

2. In the high state $\theta_t = H$,

(a) If $\left(\rho + \lambda_{HL} - r_H^0\right) j_L(f) \geq \left(\rho - r_H^0\right) j_H(f)$,

$$d_H(f) = j_L(f) \tag{28}$$

$$g_H(f) = \frac{\left(\rho - r_H^\xi\right)(p_H(f) - p_L(f)) + \left(r_H^0 - r_H^\xi\right) p_L(f)}{-f p_H'(f)}. \tag{29}$$

(b) If $\left(\rho + \lambda_{HL} - r_H^0\right) j_L(f) < \left(\rho - r_H^0\right) j_H(f)$,

$$d_H(f) = j_H(f) \tag{30}$$

$$g_H(f) = \frac{\left(r_H^0 - r_H^\xi\right) p_H(f)}{-f p_H'(f)}. \tag{31}$$

Whereas the short-term debt policy resembles that in the benchmark model, the issuance of long-term debt is different because now it considers both hedging and interest-rate management. Equation (27) shows that in the low state, when long-term creditors have a lower cost of capital, that is, $r_L^\xi < r_L^0$, the borrower will issue long-term debt. Given the lower interest rate for long-term bonds (due to an inverted yield curve), the firm replaces short-term debt for long-term debt. However, due to the price impact, the adjustment is gradual. The rate of issuance, $g_L(f) \cdot f$, is the marginal flow benefits of replacing short- with long-term debt $(r_L^0 - r_L^\xi)p_L(f)$, scaled by the price impact $-p_L'(f)$. In the high state, the issuance policy depends on whether short-term debt is risky. When short-term debt is risky, the issuance policy in equation (31) resembles the one in the low state. The firm actively trades long-term debt only to adjust due to differences in the discount rate. If $r_H^\xi > r_H^0$, the firm will buy back long-term debt to replace it with short-term debt. When short-term debt is riskless, the issuance policy in equation (29) is driven by two factors. As in the model without interest-rate risk, the numerator term includes the benefits of sharing the downside risk $(\rho - r_H^\xi)(p_H(f) - p_L(f))$. However, an additional motive to issue long-term debt exists when long-term creditors have a lower cost of capital, that is, $r_H^0 > r_H^\xi$, which is captured by the second term in the numerator of equation (29). In the presence of interest-rate risk, the borrower may optimally buy back long-term debt when long-term creditors have a higher discount rate than short-term creditors. Importantly, note that variation in interest rates alone is not sufficient to generate this maturity adjustment. In fact, the issuance policy reduces to the one in the benchmark model if $r_\theta^\xi = r_\theta^0$. Hence, market segmentation is crucial in generating buybacks of long-term debt driven by an upwards-sloping yield curve.

Empirical evidence has shown the real interest rate is countercyclical (Winberry, 2021), and the yield curve is upward (downward) sloping in upturns (downturns). Following these observations, we expect $r_H^0 < r_H^\xi$ and $r_L^0 > r_L^\xi$ to hold. Therefore, $g_L(f) > 0$ so that in downturns, the borrower would like to issue long-term debt for interest-rate risk-management purposes. In the upturn, the borrower would buy back long-term debt and issue short-term debt when she is close to default. When she is far from default, the downward risk-sharing factor continues to motivate her to borrow long. But the upward-sloping yield curve reduces the rate of long-term debt issuance.

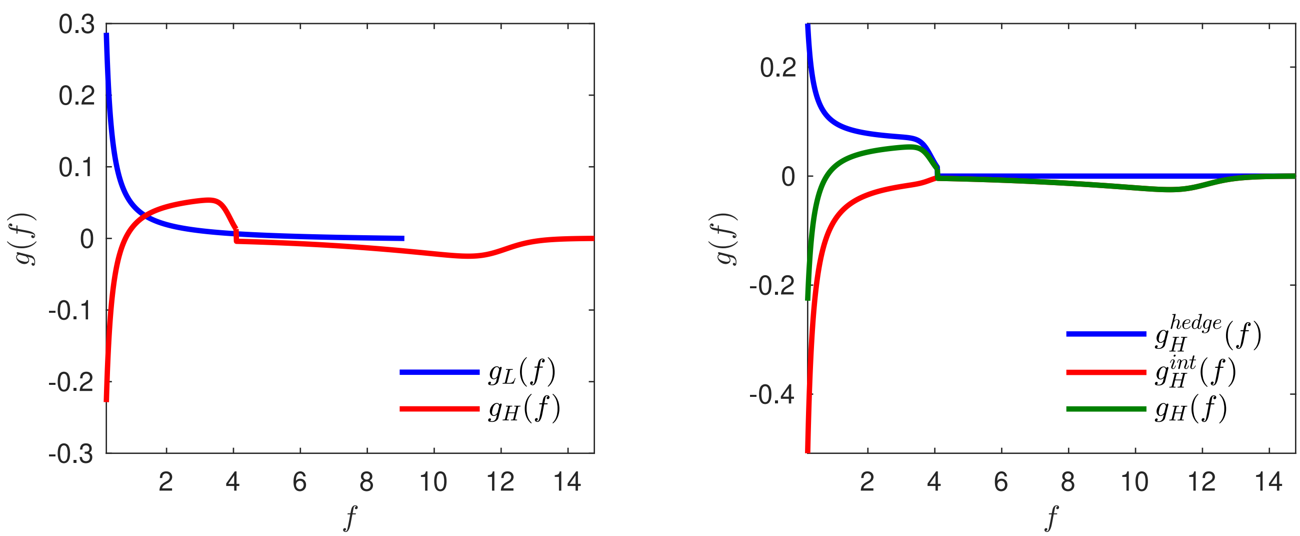


Figure 8: Equilibrium Issuance Rate

This figure plots the issuance function in the model with interest-rate risk. The right panel presents a decomposition of the issuance rate in the hight state. $g^{hedge}(f) = (\rho - r_H^\xi)\,(p_H(f) - p_L(f))\,/\,(-fp'_H\,(f))$ corresponds to the first term in (29), whereas $g^{int}(f) = (r_H^0 - r_H^\xi)p_L(f)/\,(-fp'_H\,(f))$ corresponds to the second term. The parameter values are the following: $\rho = 0.1$, $r_H^0 = 0.01$, $r_H^\xi = 0.015$, $r_L^0 = 0.02$, $r_L^\xi = 0.015$, $\mu_H = 0.015$, $\mu_L = -0.2$, $\sigma = 0.1$, $\xi = 0.1$, $\lambda_{HL} = 0.2$. With these parameters, $f_\dagger = 4.07$, $f_H^b = 14.78$, and $f_L^b = 9.10$.

Figure 8 shows the issuance function of long-term debt. The left panel plots the issuance function in the low and high state. In the low state, the firm issues long-term debt due to the inverted yield curve. In the high state, the interaction between hedging and interest-rate risk creates a nonlinear effect. In the right panel, we decompose the issuance function in the high state into two components. Whereas the first component captures hedging downside risks as in the benchmark model, the second component captures the interest-rate risk. The hedging component is positive, leading to a positive issuance of long-term debt. By contrast, the interest-rate risk component is negative (because $r_H^{\xi} > r_H^0$), leading to the buy-back of long-term debt. For very low and large values of f, the interest-rate risk component dominates and the firm actively buys back long-term debt. For intermediate values of f, the hedging component could dominate, and the firm issues long-term debt. Overall, the long-term debt issuance policy is non-monotone.

4.5 Tax Shields

Our benchmark model has assumed creditors have a lower discount rate than the borrower, which is the reason behind debt issuance. In the context of capital structure, another reason behind debt issuance is tax shields. We show the main mechanism between commitment and risk-sharing continues to hold under tax shields. The only difference is that the differential tax treatments in coupon and principal payments naturally favor long-term over short-term debt. As a result, long-term debt will be issued even in the low state L.

We keep most of the benchmark model unchanged but introduce the following modifications. Specifically, let us assume $\rho = r$ so that the borrower has the same discount rates as creditors. Moreover, let π be the corporate tax rate. The coupon payments of long-term debt are tax deductible, so that effectively the borrower makes coupon payments $r(1-\pi)F_t dt$ during a short period $[t, t+dt)$. For short-term debt, the interest payments are also tax deductible, so that the borrower effectively pays $(1-\pi)y_{t-}D_{t-}dt$ to short-term creditors. Note that for both long- and short-term debt, interest payments but not principals are tax-deductible.

In Internet Appendix B.6, we derive the issuance function of long-term debt. In both states, whereas its price still satisfies $p_\theta(f) = -j_\theta'(f)$, the issuance function is characterized by

$$g_L(f) = \frac{\pi r\,(1 - p_L(f))}{-f p_L'(f)}, \qquad g_H(f) = \begin{cases} \frac{\pi r(1-p_L(f))}{-f p_H'(f)} & f \le f_\dagger \\ \frac{\pi r(1-p_H(f)) - \pi\lambda p_H(f)}{-f p_H'(f)} & f > f_\dagger \end{cases} \tag{32}$$

Compared with Proposition 5, a main difference is long-term debt is in general issued in both states, regardless of the distance to default. The intuitive reason goes to the differential treatment of tax deductions for long- and short-term debt. For long-term debt, the coupon rates are fixed at r, independent of the borrower's default risk. By contrast, for short-term debt, the short rate y_{t-} varies with the borrower's default risk. Effectively, the tax shields for long-term debt rely on the book value of long-term debt, whereas the ones for short-term debt rely on the market value of short-term debt. Given the risk of default, the tax shields from short-term debt are lower. In fact, the expression (32) shows the size of issuance is proportional to $1 - p_L(f)$, the difference between book value and market value of long-term debt.

5 Final Remarks

Our paper offers a theory of debt maturity based on a tradeoff between commitment and risk-sharing. Short-term debt mitigates the lack-of-commitment problem but does not share any downside risk. Long-term debt suffers from dilution but shares the downside risk. In a model with a binary state, risk-sharing is not valued in the downturn or in the upturn if the borrower is close to default. If the borrower is far from default in the upturn, she borrows both kinds of debt.

Our paper has not explicitly modeled covenants, which could in principle mitigate dilutions. Introducing covenants would allow the borrower to reap more benefits from long-term debt issuance. For example, a covenant that restricts the issuance of long-term debt to be lower than some threshold can limit the extent of dilution, thereby increasing the borrower's incentive to issue long-term debt. However, covenants do not eliminate the benefits from short-term debt for two reasons. First, covenants are written on imperfect proxies of the firm's fundamentals, and therefore they don't completely rule out dilution. Second, following small and frequent shocks to cash flows, it is more costly for the borrower to adjust long-term debt. By contrast, short-term debt is more flexible to these shocks. Therefore, our main mechanism between commitment and risk sharing continues to work under covenants.

We have not modeled callable bonds, which are common tools for corporate firms to manage debt maturity in practice. In our model, callable long-term debt would not have a significant impact on the main mechanism. To illustrate, suppose the borrower has the right to call back her outstanding long-term debt at a fixed price $\bar{p}$. Following an analysis similar to that in section 3, we can easily derive that the borrower will call back all the debt if $\bar{p} + j'_\theta(f) < 0$, because the callback price $\bar{p}$ falls below the marginal increase in the continuation value $-j'_\theta(f)$. This condition holds for low levels of f, so that the borrower is expected to call back her debt when f is low but not when f is high. In any case, the tradeoff between commitment and risk-sharing in the choice of debt maturity is unaffected. An additional motive behind debt maturity management in practice is to match assets and liabilities, which we intend to study in follow-up work. Another extension is to introduce collateral and secured debt. In our model, short-term debt is essentially senior to long-term debt because it matures earlier. Fully-collateralized long-term debt is similar to short-term debt, and their payments cannot be diluted. The interaction between maturity and collateral in establishing priority is understudied and deserves more careful analysis in the future.

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Appendix

A Proofs of Section 2

Proof of Lemma 1

Proof. Let $\tau \geq t$ be the time that the state switches from θ to θ'. By the principle of dynamic programming,

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t, \tau)\}} \mathbb{E}_t \left[\int_t^{\tau_b \wedge \tau} e^{-\rho(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-} D_{s-})\, ds + p_s dG_s + dD_s \Big) + e^{-\rho \tau_b \wedge \tau} V_\tau \mathbb{1}_{\{\tau_b \geq \tau\}} \right]$$

$$= \sup_{\tau_b, \{G_s, D_s : s \in [t, \tau)\}} \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-} D_{s-})\, ds + p_s dG_s + dD_s \Big) + e^{-\rho \tau} V_\tau \mathbb{1}_{\{\tau_b \geq \tau\}} \right].$$

Given the definition of J_t, the equity value can be written as $V_t = \max\{J_t - D_{t-}, 0\}$, where the max operator takes into account the borrower's limited liability constraint. In particular, the borrower defaults at time τ if $J_\tau < D_{\tau-}$, so that $V_\tau = \max\{J_\tau - D_{\tau-}, 0\}$. Hence,

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t, \tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big((X_s - (r+\xi)F_s - y_{s-} D_{s-})\, ds + p_s dG_s + dD_s \Big)$$
$$+ e^{-\rho\tau} \max\{J_\tau - D_{\tau-}, 0\} \mathbb{1}_{\{\tau_b \geq \tau\}} \Bigg]$$
$$= \sup_{\tau_b, \{G_s, D_s : s \in [t, \tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(\big(X_s - (r+\xi)F_s - y_{s-} D_{s-} + \lambda_{\theta_s \theta'_s} \max\{J_s - D_{s-}, 0\}\big)\, ds$$
$$+ p_s dG_s + dD_s \Big) \Bigg]$$

Using the integration by parts formula for semi-martingales in Corollary 2 in Section 2.6 of Protter (2005), we get

$$\mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} dD_s \right] = \mathbb{E}_t \left[e^{-(\rho+\lambda)(\tau_b - t)} D_{\tau_b} \right] - D_{t-} + \mathbb{E}_t \left[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} (\rho+\lambda) D_{s-} ds \right].$$

At the time of default, $D_T = 0$. Hence

$$V_t = \sup_{\tau_b, \{G_s, D_s : s \in [t, \tau)\}} \mathbb{E}_t \Bigg[\int_t^{\tau_b} e^{-(\rho+\lambda)(s-t)} \Big(\big(X_s - (r+\xi)F_s + (\rho + \lambda - y_{s-}) D_{s-} + \lambda_{\theta_s \theta'_s} (J_s - D_{s-})^+ \big)\, ds$$
$$+ p_s dG_s + dD_s \Big) \Bigg] - D_{t-}.$$

□

B Proofs of Section 3

B.1 Maximum Principle

Our proofs use repeatedly the Maximum Principle for differential equations. Theorem 3 and 4 from Chapter 1 in Protter and Weinberger (1967) are particularly useful, and we state them below.

Theorem 1 (Theorem 3 in Protter and Weinberger (1967))**.** *If $u(x)$ satisfies the differential inequality*

$$u'' + g(x)u' + h(x)u \geq 0 \tag{33}$$

in an interval $(0, b)$ with $h(x) \leq 0$, if g and h are bounded on every closed subinterval, and if u assumes a nonnegative maximum value M at an interior point c, then $u(x) \equiv M$.

Theorem 2 (Theorem 4 in Protter and Weinberger (1967))**.** *Suppose that u is a nonconstant solution of the differential inequality* (33) *having one-sided derivatives at a and b, that $h(x) \leq 0$, and that g and h are bounded on every closed subinterval of (a, b). If u has a nonnegative maximum at a and if the function $g(x) + (x - a)h(x)$ is bounded from below at $x = a$, then $u'(a) > 0$. If u has a nonnegative maximum at b and if $g(x) - (b - x)h(x)$ is bounded from above at $x = b$, then $u'(b) > 0$.*

Corollary 1. *If u satisfies* (33) *in an interval (a, b) with $h(x) \leq 0$, if u is continuous on $[a, b]$, and if $u\,(a) \leq 0$, $u\,(b) \leq 0$, then $u\,(x) < 0$ in (a, b) unless $u \equiv 0$.*

B.2 Equilibrium

Proof of Proposition 1

Equation (11) is a second-order ODE, and a standard solution takes the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Plugging into the ODE, we can get

$$A_0 = \frac{1}{r - \mu_L}, \quad A_1 = 1, \quad \gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\,(r - \mu_L)}}{\sigma^2} > 1,$$
$$\gamma_2 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\,(r - \mu_L)}}{\sigma^2} < 0$$

The condition $\lim_{f\to 0} j_L(f) < \infty$ implies $A_3 = 0$. We define $\gamma \equiv \gamma_1$. Combining with value-matching and smooth-pasting condition, we get the solution to $j_L(f)$ and f_L^b.

To derive $g_L(f)$, let us first write down the HJB for $p_L(f)$:

$$(r+\xi)\,p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f), \tag{34}$$

where we have used the condition (8). The result of $g_L(f) \equiv 0$ follows from differentiating (11), applying in condition (10), and subtracting (34).

Proof of Proposition 2

We start establishing the uniqueness of the equilibrium.

Uniqueness: For an arbitrary positive function $\tilde{j}$, we define the following operator:

$$\Phi(\tilde{j})(f) \equiv \sup_{\tau\geq 0} \mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t}\left(1-(r+\xi)z_t + \pi(z_t,\tilde{j}(z_t))\right)dt \Big| z_0 = f\right]$$
$$\text{subject to} \quad dz_t = -(\xi+\mu_H)z_t dt - \sigma z_t dB_t,$$

where

$$\pi(z,\tilde{j}) \equiv \max_{d\in[0,\tilde{j}]} (\rho - r)\,d + \mathbb{1}_{\{d\leq j_L(z)\}}\cdot \lambda j_L(z) = \max\{(\rho-r)\,j_L(z) + \lambda j_L(z), (\rho-r)\,\tilde{j}\}.$$

It follows from the HJB equation that the value function j_H is a fixed point $j_H(f) = \Phi(j_H)(f)$. Hence, it is enough to show that the operator Φ is contraction to get that the solution is unique. First, we can notice that Φ is a monotone operator: For any pair of functions $\tilde{j}_1 \geq \tilde{j}_0$, we have $\pi(f,\tilde{j}_1) \geq \pi(f,\tilde{j}_0)$; thus it follows that $\Phi(\tilde{j}_1)(f) \geq \Phi(\tilde{j}_0)(f)$. Next, we can verify that Φ satisfies discounting: For $a \geq 0$, we have

$$\begin{aligned}\pi(z,\tilde{j}+a) &= \max\{(\rho-r)\,j_L(z) + \lambda j_L(z), (\rho-r)\,(\tilde{j}+a)\} \\ &\leq \max\{(\rho-r)\,j_L(z) + \lambda j_L(z) + (\rho-r)a, (\rho-r)\,(\tilde{j}+a)\} = (\rho-r)a + \pi(z,\tilde{j}),\end{aligned}$$

so letting $\tau^*(\tilde{j})$ denote the optimal stopping policy, we have

$$
\begin{aligned}
\Phi(\tilde{j}+a)(f) &= \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t)+a)\right)dt\Big|z_0=f\right] \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j}+a)} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right]+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&\leq \mathbb{E}\left[\int_0^{\tau^*(\tilde{j})} e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t))\right)dt\Big|z_0=f\right]+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \\
&= \Phi(\tilde{j})(f)+\frac{\rho-r}{\hat{\rho}}\mathbb{E}\left[1-e^{-\hat{\rho}\tau^*(\tilde{j}+a)}\Big|z_0=f\right]a \leq \Phi(\tilde{j})(f)+\frac{\rho-r}{\rho+\lambda-\mu_H}a.
\end{aligned}
$$

Thus, the operator Φ is monotone and satisfies discounting, it follows then by Blackwell's sufficiency conditions that Φ is a contraction, which means that there is a unique fixed point $j_H(f)=\Phi(j_H)(f)$.

Next, we provide a the solution to the HJB equation when both long- and short-term debt are issued. That is, when $\lambda>\bar{\lambda}$.

Solution HJB Equation $\lambda>\bar{\lambda}$: Let us first write down the HJBs in different regions as well as the boundary conditions. Specifically, the value function satisfies

$$
\begin{aligned}
(\rho+\lambda-\mu_H)\,j_H(f) &= 1-(r+\xi)\,f+(\rho+\lambda-r)\,j_L(f)-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\;\; f\in[0,f^\dagger] \\
(r+\lambda-\mu_H)\,j_H(f) &= 1-(r+\xi)\,f-(\mu_H+\xi)\,fj_H'(f)+\frac{1}{2}\sigma^2f^2j_H''(f),\;\; f\in[f^\dagger,f_H^b].
\end{aligned}
$$

The boundary conditions are

$$j_H(f\dagger-)=j_H(f\dagger+) \tag{35}$$
$$j_H'(f_\dagger-)=j_H'(f_\dagger+) \tag{36}$$
$$j_H(f_H^b)=0 \tag{37}$$
$$j_H'(f_H^b)=0 \tag{38}$$
$$\lim_{f\to0} j_H(f)<\infty \tag{39}$$
$$j_H\left(f_\dagger\right)=\left(1+\frac{\lambda}{\rho-r}\right)j_L\left(f_\dagger\right). \tag{40}$$

Next, let us supplement the expressions of the auxiliary functions. The solutions to the constants

$\{\phi, \beta_1, \beta_2\}$ and the boundaries $\left\{f_\dagger, f_H^b\right\}$ are provided as we solve the ODE system.

$$u_0(f) \equiv \frac{1}{r-\mu_L}\frac{\rho+\lambda-\mu_L}{\rho+\lambda-\mu_H} - f + \frac{(\rho+\lambda-r)}{(\rho+\lambda-r)+(\mu_H-\mu_L)(\gamma-1)}\frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma,$$

$$u_1(f) \equiv \frac{1}{r+\lambda-\mu_H} - \frac{r+\xi}{r+\xi+\lambda}f, \qquad j_L(f) = \frac{1}{r-\mu_L} - f + \frac{f_L^b}{\gamma}\left(\frac{f}{f_L^b}\right)^\gamma,$$

$$h_0\left(f, f_\dagger, f_H^b\right) = \frac{\left(\frac{f}{f_H^b}\right)^{\beta_1} - \left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}, \qquad h_1\left(f, f_\dagger, f_H^b\right) = \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\left(\frac{f}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}\left(\frac{f}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.$$

The rest of the proof includes three parts. In the first part, we detail the solutions to the ODE system (15) combined with the boundary conditions (35)-(40). In the second part, we prove a single-crossing property and therefore shows that it is optimal for the borrower to issue riskless short-term debt $d_H = j_L(f)$ if and only if $f \le f_\dagger$. Finally, we verify that $j_H(f)$ is a convex function on $\left[0, f_H^b\right]$, so that it is indeed optimal for the borrower to issue long-term debt smoothly.

Part 1: the solution to the ODE system. On $[0, f_\dagger]$, the solution to the ODE taking into condition (39) shows that

$$j_H(f) = u_0(f) + Bf^\phi,$$

where

$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H+\xi+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho+\lambda-\mu_H)}}{\sigma^2} > 1.$$

The coefficient B is pinned down from the value at $j_H(f_\dagger)$

$$B = f_\dagger^{-\phi}\left(j_H(f_\dagger) - u_0(f_\dagger)\right)$$

so that

$$j_H(f) = u_0(f) + \left(j_H(f_\dagger) - u_0(f_\dagger)\right)\left(\frac{f}{f_\dagger}\right)^\phi, \qquad \forall f \in [0, f_\dagger],$$

where $j_H(f_\dagger) = \left(1 + \frac{\lambda}{\rho-r}\right)j_L(f_\dagger)$.

On $\left[f_\dagger, f_H^b\right]$, the solution to the ODE is

$$j_H\left(f\right) = u_1(f) + D_1 f^{\beta_1} + D_2 f^{\beta_2},$$

where

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} > 1$$
$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} < 0$$

Using (37) and (35), we get

$$D_1 = \frac{j_H(f_\dagger) + u_1(f_H^b)\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2} - u_1(f_\dagger)}{(f_H^b)^{\beta_1}\left[\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}\right]}, \qquad D_2 = (f_H^b)^{-\beta_2}\left(-u_1(f_H^b) - D_1(f_H^b)^{\beta_1}\right).$$

so that

$$j_H(f) = u_1(f) + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)h_0\left(f, f_\dagger, f_H^b\right) + u_1(f_H^b)h_1\left(f, f_\dagger, f_H^b\right).$$

It remains to find equations that solve $\left\{f_\dagger, f_H^b\right\}$, which come from the smooth pasting conditions (36) and (38). These two conditions lead to the two-variable, non-linear equation system below

$$u_1(f_H^b)\left[\frac{\beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} - \frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\right] = u_1'(f_H^b)f_H^b + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}} \tag{41}$$

$$\begin{gathered}\left(u_0'(f_\dagger) - u_1'(f_\dagger)\right)f_\dagger + \phi\left(j_H(f_\dagger) - u_0(f_\dagger)\right) = \\ u_1(f_H^b)\frac{\beta_1 - \beta_2}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1+\beta_2} + \left(j_H(f_\dagger) - u_1(f_\dagger)\right)\frac{\beta_1\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \beta_2\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}{\left(\frac{f_\dagger}{f_H^b}\right)^{\beta_1} - \left(\frac{f_\dagger}{f_H^b}\right)^{\beta_2}}.\end{gathered} \tag{42}$$

Part 2: Optimality Short-term Debt Policy We start with the following result, which will be used later on.

Lemma 2. *If $\lambda > \bar{\lambda}$, then*

$$\left(1 + \frac{\lambda}{\rho - r}\right)(\rho + \lambda - \mu_H) > (\rho + \lambda - \mu_L).$$

Proof. See online appendix. □

Next, the following result shows that it is optimal for the borrower to issue $d_H = j_L(f)$ when $f \leq f_\dagger$ and $d_H = j_H(f)$ otherwise.

Lemma 3 (Single-crossing)**.** *There exists a unique* $f_\dagger \in \left(0, f_L^b\right)$ *such that* $(\rho + \lambda - r)j_L(f) \geq (\rho - r)j_H(f)$ *if and only if* $f \leq f_\dagger$*.*

Proof. See online appendix. □

Part 3: Strict convexity of $j_H(f)$ on $\left[0, f_H^b\right]$. The proof relies on a few auxiliary lemmas.

Lemma 4.

$$j'_H(f) \geq -1, \qquad \forall f \in \left[0, f_H^b\right],$$

Proof. See online appendix. □

Lemma 5.

$$f_H^b > \frac{1}{r+\xi} \qquad \textit{and} \qquad \min\left\{j''_H(0), j''_H\left(f_H^b\right)\right\} > 0,$$

Proof. See online appendix. □

Lemma 6.

$$j'''_H(f_\dagger^-) > j'''_H(f_\dagger^+).$$

Proof. See online appendix. □

Now we are ready to verify that the solution to the HJB equation is convex. We differentiate the HJB twice and let $\tilde{u} \equiv f j''_H$

$$(\rho + \lambda + \xi)\tilde{u} = (\rho + \lambda - r) f j''_L - \left(\mu_H + \xi - \sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in [0, f_\dagger] \tag{43}$$

$$(r + \lambda + \xi)\tilde{u} = -\left(\mu_H + \xi - \sigma^2\right) f\tilde{u}' + \frac{1}{2}\sigma^2 f^2 \tilde{u}'' \qquad f \in \left[f_\dagger, f_H^b\right]. \tag{44}$$

By the maximum principle in Theorem 1, $\tilde{u}$ cannot have an interior nonpositive local minimum in $(0, f_\dagger) \cup (f_\dagger, f_H^b)$. Because $\tilde{u}$ is differentiable on $(0, f_\dagger) \cup (f_\dagger, f_H^b)$, the only remaining possibility of a nonpositive minimum is that $\tilde{u}(f_\dagger) < 0$. As $\tilde{u}(0)$ and $\tilde{u}(f_H^b)$ are positive, this requires that $j''_H(f_\dagger-) + f_\dagger j'''_H(f_\dagger-) = \tilde{u}'(f_\dagger-) < \tilde{u}'(f_\dagger+) = j''_H(f_\dagger+) + f_\dagger j'''_H(f_\dagger+)$. From the HJB equation it follows that j_H is twice continuously differentiable at $f_\dagger$, so such a kink would require $j'''_H(f_\dagger^-) < j'''_H(f_\dagger^+)$, which is ruled out by Lemma 6. We can conclude that $\tilde{u}$ does not have an interior nonpositive minimum, so it follows that $\tilde{u}(f) = f j''_H(f) > 0$ on $(0, f_H^b)$.

Solution HJB Equation $\lambda \leq \bar{\lambda}$: In the case that $\lambda \leq \bar{\lambda}$, the firm never issues long term debt, so the analysis reduces to the one in Case 1 for $f' > f_{\dagger}$.

Proof of Proposition 4

Proof. In the low state, short-term debt is riskless and the borrower never defaults. The borrower chooses short-term debt $D_t = Xj_L$, so the value of the firm is

$$\tilde{J}_L(X) = \frac{X}{r - \mu_L}.$$

In the high state, there is a choice between borrowing risky and riskless debt. If she borrows risky short-term debt, again, she would like to take 100% leverage, in which case

$$\hat{J}_H(X) = \frac{X}{r + \lambda - \mu_H}.$$

On the other hand, if she borrows riskless debt up to $X_t j_L$, the firm value is

$$\widehat{J}_H(X) = \frac{X}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho - r + \lambda}{r - \mu_L}\right).$$

From here we get that the value of the firm is

$$\tilde{J}_H(X) = X \max\left\{\frac{1}{r + \lambda - \mu_H}, \frac{1}{\rho + \lambda - \mu_H}\left(1 + \frac{\rho - r + \lambda}{r - \mu_L}\right)\right\}$$

Finally, $\tilde{J}_L(X) \geq J_L(X, F) + p_L(X, F)\, F$ is straightforward given the former is the first-best firm value. In the high state, this is equivalent to proving $\tilde{j}_H \geq j_H(f) + p_H(f)\, f$. It is easily verified that $\tilde{j}_H \geq j_H(0)$ (and the equality holds for both cases no matter the value of λ). The result follows from

$$\frac{d\left[j_H(f) + p_H(f)\, f\right]}{df} = p'_H(f) f < 0.$$

□

Proof of Proposition 5

Proof. In the low state, the HJB becomes

$$\rho\tilde{V}_L = X - (r + \xi)\, F - \frac{\partial \tilde{V}_L}{\partial F}\xi F + \frac{\partial \tilde{V}_L}{\partial X}\mu_L X + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \tilde{V}_L}{\partial X^2}.$$

Again, let $\tilde{V}_L = X\tilde{v}_L$ so that $\frac{\partial \tilde{V}_L}{\partial F} = \tilde{v}'_L$, $\frac{\partial \tilde{V}_L}{\partial X} = \tilde{v}_L - f\tilde{v}'_L$, and $X\frac{\partial^2 \tilde{V}_L}{\partial X^2} = f^2\tilde{v}''_L$. The scaled HJB becomes

$$(\rho - \mu_L)\,\tilde{v}_L = 1 - (r+\xi)\,f - (\mu_L + \xi)\,f\tilde{v}'_L + \frac{1}{2}\sigma^2 f^2 \tilde{v}''_L.$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_L(f) < \infty$, $\tilde{v}_L\left(\tilde{f}^b_L\right) = 0$, and $\tilde{v}'_L\left(\tilde{f}^b_L\right) = 0$, we obtain the solution

$$\tilde{v}_L(f) = \frac{1}{\rho - \mu_L} - \frac{r+\xi}{\rho+\xi} f + \frac{r+\xi}{\rho+\xi}\frac{\tilde{f}^b_L}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}^b_L}\right)^{\tilde{\gamma}}, \qquad \tilde{f}^b_L = \frac{1}{\rho-\mu_L}\frac{\tilde{\gamma}}{\tilde{\gamma}-1}\frac{\rho+\xi}{r+\xi}$$

where

$$\tilde{\gamma} = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho - \mu_L)}}{\sigma^2} > 1.$$

In a smooth equilibrium, $\tilde{p}_L = -\tilde{v}'_L$, and $\tilde{p}_L$ satisfies

$$(r+\xi)\,\tilde{p}_L = (r+\xi) + \left(g_L - \xi - \mu_L + \sigma^2\right) f\tilde{p}'_L + \frac{1}{2}\sigma^2 f^2 \tilde{p}''_L.$$

Differentiating once the HJB for $\tilde{v}_L$, we get $\tilde{g}_L = \frac{(\rho - r)\tilde{p}_L}{f\tilde{v}''_L}$.

In the high state , the scaled HJB becomes

$$(\rho - \mu_H)\,\tilde{v}_H = 1 - (r+\xi)\,f - (\mu_H + \xi)\,f\tilde{v}'_H + \frac{1}{2}\sigma^2 f^2 \tilde{v}''_H + \lambda\left(\tilde{v}_L - \tilde{v}_H\right).$$

Using the conditions $\lim_{f\to 0}\ \tilde{v}_H(f) < \infty$, $\tilde{v}_H\left(\tilde{f}^b_H\right) = 0$, and $\tilde{v}'_H\left(\tilde{f}^b_H\right) = 0$, we obtain the solution

$$\tilde{v}_H(f) = \tilde{u}_0(f) - \tilde{u}_0\left(\tilde{f}^b_H\right)\left(\frac{f}{\tilde{f}^b_H}\right)^{\phi},$$

where

$$\tilde{u}_0(f) = \frac{1}{\rho-\mu_L}\frac{\rho+\lambda-\mu_L}{\rho+\lambda-\mu_H} - \frac{r+\xi}{\rho+\xi} f + \frac{\lambda\frac{r+\xi}{\rho+\xi}}{(\rho+\lambda-\mu_H) + \tilde{\gamma}(\mu_H+\xi) - \frac{1}{2}\sigma^2\tilde{\gamma}(\tilde{\gamma}-1)}\frac{\tilde{f}^b_L}{\tilde{\gamma}}\left(\frac{f}{\tilde{f}^b_L}\right)^{\tilde{\gamma}},$$

and

$$\phi = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(\rho + \lambda - \mu_H\right)}}{\sigma^2} > 1$$

Finally, the boundary $\tilde{f}_H^b$ is pinned down by the smooth-pasting condition

$$\tilde{f}_H^b \tilde{u}_0'\left(\tilde{f}_H^b\right) - \phi \tilde{u}_0\left(\tilde{f}_H^b\right) = 0.$$

In a smooth equilibrium, $\tilde{p}_H = -\tilde{v}_H'$, and $\tilde{p}_H$ satisfies

$$(r+\xi)\,\tilde{p}_H = (r+\xi) + \left(\tilde{g}_H - \xi - \mu_H + \sigma^2\right) f \tilde{p}_H' + \frac{1}{2}\sigma^2 f^2 \tilde{p}_H'' + \lambda\left(\tilde{p}_L - \tilde{p}_H\right).$$

Differentiating once the HJB for $\tilde{v}_H$, we get $\tilde{g}_H = \frac{(\rho-r)\tilde{p}_H}{f\tilde{v}_H''}$.

□

Proof of Proposition 6

Proof. The proof follows from the following lemmas (proofs available in online appendix):

Lemma 7. *$j_L(f) \geq \tilde{v}_L(f)$ and $f_L^b \geq \tilde{f}_L^b$*

Lemma 8. *$\tilde{p}_L(f) \leq p_L(f)$, and the inequality is strict for $\forall f > 0$.*

Lemma 9. *$j_H(f) \geq \tilde{v}_H(f)$ and $f_H^b \geq \tilde{f}_H^b$.*

Lemma 10. *There is $0 \leq \underline{f} \leq f_\dagger \leq \overline{f} \leq \tilde{f}_H^b$ such that $\tilde{p}_H(f) \leq p_H(f)$ on $[0, \underline{f}] \cup [\overline{f}, \tilde{f}_H^b]$*

□

Proof of Proposition 7

Proof. The proof proceeds in a few steps.

State $\theta_t = L$. The HJB in the low state is

$$\begin{aligned}
(\rho+\zeta-\mu_L)\, j_L(f) &= \max_{d \leq j_L(f)} 1 - (r+\xi) f + (\rho+\zeta-y)\, d - (\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
&= 1 - (r+\xi) f + (\rho - r)\, j_L(f) - (\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
\Rightarrow (r+\zeta-\mu_L)\, j_L(f) &= 1 - (r+\xi) f - (\mu_L+\xi) f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f),
\end{aligned}$$

where we have used the condition $y = r + \zeta$ that compensates the disaster risk. Take derivative of the above equation

$$(r+\zeta+\xi)\, j_L'(f) = -\,(r+\xi) - \left(\mu_L + \xi - \sigma^2\right) f j_L''(f) + \frac{1}{2}\sigma^2 f^2 j_L'''(f)\,.$$

The debt price follows

$$(r+\xi+\zeta)\, p_L(f) = (r+\xi) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)$$

From here we get,

$$g_L(f) = \frac{(r+\xi+\zeta)\, j_L'(f) + (r+\xi) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)} = 0.$$

Next, we solve for $j_L(f)$, which follows

$$(r+\zeta-\mu_L)\, j_L(f) = 1 - (r+\xi)\, f - (\mu_L+\xi)\, f j_L'(f) + \frac{1}{2}\sigma^2 f^2 j_L''(f)$$

with boundary condition

$$j_L(f_L^b) = 0, \qquad j_L'(f_L^b) = 0.$$

The solution is

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1},$$

where

$$A_0 = \frac{1}{r+\zeta-\mu_L}, \qquad A_1 = \frac{r+\xi}{r+\zeta+\xi}, \qquad A_2 = \frac{1}{\gamma_1}\frac{r+\xi}{r+\zeta+\xi}\left(f_L^b\right)^{1-\gamma_1}$$

and

$$f_L^b = \frac{\gamma_1}{\gamma_1-1}\frac{1}{r+\zeta-\mu_L}\frac{r+\zeta+\xi}{r+\xi}$$

is the default boundary. Moreover,

$$\gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\,(r+\zeta-\mu_L)}}{\sigma^2} > 1$$

State H. The function of HJB and debt price are the same as benchmark so the function of long-term debt issuance is also the same, i.e, when $f < f_{\dagger}$,

$$g_H(f) = \frac{(\rho - r)\left(j_L'(f) - j_H'(f)\right)}{f j_H''(f)}$$

and $g_H(f) = 0$ when $f > f_{\dagger}$.

Region $(\rho + \lambda - r)\, j_L(f) \geq (\rho - r)\, j_H(f)$. The HJB is

$$(\rho + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f - (\mu_H + \xi)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) + (\rho - r + \lambda) j_L(f).$$

The solution is

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1}$$

where

$$B_0 = \frac{\rho + \lambda + \zeta - \mu_L}{(\rho + \lambda - \mu_H)\,(r + \zeta - \mu_L)}, \qquad B_1 = \frac{(r + \xi)\,(\rho + \lambda + \zeta + \xi)}{(\rho + \xi + \lambda)\,(r + \zeta + \xi)},$$

$$B_2 = \frac{\rho + \lambda - r}{(\rho + \lambda - \mu_H) + (\mu_H + \xi)\,\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1\,(\gamma_1 - 1)} A_2.$$

and

$$\phi_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\,(\rho + \lambda - \mu_H)}}{\sigma^2} > 1$$

The coefficient B_3 will be determined by the boundary conditions.

Region $(\rho + \lambda - r)\, j_L(f) < (\rho - r)\, j_H(f)$. The HJB is

$$(r + \lambda - \mu_H)\, j_H(f) = 1 - (r + \xi)\, f - (\mu_H + \xi)\, f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).$$

The solution is

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2},$$

where

$$D_0 = \frac{1}{r + \lambda - \mu_H}, \qquad D_1 = \frac{r + \xi}{r + \xi + \lambda}$$

and

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} > 1,$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda - \mu_H\right)}}{\sigma^2} < 0$$

The coefficients (D_2, D_3) will be determined by the boundary conditions.

Long-term debt issuance at $f = 0$. In the region $f < f_\dagger$,

$$g_H(f)f = \frac{(\rho - r)\left(j_L'(f) - j_H'(f)\right)}{j_H''(f)} = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1\left(A_2 - B_2\right)f^{\gamma_1 - 1} - \phi_1 B_3 f^{\phi_1 - 1}\right)}{B_2\gamma_1(\gamma_1 - 1)f^{\gamma_1 - 2} + B_3\phi_1\left(\phi_1 - 1\right)f^{\phi_1 - 2}}.$$

Given that $\min\{\phi_1, \gamma_1\} > 1$ it becomes immediately clear that as long as $\min\{\phi_1, \gamma_1\} \geq 2$, $\lim_{f\to 0} g_H(f)f > 0$. To see this, assume $\phi_1 > \gamma_1$. If $\gamma_1 > 2$,

$$\lim_{f\to 0} g_H(f)f = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2\left(A_2 - B_2\right)f^{\gamma_1 - 1} - \phi_1^2 B_3 f^{\phi_1 - 1}\right)}{B_2\gamma_1(\gamma_1 - 1)^2 f^{\gamma_1 - 2} + B_3\phi_1\left(\phi_1 - 1\right)^2 f^{\phi_1 - 2}} = \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda}\right)}{0} = \infty.$$

If $\gamma_1 = 2$,

$$\begin{aligned}\lim_{f\to 0} g_H(f)f &= \frac{(\rho - r)\left(\frac{r+\xi}{r+\zeta+\xi}\frac{\zeta}{\rho+\xi+\lambda} + \gamma_1^2\left(A_2 - B_2\right)f^{\gamma_1 - 1} - \phi_1^2 B_3 f^{\phi_1 - 1}\right)}{B_2\gamma_1(\gamma_1 - 1)^2 f^{\gamma_1 - 2} + B_3\phi_1\left(\phi_1 - 1\right)^2 f^{\phi_1 - 2}} \\ &= \frac{2\left(\rho + \lambda - r - \zeta + \mu_H - \mu_L\right)(\rho - r)\,\zeta\left(r + \zeta + \xi\right)}{\left(\rho + \lambda - r\right)\left(r + \zeta - \mu_L\right)\left(r + \xi\right)\left(\rho + \xi + \lambda\right)}.\end{aligned}$$

The results are similar if $\phi_1 < \gamma_1$. The condition $\zeta > \underline{\zeta}$ is required for $\gamma_1 \geq 2$.

The condition $(\rho + \lambda - r)\, j_L(0) > (\rho - r)\, j_H(0)$. Note that the assumption $f_\dagger > 0$ requires

$$\begin{aligned}(\rho + \lambda - r)\, j_L(0) > (\rho - r)\, j_H(0) \qquad &\Rightarrow \quad (\rho + \lambda - r)\, A_0 - (\rho - r)\, B_0 > 0 \\ &\Rightarrow \quad \zeta < \frac{\lambda}{\rho - r}\left(\rho + \lambda - \mu_H\right) - \left(\mu_H - \mu_L\right).\end{aligned}$$

□

Proof of Proposition 8

Proof. Recall that $f_\dagger$ is determined from $(\rho+r-\lambda)j_L(f)-(\rho+r)j_H(f)=0$. Given that restructuring only occurs to the $\theta=H$ state, $j_L(f)$ does not depend on $R(f)$, $b(f)$, or α. For both statements to hold, using monotone comparative statics, we only need to show that $j_H^2(f)<j_H^1(f)$, where $j_H^i(f)$ is associated with restructuring $R_i(f)$, $b_i(f)$ and α_i. As in proof of Proposition 1, we let

$$\Phi^i(\tilde{j})(f) \equiv \sup_{\tau\geq 0}\mathbb{E}\left[\int_0^\tau e^{-\hat{\rho}t}\left(1-(r+\xi)z_t+\pi(z_t,\tilde{j}(z_t))\right)dt + e^{-\hat{\rho}\tau}j_H^{R_i}(f)\Big|z_0=f\right]$$
$$\text{subject to}\quad dz_t=-(\xi+\mu_H)z_t dt-\sigma z_t dB_t,$$

where

$$j_H^{R_i}(f)\equiv \alpha_i j_H(0)-\min_\Delta\left\{fR_i(f+\Delta)+b_i(f+\Delta)\right\},$$

It can be easily verify that $j_2^{R_2}(f)<j_H^{R_1}(f)$. Let Δ_i be the solution of the maximization problem above, then, noticing that $j_H^1(0)=j_H^2(0)$ (as this is the value as if the firm never issued long term debt), we obtain

$$\begin{aligned}
j_H^{R_1}(f) &\geq \alpha_1 j_H(0)-(fR_1(f+\Delta_2)+b_1(f+\Delta_2))\\
&> \alpha_2 j_H(0)-(fR_1(f+\Delta_2)+b_1(f+\Delta_2))\\
&= j_H^{R_2}(f).
\end{aligned}$$

It follows that, $\Phi^1(\tilde{j})(f)>\Phi^2(\tilde{j})(f)$ for all $f\in[0,f_L^b]$. Hence, Theorem 3 in Milgrom and Roberts (1994) implies that the fixed point $j_H^i(f)=\Phi^i(j_H^i)(f)$ satisfy $j_H^1(f)\geq j_H^2(f)$, which means that $(\rho+r-\lambda)j_L(f)-(\rho+r)j_H^1(f)<(\rho+r-\lambda)j_L(f)-(\rho+r)j_H^2(f)$. It follows from Theorem 1 in Milgrom and Roberts (1994) that $f_{2\dagger}>f_{1\dagger}$.

$\square$

Internet Appendix for "Debt Maturity Management"

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This Internet Appendix contains additional analysis to accompany the manuscript. Section A proves the proof of all lemmas that we state in the appendix. Section B provides the details for Section 4. Section C offers additional examples of how real world corporations manage debt maturity.

A Proofs of lemmas

Proof of Lemma 2

Proof. The proof of Proposition 4 makes it clear that the condition $\lambda > \bar{\lambda}$ guarantees

$$\frac{1}{r+\lambda-\mu_H} < \frac{1}{\rho+\lambda-\mu_H}\left(1+\frac{\rho+\lambda-r}{r-\mu_L}\right).$$

From here, we get

$$\begin{aligned}
&\frac{\rho+\lambda-\mu_H}{r+\lambda-\mu_H} < \frac{\rho+\lambda-\mu_L}{r-\mu_L} \\
\Rightarrow\ &\frac{r+\lambda-\mu_H}{\rho+\lambda-\mu_H} > \frac{r-\mu_L}{\rho+\lambda-\mu_L} \\
\Rightarrow\ &\frac{r-\rho}{\rho+\lambda-\mu_H} > \frac{r-\rho-\lambda}{\rho+\lambda-\mu_L} \\
\Rightarrow\ &\frac{\rho-r}{\rho+\lambda-\mu_H} < \frac{\rho+\lambda-r}{\rho+\lambda-\mu_L} \\
\Rightarrow\ &\left(1+\frac{\lambda}{\rho-r}\right)(\rho+\lambda-\mu_H) > (\rho+\lambda-\mu_L).
\end{aligned}$$

□

Proof of Lemma 3

Proof. Define $a \equiv 1 + \frac{\lambda}{\rho - r}$. The goal is to show $aj_L - j_H > 0$ for $f < f_\dagger$, and vice versa. Let us introduce two operators: for a function u let,

$$L^{0\dagger}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi) f u' - (\rho + \lambda - \mu_H) u$$

$$L^{\dagger b}u \equiv \frac{1}{2}\sigma^2 f^2 u'' - (\mu_H + \xi) f u' - (r + \lambda - \mu_H) u.$$

The HJB in state $\theta = H$ (15), can be therefore written as

$$L^{0\dagger} j_H + 1 - (r + \xi) f + (\rho + \lambda - r) j_L = 0, \ \ f \in (0, f^\dagger)$$

$$L^{\dagger b} j_H + 1 - (r + \xi) f = 0, \ \ f \in (f^\dagger, f^b_H).$$

Similarly, the HJB in state $\theta = L$, (11) can be written as

$$L^{0\dagger} a j_L + a(\mu_H - \mu_L) f j_L' + a(\rho + \lambda - r + \mu_L - \mu_H) j_L + a(1 - (r + \xi) f) = 0$$

$$L^{\dagger b} a j_L + a(\mu_H - \mu_L) f j_L' - a(\mu_H - \mu_L - \lambda) j_L + a(1 - (r + \xi) f) = 0.$$

Therefore, we have

$$L^{0\dagger}(aj_L - j_H) + H(f) = 0$$

$$L^{\dagger b}(aj_L - j_H) + H(f) = 0,$$

where the function $H(f)$ defined as

$$H(f) \equiv a(\mu_H - \mu_L) f j_L' - a(\mu_H - \mu_L - \lambda) j_L(f) + (a - 1)(1 - (r + \xi) f),$$

is convex as

$$H''(f) = \Big[(\mu_H - \mu_L) a \frac{f j_L'''}{j_L''} + (\mu_H - \mu_L) a + (\rho + \lambda - r)(a - 1)\Big] j_L''$$

$$= \Big[(\mu_H - \mu_L) a (\gamma - 1) + (\rho + \lambda - r)(a - 1)\Big] j_L'' > 0.$$

Therefore, $H(f)$ attains its maximum on $[0, f_L^b]$ is attained on the boundary 0 or f_L^b. Evaluating $H(f)$ at the two boundaries and using the hypothesis $\lambda > \bar{\lambda}$, we have

$$\left(1 + \frac{\lambda}{\rho - r}\right)(\rho + \lambda - \mu_H) > (\rho + \lambda - \mu_L),$$

from Lemma 2. Then, we get

$$H(0) = \frac{a(\rho + \lambda - \mu_H) - (\rho + \lambda - \mu_L)}{r - \mu_L} > 0$$
$$H(f_L^b) = (a-1)\left(1 - (r + \xi) f_L^b\right) < 0.$$

Therefore, there exists a unique f' such that $H(f) \geq 0$ on $[0, f']$ and $H(f) \leq 0$ on $[f', f_L^b]$. Depending on whether $f' < f_\dagger$ or not, we need to consider two cases.

- Case 1: $f' > f_\dagger$.
  - On $f \in [0, f_\dagger]$, we know $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f_\dagger]$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f_\dagger)$. Moreover, Theorem 2 and Corollary 1 imply $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$.
  - On $f \in [f', f_L^b]$, we know $H(f) \leq 0$ and $L^{\dagger b}(aj_L - j_H) \geq 0$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Given that $aj_L\left(f_L^b\right) - j_H\left(f_L^b\right) < 0$, $aj_L(f) - j_H(f) \leq 0$, $\forall f \in \left[f', f_L^b\right]$.
  - On $f \in [f_\dagger, f']$. Suppose there exists a $f'' \in (f_\dagger, f')$ such that $aj_L(f'') - j_H(f'') > 0$. Given that $aj_L(f_\dagger) - j_H(f_\dagger) = 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) < 0$, it must be that $aj_L(f) - j_H(f)$ has a nonpositive interior minimum on $[f_\dagger, f'']$. Meanwhile, from $L^{\dagger b}(aj_L(f) - j_H(f)) \leq 0$ for $f \in (f_\dagger, f'')$, we know from Theorem 1 that $aj_L(f) - j_H(f)$ cannot have a nonpositive interior minimum on $(f_\dagger, f'')$, which constitutes a contradiction.
- Case 2: $f' \leq f_\dagger$.

- On $f \in [f_\dagger, f_L^b]$, we know that $H(f) < 0$ and $L^{\dagger b}(aj_L - j_H) \leq 0$. From Theorem 1 and 2, we know $aj_L(f) - j_H(f) \leq 0$ and $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$.
- On $f \in [f', f_\dagger]$, $L^{0\dagger}(aj_L - j_H) \geq 0$ so that $aj_L(f) - j_H(f)$ cannot have a nonnegative interior maximum. Together with $aj_L'(f_\dagger) - j_H'(f_\dagger) \leq 0$, this shows $aj_L(f) - j_H(f) \geq 0$.
- On $f \in [0, f']$, we know that $H(f) > 0$ and $L^{0\dagger}(aj_L - j_H) < 0$ on $[0, f_\dagger]$. Using Theorem 1, we know that $aj_L(f) - j_H(f)$ cannot have a negative interior minimum on $[0, f']$. Given $aj_L(0) - j_H(0) > 0$, we know that $aj_L(f) - j_H(f) > 0$, $\forall f \in [0, f')$.

□

Proof of Lemma 4

Proof. Let $\hat{u} = j_H'(f) + 1$ and the goal is to show $\hat{u}(f) \geq 0$, $\forall f \in \left[0, f_H^b\right]$. We know from (17) that $\hat{u}(0) = 0$ and (38) that $\hat{u}\left(f_H^b\right) = 1$. Moreover, $\hat{u}$ satisfies

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (\rho + \lambda + \xi)\hat{u} = -(\rho + \lambda - r)(j_L' + 1) < 0, \qquad f \in [0, f_\dagger]$$

$$\frac{1}{2}\sigma^2 f^2 \hat{u}'' - \left(\mu_H + \xi - \sigma^2\right) f\hat{u}' - (r + \lambda + \xi)\hat{u} = -\lambda < 0 \qquad f \in [f_\dagger, f_H^b].$$

By Theorem 1, we know $\hat{u}(f)$ cannot admit a nonpositive interior minimum on $\left[0, f_H^b\right]$, which rules out the possibility that $\hat{u}(f) < 0$. □

Proof of Lemma 5

Proof. For any $f \leq \frac{1}{r+\xi}$, there is a naive policy that the equity holder does not issue any long-term debt, in which case the scaled net cash flow rate becomes $1 - (r + \xi) f + (\rho + \lambda - y) d > 0$. In other words, the naive policy generates positive cash flow to the borrower, so that it is never optimal to default. Therefore, it must be that $f_H^b > \frac{1}{r+\xi}$. Plugging (37) and (38) into (15), we get $j_H''\left(f_H^b\right)$ whenever $f_H^b > \frac{1}{r+\xi}$.

Next, let us turn to prove that $j_H''(0) \geq 0$. Let us define $u \equiv j_H'$ and differentiate the HJB once

$$\frac{1}{2}\sigma^2 f^2 u'' - \left(\mu_H + \xi - \sigma^2\right) f u' - (\rho + \lambda + \xi)\, u = (r + \xi) - (\rho + \lambda - r)\, j_L'.$$

Moreover, let z be the solution to

$$\frac{1}{2}\sigma^2 f^2 z'' - \left(\mu_H + \xi - \sigma^2\right) f z' - (\rho + \lambda + \xi)\, z = (r + \xi) - (\rho + \lambda - r)\, j_L'(0)$$

with boundary conditions

$$\lim_{f \downarrow 0} z(f) < \infty$$

$$z(f_\dagger) = u(f_\dagger) = j_H'(f_\dagger).$$

The solution is

$$z(f) = -\frac{r+\xi}{\rho+\lambda+\xi} + \frac{(\rho+\lambda-r)\, j_L'(0)}{\rho+\lambda+\xi} + \left(j_H'(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-r)\, j_L'(0)}{\rho+\lambda+\xi}\right)\left(\frac{f^{\omega_1}}{f_\dagger}\right)^{\omega_1},$$

where

$$\omega_1 = \frac{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu_H + \xi - \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2(\rho + \lambda + \xi)}}{\sigma^2} > 0.$$

Let $\delta(f) = z - u$. It is easily verified that $\delta(0) = 0$ and $\delta(f_\dagger) = 0$. Moreover, δ satisfies

$$\frac{1}{2}\sigma^2 f^2 \delta'' - \left(\mu_H + \xi - \sigma^2\right) f \delta' - (\rho + \lambda + \xi)\, \delta = (\rho + \lambda - r)\left(j_L'(f) - j_L'(0)\right) \geq 0.$$

By Theorem 1, δ cannot have an interior nonnegative maximum, and the maximum is attained at $f = 0$. Theorem 2 further implies $\delta'(0) < 0$ so $u'(0) > z'(0)$. Finally, we know that

$$z'(f) = \omega_1 \left(j_H'(f_\dagger) + \frac{r+\xi}{\rho+\lambda+\xi} - \frac{(\rho+\lambda-r)\, j_L'(0)}{\rho+\lambda+\xi}\right) f_\dagger^{-\omega_1} f^{\omega_1 - 1} = \omega_1 \left(j_H'(f_\dagger) + 1\right) f_\dagger^{-\omega_1} f^{\omega_1 - 1},$$

which implies $z'(f) \geq 0$ given that $j_H'(f_\dagger) \geq -1$. Therefore, $u'(0) = j_H''(0) > 0$. $\square$

Proof of Lemma 6

Proof. We differentiate the HJB (15) once and take the difference between the left limit $f_\dagger-$ and right limit $f_\dagger+$

$$\frac{1}{2}\sigma^2 f^2 (j_H'''(f_\dagger+) - j_H''''(f_\dagger-)) = (\rho - r)\left[a j_L'(f_\dagger) - j_H'(f_\dagger)\right],$$

where $a \equiv 1 + \frac{\lambda}{\rho - r}$ The proof of Proposition 3 shows $a j_L'(f_\dagger) - j_H'(f_\dagger) < 0$ so that $j_H''''(f_\dagger-) > j_H'''(f_\dagger+)$. ☐

Proof of Lemma 7

Proof.

$$\begin{aligned}
J_L(X,F) &= \sup_{\tau_b, \{G_s, D_s \le J_L(X_s,F_s)\}} \mathbb{E}\left[\int_t^{\tau_b} e^{-\rho(s-t)}\left(X_s - (r+\xi)F_s + (\rho - y_{s-})D_{s-}\right)ds + p_s dG_s\Big)\Big| X_t = X,\ F_t = F\right] \\
&> \sup_{\tau_b, \{G_s\}} \mathbb{E}\left[\int_t^{\tau_b} e^{-\rho(s-t)}\left(X_s - (r+\xi)F_s\right)ds + p_s dG_s\Big)\Big| X_t = X,\ F_t = F\right] \\
&= \sup_{\tau_b} \mathbb{E}\left[\int_t^{\tau_b} e^{-\rho(s-t)}\left(X_s - (r+\xi)F_s\right)ds\Big)\Big| X_t = X,\ F_t = F\right] = \tilde{V}_L(X,F) \qquad (45)
\end{aligned}$$

where the inequality comes from that $(\rho - y_{s-})D_{s-}$ is positive.It implies that the value function in the low state in the benchmark is higher than that in the economy with long-term debt only for any (X, F).

Let T_L and $\tilde{\tau}_{bL}$ be the default time in both economies in the low state. We have that $\tilde{v}_L(f_{\tilde{\tau}_{bL}}) = 0 \Rightarrow j_L(f_{\tilde{\tau}_{bL}}) > 0$, which means that $T_L > \tilde{\tau}_{bL}$. It follows then that $f_L^b > \tilde{f}_L^b$. ☐

Proof of Lemma 8

Proof. Given $f_L^b > \tilde{f}_L^b$ and $\tilde{g}_L(f) > g_L(f) = 0$ and $T_L > \tilde{T}_L$, the result follows from the definition

$$p_L = \mathbb{E}\left[\int_t^{\tau_\xi \wedge T_L} e^{-r(s-t)} r ds + e^{-r(T_L - t)} \mathbb{1}_{T_L > \tau_\xi}\right]$$
$$\tilde{p}_L = \mathbb{E}\left[\int_t^{\tau_\xi \wedge \tilde{\tau}_{bL}} e^{-r(s-t)} r ds + e^{-r(\tilde{\tau}_{bL} - t)} \mathbb{1}_{\tilde{\tau}_{bL} > \tau_\xi}\right].$$

□

Proof of Lemma 9

Proof. We define an auxiliary process z_t that satisfies $z_0 = f_0$ and

$$dz_t = -(\xi + \mu_H) z_t dt - \sigma z_t dB_t.$$

The value functions are equivalently

$$j_H(z) \equiv \sup_{\tau_b, d_t \in [0, j_H(z_t)]} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) z_t + (\rho - r)\, d_t + \mathbb{1}_{\{d_t \le j_L(f_t)\}} \cdot \lambda j_L(z_t)\right) dt\right]$$
$$\tilde{v}_H(z) \equiv \sup_{\tau_b} \mathbb{E}\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) z_t + \lambda \tilde{v}_L(z_t)\right) dt\right],$$

where $\hat{\rho} \equiv \rho + \lambda - \mu_H$. Note that

$$\begin{aligned} j_H(f) &\ge \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + (\rho - r)\, j_L(f_t) + \lambda j_L(f_t)\right) dt\right] \\ &> \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + \lambda j_L(f_t)\right) dt\right] \\ &\ge \sup_{\tau_b} E\left[\int_0^{\tau_b} e^{-\hat{\rho} t} \left(1 - (r+\xi) f_t + \lambda \tilde{v}_L(f_t)\right) dt\right] \\ &= \tilde{v}_H(f). \end{aligned}$$

It follows that $f_H^b > \tilde{f}_H^b$. □

Proof of Lemma 10

Proof. Define $\Delta_H(f) = p_H(f) - \tilde{p}_H(f)$. We get

- On $f \in (0, f_\dagger)$ It should be

$$(\rho + \lambda + \xi)\, \Delta_H(f) = (\rho - r + \lambda)\, \Delta_L(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) + \frac{1}{2}\sigma^2 f^2 \Delta_H''(f),$$

where we have used the condition

$$g_H f p_H' - \tilde{g}_H f \tilde{p}_H' = -(\rho - r)(p_H - p_L) + (\rho - r)(\tilde{p}_H - \tilde{p}_L) = -(\rho - r)\, \Delta_H + (\rho - r)\, \Delta_L.$$

From here we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \lambda + \xi)\, \Delta_H(f) \leq 0.$$

By the maximum principle, $\Delta_H(f)$ cannot have a nonpositive minimum. In addition, $\Delta_H(0) = 1 - \frac{r+\xi}{\rho+\xi} > 0$. Hence, $\Delta_H(f)$ single crosses 0 from above when f starts from $f = 0$.

- On $f \in (f_\dagger, \tilde{f}_H^b)$

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \xi)\, \Delta_H(f) + (\rho - r - \lambda)\, p_H(f) + \lambda\left(\tilde{p}_H(f) - \tilde{p}_L(f)\right) = 0.$$

Given $\rho > r + \lambda$, we get that

$$\frac{1}{2}\sigma^2 f^2 \Delta_H''(f) - \left(\xi + \mu_H - \sigma^2\right) f \Delta_H'(f) - (\rho + \xi)\, \Delta_H(f) \leq 0.$$

It follows that $\Delta_H(f)$ cannot have a nonpositive minimum. In addition, $\Delta_H(\tilde{f}_H^b) \geq 0$ since $f_H^b \geq \tilde{f}_H^b$. Hence, $\Delta_H(f)$ single crosses 0 from below when f goes to $f = \tilde{f}_H^b$.

$\square$

B Proofs of Section 4

B.1 Jump Risk

The value function satisfies the equation

$$(\rho+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f) \\ +\max\Big\{(\rho-r)j(\eta f)+\lambda j(\eta f),(\rho-r)j(f)\Big\}$$

In the region $(f_\dagger, f^b)$, the equation reduces to

$$(r+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f),$$

while in the region $(f_\dagger, f^b)$, the equation reduces to

$$(\rho+\lambda-\mu)\,j(f) = 1-(r+\xi)\,f-(\mu+\xi)\,fj'(f)+\frac{1}{2}\sigma^2 f^2 j''(f)+(\rho+\lambda-r)j(\eta f).$$

The threshold $f_\dagger$ is determined by the condition

$$(\rho+\lambda-r)j(\eta f_\dagger) = (\rho-r)j(f_\dagger)$$

B.2 Model solution under $\lambda_{LH} > 0$

B.2.1 Region $(\rho+\lambda_{HL}-r)\,j_L(f) \geq (\rho-r)\,j_H(f)$:

Guess a solution of the form

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2} + A_4 f^{\gamma_3} + A_5 f^{\gamma_4}$$

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2} + B_4 f^{\gamma_3} + B_5 f^{\gamma_4}.$$

Plugging into the ODE, we can get

$$A_0 = \frac{\rho + \lambda_{HL} + \lambda_{LH} - \mu_H}{(\rho + \lambda_{HL} - \mu_H)(r - \mu_L) + \lambda_{LH}(r - \mu_H)},$$

$$B_0 = \frac{1 + (\rho + \lambda_{HL} - r) A_0}{\rho + \lambda_{HL} - \mu_H}.$$

and $A_1 = B_1 = 1$.

In addition, for any $i = 1, 2, 3, 4$

$$(r + \lambda_{LH} - \mu_L) A_{i+1} = -(\mu_L + \xi) A_{i+1}\gamma_i + \lambda_{LH} B_{i+1} + \frac{1}{2}\sigma^2 A_{i+1}\gamma_i(\gamma_i - 1)$$

$$(\rho + \lambda_{HL} - \mu_H) B_{i+1} = (\rho + \lambda_{HL} - r) A_{i+1} - (\mu_H + \xi) B_{i+1}\gamma_i + \frac{1}{2}\sigma^2 B_{i+1}\gamma_i(\gamma_i - 1)$$

If we multiply the equation for A_2 by γ_1, we get

$$(r + \lambda_{LH} - \mu_L)\gamma_1 A_2 = -(\mu_L + \xi) A_2\gamma_1^2 + \lambda_{LH} B_2\gamma_1 + \frac{1}{2}\sigma^2 A_2\gamma_1^2(\gamma_1 - 1)$$

when $\lambda_{LH} \neq 0$, from the equation for A_2 we have

$$\lambda_{LH} B_2 = \left[(r + \lambda_{LH} - \mu_L) + (\mu_L + \xi)\gamma_1 - \frac{1}{2}\sigma^2\gamma_1(\gamma_1 - 1)\right] A_2.$$

Substituting in the equation for B_2, we get

$$B_2\gamma_1 = \frac{(\rho + \lambda_{HL} - r)}{(\mu_H + \xi)} A_2 - \frac{\left(\rho + \lambda_{HL} - \mu_H - \frac{1}{2}\sigma^2\gamma_1(\gamma_1 - 1)\right)}{\lambda_{LH}(\mu_H + \xi)}\left[(r + \lambda_{LH} - \mu_L) + (\mu_L + \xi)\gamma_1 - \frac{1}{2}\sigma^2\gamma_1(\gamma_1 - 1)\right] A_2.$$

Substituting back in the equation for A_2 (multiplied by γ_1), and canceling A_2, we get

$$0 = -\frac{1}{4}\sigma^4\gamma_1^4 + \left(\frac{1}{2}\sigma^2\left(\mu_L + \mu_H + 2\xi + \sigma^2\right)\right)\gamma_1^3 +$$

$$+\left[-\left(\mu_L + \xi\right)\left(\mu_H + \xi\right) + \frac{1}{2}\sigma^2\left(\rho + \lambda_{HL} + r + \lambda_{LH} - 2\left(\mu_L + \mu_H + \xi\right)\right) - \frac{1}{4}\sigma^4\right]\gamma_1^2 +$$

$$+\left[-\left(\frac{1}{2}\sigma^2 + \mu_H + \xi\right)\left(r + \lambda_{LH} - \mu_L\right) - \left(\rho + \lambda_{HL} - \mu_H\right)\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\right]\gamma_1$$

$$+\left(\mu_H - r\right)\lambda_{LH} - \left(\rho + \lambda_{HL} - \mu_H\right)\left(r - \mu_L\right)$$

This equation has four roots. Under reasonable parameters, all the four roots are real and two of them are positive. Therefore, the transversality conditions

$$\lim_{f \to 0} j_H(f) < \infty,$$

$$\lim_{f \to 0} j_L(f) < \infty,$$

implies that $A_4 = A_5 = B_4 = B_5 = 0$.

B.2.2 Region $(\rho + \lambda_{HL} - r)\, j_L(f) < (\rho - r)\, j_H(f)$:

We guess the solution follows

$$j_L(f) = C_0 - C_1 f + C_2 f^{\beta_1} + C_3 f^{\beta_2} + C_4 f^{\beta_3} + C_5 f^{\beta_4}$$

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.$$

From the HJB equation of $j_H(f)$, we get

$$D_0 = \frac{1}{r + \lambda_{HL} - \mu_H}$$

$$D_1 = \frac{r + \xi}{r + \xi + \lambda_{HL}}$$

Moreover,

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2},$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r - \lambda_{HL}\right)}}{\sigma^2}.$$

solves

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta + \mu_H - r - \lambda_{HL} = 0$$

Next, we look at the solution for j_L and we can get

$$C_0 = \frac{1 + \lambda_{LH} D_0}{r + \lambda_{LH} - \mu_L}$$

$$C_1 = \frac{r + \xi + \lambda_{LH} D_1}{r + \xi + \lambda_{LH}}$$

$$C_2 = \frac{\lambda_{LH} D_2}{r + \lambda_{LH} - \mu_L + \left(\mu_L + \xi\right)\beta_1 - \frac{1}{2}\sigma^2\left(\beta_1 - 1\right)\beta_1}$$

$$C_3 = \frac{\lambda_{LH} D_3}{r + \lambda_{LH} - \mu_L + \left(\mu_L + \xi\right)\beta_2 - \frac{1}{2}\sigma^2\left(\beta_2 - 1\right)\beta_2}.$$

Moreover,

$$\beta_3 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} - \mu_L\right)}}{\sigma^2} > 1,$$

$$\beta_4 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r + \lambda_{LH} - \mu_L\right)}}{\sigma^2} < 0.$$

solves

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\beta - \left(r + \lambda_{LH} - \mu_L\right) = 0$$

B.2.3 Equilibrium Issuance

In the region $f < f^{\dagger}$ we have (where $A_1 = B_1 = 1$)

$$j_L(f) = A_0 - f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}$$
$$j_H(f) = B_0 - f + B_2 f^{\gamma_1} + B_3 f^{\gamma_2}.$$

so

$$g_H(f) = \frac{(\rho - r)\left(j_L'(f) - j_H'(f)\right)}{f j_H''(f)} = \frac{(\rho - r)\left(\gamma_1 (A_2 - B_2) f^{\gamma_1 - 1} + \gamma_2 (A_3 - B_3) f^{\gamma_2 - 1}\right)}{B_2 \gamma_1 (\gamma_1 - 1) f^{\gamma_1 - 1} + B_3 \gamma_2 (\gamma_2 - 1) f^{\gamma_2 - 1}}$$

B.3 Shock elasticity

The next Proposition provides a formula for computing the shock elasticity.

Proposition 10. *Consider the process $Y_t = \varphi(f_t, \theta_t)$. The shock elasticity of Y_t is*

$$\varepsilon_Y^B(t, f, \theta) \equiv \frac{\mathbb{E}[\varphi'(f_t, \theta_t) \mathcal{D}_0 f_t \mathbb{1}_{\{t < \tau_b\}} | f_0 = f, \theta_0 = \theta]}{\mathbb{E}[\varphi(f_t, \theta_t) \mathbb{1}_{\{t < \tau_b\}} | f_0 = f, \theta_0 = \theta]},$$

where

$$\mathcal{D}_0 f_t = -\sigma f_t \exp\left\{ \int_0^t g_{\theta_s}'(f_s) f_s ds \right\},$$

is the Malliavin derivative of f_t. In particular,

$$\varepsilon_f^B(t, f, \theta) = -\sigma \frac{\mathbb{E}\left[f_t \exp\left\{ \int_0^t g_{\theta_s}'(f_s) f_s ds \right\} \mathbb{1}_{\{t < \tau_b\}} \middle| f_0 = f, \theta_0 = \theta \right]}{\mathbb{E}\left[f_t \mathbb{1}_{\{t < \tau_b\}} \middle| f_0 = f, \theta_0 = \theta \right]}$$

$$\varepsilon_d^B(t, f, \theta) = -\sigma \frac{\mathbb{E}\left[d_{\theta_t}'(f_t) f_t \exp\left\{ \int_0^t g_{\theta_s}'(f_s) f_s ds \right\} \mathbb{1}_{\{t < \tau_b\}} \middle| f_0 = f, \theta_0 = \theta \right]}{\mathbb{E}\left[d_{\theta_t}(f_t) \mathbb{1}_{\{t < \tau_b\}} \middle| f_0 = f, \theta_0 = \theta \right]}.$$

The shock elasticities of F_t and D_t are given by

$$\varepsilon_F^B(t,f,\theta) = \frac{\mathbb{E}\left[X_t\left(\sigma f_t + \mathcal{D}_0 f_t\right)\mathbb{1}_{\{t<\tau_b\}}\middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}{\mathbb{E}\left[X_t f_t \mathbb{1}_{\{t<\tau_b\}}\middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}$$

$$\varepsilon_D^B(t,f,\theta) = \frac{\mathbb{E}\left[X_t\left(\sigma d_t + d'_{\theta_t}(f_t)\mathcal{D}_0 f_t\right)\mathbb{1}_{\{t<\tau_b\}}\middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}{\mathbb{E}\left[X_t d_t \mathbb{1}_{\{t<\tau_b\}}\middle| f_0 = f, X_0 = X, \theta_0 = \theta\right]}.$$

Proof. We start deriving the Malliavin derivative of f_t. Using Ito's Lemma we get a stochastic differential equation for $\log(f_t)$

$$d\log(f_t) = \left(g_{\theta_t}\left(e^{\log(f_t)}\right) - \xi - \mu_{\theta_t} + \frac{1}{2}\sigma^2\right)dt - \sigma dB_t.$$

The Malliavin derivative of $\log(f_t)$ is given by the solution to the differential equation

$$d\mathcal{D}_0 \log f_t = \left(g'_{\theta_t}(f_t)f_t\right)\mathcal{D}_0 \log f_t dt,$$

with initial condition $\mathcal{D}_0 \log f_0 = -\sigma$ (Hu et al., 2019). It follows from the previous equation that

$$\mathcal{D}_0 \log f_t = -\sigma \exp\left\{\int_0^t g'_{\theta_s}(f_s)f_s ds\right\}.$$

From here, we can find the derivative of f_t using the chain rule

$$\mathcal{D}_0 f_t = f_t \mathcal{D}_0 \log f_t = -\sigma f_t \exp\left\{\int_0^t g'_{\theta_s}(f_s)f_s ds\right\}.$$

It follows that $\mathcal{D}_0 Z_t = \mathcal{D}_0\varphi(f_t,\theta_t) = \varphi'(f_t,\theta_t)\mathcal{D}_0 f_t$. Next, we look at the Malliavin derivatives of F_t and D_t. These derivatives can be obtained using the product rule, which yield $\mathcal{D}_0 F_t = X_t\mathcal{D}_0 f_t + f_t \mathcal{D}_0 X_t$. Noticing that

$$d\log X_t = \left(\mu_{\theta_t} - \frac{1}{2}\sigma^2\right)dt + \sigma dB_t,$$

we get $\mathcal{D}_0 \log X_t = \sigma$. Thus, using the chain rule we get $\mathcal{D}_0 X_t = \sigma X_t$. It follows that $\mathcal{D}_0 F_t = X_t \left(\mathcal{D}_0 f_t + \sigma f_t\right)$. Similarly, for short-term debt we have that $\mathcal{D}_0 D_t = X_t \mathcal{D}_0 d_t + d_t \mathcal{D}_0 X_t$, so $\mathcal{D}_0 D_t = \sigma\left(X_t d_t + d'_{\theta_t}(f_t)\mathcal{D}_0 f_t\right)$. $\square$

B.4 Interest-rate risk

We offer detailed solution to the model with interest-rate risk introduced in subsection 4.4.

B.4.1 Model and HJB.

Let creditors' discount rates be $\left\{r^0_L, r^0_H, r^\xi_L, r^\xi_H\right\}$, where the subscripts stand for the state $\theta_t \in \{H, L\}$ and the superscripts represent the discount rate of short- and long-term creditors . We assume $\max\left\{r^0_L, r^0_H, r^\xi_L, r^\xi_H\right\} < \rho$. Moreover, we assume $\mu_H - \mu_L$ is sufficiently large so that in equilibrium, the value function always satisfies $j_H(f) > j_L(f)$, $\forall f$. Given so, the short rate is

$$y_H\left(d, f\right) = r^0_H + \lambda_{HL}\mathbb{1}_{d > j_L(f)} \tag{46}$$

$$y_L\left(d, f\right) = r^0_L \tag{47}$$

The HJBs are

$$\begin{aligned}
\left(\rho + \lambda_{LH} - \mu_L\right) j_L(f) = \max_{d_L \le j_L(f)}\ & 1 - \left(r^\xi_L + \xi\right) f + \left(\rho + \lambda_{LH} - y_L\right) d_L - \left(\mu_L + \xi\right) f j'_L(f) + \lambda_{LH}(j_H(f) - d_L)^+ \\
& + \frac{1}{2}\sigma^2 f^2 j''_L\left(f\right) \\
\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = \max_{d_H \le j_H(f)}\ & 1 - \left(r^\xi_H + \xi\right) f + \left(\rho + \lambda_{HL} - y_H\right) d_H - \left(\mu_H + \xi\right) f j'_H(f) + \lambda_{HL}(j_L(f) - d_H)^+ \\
& + \frac{1}{2}\sigma^2 f^2 j''_H\left(f\right)
\end{aligned}$$

Given $j_H(f) > j_L(f)$, $\forall f$, it continues to hold in state L that $d_L = j_L(f)$. In state H, there are still two candidates, $d_H \in \{j_L(f), j_H\left(f\right)\}$, and $d_H = j_L(f)$ iff

$$\left(\rho + \lambda_{HL} - r^0_H\right) j_L \ge \left(\rho - r^0_H\right) j_H.$$

The HJB in the L state becomes,

$$\left(r_L^0 + \lambda_{LH} - \mu_L\right) j_L = 1 - \left(r_L^\xi + \xi\right) f - \left(\mu_L + \xi\right) f j_L' + \lambda_{LH} j_H + \frac{1}{2}\sigma^2 f^2 j_L''. \tag{48}$$

In the H state, the HJB becomes

$$\begin{aligned}\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - \left(\mu_H + \xi\right) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) \\ + \max\left\{(\rho + \lambda_{HL} - r_H^0) j_L, (\rho - r_H^0) j_H\right\}.\end{aligned}$$

Finally, the state variable evolves according to

$$df_t = \left(g_{\theta_t}(f_t) - \mu_{\theta_t} - \xi + \sigma^2\right) f_t dt - \sigma f_t dB_t.$$

B.4.2 Debt Price and Issuance.

The prices of debt satisfy the following HJB:

$$\left(r_L^\xi + \xi + \lambda_{LH}\right) p_L(f) = \left(r_L^\xi + \xi\right) + \lambda_{LH} p_H(f) + \left(g_L(f) - \xi - \mu_L + \sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f)$$

$$\left(r_H^\xi + \xi + \lambda_{HL}\right) p_H(f) = \left(r_H^\xi + \xi\right) + \lambda_{HL} p_L(f) \mathbb{1}_{d_H \le j_L(f)} + \left(g_H(f) - \xi - \mu_H + \sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)$$

From here we get,

$$\begin{aligned} g_L(f) &= \frac{\left(r_L^\xi + \xi + \lambda_{LH}\right) p_L(f) - \left(r_L^\xi + \xi\right) - \lambda_{LH} p_H(f) + \left(\xi + \mu_L - \sigma^2\right) f p_L'(f) - \frac{1}{2}\sigma^2 f^2 p_L''(f)}{f p_L'(f)} \\ &= \frac{\left(r_L^\xi + \xi + \lambda_{LH}\right) j_L'(f) + \left(r_L^\xi + \xi\right) - \lambda_{LH} j_H'(f) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)}. \end{aligned}$$

$$\begin{aligned} g_H(f) &= \frac{\left(r_H^\xi + \xi + \lambda_{HL}\right) p_H(f) - \left(r_H^\xi + \xi\right) - \lambda_{HL} p_L(f) \mathbb{1}_{d_H \le j_L(f)} + \left(\xi + \mu_H - \sigma^2\right) f p_H'(f) - \frac{1}{2}\sigma^2 f^2 p_H''(f)}{f p_H'(f)} \\ &= \frac{\left(r_H^\xi + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^\xi + \xi\right) - \lambda_{HL} j_L'(f) \mathbb{1}_{d_H \le j_L(f)} + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f)}{f j_H''(f)}. \end{aligned}$$

We differentiate the HJB in the low state

$$\left(r_L^\xi + \xi + \lambda_{LH}\right) j_L'(f) + \left(r_L^\xi + \xi\right) - \lambda_{LH} j_H'(f) + \left(\xi + \mu_L - \sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f) = \left(r_L^\xi - r_L^0\right) j_L'(f).$$

Hence, we get

$$g_L(f) = \frac{\left(r_L^\xi - r_L^0\right) j_L'(f)}{f j_L''(f)} = \frac{\left(r_L^\xi - r_L^0\right) j_L'(f)}{-f p_L'(f)}.$$

Case 1: $\left(\rho + \lambda_{HL} - r_H^0\right) j_L \geq \left(\rho - r_H^0\right) j_H$**.** In this case, $d_H(f) = j_L(f)$, and we differentiate the HJB in the high state:

$$\begin{aligned}\left(r_H^\xi + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^\xi + \xi\right) - \lambda_{HL} j_L'(f) + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) \\ = \left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^\xi\right) j_H'(f)\end{aligned}$$

Hence,

$$g_H(f) = \frac{\left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^\xi\right) j_H'(f)}{f j_H''(f)} = \frac{\left(\rho - r_H^0\right) j_L'(f) - \left(\rho - r_H^\xi\right) j_H'(f)}{-f p_H'(f)}$$

Case 2: $\left(\rho + \lambda_{HL} - r_H^0\right) j_L < \left(\rho - r_H^0\right) j_H$**.** In this case, $d_H(f) = j_H(f)$, and we differentiate the HJB in the high state:

$$\left(r_H^\xi + \xi + \lambda_{HL}\right) j_H'(f) + \left(r_H^\xi + \xi\right) + \left(\xi + \mu_H - \sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f) = \left(r_H^\xi - r_H^0\right) j_H'(f)$$

so

$$g_H(f) = \frac{\left(r_H^\xi - r_H^0\right) j_H'(f)}{f j_H''(f)} = \frac{\left(r_H^\xi - r_H^0\right) j_H'(f)}{-f p_H'(f)}.$$

B.4.3 Solve the value function under $\lambda_{LH} = 0$

State $\theta = L$. Under $\lambda_{LH} = 0$,

$$\left(r_L^0 - \mu_L\right) j_L = 1 - \left(r_L^\xi + \xi\right) f - \left(\mu_L + \xi\right) f j_L' + \frac{1}{2}\sigma^2 f^2 j_L''. \tag{49}$$

Let

$$j_L(f) = A_0 - A_1 f + A_2 f^{\gamma_1} + A_3 f^{\gamma_2}.$$

Then we can get

$$A_0 = \frac{1}{r_L^0 - \mu_L},\ A_1 = \frac{r_L^\xi + \xi}{r_L^0 + \xi}$$

and

$$\gamma_1 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r_L^0 - \mu_L\right)}}{\sigma^2} > 1,$$
$$\gamma_2 = \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\left(r_L^0 - \mu_L\right)}}{\sigma^2} < 0,$$

solves

$$\frac{1}{2}\sigma^2\gamma^2 - \left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)\gamma - \left(r_L^0 - \mu_L\right) = 0$$

The transversality conditions

$$\lim_{f\to 0} j_L(f) < \infty$$

imply that $A_3 = 0$.

State $\theta = H$ and $(\rho + \lambda_{HL} - r_H^0)j_L \geq (\rho - r_H^0)j_H$. The HJB becomes

$$\left(\rho + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - \left(\mu_H + \xi\right) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f) + (\rho + \lambda_{HL} - r_H^0) j_L \tag{50}$$

Guess a solution of the form

$$j_H(f) = B_0 - B_1 f + B_2 f^{\gamma_1} + B_3 f^{\phi_1} + B_4 f^{\phi_2}.$$

From here we get

$$B_0 = \frac{1 + \left(\rho + \lambda_{HL} - r_H^0\right) A_0}{\rho + \lambda_{HL} - \mu_H} = \frac{1 + \left(\rho + \lambda_{HL} - r_H^0\right) \frac{1}{r_L^0 - \mu_L}}{\rho + \lambda_{HL} - \mu_H},$$

$$B_1 = \frac{(r_H^\xi + \xi) + \left(\rho + \lambda_{HL} - r_H^0\right) A_1}{\rho + \xi + \lambda_{HL}} = \frac{(r_H^\xi + \xi) + \left(\rho + \lambda_{HL} - r_H^0\right) \frac{r_L^\xi + \xi}{r_L^0 + \xi}}{\rho + \xi + \lambda_{HL}},$$

$$B_2 = \frac{\rho + \lambda_{HL} - r_H^0}{(\rho + \lambda_{HL} - \mu_H) + (\mu_H + \xi)\gamma_1 - \frac{1}{2}\sigma^2 \gamma_1 (\gamma_1 - 1)} A_2$$

The function

$$\frac{1}{2}\sigma^2 \phi_1^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\phi_1 - (\rho + \lambda_{HL} - \mu_H) = 0$$

has two roots where

$$\phi_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (\rho + \lambda_{HL} - \mu_H)}}{\sigma^2} > 1,$$

$$\phi_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (\rho + \lambda_{HL} - \mu_H)}}{\sigma^2} < 0.$$

The transversality conditions

$$\lim_{f \to 0} j_H(f) < \infty$$

imply that $B_4 = 0$.

State $\theta = H$ and $(\rho + \lambda_{HL} - r_H^0) j_L < (\rho - r_H^0) j_H$. The HJB becomes

$$\left(r_H^0 + \lambda_{HL} - \mu_H\right) j_H(f) = 1 - \left(r_H^\xi + \xi\right) f - (\mu_H + \xi) f j_H'(f) + \frac{1}{2}\sigma^2 f^2 j_H''(f).$$

We can also look for a solution of the form

$$j_H(f) = D_0 - D_1 f + D_2 f^{\beta_1} + D_3 f^{\beta_2}.$$

From the equation for $j_H(f)$, we get

$$D_0 = \frac{1}{r_H^0 + \lambda_{HL} - \mu_H}$$

$$D_1 = \frac{r_H^\xi + \xi}{r_H^0 + \xi + \lambda_{HL}}$$

Moreover,

$$\beta_1 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r_H^0 - \lambda_{HL}\right)}}{\sigma^2} > 1,$$

$$\beta_2 = \frac{\mu_H + \xi + \frac{1}{2}\sigma^2 - \sqrt{\left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)^2 - 2\sigma^2\left(\mu_H - r_H^0 - \lambda_{HL}\right)}}{\sigma^2} < 0.$$

solves

$$\frac{1}{2}\sigma^2\beta^2 - \left(\mu_H + \xi + \frac{1}{2}\sigma^2\right)\beta + \mu_H - r_H^0 - \lambda_{HL} = 0.$$

B.4.4 Numerical Example

Parameters are as follows:

$$\rho = 0.1,\ r_H^0 = 0.015,\ r_H^\xi = 0.017,\ r_L^0 = 0.02,\ r_L^\xi = 0.018,\ \mu_H = 0.015,\ \mu_L = -0.2,\ \sigma = 0.1,\ \lambda_{HL} = 0.2.$$

Under these parameters, we get

$$f_L^b = 8.88$$

$$f_H^b = 14.41$$

$$f_\dagger = 4.16.$$

Numerically, the condition

$$(\rho + \lambda_{HL} - r^0_H)j_L \geq (\rho - r^0_H)j_H$$

holds in state H if and only if $f \leq f_\dagger$. Moreover, all the value functions are convex, confirming the equilibrium.

B.5 Upward Regime Shift

Suppose $\theta \in \{L, H, G\}$, where $\mu_G > \mu_H > \mu_L$. We use the notation G so that we do not need to change the notation in the benchmark model. The transitional intensity is $\lambda_{HG} = \lambda_G$ and $\lambda_{HL} = \lambda_L$, and the other intensities are zero.

In both $\theta = G$ and $\theta = L$, there is no long-term debt issuance. The value functions are

$$\begin{aligned}
j_L(f) &= \frac{1}{r - \mu_L} - f + \frac{f^b_L}{\gamma_L}\left(\frac{f}{f^b_L}\right)^{\gamma_L} \\
j_G(f) &= \frac{1}{r - \mu_G} - f + \frac{f^b_G}{\gamma_G}\left(\frac{f}{f^b_G}\right)^{\gamma_G},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_L &= \frac{\mu_L + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_L + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_L)}}{\sigma^2} > 1 \\
\gamma_G &= \frac{\mu_G + \xi + \frac{1}{2}\sigma^2 + \sqrt{\left(\mu_G + \xi + \frac{1}{2}\sigma^2\right)^2 + 2\sigma^2 (r - \mu_G)}}{\sigma^2} > 1 \\
f^b_L &= \frac{\gamma_L}{\gamma_L - 1}\frac{1}{r - \mu_L} \\
f^b_G &= \frac{\gamma_G}{\gamma_G - 1}\frac{1}{r - \mu_G}.
\end{aligned}$$

The HJB in the high state is

$$(\rho+\lambda_G+\lambda_L-\mu_H)\,j_H(f)=\max_{0\le d_H\le j_H(f)} 1-(r+\xi)\,f+(\rho-r)\,d_H+\mathbb{1}_{\{d_H\le j_L(f)\}}\cdot\lambda_L j_L(f)+\lambda_G j_G(f)$$
$$-(\xi+\mu_H)\,fj_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f)\,.$$

Equivalently,

$$(\rho+\lambda_G+\lambda_L-\mu_H)\,j_H(f)=\max_{0\le d_H\le j_H(f)} 1-(r+\xi)\,f-(\xi+\mu_H)\,fj_H'(f)+\frac{1}{2}\sigma^2 f^2 j_H''(f)$$
$$+\max\left\{(\rho-r)\,j_L(f)+\lambda j_L(f),(\rho-r)j_H(f)\right\}.$$

The price satisfies

$$(r+\xi+\lambda_G+\lambda_L)\,p_H(f)=r+\xi+\lambda_G p_G(f)+\mathbb{1}_{\{f\le f_\dagger\}}\lambda_L p_L(f)+\left(g_H(f)-\xi-\mu_H+\sigma^2\right)fp_H'(f)+\frac{1}{2}\sigma^2 f^2 p_H''(f)$$

Again,

$$g_H(f)=\begin{cases}\frac{(\rho-r)(p_H(f)-p_L(f))}{-fp_H'(f)} & f\le f_\dagger\\ 0 & f>f_\dagger\end{cases}.$$

Finally, we need the condition that

$$(\rho-r+\lambda_G)\,j_L(0)>(\rho-r)j_H(0)$$
$$\Rightarrow\frac{\rho-r+\lambda_L}{r-\mu_L}>(\rho-r)\frac{1+\frac{\lambda_G}{r-\mu_G}+\frac{(\rho-r+\lambda_L)}{r-\mu_L}}{\rho+\lambda_G+\lambda_L-\mu_H}$$
$$\frac{\rho-r+\lambda_L}{\rho-r}>\frac{\rho-\mu_L+\lambda_L+\lambda_G\frac{\rho-r+\lambda_L}{r-\mu_G}}{\rho+\lambda_G+\lambda_L-\mu_H}.$$

If $\lambda_G=0$, the condition goes back to the one we had in the paper. With the upside, the functional form of $g_H(\cdot)$ stays unchanged. However, $p_H(\cdot)$ is different, $f_\dagger$ is different, and the condition that $(\rho-r+\lambda_G)\,j_L(0)>(\rho-r)j_H(0)$ is also different.

B.6 Formal Analysis with Tax shields

The HJB equations are

$$
\begin{aligned}
(r+\lambda_{LH}-\mu_L)\, j_L(f) = \max_{d\le j_L(f)}\ & (1-\pi) - (r(1-\pi)+\xi)\, f + (r+\lambda_{LH}-(1-\pi)\, y)\, d - (\mu_L+\xi)\, f j_L'(f) \\
& + \lambda_{LH}(j_H(f)-d)^+ + \frac{1}{2}\sigma^2 f^2 j_L''(f) \\
(r+\lambda_{HL}-\mu_H)\, j_H(f) = \max_{d\le j_H(f)}\ & (1-\pi) - (r(1-\pi)+\xi)\, f + (r+\lambda_{HL}-(1-\pi)\, y)\, d - (\mu_H+\xi)\, f j_H'(f) \\
& + \lambda_{HL}(j_L(f)-d)^+ + \frac{1}{2}\sigma^2 f^2 j_H''(f)
\end{aligned}
$$

where interest rates are

$$y_H(d,f) = r + \lambda_{HL}\mathbf{1}_{d>j_L(f)},\ y_L(d,f) = r.$$

The debt price satisfies

$$
\begin{aligned}
(r+\xi+\lambda_{LH})\, p_L(f) &= (r+\xi) + \lambda_{LH} p_H(f) + \left(g_L(f)-\xi-\mu_L+\sigma^2\right) f p_L'(f) + \frac{1}{2}\sigma^2 f^2 p_L''(f) \\
(r+\xi+\lambda_{HL})\, p_H(f) &= (r+\xi) + \lambda_{HL} p_L(f)\mathbf{1}_{\{d_H(f)\le j_L(f)\}} + \left(g_H(f)-\xi-\mu_H+\sigma^2\right) f p_H'(f) + \frac{1}{2}\sigma^2 f^2 p_H''(f)
\end{aligned}
$$

From here we get,

$$
\begin{aligned}
g_L(f) &= \frac{(r+\xi+\lambda_{LH})\, p_L(f) - (r+\xi) - \lambda_{LH} p_H(f) + \left(\xi+\mu_L-\sigma^2\right) f p_L'(f) - \frac{1}{2}\sigma^2 f^2 p_L''(f)}{f p_L'(f)} \\
g_H(f) &= \frac{(r+\xi+\lambda_{HL})\, p_H(f) - (r+\xi) - \lambda_{HL} p_L(f)\mathbf{1}_{\{d_H(f)\le j_L(f)\}} + \left(\xi+\mu_H-\sigma^2\right) f p_H'(f) - \frac{1}{2}\sigma^2 f^2 p_H''(f)}{f p_H'(f)}
\end{aligned}
$$

From here we get

$$
\begin{aligned}
g_L(f) &= \frac{(r+\xi+\lambda_{LH})\, j_L'(f) + (r+\xi) - \lambda_{LH} j_H'(f) + \left(\xi+\mu_L-\sigma^2\right) f j_L''(f) - \frac{1}{2}\sigma^2 f^2 j_L'''(f)}{f j_L''(f)} \\
g_H(f) &= \frac{(r+\xi+\lambda_{HL})\, j_H'(f) + (r+\xi) - \lambda_{HL} j_L'(f)\mathbf{1}_{\{d_H(f)\le j_L(f)\}} + \left(\xi+\mu_H-\sigma^2\right) f j_H''(f) - \frac{1}{2}\sigma^2 f^2 j_H'''(f)}{f j_H''(f)}
\end{aligned}
$$

When $\theta = L$, we take the first order derivative of HJB with respect to f and find

$$(r+\xi+\lambda_{LH})\, j_L'(f)+(r+\xi)-\lambda_{LH} j_H'(f)+\left(\mu_L+\xi-\sigma^2\right) f j_L''(f)-\frac{1}{2}\sigma^2 f^2 j_L'''(f)=\pi r\left(j_L'(f)+1\right).$$

Hence, we get

$$g_L(f)=\frac{\pi r\,(j_L'(f)+1)}{f j_L''(f)}=\frac{\pi r\,(1-p_L(f))}{f j_L''(f)}\geq 0. \tag{51}$$

We need to consider the following two cases when we derive the long-term debt issuance at $\theta = H$.

Case 1: $(\pi r+\lambda_{HL})\, j_L(f) \geq \pi\,(r+\lambda_{HL})\, j_H(f)$ In this case, $d_H(f) = j_L(f)$. We take the first order derivative of HJB with respect to f and find

$$(r+\lambda_{HL}+\xi)\, j_H'(f)+(r+\xi)-\lambda_{HL} j_L'(f)+\left(\mu_H+\xi-\sigma^2\right) f j_H''(f)-\frac{1}{2}\sigma^2 f^2 j_H'''(f)=\pi r j_L'(f)+\pi r.$$

Hence,

$$g_H(f)=\frac{\pi r\,(j_L'(f)+1)}{f j_H''(f)}=\frac{\pi r\,(1-p_L(f))}{f j_H''(f)}\geq 0 \tag{52}$$

Case 2: $(\pi r+\lambda_{HL})\, j_L(f) < \pi\,(r+\lambda_{HL})\, j_H(f)$ In this case, $d_H(f) = j_H(f)$. We take the first order derivative with respect to f and find

$$(r+\lambda_{HL}+\xi)\, j_H'(f)+(r+\xi)+\left(\mu_H+\xi-\sigma^2\right) f j_H''(f)-\frac{1}{2}\sigma^2 f^2 j_H'''(f)=\pi\,(r+\lambda_{HL})\, j_H'(f)+\pi r$$

Hence,

$$g_H(f)=\frac{\pi\,(r+\lambda_{HL})\, j_H'(f)+\pi r}{f j_H''(f)} \tag{53}$$

Numerically, the condition

$$(\pi r+\lambda_{HL})\, j_L(f) \geq \pi\,(r+\lambda_{HL})\, j_H(f)$$

holds in state H if and only if $f \leq f_\dagger$. Moreover, all the value functions are convex, confirming the equilibrium.

C Additional Examples for Debt Maturity Management

We present two additional examples to supplement the empirical patterns of PG&E presented in subsection 4.1. Figure 1 plots the debt maturity patterns of General Motors (GM) and Figure 2 American Airlines (AA). In both figures, the left panel plots the maturity of newly issued debt (weighted by the market value of debt at issuance), whereas the right panel describes the ratio of long-term debt due within a year to the sum of current and long-term liabilities. As before, the red-shaded areas mark bankrupcies, and the gray areas are NBER recessions.

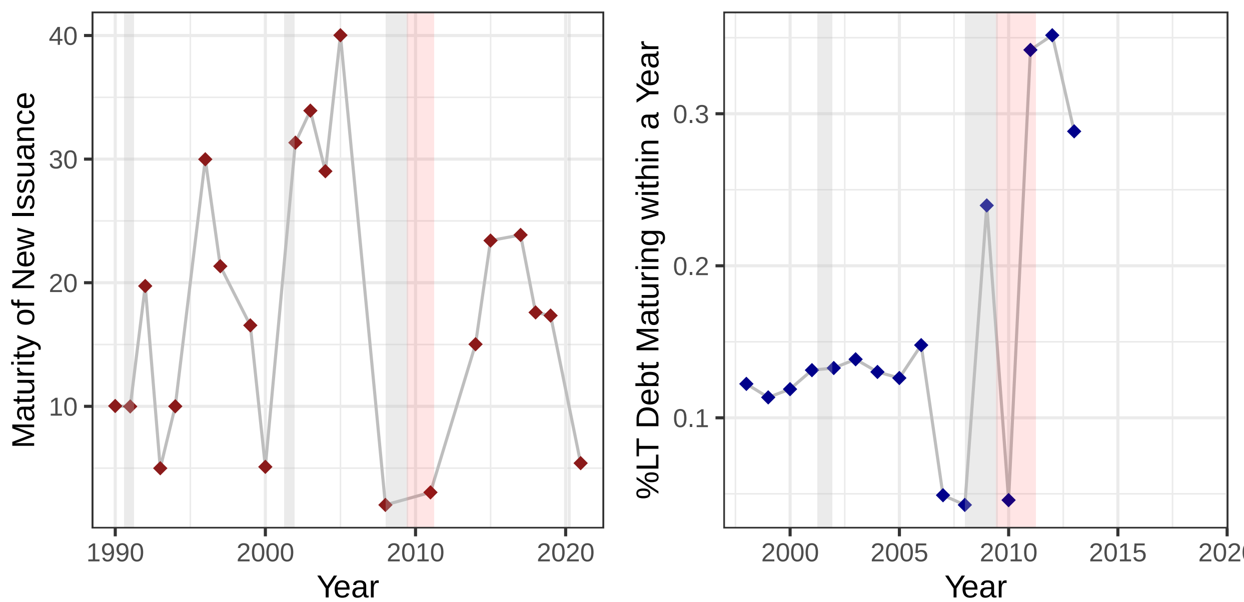

Figure 1: Maturity Structure of General Motors Company

This figure shows the average maturity of bonds in a year (weighted by market value at issuance) and the share of long-term debt maturing within one year for General Motors Company (GM). The gray shaded area indicates NBER recession while the red shaded area indicates periods over which GM was in bankruptcy procedures. Source: Compustat and Mergent FISD.

Clearly, the maturity of newly issued debt is short for both firms during NBER recessions as well as the bankruptcy period. In the case of GM, the maturity of newly-issued debt was also short in 1992, which might be due to the record-level loss reported. That year, GM reported a $23.5-billion loss for 1992–the largest ever by a U.S. company by then.[1] Even though this loss didn't push GM into bankruptcy, one could expect that this loss pushed GM closer to default. Moreover, the right panels of both figure make it clear that the ratio of long-term debt due within a year were high for both firms prior to the bankruptcy. Again, these patterns are in-line with the prediction of our theoretical model.

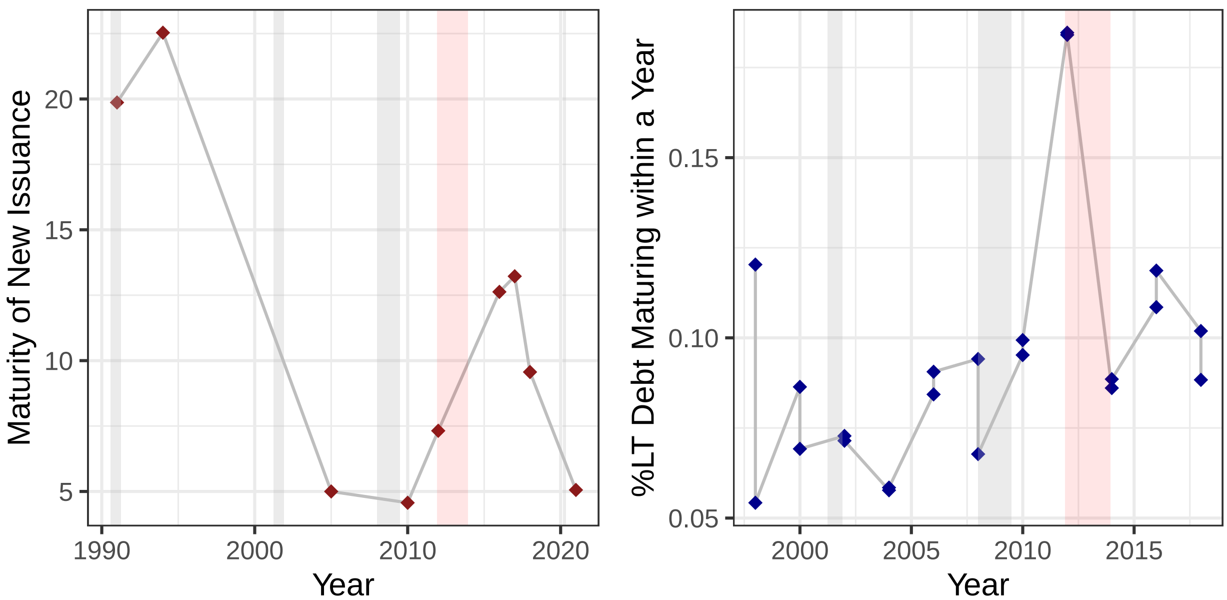

Figure 2: Maturity Structure of American Airlines, Inc.

This figure shows the average maturity of bonds in a year (weighted by market value at issuance) and the share of long-term debt maturing within one year for American Airlines, Inc. (AA). The gray shaded area indicates NBER recession while the red shaded area indicates periods over which AA was in bankruptcy procedures. Source: Compustat and Mergent FISD.

[1]https://www.latimes.com/archives/la-xpm-1993-02-12-fi-1334-story.html