

# Intermediary Financing without Commitment

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## Abstract

Intermediaries can reduce agency frictions in the credit market through monitoring. To be a credible monitor, an intermediary needs to retain a fraction of its loans; we study the credit market dynamics when it cannot commit to doing so. We compare the role of certification – monitoring to increase repayment – with the role of intermediation – channeling funds from depositors to the borrower. With commitment to retentions, certification and intermediation are equivalent. Without commitment, they lead to very different dynamics in loan sales and monitoring. A certifying bank sells its loans and reduces monitoring over time. By contrast, an intermediating bank issues short-term deposits to internalize the monitoring externalities and retain its loans. While the borrowing capacity is higher under intermediation, an entrepreneur may prefer to borrow from a certifying-only intermediary.

**Keywords:** commitment; certification; intermediation; trading; dynamic models

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# 1 Introduction

Financial intermediaries conduct valuable services and therefore benefit the real economy. For example, venture capitalists (VC) act as mentors to entrepreneurs: by monitoring to reduce agency frictions, they *certify* the venture so that others are willing to contribute funding as well. Intermediaries can also be commercial banks and savings institutions, which not only certify their borrowers but also *intermediate* funds from depositors to borrowers. How do these two roles of certification and intermediation relate to each other? Do intermediaries' liability structures matter for the services provided?

To appropriately align intermediaries' incentives, the optimal financing arrangement typically involves the retention of a fraction of loans by them as the skin in the game. If intermediaries can commit to retaining their loans, certification and intermediation are equivalent, and non-intermediary investors are indifferent between direct investment into the project (e.g., VC financing) or indirect investment through a bank. In other words, the liabilities of banks (i.e., how banks are financed) are largely irrelevant. In practice, however, intermediaries' commitment to loan retentions is limited: 60% of the loans are sold within one month after origination, and nearly 90% are sold within one year (Drucker and Puri, 2009).<sup>1</sup> Moreover, evidence shows that securitization and the "originate-to-distribute" model have led to lax screening and monitoring (Purnanandam, 2011; Keys et al., 2010, 2012), thereby reducing the loan quality.<sup>2</sup> This paper studies the equilibrium dynamics in loan sales and monitoring when intermediaries *cannot* commit to their retentions. We show that an appropriately-chosen liability structure is crucial in the alignment of banks' incentives for both monitoring and loan sales.

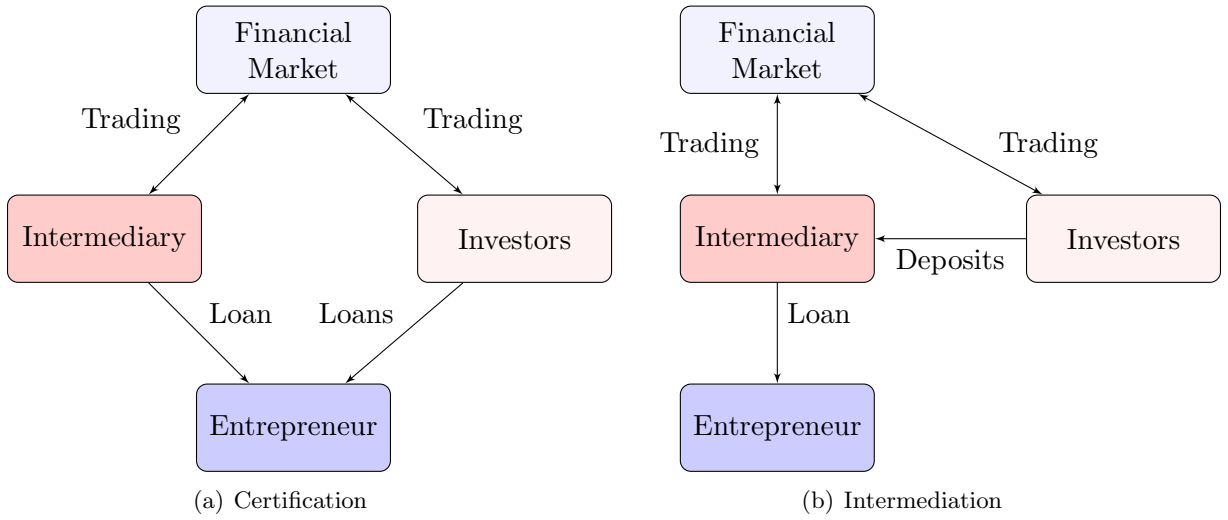
We build on the classic model of Holmstrom and Tirole (1997) in which banks can monitor to increase an entrepreneur's borrowing capacity. Being a credible monitor requires the bank to retain a sufficient fraction of loans on its balance sheet. However, in the presence of the financial market, the bank has incentives to sell its loans, due to its high cost of capital. The more it sells, the less likely it will monitor, and the price of the loans will drop more as well. We study two types of intermediary structures: certification and intermediation. As shown in Figure 1, in certification, both the bank and investors directly invest in the borrower's venture, and the role of the bank is to certify that it will monitor. In intermediation, investors deposit in the bank, which then invests a collection of its own funds and the deposits into the borrower's venture. Although the two structures are equivalent in the static framework with the bank's commitment to

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<sup>1</sup>More recently, Blickle et al. (2020) show that in loan syndications, the lead bank sells its entire share for 12% of all loans and for 27% of all term loans shortly after loan origination.

<sup>2</sup>Meanwhile, Begley and Purnanandam (2017) show securitization, if well structured, does not necessarily lead to a reduction in loan quality.

its retention, without commitment they lead to very different dynamics in loan sales and monitoring. In certification, the lack of commitment induces the bank to sell its loans gradually, and the bank's monitoring intensity declines over time. In intermediation, the bank is able to issue short-term deposits, which helps it commit to the retention and the decision to monitor. As a result, an intermediating bank never sells its loans, and the entrepreneur is always able to borrow more. However, if both structures enable the entrepreneur to borrow enough to invest, she may end up choosing certification. In other words, the intermediary structure that maximizes the initial borrowing may not be the one that maximizes the borrower's expected payoff.



**Figure 1: Certification vs. Intermediation**

More specifically, an entrepreneur is endowed with an investment opportunity, which requires a fixed-size of investment and pays off some final cash flows at a random time in the future. She has limited personal wealth and needs to borrow to make up the investment shortfall. Due to moral hazard in effort choices, she can only pledge a fraction of the final output to outside creditors, including banks and investors. Banks have a higher cost of capital, but only they can monitor to reduce the entrepreneur's private benefits. Although monitoring increases the project's pledgeable income and enables the entrepreneur to borrow more, it also entails a physical cost. Therefore, a credible monitor needs to retain a sufficient fraction of loans as its skin in the game.

We depart from [Holmstrom and Tirole \(1997\)](#) by introducing a competitive financial market, in which the bank is allowed to trade its loans. Loans are rationally priced, and therefore the prices depend on the bank's incentives to monitor both contemporaneously and in the future. If the bank has sold or is expected to sell a large fraction of the loans, it will monitor less often,

and consequently the price of loans will fall substantially. This price impact deters the bank from selling the loans too fast and too aggressively.

Although certification and intermediation are equivalent in a two-period model under bank's commitment to its retention, they lead to different dynamics in loan sales and monitoring if there is no such a commitment. Specifically, the lack of commitment hurts a certifying bank as in the standard durable-goods monopoly problem (Coase, 1972; Fudenberg et al., 1985; Gul et al., 1986).<sup>3</sup> In our context, one can interpret the bank as the monopolist, and the durable goods as claims to the cash flows of the entrepreneur's project. A similar result arises when a borrower cannot commit to its debt level (Bulow and Rogoff, 1989; Bizer and DeMarzo, 1992; Admati et al., 2018; DeMarzo and He, forthcoming), or when a large shareholder cannot commit to its future stake (DeMarzo and Urošević, 2006). Indeed, a certifying bank has incentives to sell, because its marginal valuation is below that of investors. After its initial loan sell, the bank has incentives to keep selling to exploit the remaining gains of trade. The price of the loans drops following the expectation that the bank will keep selling, reducing the likelihood of monitoring, which in turn reduces the bank's proceeds from selling. Hence, the bank trades off the immediate trading gains versus the drop in its future payoff, including the drop in the loan's valuation as well as the decline in the expected payments it can collect upon project maturity. Similar to the result in the Coase conjecture, the certifying bank does not benefit from its ability to trade loans at all.

By contrast, an intermediating bank does not have incentives to sell, due to its ability to issue deposits modeled as instantly-maturing debt. Indeed, when the bank has access to deposits as a source of cheap financing, selling loans will not only depress the valuation of the loan, but also increase the interest rate of deposits. The elevated deposit rate acts as a mechanism that deters the bank from selling. As a result, the bank finds it optimal to retain the entire loan on its balance sheet even though it has not committed to doing so. This result is related to the literature on the commitment role of short-term debt (Calomiris and Kahn, 1991; Diamond and Rajan, 2001); however, the channel is different. Whereas this literature emphasizes the demandable feature of debt and the externalities from creditor runs, our result does not rely on the first-come-first-serve constraint. Instead, our mechanism depends on the endogenous deposit rate as the discipline device. Deposits are modeled as pari-passu debt, so that if the bank fails, all depositors receive an equal amount.<sup>4</sup> In the case of a durable-goods monopoly, it is well-known that a monopolist

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<sup>3</sup>In traditional durable-goods monopoly and bargaining models, all trade is immediate in the continuous-trading limit. However, Fuchs and Skrzypacz (2010) show the equilibrium can involve delay with smooth trading when competitive buyers arrive over time, or when the production technology exhibits increasing returns to scale.

<sup>4</sup>The commitment problem is also mitigated when claims are collateralized. Rampini and Viswanathan (2019) emphasize the advantage of intermediaries in collateralizing claims. In their paper, certification and intermediation are still equivalent.

can overcome the commitment problem by renting the good, rather than selling it (Bulow, 1982). The short-term nature of the rental contract does not allow the monopolist to take advantage of early buyers. Instead, any change in rental prices simultaneously affect all buyers, eliminating the monopolist’s temptation to discriminate buyers over time. The associated commitment problem is therefore resolved. In our context, an intermediating bank who issues short-term deposits against the project’s cash flows can be thought as a rentor of the claims. The short term nature of the contract implies that any change in retention is immediately priced by all depositors, which eliminates the incentives to trade over time.

This distinction between certification and intermediation points out an important externality in bank monitoring. As in Diamond (1984), monitoring suffers from the free-rider problem in that investors enjoy the benefits but do not share the cost with the bank. Therefore, the equilibrium monitoring effort is too low. In certification, this externality leads to the bank reducing its probability of monitoring over time. In intermediation, however, deposits help the bank internalize the externalities. Indeed, because these deposits are fairly priced, the bank internalizes the cost of monitoring through issuing deposits. As a result, the bank and its depositors share both the benefits and costs of monitoring. Essentially, deposits create a market for the monitoring services the bank offers to be fairly priced, and the rate of deposits would have immediately increased, had the bank sold its retention. Therefore, our paper emphasizes the importance of informed short-term debt in disciplining the bank’s lack of commitment, which complements literature on the creation of safe and uninformed deposits by banks for transactional and liquidity reasons (Dang et al., 2017).

Next, we turn to the entrepreneur’s initial choice of the intermediary structure. We show that if the entrepreneur’s net worth is high enough, she may end up choosing certification, even though she borrows less. The reason is that intermediation can lead to too much monitoring (Pagano and Röell, 1998). Intuitively, the entrepreneur cares about not only the value of the project, but also her future benefits. The level of monitoring if the bank retains the entire loan may be excessive. Although bank monitoring introduces a positive externality to investors, it also imposes a negative externality on the entrepreneur by reducing her private benefits. This effect dominates if private benefits are sufficiently large, which is why the entrepreneur prefers the certification structure. While the existing literature on VC vs. bank emphasizes factors including the business prospect, the availability of collaterals, and the choice of equity and debt contracts, we focus on the intermediary’s tradeoff between commitment and monitoring. In particular, our result implies that entrepreneurs with higher private benefits – such as those value corporate perks and enjoy control rents – tend to borrow from certifying intermediaries such as VCs, whereas entrepreneurs that are more financially-constrained – such as small-business owners – prefer to rely on commercial banks and community banks.

Finally, we explore the impact of common policies designed to address the bank commitment problem. In particular, we look at the impact of a lock-up period and minimum retention levels.<sup>5</sup> Although the qualitative nature of the equilibrium outcome remains unchanged, these policies allow the entrepreneur to borrow more upfront. Moreover, if an intermediating bank’s deposits have long maturity or are subsidized by the government, the bank’s commitment to retention is also impaired.

## 2 The Model

Our model builds on the fixed-size investment setup in [Holmstrom and Tirole \(1997\)](#). A key innovation is to introduce a competitive financial market in which the bank can trade its loans. As a result, the bank cannot commit to its loan retention, and thus the decisions to monitor. The more it sells, the less likely it will monitor, and the loans will be valued lower as well. We explore how the bank’s equilibrium trading behavior interacts with its monitoring decisions, in the dynamic context of certification and intermediation.

### 2.1 Agents and Technology

Time is continuous and goes to infinity:  $t \in [0, \infty)$ . There are three groups of agents: one entrepreneur (she) – the borrower; competitive intermediaries – banks; and investors. All agents are risk neutral and have limited liabilities. The entrepreneur starts out with cash level  $A$ , whereas banks and investors have deep pockets. We assume investors do not discount future cash flows, whereas the entrepreneur and intermediaries discount the future at a rate  $\rho > 0$ .

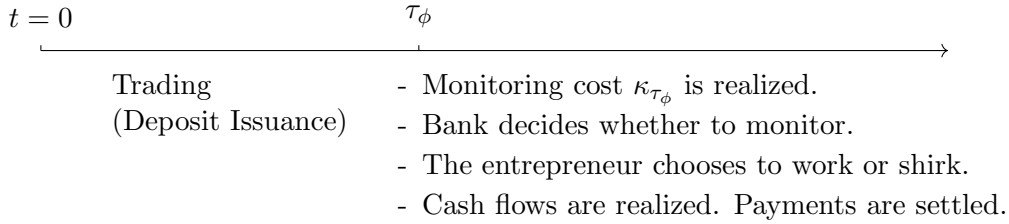
At time 0, the entrepreneur has access to a project that requires a fixed investment size  $I > A$ . Thus, she needs to borrow at least  $I - A$ . The project matures at a random time  $\tau_\phi$ , which arrives upon a Poisson event with intensity  $\phi > 0$ . Define  $\Phi = \frac{\phi}{\rho + \phi}$  as the effective time discount the entrepreneur and banks apply to the project’s final cash flows. At  $\tau_\phi$ , the project generates the final cash flows  $R$  in the case of success and 0 in the case of failure. The probability of success is  $p_H$  if the entrepreneur works at  $\tau_\phi$ , and  $p_L = p_H - \Delta$  if she shirks. Two options of shirking are available: the high option brings private benefit  $B$ , which exceeds  $b$ , the private benefit associated with the low option.

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<sup>5</sup>The Dodd-Frank Act imposes a minimum of 5% credit risk retention – known as the “risk-retention rule.” The EU has had similar restrictions in place since 2019. However, the D.C. Circuit Court invalidated the risk-retention rule for collateralized loan obligations (CLOs) in 2018.

## 2.2 Monitoring, Financial Structures, and Contracts

A competitive set of banks are present at  $t = 0$ , and the entrepreneur signs a contract with one of them. One can think about this bank as the relationship lender. Note we do not allow for multiple banking relationships to avoid duplication of monitoring efforts and the free-rider problem (Diamond, 1984). At  $\tau_\phi$ , the project matures, and the bank can monitor to eliminate the high shirking option.<sup>6</sup> To do so, it needs to pay a private monitoring cost  $\tilde{\kappa} > 0$ , where  $\tilde{\kappa} \in [0, \bar{\kappa}]$  has a distribution with  $F(\cdot)$  and  $f(\cdot)$  being the cumulative distribution function (CDF) and probability density function (PDF), respectively. Figure 2 describes the timing.



**Figure 2: Timing**

We study two types of financial structures: certification and intermediation. In certification, the bank puts its own funds in the entrepreneur's venture to ensure it will monitor, which then attracts investors to *directly* invest in the venture as well. One can think of this type of bank as either a venture capitalist or the lead investment bank in loan syndications. In certification, the entrepreneur directly signs the initial contract with the bank and investors. Under limited liability, no agent receives anything if the project fails. If the project succeeds, let  $R_f$ ,  $R_m$ , and  $R_u$  be the scheduled payments to the entrepreneur, the bank, and investors, which sum up to  $R$ . We use  $R_o = R_m + R_u$  to denote the total claims held by outside creditors, namely, the bank and investors. For the remainder of this paper, we also refer to  $R_o$  as *loans*.

In intermediation, investors do not directly invest in the entrepreneur's project. Instead, they make deposits in the bank, which in turn lends to the entrepreneur a collection of its own funds and these deposits.<sup>7</sup> This type of bank resembles a commercial bank. In intermediation, the entrepreneur signs an initial contract with a bank and promises to repay  $R_o = R - R_f$  if the project succeeds. The bank, in turn, offers deposit contracts  $\{D_t, y_t\}$  to investors/depositors over

<sup>6</sup>Monitoring can also be interpreted as external corporate governance.

<sup>7</sup>These deposits should be interpreted as non-core deposits, which are not fully insured. Such deposits are sensitive to information. Examples include brokered deposits, repos, wholesale lending, and commercial papers.

time, where  $D_t$  is the amount of deposits and  $y_t$  is the associated interest payment (henceforth, the deposit rate).<sup>8</sup> We model the deposit contract as debt with *instant* maturity. Whenever the bank fails to honor its deposit payments, the remaining loans are sold to repay its depositors. For simplicity, we assume no bankruptcy cost (or loss of charter value) is incurred if the bank fails, and a positive bankruptcy cost will change the results only quantitatively.

Given that the entrepreneur has (weakly) the highest cost of capital among all the agents, she should retain as little stake as possible. Therefore, in both certification and intermediation, it is optimal to let the entrepreneur retain  $R_f = b/\Delta$ , which guarantees she will work if the bank monitors. For the rest of this paper, we always set  $R_o = R - b/\Delta$ .

### 2.3 Trading and Pricing in the Financial Market

A competitive financial market opens in which loans can be traded.<sup>9</sup> We normalize the total share of loans outstanding to one and use  $\theta_t$  to denote the bank's retention at time  $t$ . In our model,  $\theta_t$  will be the payoff-relevant state variable. Sometimes  $\theta_t$  is also referred to as the bank's skin in the game, which is publicly observable. Before trading starts, the bank's initial retention is  $\theta_{0-} = R_m/(R_m + R_u)$  in certification and  $\theta_{0-} = 1$  in intermediation. We consider trading strategies that admit both smooth and atomic trading, as well as mixed strategies over the time of atomic trades. A Markov trading strategy is defined as  $(\theta_t)_{t \geq 0}$  being a Markov process.

The price of loans depends on whether the entrepreneur works or shirks, which in turn depends on the probability of bank monitoring. Conditional on the project maturing, let  $p(\theta)$  be the equilibrium probability that the bank with retention  $\theta$  monitors; investors of the loan receive

$$d(\theta) = p(\theta) R_o \quad (1)$$

per share. Let  $q(\theta)$  be the price of the loan per share when  $\theta_t = \theta$ . In a competitive financial market, the price is given by the expected present value of the asset:

$$q(\theta) = \mathbb{E} \left[ d(\theta_{\tau_\phi}) \mid \theta_t = \theta \right], \quad (2)$$

where the expectation operator is taken with respect to the equilibrium path of  $\{\theta_s\}_{t \leq s \leq \tau_\phi}$ .

The probability of monitoring  $p(\theta)$  will differ in certification and intermediation. Let  $\kappa$  be the realization of the stochastic monitoring cost  $\tilde{\kappa}$ . In certification, the bank with retention  $\theta$  chooses

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<sup>8</sup>The deposits should be interpreted as uninsured, non-core deposits. Subsection 4.2 studies subsidized deposits.

<sup>9</sup>We assume the entrepreneur's retention  $R_f$  is not tradable, or equivalently, the entrepreneur can commit to hold onto  $R_f$  on the balance sheet.



to monitor if and only if

$$\kappa \leq \kappa_c := \Delta R_o \theta, \quad (3)$$

where  $\Delta := p_H - p_L$ . In intermediation, a bank with retention  $\theta$  and deposits  $D$  monitors if and only if

$$\kappa \leq \kappa_i := \Delta (R_o \theta - D). \quad (4)$$

A comparison between (3) and (4) illustrates the static tradeoff between certification and intermediation. For the same retention  $\theta$ , a higher  $D$  reduces an intermediating bank's incentive to monitor: this result is the standard debt-overhang effect.

Let  $G(\theta)$  be the bank's instant trading gains. In the case of continuous trading,  $dG(\theta) = -q(\theta) \dot{\theta} dt$ . In the case of atomic trading, the bank's holding jumps to  $\theta^+$  and the associated trading gain is  $dG(\theta) = q(\theta^+) (\theta - \theta^+)$ . Note that trading is settled at price  $q(\theta^+)$  to reflect the price impact.

## 2.4 Equilibrium Definition

### 2.4.1 Certification

If the project matures at time  $t$ , the bank's expected payoff is

$$\pi_c(\theta) = p_c(\theta) R_o \theta - \int_0^{\kappa_c} \kappa dF(\kappa), \quad (5)$$

where the project succeeds with probability

$$p_c(\theta) := p_L + F(\kappa_c) \Delta, \quad (6)$$

upon which the bank receives  $R_o \theta$ . The bank maximizes the sum of its payoff upon the project's maturation  $e^{-\rho(\tau_\phi - t)} \pi_c(\theta_{\tau_\phi})$  and the cumulative trading gains  $\int_0^{\tau_\phi} e^{-\rho(s-t)} dG(\theta_s)$ . Because  $\tau_\phi$  follows the exponential distribution, the maximization problem is equivalent to

$$\max_{\{\theta_t\}_{t \geq 0}} \mathbb{E} \left[ \int_0^\infty e^{-(\rho + \phi)t} \left( \phi \pi_c(\theta_t) dt + dG(\theta_t) \right) \right], \quad (7)$$

where the expectation operator allows for mixed strategies in  $\{\theta_t\}$ .

We consider a Markov perfect equilibrium in which the state variable is the bank's retention  $\theta$ , henceforth, the certification equilibrium.<sup>10</sup>

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<sup>10</sup>If  $\{\theta_t\}$  is not restricted to the class of Markov processes, one may construct equilibria that are close to the

**Definition 1.** A *certification equilibrium* is a Markov perfect equilibrium consisting of a price function  $q: [0, 1] \rightarrow \mathbb{R}_+$  and a trading strategy  $(\theta_t)_{t \geq 0}$  that satisfy the following:

1. For all  $\theta_0 \in [0, 1]$ ,  $(\theta_t)_{t \geq 0}$  is a Markov trading strategy with initial value  $\theta_0$  that maximizes (7).
2. For all  $\theta \in [0, 1]$ , the price  $q(\theta)$  satisfies the break-even condition (2).

### 2.4.2 Intermediation

If the project matures at time  $t$ , the bank's expected payoff is

$$\hat{\pi}_i(\theta, D) = \hat{p}(\theta, D)(\theta R_o - D) - \int_0^{\kappa_i} \kappa dF(\kappa), \quad (8)$$

where the project succeeds with probability

$$\hat{p}_i(\theta, D) := p_L + F(\kappa_i) \Delta, \quad (9)$$

at which time the bank's equity holder receives  $\theta R_o - D$  after paying off its depositors. Besides trading gains, an intermediating bank also receives income from deposit issuance. In particular, let  $D_0$  be the value of deposits issued at  $t = 0$ . The bank's net income from deposit issuance at time  $t$  is  $dD_t - y_t D_t dt$ , where

$$y_t = \hat{y}(\theta, D) = \phi(1 - \hat{p}(\theta, D)) \quad (10)$$

compensates the default risk borne by depositors.<sup>11</sup> In intermediation, the bank trades loans and issues deposits to maximize the the expected payoff upon the project maturing, together with the

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commitment solution. In the context of a durable-goods monopoly, [Ausubel and Deneckere \(1989\)](#) show that in the no-gap case, non-Markov equilibria exist in which the seller can achieve payoffs close to the commitment solution. The logic behind the construction is similar to the one in the folk theorem for repeated games. The no-gap case corresponds to the version of our model under  $p_L = 0$ . At  $\theta = 0$ , the marginal valuation of investors coincides with the bank's under  $p_L = 0$ .

<sup>11</sup>Implicitly, the bank will honor its debt payments whenever  $t < \tau_\phi$ , that is, if the project does not mature. In the case of strategic default, depositors acquire the remaining loans and immediately resell them, which necessarily leads to zero payoff to the bank. Thus, strategic default never happens for  $t < \tau_\phi$ . Strategic defaults won't happen at  $t = \tau_\phi$  either, under the assumption that the bank cannot renegotiate with its depositors or the loan investors. [Diamond and Rajan \(2001\)](#) show renegotiation with depositors will trigger runs. Renegotiation with dispersed public investors is also costly due to free-rider problems ([Bolton and Scharfstein, 1996](#)).

net income from deposit issuance and trading gains; that is,

$$\max_{\{\theta_t, D_t\}} \mathbb{E} \left[ \int_0^\infty e^{-(\rho+\phi)t} \left( \phi \hat{\pi}_i(\theta_t, D_t) dt + dD_t - \hat{y}(\theta_t, D_t) D_t dt + dG(\theta_t) \right) \right]. \quad (11)$$

The choice of  $D_t$  in (11) is restricted by the bank's limited liability, which imposes a borrowing constraint illustrated below in (12). Lemma 1 shows the choice of  $D_t$  is essentially a static decision. Therefore, we can use  $\theta_t$  as the state variable and suppress the problem's dependence on  $D_t$ .

**Lemma 1.** *The maximization problem (11) is equivalent to solving*

$$\phi \pi_i(\theta) := \max_{D \leq \Pi_i(\theta)} \left\{ \phi \left[ \hat{p}_i(\theta, D) \theta R_o - \int_0^{\kappa_i} \kappa dF(\kappa) \right] + \rho D \right\}, \quad (12)$$

and

$$\Pi_i(\theta_0) = E(\theta_0, D_0) + D_0 = \max_{(\theta_t)_{t \geq 0}} \int_0^\infty \mathbb{E} \left[ e^{-(\rho+\phi)t} \left( \phi \pi_i(\theta_t) dt + dG(\theta_t) \right) \right]. \quad (13)$$

Lemma 1 implies we can solve deposit issuance and loan trading separately. From (12), the choice of deposit  $D$  only involves static tradeoffs. A higher  $D$  reduces the probability of monitoring  $\hat{p}(\theta, D)$  (see equation (4)), thereby reducing the value of loans. On the other hand,  $\rho D$  shows that deposits are a cheaper means of funding. Given that all the functions on the right-hand side of (12) are continuous, the optimal deposit issuance is also continuous, that is, does not admit jumps. Deposit issuance is bounded by the endogenous constraint  $D \leq \Pi_i(\theta)$ , which arises from the bank's limited liability. Here,  $\Pi_i(\theta)$  is the bank's value function given its retention  $\theta$ , which implicitly assumes deposit issuance has been chosen at the optimal level. Therefore, this constraint involves a fixed point for the value function  $\Pi_i(\theta)$ . In (13), the left-hand side  $\Pi_i(\theta_0)$  includes  $E(\theta_0, D_0)$ , the value to the bank's equity holders, and  $D_0$ , the value to the depositors. For the remainder of this paper, we sometimes refer to  $\Pi_i$ ,  $E$ , and  $D$  as the bank value, the equity value, and the deposit value, respectively. One implication of Lemma 1 is that even though the bank's equity holders decide its trading strategy, maximizing the bank's equity value  $E(\theta_0, D_0)$  is equivalent to maximizing the total bank value  $\Pi_i(\theta_0)$ , because deposits  $D_0$  are fairly priced.

We look for a Markov perfect equilibrium in state variable  $\theta_t$ , henceforth, the intermediation equilibrium.

**Definition 2.** *An **intermediation equilibrium** is a Markov perfect equilibrium consisting on a price function  $q: [0, 1] \rightarrow \mathbb{R}_+$ , a trading strategy  $(\theta_t)_{t \geq 0}$ , a deposit-issuance policy  $D^*: [0, 1] \rightarrow \mathbf{R}_+$ , and the deposit-rate function  $y: [0, 1] \rightarrow \mathbf{R}_+$  that satisfy the following:*

1. For all  $\theta \in [0, 1]$ , the deposit-issuance policy  $D^*(\theta)$  solves (12).
2. For all  $\theta_0 \in [0, 1]$ ,  $(\theta_t)_{t \geq 0}$  is Markov trading strategy with initial value  $\theta_0$  that maximizes (13).
3. For all  $\theta \in [0, 1]$ , the price  $q(\theta)$  satisfies the break-even condition (2).
4. For all  $\theta \in [0, 1]$ , the deposit rate  $y(\theta) := \hat{y}(\theta, D^*(\theta))$  satisfies (10).

## 2.5 Modeling Discussion and Parametric Assumptions

**Continuous-time framework.** In principle, one can construct a four-period (i.e., one contracting period, two trading periods, and one final period with realization of cash flows) model to analyze the dynamics of trading in certification and intermediation. For tractability reasons, our model is built in continuous time. As illustrated in DeMarzo and Urošević (2006) and Fuchs and Skrzypacz (2019), discrete-time models of bargaining and durable-goods monopolies are rarely solvable in closed form, unless they are restricted to a linear-quadratic specification, which admits a linear equilibrium. Moreover, even in models with three periods, clean and analytical results are scarcely obtainable. By contrast, formulating the problem in continuous time will allow for a clean characterization of the equilibrium and associated trading dynamics.

**Stochastic monitoring cost.** The stochastic-cost assumption smooths bank's equilibrium monitoring decisions, which become a continuous function of its loan retention. Stochastic costs can be interpreted as variations in legal and enforcement costs, or simply fluctuations in the costs of hiring loan officers. Two alternative formulations will generate results equivalent to that of a stochastic monitoring cost: The first is to introduce a continuous distribution of private benefits, and the second one follows subsection 9.2 in Tirole (2010) by assuming that monitoring is only effective with some probability. This probability is assumed to vary continuously with the bank's monitoring effort (which is increasing in  $\theta$ ).

**Effort, monitoring and cash flows.** We have assumed that the project does not generate any interim cash flows, and that the entrepreneur's effort and the bank's monitoring are only required at  $\tau_\phi$  (when the project matures). Due to the exponential arrival of final cash flows, this model is equivalent to a model in which independent cash flows are generated over time, and the levels of interim cash flows require continuous monitoring and entrepreneur effort. If one takes the terms of the lending contract as exogenously given, our analysis can be directly extended to include this case. The purpose of only considering a single round of effort and cash flow realizations is to keep the contract design problem static, so that we can focus on the dynamics resulting from loan sales.

Introducing long-term effort and monitoring would require the consideration of dynamic contracts which specify payments contingent on the history of cash flow realizations.

To make the problem non-trivial, we impose the following parametric assumptions. The first assumption says the project's expected payoff is always higher if the entrepreneur works.

**Assumption 1.**

$$p_H R > p_L R + B.$$

Assumption 2 ensures the problem is not driven by some particular feature of the monitoring cost's distribution function. It is satisfied by most commonly-used distribution functions such as the uniform distribution.

**Assumption 2.**

$$\frac{f'}{f^2} < \frac{\Phi}{1 - \Phi} \Delta, \quad \forall \kappa \in [0, \bar{\kappa}].$$

Finally, we restrict the (expected) monitoring cost to be sufficiently low.

**Assumption 3.**

$$\int_0^{\Delta R - b} \kappa dF(\kappa) \leq \Phi F(\Delta R - b)(\Delta R - b) - (1 - \Phi)p_L(\Delta R - b).$$

This assumption leads to the following result. If the bank always retains the entire loan (i.e.,  $\theta_t \equiv 1, \forall t \leq \tau_\phi$ ), the bank's payoff exceeds that if it immediately sells the entire loan and never monitors. If this assumption is violated, bank monitoring is never needed in equilibrium.

## 2.6 Two Benchmarks

### 2.6.1 Static Benchmark: Equilibrium without the Financial Market

We first solve the static model without the financial market. The goal of this subsection is to establish two equivalence results. First, the maximum amount that the entrepreneur is able to borrow is equivalent under certification and intermediation. Second, maximizing the initial borrowing amount is equivalent to maximizing the entrepreneur's expected payoff subject to lenders' break-even.

With slight abuse of notation, let  $\Phi$  be the bank's one-period discount rate in this subsection. The optimization problem in certification is

$$L_c = \max_{\{R_f, R_m, R_u\}} p_c(\Phi R_m + R_u) - \Phi \int_0^{R_m \Delta} \kappa dF(\kappa)$$

$$s.t. \quad R_m + R_u = R - \frac{b}{\Delta},$$

where  $L_c$  is the entrepreneur's maximum borrowing amount. The optimization problem in intermediation includes two steps. First, the bank chooses  $R_u$ ,<sup>12</sup> taken  $R_o = R_m + R_u$  as given. In the second step, the entrepreneur chooses  $R_o$  to maximize her borrowing amount  $L_i$ :

$$L_i = \max_{R_o} \left\{ \max_{R_u} p_i(\Phi R_m + R_u) - \Phi \int_0^{R_m \Delta} \kappa dF(\kappa) \right\}.$$

Proposition 1 shows certification and intermediation are equivalent here.

**Proposition 1** (Static Benchmark). *In the static model,  $L_c = L_i$  so that the equilibrium outcomes in certification and intermediation are equivalent. Moreover, the optimal contract that leads to maximum borrowing also maximizes the borrower's expected payoff.*

The proof is straightforward and therefore omitted. We offer intuitions for the second equivalence results. Because lenders always break even, and the project's expected payoff on the equilibrium path is always  $p_H R$ , maximizing the borrowing amount is equivalent to maximizing the borrower's expected payoff. We show in subsection 3.3 that this result is no longer true in the dynamic setup without commitment.

### 2.6.2 The Constrained Social Planner's Problem

The social planner chooses the bank's retention  $\{\theta_t\}_{t \geq 0}$  to maximize the aggregate social welfare, subject to the constraint that the bank decides whether to monitor given its retention. Under the commitment to  $\{\theta_t\}_{t \geq 0}$ , the choices of deposit  $D_t$  are redundant, since depositors can directly invest in the loan.

**Proposition 2** (Constrained Social Planner's Problem). *A social planner subject to constraint (3)*

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<sup>12</sup>Note the deposit choice in the static model is  $D = R_u$ .

chooses  $\{\theta_t\}_{t \geq 0}$  to maximize the social welfare:

$$W = \max_{(\theta_t)_{t \geq 0}} \int_0^\infty \phi e^{-\phi t} \left\{ (1 - e^{-\rho t}) (1 - \theta_t) d(\theta_t) + e^{-\rho t} \left[ p(\theta_t) R + (1 - F(\Delta R_o \theta_t)) B - \int_0^{\Delta R_o \theta_t} \kappa dF(\kappa) \right] \right\} dt. \quad (14)$$

The optimal retention always satisfies  $\theta_t < 1$ .

Due to differences in time discounting, the flow payoff in (14) is a weighted sum of the payoff to investors  $((1 - \theta_t) d(\theta_t))$  and the payoff to the bank and the entrepreneur. The project succeeds with probability  $p(\theta_t) R$ , and the entrepreneur shirks to receive the high private value  $B$  with probability  $(1 - F(\Delta R_o \theta_t))$ . Note that in this case, the optimal retention  $\theta_t$  is essentially a static choice that balances the benefits and costs of bank monitoring. Moreover, the difference in time discounting makes this tradeoff time-varying, which implies the optimal retention is in general time-varying as well.

Before concluding this section, note that given all parties are risk neutral and transfers can be made at the initial date, the constrained social planner's solution is identical to one in which the entrepreneur chooses  $\{\theta_t\}_{t \geq 0}$  to maximize her payoff, subject to the lenders' participation constraints as well as the constraint that the bank decides whether to monitor.

### 3 Equilibrium

In this section, we solve the model. Subsections 3.1 and 3.2 respectively derive the certification and intermediation equilibrium. Subsection 3.3 presents a special case in which the monitoring cost  $\tilde{\kappa}$  follows the uniform distribution and  $p_L = 0$ , where we obtain closed-form solutions in primitives. The entrepreneur's initial choices are therefore derived by comparing the two equilibria.

#### 3.1 Certification Equilibrium

In general, the bank can trade loans smoothly or atomically. We show both types of trading can occur in equilibrium. Let  $\Pi_c(\theta)$  be the bank's value function with retention  $\theta$ .<sup>13</sup> In the

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<sup>13</sup>A certifying bank does not issue deposits, so there is no distinction between the bank value and the equity holder's value.

smooth-trading region,  $\Pi_c(\theta)$  satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\rho \Pi_c(\theta) = \max_{\dot{\theta}} \left[ \phi \left[ \pi_c(\theta) - \Pi_c(\theta) \right] + \dot{\theta} \left[ \Pi'_c(\theta) - q_c(\theta) \right] \right]. \quad (15)$$

Whereas the left-hand side stands for the bank's required return, the first term on the right-hand side represents the event of the project maturing, in which case the bank receives  $\pi_c(\theta)$  defined in (5). The second term captures the overall benefit of trading, which includes the change to the bank's continuation value as well as the trading gain. A necessary condition for smooth trading is

$$\Pi'_c(\theta) = q_c(\theta), \quad (16)$$

so that the bank is indifferent between trading or not. In this case, the per-share trading gain  $q_c(\theta)$  is offset by the drop in the bank's continuation value  $\Pi'_c(\theta)$ . Substituting the indifference condition (16) into (15), we immediately get that in the region of smooth trading,

$$\Pi_c(\theta) = \Phi \pi_c(\theta). \quad (17)$$

Note the bank value is equal to the payoff it receives conditional on the project maturing, times the bank's effective discount rate  $\Phi = \frac{\phi}{\rho + \phi}$ . Surprisingly, the bank does not benefit from its ability to trade these loans in the financial market at all; its payoff is identical to the one if it retains  $\theta$  until the project finally matures at  $\tau_\phi$ . The observation that lack of commitment fully offsets the trading gains has already been noted in previous bargaining models (Fuchs and Skrzypacz, 2010; Daley and Green, 2020) and in other corporate finance settings (DeMarzo and Urošević (2006) in the context of trading by a large shareholder, and DeMarzo and He (forthcoming) in the context of leverage dynamics).

Even though the bank's equilibrium payoff is identical to the one if it does not trade at all, it doesn't imply the bank will not trade loans on the equilibrium path. In fact, the price of loans following a no-trade strategy will be too high, which gives the bank strict incentives to sell. Indeed, the bank sells the loans for two reasons. First, due to the higher cost of capital, the bank's marginal valuation is below that of investors. Second, after the initial sell and the reduction of retention, the bank is willing to sell again because the price impact only accrues to a smaller number of shares. Now, we characterize the bank's equilibrium trading strategy, which comes from the determination of the equilibrium loan prices. Because investors do not discount future cash flows,  $q_c(\theta)$  must



satisfy the following asset-pricing equation whenever the bank trades smoothly:

$$0 = \phi \left[ d_c(\theta) - q_c(\theta) \right] + \dot{\theta} q'_c(\theta), \quad (18)$$

where  $\phi \left[ d_c(\theta) - q_c(\theta) \right]$  resembles the dividend income and  $\dot{\theta} q'_c(\theta)$  the capital gain.<sup>14</sup> Combining (16), (17), and (18), and using the relation  $d_c(\theta) = \pi'_c(\theta)$ , one can derive the following dynamic trading strategies in equilibrium:

$$\dot{\theta} = -\phi \frac{(1 - \Phi) \pi'_c(\theta)}{\Phi \pi''_c(\theta)} < 0. \quad (19)$$

Clearly, in the smooth-trading region, the bank sells loans over time and its retentions declines continuously, even though it is indifferent between selling the loan or not. Intuitively, the equilibrium loan price is forward-looking, and therefore takes into account the bank's decisions regarding future monitoring. To satisfy the bank's indifference condition, the equilibrium price of the loan cannot be too high, implying the probability of monitoring must decline over time. Therefore, the only trading strategy consistent with this price requires the bank to sell its loans over time.

So far, we have focused only on the case of smooth trading. Meanwhile, the bank also has the option to sell an atom of loans. In general, the bank can sell either a fraction or all the remaining loans. Lemma 3 in the appendix proves the bank will never sell a fraction. This result follows the intuition in standard Coasian dynamic models. Atomic trading arises whenever the bank has strict incentives to sell. If so, it prefers to sell as fast as possible. Therefore, we are left to check when the bank decides to sell off all its retention at a price  $q_c(0)$ , where  $q_c(0) = p_L R_o$  is the per-share loan price without monitoring. Indeed, we show a unique  $\theta_*$  exists such that  $\theta_* q_c(0) = \Phi \pi_c(\theta_*)$ , and  $\theta_* q_c(0) < \Phi \pi_c(\theta_*)$  if and only if  $\theta > \theta_*$ . In other words, a unique cutoff  $\theta_*$  exists below which the bank finds it optimal to sell off all the remaining loans.

The final step in the equilibrium construction is to derive the trading strategy at  $\theta = \theta_*$ . First, note the bank cannot hold onto the loans forever, because the resulting loan price will be too high to induce the bank to sell. The bank cannot sell smoothly either; shortly afterwards, the bank will have strict incentives to sell off the rest of the loan. Suppose the bank sells off immediately after  $\theta_t$  reaches  $\theta_*$ . The price of the loan will then experience a deterministic downward jump, which is inconsistent with the asset-pricing equation (2). Mathematically, let  $T_* := \inf \{t > 0 : \theta_t = \theta_*\}$ : whereas  $q(\theta_{T_*-})$  satisfies (16), the price at time  $T_*$  would satisfy  $q(\theta_{T_*}) = q(0) = p_L R_o$ . Therefore, the equilibrium necessarily involves some delay before the entire selloff. The only (stationary)

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<sup>14</sup>The terms without subscripts  $c$  have been defined in equations (1) and (2).

trading strategy at  $\theta_*$  consistent with (2) is for the bank to adopt a mixed strategy<sup>15</sup>: the bank sells off all its remaining loans at  $\tau_\lambda$ , which arrives upon a Poisson event at intensity  $\lambda$  that satisfies

$$q_c(\theta_*) = \mathbb{E} [d_c(\theta_{\tau_\phi}) | \theta_t = \theta_*] = \frac{\lambda}{\phi + \lambda} d_c(0) + \frac{\phi}{\phi + \lambda} d_c(\theta_*).$$

Simple derivation shows  $\lambda$  is determined by

$$p_L + \frac{\phi}{\lambda + \phi} F(\Delta R_o \theta_*) \Delta = \Phi [p_L + F(\Delta R_o \theta_*) \Delta]. \quad (20)$$

Proposition 3 summarizes the previous discussion and describes the equilibrium outcome. The formal proof requires verification that the bank's trading strategy is optimal, which is supplemented in the appendix using results from the theory of optimal control in stratified domains.<sup>16</sup>

**Proposition 3** (Certification Equilibrium). *A unique **certification equilibrium** exists. Given the bank's initial retention  $\theta_0$ , the bank sells its loans smoothly at a rate given by equation (19) until  $T_*$ , after which it sells off its remaining loans at some Poisson rate  $\lambda$  that satisfies (20). The equilibrium loan price is*

$$q_c(\theta_t) = \begin{cases} \Phi(p_L + F(\Delta R_o \theta_t) \Delta) R_o & t < T_* \\ \left(p_L + \frac{\phi}{\lambda + \phi} F(\Delta R_o \theta_*) \Delta\right) R_o & T_* \leq t < \tau_\lambda \\ p_L R_o & t \geq \tau_\lambda. \end{cases} \quad (21)$$

The contract that maximizes the initial borrowing amount has  $\theta_0 = 1$ , in which case the borrowing amount is

$$L_c = \Phi \pi_c(1). \quad (22)$$

A contract that has the entrepreneur borrow exclusively from the bank at  $t = 0$  enables the most upfront borrowing, even though bank capital is costly. Intuitively, under this contract, the bank needs to spend the longest period to fully offload its loans. This result is in contrast to that in Holmstrom and Tirole (1997), where the entrepreneur prefers to use as little bank capital as possible, because bank capital is expensive. Our setup, however, contains a second channel

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<sup>15</sup>The delay can also be deterministic, but the equilibrium is no longer within the class of a Markov perfect equilibrium. The equilibrium would also depend on the time since the bank's retention reached  $\theta_*$ . The price  $q_t$  would not be stationary, and it would depend on the trading history before time  $t$ .

<sup>16</sup>Due to the discontinuity in the price function  $q_c(\theta)$ , the HJB equation (15) is discontinuous at  $\theta_*$ . This technical problem can be sidestepped using (discontinuous) viscosity solution methods.

whereby higher bank retention slows down the bank's selling process. The longer the bank takes to sell off the entire loan, the more likely the bank will monitor, and the value of the loan is also higher. As a result, the second channel necessarily dominates.

We end this subsection by describing the borrower's expected payoff  $V_c$ . Specifically, given that the entrepreneur retains  $R_f$ , her payoff is

$$V_c = \mathbb{E} \left[ \int_0^\infty \phi e^{-(\rho+\phi)t} \left\{ \mathbb{1}_{\{\kappa \leq \kappa_c\}} p_H R_f + \mathbb{1}_{\{\kappa > \kappa_c\}} (p_L R_f + B) \right\} dt \right], \quad (23)$$

where  $\kappa$  is the realization of  $\tilde{\kappa}$ . The expectation operator is taken with respect to the bank's equilibrium trading strategy, which involves mixed strategies. Intuitively, if the realized monitoring cost is lower than the threshold  $\kappa_c$  defined in (3), the bank monitors, and the entrepreneur receives  $p_H R_f$  in expectation. Otherwise, the bank chooses not to monitor, and the entrepreneur receives the expected return  $p_L R_f$  together with the private benefits  $B$ .

## 3.2 Intermediation Equilibrium

The analysis of the intermediation equilibrium has two steps: deposit issuance and loan trading.

### 3.2.1 Deposit Issuance

Plugging (10) and (8) into (12) and dividing both sides by  $\rho + \phi$ , we can rewrite the bank's objective function without trading gains as follows:

$$\mathcal{V}(D, \theta) := \Phi \left[ \hat{p}_i(\theta, D) \theta R_o - \int_0^{\kappa_i} \kappa dF(\kappa) \right] + (1 - \Phi) D. \quad (24)$$

This objective function includes several terms. The term in the bracket is the net payoff to the bank and its depositors: with probability  $\hat{p}(\theta, D)$ , the project succeeds so that they receive  $\theta R_o$ ;  $\int_0^{\kappa_i} \kappa dF(\kappa)$  is the expected monitoring cost. The last term in (24) is the value from issuing deposits. An increase in  $D$  reduces the bank's monitoring incentive and therefore reduces the first term. Meanwhile, an increase in  $D$  also reduces the bank's funding cost and therefore increases the last term. The optimal  $D$  is chosen to balance the two effects.

However, note that the bank's equity holders' limited liability constraint requires that  $E(\theta, D) = \Pi_i(\theta) - D \geq 0$ , so that for any  $\theta$ ,  $D \leq \Pi_i(\theta)$ . This constraint implicitly defines a maximum value of deposit that the bank can issue for any choice of  $\theta$ , which we denote as  $D^{max}(\theta)$ .<sup>17</sup> Because  $\Pi_i(\theta)$ , the bank's maximal value given its retention  $\theta$  has implicitly assumed that deposit issuance

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<sup>17</sup>Note in the case the bank decides to sell off all its remaining loans,  $D^{max}(\theta)$  is defined as 0.

is optimally chosen, solving for  $D^{\max}(\theta)$  is therefore equivalent to looking for a fixed-point in the constraint.<sup>18</sup>

Let  $\mathcal{L} = \mathcal{V}(D, \theta) + \Phi z(\theta) [D^{\max}(\theta) - D]$ , where  $\Phi z(\theta)$  is the Lagrange multiplier of the deposit issuance constraint. From the first order condition,  $\mathcal{L}_D = 0$ , we can easily show that the optimal solution  $D^*(\theta)$  solves

$$\phi [f(\kappa_i) \Delta^2] D^*(\theta) = \rho - \phi z(\theta), \quad (25)$$

To finish the derivation of optimal deposit choice  $D^*(\theta)$ , we need one more assumption to guarantee that the second order condition of the constrained maximization problem is satisfied.

**Assumption 4.** *The distribution function  $f(\tilde{\kappa})$  satisfies*

$$-\Phi [\hat{p}_i(\theta, D) f(\kappa) + f^2(\kappa) \Delta^2 D] < \Delta f'(\kappa) D < f(\kappa), \quad \forall D \leq p_H R_o \theta, \forall \theta \in [0, 1], \forall \kappa \in [0, \bar{\kappa}].$$

This assumption imposes lower and upper bounds on  $f'(\kappa)$ , and will be satisfied in the uniform-distribution case. The upper bound guarantees that, for any given  $\theta$ , the second-order partial derivative of (24) will satisfy  $\mathcal{L}_{DD} < 0$ . The lower bound guarantees that, in the region where the equity holders' limited liability binds ( $D = D^{\max}(\theta)$ ), the value function  $\Pi_i(\theta)$  will be convex in  $\theta$ .

### 3.2.2 Trading

Next, we turn to the maximization problem (13) and study how an intermediating bank trades its loans over time. Following similar steps in the certification equilibrium, the term  $\dot{\theta} (\Pi'_i(\theta) - q_i(\theta))$  must vanish in the smooth-trading region, so the bank's continuation value satisfies the HJB equation:

$$\rho \Pi_i(\theta) = \phi \left[ \pi_i(\theta) - \Pi_i(\theta) \right], \quad (26)$$

and the equilibrium price is determined by the indifference condition  $\Pi'_i(\theta) = q_i(\theta)$ .

As in the certification case, given  $q_i(\theta)$ , the trading strategy follows from the asset-pricing equation

$$0 = \phi \left[ d_i(\theta) - q_i(\theta) \right] + \dot{\theta} q'_i(\theta) \implies \dot{\theta} = -\phi \frac{d_i(\theta) - q_i(\theta)}{q'_i(\theta)}. \quad (27)$$

The functions  $d_i(\theta)$  and  $\pi_i(\theta)$  are defined under the optimal deposit issuance  $D^*(\theta)$ . Applying the

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<sup>18</sup> A similar problem is analyzed in [Abel \(2018\)](#).

envelope theorem in Milgrom and Segal (2002), we get

$$\pi'_i(\theta) = d_i(\theta) + \left[ f(\kappa_i) \frac{\partial \kappa_i}{\partial \theta} \Delta \right] D^*(\theta) + z(\theta) D^{\max'}(\theta). \quad (28)$$

Using condition (25), the equilibrium trading rate is<sup>19</sup>

$$\dot{\theta} = \phi \frac{(1 - \Phi)(1 - p(\theta))R_o + \Phi z(\theta) (D^{\max'}(\theta) - R_o)}{\Phi \pi''_i(\theta)}. \quad (29)$$

**Lemma 2.** *1. There exists a threshold where  $D^*(\theta) = D^{\max}(\theta)$  if and only if  $\theta$  falls below this threshold.*

*2. For any  $\theta$ ,  $\dot{\theta} > 0$  holds under the optimal deposit issuance policy  $D(\theta) = D^*(\theta)$ .*

The first statement of Lemma 2 shows that the equity holder's limited liability constraint is monotonic, i.e., it only binds when  $\theta$  is low. In other words, the equity holder's constraint in issuing deposits becomes less and less binding as the bank holds more retention. The second statement shows that in contrast to the certification equilibrium, a bank would prefer to buy more shares in the intermediation equilibrium. A comparison between (29) and (19) highlights a crucial difference between the two implementation structures. In certification, the bank has incentives to sell because investors have a higher valuation of these loans. In intermediation, this incentive disappears. In fact, the bank has incentives to increase its retentions. This distinction arises because an intermediating bank issues deposits to finance its loans. To see this, compare  $\pi'_c(\theta) = d_c(\theta)$  with (28), which clearly shows an increase in  $\theta$  leads to two additional benefits in intermediation. The first benefit is characterized by the term  $\left[ f(\kappa_i) \frac{\partial \kappa_i}{\partial \theta} \Delta \right] D^*$ . Intuitively,  $f(\kappa_i) \frac{\partial \kappa_i}{\partial \theta}$  is the marginal effect of retention  $\theta$  on the incremental probability of bank monitoring, so that  $f(\kappa_i) \frac{\partial \kappa_i}{\partial \theta} \Delta$  is the equilibrium probability that deposits will be repaid. Given that deposits are valued fairly, more retention enables the bank to issue *cheaper* deposits. The second benefit is captured by the last term  $z(\theta) D^{\max'}(\theta)$ , which is only positive if  $D^*(\theta) = D^{\max}(\theta)$ . Intuitively, an increase in  $\theta$  also relaxes the constraint on deposit issuance derived from the bank equity holders' limited liability protection. Therefore, more retention has an add-on benefit by further allowing the bank to issue *more* deposits. Note this second benefit disappears whenever the deposit-issuance constraint is slack, that is,  $z(\theta) = 0$ . Even though the bank has a higher cost of capital and deposits reduce the bank's incentive to monitor, the availability of cheap deposits as a source of financing offers the bank sufficient incentives to increase its retention. Recall that by definition,  $\theta_0 = 1$  in intermediation. In

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<sup>19</sup> Assumption 2 guarantees  $\pi''_i(\theta) > 0$ .

equilibrium, the bank never sells its loans and  $\theta_t \equiv 1$  for all  $t \geq 0$ . In other words, an intermediating bank retains the entire loan until the project matures.

The result that short-term deposits (or in general short-term debt) could help with the commitment problem has been introduced in Calomiris and Kahn (1991), Diamond and Rajan (2001), and Diamond (2004). Whereas the mechanism of these papers relies on the demandable feature of deposits and run externalities, the channel in our paper is different. In particular, the bank retention choice directly affects the cost (and also the amount) of deposits. The rates of deposits enable the bank to internalize the externalities, such that monitoring is “awarded” in the form of cheaper deposits. In this sense, our paper is related to Myers (1977), who suggests short-term debt as a possible solution to the debt-overhang problem.<sup>20</sup> Another difference is the nature of the commitment problem. Whereas these papers study the bank’s commitment to enforce claims and collect payments, we focus on how the bank commits to retain its loans over time.

Finally, to complete the characterization of the equilibrium, we consider the case in which the bank trade loans atomically. Following the same logic as in the certification equilibrium, we can show a unique  $\theta_*$  exists such that the bank sells off all the remaining loans at price  $q(0) = p_L R_o$  if  $\theta < \theta_*$ . However, when  $\theta = \theta_*$ , unlike in the certification equilibrium, the mixed strategy is no longer needed. Instead, the bank chooses to buy the loan smoothly so that  $\theta$  will increase above  $\theta_*$ . Therefore, the price of the loan satisfies  $q_i(\theta_*) = \Phi\pi'(\theta_*)$ .<sup>21</sup> The following proposition summarizes the results.

**Proposition 4** (Intermediation Equilibrium). *1. There exists an unique  $\theta_\dagger$ . If  $\theta \geq \theta_\dagger$ , the bank buys loans smoothly at the rate given by equation (29), and the price of the loan satisfies  $q_i(\theta) = \Phi\pi'_i(\theta)$ , where  $\pi'_i(\theta)$  satisfies (28). If  $\theta < \theta_\dagger$ , the bank sells off the rest of the loan immediately at the price  $q_i(\theta) = p_L R_o$ .*

*2. In the unique **intermediation equilibrium** where  $\theta_0 = 1$ , the bank holds  $\theta_t = 1$  until the project matures, and maintains a constant debt level  $D_t = D^*(1)$ . The equilibrium loan price is  $q_i(1) = \Phi\pi'_i(1)$ .*

Similar to certification, we use  $V_i$  to denote the entrepreneur’s expected payoff in intermediation:

$$V_i = \mathbb{E} \left[ \int_0^\infty \phi e^{-(\rho+\phi)t} \left\{ \mathbb{1}_{\{\kappa \leq \kappa_i\}} p_H R_f + \mathbb{1}_{\{\kappa > \kappa_i\}} (p_L R_f + B) \right\} dt \right].$$

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<sup>20</sup>One can think of monitoring as value-enhancing investment.

<sup>21</sup>Both in the case of certification and intermediation, the price function  $q(\theta)$  is discontinuous at  $\theta_*$ . However, whereas in certification the bank trades toward the discontinuity point (i.e.,  $\dot{\theta}(\theta_*+) < 0$ ), in intermediation, the bank trades away from the discontinuity point (i.e.,  $\dot{\theta}(\theta_*+) > 0$ ). The construction of the equilibrium (and the analysis of the bank’s optimal control problem) is simpler in this latter case because the trajectory of  $\theta_t$  does not “see” the discontinuity.

The expressions differs from (23) in that the threshold cost for monitoring is replaced by  $\kappa_i$ .

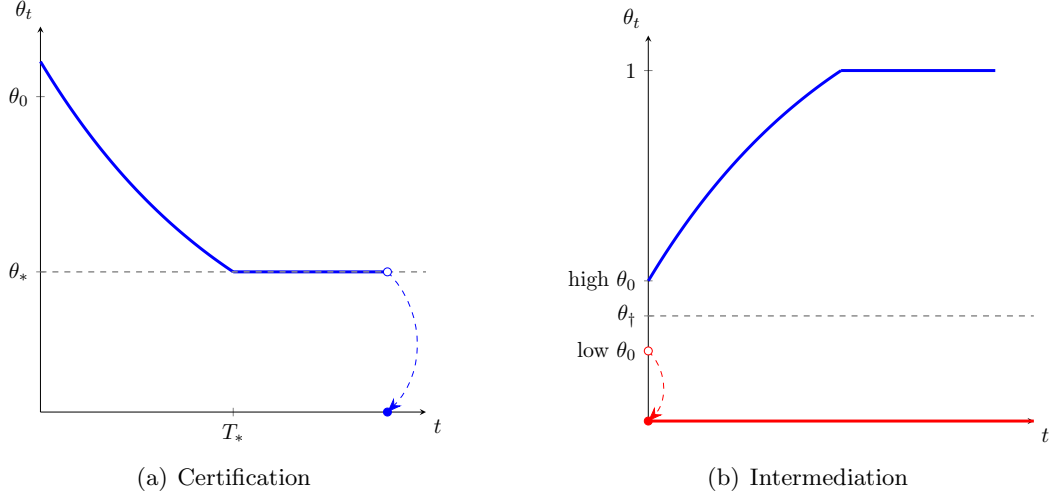
**Discussion: Trading Dynamics in Certification and Intermediation.** We have seen the equilibrium trading dynamics are remarkably different in certification and intermediation. We offer some further discussions regarding the fundamental mechanisms behind the difference. In general, monitoring has the property of a public good in the sense that investors can free ride the bank. All creditors commonly share the benefits from monitoring, whereas the bank exclusively bears the cost. In other words, by monitoring the borrower, the bank brings positive externalities to investors and therefore benefits the entrepreneur indirectly. Therefore, the equilibrium monitoring effort is inefficiently low, and in a dynamic framework with certification, the bank reduces its probability of monitoring over time. Price impacts deter the bank from selling the loan too fast and too aggressively.

In intermediation, the role of deposits is to help the bank internalize the externality from monitoring. Indeed, these deposits are fairly priced, which reflects the probability of monitoring. The income that the bank receives from deposit insurance thus compensates its cost incurred during monitoring. In this case, the bank and its depositors share both the benefits and costs of monitoring. Essentially, deposits create a market for the services offered by the bank, that is, monitoring, to be fairly priced.

The above argument also explains the importance of short-term deposits to align the bank's incentive in monitoring. Indeed, in the case of instantly maturing deposits, the value and the issuance of deposits are continuously adjusted without any friction. If, instead, depositors have a positive maturity, the bank will not completely internalize the externalities from monitoring.

Our paper therefore emphasizes the complementarity between loan retention and deposit issuance. We offer another mechanism whereby the asset side of banks is closely linked to the liability side. In contrast with others in the literature (Kashyap et al., 2002; Diamond, 1984; Dang et al., 2017), our mechanism is dynamic and highlights the pricing role of short-term debt. In contrast with Dang et al. (2017), we emphasize the role of informed depositors in disciplining banks when banks have limited commitment.

Figure 3 compares the bank's trading dynamics in certification (left) and intermediation (right). In the certification equilibrium, the bank first sells the loan smoothly, so long as its initial retention satisfies  $\theta_0 > \theta_*$ . After its retention  $\theta_t$  reaches  $\theta_*$ , the bank sells off the entire rest of the loan retention, following a stochastic delay, and has zero retention afterwards. By contrast, the bank will buy loans smoothly over time, so long as its retention satisfies  $\theta_0 \in (\theta_+, 1)$ , illustrated by the blue pattern in the right panel. On the other hand, if  $\theta_0 < \theta_+$ , an intermediating bank immediately sells off the entire loan and has zero retention afterwards, as illustrated by the red pattern.



**Figure 3: Retention Patterns**

### 3.3 A Special Case: Uniform Distribution

We present a special case of the model in which the monitoring cost  $\tilde{\kappa}$  follows the uniform distribution. This assumption not only simplifies the analysis but also helps us focus on the insights that do not rely on the particular features of the distribution function. Under the uniform distribution, the marginal effect of retention  $\theta_t$  on the probability of monitoring is always a constant. Moreover, we assume the probability of success is  $p_L = 0$  if the entrepreneur shirks. This assumption naturally leads to a result that the bank will never sell off all its loans atomically, because the resulting price will be zero.<sup>22</sup> As a result,  $\theta_* = 0$  and  $T_* \rightarrow \infty$  in the certification equilibrium.

**Assumption 5.**  $f(\kappa) = \frac{1}{\bar{\kappa}}$  for  $\kappa \in [0, \bar{\kappa}]$ , and  $p_L = 0$ .

The constrained optimal solution turns out very straightforward under Assumption 5.

**Corollary 1.** *Under Assumption 5, the constrained social planner's solution in Proposition 2 becomes*

$$\theta_t = \frac{1}{2 - e^{-\rho t}} \left[ (1 - e^{-\rho t}) + e^{-\rho t} \frac{\Delta R - B}{\Delta R - b} \right].$$

Moreover,  $\theta_0 = \frac{\Delta R - B}{\Delta R - b}$  and  $\lim_{t \rightarrow \infty} \theta_t = \frac{1}{2}$ .

<sup>22</sup>In the durable-goods monopoly literature, this is referred to as the “no-gap” case.



### 3.3.1 Certification Equilibrium

We first describe the certification equilibrium. Simple calculation shows  $p_c(\theta) = (\kappa_c/\bar{\kappa})\Delta = \Delta^2 R_o \theta / \bar{\kappa}$ . Equation (5), the bank's expected payoff when the project matures, becomes  $\pi_c(\theta) = \frac{1}{2\bar{\kappa}} (\Delta R_o \theta)^2$ . The evolution of  $\theta_t$  in (19) follows  $\dot{\theta} = -\rho\theta$ , which implies  $\theta_t = \theta_0 e^{-\rho t}$ .

**Corollary 2.** *In the certification equilibrium, given any  $\theta_0$ , the firm is able to borrow*

$$L_c(\theta_0) = \frac{\Phi}{2\bar{\kappa}} (\Delta R_o \theta_0)^2 + \frac{\Phi}{\bar{\kappa}} (\Delta R_o)^2 \theta_0 (1 - \theta_0),$$

and the entrepreneur's payoff is

$$V_c(\theta_0) = \Phi B - \frac{\phi}{2\rho + \phi} \frac{\Delta R_o \theta_0}{\bar{\kappa}} (B - b).$$

Let  $\theta_0^* = 1 - \frac{\rho + \phi}{2\rho + \phi} \frac{B - b}{\Delta R_o}$ .

1. If  $I - A \leq L_c(\theta_0^*)$ , the entrepreneur chooses  $\theta_0 = \theta_0^*$ .
2. If  $I - A \in (L_c(\theta_0^*), L_c(1)]$ , the entrepreneur chooses  $\theta_0 = \theta_0^{\min}$ , where  $\theta_0^{\min}$  is the unique solution to  $L_c(\theta_0^{\min}) = I - A$ .
3. If  $I - A > L_c(1)$ , the entrepreneur is unable to borrow enough.

The expressions behind  $L_c(\theta_0)$  and  $V_c(\theta_0)$  have straightforward interpretations. Specifically,  $L_c(\theta_0)$  includes  $\Pi_c(\theta_0)$ , the contribution from the bank, and  $(1 - \theta_0) q_c(\theta_0)$ , the amount lent by investors. The entrepreneur's payoff is also clear. Whereas  $\Phi B$  captures the discounted payoff if she always shirks and obtains the higher private benefit, the second term takes into account bank monitoring, which eliminates the high private benefit  $B$ . Instead, under  $p_L = 0$  and  $R_f = b/\Delta$ , the manager effectively receives an expected payoff  $b$ .  $\Delta R_o \theta_0 / \bar{\kappa}$  captures the probability of monitoring under  $\theta = \theta_0$ , and the term  $\frac{\phi}{2\rho + \phi}$  is the effective time discount, which includes random project maturing  $(\rho + \phi)$ , as well as the bank's equilibrium trading strategy so that its retention  $\theta_t$  declines exponentially at a rate  $\rho$ . Clearly, a higher initial retention  $\theta_0$  increases bank monitoring and the amount of total lending  $L_c(\theta_0)$ ; it also reduces the entrepreneur's expected payoff  $V_c(\theta_0)$ . Although the borrowing amount  $L_c(\theta_0)$  is maximized at  $\theta_0 = 1$ , the retention that maximizes the firm's overall payoff  $V_c(\theta_0) + L_c(\theta_0)$  has an interior solution  $\theta_0^* \in (0, 1)$ . If the borrowing constraint is slack so that  $L_c(\theta_0^*) > I - A$ , the entrepreneur chooses  $\theta_0^*$ . Otherwise, she chooses the minimum  $\theta_0$  that enables her to borrow just enough to invest.

### 3.3.2 Intermediation Equilibrium

Next, we describe the intermediation equilibrium under Assumption 5. As described in Proposition 4,  $\theta_t \equiv \theta_0 = 1$  always holds. Simple calculations show  $\tilde{D}(\theta) = \frac{\rho\bar{\kappa}}{\phi\Delta^2}$  is a constant, whereas

$$D^{\max}(\theta) = -\frac{\bar{\kappa}}{\Delta^2} + \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o\theta)^2}.$$

At  $\theta = 1$ , the optimal choice of deposit is an interior solution if and only if  $\tilde{D}(1) < D^{\max}(1)$ , which requires  $\Phi$  to be sufficiently high or, equivalently, the bank's cost of capital  $\rho$  to be sufficiently low. Otherwise, the bank's cost of capital is too high compared to that of investors. As a result, the bank will issue deposits up to the level constrained by its limited liability.

**Corollary 3.** *In the intermediation equilibrium, the firm is able to borrow*

$$L_i(1) = \Phi \left[ \frac{\Delta^2 [R_o^2 - (D^*(1))^2]}{2\bar{\kappa}} + \frac{\rho}{\phi} D^*(1) \right],$$

where  $D^*$  is available in the appendix. The entrepreneur's payoff is

$$V_i = \Phi B - \Phi \frac{\Delta (R_o - D^*(1))}{\bar{\kappa}} (B - b).$$

There exists a unique  $\underline{\Phi}$  where the constraint in deposit issuance binds at  $\theta = 1$  if and only if  $\Phi < \underline{\Phi}$ .

Similar to  $V_c$ , the entrepreneur's payoff  $V_i$  also has an intuitive interpretation. Whereas the first term considers the discounted payoff if she always shirks, the second term takes into account bank monitoring, which happens with conditional probability  $\frac{\Delta R_o - D^*(1)}{\bar{\kappa}}$ . Indeed, this expression is obtained by evaluating (4) at  $\theta = 1$  and  $D = D^*(1)$ .

### 3.3.3 Initial Choice

We study the borrower's initial choice between certification and intermediation. Note the entrepreneur is always able to borrow more in intermediation, regardless of the deposit-issuance constraint. Therefore, if  $I - A \in (L_c(1), L_i(1))$ , the entrepreneur is only able to borrow through intermediation at  $t = 0$ . The most interesting comparison is under the case  $I - A < L_c(\theta^*)$ , so that the borrowing constraint is slack in both certification and intermediation. Proposition 5 shows certification could dominate intermediation.

**Proposition 5.** *Under Assumption 5 and  $L_c(\theta_0^*) > I - A$ , a threshold  $B^*$  exists such that the entrepreneur chooses certification if and only if  $B > B^*$ .*

We explain the mechanism behind Proposition 5. Given  $L_i(1) > L_c(\theta_0)$ ,<sup>23</sup> the entrepreneur may prefer certification only if  $V_c(\theta_0^*) > V_i$ , that is, if the entrepreneur obtains a higher expected payoff under certification. Simple calculation shows

$$V_c - V_i = \frac{\phi\Delta}{\bar{\kappa}} (B - b) \left( \frac{R_o - D^*(1)}{\rho + \phi} - \frac{R_o\theta_0}{2\rho + \phi} \right).$$

The term  $\frac{R_o\theta_0}{2\rho + \phi}$  captures the fact that in certification, the bank starts with  $\theta_0$  but sells at rate  $\rho$  over time. By contrast,  $\frac{R_o - D^*(1)}{\rho + \phi}$  shows that an intermediating bank always retains  $\theta = 1$ , and the benefit from issuing deposits  $D^*(1)$  will ultimately accrue to the entrepreneur. Clearly,  $V_c > V_i$  if and only if

$$\frac{R_o - D^*(1)}{\rho + \phi} > \frac{R_o\theta_0}{2\rho + \phi},$$

which is necessarily the case if  $\rho$ ,  $\theta_0$ , and  $D^*(1)$  are small. Conditional on  $V_c > V_i$ , a higher  $B$  leads to a larger difference in the entrepreneur's payoff. If the difference becomes sufficiently large, it can offset the difference in the amount of borrowing ( $L_i(1) - L_c(\theta_0)$ ). In this case, the entrepreneur ends up choosing certification.

This result is reminiscent of the literature on the potential of overmonitoring (Pagano and Röell, 1998).<sup>24</sup> From the perspective of the entrepreneur, she cares not only about the market value of the project, but also her future benefits as the manager. Therefore, a level of monitoring that maximizes the firm value may be excessive. Although bank monitoring introduces a positive externality to investors, it also imposes a negative externality on the entrepreneur by restricting her from choosing the high private benefit  $B$ .

Our result implies more financially-constrained entrepreneurs borrows from intermediating banks, such as commercial and community banks, whereas entrepreneurs with more net worth and who potentially enjoy higher private benefits (or control rents) tend to borrow from certifying banks, such as venture capitalists (VCs). To the best of our knowledge, this result has not been covered by previous theoretical papers. De Bettignies and Brander (2007) argues the critical distinction between VCs and banks is that only VCs provide value-adding services. Winton and Yerramilli (2008) argues that compared with banks, VCs are more capable monitors but have higher costs of capital. Similarly, Ueda (2004) assumes VCs can screen borrowers better than banks before the investment is made. Her model predicts entrepreneurs will approach VCs when they require large investment amounts. Landier (2005) introduces two-side hold-up problems and the stigma associated with a failed entrepreneur. He argues that if the entrepreneur is held up by investors,

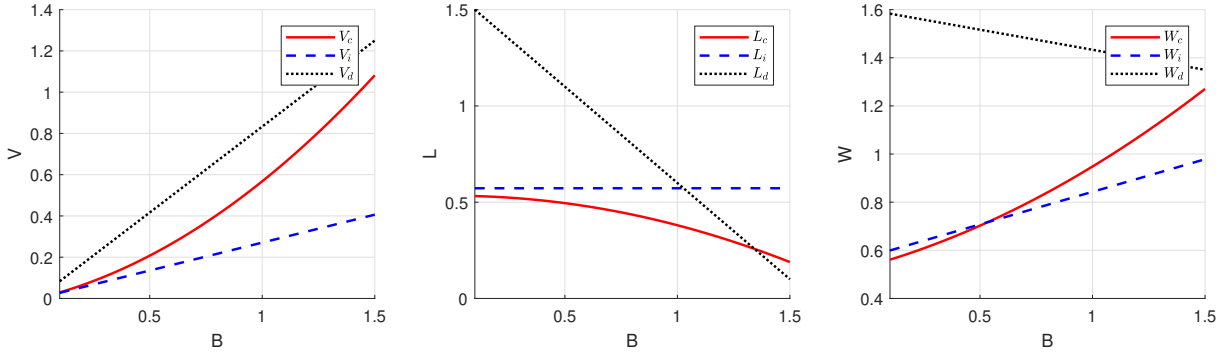
<sup>23</sup>The proof follows because  $L_i(1) \geq L_c(1) > L_c(\theta_0)$  and  $D^*(1) \leq \tilde{D}(1) = \frac{\rho\bar{\kappa}}{\phi\Delta^2}$ .

<sup>24</sup>See section 9.2.2 in Tirole (2010) for a summary of this literature.

bank financing is more preferable. By contrast, VC financing is more preferable if investors can be held up by the entrepreneur.

One may wonder whether the entrepreneur may benefit by simultaneously borrowing from both an intermediating bank and investors. The answer is yes. Under some conditions, the constrained-optimal allocation features the bank's retention  $\theta_t$  increasing over time. An example is illustrated in Corollary 1, if  $\theta_0 = \frac{\Delta R - B}{\Delta R - b} < \frac{1}{2}$ . This solution can be better approximated by combining intermediation and direct borrowing from outside investors. Over time, the intermediating bank buys the loan from outside investors. This arrangement can potentially lead to a higher entrepreneur's payoff than both certification and intermediation.

**Numerical Example** We illustrate the comparison using the following numerical example. Let  $\phi = 1$ ,  $b = 0$ ,  $R = 2$ ,  $p_H = 0.8$ ,  $p_L = 0$ ,  $\rho = 0.01$ , and  $\kappa$  follow the uniform distribution on  $[0, 2]$ . Under given parameters, we can easily calculate that  $\tilde{D}(1) = 0.0312$ ,  $D^{\max} = 0.3122$ , and  $\kappa_i = 1.5750$ . We also include the option of directly borrowing from investors, in which case the payoffs are  $V_d = \Phi B$ ,  $L_d = p_H R - B$ , and  $W_d = p_H R - (1 - \Phi) B$ .



**Figure 4: Valuation under Certification, Intermediation, and Direct Lending**

Figure 4 illustrates the payoffs when the private benefit  $B$  varies. The red (solid), blue (dashed), and black (dotted) lines, respectively, stand for certification, intermediation, and directly lending. The left panel describes the entrepreneur's payoff, which increases with  $B$  in all three cases. The entrepreneur always obtains the highest payoff in direct lending. This result illustrates the negative externalities of bank monitoring on the entrepreneur. Moreover, her payoff is convex in  $B$  in certification but linear in intermediation. The middle panel shows the maximum amount of borrowing, which is independent of  $B$  in intermediation but decreases in the other two cases, because  $\theta_t \equiv 1$  in intermediation but  $\theta_t$  declines in certification. Finally, the right panel compares the overall payoff. Clearly, certification has a higher overall payoff than intermediation once  $B$  becomes sufficiently high.

## 4 Extensions

### 4.1 Certification: a Lockup Period and Minimum Retention

#### Lockup Period

We introduce a lockup arrangement that allows the bank to commit to  $\theta_0$  for a period  $[0, t_\ell]$ . Due to the stationary environment, the subgame starting from  $t_\ell$  and the associated equilibrium are unchanged from those in 3.1. Between  $[0, t_\ell]$ , no trading occurs, and the bank's flow payoff is  $\phi\pi_c(\theta_0)$ . Let  $L_\ell$  be the total lending at  $t = 0$ . Under Assumption 5, it is easily shown that

$$L_\ell = \Pi_c(\theta_0) + (1 - \theta_0) \left[ d_c(\theta_0) \left( 1 - e^{-\phi t_\ell} \right) + e^{-\phi t_\ell} q_c(\theta_0) \right].$$

Again, this expression confirms the earlier result that due to the lack of commitment, the certifying bank does not benefit from its ability to trade loans at all: it is able to lend exactly  $\Pi_c(\theta_0)$  regardless of  $t_\ell$ . Meanwhile, investors are willing to lend more as the lockup period  $t_L$  becomes longer, because  $q_c(\theta_0) = \Phi d_c(\theta_0)$ . Therefore,  $\frac{\partial L_\ell}{\partial t_\ell} > 0$  so that the lockup period increases the total amount of lending: the incremental lending comes from investors' willingness to lend due to the bank's improved commitment.

Moreover, we use  $V_\ell$  to denote the entrepreneur's overall payoff. Under Assumption 5,

$$V_\ell = \Phi \left[ \frac{\Delta R_o \theta_0}{\bar{\kappa}} b + \left( 1 - \frac{\Delta R_o \theta_0}{\bar{\kappa}} \right) B \right] \left( 1 - e^{-(\rho+\phi)t_\ell} \right) + e^{-(\rho+\phi)t_\ell} V_c(\theta_0).$$

Intuitively, during the lockup period  $[0, t_\ell]$ , the entrepreneur is monitored and receives  $b$  with probability  $\Delta R_o \theta_0 / \bar{\kappa}$ ; otherwise, she receives  $B$ . It is easily derived that  $\frac{\partial V_\ell}{\partial t_\ell} < 0$ , so that a longer lockup period leads to a lower payoff to the entrepreneur. Intuitively, the entrepreneur receives  $b$  with monitoring but  $B$  without monitoring. Although a longer lockup period increases monitoring, it also reduces the entrepreneur's equilibrium payoff.

Let  $W_\ell = V_\ell + L_\ell$  be the aggregate social welfare, which is also the objective function that the entrepreneur tries to maximize at  $t = 0$ . Under Assumption 5, a longer lock-up period always increases the aggregate social welfare.

**Corollary 4** (Optimal Lockup Period). *For  $\theta_0 = \theta_0^*$  as in Corollary 2, the aggregate social welfare attains the maximum as  $t_\ell \rightarrow \infty$ .*

## Minimum Retention

According to section 941 of the Dodd-Frank Act, securitizers are required to retain no less than 5% of the credit risks associated with any securitization to perform intermediation services. This rule is commonly known as risk retention. In February 2018, the circuit court exempted CLO funds, We evaluate such a policy by imposing a minimum retention requirement on the bank.<sup>25</sup>

Suppose the bank must hold at least a fraction  $\underline{\theta}$  of the loans on its balance sheet. The equilibrium is qualitatively similar to the one in Proposition 3: some threshold  $\tilde{\theta}$  exists such that smooth trading occurs for  $\theta \in (\tilde{\theta}, 1]$ , and an atom  $\tilde{\theta}$  exists where the holdings jump to  $\underline{\theta}$ . For the same reasons that liquidation has to be random in the certification case, the jump from  $\tilde{\theta}$  to  $\underline{\theta}$  must also happen with a random delay.

The construction of the equilibrium is similar to the one without minimum retention, with details available in the appendix. The only difference is that we need to consider the incentives to jump to the constant  $\underline{\theta}$  rather than the incentives to sell off the entire loan (i.e., to jump to  $\theta = 0$ ).

## 4.2 Intermediation: Alternative Bank Liability Structures

In this subsection, we study the intermediation equilibrium under alternative liability structures. This exercise highlights the crucial feature of short-term deposits: that the deposit rate can adjust as soon as the bank changes its retention, to reflect the credit risks borne by depositors. Specifically, we are going to show that the bank's commitment to retention will no longer hold under long-term deposits, nor under deposits that are sufficiently subsidized.

### Long-term Deposits/Outside Equity

We solve the intermediation equilibrium in which the bank can only issue long-term deposits at  $t = 0$ , that is, deposits that only mature with the project at  $\tau_\phi$ . Given that the final cash flow has a binary outcome of either  $R$  or  $0$ , long-term deposits are identical to outside equity. This extension highlights the importance of short-term deposits in commitment. Specifically, we show that under long-term deposits, the equilibrium is identical to the certification equilibrium, in which the bank trades loans over time. Therefore, a certifying bank can be equivalently understood as an intermediating bank but issues long-term debt.

Let  $D_0 = D$  be the amount of long-term deposits that the bank issues at  $t = 0$ . Note that after  $t = 0$ , the bank can no longer raise any deposits. The definitions for  $\kappa_i$ ,  $\hat{p}(\theta, D)$ , and  $\hat{\pi}_i(\theta, D)$  follow from those in section 2. Because  $D_0 = D$  is only chosen at  $t = 0$ , we suppress these

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<sup>25</sup>Securitization is identical to loan sales in our setup, given the binary outcome of the final cash flows.

functions' dependence on  $D$ , and therefore refer to  $\kappa_i(\theta)$ ,  $p(\theta)$ , and  $\pi_i(\theta)$ , respectively. The bank's maximization problem becomes

$$\max_{\{\theta_t, D_0\}} (1 - y_0) D_0 + \mathbb{E} \left[ \int_0^\infty e^{-(\rho+\phi)t} \left( \phi \pi_i(\theta_t) dt + dG(\theta_t) \right) \right],$$

where  $y_0$  compensates the depositors' credit risk. Note that after  $t = 0$ , the bank only chooses its trading strategy, and the solution method as well as the equilibrium outcomes naturally follow the certification equilibrium in subsection 3.1. In fact, they two are identical because depositors are effectively investors who directly invest in the venture.<sup>26</sup>

## Deposit Subsidies

Next, we introduce an extension where the deposit rates are partially subsidized by the government. One can think about this subsidy as either deposit insurance or the implicit guarantee from a government bailout.<sup>27</sup> We show that once the subsidy becomes sufficiently high, an intermediating bank can no longer commit to its retention but instead sells loans over time, just as a certifying bank. This exercise highlights the importance of the deposit rate in aligning the bank's incentives to commit to its retentions.

Specifically, we assume the bank only needs to pay a fraction  $\xi$  of the deposit rate so that equation (10) becomes

$$y_t = \phi \xi (1 - \hat{p}(\theta, D)),$$

where  $\xi \in (0, 1)$ . The analysis follows that in subsection 3.2, in which deposit issuance and trading are solved sequentially.

**Proposition 6.** *A  $\xi_\dagger$  exists such that in the intermediation equilibrium,  $\dot{\theta} < 0$  if  $\xi < \xi_\dagger$  for  $\theta$  sufficiently large.*

Intuitively, the bank no longer has the incentives to retain its loans if the deposit is mostly subsidized by the government and when  $\theta_t$  is still high such that the deposit-issuance constraint is

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<sup>26</sup>A subtle difference is that long-term depositors cannot trade their deposits, whereas investors in certification can sell the loans. However, given that in the certification equilibrium, the bank sells the loan (and, equivalently, investors buy the loan) over time, this difference does not affect the result. In other words, in the equilibrium with long-term deposits, depositors buy loans in the secondary market after  $t = 0$ .

<sup>27</sup>In the U.S., deposit insurance takes the form of a maximum guaranteed amount that has been \$250,000 since 2010. There is a one-to-one mapping between the maximum insurance amount and the parameter  $\xi$  introduced later on. To see this, note one can think about the deposit rate as  $y_t = 0$  for deposits below \$250,000 but following (10) for deposits above the limit. Our parameter  $\xi$  captures the fraction of deposits that are above the limit.

slack. If  $\theta_t$  is low, this result is no longer true, because the deposit-issuance constraint binds; that is,  $D^*(\theta) = D^{\max}(\theta)$ . In this case, the bank will have incentives to retain loans. The overall effect in this case combined the two.

### Bail-in vs. Bailout

Let us modify the model by assuming if the project fails and generates nothing, the bank is able to pay the depositors up to  $X > 0$ . One can think about  $X$  as the level of the bank's risk-absorbing equity or the liquidity required to put aside in case of bank failure.<sup>28</sup> The bank's incentive compatibility constraint in monitoring becomes  $\kappa_i = \Delta(R_o\theta + X - D)$ . Moreover, it is never optimal for the bank to issue risk-less deposits. In other words, the endogenous choice of deposits always satisfies  $D > X$ , with an interest rate  $\hat{y}(\theta, D) = \phi(1 - \hat{p}(\theta, D - X))(1 - \frac{X}{D})$ .<sup>29</sup> The equilibrium in this case is qualitatively unchanged from the intermediation equilibrium described in Proposition 4, instead of the one highlighted in Proposition 6.

The difference between  $X$  and  $\xi$  can be interpreted as the bail-ins vs. bailouts. Once again, this difference highlights the importance of monitoring externalities. When deposits are subsidized, only a fraction of the externalities is internalized, so the bank's incentives to monitor are also reduced. In the extreme case where the deposit rate is independent of the bank's retention (and therefore monitoring), the results go back to the case of certification. By contrast, if a bail-in occurs and the bank compensates the depositors' part of the losses ( $D/X$ ) in the case of a bank failure, the monitoring externalities are still internalized.

## 5 Conclusion and Implications

This paper presents a dynamic theory of intermediary financing when banks cannot commit to its retentions. We show two equivalence results fail in the dynamic framework without commitment. First, the implementation of the financial structure leads to different credit market dynamics. Whereas a certifying bank sells its loans over time, an intermediating bank retains its loans through issuing short-term deposits. As a result, certification in general leads to a lower borrowing amount than intermediation. Second, maximizing the borrowing amount is different from maximizing the

<sup>28</sup>One example is the liquidity-coverage ratio (LCR).

<sup>29</sup>Following the same step as in the derivation of equation (24), we can write the bank's payoff function in terms of net debt  $\tilde{D} \equiv D - X$ :

$$\mathcal{V}(\tilde{D}, \theta) := X + \Phi \left[ \hat{p}_i(\theta, \tilde{D})\theta R_o - \int_0^{\tilde{\kappa}_i} \kappa dF(\kappa) \right] + (1 - \Phi) \tilde{D}.$$



entrepreneur’s expected payoff. In particular, if the entrepreneur can borrow enough under both structures, she may prefer certification to intermediation for a higher expected payoff.

While we focus on monitoring, we have not modeled screening (Ramakrishnan and Thakor, 1984) – another valuable banking service that gives the relationship bank upfront private information. As shown by Parlour and Plantin (2008), this informational advantage will lead to illiquidity in the secondary loan market.<sup>30</sup> Initial screening will allow for the relationship bank to signal through either dynamic retention or debt issuance.<sup>31</sup> Given that our focus is on how the bank’s liability structure enables commitment to retention, we chose to stay away from these two complications introduced from screening.

In the paper, we only allow for one bank to monitor after  $t = 0$ . If we allow the entrepreneur to sign contracts with multiple banks that can all monitor after time 0, two additional effects will arise. First, if the monitoring costs are *i.i.d.* across banks, introducing multiple banks naturally leads to a diversification effect, which increases the chances of monitoring. However, a pecuniary externality exists when one bank trades its loans over time in the sense that it only has a partial price impact. As a result, banks will sell faster. The overall effect is therefore unclear.

We have not discussed the important role of bank equity. Mehran and Thakor (2011) shows that better-capitalized banks conduct more monitoring. Berger and Bouwman (2013) provide related evidence. We have not modeled the intermediary’s concern for reputation, as in Chemmanur and Fulghieri (1994b) and Chemmanur and Fulghieri (1994a). While reputation does not directly affect returns to monitoring, it nevertheless can be important in the certification equilibrium. In particular, one can construct an equilibrium in which the commitment problem is mitigated if the intermediary has long-run reputation. Given that commitment is already achieved in the intermediation equilibrium, our paper implies that 1) reputation is especially value-adding in certifying intermediaries, and 2) that reputation and appropriately-chosen liability structures can be substitutes in implementing loan retention. Another missing element is the diversification of banks’ financing projects.<sup>32</sup>

For tractability reasons, we have assumed the entrepreneur is able to commit to her retention over time. Relaxing this assumption and studying the interaction between the commitment problems between the borrower and the bank can be an interesting extension in the future. In particular, we can study the role of many sophisticated contracts such as covenants in preventing the commitment issues. Similarly, one can also introduce the role of a monitoring blockholder, and study the interaction between activism and bank monitoring (Admati et al., 1994; DeMarzo and Urošević,

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<sup>30</sup>More recently, Hu and Varas (forthcoming) showed that zombie lending will emerge in this context when screening takes time.

<sup>31</sup>See Leland and Pyle (1977) and Ross (1977) for related issues in the static environment.

<sup>32</sup>In Holmstrom and Tirole (1997) and Parlour and Plantin (2008), risks also cannot be diversified. Perfect diversification will crowd out the need for retention, as shown in Diamond (1984).

2006).

Another interesting extension is along the lines of [Gorton and Pennacchi \(1990\)](#). One can imagine information on the entrepreneur's asset payoffs arrives gradually so that uncertainty is resolved gradually. In this case, although instantly maturing deposits (or, in general, short-term debt) improve the commitment problem and eliminate the dilution effect ([DeMarzo and He, forthcoming](#)), they also expose the borrowers to rollover risks ([Diamond, 1991](#)). Studying how a carefully designed security could trade off between commitment and risk hedging would be interesting.

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## Appendix

### Proof of Proposition 2 and Corollary 1

*Proof.* Given any  $\{\theta_t\}_{t \geq 0}$ , the entrepreneur's payoff is  $\int_0^\infty e^{-(\rho+\phi)t} \phi v(\theta_t) dt$ , whereas the bank receives  $\int_0^\infty e^{-(\rho+\phi)t} \phi \pi(\theta_t) dt$  and investors receive  $\int_0^\infty e^{-\phi t} \phi (1 - \theta_t) d(\theta_t) dt$ .

$$\begin{aligned} v(\theta) &= p(\theta) (R - R_o) + (1 - F(\kappa_c)) B \\ \pi(\theta) &= p(\theta) R_o \theta - \int_0^{\kappa_c} \kappa dF(\kappa) \\ d(\theta) &= p(\theta) R_o. \end{aligned}$$

From here, we get that the planners problem is

$$\begin{aligned} W &= \max_{(\theta_t)_{t \geq 0}} \int_0^\infty \phi e^{-\phi t} \left[ (1 - \theta_t) d(\theta_t) + e^{-\rho t} (v_c(\theta) + \pi_c(\theta)) \right] dt \\ &= \max_{(\theta_t)_{t \geq 0}} \int_0^\infty \phi e^{-\phi t} \left\{ (1 - e^{-\rho t}) [(1 - \theta_t) p(\theta_t) R_o] + e^{-\rho t} \left[ p(\theta_t) R + (1 - F(\kappa_c)) B - \int_0^{\kappa_c} \kappa dF(\kappa) \right] \right\} dt. \end{aligned}$$

We can maximize the previous expression pointwise to get the first order condition

$$(1 - \theta_t) f(\kappa_c) \Delta R_o (1 - e^{-\rho t}) \frac{\partial \kappa_c}{\partial \theta} - p(\theta_t) (1 - e^{-\rho t}) R_o + e^{-\rho t} f(\kappa_c) (\Delta R - B - \kappa_c) \frac{\partial \kappa_c}{\partial \theta} = 0$$

Substituting  $\kappa_c = \Delta R_o \theta$  we get

$$(1 - \theta_t) f(\kappa_c) (1 - e^{-\rho t}) (\Delta R_o)^2 - (p_L + F(\kappa_c) \Delta) (1 - e^{-\rho t}) R_o + e^{-\rho t} f(\kappa_c) (\Delta R - B + \kappa_c) \Delta R_o = 0$$

Under Assumption 5, this simplifies to

$$\theta_t = \frac{\Delta R - b - e^{-\rho t} (B - b)}{(\Delta R - b) (2 - e^{-\rho t})}.$$

Differentiating we get

$$\dot{\theta}_t = \frac{\rho e^{\rho t} (2B - b - \Delta R)}{(\Delta R - b) (2 - e^{\rho t})^2}.$$

Therefore,

$$\dot{\theta}_t < 0 \Leftrightarrow \frac{\Delta R - B}{B - b} > 1.$$

At time zero we have

$$\theta_0 = \frac{\Delta R - B}{\Delta R - b},$$

and as  $t \rightarrow \infty$

$$\lim_{t \rightarrow \infty} \theta_t = \frac{\Delta R - b}{2(\Delta R - b)} = \frac{1}{2}.$$

The expression for the welfare function is straightforward. The optimal retention satisfies the first order condition

$$(1 - e^{-\rho t}) [(1 - \theta_t) f(\kappa_c) (\Delta R_o)^2 - p(\theta_t) R_o] + e^{-\rho t} f(\kappa_c) (\Delta R - B - \kappa_c) \Delta R_o = 0.$$

Plugging in  $\theta_t = 1$ , we can show that the FOC satisfies

$$(1 - e^{-\rho t}) [-p(1) R_o] + e^{-\rho t} f(\kappa_c) (b - B) \Delta R_o < 0.$$

The solution under Assumption 5 is straightforward, as well as the verification that the second-order condition is satisfied.  $\square$

## A Certification

In this section, we prove Proposition 3.

**Lemma 3.** *The bank with retention  $\theta$  never sells a fraction of the loans.*

*Proof.* Suppose the bank with retention  $\theta$  sells  $\theta^+ - \theta$ , where  $\theta^+ > 0$ , and that after this it continues trading smoothly. Multiple jumps are ruled out without loss of generality. In this case, the overall trading gains are  $dG(\theta) + \Pi_c(\theta^+) - \Pi_c(\theta) = (\theta - \theta^+) q_c(\theta^+) + \Pi_c(\theta^+) - \Pi_c(\theta)$ , where  $dG(\theta)$  is the instant trading gain and  $\Pi(\theta^+) - \Pi(\theta)$  are the gains (negative loss) in its continuation value. Block trading is suboptimal as long as

$$\theta = \arg \max_{\theta^+} \left\{ \Pi_c(\theta^+) + (\theta - \theta^+) q_c(\theta^+) \right\}. \quad (30)$$

It is easy to verify that the first order condition is always satisfied at  $\theta^+ = \theta$ , thus it suffices to show that the second order condition for global optimality is satisfied.  $\square$

## Verification of Optimality Trading Strategy

In this section, we complete the characterization of the equilibrium by verifying that the equilibrium trading strategy maximizes the bank's payoff given the price function  $q(\theta)$ . Because the payoff in a mixed strategy equilibrium is given by the payoff of any pure strategy in its support, we can restrict attention to pure strategies in the verification of optimality. A trading strategy for the bank is given by right continuous function with left limits. A trading strategy is admissible if it can be decomposed as

$$\theta_t = \int_0^t \dot{\theta}_t^c dt + \sum_{k \geq 0} (\theta_{t_k}^d - \theta_{t_k-}^d),$$



for some bounded function  $\dot{\theta}_t^c$ . We denote the set of admissible trading strategies by  $\Theta$ . The bank's optimization problem is to choose  $\theta \in \Theta$  to maximize its payoff

$$\Pi^*(\theta_0) = \sup_{\theta \in \Theta} \int_0^\infty e^{-(\rho+\phi)t} (\phi\pi(\theta_t) - \dot{\theta}_t^c q(\theta_t)) dt - \sum_{k \geq 0} e^{-(\rho+\phi)t_k} q(\theta_{t_k}) (\theta_{t_k}^d - \theta_{t_k-}^d). \quad (31)$$

Due to the discontinuity in the price function  $q(\theta)$ , the Hamilton-Jacobi-Bellman (HJB) equation is discontinuous at  $\theta_*$ , so we need to resort to the theory of viscosity solutions for the analysis of the bank's problem. Our problem is a particular case of the general class of optimal control problems in stratified domains studied by [Barles et al. \(2018\)](#). Our proof relies on their characterization of the value function using viscosity solution methods. The analysis in [Barles et al. \(2018\)](#) does not consider the case in which the trajectory of the state variable can be discontinuous (impulse control). However, as we show below, we can approximate a trading  $\theta_t \in \Theta$  by an absolutely continuous trading strategy with derivative  $|\dot{\theta}_t| \leq N$  for some  $N$  large enough (the approximation is in the sense that it yields a similar payoff). Thus, we can consider a sequence of optimization problems

$$\Pi_N^*(\theta_0) = \sup_{|\dot{\theta}_t| \leq N} \int_0^\infty e^{-(\rho+\phi)t} (\phi\pi(\theta_t) - \dot{\theta}_t q(\theta_t)) dt, \quad (32)$$

and verify that, for any  $\theta \in [0, 1]$ ,  $\Pi_N^*(\theta) \rightarrow \Pi(\theta)$ , where

$$\Pi(\theta) = \begin{cases} \Phi\pi(\theta) & \text{if } \theta \in [\theta_*, 1] \\ q(0)\theta & \text{if } \theta \in [0, \theta_*) \end{cases}.$$

The following Lemma establishes that we can indeed consider the limit of bounded absolutely continuous strategies.

**Lemma 4.** *For any  $\theta_0 \in [0, 1]$ ,  $\lim_{N \rightarrow \infty} \Pi_N^*(\theta_0) = \Pi^*(\theta_0)$ .*

*Proof.* Let  $\theta_t^{\epsilon*}$  be an  $\epsilon$ -optimal policy (at this point in the proof we have not established existence of an optimal policy). For any  $k \geq 0$ , let  $\Delta_k \equiv \inf\{\Delta > 0 : \theta_{t_k-\Delta}^{\epsilon*} + \text{sgn}(\theta_{t_k}^{\epsilon*} - \theta_{t_k-}^{\epsilon*})N\Delta = \theta_{t_k}^{\epsilon*}\}$  (we can find  $\Delta_k$  if  $N$  is large enough as  $|\dot{\theta}_t^{\epsilon*}| \leq M$  for some finite  $M$ ). Consider the policy  $\hat{\theta}_t^N = \theta_t^{\epsilon*}$  if  $t \notin \cup_{k \geq 0} (t_k - \Delta_k, t_k)$ , and  $\hat{\theta}_t^N = \theta_{t_k-\Delta_k}^{\epsilon*} + \text{sgn}(\theta_{t_k}^{\epsilon*} - \theta_{t_k-}^{\epsilon*})N(t - t_k + \Delta_k)$  if  $t \in \cup_{k \geq 0} (t_k - \Delta_k, t_k)$ . The difference between the payoff

of  $\theta_t^{\epsilon*}$  and  $\hat{\theta}_t^N$  is

$$\begin{aligned}\Pi^{\epsilon*}(\theta_0) - \hat{\Pi}_N &= \sum_{k \geq 0} \left\{ \int_{t_k - \Delta_k}^{t_k} e^{-(\rho+\phi)t} (\phi\pi(\theta_t^{\epsilon*}) - \dot{\theta}_t^{\epsilon c*} q(\theta_t^{\epsilon*}) - \phi\pi(\hat{\theta}_t^N)) dt \right. \\ &\quad \left. + \int_{t_k - \Delta_k}^{t_k} e^{-(\rho+\phi)t} \text{sgn}(\theta_{t_k}^{\epsilon*} - \theta_{t_k-}^{\epsilon*}) N q(\hat{\theta}_t^N) dt - e^{-(\rho+\phi)t_k} q(\theta_{t_k}^{\epsilon*}) (\theta_{t_k}^{\epsilon d*} - \theta_{t_k-}^{\epsilon d*}) \right\} - \epsilon \\ &= \sum_{k \geq 0} \left\{ \int_{t_k - \Delta_k}^{t_k} e^{-(\rho+\phi)t} (\phi\pi(\theta_t^*) - \dot{\theta}_t^{\epsilon c*} q(\theta_t^*) - \phi\pi(\hat{\theta}_t^N)) dt \right. \\ &\quad \left. + \frac{\theta_{t_k}^{\epsilon*} - \theta_{t_k - \Delta_k}^{\epsilon*}}{\Delta_k} \int_{t_k - \Delta_k}^{t_k} e^{-(\rho+\phi)t} q(\hat{\theta}_t^N) dt - e^{-(\rho+\phi)t_k} q(\theta_{t_k}^{\epsilon*}) (\theta_{t_k}^{\epsilon d*} - \theta_{t_k-}^{\epsilon d*}) \right\} - \epsilon.\end{aligned}$$

For all  $k \geq 0$ , we have that  $\Delta_k \downarrow 0$  as  $N \rightarrow \infty$ . It follows that

$$\lim_{\Delta_k \downarrow 0} \frac{1}{\Delta_k} \int_{t_k - \Delta_k}^{t_k} e^{-(\rho+\phi)t} q(\hat{\theta}_t^N) dt = \begin{cases} e^{-(\rho+\phi)t_k} q(\theta_{t_k}^{\epsilon*} -) & \text{if } \theta_{t_k}^{\epsilon*} > \theta_{t_k-}^{\epsilon*} \\ e^{-(\rho+\phi)t_k} q(\theta_{t_k}^{\epsilon*} +) & \text{if } \theta_{t_k}^{\epsilon*} < \theta_{t_k-}^{\epsilon*}. \end{cases}$$

The price function is right continuous so  $q(\theta_{t_k}^{\epsilon*} +) = q(\theta_{t_k}^{\epsilon*})$ . We can conclude that

$$\lim_{N \rightarrow \infty} (\Pi^{\epsilon*}(\theta_0) - \hat{\Pi}_N(\theta_0)) = \sum_{k \geq 0} e^{-(\rho+\phi)t_k} (q(0) - q(\theta_*)) (\theta_{t_k}^* - \theta_{t_k-}^*)^+ \mathbf{1}_{\{\theta_{t_k}^* = \theta_*\}} - \epsilon \leq 0.$$

Because this holds for any  $\epsilon > 0$ , we can conclude that  $\lim_{N \rightarrow \infty} (\Pi^*(\theta_0) - \hat{\Pi}_N(\theta_0)) \leq 0$ , and given that  $\Pi^*(\theta_0) \geq \hat{\Pi}_N(\theta_0)$ , we get  $\lim_{N \rightarrow \infty} \hat{\Pi}_N(\theta_0) = \Pi^*(\theta_0)$ . For  $N$  large enough, the policy  $\hat{\theta}_t^N$  satisfies  $|\dot{\hat{\theta}}_t^N| \leq N$  (this can be guaranteed because for any  $\epsilon$  there is  $M$  such that  $|\dot{\theta}_t^{\epsilon c*}| \leq M$ ), so its payoff,  $\hat{\Pi}_N(\theta_0)$  provides a lower bound to  $\Pi_N^*(\theta_0)$ , which means that  $\lim_{N \rightarrow \infty} \Pi_N^*(\theta_0) = \Pi^*(\theta_0)$ .  $\square$

This shows that the value function converges (pointwise) to the one in the equilibrium under consideration. Hence, we can verify the optimality of the bank's strategy by analyzing the control problem (32). For future reference, recall that the price function in the control problem (32) is given by

$$q(\theta) = \begin{cases} \Phi\pi'(\theta) & \text{if } \theta \geq \theta_* \\ p_L R(1 - \alpha) & \text{if } \theta < \theta_* \end{cases}, \quad (33)$$

where the threshold  $\theta_*$  is given by  $\Phi\pi(\theta_*) = q(0)\theta_*$ . Notice that we are not computing the equilibrium in a model in which the bank is restricted to use absolutely continuous trading strategies with bounded derivative  $\dot{\theta}_t$ , but rather considering the equilibrium price function in the general case, and then considering a sequence of auxiliary optimization problems to construct the value function. Because the expected payoff of the candidate equilibrium strategy is equal to the value function, it is necessarily optimal.

The Hamilton-Jacobi-Bellman equation (HJB) for the optimization problem (32) is

$$(\rho + \phi)\Pi_N(\theta) - H(\theta, \Pi'_N(\theta)) = 0, \quad (34)$$

where  $H$

$$H(\theta, \Pi'_N) \equiv \phi\pi(\theta) + \max_{|\theta| \leq N} \left\{ \dot{\theta}(\Pi'_N - q(\theta)) \right\}. \quad (35)$$

We guess and verify that, for  $N$  large enough, the solution (in the viscosity sense) of the previous equation is

$$\Pi_N(\theta) = \begin{cases} \Phi\pi(\theta) & \text{if } \theta \in [\theta_*, 1] \\ e^{-\frac{\rho+\phi}{N}(\theta_*-\theta)}\Phi\pi(\theta_*) + \frac{(\rho+\phi)}{N} \int_{\theta}^{\theta_*} e^{-\frac{\rho+\phi}{N}(y-\theta)} \left( \Phi\pi(y) - \frac{N}{\rho+\phi}q(0) \right) dy & \text{if } \theta \in [\tilde{\theta}_N, \theta_*] \\ \frac{N}{\rho+\phi} \left( 1 - e^{-\frac{(\rho+\phi)}{N}\theta} \right) q(0) + \frac{(\rho+\phi)}{N} \int_0^{\theta} e^{-\frac{(\rho+\phi)}{N}(\theta-y)} \Phi\pi(y) dy & \text{if } \theta \in [0, \tilde{\theta}_N], \end{cases} \quad (36)$$

where  $\tilde{\theta}_N$  is the unique solution on  $[0, 1]$  to the equation

$$\begin{aligned} \frac{N}{\rho+\phi} \left( 1 - e^{-\frac{(\rho+\phi)}{N}\tilde{\theta}_N} \right) q(0) + \frac{(\rho+\phi)}{N} \int_0^{\tilde{\theta}_N} e^{-\frac{(\rho+\phi)}{N}(\tilde{\theta}_N-y)} \Phi\pi(y) dy = \\ e^{-\frac{\rho+\phi}{N}(\theta_*-\tilde{\theta}_N)}\Phi\pi(\theta_*) + \frac{(\rho+\phi)}{N} \int_{\tilde{\theta}_N}^{\theta_*} e^{-\frac{\rho+\phi}{N}(y-\tilde{\theta}_N)} \left( \Phi\pi(y) - \frac{N}{\rho+\phi}q(0) \right) dy \end{aligned} \quad (37)$$

## A.1 Auxiliary Lemmas

Before proceeding with the verification theorem, we provide several Lemmas providing properties of our candidate value function  $\Pi_N(\theta)$  that will be later used in the verification

**Lemma 5.** *If  $\Phi\pi_c(1) > p_L R_o > 0$ , then there exists a unique  $\theta_* \in (0, 1)$  solving the equation*

$$\theta_* q_c(0) = \Phi\pi_c(\theta_*) \quad (38)$$

*If  $p_L = 0$ , then  $\theta_* = 0$  is the unique solution to (38) on  $[0, 1]$ .*

*Proof.* As  $\Phi\pi_c(0) = 0$ , equation (38) is trivially satisfied at  $\theta_* = 0$ , we want to show that if  $\Phi\pi_c(1) > p_L R_o = q_c(0)$ , then there is a non trivial solution  $\theta_* > 0$  that also satisfies equation (38). First, if  $\Phi\pi_c(1) > p_L R_o$ , then the right hand side of equation (38) is strictly larger than its left hand side evaluated at  $\theta_* = 1$ . Second, as  $\Phi\pi'_c(0) < q_c(0)$  it follows that for  $\varepsilon$  small enough  $\varepsilon q_c(0) > \Phi\pi_c(\varepsilon)$ . Thus, it follows from continuity that a non trivial solution exists on  $(0, 1)$ . Uniqueness follows because

$$\begin{aligned} q_c(0) - \Phi\pi'_c(\theta_*) &= \frac{\Phi\pi_c(\theta_*)}{\theta_*} - \Phi p_c(\theta_*) R_o \\ &= \Phi \left[ p_c(\theta_*) R_o - \frac{1}{\theta_*} \int_0^{\kappa_c(\theta_*)} \kappa dF(\kappa) \right] - \Phi p_c(\theta_*) R_o < 0, \end{aligned}$$

so the function  $\theta q_c(0) - \Phi\pi_c(\theta)$  single crosses 0 from above, which implies  $\theta q_c(0) > \Phi\pi_c(\theta)$  on  $\theta \in (0, \theta_*)$  and  $\theta q_c(0) < \Phi\pi_c(\theta)$  on  $\theta \in (\theta_*, 1]$ . Finally, if  $p_L = 0$ , then  $\Phi\pi'_c(0) = q_c(0) = 0$ . It follows then from the convexity of  $\pi_c(\theta)$  that  $\theta_* = 0$  is a global maximum of  $\theta q_c(0) - \Phi\pi_c(\theta)$ , which means that  $\theta q_c(0) < \Phi\pi_c(\theta)$  for all  $\theta > 0$ .  $\square$

**Lemma 6.** *There is a unique solution  $\tilde{\theta}_N \in (0, \theta_*)$  to equation (37).*

*Proof.* First, we show existence. Given the definition of  $\theta_*$  and the convexity of  $\pi(\theta)$  we have that  $\Phi\pi(\theta) < \theta q(0)$  for all  $\theta < \theta_*$ . Hence,

$$\frac{N}{\rho + \phi} \left(1 - e^{-\frac{(\rho + \phi)}{N}\theta_*}\right) q(0) + \frac{(\rho + \phi)}{N} \int_0^{\theta_*} e^{-\frac{(\rho + \phi)}{N}(\theta - y)} \Phi\pi(y) dy < \Phi\pi(\theta_*).$$

We also have that

$$\begin{aligned} e^{-\frac{\rho + \phi}{N}\theta_*} \Phi\pi(\theta_*) + \frac{(\rho + \phi)}{N} \int_0^{\theta_*} e^{-\frac{\rho + \phi}{N}y} \left( \Phi\pi(y) - \frac{N}{\rho + \phi} q(0) \right) dy &\leq \Phi\pi(\theta_*) - \frac{N}{(\rho + \phi)} \left(1 - e^{-\frac{\rho + \phi}{N}\theta_*}\right) q(0) \\ &\leq \Phi\pi(\theta_*) - \theta_* q(0) = 0. \end{aligned}$$

The existence of a solution follows from the intermediate value theorem. To show uniqueness we consider the derivative of the difference between the left and the right hand sides of equation (37) evaluated at  $\theta$ , which we denote by  $G'(\theta)$ .

$$\begin{aligned} G'(\theta) &= e^{-\frac{(\rho + \phi)}{N}\theta} q(0) + \frac{(\rho + \phi)}{N} \Phi\pi(\theta) - \frac{(\rho + \phi)^2}{N^2} \int_0^\theta e^{-\frac{(\rho + \phi)}{N}(\theta - y)} \Phi\pi(y) dy \\ &\quad - \frac{(\rho + \phi)}{N} e^{-\frac{\rho + \phi}{N}(\theta_* - \theta)} \Phi\pi(\theta_*) + \frac{(\rho + \phi)}{N} \left( \Phi\pi(\theta) - \frac{N}{\rho + \phi} q(0) \right) \\ &\quad - \frac{(\rho + \phi)^2}{N^2} \int_\theta^{\theta_*} e^{-\frac{\rho + \phi}{N}(y - \theta)} \left( \Phi\pi(y) - \frac{N}{\rho + \phi} q(0) \right) dy \end{aligned}$$

From here we get that

$$\begin{aligned} G'(\theta_*) &= - \left(1 - e^{-\frac{(\rho + \phi)}{N}\theta_*}\right) q(0) - \frac{(\rho + \phi)^2}{N^2} \int_0^{\theta_*} e^{-\frac{(\rho + \phi)}{N}(\theta_* - y)} \Phi\pi(y) dy + \frac{(\rho + \phi)}{N} \Phi\pi(\theta_*) \\ &\leq \frac{(\rho + \phi)}{N} (\Phi\pi(\theta_*) - \theta_* q(0)) = 0 \\ G'(0) &= - \frac{(\rho + \phi)}{N} e^{-\frac{\rho + \phi}{N}\theta_*} \Phi\pi(\theta_*) - \frac{(\rho + \phi)^2}{N^2} \int_0^{\theta_*} e^{-\frac{\rho + \phi}{N}y} \left( \Phi\pi(y) - \frac{N}{\rho + \phi} q(0) \right) dy \\ &\geq - \frac{(\rho + \phi)}{N} \Phi\pi(\theta_*) + \left(1 - e^{-\frac{\rho + \phi}{N}\theta_*}\right) q(0) \\ &\geq \frac{(\rho + \phi)}{N} (\theta_* q(0) - \Phi\pi(\theta_*)) = 0 \end{aligned}$$

Moreover, we get that, for any  $\theta \in (0, \theta_*)$ ,  $G(\theta) = 0$  implies

$$\begin{aligned} G'(\theta) &= \frac{2(\rho + \phi)}{N} \left[ \Phi\pi(\theta) - \left( \frac{N}{\rho + \phi} \left(1 - e^{-\frac{(\rho + \phi)}{N}\theta}\right) q(0) + \frac{(\rho + \phi)}{N} \int_0^\theta e^{-\frac{(\rho + \phi)}{N}(\theta - y)} \Phi\pi(y) dy \right) \right] \\ &\leq \frac{2(\rho + \phi)}{N} [\Phi\pi(\theta) - \theta q(0)] < 0. \end{aligned}$$

It follows that  $G(\theta)$  single crosses 0, so there is a unique solution to the equation  $G(\theta) = 0$ .  $\square$

**Lemma 7.** *There is  $\tilde{N}$  such that, for all  $N > \tilde{N}$ ,  $\Pi'_N(\theta) < q(0)$  on  $(0, \tilde{\theta}_N)$  and  $\Pi'_N(\theta) > q(0)$  on  $(\tilde{\theta}_N, \theta_*)$ .*

*Proof.* First, we verify that  $\Pi'_N(\theta) < q(0)$  on  $(0, \tilde{\theta}_N)$ . The derivative of  $\Pi_N(\theta) - \theta q(0)$  on  $(0, \tilde{\theta}_N)$  is given by

$$\begin{aligned}\Pi'_N(\theta) - q(0) &= \frac{\rho + \phi}{N} \Phi\pi(\theta) - \left(1 - e^{-\frac{(\rho + \phi)}{N}\theta}\right) q(0) - \frac{(\rho + \phi)^2}{N^2} \int_0^\theta e^{-\frac{(\rho + \phi)}{N}(\theta - y)} \Phi\pi(y) dy \\ &\leq \frac{\rho + \phi}{N} \Phi\pi(\theta) - \left(1 - e^{-\frac{(\rho + \phi)}{N}\theta}\right) q(0) \leq \frac{\rho + \phi}{N} (\Phi\pi(\theta) - \theta q(0)) < 0.\end{aligned}$$

The derivative of  $\Pi_N(\theta) - \theta q(0)$  on  $(\tilde{\theta}_N, \theta_*)$  is given by

$$\Pi'_N(\theta) - q(0) = \frac{\rho + \phi}{N} \left[ e^{-\frac{\rho + \phi}{N}(\theta_* - \theta)} \Phi\pi(\theta_*) - \Phi\pi(\theta) + \frac{(\rho + \phi)}{N} \int_\theta^{\theta_*} e^{-\frac{\rho + \phi}{N}(y - \theta)} \left( \Phi\pi(y) - \frac{N}{\rho + \phi} q(0) \right) dy \right]$$

Differentiating the HJB equation we get that

$$\begin{aligned}\Pi''_N(\theta) &= \frac{(\rho + \phi)}{N} (\Pi'_N(\theta) - \Phi\pi'(\theta)) \\ \Pi'''_N(\theta) &= \frac{(\rho + \phi)}{N} (\Pi''_N(\theta) - \Phi\pi''(\theta)).\end{aligned}$$

From here we get that  $\Pi''_N(\theta) = 0 \implies \Pi'''_N(\theta) < 0$ , so the function  $\Pi'_N(\theta)$  is quasi-concave on  $(\tilde{\theta}_N, \theta_*)$ . Moreover,  $\Pi'_N(\theta_* -) = q(0)$ , and

$$\Pi'_N(\tilde{\theta}_N +) = q(0) + \frac{(\rho + \phi)}{N} \left( \Pi_N(\tilde{\theta}_N +) - \Phi\pi(\tilde{\theta}_N) \right) > q(0),$$

so we can conclude that  $\Pi'_N(\theta) > q(0)$  on  $(\tilde{\theta}_N, \theta_*)$  as long as  $\Pi_N(\tilde{\theta}_N +) > \Phi\pi(\tilde{\theta}_N)$ , which follows from

$$\begin{aligned}\Pi_N(\tilde{\theta}_N +) - \Phi\pi(\tilde{\theta}_N) &= \Pi_N(\tilde{\theta}_N -) - \Phi\pi(\tilde{\theta}_N) \\ &= \frac{N}{\rho + \phi} \left( 1 - e^{-\frac{(\rho + \phi)}{N}\tilde{\theta}_N} \right) q(0) - \Phi\pi(\tilde{\theta}_N) + \frac{(\rho + \phi)}{N} \int_0^{\tilde{\theta}_N} e^{-\frac{(\rho + \phi)}{N}(\tilde{\theta}_N - y)} \Phi\pi(y) dy \\ &\geq \frac{N}{\rho + \phi} \left( 1 - e^{-\frac{(\rho + \phi)}{N}\tilde{\theta}_N} \right) q(0) - \Phi\pi(\tilde{\theta}_N) \\ &= \tilde{\theta}_N q_0 - \frac{(\rho + \phi)\tilde{\theta}_N^2}{N} - \Phi\pi(\tilde{\theta}_N) + O(1/N^2).\end{aligned}$$

$\tilde{\theta}_N q_0 > \Phi\pi(\tilde{\theta}_N)$  because  $\theta q_0$  single crosses  $\Phi\pi(\theta)$  at  $\theta_* \geq \tilde{\theta}_N$ . Hence, there is  $\tilde{N}$  such that, for all  $N \geq \tilde{N}$ , we have  $\Pi_N(\tilde{\theta}_N +) > \Phi\pi(\tilde{\theta}_N)$ .  $\square$

**Lemma 8.** *Let*

$$\Pi(\theta) = \begin{cases} \Phi\pi(\theta) & \text{if } \theta \in [\theta_*, 1] \\ q(0)\theta & \text{if } \theta \in [0, \theta_*) \end{cases}.$$

Then, for any  $\theta \in [0, 1]$

$$\lim_{N \rightarrow 0} \Pi_N(\theta) = \Pi(\theta).$$

*Proof.* For all  $\theta \geq \theta_*$ ,  $\Pi_N(\theta) = \Pi(\theta)$ , and, for any  $\theta < \theta_*$ ,  $\lim_{N \rightarrow \infty} \Pi_N(\theta) = \theta q(0) = \Pi(\theta)$  by L'Hopital's rule.  $\square$

## A.2 Verification of Optimality

We start providing the necessary definitions from the theory of viscosity solutions, together with the relevant results from the theory of optimal control in stratified domains in [Barles et al. \(2018\)](#). We make some changes in notation to make it consistent with our setting, and to translate their minimization problem into a maximization one. While [Barles et al. \(2018\)](#) considers the state space to be the complete real line, the state space in our case is  $[0, 1]$ . However, we can extend the state space by letting the payoff on the complement of  $[0, 1]$  to be sufficiently low. This can be achieved by adding a penalization term, and setting the flow payoff equal to  $\phi\pi(1) - \dot{\theta}q(1) - k|\theta - 1|$  for  $\theta > 1$ , and  $\phi\pi(0) - \dot{\theta}q(0) - k|\theta|$  for  $\theta < 0$ . By choosing  $k$  large enough, we can ensure that the optimal solution never exits the interval  $[0, 1]$ . Due to the discontinuity in the Hamiltonian at  $\theta_*$ , a viscosity solution might fail to be unique. In order to fully characterize the value function we need to specify its behavior at  $\theta_*$ . This is done in [Barles et al. \(2018\)](#) by considering the concept of *Flux-limited* sub- and supersolutions. Letting  $\Omega_0 = (-\infty, \theta_*)$  and  $\Omega_1 = (\theta_*, \infty)$ , we consider the equation

$$\begin{cases} (\rho + \phi)\Pi - H_0(\theta, \Pi) = 0 & \text{in } \Omega_0 \\ (\rho + \phi)\Pi - H_1(\theta, \Pi) = 0 & \text{in } \Omega_1 \\ (\rho + \phi)\Pi - \phi\pi(\theta_*) = 0 & \text{in } \{\theta_*\}, \end{cases} \quad (39)$$

where

$$\begin{aligned} H_0(\theta, \Pi') &= \phi\pi(\theta) + k \min\{0, \theta\} + \max_{|\dot{\theta}| \leq N} \left\{ \dot{\theta}(\Pi' - q(0)) \right\} \\ H_1(\theta, \Pi') &= \phi\pi(\theta) - k \max\{0, \theta - 1\} + \max_{|\dot{\theta}| \leq N} \left\{ \dot{\theta}(\Pi' - q(\theta)) \right\} \end{aligned}$$

In  $\Omega_0 \cup \Omega_1$ , the definitions are just classical viscosity sub- and supersolution, which we provide next for completeness.

**Definition 3** ([Bardi and Capuzzo-Dolcetta \(2008\)](#), Definition 1.1). *A function  $u \in C(\mathbb{R})$  is a viscosity subsolution of (34) if, for any  $\varphi \in C^1(\mathbb{R})$ ,*

$$(\rho + \phi)u(\theta_0) - H(\theta_0, \varphi'(\theta_0)) \leq 0, \quad (40)$$

*at any local maximum point  $\theta_0 \in \mathbb{R}$  of  $u - \varphi$ . Similarly,  $u \in C(\mathbb{R})$  is a viscosity supersolution of (34) if, for any  $\varphi \in C^1(\mathbb{R})$ ,*

$$(\rho + \phi)u(\theta_1) - H(\theta_1, \varphi'(\theta_1)) \geq 0, \quad (41)$$

at any local minimum point  $\theta_1 \in \mathbb{R}$  of  $u - \varphi$ . Finally,  $u$  is a viscosity solution of (34) if it is simultaneously a viscosity sub- and supersolution.

Before providing the definition of sub- and supersolution on  $\{\theta_*\}$ , we introduce the following space  $\mathfrak{S}$  of real valued test functions:  $\varphi \in \mathfrak{S}$  if  $\varphi \in C(\mathbb{R})$  and there exist  $\varphi_0 \in C^1(\overline{\Omega}_0)$  and  $\varphi_1 \in C^1(\overline{\Omega}_1)$  such that  $\varphi = \varphi_0$  in  $\overline{\Omega}_0$ , and  $\varphi = \varphi_1$  in  $\overline{\Omega}_1$ . Next, we introduce two Hamiltonians that are needed to define a flux-limited sub- and supersolution at  $\{\theta_*\}$ .

$$\begin{aligned} H_1^+(\theta_*, \Pi') &\equiv \phi\pi(\theta) + \sup_{0 < \dot{\theta} \leq N} \left\{ \dot{\theta}(\Pi' - q(\theta_*)) \right\} \\ H_0^-(\theta_*, \Pi') &\equiv \phi\pi(\theta) + \sup_{0 > \dot{\theta} \geq -N} \left\{ \dot{\theta}(\Pi' - q(0)) \right\}. \end{aligned}$$

**Definition 4** (Barles et al. (2018), Definition 2.1). *An upper semi-continuous, bounded function  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a flux-limited subsolution on  $\{\theta_*\}$  if for any test function  $\varphi \in \mathfrak{S}$  such that  $u - \varphi$  has a local maximum at  $\theta_*$ , we have*

$$(\rho + \phi)u(\theta_*) - \max\{\phi\pi(\theta_*), H_0^-(\theta_*, \varphi'_0(\theta_*)), H_1^+(\theta_*, \varphi'_1(\theta_*))\} \leq 0. \quad (42)$$

*A lower semi-continuous, bounded function  $v : \mathbb{R} \rightarrow \mathbb{R}$  is a flux-limited supersolution on  $\{\theta_*\}$  if for any test function  $\varphi \in \mathfrak{S}$  such that  $u - \varphi$  has a local minimum at  $\theta_*$ , we have*

$$(\rho + \phi)v(\theta_*) - \max\{\phi\pi(\theta_*), H_0^-(\theta_*, \varphi'_0(\theta_*)), H_1^+(\theta_*, \varphi'_1(\theta_*))\} \geq 0. \quad (43)$$

The Hamiltonians  $H_0^-$  and  $H_1^+$  are needed to express the optimality conditions at the discontinuity  $\theta_*$ .  $H_1^+$  consider controls that starting at  $\theta_*$  take  $\theta_t$  towards the interior of  $[\theta_*, 1]$ , and  $H_0^-$  considers controls that starting at  $\theta_*$  take  $\theta_t$  towards the interior of  $[0, \theta_*]$ . The use of the Hamiltonians  $H_0^-$  and  $H_1^+$  at  $\{\theta_*\}$ , instead of  $H_0$  and  $H_1$ , distinguishes *flux-limited* viscosity solutions from the traditional (discontinuous) viscosity solutions.

We consider the following control problem, equivalent to the one defined in (32),

$$\begin{aligned} \tilde{\Pi}_N^*(\theta_0) &= \sup_{|\theta_t| \leq N} \int_0^\infty e^{-(\rho+\phi)t} \left( \phi\tilde{\pi}(\theta_t) - \theta_t\tilde{q}(\theta_t)\mathbf{1}_{\{\theta_t \neq \theta_*\}} - k(\max\{0, \theta_t - 1\} - \min\{0, \theta_t\}) \right) dt \\ \tilde{\pi}(\theta) &= \pi(0)\mathbf{1}_{\{\theta < 0\}} + \pi(\theta)\mathbf{1}_{\{\theta \in [0, 1]\}} + \pi(1)\mathbf{1}_{\{\theta > 1\}} \\ \tilde{q}(\theta) &= q(0)\mathbf{1}_{\{\theta < 0\}} + q(\theta)\mathbf{1}_{\{\theta \in [0, 1]\}} + q(1)\mathbf{1}_{\{\theta > 1\}}. \end{aligned}$$

The following Theorem characterizes the value function  $\tilde{\Pi}_N^*$  in terms of flux-limited viscosity solutions.

**Theorem 1** (Barles et al. (2018), Theorem 2.9). *The value function  $\tilde{\Pi}_N^*$  is the unique flux-limited viscosity solution to equation (39).*

We can now proceed to apply Theorem 1 to verify that  $\Pi_N$  defined in (36) is the value function of the control problem (32).

**Verification** Lemmas 6 and 7 imply that  $\Pi_N$  is a classical solution on  $\Omega \setminus \{\tilde{\theta}_N, \theta_*\}$  so we only need to verify the conditions for a viscosity solution on  $\{\tilde{\theta}_N, \theta_*\}$ .  $\Pi_N$  defined in (36) is a classical solution on  $(\theta_*, 1)$ . At  $\theta = \theta_*$ ,  $\Pi_N$  has a convex kink so we only need to verify the supersolution property. That is, that for any  $\varphi'(\theta_*)$  in the subdifferential of  $\Pi_N(\theta)$  at  $\theta_*$ , which is  $[\Pi'_N(\theta_*-), \Pi'_N(\theta_*+)]$ , inequality (43) is satisfied.  $H_1^+(\theta_*, \varphi'(\theta_*))$  is nondecreasing in  $\varphi'(\theta_*)$  and  $H_0^-(\theta_*, \varphi'(\theta_*))$  is nonincreasing in  $\varphi'(\theta_*)$ ; thus, the supersolution property follows from

$$\begin{aligned} (\rho + \phi)\Pi_N(\theta_*) - H_1^+(\theta_*, \Pi'_N(\theta_*+)) &= (\rho + \phi)\Pi_N(\theta_*) - \phi\pi(\theta_*) = 0 \\ (\rho + \phi)\Pi_N(\theta_*) - H_0^-(\theta_*, \Pi'_N(\theta_*-)) &= (\rho + \phi)\Pi_N(\theta_*) - \phi\pi(\theta_*) = 0. \end{aligned}$$

As  $\Pi'_N(\tilde{\theta}_N-) < q(0) < \Pi'_N(\tilde{\theta}_N+)$ ,  $\Pi_N(\theta)$  has a convex kink at  $\tilde{\theta}_N$ , we only need to verify the property for a supersolution. Thus, we need to verify that for any  $\varphi'(\tilde{\theta}_N) \in [\Pi'_N(\tilde{\theta}_N-), \Pi'_N(\tilde{\theta}_N+)]$ , inequality (41) is satisfied. This amounts to verify that

$$(\rho + \phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) \geq N \max \left\{ |\Pi'_N(\tilde{\theta}_N-) - q(0)|, |\Pi'_N(\tilde{\theta}_N+) - q(0)| \right\}$$

By definition of  $\tilde{\theta}_N$ , we have

$$(\rho + \phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) = N(\Pi'_N(\tilde{\theta}_N+) - q(0)) = N(q(0) - \Pi'_N(\tilde{\theta}_N-)),$$

so it follows that

$$(\rho + \phi)\Pi_N(\tilde{\theta}_N) - \phi\pi(\tilde{\theta}_N) = N|\Pi'_N(\tilde{\theta}_N-) - q(0)| = N|\Pi'_N(\tilde{\theta}_N+) - q(0)|.$$

Finally, at  $\theta = 1$ , by choosing  $k$  large enough, we have that the solution of the HJB equation on  $\{\theta > 1\}$  entails  $\dot{\theta}(\theta) = -N$ . Moreover,  $\Pi'(1-) = q(1)$  implies that the value function is differentiable at  $\theta = 1$  (in the extended problem) and that  $\dot{\theta}(1) \leq 0$  is optimal. Thus, the state constraint is satisfied at  $\theta = 1$ . A similar argument applies at  $\theta = 0$ . Thus, we can conclude that  $\Pi_N(\theta)$  is a flux-limited viscosity solution, so, by Theorem 1, it is the value function of the optimal control problem.

The uniqueness proof follows from the fact that the HJB has a unique solution in the smooth trading region, under the given boundary conditions. In the region  $[0, \theta_*)$ , smooth trading and no trading are both ruled out. This implies that trading must be atomic. Under atomic trading, the price must be at least  $p_L R_o$ , but the bank will sell everything, even below this price. The solution  $\theta_*$  is easily verified as being unique.



## B Intermediation

### Proof of Lemma 1

*Proof.* Using integration by parts and a Transversality condition  $\lim_{t \rightarrow \infty} e^{-(\rho+\phi)t} D_t = 0$ , we have

$$\int_0^\infty e^{-(\rho+\phi)s} dD_s = \int_0^\infty e^{-(\rho+\phi)s} (\rho + \phi) D_s ds - D_0.$$

(11) can be rewritten as

$$\max_{(D_t, \theta_t)_{t \geq 0}} \mathbb{E} \left[ \int_0^\infty e^{-(\rho+\phi)s} [(\phi \hat{\pi}(\theta_s, D_s) - y(\theta_s, D_s) D_s + (\rho + \phi) D_s) ds + dG(\theta_s)] \right] - D_0$$

The optimization in the lemma follows directly from the definition of  $\phi\pi(\theta)$  □

### Proof of Lemma 2

*Proof.* The first statement follows from the fact that  $z'(\theta) < 0$ , which is necessarily the case given Assumption 4 and equation (25). Moreover, the constraint clearly binds at  $\theta = 0$  with  $D^*(0) = D^{\max}(0) = \Pi_i(0) = 0$ .

There are two cases to be considered; when the borrowing constraint is binding and when it is not. When the borrowing constraint is binding,  $z(\theta) = 0$ , so the conclusion follows directly. Thus, we only need to verify the case in which the borrowing constrain,  $D \leq D^{\max}(\theta)$  is binding.

Note that from the solution to  $D^*(\theta)$ , it is immediately clear that  $z(\theta) \leq \frac{\rho}{\phi}$ . In the constrained region,  $D^{\max}(\theta) = \mathcal{V}(D^{\max}(\theta), \theta)$ . Using the Envelope Theorem to solve for  $D^{\max'}(\theta)$ , we get

$$\begin{aligned} D^{\max'}(\theta) &= \frac{d\mathcal{V}(D^{\max}(\theta), \theta)}{d\theta} \\ &= \frac{\partial \mathcal{L}}{\partial \theta} \Big|_{D=D^{\max}(\theta)} \\ &= \frac{\Phi}{1 - \Phi z(\theta)} [p(\theta) R_o + f(\kappa_i) \Delta^2 R_o D^{\max}(\theta)]. \end{aligned}$$

Since the *F.O.C.* implies

$$f(\kappa_i) \Delta^2 D^{\max}(\theta) = \frac{\rho - \phi z(\theta)}{\phi},$$

we can show

$$D^{\max'}(\theta) = \frac{\Phi}{1 - \Phi z(\theta)} \left[ p(\theta) + \frac{\rho - \phi z(\theta)}{\phi} \right] R_o.$$

Substituting this expression in equation (29), we get

$$\begin{aligned}
\dot{\theta} &= \phi \frac{(1 - \Phi)(1 - p(\theta))R_o + \Phi z(\theta)(D^{\max'}(\theta) - R_o)}{\Phi \pi_i''(\theta)} \\
&= \phi \frac{(1 - \Phi)(1 - p(\theta)) + \Phi z(\theta) \left( \phi \frac{p(\theta) - 1}{(\rho + \phi) - \phi z(\theta)} \right)}{\Phi \pi_i''(\theta)} R_o \\
&= \phi \frac{(1 - \Phi) - \frac{\Phi z(\theta) \phi}{(\rho + \phi) - \phi z(\theta)}}{\Phi \pi_i''(\theta)} (1 - p(\theta)) R_o.
\end{aligned}$$

Clearly,  $\dot{\theta} > 0$  if and only if

$$(1 - \Phi) - \frac{\Phi z(\theta) \phi}{(\rho + \phi) - \phi z(\theta)} > 0 \implies \frac{\rho}{\phi} > \frac{\phi}{(\rho + \phi) - \phi z(\theta)} z(\theta),$$

which always holds because of the upper bound on the Lagrange multiplier,  $z(\theta) \leq \frac{\rho}{\phi}$ . □

## Proof of Proposition 4

*Proof.* The convexity of  $\Pi(\theta)$  in the unconstrained region  $D^*(\theta) < D^{\max}(\theta)$  follows under Assumption 2. In the constrained region where  $D^*(\theta) = D^{\max}(\theta) = \Pi_i(\theta)$ , we differentiate  $\Pi_i'(\theta)$  and get

$$\Pi_i''(\theta) = \frac{\Phi}{1 - \Phi z(\theta)} [d_i'(\theta) + z'(\theta)(\Pi_i'(\theta) - R_o)].$$

Substituting the first order condition for  $D$  in  $\Pi_i'(\theta)$ , we get that

$$\Pi_i'(\theta) - R_o = -\frac{\Phi}{1 - \Phi z(\theta)} (1 - p(\theta)) < 0.$$

Hence, it suffices to show that  $z'(\theta) \leq 0$ . Note that (25) implies

$$z'(\theta) = - \left[ f' \Delta^2 D^{\max} \frac{\partial \kappa_i}{\partial \theta} + f \Delta^2 D^{\max'} \right].$$

Note that  $D^{\max'}(\theta) = \Pi_i'(\theta) = \Phi \pi_i'(\theta)$ , so that

$$D^{\max'}(\theta) = \frac{\Phi}{1 - \Phi z(\theta)} [p(\theta) R_o + f \Delta^2 R_o D^{\max}].$$

Therefore, for  $z'(\theta) < 0$ , it suffices to have  $\Delta f' D^{\max} + \frac{\Phi}{1 - \Phi z(\theta)} [p(\theta) + f \Delta^2 D^{\max}] > 0$ , which always holds under the lower bound imposed by Assumption 4.

Next, at the boundary  $\theta_D$  where the equity holder's limited liability constraint binds,  $\Pi_i'(\theta)$  is continuous, which follows from (28), and the continuity of  $d_i(\theta)$ ,  $D^*(\theta)$ , as well as the fact that  $\lim_{\theta \uparrow \theta_D} z(\theta) = z(\theta_D) =$

0. Therefore, we conclude that  $\Pi_i(\theta)$ , the value function in the smooth-trading case is globally convex.

Finally, note that at  $\theta = 0$ ,  $\Pi'_i(\theta) < p_L R_o$ . At  $\theta = 1$ ,  $\Pi_i(1) > \Pi_c(1)$ , where  $\Pi_c(1) > p_L R_o$  follows from Assumption 3. The intersection between  $\Pi_i(\theta)$  and  $p_L R_o \theta$  then admits the unique solution  $\theta_\dagger$ . Moreover,  $\Pi_i(\theta) < p_L R_o \theta$  if and only if  $\theta < \theta_\dagger$ .

□

## C Special Case: Uniform Distribution

### Proof of Corollary 2

*Proof.* Let us define

$$v(\theta) = F(\kappa_c) p_H(R - R_o) + (1 - F(\kappa_c)) B \quad (44)$$

as the borrower's expected payoff if the asset matures. Therefore, the borrower's expected payoff at  $t = 0$  is

$$V_c(\theta_0) = \int_0^\infty e^{-(\rho+\phi)t} \phi v(\theta_t) dt = \Phi B + \frac{\phi(b-B)}{\bar{\kappa}} \frac{\Delta R_o \theta_0}{2\rho + \phi}, \quad (45)$$

where we have substituted  $R_o = R - \frac{b}{\Delta}$  and therefore  $p_H(R - R_o) = b$ . Finally, the borrower's problem at time 0 is

$$W_c = \max_{\{\theta_0, R_o\}} V_c(\theta_0) + L_c(\theta_0) \quad (46)$$

$$s.t. \quad L_c(\theta_0) \geq I - A. \quad (47)$$

It is easily verified that,  $L_c(\theta_0)$  is maximized at  $\theta_0 = 1$ . However,  $V_c(\theta_0) + L_c(\theta_0)$  is maximized at

$$\theta_0 = 1 - \frac{\rho + \phi}{2\rho + \phi} \frac{B - b}{\Delta R_o}. \quad (48)$$

□

### Proof of Corollary 3

*Proof.* In the uniform case with  $p_L = 0$  we have that  $D^{\max}(\theta)$  solves

$$\frac{\Delta^2}{2\bar{\kappa}} [(R_o \theta)^2 - D^2] = D.$$

The previous equation has two roots, and the positive root is

$$D^{\max}(\theta) = -\frac{\bar{\kappa}}{\Delta^2} + \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o \theta)^2}.$$

1. If  $\Phi \geq \underline{\Phi} := \sqrt{\frac{(\bar{\kappa}/\Delta^2)^2}{(\bar{\kappa}/\Delta^2)^2 + (R_o)^2}}$ , the bank's deposit choice satisfies  $D^*(1) = \tilde{D}(1) = \frac{\rho\bar{\kappa}}{\phi\Delta^2}$ .
2. Otherwise, the bank's deposit choice satisfies  $D^*(1) = D^{\max}(1) = -\frac{\bar{\kappa}}{\Delta^2} + \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o)^2}$ .

Moreover,  $L_i(1) = \Pi_i(1) = \Phi\pi_i(1)$ . Let us also supplement the relevant expressions

$$D^*(1) = \min \left\{ \frac{\rho\bar{\kappa}}{\phi\Delta^2}, -\frac{\bar{\kappa}}{\Delta^2} + \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + (R_o)^2} \right\}.$$

Next, let us look for conditions where

$$\tilde{D}(1) = \frac{\rho\bar{\kappa}^2}{\phi\Delta^2} \leq D^{\max}(1) = -\frac{\bar{\kappa}}{\Delta^2} + \sqrt{\left(\frac{\bar{\kappa}}{\Delta^2}\right)^2 + R_o^2}.$$

Simple derivation shows this is satisfied if and only if

$$\Phi > \sqrt{\frac{(\bar{\kappa}/\Delta^2)^2}{R_o^2 + (\bar{\kappa}/\Delta^2)^2}},$$

which holds if and only if  $\rho$  is sufficiently low. If  $\tilde{D}(1) < D^{\max}(1)$ , then we can plug in  $D^*(1)$  and get

$$\pi_i(1) = \frac{\Delta^2}{2\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)^2 + \frac{\rho + \phi \frac{\Delta^2}{\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)}{\phi} \frac{\rho\bar{\kappa}}{\phi\Delta^2}.$$

According to (4),  $\kappa_i = \Delta \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)$  and  $p_i(1) = \frac{\kappa_i}{\bar{\kappa}} \Delta = \frac{\Delta^2}{\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)$ . Consequently,  $\hat{\pi}_i(1, D^*)$  and  $\pi_i(1)$  defined in (8) and (12) become

$$\hat{\pi}_i(1, D^*) = \frac{\Delta^2}{2\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)^2 \quad (49)$$

$$\pi_i(1) = \frac{\Delta^2}{2\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)^2 + \frac{\rho + \phi \frac{\Delta^2}{\bar{\kappa}} \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)}{\phi} \frac{\rho\bar{\kappa}}{\phi\Delta^2}. \quad (50)$$

As in the certification case, we can similarly define

$$v(1) = F(\kappa_i) p_H(R - R_o) + (1 - F(\kappa_i)) B = B + \frac{\Delta \left( R_o - \frac{\rho\bar{\kappa}}{\phi\Delta^2} \right)}{\bar{\kappa}} (b - B) \quad (51)$$

$$V_i(1) = \int_0^\infty e^{-(\rho+\phi)t} \phi v(\theta_t) dt = \Phi v(1) \quad (52)$$

$$L_i(1) = \Pi_i(1) = \Phi\pi_i(1). \quad (53)$$

Finally, the borrower's payoff at time 0 is

$$W_i = V_i(1) + L_i(1). \quad (54)$$

□

## Proof of Proposition 5

*Proof.* Note that  $L_c(\theta_0) = \Pi_c(\theta_0) + (1 - \theta_0)q_c(\theta_0)$ . In the proposition,  $L_c(\theta_0^*) > I - A$  guarantees that the borrowing constraints are slack in both certification and intermediation. Under certification

$$W_c = \Phi B + \frac{\phi(b-B)}{\bar{\kappa}} \frac{\Delta R_o}{2\rho + \phi} \theta_0 + \frac{\Phi}{\bar{\kappa}} (\Delta R_o)^2 \theta_0 \left(1 - \frac{\theta_0}{2}\right),$$

where  $\theta_0$  is evaluated at

$$\theta_0^* = 1 - \frac{\rho + \phi}{2\rho + \phi} \frac{B - b}{\Delta R_o} = 1 - \frac{1}{2 - \Phi} \frac{B - b}{\Delta R_o},$$

Under intermediation,

$$W_i = \Phi \left[ B + \frac{\Delta \left( R_o - \frac{\rho \bar{\kappa}}{\phi \Delta^2} \right)}{\bar{\kappa}} (b - B) \right] + \frac{\Phi}{2\bar{\kappa}} \Delta^2 \left( R_o - \frac{\rho \bar{\kappa}}{\phi \Delta^2} \right)^2 + \frac{\rho + \phi \frac{\Delta^2}{\bar{\kappa}} \left( R_o - \frac{\rho \bar{\kappa}}{\phi \Delta^2} \right)}{\rho + \phi} \frac{\rho \bar{\kappa}}{\phi \Delta^2}.$$

Certification dominates intermediation if  $W_c > W_i$ , letting  $\Delta W \equiv W_c - W_i$ ,

$$\begin{aligned} \Delta W &= \Phi \left[ \frac{1}{2 - \Phi} \frac{(b - B)}{\bar{\kappa}} \left( \Delta R_o - \frac{1}{2 - \Phi} (B - b) \right) + \frac{1}{2\bar{\kappa}} \left( \Delta R_o - \frac{1}{2 - \Phi} (B - b) \right) \left( \Delta R_o + \frac{1}{2 - \Phi} (B - b) \right) \right. \\ &\quad \left. - \frac{(b - B)}{\bar{\kappa}} \left( \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) - \frac{1}{2\bar{\kappa}} \left( \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right)^2 - \left( \frac{1 - \Phi}{\Phi} + \frac{\Delta}{\bar{\kappa}} \left( \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) \right) \frac{\rho \bar{\kappa}}{\phi \Delta^2} \right] \\ &= \frac{\Phi}{2\bar{\kappa}} \left[ \left( \frac{B - b}{2 - \Phi} \right)^2 + 2 \left( \frac{1 - \Phi}{2 - \Phi} \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) (B - b) - \left( \frac{\rho \bar{\kappa}}{\phi \Delta} \right)^2 \right] \end{aligned}$$

The previous equation is negative for  $B$  on  $(\underline{B}, \bar{B})$ , where

$$\begin{aligned} \underline{B} &= b - (2 - \Phi)^2 \left[ \sqrt{\left( \frac{1 - \Phi}{2 - \Phi} \Delta R_o + \frac{\rho \bar{\kappa}}{\phi \Delta} \right)^2 + \left( \frac{\rho \bar{\kappa}}{\phi \Delta (2 - \Phi)} \right)^2} - \left( \frac{1 - \Phi}{2 - \Phi} \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) \right] \\ \bar{B} &= b + (2 - \Phi)^2 \left[ \sqrt{\left( \frac{1 - \Phi}{2 - \Phi} \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right)^2 + \left( \frac{\rho \bar{\kappa}}{\phi \Delta (2 - \Phi)} \right)^2} - \left( \frac{1 - \Phi}{2 - \Phi} \Delta R_o - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) \right] \end{aligned}$$

Because,  $\underline{B} < b$ , we only need to consider the upper bound, so, after substituting  $R_o$ , we get that the expression is negative as long as  $B$  satisfies

$$b < B < b + (2 - \Phi)^2 \left[ \sqrt{\left( \frac{1 - \Phi}{2 - \Phi} (\Delta R - b) - \frac{\rho \bar{\kappa}}{\phi \Delta} \right)^2 + \left( \frac{\rho \bar{\kappa}}{\phi \Delta (2 - \Phi)} \right)^2} - \left( \frac{1 - \Phi}{2 - \Phi} (\Delta R - b) - \frac{\rho \bar{\kappa}}{\phi \Delta} \right) \right].$$

Thus, we can conclude that certification dominates if

$$B > b + (2 - \Phi)^2 \left[ \sqrt{\left( \frac{1 - \Phi}{2 - \Phi} (\Delta R - b) - \frac{\rho \kappa}{\phi \Delta} \right)^2 + \left( \frac{\rho \bar{\kappa}}{\phi \Delta (2 - \Phi)} \right)^2} - \left( \frac{1 - \Phi}{2 - \Phi} (\Delta R - b) - \frac{\rho \kappa}{\phi \Delta} \right) \right].$$

We need to compare now  $W_c$  and  $W_d$ .

$$W_c - W_d = B + \frac{\phi(b - B)}{\bar{\kappa}} \frac{\Delta R_o}{2\rho + \phi} \theta_0 + \frac{\Phi}{\bar{\kappa}} (\Delta R_o)^2 \theta_0 \left( 1 - \frac{\theta_0}{2} \right) - p_H R$$

□

## Proof of Corollary 4

*Proof.* For any  $\theta_0$ , it is easily verified that

$$\frac{\partial W_\ell}{\partial t_\ell} = 0 \implies \frac{\partial^2 W_\ell}{\partial t_\ell^2} > 0,$$

which means that  $W_\ell(t_\ell)$  is quasi-convex, so it is maximized on  $\{0, \infty\}$ . Moreover, it is easily verified that

$$\lim_{t_\ell \rightarrow \infty} W_\ell(t_\ell) - W_\ell(0) = (1 - \theta_0^*) (1 - \Phi) \frac{(\Delta R_o)^2 \theta_0^*}{\bar{\kappa}} - \Phi \frac{\rho}{2\rho + \phi} \frac{\Delta R_o \theta_0^*}{\bar{\kappa}} (B - b) > 0,$$

so  $\arg \max_{\{t_\ell \geq 0\}} W_\ell(t_\ell) = \infty$ .

□

## Details for Minimum Retention

Next, we construct the equilibrium in the case of certification with minimum retention. First, let's consider the price of the loan once the bank hits the minimum retention level  $\underline{\theta}$ . Because  $\theta = \underline{\theta}$  is an absorbing state, the price of the loan satisfies  $q_c(\underline{\theta}) = d_c(\underline{\theta}) = \pi'_c(\underline{\theta})$ . For any  $\theta > \underline{\theta}$  with smooth trading, the HJB is (15) the loan price satisfies equation (18) are unchanged. Therefore, in any region with smooth-trading, the bank's value function, the equilibrium trading strategy, and the price of loans is unchanged. It remains to check if at some  $\theta$ , the bank has incentives to trade atomically to  $\theta^+ = \underline{\theta}$ . The payoff of jumping from  $\theta$  to  $\underline{\theta}$  is  $\Phi \pi_c(\underline{\theta}) + q_c(\underline{\theta})(\theta - \underline{\theta})$  while the payoff in the smooth trading region is  $\Pi_c(\theta) = \Phi \pi_c(\theta)$ . If  $\Phi \pi_c(\theta) + q_c(\theta)(1 - \theta) > \Phi \pi_c(1)$ , the certifying bank always sells immediately to  $\theta^+ = \underline{\theta}$ . Otherwise, as in the proof of the case without minimum requirements, there is a unique  $\tilde{\theta}$  satisfying

$$\Phi \pi_c(\tilde{\theta}) = \Phi \pi_c(\underline{\theta}) + q(\underline{\theta})(\tilde{\theta} - \underline{\theta}) = \Phi \pi_c(\underline{\theta}) + \pi'_c(\underline{\theta})(\tilde{\theta} - \underline{\theta}). \quad (55)$$

By no-arbitrage, the price function must be upper semi-continuous (that is  $q(\tilde{\theta}) = q(\tilde{\theta}+)$ ), which means that  $q(\tilde{\theta}) = \Phi\pi'_c(\tilde{\theta})$ .<sup>33</sup> At the same time, no-arbitrage also requires to be equal to the expected dividend  $\mathbb{E}[d_c(\theta_{\tau_\phi})|\theta_t = \tilde{\theta}]$ . If the bank stops trading at  $\tilde{\theta}$ , and remains there for an exponential time with mean arrival rate  $\lambda$ , at which time sells  $\tilde{\theta} - \underline{\theta}$ , the expected dividend is

$$\mathbb{E}[d_c(\theta_{\tau_\phi})|\theta_t = \tilde{\theta}] = \frac{\lambda}{\phi + \lambda}d_c(\underline{\theta}) + \frac{\phi}{\phi + \lambda}d_c(\tilde{\theta}).$$

Combining the previous conditions, we get that  $\lambda$  is implicitly given by

$$\Phi\pi'_c(\tilde{\theta}) = \frac{\lambda}{\phi + \lambda}\pi'_c(\underline{\theta}) + \frac{\phi}{\phi + \lambda}\pi'_c(\tilde{\theta}) \quad (56)$$

The next proposition, summarize the previous discussion and describe the equilibrium in the certification case with minimum retention requirements.

**Proposition 7** (Equilibrium with Minimum Retention). *There is a unique **Certification Equilibrium with Minimum Retention**. Given the bank's initial retention  $\theta_0$ , the bank sells its loans smoothly at a rate given by equation (19) until  $\tilde{T} = \min\{t > 0 : \theta_t = \tilde{\theta}\}$ , where  $\tilde{\theta}$  is the unique solution to equation (55). After time  $\tilde{T}$ , the bank holds  $\theta_t = \tilde{\theta}$  until an exponentially distributed random time  $\tau_\lambda$ , at which time it sells off  $\tilde{\theta} - \underline{\theta}$ . The exponential time  $\tau_\lambda$  has a mean arrival rate  $\lambda \mathbf{1}_{\{\theta_t = \tilde{\theta}\}}$ , where  $\lambda$  satisfies (56). After time  $\tau_\lambda$ , the bank holds  $\underline{\theta}$  until the projects maturity. The equilibrium loan price is*

$$q_c(\theta_t) = \begin{cases} \Phi(p_L + F(\Delta R_o \theta_t) \Delta) R_o & t < \tau_\lambda \\ \left(p_L + \frac{\phi}{\lambda + \phi} F(\Delta R_o \tilde{\theta}) \Delta\right) R_o & T_* \leq t < \tau_\lambda \\ p_L + F(\Delta R_o \underline{\theta}) \Delta & t \geq \tau_\lambda \end{cases} \quad (57)$$

We omit a formal proof of this proposition as it follows the steps of the proof of Proposition 3.

## Proof of Proposition 6

*Proof.* We can similarly define  $\mathcal{V}$ , the bank's objective function without trading gains as

$$\mathcal{V}(D, \theta) := \Phi \left[ \hat{p}_i(\theta, D) \theta R_o - \int_0^{\kappa_i} \kappa dF(\kappa) \right] + (1 - \Phi) D + \Phi (1 - \hat{p}_i(\theta, D)) (1 - \xi) D,$$

where the new term  $\Phi (1 - \hat{p}_i(\theta, D)) (1 - \xi) D$  stands for the benefit of the government subsidy.

If  $\theta$  is sufficiently large such that the deposit issuance constraint is slack, simple derivation shows that

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<sup>33</sup>Otherwise, there would be a deterministic downward jump in the price at the time  $\theta_t$  reaches  $\tilde{\theta}$  which would be inconsistent with no-arbitrage.

the optimal deposit issuance satisfies

$$\tilde{D}(\theta) = \frac{(1 - \Phi) - \Phi(1 - p(\theta))(1 - \xi)}{\Phi[1 - (1 - p(\theta))(1 - \xi)]f(\kappa_i)\Delta^2}.$$

Recall that in Lemma 2, the constraint is slack whenever  $\theta$  is sufficiently high.

In the region that the deposit-issuance constraint is slack, the HJB equation implies

$$\dot{\theta} = \phi \frac{R_o \xi \frac{(1-\Phi) - \Phi(1-p(\theta))(1-\xi)}{1 - (1-p(\theta))(1-\xi)} - (1 - \Phi)p(\theta)R_o}{\Phi \pi_i''(\theta)}.$$

Clearly, when  $\xi = 1$ , we get the results in subsection 3.2 that  $\dot{\theta} = \phi \frac{R_o(1-\Phi)(1-p(\theta))}{\Phi \pi_i''(\theta)} > 0$ . Moreover, when  $\xi = 0$  so that the entire deposit rate is subsidized by the government,  $\dot{\theta} = \phi \frac{-(1-\Phi)p(\theta)R_o}{\Phi \pi_i''(\theta)} < 0$ , implying the bank sells loans over time. In general, there exists a  $\xi_{\dagger}$  and  $\dot{\theta} < 0$  if  $\xi < \xi_{\dagger}$ .  $\square$