# “Implementation of the Dynamic Time-History Non-Linear Seismic Analysis in MatLab for 2D frames”

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**Overview**

This software is the implementation of the Dynamic Non-Linear Time-History Seismic Analysis for 2D frames fully in MatLab. The functions involve the modal analysis and the Newmark-$\beta$ method for the numerical solution of Second Order differential equations in time steps. Two structural examples are presented, both of which are analysed with damping effects and without them. An artificial acceleration history vector is given as entry data to assess the equivalent inertial forces in the different DOF of the structures at each time step from the modes and frequencies obtained by solving the system $[K - \omega^2 M]\phi = 0$. The numerical solution is non-linear, that is, plastic hinges formations at the end of the elements are considered through the execution of the Static Non-Linear Pushover method at each time-step of analysis.

**Keywords:** Dynamic Time-History, Seismic Analysis, Non-Linear solution, Plastic Hinges Formations, Newmark-$\beta$

## 1 The Dynamic Non-Linear Time-History Seismic Analysis

This analysis is considered as the most sophisticated one of any other analysis given that the non-linear response of materials and/or geometrical non-linearities are considered here in every time-step of assessment so that a more accurate analysis of the structural response can be carried out for a given ground acceleration history (1). The equivalent inertial forces at any of the system DOF's are obtained by solving the classical eigenvalue and eigenvector problem without considering the damping matrix (2), even when damping effects are considered. Thus, the displacement response before such obtained equivalent inertial forces can be carried out through a quasi Non-linear Pushover analysis at each time-step so that the structure's stiffness degradation and damping degradation can be considered.

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{F(t)\} = \frac{\phi_i^T M[1]}{M*} a(t) M\phi_i \quad (1)$$

$$([K] - \omega^2[M])[\phi] = \{0\} \quad (2)$$

### 1.1 Material Non-Linearities - Quasi Pushover analysis

By referring to executing a *quasi* Non-Linear Pushover analysis to consider the material non-linearities of the structure's response it means that no incremental load is computed but only those induced by the ground acceleration at a given time-step $F(t)$ so that the main purpose would be to identify if any plastic hinge formation takes place due to such external loads so that a new stiffness matrix and an equivalent internal force vector is obtain to consider such damage in the following time-steps and in the structural response in general. The process of this such Quasi Pushover analysis is described below in pseudo-code:

**Algorithm 1.1:** Pseudo-code for the Quasi Pushover to consider material non-linearities in the Newmark-$\beta$ method.

1. Input materials and geometry properties of the structure, E,A,G,J,Iz,Iy, coord-xy-nodes
2. Input current plastic hinge state and Resistant Bending moments at each element's ends $Mp$
3. Perform a static linear analysis with the current plastic hinge state and check for new plastic hinge formations
4. Build a new global stiffness matrix and the corresponding equivalent internal force vector in case a new plastic hinge formation took place

For the transformation process of equivalent non-plastic structures to plastic ones, the liberation of the corresponding DOFs and the proper distribution of internal loads must be done. For this purpose the following mechanisms of **Fig. 1**, **Fig. 2** and **Fig. 3** apply with their corresponding equivalent numerical matricial representations (3), (4) and (5):

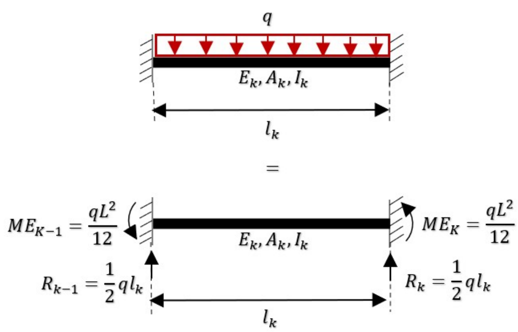



**Figure 1.** Equivalent structural system for a beam element without plastic hinge formations at the ends.

$$[K] = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{12EI_z}{l^3} & \frac{6EI_z}{l^2} & 0 & -\frac{12EI_z}{l^3} & \frac{6EI_z}{l^2} \\ 0 & \frac{6EI_z}{l^2} & \frac{4EI_z}{l} & 0 & -\frac{6EI_z}{l^2} & \frac{2EI_z}{l} \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{12EI_z}{l^3} & -\frac{6EI_z}{l^2} & 0 & \frac{12EI_z}{l^3} & -\frac{6EI_z}{l^2} \\ 0 & \frac{6EI_z}{l^2} & \frac{2EI_z}{l} & 0 & -\frac{6EI_z}{l^2} & \frac{4EI_z}{l} \end{bmatrix} \tag{3}$$

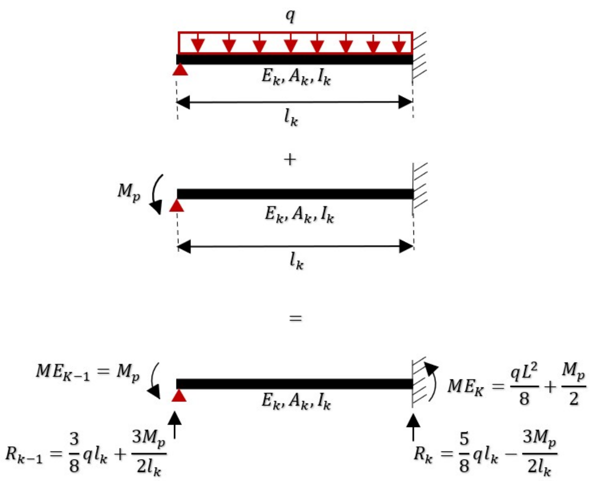



**Figure 2.** Equivalent structural for a beam element with a plastic hinge formation at one end.

$$[K] = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & \frac{3EI_z}{l^3} & 0 & 0 & -\frac{3EI_z}{l^3} & \frac{3EI_z}{l^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & -\frac{3EI_z}{l^3} & 0 & 0 & \frac{3EI_z}{l^3} & -\frac{3EI_z}{l^2} \\ 0 & \frac{3EI_z}{l^2} & 0 & 0 & -\frac{3EI_z}{l^2} & \frac{3EI_z}{l} \end{bmatrix} \tag{4}$$

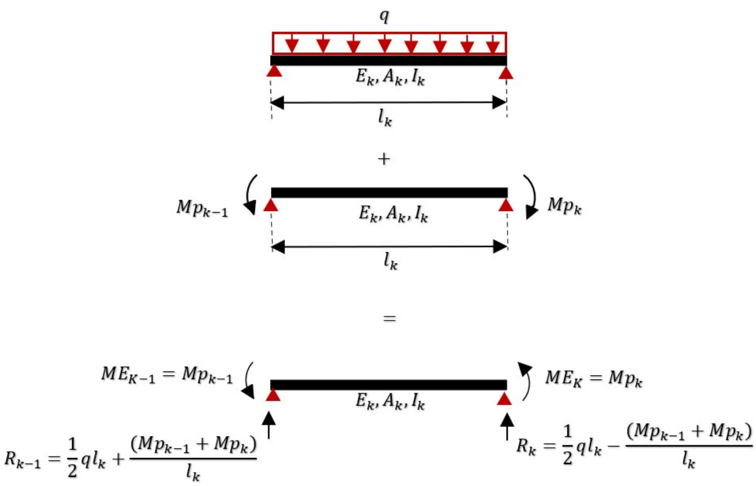



**Figure 3.** Equivalent structural system for a beam element with two plastic hinge formations at the ends.

$$
[K] = \begin{bmatrix} \frac{EA}{l} & 0 & 0 & -\frac{EA}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{EA}{l} & 0 & 0 & \frac{EA}{l} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}
$$

## 1.2 The consistent mass method

The construction of the mass matrix is done through a Finite Element method approach in which an element mass matrix (6) is built for all elements and assembled into a global one as for the stiffness matrix.

$$
M_e = \frac{\gamma AL}{420g} \begin{bmatrix} 440 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix} \tag{6}
$$

It must be stressed that not only the stiffness matrix should be updated for the material non-linearity consideration but also the internal force system, so that the hypothesis of perfect $Elasto-Plastic$ behaviour is complied. Thus, according to the previous equivalent mechanical mechanisms the following element force vectors must be included in the non-linear Newmark-$\beta$ response:

**No plastic formation at the ends:**

$$
f_e = \begin{bmatrix} \frac{q_x \cdot L}{2} & \frac{q_y \cdot L}{2} & \frac{q_y \cdot L^2}{12} & \frac{q_x \cdot L}{2} & \frac{q_y \cdot L}{2} & -\frac{q_y \cdot L^2}{12} \end{bmatrix} \tag{7}
$$

**One plastic formation at on end:**

$$
f_e = \begin{bmatrix} 0 & -\frac{3M_p}{2L} & \frac{M_p}{2} & 0 & \frac{3M_p}{2L} & M_p \end{bmatrix} + \begin{bmatrix} \frac{q_x \cdot L}{2} & \frac{5 \cdot q_y \cdot L}{8} & \frac{q_y \cdot L^2}{8} & \frac{q_x \cdot L}{2} & \frac{3q_y \cdot L}{8} & 0 \end{bmatrix} \tag{8}
$$

**One plastic formation at each end:**

$$
f_e = \begin{bmatrix} 0 & \frac{(Mp_1+Mp_2)}{L} & Mp_1 & 0 & -\frac{(Mp_1+Mp_2)}{L} & Mp_2 \end{bmatrix} + \begin{bmatrix} \frac{q_x \cdot L}{2} & \frac{q_y \cdot L}{2} & 0 & \frac{q_x \cdot L}{2} & \frac{q_y \cdot L}{2} & 0 \end{bmatrix} \tag{9}
$$

## 1.3 Structural Rayleigh damping

For MDOF the damping matrix $C$ can be computed as a function of the mass matrix and the stiffness matrix by using the Rayleigh's coefficients $\alpha$ and $\beta$ as in (10), where $\alpha$ and $\beta$ are determined by solving the linear system of equations (11) from the selection of a pair of chosen modal damping factors $\xi_1, \xi_2$ and a corresponding pair of frequencies $\omega_1, \omega_2$:

$$C_e = \alpha \cdot M_e + \beta \cdot K_e \tag{10}$$

$$\begin{aligned} \xi_1 &= \frac{1}{2\omega_1}\alpha + \frac{\omega_1}{2}\beta \\ \xi_2 &= \frac{1}{2\omega_2}\alpha + \frac{\omega_2}{2}\beta \end{aligned} \tag{11}$$

Even though Rayleigh damping has the clear disadvantage of not allowing realistic damping for all modes of interest (in case there are far more than two in question) it provides a great solution approach when only a few modes are sought to be analysed.

## 2 Dynamic acceleration

When aiming at solving a dynamic system subject to an earthquake excitation the accelerations in time are required, for which purpose, a real or artificial accelerogram as the one shown in **Fig. 4** should be given as entry data.

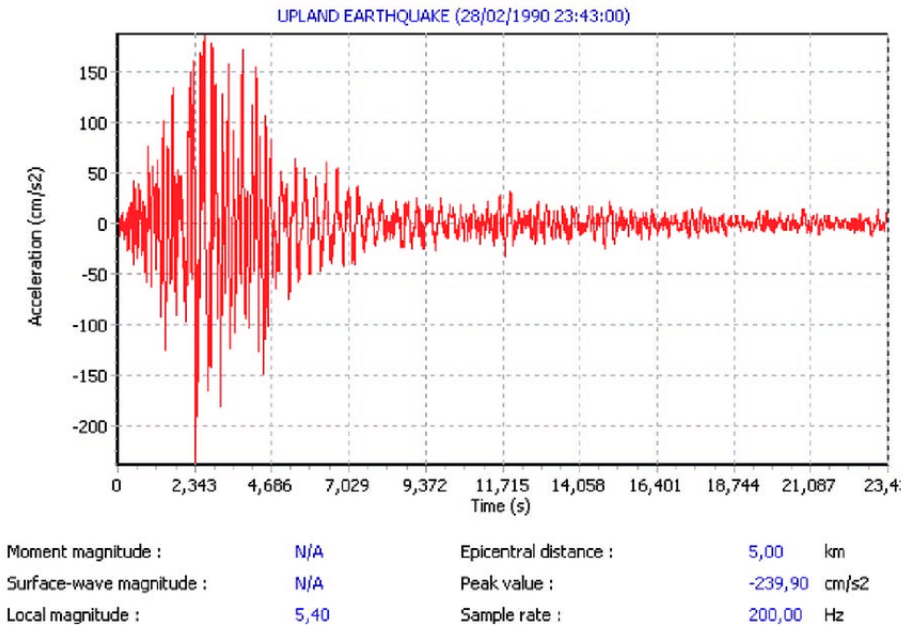



**Figure 4.** Example of an accelerogram for an earthquake.

From such accelerations in time the vector $f(t)$ in (1) corresponding to the DOF forces in time could be obtained by solving the eigenvalue/eigenvector problem for the modes of vibration in function of such accelerations.

# 3 Numerical evaluation of dynamic response

When considering coupled equations of motion in which damping is integrated then one must opt to apply other procedures different than *mode-superposition* methods to solve them, such as **direct integration** methods that aim mainly at the solution of second-order differential equations (see [2]).

## 3.1 Beta-Newmark method

The Newmark-$\beta$ method initially introduced as the *Constant average acceleration method* is presented in a form appropriate for direct integration of linear second-order systems, such as those whose damping matrix cannot be diagonalized by the system of normal modes. This method in contrast with the *Central difference method* - which is probably the most known and fundamental algorithm for the approximate numerical solution second-order differential ordinary equations - provides **unconditionally stable** solutions, that is, that it is not entirely required to have qualitative knowledge a priori of the frequencies presented in the system's response to select a proper time step [2].

The basis of the Newmark-$\beta$ is the assumption that the system displacement vector $u_{n+i}$ and its derivative $\dot{u}_{n+1}$ can be written in the form (12), which can be viewed as a type of generalized Taylor series approximation. Thus, the Newmark-$\beta$ method is said to be an implicit method with the discrete equation of motion being enforced at the time step $t_{n+1}$ as (13):

$$\begin{aligned} \dot{u}_{n+1} &= \dot{u}_n + ((1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1})h \\ u_{n+1} &= u_n + \dot{u}_n h + (1-2\beta)\frac{h^2}{2}\ddot{u}_n + \beta h^2 \ddot{u}_{n+1} \end{aligned} \tag{12}$$

$$M\ddot{u}_{n+1} + C\dot{u}_{n+1} + Ku_{n+1} = p(t_{n+1}) \tag{13}$$

### 3.1.1 The Non-Linear Beta-Newmark method

One approach to compute the Non-Linear Newmark-$\beta$ method is to consider the formation of plastic hinges at the ends of the structure's elements through the Static Non-Linear Pushover method, for instance, so that the stiffness matrix and therefore, the damping matrix may be modified at a given time-step of analysis due to the stiffness degradation that was caused by the plastic hinges formations. The following algorithm describes this such method in detail:

### 3.1.2 Algorithm for the Non-Linear Beta-Newmark method

**Algorithm 3.1:** Pseudo-code for the Non-Linear Newmark-$\beta$ method for 2nd Order Ordinary Differential Equations.

1. Input M,K and C (if any)
2. Input initial conditions $u_0, v_0$ and compute the initial acceleration $a_0$ as:
   $a_0 = M^{-1}(p_0 - Cv_0 - Ku_0)$
3. Set the time step length $h$
4. Initialize loop iteration:

   **For** $n = 1$ **To** $n = n - steps$

   **4.1** Compute the right hand side term RHS as:
   $RHS_n = -Ku_n - (C + hK)\dot{u}_n - [h(1-\gamma)C + \frac{h^2}{2}(1-2\beta)K]\ddot{u}_n$

   **4.2** Solve for the second derivatives at the next time step:
   $[M + \gamma C + \beta h^2 K]\ddot{u}_{n+1} = RHS_n$

   **4.3** Evaluate the set of displacements and velocities for the next time step:
   $\dot{u}_{n+1} = \dot{u}_n + ((1-\gamma)\ddot{u}_n + \gamma\ddot{u}_{n+1})h$
   $u_{n+1} = u_n + \dot{u}_n h + (1-2\beta)\frac{h^2}{2}\ddot{u}_n + \beta h^2 \ddot{u}_{n+1}$

   **4.4** Assess if one or more plastic hinge formation took place by executing the Quasi Pushover method. If so, update $K$

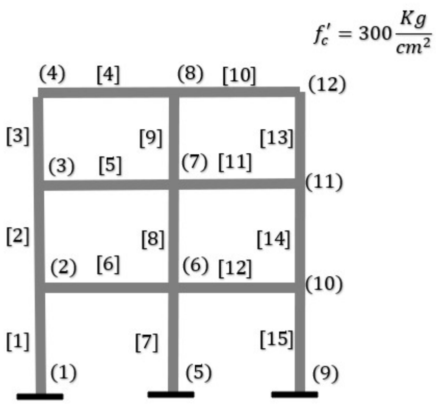
   **4.5** Update the damping matrix $C$ (with the Rayleigh coefficients, for instance) as:
   $C = \alpha_1 M + \alpha_2 K$

   **End For**

# 4 Illustrative example

## 4.1 Problem

It is required to perform a non-linear dynamic analysis for the following structural model of **Fig. ??** subject to a given ground seismic acceleration history.



## 4.2 Solution

Let us start by setting the material, mechanical properties and geometry:

```
nnodes=12;
nbars=15;

%% Materials
fpc=[300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300;
     300];

% Modulus of Elasticity of each element
E=14000*(fpc).^0.5;

%% Geometry
dimensions=[20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50;
            20 50];

A=dimensions(:,1).*dimensions(:,2);
I=1/12.*dimensions(:,1).*dimensions(:,2).^3;
```

```
% Coordinates of each node for each bar
coordxy=[0 0;
         0 300;
         0 600;
         0 900;
         600 0;
         600 300;
         600 600;
         600 900;
         1200 0;
         1200 300;
         1200 600;
         1200 900];
```

Next, the node connectivity and topology:

```
%% Topology
% Node conectivity
ni=[1;2;3;4;3;2;5;6;7;8; 7; 6; 9; 10;11];
nf=[2;3;4;8;7;6;6;7;8;12;11;10;10;11;12];

% Topology matrix
Edof=zeros(nbars,7);
for i=1:nbars
    Edof(i,1)=i;
    Edof(i,2)=ni(i)*3-2;
    Edof(i,3)=ni(i)*3-1;
    Edof(i,4)=ni(i)*3;

    Edof(i,5)=nf(i)*3-2;
    Edof(i,6)=nf(i)*3-1;
    Edof(i,7)=nf(i)*3;
end
```

Now, for the boundary conditions, it is considered that the three nodes at the base are fixed, therefore:

```
%% Boundary conditions
% Prescribed DOF - [N-DOF, displacement]
bc=[1 0;
    2 0;
    3 0;
    13 0;
    14 0;
    15 0;
    25 0;
```

```
    26 0;
    27 0];
```

*Note that the DOF numbering was made according to the node number so that for a node $i$:*

$$DOF_x = 3 \cdot i - 2$$
$$DOF_y = 3 \cdot i - 1$$
$$DOF_\theta = 3 \cdot i$$

Now, a uniformly distributed load over each beam is applied with a magnitude of $q = 50Kg/cm$ an assigned through the array $qbarxy$ as:

```
%% Loads

type_elem=[1 "Col";
           2 "Col";
           3 "Col";
           4 "Beam";
           5 "Beam";
           6 "Beam";
           7 "Col";
           8 "Col";
           9 "Col";
           10 "Beam";
           11 "Beam";
           12 "Beam";
           13 "Col";
           14 "Col";
           15 "Col"];

% Distributed Loads on beams
beamsLoads=[1 -50;
          2 -50;
          3 -50;
          4 -50;
          5 -50;
          6 -50];

elem_cols=[];
elem_beams=[];
beams=0;
cols=0;
for j=1:nbars % To identify which elements are beams and which are columns
    if type_elem(j,2)=="Beam"
        beams=beams+1;
        elem_beams=[elem_beams,j];
```

```
    elseif type_elem(j,2)=="Col"
        cols=cols+1;
        elem_cols=[elem_cols,j];
    end
end
qbarxy=zeros(nbars,2);
qbarxy(elem_beams',2)=1.1*(beamsLoads(:,2));
```

*Note that an array called $type\_elem$ was first created to label each element as a beam or column to then automate the load assignation through the array $beamsLoads$. However, there are many other ways to carry out this load assignation. The user should feel free to modify this block of code to his own preferences.*

Next, the mode of vibration of interest and the Rayleigh coefficients for the computation of the damping matrix are set:

```
%% Mode of vibration of interest
modal=1;

%% Damping matrix (for the damped case)
omega1=5; % frequencies
omega2=10;

zeta1=0.25; % damping factors for each DOF
zeta2=0.2;

D=[1/(2*omega1) omega1/2;
    1/(2*omega2) omega2/2];

theta=[zeta1;
    zeta2];

AlfaBeta=D\theta; % Rayleigh coefficients
```

Now, the computation of the damping matrix, the mass matrix and the stiffness matrix are computed as:

```
%% Modal analysis
pvconc=0.0024; % unit weight of concrete
unitWeightElm=zeros(nbars,1)+pvconc;

% Consistent mass method
[Cgl,Mgl,Kgl]=SeismicModalMDOF2DFrames2...
(coordxy,A,unitWeightElm,qbarxy,Edof,E,I,ni,nf,AlfaBeta,g);
```

Finally, the dynamic analysis is executed. For this purpose an artificial acceleration history is set. Such acceleration history will be based on a max one given by seismic response spectrum $sa$ and then integrated in a cos function as:

```
%% Dynamic analysis
% Max Seismic response from the CFE-15 spectrum

g=981; % gravity acceleration (cm/sec^2)
Fsit=2.4; FRes=3.8; % Factores de sitio y de respuesta
a0_tau=200; % cm/sec^2

ro=0.8; % Redundance factor
alf=0.9; % Irregularity factor
Q=4; % Seismic behaviour factor

Ta=0.1;
Tb=0.6;
Te=0.5; % Structure's period
k=1.5; % Design spectrum slope
Qp=1+(Q-1)*sqrt(Te/(k*Tb)); % Ductility factor

Ro=2.5; % Over-resistance index
R=Ro+1-sqrt(Te/Ta); % Over-resistance factor

sa=a0_tau*Fsit*FRes/(R*Qp*alf*ro); % Reduced pseudo-acceleration (cm/sec^2)

% Time discretization
dt=0.05;
ttotal=8;
t=0:dt:ttotal;
npoints=length(t);

% Artificial ground acceleration history
tload=1.5; % duration of external excitation

g=sa*cos(5*t); % Acceleration in time
for i=1:length(g)
    if t(i)>tload
        g(i)=20*cos(30*t(i));
    end
end
```

Once this such acceleration history, then a force history $f(t_i)$ array is computed by executing a modal analysis through the solution of the eigenvalue/eigenvector problem $[K - \lambda M]\phi = 0$ for each time step:

```
% Forces history
```

```
f(:,1)=zeros(3*nnodes,1);
for i=1:npoints
    % Modal analysis without Damping
    [f(:,i+1),T(:,i),La(:,i),Egv]=ModalsMDOF2DFrames2(Mgl,Kgl,...
        bc,g(i),modal);

end
```

Now, the parameters and variables for the execution of the Non-Linear Newmark-$\beta$ method are set.

```
% Analysis in time with/without viscous damping
beta=0.25; % parameters of the algorithm Newmark-Beta
gamma=0.5;

d0=zeros(3*nnodes,1); % initial values of displacement and velocity
v0=zeros(3*nnodes,1);

% Initial elements' end support conditions
support=[1 "Fixed" "Fixed";
         2 "Fixed" "Fixed";
         3 "Fixed" "Fixed";
         4 "Fixed" "Fixed";
         5 "Fixed" "Fixed";
         6 "Fixed" "Fixed";
         7 "Fixed" "Fixed";
         8 "Fixed" "Fixed";
         9 "Fixed" "Fixed";
         10 "Fixed" "Fixed";
         11 "Fixed" "Fixed";
         12 "Fixed" "Fixed";
         13 "Fixed" "Fixed";
         14 "Fixed" "Fixed";
         15 "Fixed" "Fixed"];

% Elastic resistant bending moments for each element's ends
Mp=[9680000 9680000;
    8490000 8490000;
    8490000 8490000;
    3363000 3276940;
    3363000 3276940;
    3363000 3276940;
    9680000 9680000;
    8490000 8490000;
    8490000 8490000;
    3363000 3276940;
    3363000 3276940;
```

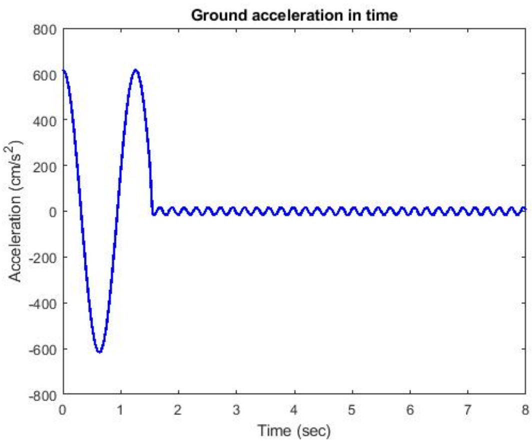
```
    3363000 3276940;
    9680000 9680000;
    8490000 8490000;
    8490000 8490000]; % Kg-cm

mpbar=zeros(nbars,2); % to save the plastic moments at each articulation
                      % of each bar as plastifications occur
plastbars=zeros(2,nbars);
dofhist=[16 19 22]; % dof to evaluate

% Non-Linear Newmark-Beta
[Dsnap,D,V,A]=NewmarkBetaNonLinearMDOF2(Kgl,Cgl,Mgl,d0,v0,dt,beta,gamma,...
 t,f,dofhist,bc,AlfaBeta,qbarxy,A,Mp,E,I,coordxy,ni,nf,support,mpbar,...
 plastbars);
```

*Note that both the elements' end supports and resistant elastic bending moments have to be given, as the Pushover analysis is performed for each time step to verify if plastic hinge formation took place.*

Note also that the DOF of interest to assess the displacements, velocities and accelerations were the $16, 19, 22$ corresponding to the horizontal displacement of each floor.

### 4.3 Results

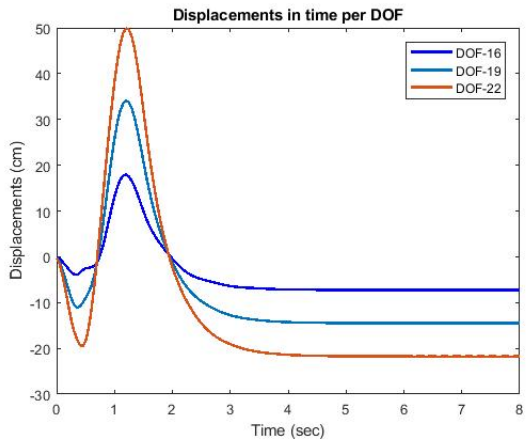
The artificial acceleration history entered is shown in the following plot:

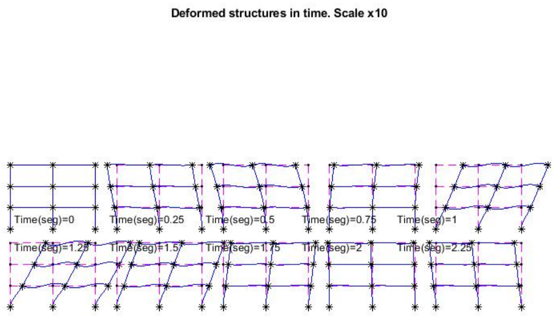



The following displacement history was recorded for each DOF in question $(16, 19, 22)$:

*Note that at the end the DOF's displacements remain constant and non-zero, even though the acceleration is very low, indicating that the structure underwent the plastic range and therefore, will remain deformed.*

Finally, the deformed structure at 10 time steps are shown at the end as:

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[1] Comisión Federal de Electricidad CFE-2015, Manual de Diseño de Obras Civiles, Capítulo C.1.3 Diseño por sismo, México, 2015. Spanish version.

[2] Fundamentals of Structural Dynamics, Roy R. Craig Jr., Andrew J. Kurdila, 2nd Ed., John Wiley & Sons, Inc., 2006