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Stationary and Transient 2D Heat-Flow in a concrete wall in MatLab

Contents

1 Overview

The present software is the implementation of the Finite Element Method in MatLab for the 2D Stationary and Transient Heat-Flow analysis of a concrete wall. The wall is modelled to be not only of concrete but with additional inner layers of wool and plaster. For the initial temperature distribution in time 0, a stationary heat-flow analysis is first carried out for the wall subject to an inner temperature of $30C(303.15K)$ and an outer temperature of $5C(278.15K)$. Thereafter, a transient heat-flow analysis is performed for a sudden increase in the outer temperature to $100C(373.15K)$. A mesh of linear triangular finite elements is used for discretization. Numerical integration and interpolation is carried out in natural coordinates.
The final outcomes were quite acceptable in comparison with those obtained from COMSOL Multiphysics. For the transient heat-flow analysis an amount of 1400 time-steps were required for the wall to reach equilibrium again once the increase in the outer temperature had occurred, which accounts in favour for the use of concrete as a construction material for infrastructure subject to fire exposure, explosions or extreme heat conditions.

2 Introduction

The analysis of heat-flow in structures that are or may be subject to fire, explosions or intense heat conditions is of primary importance for their proper design and assessment. An excessive amount of heat may damage a structure and its non-structural components, due to material dilatations that may cause certain amount of induced internal stresses in the structure or by deterioration (peeling off, crumbling, etc.).

In this sense, designers and structural engineers have to take decisions regarding the type of materials to use, either to reinforce an existing structure or to build a new structure. Reinforced Concrete is the primary material that comes to mind when designing a new structure that may be subject to critical conditions of heat or fire, given that it possesses certain mechanical and chemical properties that are advantageous for such scenarios, in comparison with other materials such as structural steel or wood. However, it is also common to reinforce a concrete structure with fireproof materials such as wool, glass or plaster, among others, to help the whole concrete structure to better withstand the fire and heat conditions. In **Fig. 1** for example, a concrete wall over which two layers of fire resistant materials are put on its outer boundary is shown.

Nevertheless, in order for structural engineers to determine the width of the walls or dimension of a structure in general to withstand properly the heat and fire conditions that the structure may be subject to with the less usage of materials a thorough analysis must be carried on through the Heat equation, usually with the Finite Element Method or the Finite Difference Method, among others, according also to the budget and computational power available.

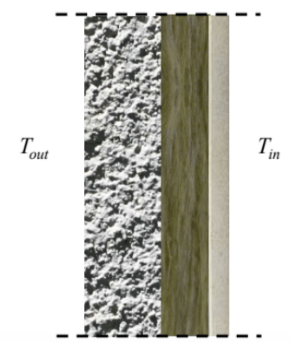

Figure 1: *Representation of house's wall whose both surfaces are subject to different temperature values in the outside and inside (but constant through time), therefore having a steady inner heat flow.*

By using such computational techniques of analysis, the whole distribution of heat in the whole established domain can be assessed (heat-flow) from which certain useful information can be thereafter known, such as the max rate of heat loss for an extended period of time. In this sense, it becomes important also to establish the dimension of analysis, either 1D, 2D or 3D, depending on the relative magnitudes of heat transfer rates in the different directions and the level of accuracy desired. For the given example of the concrete wall, the numerical model could be assessed in 2D with quite accurate results, given that any of two whole transversal faces of the wall are usually subject to the same amount of heat.

2.1 Definition of the problem

The following wall of **Fig. 2 Left** with an outer layer of concrete (thickness $h_1 = 300mm$), a middle layer of mineral wool (thickness $h_2 = 150mm$) and an internal layer of plaster (thickness $h_3 = 18mm$), subject to an inner air temperature of $T_{in} = 30C$ and an outer air temperature of $T_{out} = 5C$ is to be analysed with the Finite Element Method in a 2D dimension by using **triangular elements**. The wall penetrates a concrete slab of thickness $h_4 = 150mm$ which separates two floors inside a building. By symmetry, the domain of the whole model can be reduced as **Fig. 2 (Right)**.

After the previous steady state definition a sudden increase of temperature to $100C$ of the outside air occurs, from which a transient heat-transfer event takes place (warming). Thus, the assessment of the warming process is also analysed, that is, how the temperature varies from this occurrence over the whole domain of the structure through time.

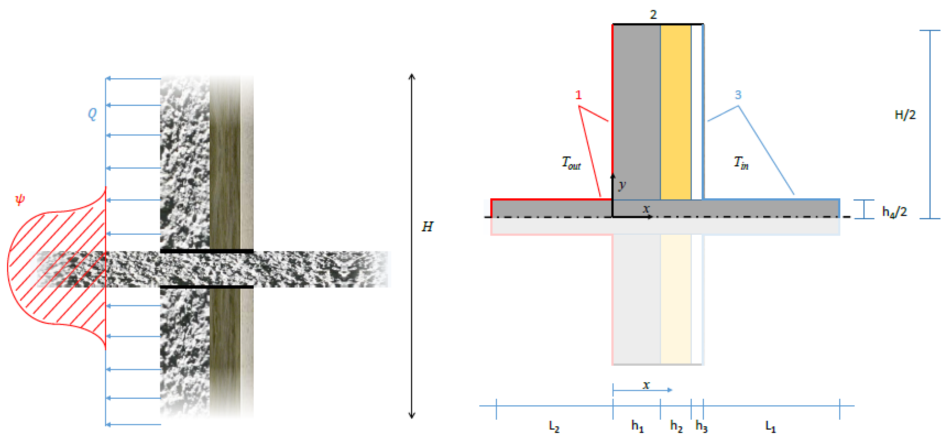

Figure 2: *Structure geometry description Wall-Slab subject to different temperatures at its boundaries.*

3 The Heat-Flow equation

Heat flow or Heat conduction is defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones. Unlike temperature, heat transfer or heat flow has direction as well as magnitude. The specification of the temperature at a point in a medium first requires the specification of the location of that point.

Heat flow problems are often classified as being steady or transient (unsteady). The term steady implies no change with time at an point within the medium, therefore the temperature or heat flux remains unchanged with time during steady transfer throughout the medium at any point, although both quantities may vary from one location to another (see **Fig. 3**).

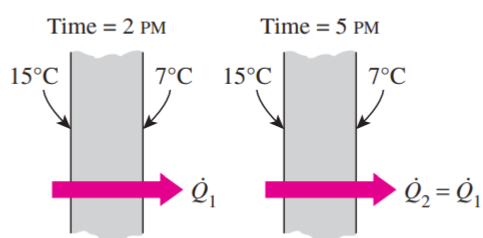

Figure 3: *Representation of a steady-state heat flow problem. No variation of temperature or heat flux through time is observed, although their values may vary from location to location in the medium.*

For example, heat transfer through the walls of a house or a building is steady when the conditions inside the house and the outdoors remain constant for several hours. Even for this case, the temperatures on the inner and outer surfaces of the wall will be different unless the temperatures inside and outside the house are the same. On the other hand, however, when there is an abrupt change or drop in temperature surrounding the wall, then a transient heat transfer problem takes place over the whole domain of the medium (for cooling or warming), that is, the temperature would vary with time as well as with position on the wall's domain.

The driving force of heat-flow q in one direction x is proportional to the *temperature difference* dT/dx, the larger the temperature difference the larger the *rate* of heat transfer (dq/dx). The area normal to the direction of heat-flow also contributes proportionally to the rate of heat conduction, whereas the distance dx in that direction is inversely proportional, as it is expressed in the differential form by the **Fourier's law of heat conduction** (1):

$$\dot{Q} = q = -kA\frac{dT}{dx} \tag{1}$$

where k is the thermal conductivity of the material, which is measure of the ability of a material to conduct heat, and A is the heat conduction area normal to the x direction. Such a value varies in general with temperature, although sufficient accurate results can be obtained by using a constant value for an average temperature. The negative signs indicates that the heat flow goes naturally from the higher temperature's location to the lowest one.

3.1 2D Heat-Flow equation

To obtain a more general form of the Fourier's law of heat conduction for a 2D problem (x, y), then both direction can be simply considered separately as the components of a vector by supposing that the material is *isotropic*, that is, that it has the same properties (k_x and k_y) in all directions, therefore (2):

$$\dot{Q} = q = -D_{xy}\nabla T \tag{2}$$

where D_{xy} is equal to (3):

$$D = \begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \tag{3}$$

3.1.1 Boundary Conditions

It is known that when a heated surface is exposed to a cooling medium, the surface will cool until reach equilibrium of heat with such medium. It is said then, that the heat is convected away. Thus, the *convection heat transfer* between the surface and the medium in contact is given by the *Newton's law of cooling* (4):

$$q_s = \alpha(T_s - T_m) = \beta A(T_s - Tm) \tag{4}$$

where T_s is the surface temperature, T_m is the temperature of the surrounding medium, and α is the thermal conductivity (a material property), also expressed as βA where β is the convection heat transfer coefficient.

Now, when a heat-flow Q_0 is applied, the heat-flow due to conduction and convection at a boundary point must be in balance as (5) in 1D:

$$kA\frac{\partial T}{\partial x} + \alpha(T - T_m) + Q_0 = 0 \tag{5}$$

3.1.2 Finite Element formulation

When generalizing the previous equation for a continuous medium, we obtain the *Weak form* of the of the stationary heat-flow (6):

$$\int_A (\nabla v) q dA + \int_L v\alpha(T_L - Tm) = \int_A vQdA \tag{6}$$

where v equal to NC is the approximation with Galerkin's method, with $(\nabla v)^T = \nabla N^T C^T$ which is equal to (7):

$$\nabla N^T C^T = B^T C^T \tag{7}$$

where N are the shape functions and where B in Cartesian coordinates is equal to (8):

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \dots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_2}{\partial y} & \dots & \frac{\partial N_n}{\partial y} \end{bmatrix} \tag{8}$$

Also, the temperature over an element from element nodal temperatures a can be also approximated through the same shape functions N as $T = Na$, which in matrix form it yields (9):

$$T = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} \tag{9}$$

Thus, the gradient of temperature of an element ∇T can be represented as (10):

$$\nabla T = \nabla(Na) = Ba \tag{10}$$

Hence, the weak form (6) can be rewritten generally in the FE form as (11):

$$C^T \int_A B^T D \nabla T B dA \cdot a = C^T \int_A N^T Q dA - C^T \int_L N^T \alpha(Na - T_m) dL \tag{11}$$

For our present problem, given that we have two opposite boundaries L_1 and L_2 and given that the heat-flux $Q = 0$, the previous FE form should be expressed as (12):

$$\int_A B^T DBdA \cdot a = -\int_{L_1} N^T \alpha(Na - T_1)dL_1 + \int_{L_2} N^T \alpha(Na - T_2)dL_2 \tag{12}$$

Thus, by re-ordering of terms, we obtain (13):

$$\int_A B^T DBdA \cdot a + \int_{L_1} N^T \alpha N \cdot a + \int_{L_2} N^T \alpha N \cdot a = -\int_{L_1} N^T \alpha(-T_1)dL_1 + \int_{L_2} N^T \alpha(-T_2)dL_2 \tag{13}$$

Which in matricial form it produces (14), where K is the heat-conduction matrix and K_c is the convection matrix:

$$([K] + [K_c]) \cdot a = F_b \tag{14}$$

3.2 2D Transient Heat-Flow

The balance of energy requires that:

$$\frac{\partial}{\partial x}(kA\frac{\partial T}{\partial x}) + Aq = \rho c A \frac{\partial T}{\partial t} \tag{15}$$

where t is time, ρ is the density, c is the specific heat of the material, T is the temperature and A is the cross-section area perpendicular to the heat-flow.

3.2.1 Finite Element formulation

Such equation could be expressed in the *Strong Form* as (16):

$$QA + \nabla^T qA = \rho c A\dot{T} \tag{16}$$

In general way, we could also express the previous equation in the Weak form as (17):

$$\int_A vQdA + \int_A \nabla^T vqdA = \int_A \nabla^T v\rho cTdA \tag{17}$$

Now, by considering the boundary conditions as (18) and adding them in the Weak Form equation (similar as for the steady-state problem), we obtain (19):

$$\begin{aligned} q_n &= \alpha(T_L - T_m) \\ T(x, y, \tau) &= T_0 \end{aligned} \tag{18}$$

$$\int_A QdA + \int_A (\nabla v)^T qdA - \int_L v\alpha(T_L - T_m)dL = \int_A v\rho c\nabla TdA \tag{19}$$

And finally, the FE form:

$$\int_A Q dA + \int_A (\nabla v)^T D \nabla T dA - \int_L v\alpha(T_L - T_m) dL = \int_A v\rho c\dot{T} dA \tag{20}$$

where T and $\dot{T}$ can be expressed as:

$$\begin{aligned} T(x, y, \tau) &= N \cdot a(\tau) \\ \dot{T}(x, y, \tau) &= N\dot{a}(\tau) \end{aligned} \tag{21}$$

and c as (22):

$$c = \int_A N^T \rho c N dA \tag{22}$$

And similar as for the 2D steady-state FE equation, we could substitute $v = NC$ and $(\nabla v)^T = \nabla N^T C^T$ with the approximation to yield (23):

$$C^T \int_A N^T Q dA + C^T \int_A B^T D \nabla T B dA \cdot a - \int_L C^T N^T \alpha(Na - T_m) dL = C^T \int_A N^T \rho c\dot{a} dA \tag{23}$$

Then, similar a for the steady-state formulation, the two opposite boundaries L_1 and L_2 must be considered, as well as the fact that there is no heat-flux $Q = 0$. Therefore, the previous general FE equation should be expressed as (24):

$$\begin{aligned} \int_A B^T D \nabla T B dA \cdot a - \int_{L_1} N^T \alpha(Na - T_1) dL_1 + ... \\ ... + \int_{L_2} N^T \alpha(Na - T_2) dL_2 = \int_A N^T \rho c\dot{a} dA \end{aligned} \tag{24}$$

Finally, by re-ordering terms, we obtain (25):

$$\begin{aligned} \int_A N^T \rho c dA \cdot \dot{a} + \int_A B^T D B dA \cdot a + \int_{L_1} N^T \alpha N dL_1 \cdot a - \int_{L_2} N^T \alpha N \cdot a = ... \\ ... = \int_{L_1} N^T \alpha N T_1 dL_1 - \int_{L_2} N^T \alpha(T_2) dL_2 \end{aligned} \tag{25}$$

Which could be expressed in matrix form as (26), where C is *Capacity matrix*, K is the *Conduction matrix* and K_c is the *Convection matrix*.

$$C \cdot \dot{a} + (K + K_c) \cdot a = F_b \tag{26}$$

3.3 Time analysis

3.3.1 The generalized midpoint rule

The previous equation is a First Order differential equation that could be re-arranged as in (27), where $K \to K + K_c$ and $f = F_b$:

$$c\dot{a} + Ka = f \tag{27}$$

whose solution could be approximated iteratively in time by considering a linear variation of the solution between each time-step, such that (28) and (29):

$$a(t) \approx \theta a_{n+1} + (1-\theta)a_n \tag{28}$$

$$\dot{a}(t) \approx \frac{a_{n+1} - a_n}{\Delta t} \tag{29}$$

where $\Delta t = t_{n+1} - t_n$ and $0 \leq \theta \leq 1$ equal to (30):

$$\theta = \frac{t - t_n}{\Delta t} \tag{30}$$

Therefore, the solution of (27) could be expressed as (31):

$$(1 + \theta\Delta t\lambda)a_{n+1} = (1 - (1-\theta)\Delta t\lambda)a_n + \Delta t f(t_n + \theta\Delta t) \tag{31}$$

where $f(t_n + \theta\Delta t)$ can be expressed also as (32):

$$f(t_n + \theta\Delta t) = \theta f(t_{n+1}) + (1-\theta)f(t_n) \tag{32}$$

Now, the choice of the value of θ can significantly influence the solution and its convergence. Two cases are, in general, present: if $\theta \geq 0.5$, then the solution of (31) becomes unconditionally stable. On the other hand, if $\theta < 0.5$ the length of the time-step value Δt must be chosen as (33), for which cases the solution will be conditionally stable:

$$\Delta \leq \frac{2}{(1-2\theta)\lambda} \tag{33}$$

where λ is the largest eigenvalue to $det(K - \lambda c) = 0$.

This is the method used in this software to solve such differential equation and perform the transient heat-flow analysis.

4 Finite Element Discretization

When modelling a continuum with the Finite Element Method it is important to take on account which sort of discretization would be more fit for the numerical solution. Not only it is crucial to consider the geometry of the continuum but also the physical problem or phenomenon of study, given that the degree of refinement of the mesh may influence significantly the results (**Fig. 4**). The computational power available is another vital factor to consider when meshing a FE model; the more mesh density the highest the cost of computation. One may also need to choose weather to work with natural coordinates or isoparametric formulations. Isoparametric formulations in contrast to natural coordinates formulations make it possible to generate elements that are non-rectangular or have curved sides. Therefore, they have far more preference of use when grading meshes for irregular shapes.

In the present work, the Finite Element formulation will be based on natural coordinates.

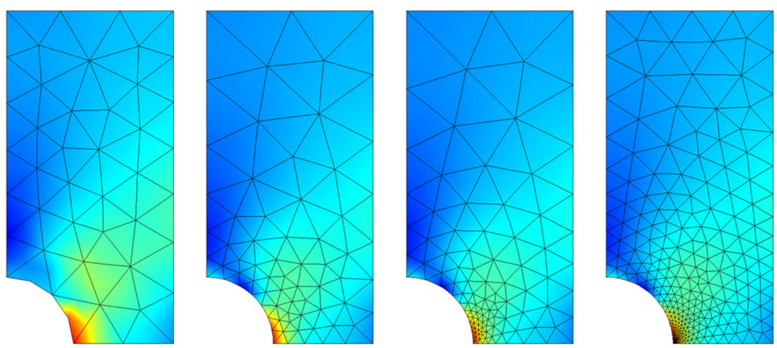

Figure 4: *Illustration of the influence in results of a Finite Element Model by the definition of the mesh discretization.*

4.1 Straight sided triangles - Natural coordinates

The Cartesian coordinates of a straight-sided triangle can be defined in function of the natural area coordinates of such triangle. Let us consider for instance, the triangle of **Fig. 5** whose area matrix is (34) from which the area can be computed as (35), and with which the equation (36) is satisfied:

$$[A] = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \tag{34}$$

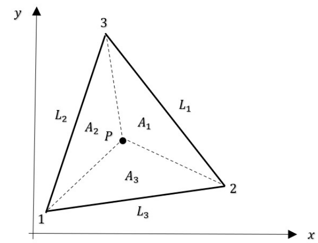

Figure 5: *Natural area coordinates for a triangle.*

$$A = \frac{1}{2}det[A] = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) \tag{35}$$

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = [A] \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{36}$$

where $\xi_1 = \xi_2 = \xi_3 = 1/3$.

This way, a point in the triangle is uniquely located by specifying any two of its area coordinates and the matrix B can be expressed as (37) and the shape functions N as (38):

$$[B] = \frac{1}{2A} \begin{bmatrix} (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \tag{37}$$

$$[N] = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \tag{38}$$

Thus, the conductive matrix K can be expressed as (39):

$$K = \alpha A B^T B \tag{39}$$

Thus, the force vector by heat-flux, if there were any, could be computed as (40), whereas for the convective matrix K_c (41) and the convective force vectors (42), given that $N^{\cdot} N = 1$:

$$f_l = \frac{QA}{3} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T \tag{40}$$

$$K_c = \alpha L \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

$$f_l = \alpha T L \begin{bmatrix} 1 & 1 \end{bmatrix}^T \tag{42}$$

Finally, the Capacity matrix in the transient heat-transfer situation, could be computed as (43) by considering only one point of integration:

$$C = \rho c A N^T N \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{43}$$

5 Computational implementation

5.1 Steady Heat-Flow

5.1.1 Pseudo-code Algorithm

Algorithm 5.1: Algorithm for the Steady-state 2D Heat-transfer analysis.

BEGIN

1.- Mesh: discretization

1.1 Determine which nodes are at each boundary to apply properly the boundary conditions.

2.- Set thermal material properties: thermal conductivities

3.- Build numerical FE model.

3.1 Compute the heat conduction matrix K_e for each finite element.

3.2 Assemble all heat conduction matrices of all elements into one global matrix, according to their respective DOF.

3.3 Compute the Heat-Flux vector F_q (if any).

2.4 Compute the heat convection matrix K_c and vectors F_b

4.- Solve system of equations $(K + K_c)a = Fq + Fb$ **for the unknown** a**.**

END

5.2 Transient Heat-Flow

5.2.1 Pseudo-code Algorithm

Algorithm 5.2: Algorithm for the Transient 2D Heat-transfer analysis.

BEGIN

1- Set thermal properties of materials: Density Heat Capcity.

2- Compute the Heat Capacity matrix C.

3- Compute the Stead-state solution of Algorithm 5.2 as the temperature of τ_0.

4- Set the number of time-steps or total time.

5- Set the stability parameters for the solution of the differential equation in time: θ and λ_{max} (largest eigenvalue of $det[K - \lambda C] = 0$).

5.1 Compute the step-length or Δt for the solution of the differential equation

6.- Set the new boundary conditions F_b and the respective heat-convection matrix.

7.- Establish the differential equation $C\dot{a} + (K + K_c)a = F_b$ to be solved for $a(t)$ by applying the generalized midpoint rule or the Crank-Nicholson method.

END

6 Results and discussion

The following table resumes the parameters considered as material properties:

Table 1: *Material properties for analysis.*

Property	Concrete	Wool	Plaster
Thermal-conductivity ($Watt/m \cdot K$)	1.7	0.05	0.5
Density ($J/m^3 \cdot K$)	$2.112x10^6$	$1.7952x10^6$	$700x10^3$

With a Heat-Transfer convection coefficient of $\alpha = 10\frac{Watt}{m \cdot K}$.

For the time-stepping analysis of transient heat-transfer, the following parameters were considered:

$$\theta = 0.45$$
$$dt = 0.9(\frac{2}{(1-\theta)\lambda}) = 1623.5$$
$$n - steps = 1400$$

Note: The Power method was used to determine the largest eigenvalue of $det(K - \lambda C) = 0$.

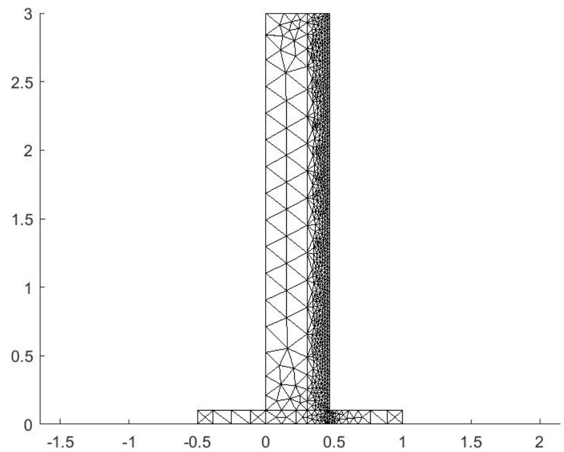

Figure 6: *Linear triangular mesh for the wall's domain of analysis.*

6.1 Mesh

6.2 Steady-state Heat transfer

6.2.1 Solution with COMSOL Multiphysics

The wall's domain and boundary conditions were modelled in COMSOL Multi-physics for the comparison of results. The heat distribution obtained in COMSOL is presented next:

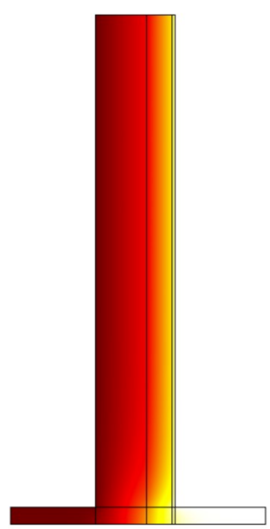

Figure 7: *Real solution obtained with COMSOL Multi-physics for the heat distribution of the wall.*

6.2.2 Own solution in MatLab

The solution generated through programming in this work is the one shown below in **Fig. 8** which is quite similar to that obtained in COMSOL. The total Heat-Flux at the outer boundary was calculated as the sum of heat-flux of each finite element at their respective boundary in contact with the outer temperature, as (44):

$$Hc = \sum_{1}^{n-elem} \alpha L_e(\sqrt{(T_x^2 + T_y^2)} - T_{out}) = 1,890.3Watts \tag{44}$$

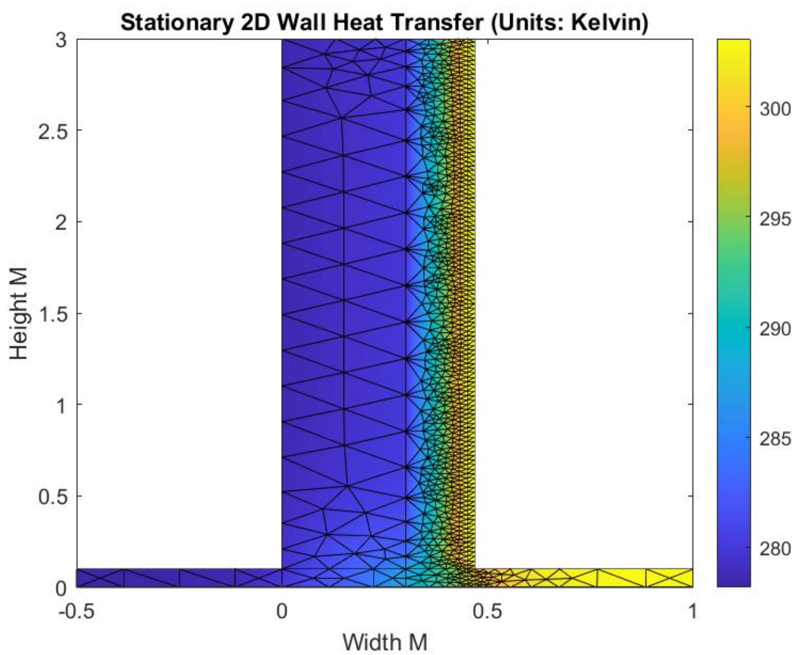

Figure 8: *Solution generated through programming for the heat distribution of the wall.*

6.3 Transient Heat transfer

From the steady-state heat-transfer solution the following transient states are derived by solving the First Order differential equation in time with the Midpoint rule method.

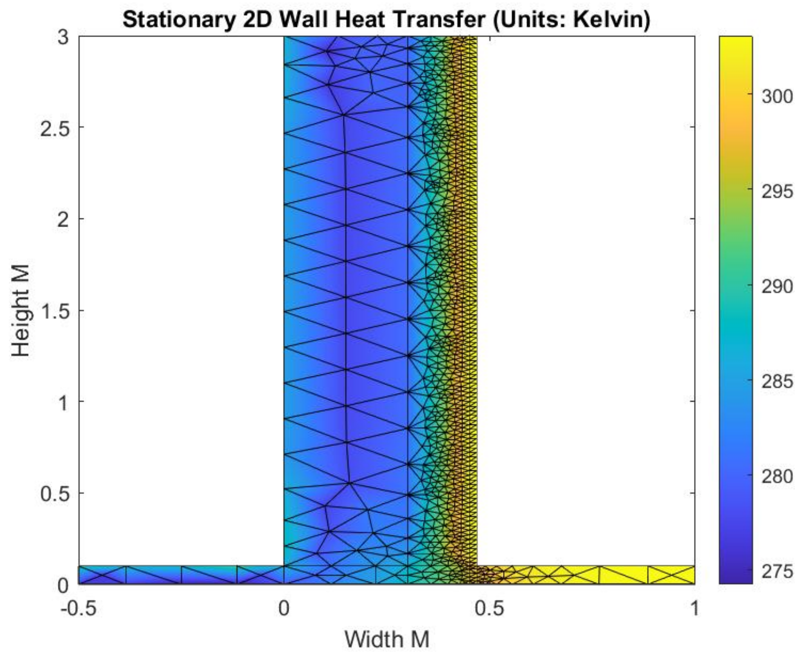

Figure 9: *Transient heat-flow state at time step 3.*

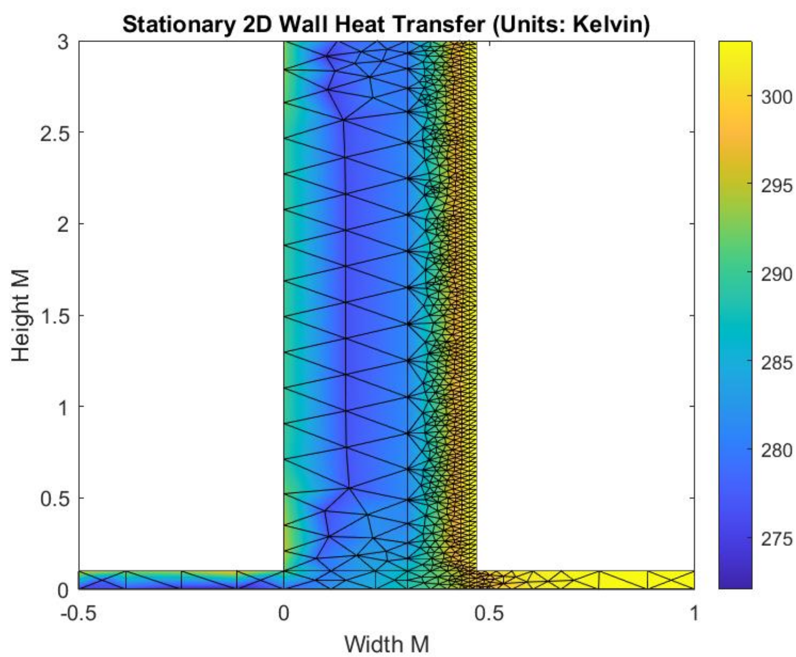

Figure 10: *Transient heat-flow state at time step 5.*

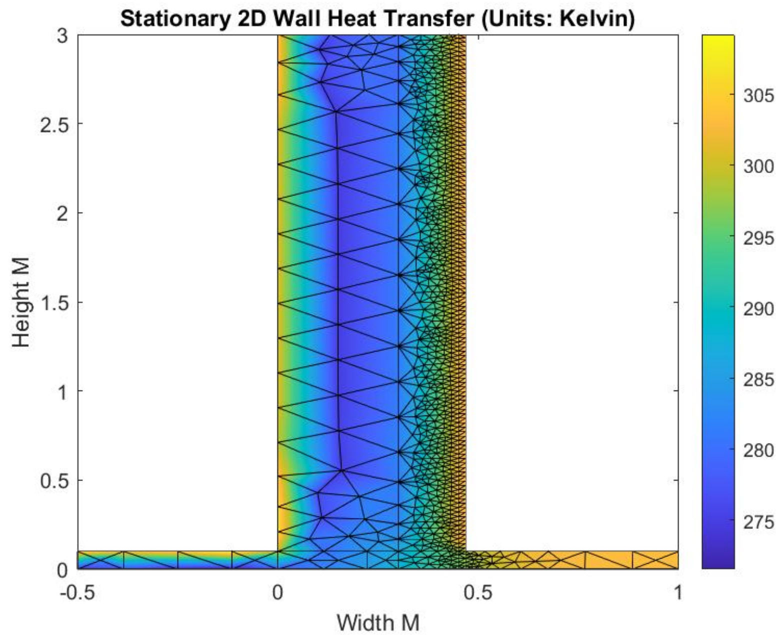

Figure 11: *Transient heat-flow state at time step 10.*

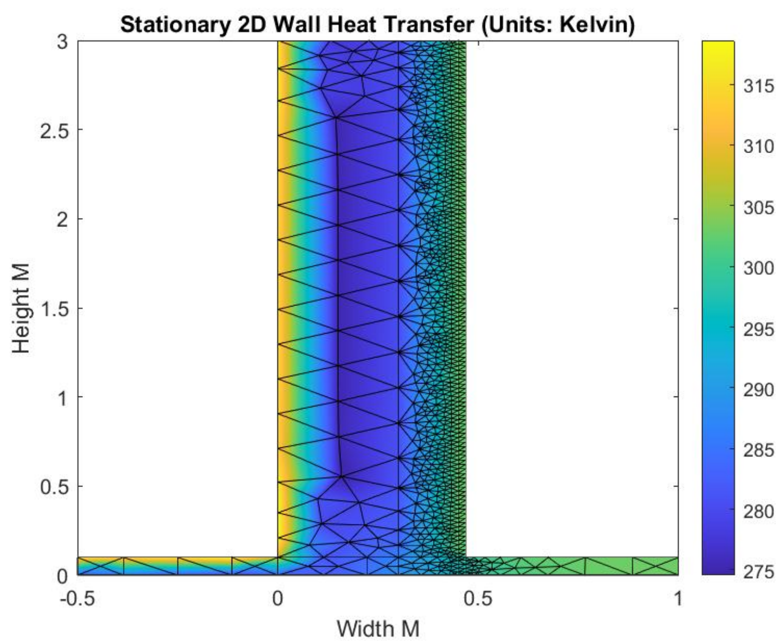

Figure 12: *Transient heat-flow state at time step 20.*

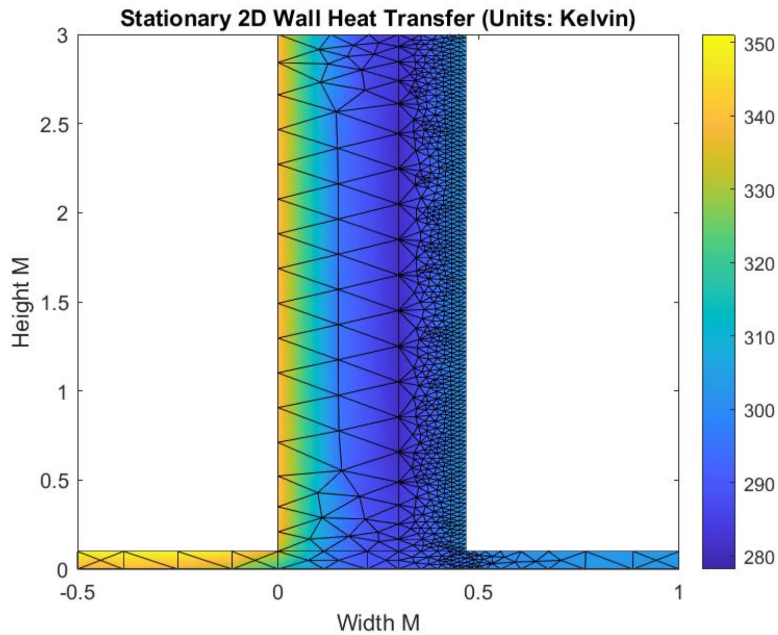

Figure 13: *Transient heat-flow state at time step 100.*

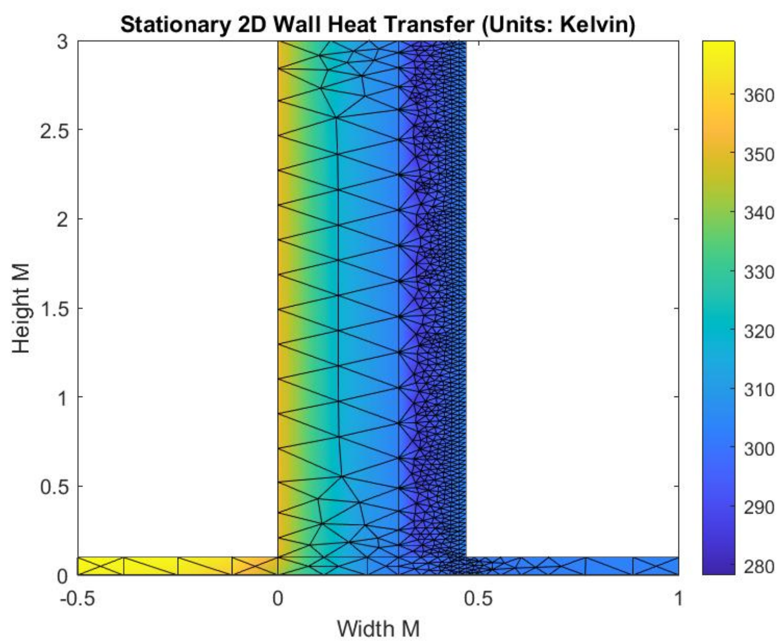

Figure 14: *Transient heat-flow state at time step 250.*

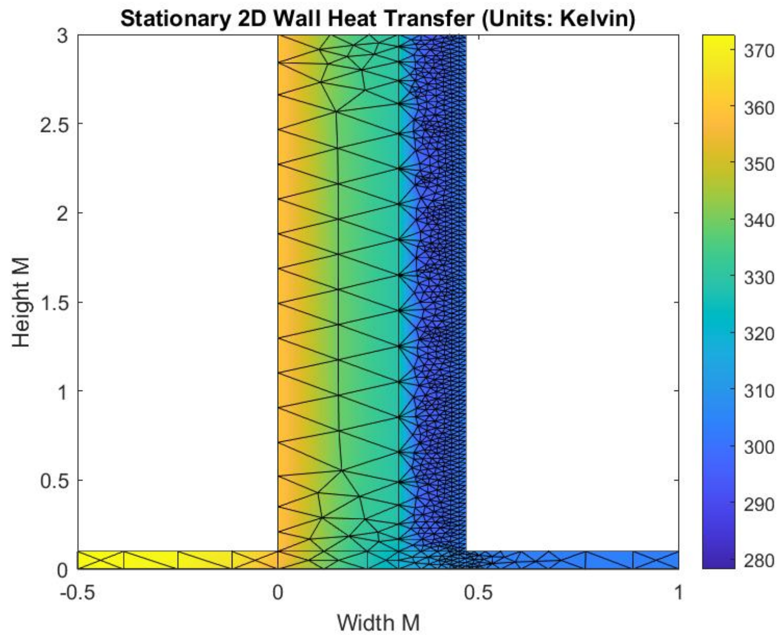

Figure 15: *Transient heat-flow state at time step 500.*

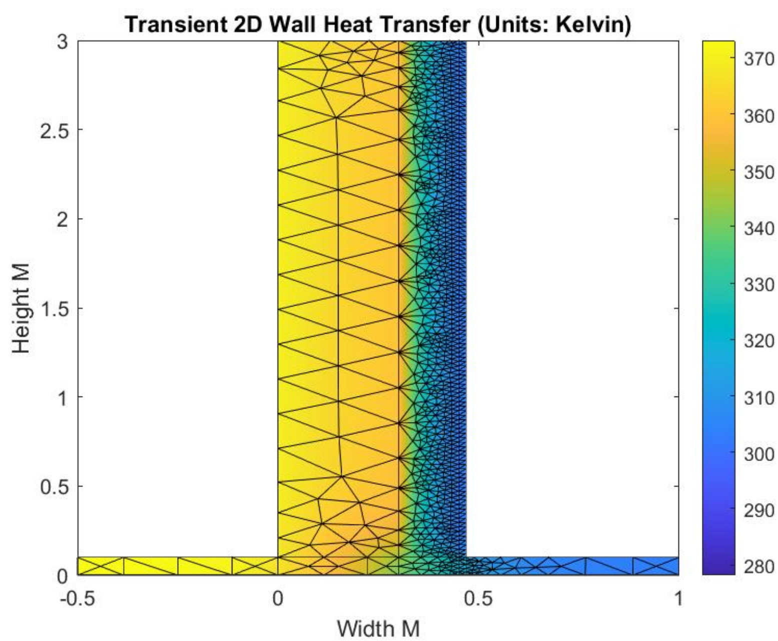

Figure 16: *Transient heat-flow state at time step 1400.*

7 Observations and commentaries

The temperature inside the wall started to change quite visibly from the very moment at which the temperature in the outer boundary increased to the 100°C. The rate of conduction was rapid initially, as it only took 10 time-steps for that boundary to reach approximately a 40°C temperature (considering that the initial temperature at time 0 was of just 5°C). From the 80th time-step, the rate of the temperature variation over that boundary was considerably slow (see **Fig. 17**):

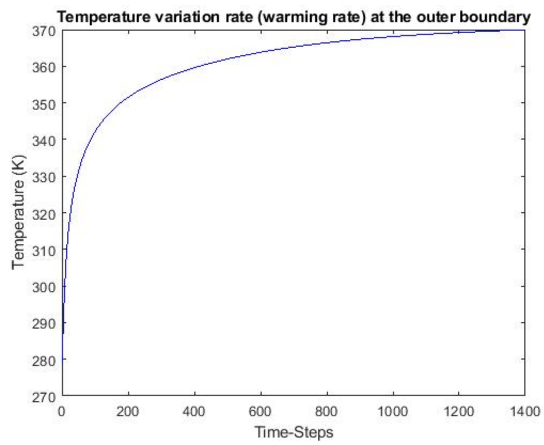

Figure 17: *Rate of temperature's variation at the outer wall's boundary from the moment at which the outer temperature increases from 5°C to 100°C.*

On the other hand, as it can be observed from **Fig. 9** to **Fig. 16** the insulating materials (wool and plaster) maintained their temperature practically constant through out the whole warming process of the wall, in contrast to what is observed in the heat-conduction of the concrete wall's portion.

8 Conclusions

It was demonstrated the great performance that the wool and plaster as insulating materials have. On the other hand, referring to the whole wall, concrete turned out to be quite efficient regarding heat-conduction, as it took for a lot of time for the whole wall (specially for the concrete wall portion) to reach equilibrium for the increment to 100°C, in other words, concrete can be quite efficient to preserve the temperature against fire or extreme heat for a long time, which is quite advantageous for buildings and infrastructure projects in general.

References

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