上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

Cooperative Control of Linear Systems with Coupled Constraints via Distributed Model Predictive Control

Lifeng Zhou, Shaoyuan Li

Outline

➢ **Cooperative Control of Linear Systems with Coupled Constraints via Distributed Model Predictive Control**

- Problem formulation
- Main results
- Illustrative examples

1. Problem Formulation
2. Main Results
3. Future Work

- A system composed of $\mathcal{N}$ linear systems, for each subsystem i:

$$\boldsymbol{x}_i(\boldsymbol{k}+\boldsymbol{1}) = \boldsymbol{A}\boldsymbol{x}_i(\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u}_i(\boldsymbol{k}) \tag{1}$$

- Definition of cooperative set:

1. Define $\mathcal{I}_c$ as the set of subsystems involved in constraint c, and $\mathcal{C}_i$ as the set of constraints involving subsystem i:

$$\begin{aligned} \mathcal{I}_c &\triangleq \{i \in \mathcal{N}: [E_{ci}F_{ci} \neq 0]\} \\ \mathcal{C}_i &\triangleq \{c \in \mathcal{C}: [E_{ci}F_{ci} \neq 0]\} \end{aligned} \tag{2}$$

2. Then the set of all other subsystems coupled to i is

$$Q_i = \left(\bigcup_{c \in \mathcal{C}_i} \mathcal{I}_c \right) \backslash \{i\} \tag{3}$$

3. Cooperative set for i is

$$\delta_i = \bigcup_{c \in \mathcal{C}_i} \mathcal{I}_c \tag{4}$$

And define the number of subsystems in cooperative set for i is $|\delta_i|$.

DMPC method for consensus

For subsystem i, the finite horizon optimization problem is formally described as below:

Problem $\mathcal{P}_i$: At instant k,

$$\min_{\boldsymbol{u_i}(\boldsymbol{k})} J_i(\boldsymbol{\varepsilon_i}(\boldsymbol{k}), \boldsymbol{u_i}\,(\boldsymbol{k}))$$

$$= \sum_{t=0}^{T-1} \left(\|\boldsymbol{u_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k})\|_{R_i}^2 + \|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k})\|_{Q_i}^2 \right) \quad (5)$$

$$+\|\boldsymbol{\varepsilon_i}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|_{P_i}^2$$

with $\boldsymbol{\varepsilon_i}(\boldsymbol{k}) \triangleq \boldsymbol{x_i}(\boldsymbol{k}) - \frac{1}{|\delta_i|} \sum_{j \epsilon \delta_i} \boldsymbol{x_j}(\boldsymbol{k})$

Subject to $\forall t \in \{0, \dots, T-1\}$:

$$\boldsymbol{x_i}(\boldsymbol{k}+\boldsymbol{t}+\mathbf{1}|\boldsymbol{k}) = \boldsymbol{A}\boldsymbol{x_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k}) + \boldsymbol{B}\boldsymbol{u_i}(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k}) \quad (5a)$$

$$\boldsymbol{u}_i(\boldsymbol{k}+\boldsymbol{t}|\boldsymbol{k}) \in \mathcal{U}_i \tag{5b}$$

$$\boldsymbol{x}_i(\boldsymbol{k}+\boldsymbol{t}+\mathbf{1}|\boldsymbol{k}) \in \mathcal{X}_i \tag{5c}$$

$$\boldsymbol{x}_i(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k}) \in \mathcal{X}_{F_i} \tag{5d}$$

$$q_c\left(\left(\boldsymbol{x}_i(\boldsymbol{k}+\boldsymbol{t}+\mathbf{1}|\boldsymbol{k}), \left\{\boldsymbol{x}_j(\boldsymbol{k}+\boldsymbol{t}+\mathbf{1}|\boldsymbol{k})\right\}_{j\in\mathcal{N}_i}\right) \subseteq \mathcal{Q}_c \tag{5e}$$

Assumption 1: There exists a terminal region $\mathcal{X}^T \in \mathcal{X}$, and terminal control law $\boldsymbol{u_i}^{\boldsymbol{T}} = \boldsymbol{k}_{\boldsymbol{i}}^{\boldsymbol{T}}\left(\boldsymbol{x_i}, \{\boldsymbol{x_j}\}_{j\in\mathcal{N}_i}\right)$, such that the terminal region $\mathcal{X}^T$ is invariant with respect to the overall closed-up system $\boldsymbol{x}(\boldsymbol{k}+\mathbf{1}) = (\boldsymbol{I_N}\otimes\boldsymbol{A})\boldsymbol{x}(\boldsymbol{k}) + (\boldsymbol{I_N}\otimes\boldsymbol{A})\boldsymbol{u^T}$ with $\boldsymbol{u^T} = [\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{T}^T}, \dots, \boldsymbol{u}_{\boldsymbol{N}}^{\boldsymbol{T}^T}]^T$. And the following holds for all $x \in \mathcal{X}^T$ and for all $i \in \mathcal{I}$:

$$\boldsymbol{k}_{\boldsymbol{i}}^{\boldsymbol{T}}\left(\boldsymbol{x_i}, \{\boldsymbol{x_j}\}_{j\in\mathcal{N}_i}\right) \in \mathcal{U}_i \tag{6a}$$

$$q_c\left(\boldsymbol{x_i}, \{\boldsymbol{x_j}\}_{j\in\mathcal{N}_i}\right) \subseteq \mathcal{Q}_c, c \in \{1, \dots, \mathcal{C}\} \tag{6b}$$

$$\sum_{i=1}^{N} \|\boldsymbol{\varepsilon}_{\boldsymbol{i}}^{+}\|_{P_i}^2 + \|\boldsymbol{\varepsilon_i}\|_{Q_i}^2 - \|\boldsymbol{\varepsilon_i}\|_{P_i}^2 + \left\|\boldsymbol{u^T}\right\|_{R_i}^2 \leq 0 \tag{6c}$$

with $\boldsymbol{\varepsilon}_{\boldsymbol{i}}^{+} := \boldsymbol{\varepsilon_i}(\boldsymbol{k}+\mathbf{1}+\boldsymbol{T}|\boldsymbol{k}+\mathbf{1})$ at instant k.

Here we choose: $\boldsymbol{u_i}^{\boldsymbol{T}} = \boldsymbol{k}_{\boldsymbol{i}}^{\boldsymbol{T}}\left(\boldsymbol{x}_{\boldsymbol{i}}, \{\boldsymbol{x}_{\boldsymbol{j}}\}_{j\in\mathcal{N}_i}\right) := -\boldsymbol{k}_{\boldsymbol{i}}\boldsymbol{\varepsilon}_{\boldsymbol{i}}$ (7)

$$\sum_{i=1}^{N} \|\boldsymbol{\varepsilon}_{\boldsymbol{i}}^{+}\|_{P_i}^2 + \|\boldsymbol{\varepsilon}_{\boldsymbol{i}}\|_{Q_i}^2 - \|\boldsymbol{\varepsilon}_{\boldsymbol{i}}\|_{P_i}^2 + \left\|\boldsymbol{u_i}^{\boldsymbol{T}}\right\|_{R_i}^2 \le 0 \quad \text{(6c)}$$

$$\Longrightarrow \boldsymbol{\varepsilon}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})^T\left(\left(\widetilde{\boldsymbol{A}}+\widetilde{\boldsymbol{\mathcal{L}}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}}\right)^T P\left(\widetilde{\boldsymbol{A}}+\widetilde{\boldsymbol{\mathcal{L}}}\widetilde{\boldsymbol{B}}\widetilde{\boldsymbol{K}}\right)\right.$$

$$\left. -P + \widetilde{\boldsymbol{K}}^{\boldsymbol{T}} R \widetilde{\boldsymbol{K}} + Q\right)\boldsymbol{\varepsilon}(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k}) \le 0 \quad \text{(8)}$$

LMI:

$$\Longrightarrow \begin{bmatrix} \boldsymbol{X} & \boldsymbol{X}\widetilde{\boldsymbol{A}}^T + \boldsymbol{Y}^{\boldsymbol{T}}\widetilde{\boldsymbol{B}}^T & \boldsymbol{X}Q^{1/2} & \boldsymbol{Y}^{\boldsymbol{T}}R^{1/2} \\ \boldsymbol{X}+\widetilde{\boldsymbol{B}}\boldsymbol{Y} & \boldsymbol{X} & 0 & 0 \\ Q^{1/2}\boldsymbol{X} & 0 & \boldsymbol{I} & 0 \\ R^{1/2}\boldsymbol{Y} & 0 & 0 & \boldsymbol{I} \end{bmatrix} \ge 0$$

DMPC algorithm for a subsystem i:

0): Initialization: Set $k = 0$. Wait for a feasible solution $\hat{\boldsymbol{u}}_i(\mathbf{0})$ with corresponding state sequence $\hat{\boldsymbol{x}}_i(\mathbf{0})$.

1): Sample current state $\boldsymbol{x}_i(\boldsymbol{k})$.

2): Update plan. If $i_k = k$.

(a) Choose cooperating set δ_i. Subsystem i receives
states $\hat{\boldsymbol{x}}_j(\boldsymbol{k}|\boldsymbol{k}) = \boldsymbol{x}_j\,(\boldsymbol{k}|\boldsymbol{k}-\mathbf{1})$, $j \in \delta_i$ for subsystem(s) j which
have
not yet calculated its optimal input.
Or $\hat{\boldsymbol{x}}_j(\boldsymbol{k}|\boldsymbol{k}) = \boldsymbol{x}_j^*(\boldsymbol{k}|\boldsymbol{k}-\mathbf{1})$, $j \in \delta_i$ for subsystem(s) j which
already have calculated its optimal input at instant k.

(b) Obtain new plan $\boldsymbol{u_i(k)} = \boldsymbol{u_i^*(k)}$ by optimizing the distributed MPC cost function (5) and the corresponding state by $\boldsymbol{x_i^*(k)}$.

(c) Transmit optimal states $\boldsymbol{x_i^*(k)}$ to other subsystems $j, i \in \delta_j$.

else

(a) Renew current plan: $\boldsymbol{u_i(k)} = \boldsymbol{\hat{u}_i(k)} := \{\boldsymbol{u_i(k|k-1)}, \ldots, \boldsymbol{u_i(k+T-2|k-1)}, \boldsymbol{u_i}^{\boldsymbol{T}}\boldsymbol{(k+T-1|k-1)}\}$

(b) Obtain the corresponding state $\boldsymbol{\hat{x}_i(k)}$.

(c) Transmit states $\boldsymbol{\hat{x}_i(k)}$ to other subsystems $j, i \in \delta_j$.

3): Subsystem i applies $\boldsymbol{u_i(k)}$ as its actual control input at instant k. Wait one time step, increment k, go to step 1).

Feasibility

Suppose a feasible solution to the **Problem** $\mathcal{P}_i$ at instant $k-1$ for all subsystem $i \in \mathcal{I}$,

- if subsystem $i \neq i_k$

Renew ⟹ $\boldsymbol{u_i(k)} = \{\boldsymbol{u_i(k|k-1)},\ldots,\boldsymbol{u_i(k+T-2|k-1)}, \boldsymbol{u_i}^{\boldsymbol{T}}\boldsymbol{(k+T-1|k-1)}\}$

Due to Assumption (6a), $\boldsymbol{u_i(k+t|k)} \in \mathcal{U}_i$ (5b)

Associated state $\boldsymbol{x_i(k+t+1|k)} \in \mathcal{X}_i$ (5c),(5d)

Due to Assumption (6b),

$q_c((\boldsymbol{x_i(k+t+1|k)}, \{\boldsymbol{x_j(k+t+1|k)}\}_{j\in\mathcal{N}_i}) \subseteq \mathcal{Q}_c$ (5e)

⟹ Feasibility for subsystem $i \neq i_k$

- if subsystem $i = i_k$

Optimize

⟹ Solve **Problem** $\mathcal{P}_i$ at instant k, obviously, all the constraints (5a)-(5e) should be guaranteed

⟹ Feasibility for subsystem $i = i_k$

For all $i \in \mathcal{I}$

⟹ Feasibility is established

Stability of the whole system

$$V(k+1)-V(k)$$

$$=\sum_{i\in\mathcal{I}\setminus\{i_k\}} J_i(\hat{\boldsymbol{\varepsilon}}_i(\boldsymbol{k}+\mathbf{1}),\hat{\boldsymbol{u}}_i(\boldsymbol{k}+\mathbf{1}))+J_{i_k}\big(\boldsymbol{\varepsilon}_i^*(\boldsymbol{k}+\mathbf{1}),\boldsymbol{u}_i^*(\boldsymbol{k}+\mathbf{1})\big)$$

$$-\sum_{i\in\mathcal{I}} J_i(\boldsymbol{\varepsilon}_i(\boldsymbol{k}),\boldsymbol{u}_i(\boldsymbol{k}))$$

$$\leq\sum_{i\in\mathcal{I}} J_i\big(\hat{\boldsymbol{\varepsilon}}_i(\boldsymbol{k}+\mathbf{1}),\hat{\boldsymbol{u}}_i(\boldsymbol{k}+\mathbf{1})\big)-\sum_{i\in\mathcal{I}} J_i(\boldsymbol{\varepsilon}_i(\boldsymbol{k}),\boldsymbol{u}_i(\boldsymbol{k}))$$

$$=\sum_{i\in\mathcal{I}}(\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}+\mathbf{1}+\boldsymbol{T}|\boldsymbol{k}+\mathbf{1})\|_{P_i}^2+\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|_{Q_i}^2$$

$$-\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}+\boldsymbol{T}|\boldsymbol{k})\|_{P_i}^2+\left\|\boldsymbol{u}^T(\boldsymbol{k}+\boldsymbol{T})\right\|_{R_i}^2-\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}|\boldsymbol{k})\|_{P_i}^2-\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}|\boldsymbol{k})\|_{Q_i}^2)$$

$$\overset{(6c)}{\leq}\sum_{i\in\mathcal{I}}-\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}|\boldsymbol{k})\|_{P_i}^2-\|\boldsymbol{\varepsilon}_i(\boldsymbol{k}|\boldsymbol{k})\|_{Q_i}^2\leq 0$$

convergence

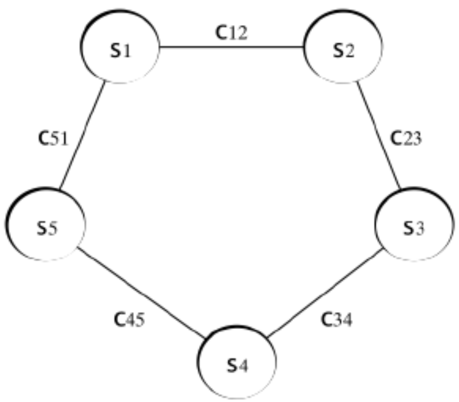
- For any sequence $\{a_0, a_1, \ldots\}$

$$S = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \,, S < \infty \to \lim_{k\to\infty} a_k = 0$$

- Apply this result to convergence analysis:

If $V(k+1) - V(k) \leq -l(\varepsilon_k)$
then $\sum_{k=0}^{\infty} l(\varepsilon_k) \leq V(x_0) - \lim_{k\to\infty} V(\varepsilon_k)$
therefore $l(\varepsilon_k) \to 0$ as $k \to \infty$

Illustrative examples



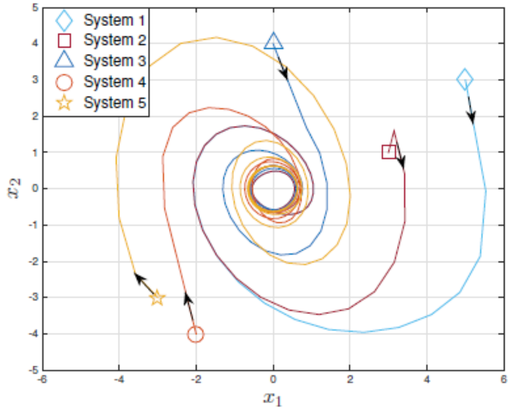
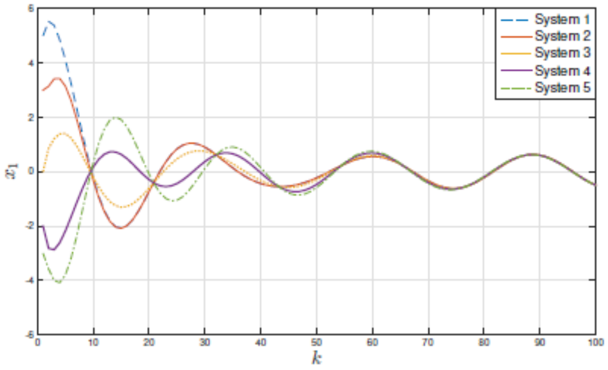
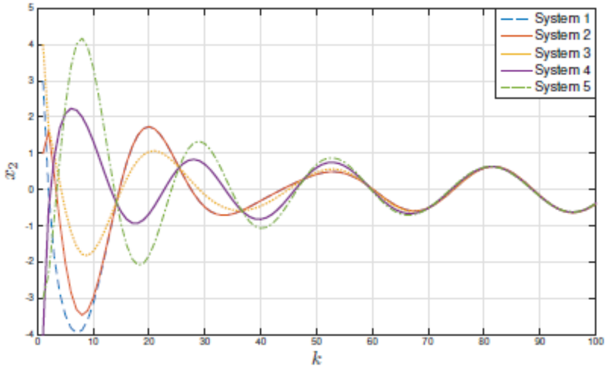


Fig. 3 *State trajectories - phase plane*



(a) x_1 - *time domain*



(b) x_2 - *time domain*

Fig. 4 *State trajectories - time domain*

Thank you!

Q & A