Homework 4

September 26, 2017

Proofs Required

Theorem. *Let G be a graph and $(T = (I, F), \mathcal{X})$ a tree decomposition of width at most k. Prove that if there are at least $k+1$ vertex-disjoint paths between vertices x and y in G, some bag of $(T, \mathcal{X})$ contains both x and y.*

Proof. We prove by contradiction. We assume that there is no bag containing both x and y, i.e., we assume that x and y are in two different bags.

Since the tree decomposition of width at most k, then we have $\mathsf{max}|X_i| \leq k + 1$ due to the definition of tree width.

We assume there are n bags containing x and m bags containing y. We denote those bags containing x as $\{B_i\}_{i\in[n]}$. Similarly, we denote those bags containing y as $\{B_j\}_{j\in[n]}$. Note that we do not want to consider those bags that consists of neither x nor y and it is safe.

We note that the bags $\mathbf{B}_x = \{B_i\}_{i\in[n]}$ containing x and the bags $\mathbf{B}_y = \{B_j\}_{j\in[m]}$ containing y comprise of two connected components respectively, and these two connected components are connected by only one edge between some bag $B_u \in \mathbf{B}_x$ and another bag $B_v \in \mathbf{B}_y$, $u \in [n]$, $v \in [m]$, since T is a tree decomposition and no bag contains both x and y.

Now we prove the following lemma.

Lemma. *Given a tree decomposition T of the graph G, let edge ab be an arbitrary edge in the tree decomposition T. The forest $T - ab$ obtained from T by deleting edge ab consists of two connected components T_a (containing a) and T_b (containing b). Let X_a be an arbitrary bag in T_a and X_b be an arbitrary bag in T_b. X_a and X_b are connected by edge ab. Then $X_a \cap X_b$ is a separator of G.*

Proof. To prove the lemma above, we denote vertices in T_a as $V_a = \{v|v \in T_a, v \notin T_b\}$, and vertices in T_b as $V_b = \{v|v \in T_b, v \notin T_a\}$. We first show that $T_a \cap T_b$ is a separator of G. We prove by contradiction. We assume that $T_a \cap T_b$ is not a separator, then there exists an edge zw, where $z \in V_a$ and $w \in V_b$. However, there must exist a bag covering z and w due to properties of tree decomposition, then either T_a consists of both z and w or T_b consists of both z and w. This contradicts to the fact that $z \in V_a$ and $w \in V_b$ and $V_a \cap V_b = \emptyset$. This proves $T_a \cap T_b$ is a separator of G. Next, since the continuity property of tree-decomposition saying that if $v \in X_a$ and $v \in X_b$ then $v \in X_k$ for all k on the path from a and b, and ab is the only edge between T_a and T_b, then we have $T_a \cap T_b = X_a \cap X_b$. Therefore, $X_a \cap X_b$ is also a separator. This finishes proving the lemma. $\square$

Then we consider the two connected components $\mathbf{B}_x = \{B_i\}_{i\in[n]}$ and $\mathbf{B}_y = \{B_j\}_{j\in[m]}$ we just described. Consider the bag $B_u \in \mathbf{B}_x$ containing x connecting to a bag $B_v \in \mathbf{B}_y$ containing y. Since $|B_u| \leq k + 1$ and $|B_v| \leq k + 1$ and the fact that the two connected components $\mathbf{B}_x$ and

$\mathbf{B}_y$ are connected by only one edge between $B_u \in \mathbf{B}_x$ and $B_v \in \mathbf{B}_y$, we have that $B_u \cap B_v$ is a separator of G using the lemma we proved above and that $|B_u \cap B_v| \leq k$. However, there are $k + 1$ vertex-disjoint paths from x to y, therefore there is at least 1 edge connecting x and y and this edge should be covered by some bag in the tree decomposition. This contradicts to our assumption that there is no bag containing both x and y. This finishes the proof. $\square$