2801.001 Spring 2018 Homework 1

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Submission instructions: Same groups as HW1. Refer to instructions from TA.

Note: Problems 1 and 2 require the datafile posted online which contains prices and implied volatilities of one-year (349 calendar days) listed options on the Nasdaq 100 index (NDX), as well as other market data.

Problem 1

(a) Estimate the market price of the 5% call spread (i.e. with strikes ATM and 5% OTM). What about the 5% put spread?

Solution.

The Nasdaq 100 data provided allows us to get the buy and sell prices of ATM (strike 5050) and 5% OTM (strikes 5200 and 5750) call and put prices (we use midpoints).

- Selling ATM Call: 337.1.
- Selling ATM Put: 326.1.
- Buying 5% OTM Call: 478.1.
- Buying 5% OTM Put: 428.3.

Therefore, the buy prices of the 5% call and put spreads are:

- 5% call spread: $478.1 - 337.1 = 141.0$.
- 5% put spread: $428.3 - 326.1 = 100.2$.

(b) If you were to price the spreads in the Black-Scholes model using a single volatility parameter σ, what value of σ would match the theoretical price with the market price? Comment on your results.

Solution.

Applying a Black-Scholes pricing formula would lead to the following theoretical prices for the 5% call spread (5% put spread resp.):

$$
\begin{aligned}
CS(K, 0.95K) &= C(0.95K) - C(K) \\
&= S_t N(d_1(0.95K)) - 0.95Ke^{-r(T-t)}N(d_2(0.95K)) \\
&\quad - (S_t N(d_1(K)) - Ke^{-r(T-t)}N(d_2(K)))
\end{aligned}
$$

$$
\begin{aligned}
PS(K, 1.05K) &= P(1.05K) - P(K) \\
&= 1.05Ke^{-r(T-t)}N(-d_2(1.05K)) - S_t N(-d_1(1.05K)) \\
&\quad - (Ke^{-r(T-t)}N(-d_2(K)) - S_t N(-d_1(K)))
\end{aligned}
$$

where

$$
\begin{aligned}
d_1(K) &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\
d_2(K) &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

σ is the implied volatility that one can choose to equate theoretical prices to market prices.

Using a solver, we get $\sigma_{CS} = 14.13\%$ using the 5% call spread and $\sigma_{PS} = 17.10\%$ using the 5% put spread.

These levels are relatively close to the implied volatility quoted for ATM (15.4 %) and 5% OTM options (14% to 16.7%)

Problem 2

(a) Using the numerical package of your choice, calibrate the parameters of the SVI model against the market-implied volatility data. Show a comparative graph of the SVI curve and the actual implied volatility data points.

Solution.

The SVI model reads:

$$\sigma^* = \sqrt{a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + s^2}\right)}$$

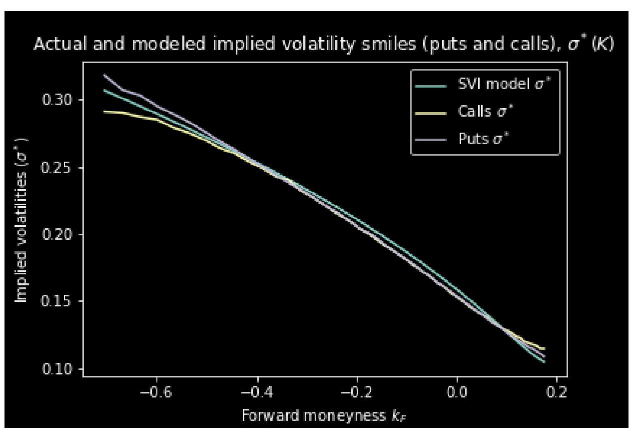
To calibrate its parameters (a, b, ρ, m, s) we perform the following least square optimization against market data (both puts and calls), with the following constraints to avoid arbitrage:

$$\begin{aligned} \min_{a,b,\rho,m,s} \quad & \sum_{i=1}^{n} \left[\sigma^{\star 2}_{\text{SVI}}(k_i, T; a, b, \rho, m, s) - \sigma^{\star 2}_{\text{Market}}(k_i, T)\right]^2 \\ \text{subject to} \quad & a, b \geq 0, \\ & -1 \leq \rho \leq 1, \\ & s > 0, \\ & b(1 + |\rho|)T \leq 4. \end{aligned}$$

Using a Sequential Least Squares Programming method in `scipy.minimize`, we obtain:

$$a = 0.000364, b = 2.55, \rho = 0.961, m = 0.239, s = 0.0149$$

The quality of the fit is very good:



(b) Compute or estimate the price of an at-the-money digital call option paying off \$1 if in one year NDX is greater than its current spot level, and zero otherwise: (i) in the Black-Scholes model, (ii) using $\pm 1\%$ call spreads, (iii) using the smile-adjusted formula on page 19.

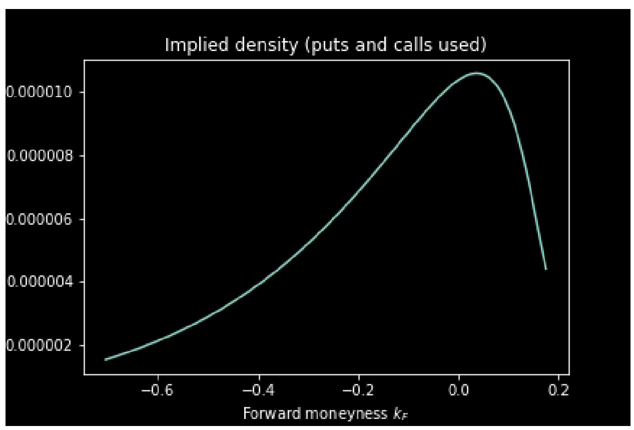
Solution.

- Black-Scholes model: $D_{BS}(S_t, K, r, T, \sigma) = e^{-rT} N(d_2)$. We obtain: 0.495.
- Using $\pm 1\%$ call spreads. We obtain: 0.419.
- Using smile-adjusted formula: $D(S_t, K, r, T) = (D_{BS} - V_{BS} \times \frac{\partial \sigma^*}{\partial K})(S_t, K, r, T, \sigma^*(K, T))$, where V_{BS} is the Black-Scholes vega, $V_{BS} = S_t \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \sqrt{T-t}$. We obtain: 0.590.

(c) Graph the implied distribution corresponding to the SVI model calibration.

Solution.

As expected, the density obtained from options prices is right skewed and has fatter tails than the Black-Scholes lognormal random variable:



(d) Use the implied distribution to compute the price of the following European exotic options, where X_0 is the current index level and X_T is the final index level:

Solution.

Knowing the implied distribution of an underlying, the value of a derivative f_t at time t with a certain payoff $f(S_T)$ at maturity T is given by discounted the expected payoff under this measure: $f_t = e^{-r(T-t)} \mathbb{E}(f(S_T))$. We apply this to the following cases:

(i) Digital call defined in question (b);
Its payoff is just $\mathbb{I}(X_T > K)$. We obtain: 0.0131.

(ii) "Reverse convertible" paying off $\max\left(100\%, 100\% + p \times \frac{X_T - X_0}{X_0}\right)$ if $\frac{X_T}{X_0} > 75\%$ and $\frac{X_T}{X_0}$ otherwise, where $p = 50\%$. Then solve for p to get a price of 100%;

With $p = 0.5$, we obtain: 0.0230.

To get a price of 1.0, we just apply a rootfinding technique to the payoff to which we substract 1.0, we get: $p = 0.5$.

(iii) Option paying off $\max\left(0, \frac{X_T - X_0}{X_T}\right)$;

We get: 0.000796.

(iv) Log-contract paying off $-2\log\left(\frac{X_T}{X_0}\right)$. Price interpretation;

We get: 0.00429.

Assuming arbitrage (it is the case since we fitted the SVI model), this can be interpreted as the current expected total return gained on the stock over the life-time of the option, i.e. 0.429 % over 1 year.

Problem 3

Find conditions on the SVI model parameters to satisfy Lees asymptotic bounds on p. 22:

$$\sigma^{\star^2}(k_F, T) \leq \frac{\beta}{T}|\log k_F|,\ \beta \in [0, 2]$$

Solution.
TODO

Problem 4

(Problem 4.3 p. 56 in textbook, with corrections): Consider an underlying stock S currently trading at $S_0 = 100$ which does not pay any dividend. Assume the local volatility function is $\sigma_{loc}(t, S) = \frac{0.1 - 0.15 \times \log\left(\frac{S}{S_0}\right)}{\sqrt{t}}$, and that interest rates are zero.

(a) Produce the graph of the local volatility surface for spots 0 to 200 and maturities 0 to 5 years.

Solution.

TODO

(b) Write a Monte-Carlo algorithm to price the following 1-year payoffs using 252 time steps and e.g. $10,000$ paths:

Solution.

(i) "Capped quadratic" option: $\min\left(1, \frac{S_1^2}{S_0^2}\right)$;

TODO

(ii) Asian at-the-money-call: $\max\left(0, \frac{S_{0.25}+S_{0.5}+S_{0.75}+S_1}{4 \times S_0} - 1\right)$;

TODO

(iii) Barrier call: $\max(0, S_1 - S_0)$ if S always traded above 80 using 252 daily observations, 0 otherwise;

TODO

Problem 5

The payoff of a 1-year at-the-money call on the geometric average return of two non-dividend paying stocks X, Y is given as:

$$f(X_T, Y_T) = \max\left(0, \sqrt{\frac{X_T Y_T}{X_0 Y_0}} - 1\right) = \frac{1}{\sqrt{X_0 Y_0}} \max\left(0, \sqrt{X_T Y_T} - \sqrt{X_0 Y_0}\right)$$

where $T = 1$ year and X_t, Y_t are the respective underlying spot prices of X, Y at any time t.

(a) Derive analytical formulas for the call value at any time $0 \leq t \leq T$ in the Black-Scholes model with constant correlation ρ (cf. Section $6-4$ in the textbook, to be covered during Session 5.)

Solution.

In a Black-Scholes world, these two correlated stocks have the dynamics:

$$dX_t = (r - \mu_1)X_t dt + \sigma_1 X_t dW_t$$

$$dY_t = (r - \mu_2)Y_t dt + \sigma_2 Y_t(\rho dW_t + \sqrt{1-\rho^2} dZ_t)$$

where r is the risk-free rate, μ_1, μ_2 are the dividend yields (equal to 0 here by assumption), ρ is the constant correlation between X and Y, and W, Z are two uncorrelated Brownian motions.

The value of this call is given by: $f_t = \frac{e^{-r(T-t)}}{\sqrt{X_0 Y_0}} \mathbb{E}\left(\max\left(0, \sqrt{X_T Y_T} - \sqrt{X_0 Y_0}\right) | X_t, Y_t\right)$

The geometric average of two lognormal is still lognormal. Indeed, for $\alpha > 0$, let us apply It-Doeblin formula to the process $V_t = (X_t Y_t)^\alpha$:

We obtain:

$$\begin{aligned} dV_t = &\left(r - \left(\mu_{X,Y} + (1-\alpha)\left(r - \mu_{X,Y} + \frac{1}{2}\sigma_{X,Y}^2\right)\right)\right) V_t dt \\ &+ \alpha V_t \left(\sigma_1 dW_t + \sigma_2 \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right)\right) \end{aligned}$$

where $\mu_{X,Y} = \mu_1 + \mu_2 - r - \rho\sigma_1\sigma_2$ and $\sigma_{X,Y} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$.

In our case, we assume no dividend ($\mu_1 = \mu_2 = 0$) and $\alpha = \frac{1}{2}$, so the dynamics of V_t simplifies to:

$$dV_t = \left(r - \frac{1}{4}\left(\sigma_1^2 + \sigma_2^2\right)\right) V_t dt + \frac{1}{2} V_t \left(\sigma_1 dW_t + \sigma_2 \left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right)\right)$$

If we call $dB_t = \frac{\sigma_1 dW_t + \sigma_2\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right)}{\sigma_{X,Y}}$, where again $\sigma_{X,Y} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$, B_t is a Brownian and we have:

$$dV_t = \left(r - \frac{1}{4}\left(\sigma_1^2 + \sigma_2^2\right)\right) V_t dt + \frac{1}{2}\sigma_{X,Y} V_t dB_t$$

Under this form, we see that $V_t = \sqrt{X_t Y_t}$ follows a lognormal distribution.

Therefore, the price of the option is given by the following Black-Scholes formula:

$$f_t = \frac{1}{\sqrt{X_0 Y_0}}\left(e^{-\frac{1}{4}\left(\sigma_1^2+\sigma_2^2\right)(T-t)}\sqrt{X_t Y_t} N(d_1^*) + \sqrt{X_0 Y_0} e^{-r(T-t)} N(d_2^*)\right)$$

where

$$d_1^*(K) = \frac{2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\sqrt{T-t}}\left[\ln\left(\frac{\sqrt{X_t Y_t}}{\sqrt{X_0 Y_0}}\right) + \left(r - \frac{1}{4}\left(\sigma_1^2 + \sigma_2^2\right) + \frac{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}{8}\right)(T-t)\right]$$

$$d_2^*(K) = d_1^* - \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\sqrt{T-t}$$

(b) Compute the value of the call using a 5% interest rate, 20% volatility for X, 30% volatility for Y, and $\rho = 0.4$. Use finite differences to estimate the deltas, gammas and cross-gamma of the call.

Solution.

Using unit values for X_0, Y_0, X_t, Y_t, and $\sigma_1 = 20\%$, $\sigma_2 = 30\%$ and $\rho = 0.4$, we get: 1.06.

For the same values, finite differences (although explicit greeks could be computed) give: $\delta_X = 2.156, \delta_Y = 2.156, \gamma_X = -1.428, \gamma_Y = -1.428, \gamma_{XY} = 1.428$.

(c) You purchased the call on a \$10,000,000 notional. What actions would you take to delta-hedge your position? What would then be your instant $P\&L$ in the following matrix of scenarios. Generally, graph your instant $P\&L$ against percent changes x, y in underlying stock prices.

Solution.

The first-order underlying risks are represented by the delta vector. It can be cancelled by buying/selling shares of each underlying by an amount equal to the corresponding delta.

One must realize that these positions DO NOT hedge the first-order correlation risk that is rather cumbersome to deal with. We would have to cancel our correlation vega, but it changes as cross-gammas (and eventually cross-volgas) are neutralized.

Here are the table and the plot of our instant $P\&L$ against percent changes x, y in underlying stock prices (we used the values of question b) for the volatilities, the correlation, the maturity and the risk-free rate).

Note that we assumed that everything happened on a short timeframe, hence we neglected interest rate effects (the true $P\&L$ would be $d\Pi - r\Pi dt$, where Π is the money borrowed to set up initially our delta-hedged portfolio).

With these parameters:

- Price of the call at time 0: 5937170.
- Initial deltas (X,Y): $1.2564, 1.2564$
- Delta-hedged portolio setup cost: 19192049.204

X \ Y	-5%	$+1\%$	$+5\%$
-5%	1268867	509080	14286
$+1\%$	509080	-250705	-745500
$+5\%$	14286	-745500	-1240294

Rebalancing this delta-hedged portfolio costs money when the stocks go up, and brings money when they go down. This is consistent since we are long the call.