

2801.001 Spring 2018 Homework 1

Martin Arienmughare – moa258

Madhur Bhattad – mb6854

Louis Guigo – lg2894

Mario Zhu – mz833

April 2018

Submission instructions: Groups of three to four, due 14 calendar days after the homework is posted, in electronic format. Submission must contain: (a) master answer sheet (typewritten or handwritten) with header containing your names and NetIDs, and (b) all relevant computer files such as code source, Excel files etc.

Problem 1

You are short 100 at-the-money call option contracts on the S&P 500 index expiring in one year with a contract multiplier of 100. The current index level is 2600, the interest and dividend rates are zero.

- (a) Simulate the evolution of the index level at periods t as a geometric Brownian motion with volatility σ (free parameter) and calculate the corresponding call value, delta, gamma and theta using a fixed 20% implied volatility.

Solution.

[TODO]

- (b) Then simulate your actual cumulative P&L when periodically delta-hedging your position (assuming you can trade the index as an asset) and compare it against the proxy formula on slide 11 for the following matrix of parameters:

$\sigma \backslash \Delta t$	Monthly (12 per year)	Weekly (52 per year)	Daily (252 per year)
$\sigma = 25\%$			
$\sigma = 20\%$			
$\sigma = 15\%$			

Solution.

[TODO]

- (c) Provide a statistical analysis of your results over 10,000 simulations, where one simulation is an entire index path.

Solution.

[TODO]

Problem 2

Use your knowledge of how the VIX is calculated to show that the VIX is not the price of an investable asset. What about the square of the VIX?

Solution.

[TODO]

Problem 3

On March 29, 2018 the S&P 500 index (SPX) is 2611.53 and the 12-month VIX is 21.28. The implied volatility smile of SPX options expiring on March 15, 2019 is given as:

$$\sigma^* = \sqrt{a + b \left(\rho(x - m) + \sqrt{(x - m)^2 + s^2} \right)}$$

where $a = 0.009$, $b = 0.11$, $\rho = 0.12$, $m = 0.2$, $s = 0.05$, $x = \log \frac{K}{F}$ is log-moneyness and $F = 2625.10$ is the forward price. The continuous interest rate is 2.09% p.a.

- (a) Draw the implied volatility smile curve for strikes $500 \leq K \leq 5000$.

Solution.

[TODO]

- (b) Calculate the fair strike K var of a variance swap expiring on March 15, 2019 with the method of your choice. How close is your calculation to the 12-month VIX? Why is it not exactly the same?

Solution.

[TODO]

Problem 4

Consider a real symmetric matrix A . Show that $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is symmetric positive-definite. Hint: Use spectral decomposition.

Solution.

Let $A \in S_n(\mathbb{R})$. Let us prove that $\exp(A) \in S_n^{++}(\mathbb{R})$.

1. The linear application $M \mapsto M^T$ is continuous on $\mathcal{M}_n(\mathbb{R})$, so if A is symmetric, we have $\exp(A)^T = \exp(A)$; $\exp(A)$ is symmetric.
2. Since A is symmetric, there exists an orthonormal basis of diagonalization of A , i.e.

$$A = P \text{Diag}(\lambda_1, \dots, \lambda_n) P^{-1}$$

with $P \in O_n(\mathbb{R})$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Taking the exponential of A , we therefore have:

$$\exp(A) = P \text{Diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) P^{-1}$$

With this writing, we clearly see that all the eigenvalues of $\exp(A)$ are strictly positive. Therefore $\exp(A)$ is positive-definite.

With these two points, $\exp(A)$ is a symmetric positive-definite matrix: $\exp(A) \in S_n^{++}(\mathbb{R})$.

Problem 5

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and E the vector space of random variables with finite second moment and mean zero. Define $\langle X, Y \rangle = \text{Cov}(X, Y)$ and, for any event $A \in \mathcal{A}$, $Z_A = I_A - \mathbb{P}(A)$ where I_A is the indicator variable of A .

- (a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on E . What is the induced norm?

Solution.

Let us recall that, for X and Y random variables, we define the covariance by:

$$\text{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

Relation from which we deduce by some algebra:

$$\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Over E , it simplifies to $\text{Cov}(X, Y) = \mathbb{E}(XY)$

Let us check that $X, Y \mapsto \langle X, Y \rangle$ is an inner product on E , the vector space of real random variables with zero mean and finite second moment.

$\text{Cov}(\cdot, \cdot)$ is an application from $E \times E$ to \mathbb{R} .

Let $X, Y, Z \in E$ and $\lambda \in \mathbb{R}$.

- i) (*Symmetry*) $\text{Cov}(X, Y) = \text{Cov}(Y, X)$, clearly, using the symmetry in X, Y in the definition of the covariance.
- ii) (*Linearity*) $\text{Cov}(\lambda X, Y) = \mathbb{E}((\lambda X - \mathbb{E}(\lambda X))(Y - \mathbb{E}(Y))) = \lambda \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = \lambda \text{Cov}(X, Y)$, by linearity of the expectation.
 Besides: $\text{Cov}(X + Z, Y) = \mathbb{E}((X + Z - \mathbb{E}(X + Z))(Y - \mathbb{E}(Y))) = \mathbb{E}((X - \mathbb{E}(X) + Z - \mathbb{E}(Z))(Y - \mathbb{E}(Y))) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) + \mathbb{E}((Z - \mathbb{E}(Z))(Y - \mathbb{E}(Y))) = \text{Cov}(X, Y) + \text{Cov}(Z, Y)$.
- iii) (*Positive*) $\text{Cov}(X, X) = \mathbb{E}((X - \mathbb{E}(X))^2) \geq 0$ by positivity of the random variable $(X - \mathbb{E}(X))^2$ and monotony of the expectation.
- iv) (*Definite*) $\text{Cov}(X, X) = 0$ iff $(X - \mathbb{E}(X))^2 = 0$ since it is a positive random variable. This implies that $X - \mathbb{E}(X) = 0$, i.e. $X = 0$ since $\mathbb{E}(X) = 0$ by definition of E .

With these elements, $\text{Cov}(\cdot, \cdot)$ is an inner product on E . The induced norm is:

$$\|X\| = \sqrt{\mathbb{E}(X^2)}$$

- (b) Show that $|\rho(X, Y)| \leq 1$ (correlation coefficient).

Solution.

By definition, $\rho(X, Y) = \frac{\langle X, Y \rangle}{\|X\| \|Y\|}$. But $\langle X, Y \rangle \leq \|X\| \|Y\|$ by Cauchy-Schwarz inequality, so $\rho \leq 1$.

- (c) Show that if $X, Y \in E \setminus \{0\}$ are probabilistically independent then X, Y are linearly independent within E . Converse?

Solution.

Let us reason by contraposition. Let us assume that X, Y are not linearly independent, i.e. there exists $\lambda \in \mathbb{R}$ such that $Y = \lambda X$. Necessarily, $\lambda \neq 0$ since $X, Y \neq 0$.

So for $x, y \in \mathbb{R}$, $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x, X = \frac{y}{\lambda})$. Therefore, $\mathbb{P}(X = x, Y = y) = 0$ if $y \neq \frac{x}{\lambda}$ and $\mathbb{P}(X = x)$ else. In that second case, we see that $\mathbb{P}(X = x, Y = y) \neq \mathbb{P}(X = x)\mathbb{P}(Y = y) = \mathbb{P}(X = x)^2$, unless X is constant equal to x which is impossible by assumption since $X \neq 0$ and $\mathbb{E}(X) = 0$ by definition of the vector space E .

Therefore, X and Y are not probabilistically independent.

This proves the direct implication over E .

The converse is false. As a counterexample, consider the random variable X taking values in the set $\{-1, 0, 1\}$ with equal probabilities $\frac{1}{3}$, and take $Y = X^2$.

Clearly, X and Y are linearly independent (since the probability affected to the outcome -1 is non zero in the law of X). However, X and Y are probabilistically dependent since for instance $\mathbb{P}(X = 0, Y = 1) = 0 \neq \mathbb{P}(X = 0) \cdot \mathbb{P}(Y = 1) = \frac{1}{3} \cdot \frac{2}{3}$.

- (d) Let $X, Y \in E \setminus \{0\}$. What is the statistical interpretation of the orthogonal projection of Y on $\text{Span}(X)$?

Solution.

The orthogonal projection of Y on the line $\text{Span}(X)$ is exactly given by:

$$\Pi_{\text{Span}(X)}(Y) = \frac{\langle X, Y \rangle}{\|X\|^2} X = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} X$$

This orthogonal projection is a random variable linearly dependent with X , which has the same correlation with X as Y has. Additionally, $\frac{\text{Cov}(X, Y)}{\text{Var}(X)}$ is the λ coefficient that minimizes the variance of any random variable $Y - \lambda X$.

In statistical terms, it means that if we wanted to predict Y using X with a linear model, i.e. $\hat{Y} = \alpha X + \beta$, $\frac{\text{Cov}(X, Y)}{\text{Var}(X)}$ would be the α coefficient that minimizes the norm of the prediction error $\epsilon = \mathbb{E}((\hat{Y} - Y)^2)$.

Indeed the first order conditions of this simple minimization problem are:

$$\frac{\partial \epsilon}{\partial \alpha} = -2\mathbb{E}((\hat{Y} - Y)\frac{\partial \hat{Y}}{\partial \alpha}) \text{ and } \frac{\partial \epsilon}{\partial \beta} = -2\mathbb{E}((\hat{Y} - Y)\frac{\partial \hat{Y}}{\partial \beta})$$

which can be rewritten:

$$\mathbb{E}((\hat{Y} - Y)X) = 0 \text{ and } \mathbb{E}(\hat{Y} - Y) = 0$$

This means that the prediction error $\hat{Y} - Y$ has to be orthogonal to X and that its expected value must be zero. Note that the first order conditions correspond to a minimum given the positive-definiteness of the Hessian.

Substituting $\hat{Y} = \alpha X + \beta$, we obtain $\alpha = \frac{\text{Cov}(X, Y)}{\text{Var}(X)}$ and $\beta = \mathbb{E}(Y) - \frac{\text{Cov}(X, Y)}{\text{Var}(X)}\mathbb{E}(X)$. As stated above, it means that $\frac{\text{Cov}(X, Y)}{\text{Var}(X)}X$ is the closest random variable (i.e. the one that minimizes the variance of the prediction error) to Y in $\text{Span}(X)$.

Besides, the prediction error is given by $\epsilon = (1 - \rho^2)\text{Var}(Y)$. If $\rho = \pm 1$, a perfect prediction can be made. When $\rho = 0$, the variance in the prediction is as large as the variation in Y , which implies that the predictor is very bad. For intermediate values of ρ , the predictor reduces the error.

- (e) Verify that $Z_A \in E$ and calculate $\langle Z_A, Z_B \rangle$ for any $A, B \in \mathcal{A}$. When is $Z_A \perp Z_B$? Are $Z_A, Z_{\bar{A}}$ linearly independent in E ?

Solution.

It is a fact that $Z_A \in E$ for any $A \in \mathcal{A}$ since this is a random variable with first and second moment respectively such that: $\mathbb{E}(Z_A) = \mathbb{E}(I_A - \mathbb{P}(A)) = \mathbb{E}(I_A) - \mathbb{P}(A) = 0$ and $\mathbb{E}(Z_A^2) = \mathbb{E}(I_A^2 - 2\mathbb{P}(A)I_A + \mathbb{P}(A)^2) = \mathbb{E}(I_A^2) - 2\mathbb{P}(A)\mathbb{E}(I_A) + \mathbb{P}(A)^2 = \mathbb{P}(A)(1 - \mathbb{P}(A))$, so $\mathbb{E}(Z_A^2) \leq \infty$ since $I_A^2 = I_A$.

Besides, for $A, B \in \mathcal{A}$, $\langle Z_A, Z_B \rangle = \langle I_A - \mathbb{P}(A), I_B - \mathbb{P}(B) \rangle = \langle I_A, I_B \rangle - \langle I_A, \mathbb{P}(B) \rangle - \langle \mathbb{P}(A), I_B \rangle + \langle \mathbb{P}(A), \mathbb{P}(B) \rangle = \langle I_A, I_B \rangle$

So, $\langle Z_A, Z_B \rangle = \mathbb{E}((I_A - \mathbb{P}(A))(I_B - \mathbb{P}(B))) = \mathbb{E}(I_{A \cap B} - \mathbb{P}(B)I_A - \mathbb{P}(A)I_B + \mathbb{P}(A)\mathbb{P}(B))$.

Id est, $\langle Z_A, Z_B \rangle = \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)$.

We can see that $Z_A \perp Z_B$ when $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, that is, when events A and B are stochastically independent.

$Z_A, Z_{\bar{A}}$ are not linearly independent in E since for any $Z_{\bar{A}} = 1 - Z_A$.

- (f) Suppose $A, B \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ are disjoint, and $B \neq \bar{A}$. Show that Z_A, Z_B are linearly independent.

Solution.

Let $A, B \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ such that $A \cap B = \emptyset$. Let $\lambda, \mu \in \mathbb{R}$ such that $\lambda Z_A + \mu Z_B = 0$.

Since $A \cap B = \emptyset$, plugging in any two elementary events $x \in A$ and $y \in B$ yields:

$$\begin{cases} -\lambda\mathbb{P}(A) + \mu(1 - \mathbb{P}(B)) = 0 \\ \lambda(1 - \mathbb{P}(A)) - \mu\mathbb{P}(B) = 0 \end{cases}$$

Taking the difference between these equations, we obtain the condition: $\lambda = \mu$.

Therefore, $\lambda(Z_A + Z_B) = 0$. Since $B \neq \bar{A}$, this implies $\lambda = \mu = 0$.

Thus, Z_A and Z_B are linearly independent.

- (g) Suppose $A \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ and define $B = \{\emptyset, A, \bar{A}, \Omega\} \subseteq \mathcal{A}$. Let $Y = E(X|B)$. Verify that $Y \in E$ and show that Y is the orthogonal projection of X on $\text{Span}(Z_A)$.

Solution.

Note that B is a σ -Algebra, therefore it makes sense to consider $Y = \mathbb{E}(X|B)$.

Let us recall that Y is defined as the almost surely unique random variable such that for any bounded and B -measurable random variable U , $\mathbb{E}(XU) = \mathbb{E}(YU)$.

We have: $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(X|B)) = \mathbb{E}(X) = 0$ using the tower property of conditional expectation and the fact that $X \in E$.

Besides, $\mathbb{E}(Y^2) = \mathbb{E}(\mathbb{E}(X|B)^2) \leq \mathbb{E}(\mathbb{E}(X|\sigma(X))^2) \leq \mathbb{E}(X^2) \leq \infty$ since $X \in E$.

Let us check that Y is the orthogonal projection of X on $\text{Span}(Z_A)$. For that matter, let us show that it is such that for all $U \in \text{Span}(Z_A)$, $\langle X - Y, U \rangle = 0$.

Let $U \in \text{Span}(Z_A)$. Since Z_A is B -measurable, U is also B -measurable, and it follows that $\mathbb{E}(UX|B) = U\mathbb{E}(X|B)$.

Therefore: $\mathbb{E}(U\mathbb{E}(X|B)) = \mathbb{E}(\mathbb{E}(UX|B)) = \mathbb{E}(UX)$ i.e. $\langle U, Y \rangle = \langle U, X \rangle$. That is exactly $\langle U, Y - X \rangle = 0$.

Since this is true for any $U \in \text{Span}(Z_A)$, we conclude that $Y - X = \mathbb{E}(X|B) - X$ realizes the minimum distance from X to $\text{Span}(Z_A)$.

Since $(E, \text{Cov}(\cdot, \cdot))$ is a Hilbert space, the minimum distance is realized by the orthogonal projection. Thus, we have $Y = \Pi_{\text{Span}(Z_A)}(X)$.

Problem 6

Consider a vector space E equipped with a norm $N(x)$.

- (a) Show that N is Euclidean (i.e. induced by some inner product) if and only if N satisfies the parallelogram law:

$$N(x+y)^2 + N(x-y)^2 = 2 \cdot [N(x)^2 + N(y)^2]$$

Hint: Define the inner product through N .

Solution.

If N is Euclidean, it derives from a certain inner product $\langle \cdot, \cdot \rangle$. And we have:

$$N(x+y)^2 + N(x-y)^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle = 2\langle x, x \rangle + 2\langle y, y \rangle + 2\langle x, y \rangle - 2\langle x, y \rangle$$

$$\text{Therefore, } N(x+y)^2 + N(x-y)^2 = 2 \cdot [N(x)^2 + N(y)^2]$$

Let's now assume that N checks the parallelogram law, and let us prove that the bilinear form $\langle \cdot, \cdot \rangle$ defined by polarization as follows is an inner product over E from which N derives:

$$\langle x, y \rangle = \frac{1}{4} \cdot [N(x+y)^2 - N(x-y)^2]$$

We clearly have, by homogeneity, $N(x) = \sqrt{\langle x, x \rangle}$.

$\langle \cdot, \cdot \rangle$ is an application from $E \times E$ to \mathbb{R} .

Let $x, y, z \in E$ and $\lambda \in \mathbb{R}$.

- i) (*Symmetry*) $\langle x, y \rangle = \langle y, x \rangle$, clearly, using the symmetry in x, y in its definition.
- ii) (*Positive*) $\langle x, x \rangle = \frac{1}{4} \cdot N(2x)^2 = N(x)^2 \geq 0$ since $N(0) = 0$ by separation.
- iii) (*Definite*) $\langle x, x \rangle = 0$ iff $\frac{1}{4}N(x)^2 = 0$ iff $x = 0$ by separation.
- iv) (*Linearity*) Observe that:

$$\langle x+y, z \rangle + \langle x-y, z \rangle = \frac{1}{4} [N(x+y+z)^2 - N(x+y-z)^2 + N(x-y+z)^2 - N(x-y-z)^2]$$

Therefore,

$$\langle x+y, z \rangle + \langle x-y, z \rangle = \frac{1}{4} [2 \cdot [N(x+y)^2 - N(z)^2] + 2 \cdot [N(x-y)^2 + N(z)^2]]$$

Thus,

$$\langle x+y, z \rangle + \langle x-y, z \rangle = 2 \cdot [N(x)^2 + N(y)^2]$$

Which can be rewritten:

$$\langle x+y, z \rangle + \langle x-y, z \rangle = 2 \cdot \langle x, y \rangle$$

From which we deduce, taking $y = x$:

$$\langle 2 \cdot x, y \rangle = 2 \cdot \langle x, y \rangle$$

By induction, we prove that this it is true for any power n that:

$$\langle 2^n \cdot x, y \rangle = 2^n \cdot \langle x, y \rangle$$

Taking $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(x - y)$, we have:

$$\langle x + y, z \rangle = \langle 2 \cdot u, z \rangle = 2 \cdot \langle u, z \rangle = \langle u + v, z \rangle + \langle u - v, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

.

By induction we conclude that for any $n \in \mathbb{N}$ (we've seen that it's true for $n = 0$), and any $x, y \in E$ we must have:

$$\langle nx, y \rangle = n \langle x, y \rangle$$

We can extend the linearity to \mathbb{Z} using $\langle -x, y \rangle = -\langle x, y \rangle$, and then to \mathbb{Q} using $\langle x, y \rangle = n \cdot \langle \frac{1}{n}x, y \rangle$.

We conclude that the linearity is true over \mathbb{R} using the well-known density of \mathbb{Q} in \mathbb{R} and the continuity of the application $(x, y) \mapsto \frac{1}{4} \cdot [N(x + y)^2 - N(x - y)^2]$, since the norm N is continuous.

Therefore, the linearity of $\langle \cdot, \cdot \rangle$ is true.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product over E from which N derives since $N(x) = \sqrt{\langle x, x \rangle}$.

Conclusion: N is a Euclidean norm if and only if it checks the parallelogram identity.

(b) Which of the following are Euclidean norms on $E = \mathbb{R}^n$?

$$N_p(x) = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$$

$$Q_p(x) = \sqrt{N_2(x)^2 + \frac{1}{p} \sum_{i < j} x_i x_j}$$

where $p \in [1, \infty)$.

Solution.

Given the previous question, we just need to verify if these norms check the parallelogram identity to know if they are Euclidean.

For $p = 2$, N_p is nothing but the so called Euclidean norm over \mathbb{R}^n , which as its name tells us, is Euclidean and derives from the canonical inner product of \mathbb{R}^n : $\langle x, y \rangle = x^T y$.

However, **for** $p \geq 1$ **but** $p \neq 2$, N_p is not Euclidean. As a counterexample, one can consider $x = (1, 1, 0, \dots, 0)$ and $y = (1, -1, 0, \dots, 0)$. We have $N_p(x) = N_p(y) = \sqrt[p]{2}$ but $N_p(x+y) = N_p(x-y) = 2$. So $N_p^2(x+y) + N_p^2(x-y) = 8 \neq N_p^2(x) + N_p^2(y) = 4 \sqrt[p]{2}$.

Since it doesn't check the parallelogram identity, N_p is not Euclidean **for** $p \geq 1$ **but** $p \neq 2$.

Q_p is not a norm for any $p \in [1, \infty)$ since it doesn't check the triangle inequality: for $x = (1, 0, \dots, 0)$ and $y = (0, 1, 0, \dots, 0)$, we have:

$$Q_p(x+y)^2 = 2^2 + \frac{1}{p} > Q_p(x)^2 + Q_p(y)^2 = 1^2 + 1^2 = 2$$

A fortiori, it can't be a Euclidean norm.

- (c) Define $A^2(x, y) = [N(x) \cdot N(y)]^2 - [N(x+y)^2 - N(x-y)^2]^2/16$. Show that if N is Euclidean then $A^2(x, y) \geq 0$ and $A(x+y, x-y) = 2 \cdot A(x, y)$.

Geometric interpretation for $E = \mathbb{R}^2$?

Solution.

[TODO]