

2801.001 Spring 2018 Homework 1

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Submission instructions: Same groups as HW1. Refer to instructions from TA.

Note: Problems 1 and 2 require the datafile posted online which contains prices and implied volatilities of one-year (349 calendar days) listed options on the Nasdaq 100 index (NDX), as well as other market data.

Problem 1

- (a) Estimate the market price of the 5% call spread (i.e. with strikes ATM and 5% OTM). What about the 5% put spread?

Solution.

The Nasdaq 100 data provided allows us to get the buy and sell prices of ATM (strike 5050) and 5% OTM (strikes 5200 and 5750) call and put prices (we use midpoints).

- Selling ATM Call: 337.1.
- Selling ATM Put: 326.1.
- Buying 5% OTM Call: 478.1.
- Buying 5% OTM Put: 428.3.

Therefore, the buy prices of the 5% call and put spreads are:

- 5% call spread: $478.1 - 337.1 = 141.0$.
- 5% put spread: $428.3 - 326.1 = 100.2$.

- (b) If you were to price the spreads in the Black-Scholes model using a single volatility parameter σ , what value of σ would match the theoretical price with the market price? Comment on your results.

Solution.

Applying a Black-Scholes pricing formula would lead to the following theoretical prices for the 5% call spread (5% put spread resp.):

$$\begin{aligned} CS(K, 0.95K) &= C(0.95K) - C(K) \\ &= S_t N(d_1(0.95K)) - 0.95K e^{-r(T-t)} N(d_2(0.95K)) \\ &\quad - (S_t N(d_1(K)) - K e^{-r(T-t)} N(d_2(K))) \end{aligned}$$

$$\begin{aligned} PS(K, 1.05K) &= P(1.05K) - P(K) \\ &= 1.05K e^{-r(T-t)} N(-d_2(1.05K)) - S_t N(-d_1(1.05K)) \\ &\quad - (K e^{-r(T-t)} N(-d_2(K)) - S_t N(-d_1(K))) \end{aligned}$$

where

$$\begin{aligned} d_1(K) &= \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right] \\ d_2(K) &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

σ is the implied volatility that one can choose to equate theoretical prices to market prices.

Using a solver, we get $\sigma_{CS} = 14.13\%$ using the 5% call spread and $\sigma_{PS} = 17.10\%$ using the 5% put spread.

These levels are relatively close to the implied volatility quoted for ATM (15.4 %) and 5% OTM options (14% to 16.7%)

Problem 2

- (a) Using the numerical package of your choice, calibrate the parameters of the SVI model against the market-implied volatility data. Show a comparative graph of the SVI curve and the actual implied volatility data points.

Solution.

The SVI model reads:

$$\sigma^* = \sqrt{a + b \left(\rho(x - m) + \sqrt{(x - m)^2 + s^2} \right)}$$

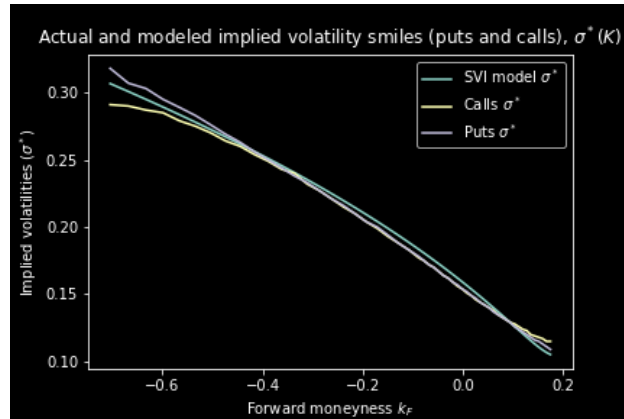
To calibrate its parameters (a, b, ρ, m, s) we perform the following least square optimization against market data (both puts and calls), with the following constraints to avoid arbitrage:

$$\begin{aligned} \min_{a, b, \rho, m, s} \quad & \sum_{i=1}^n \left[\sigma_{\text{SVI}}^*(k_i, T; a, b, \rho, m, s) - \sigma_{\text{Market}}^*(k_i, T) \right]^2 \\ \text{subject to} \quad & a, b \geq 0, \\ & -1 \leq \rho \leq 1, \\ & s > 0, \\ & b(1 + |\rho|)T \leq 4. \end{aligned}$$

Using a Sequential Least Squares Programming method in `scipy.minimize`, we obtain:

$$a = 0.000364, b = 2.55, \rho = 0.961, m = 0.239, s = 0.0149$$

The quality of the fit is very good:



- (b) Compute or estimate the price of an at-the-money digital call option paying off \$1 if in one year NDX is greater than its current spot level, and zero otherwise: (i) in the Black-Scholes model, (ii) using $\pm 1\%$ call spreads, (iii) using the smile-adjusted formula on page 19.

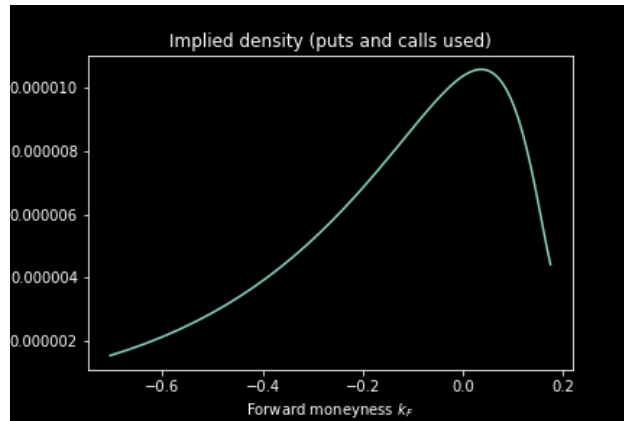
Solution.

- Black-Scholes model: $D_{BS}(S_t, K, r, T, \sigma) = e^{-rT} N(d_2)$. We obtain: 0.495.
- Using $\pm 1\%$ call spreads. We obtain: 0.419.
- Using smile-adjusted formula: $D(S_t, K, r, T) = (D_{BS} - V_{BS} \times \frac{\partial \sigma^*}{\partial K})(S_t, K, r, T, \sigma^*(K, T))$, where V_{BS} is the Black-Scholes vega, $V_{BS} = S_t \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \sqrt{T-t}$. We obtain: 0.590.

- (c) Graph the implied distribution corresponding to the SVI model calibration.

Solution.

As expected, the density obtained from options prices is right skewed and has fatter tails than the Black-Scholes lognormal random variable:



- (d) Use the implied distribution to compute the price of the following European exotic options, where X_0 is the current index level and X_T is the final index level:

Solution.

Knowing the implied distribution of an underlying, the value of a derivative f_t at time t with a certain payoff $f(S_T)$ at maturity T is given by discounted the expected payoff under this measure: $f_t = e^{-r(T-t)} \mathbb{E}(f(S_T))$. We apply this to the following cases:

- (i) Digital call defined in question (b);
Its payoff is just $\mathbb{I}(X_T > K)$. We obtain: 0.0131.

- (ii) "Reverse convertible" paying off $\max\left(100\%, 100\% + p \times \frac{X_T - X_0}{X_0}\right)$ if $\frac{X_T}{X_0} > 75\%$ and $\frac{X_T}{X_0}$ otherwise, where $p = 50\%$. Then solve for p to get a price of 100%;

With $p = 0.5$, we obtain: 0.0230.

To get a price of 1.0, we just apply a rootfinding technique to the payoff to which we subtract 1.0, we get: $p = 0.5$.

- (iii) Option paying off $\max\left(0, \frac{X_T - X_0}{X_T}\right)$;

We get: 0.000796.

- (iv) Log-contract paying off $-2 \log\left(\frac{X_T}{X_0}\right)$. Price interpretation;

We get: 0.00429.

Assuming arbitrage (it is the case since we fitted the SVI model), this can be interpreted as the current expected total return gained on the stock over the life-time of the option, i.e. 0.429 % over 1 year.

Problem 3

Find conditions on the SVI model parameters to satisfy Lees asymptotic bounds on p. 22:

$$\sigma^{\star^2}(k_F, T) \leq \frac{\beta}{T} |\log k_F|, \beta \in [0, 2]$$

Solution.

TODO

Problem 4

(Problem 4.3 p. 56 in textbook, with corrections): Consider an underlying stock S currently trading at $S_0 = 100$ which does not pay any dividend. Assume the local volatility function is $\sigma_{loc}(t, S) = \frac{0.1 - 0.15 \times \log\left(\frac{S}{S_0}\right)}{\sqrt{t}}$, and that interest rates are zero.

- (a) Produce the graph of the local volatility surface for spots 0 to 200 and maturities 0 to 5 years.

Solution.

TODO

- (b) Write a Monte-Carlo algorithm to price the following 1-year payoffs using 252 time steps and e.g. 10,000 paths:

Solution.

- (i) "Capped quadratic" option: $\min\left(1, \frac{S_1^2}{S_0^2}\right)$;

TODO

- (ii) Asian at-the-money-call: $\max\left(0, \frac{S_{0.25} + S_{0.5} + S_{0.75} + S_1}{4 \times S_0} - 1\right)$;

TODO

- (iii) Barrier call: $\max(0, S_1 - S_0)$ if S always traded above 80 using 252 daily observations, 0 otherwise;

TODO

Problem 5

The payoff of a 1-year at-the-money call on the geometric average return of two non-dividend paying stocks X, Y is given as:

$$f(X_T, Y_T) = \max \left(0, \sqrt{\frac{X_T Y_T}{X_0 Y_0}} - 1 \right)$$

where $T = 1$ year and X_t, Y_t are the respective underlying spot prices of X, Y at any time t .

- (a) Derive analytical formulas for the call value at any time $0 \leq t \leq T$ in the Black-Scholes model with constant correlation ρ (cf. Section 6 – 4 in the textbook, to be covered during Session 5.)

Solution.

In a Black-Scholes world, these two correlated stocks have the dynamics:

$$dX_t = (r - \mu_1)X_t dt + \sigma_1 X_t dW_t$$

$$dY_t = (r - \mu_2)Y_t dt + \sigma_2 Y_t (\rho dW_t + \sqrt{1 - \rho^2} dZ_t)$$

where r is the risk-free rate, μ_1, μ_2 are the dividend yields (equal to 0 here by assumption), ρ is the constant correlation between X and Y , and W, Z are two uncorrelated Brownian motions.

The value of this call is given by: $f_t = e^{-r(T-t)} \mathbb{E} \left(\max \left(0, \sqrt{\frac{X_T Y_T}{X_0 Y_0}} - 1 \right) | X_t, Y_t \right)$

The geometric average of two lognormal is still lognormal. Indeed, for $\alpha > 0$, let us apply It-Doebelin formula to the process $V_t = (X_t Y_t)^\alpha$:

We obtain:

$$\begin{aligned} dV_t = & \left(r - \left(\mu_{X,Y} + (1 - \alpha) \left(r - \mu_{X,Y} + \frac{1}{2} \sigma_{X,Y}^2 \right) \right) \right) V_t dt \\ & + \alpha V_t \left(\sigma_1 dW_t + \sigma_2 \left(\rho dW_t + \sqrt{1 - \rho^2} dZ_t \right) \right) \end{aligned}$$

where $\mu_{X,Y} = \mu_1 + \mu_2 - r - \rho \sigma_1 \sigma_2$ and $\sigma_{X,Y} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho \sigma_1 \sigma_2}$.

In our case, we assume no dividend ($\mu_1 = \mu_2 = 0$) and $\alpha = \frac{1}{2}$, so the dynamics of V_t simplifies to:

$$dV_t = \left(r - \frac{1}{4} (\sigma_1^2 + \sigma_2^2) \right) V_t dt + \frac{1}{2} V_t \left(\sigma_1 dW_t + \sigma_2 \left(\rho dW_t + \sqrt{1 - \rho^2} dZ_t \right) \right)$$

If we call $dB_t = \frac{\sigma_1 dW_t + \sigma_2 (\rho dW_t + \sqrt{1-\rho^2} dZ_t)}{\sigma_{X,Y}}$, where again $\sigma_{X,Y} = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$, B_t is a Brownian and we have:

$$dV_t = \left(r - \frac{1}{4} (\sigma_1^2 + \sigma_2^2) \right) V_t dt + \frac{1}{2} \sigma_{X,Y} V_t dB_t$$

Under this form, we see that $V_t = \sqrt{X_t Y_t}$ follows a lognormal distribution.

Therefore, the price of the option is given by the following Black-Scholes formula:

$$f_t = \frac{1}{\sqrt{X_0 Y_0}} \left(e^{-\frac{1}{4}(\sigma_1^2 + \sigma_2^2)(T-t)} \sqrt{X_t Y_t} N(d_1^*) + \sqrt{X_0 Y_0} e^{-r(T-t)} N(d_2^*) \right)$$

where

$$d_1^*(K) = \frac{2}{\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}\sqrt{T-t}} \left[\ln \left(\frac{\sqrt{X_t Y_t}}{\sqrt{X_0 Y_0}} \right) + \left(r - \frac{1}{4} (\sigma_1^2 + \sigma_2^2) + \frac{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}{8} \right) (T-t) \right]$$

$$d_2^*(K) = d_1^* - \frac{1}{2} \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \sqrt{T-t}$$

- (b) Compute the value of the call using a 5% interest rate, 20% volatility for X , 30% volatility for Y , and $\rho = 0.4$. Use finite differences to estimate the deltas, gammas and cross-gamma of the call.

Solution.

Using unit values for X_0, Y_0, X_t, Y_t , and $\sigma_1 = 20\%$, $\sigma_2 = 30\%$ and $\rho = 0.4$, we get: 1.06.

For the same values, finite differences (although explicit greeks could be computed) give: $\delta_X = 2.156$, $\delta_Y = 2.156$, $\gamma_X = -1.428$, $\gamma_Y = -1.428$, $\gamma_{XY} = 1.428$.

- (c) You purchased the call on a \$10,000,000 notional. What actions would you take to delta-hedge your position? What would then be your instant $P\&L$ in the following matrix of scenarios. Generally, graph your instant $P\&L$ against percent changes x, y in underlying stock prices.

Solution.

TODO

$\begin{matrix} & Y \\ X \end{matrix}$	-5%	+1%	+5%
-5%			
+1%			
+5%			