

# 2801.001 Spring 2018 Homework 1

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**Submission instructions:** Same groups as HW1. Refer to instructions from TA.

**Note:** Problems 1 and 2 require the datafile posted online which contains prices and implied volatilities of one-year (349 calendar days) listed options on the Nasdaq 100 index (NDX), as well as other market data.

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## Problem 1

- (a) Estimate the market price of the 5% call spread (i.e. with strikes ATM and 5% OTM). What about the 5% put spread?

### **Solution.**

The Nasdaq 100 data provided allows us to get the buy and sell prices of ATM (strike 5050) and 5% OTM (strikes 5200 and 5750) call and put prices (we use midpoints).

- Selling ATM Call: 337.1.
- Selling ATM Put: 326.1.
- Buying 5% OTM Call: 478.1.
- Buying 5% OTM Put: 428.3.

Therefore, the buy prices of the 5% call and put spreads are:

- 5% call spread:  $478.1 - 337.1 = 141.0$ .
- 5% put spread:  $428.3 - 326.1 = 100.2$ .

- (b) If you were to price the spreads in the Black-Scholes model using a single volatility parameter  $\sigma$ , what value of  $\sigma$  would match the theoretical price with the market price? Comment on your results.

**Solution.**

Applying a Black-Scholes pricing formula would lead to the following theoretical prices for the 5% call spread (5% put spread resp.):

$$\begin{aligned} CS(K, 0.95K) &= C(0.95K) - C(K) \\ &= S_t N(d_1(0.95K)) - 0.95K e^{-r(T-t)} N(d_2(0.95K)) \\ &\quad - (S_t N(d_1(K)) - K e^{-r(T-t)} N(d_2(K))) \end{aligned}$$

$$\begin{aligned} PS(K, 1.05K) &= P(1.05K) - P(K) \\ &= 1.05K e^{-r(T-t)} N(-d_2(1.05K)) - S_t N(-d_1(1.05K)) \\ &\quad - (K e^{-r(T-t)} N(-d_2(K)) - S_t N(-d_1(K))) \end{aligned}$$

where

$$\begin{aligned} d_1(K) &= \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S_t}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) (T-t) \right] \\ d_2(K) &= d_1 - \sigma \sqrt{T-t} \end{aligned}$$

$\sigma$  is the implied volatility that one can choose to equate theoretical prices to market prices.

Using a solver, we get  $\sigma_{CS} = 14.13\%$  using the 5% call spread and  $\sigma_{PS} = 17.10\%$  using the 5% put spread.

These levels are relatively close to the implied volatility quoted for ATM (15.4 %) and 5% OTM options (14% to 16.7%)

## Problem 2

- (a) Using the numerical package of your choice, calibrate the parameters of the SVI model against the market-implied volatility data. Show a comparative graph of the SVI curve and the actual implied volatility data points.

### Solution.

The SVI model reads:

$$\sigma^* = \sqrt{a + b \left( \rho(x - m) + \sqrt{(x - m)^2 + s^2} \right)}$$

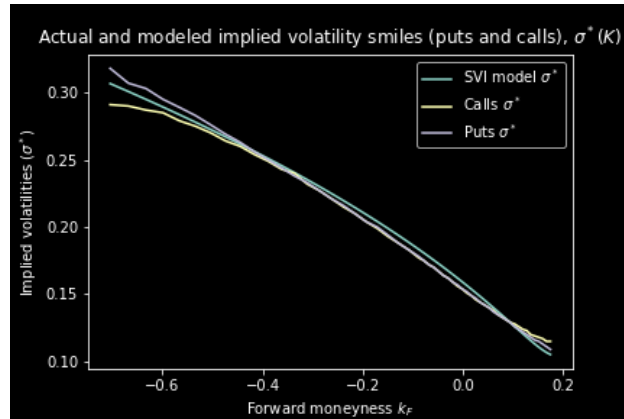
To calibrate its parameters  $(a, b, \rho, m, s)$  we perform the following least square optimization against market data (both puts and calls), with the following constraints to avoid arbitrage:

$$\begin{aligned} \min_{a, b, \rho, m, s} \quad & \sum_{i=1}^n \left[ \sigma_{\text{SVI}}^*(k_i, T; a, b, \rho, m, s) - \sigma_{\text{Market}}^*(k_i, T) \right]^2 \\ \text{subject to} \quad & a, b \geq 0, \\ & -1 \leq \rho \leq 1, \\ & s > 0, \\ & b(1 + |\rho|)T \leq 4. \end{aligned}$$

Using a Sequential Least Squares Programming method in `scipy.minimize`, we obtain:

$$a = 0.000364, b = 2.55, \rho = 0.961, m = 0.239, s = 0.0149$$

The quality of the fit is very good:



- (b) Compute or estimate the price of an at-the-money digital call option paying off \$1 if in one year NDX is greater than its current spot level, and zero otherwise: (i) in the Black-Scholes model, (ii) using  $\pm 1\%$  call spreads, (iii) using the smile-adjusted formula on page 19.

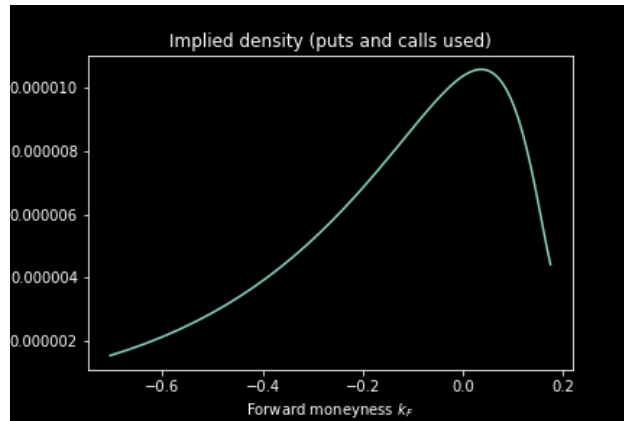
**Solution.**

- Black-Scholes model:  $D_{BS}(S_t, K, r, T, \sigma) = e^{-rT} N(d_2)$ . We obtain: 0.495.
- Using  $\pm 1\%$  call spreads. We obtain: 0.419.
- Using smile-adjusted formula:  $D(S_t, K, r, T) = (D_{BS} - V_{BS} \times \frac{\partial \sigma^*}{\partial K})(S_t, K, r, T, \sigma^*(K, T))$ , where  $V_{BS}$  is the Black-Scholes vega,  $V_{BS} = S_t \frac{e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} \sqrt{T-t}$ . We obtain: 0.590.

- (c) Graph the implied distribution corresponding to the SVI model calibration.

**Solution.**

As expected, the density obtained from options prices is right skewed and has fatter tails than the Black-Scholes lognormal random variable:



- (d) Use the implied distribution to compute the price of the following European exotic options, where  $X_0$  is the current index level and  $X_T$  is the final index level:

**Solution.**

Knowing the implied distribution of an underlying, the value of a derivative  $f_t$  at time  $t$  with a certain payoff  $f(S_T)$  at maturity  $T$  is given by discounted the expected payoff under this measure:  $f_t = e^{-r(T-t)} \mathbb{E}(f(S_T))$ . We apply this to the following cases:

- (i) Digital call defined in question (b);  
Its payoff is just  $\mathbb{I}(X_T > K)$ . We obtain: 0.0131.

- (ii) "Reverse convertible" paying off  $\max\left(100\%, 100\% + p \times \frac{X_T - X_0}{X_0}\right)$  if  $\frac{X_T}{X_0} > 75\%$  and  $\frac{X_T}{X_0}$  otherwise, where  $p = 50\%$ . Then solve for  $p$  to get a price of 100%;

With  $p = 0.5$ , we obtain: 0.0230.

To get a price of 1.0, we just apply a rootfinding technique to the payoff to which we subtract 1.0, we get:  $p = 0.5$ .

- (iii) Option paying off  $\max\left(0, \frac{X_T - X_0}{X_T}\right)$ ;

We get: 0.000796.

- (iv) Log-contract paying off  $-2 \log\left(\frac{X_T}{X_0}\right)$ . Price interpretation;

We get: 0.00429.

Assuming arbitrage (it is the case since we fitted the SVI model), this can be interpreted as the current expected total return gained on the stock over the life-time of the option, i.e. 0.429 % over 1 year.

### Problem 3

Find conditions on the SVI model parameters to satisfy Lees asymptotic bounds on p. 22:

$$\sigma^{\star^2}(k_F, T) \leq \frac{\beta}{T} |\log k_F|, \beta \in [0, 2]$$

**Solution.**

TODO

## Problem 4

(Problem 4.3 p. 56 in textbook, with corrections): Consider an underlying stock  $S$  currently trading at  $S_0 = 100$  which does not pay any dividend. Assume the local volatility function is  $\sigma_{loc}(t, S) = \frac{0.1 - 0.15 \times \log\left(\frac{S}{S_0}\right)}{\sqrt{t}}$ , and that interest rates are zero.

- (a) Produce the graph of the local volatility surface for spots 0 to 200 and maturities 0 to 5 years.

**Solution.**

TODO

- (b) Write a Monte-Carlo algorithm to price the following 1-year payoffs using 252 time steps and e.g. 10,000 paths:

**Solution.**

- (i) "Capped quadratic" option:  $\min\left(1, \frac{S_1^2}{S_0^2}\right)$ ;

TODO

- (ii) Asian at-the-money-call:  $\max\left(0, \frac{S_{0.25} + S_{0.5} + S_{0.75} + S_1}{4 \times S_0} - 1\right)$ ;

TODO

- (iii) Barrier call:  $\max(0, S_1 - S_0)$  if  $S$  always traded above 80 using 252 daily observations, 0 otherwise;

TODO

## Problem 5

The payoff of a 1-year at-the-money call on the geometric average return of two non-dividend paying stocks  $X, Y$  is given as:

$$f(X_T, Y_T) = \max \left( 0, \sqrt{\frac{X_T Y_T}{X_0 Y_0}} - 1 \right)$$

where  $T = 1$  year and  $X_t, Y_t$  are the respective underlying spot prices of  $X, Y$  at any time  $t$ .

- (a) Derive analytical formulas for the call value at any time  $0 \leq t \leq T$  in the Black-Scholes model with constant correlation  $\rho$  (cf. Section 6 – 4 in the textbook, to be covered during Session 5.)

**Solution.**

TODO

- (b) Compute the value of the call using a 5% interest rate, 20% volatility for  $X$ , 30% volatility for  $Y$ , and  $\rho = 0.4$ . Use finite differences to estimate the deltas, gammas and cross-gamma of the call.

**Solution.**

TODO

- (c) You purchased the call on a \$10,000,000 notional. What actions would you take to delta-hedge your position? What would then be your instant  $P\&L$  in the following matrix of scenarios. Generally, graph your instant  $P\&L$  against percent changes  $x, y$  in underlying stock prices.

**Solution.**

TODO

$X \backslash Y$	−5%	+1%	+5%
−5%			
+1%			
+5%			