2801.001 Spring 2018 Homework 1

Martin Arienmughare – moa258
Madhur Bhattad– mb6854
Louis Guigo – lg2894
Mario Zhu – mz833

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Submission instructions: Groups of three to four, due 14 calendar days after the homework is posted, in electronic format. Submission must contain: (a) master answer sheet (typewritten or handwritten) with header containing your names and NetIDs, and (b) all relevant computer files such as code source, Excel files etc.

Problem 1

You are short 100 at-the-money call option contracts on the S&P 500 index expiring in one year with a contract multiplier of 100. The current index level is 2600, the interest and dividend rates are zero.

(a) Simulate the evolution of the index level at periods t as a geometric Brownian motion with volatility σ (free parameter) and calculate the corresponding call value, delta, gamma and theta using a fixed 20% implied volatility.

Solution.

[TODO]

(b) Then simulate your actual cumulative P&L when periodically delta-hedging your position (assuming you can trade the index as an asset) and compare it against the proxy formula on slide 11 for the following matrix of parameters:

σ \ Δt	Monthly (12 per year)	Weekly (52 per year)	Daily (252 per year)
$\sigma = 25\%$			
$\sigma = 20\%$			
$\sigma = 15\%$			

Solution.

[TODO]

(c) Provide a statistical analysis of your results over 10,000 simulations, where one simulation is an entire index path.

Solution.

[TODO]

Problem 2

Use your knowledge of how the VIX is calculated to show that the VIX is not the price of an investable asset. What about the square of the VIX?

Solution.
[TODO]

Problem 3

On March 29, 2018 the S&P 500 index (SPX) is 2611.53 and the 12-month VIX is 21.28. The implied volatility smile of SPX options expiring on March 15, 2019 is given as:

$$\sigma^* = \sqrt{a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + s^2}\right)}$$

where $a = 0.009$, $b = 0.11$, $\rho = 0.12$, $m = 0.2$, $s = 0.05$, $x = \log \frac{K}{F}$ is log-moneyness and $F = 2625.10$ is the forward price. The continuous interest rate is 2.09% p.a.

(a) Draw the implied volatility smile curve for strikes $500 \le K \le 5000$.

Solution.

[TODO]

(b) Calculate the fair strike K var of a variance swap expiring on March 15, 2019 with the method of your choice. How close is your calculation to the 12-month VIX? Why is it not exactly the same?

Solution.

[TODO]

Problem 4

Consider a real symmetric matrix A. Show that $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is symmetric positive-definite. Hint: Use spectral decomposition.

Solution.
Let $A \in S_n(\mathbb{R})$. Let us prove that $\exp(A) \in S_n^{++}(\mathbb{R})$.

1. The linear application $M \mapsto M^T$ is continuous on $\mathcal{M}_n(\mathbb{R})$, so if A is symmetric, we have $\exp(A)^T = \exp(A)$; $\exp(A)$ is symmetric.

2. Since A is symmetric, there exists an orthonormal basis of diagonalization of A, i.e.

$$A = P\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)P^{-1}$$

with $P \in O_n(\mathbb{R})$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Taking the exponential of A, we therefore have:

$$\exp(A) = P\mathrm{Diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})P^{-1}$$

With this writing, we clearly see that all the eigenvalues of $\exp(A)$ are strictly positive. Therefore $\exp(A)$ is positive-definite.

With these two points, $\exp(A)$ is a symmetric positive-definite matrix: $\exp(A) \in S_n^{++}(\mathbb{R})$.

Problem 5

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and E the vector space of random variables with finite second moment and mean zero. Define $\langle X, Y\rangle = \mathrm{Cov}(X, Y)$ and, for any event $A \in \mathcal{A}$, $Z_A = I_A - \mathbb{P}(A)$ where I_A is the indicator variable of A.

(a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on E. What is the induced norm?

Solution.

Let us recall that, for X and Y random variables, we define the covariance by:

$$\mathrm{Cov}(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))$$

Relation from which we deduce by some algebra:

$$\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Over E, it simplifies to $\mathrm{Cov}(X, Y) = \mathbb{E}(XY)$

Let us check that $X, Y \mapsto \langle X, Y\rangle$ is an inner product on E, the vector space of real random variables with zero mean and finite second moment.

$\mathrm{Cov}(\cdot, \cdot)$ is an application from $E \times E$ to $\mathbb{R}$.

Let $X, Y, Z \in \mathbb{E}$ and $\lambda \in \mathbb{R}$.

i) *(Symmetry)* $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$, clearly, using the symmetry in X, Y in the definition of the covariance.

ii) *(Linearity)* $\mathrm{Cov}(\lambda X, Y) = \mathbb{E}((\lambda X - \mathbb{E}(\lambda X))(Y - \mathbb{E}(Y))) = \lambda \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = \lambda \mathrm{Cov}(X, Y)$, by linearity of the expectation.

   Besides: $\mathrm{Cov}(X + Z, Y) = \mathbb{E}((X + Z - \mathbb{E}(X + Z))(Y - \mathbb{E}(Y))) = \mathbb{E}((X - \mathbb{E}(X) + Z - \mathbb{E}(Z))(Y - \mathbb{E}(Y))) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) + \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))) = \mathrm{Cov}(X, Y) + \mathrm{Cov}(Z, Y)$.

iii) *(Positive)* $\mathrm{Cov}(X, X) = \mathbb{E}((X - \mathbb{E}(X))^2) \geqslant 0$ by positivity of the random variable $(X - \mathbb{E}(X))^2$ and monotony of the expectation.

iv) *(Definite)* $\mathrm{Cov}(X, X) = 0$ *iff* $(X - \mathbb{E}(X))^2 = 0$ since it is a positive random variable. This implies that$X - \mathbb{E}(X) = 0$, i.e. $X = 0$ since $\mathbb{E}(X) = 0$ by definition of E.

With these elements, $\mathrm{Cov}(\cdot, \cdot)$ is an inner product on E. The induced norm is:

$$\|X\| = \sqrt{\mathbb{E}(X^2)}$$

(b) Show that $|\rho(X,Y)| \leq 1$ (correlation coefficient).

Solution.

By definition, $\rho(X,Y) = \frac{\langle X,Y \rangle}{\|X\|\|Y\|}$. But $\langle X,Y \rangle \leq \|X\| \, \|Y\|$ by Cauchy-Schwarz inegality, so $\rho \leq 1$.

(c) Show that if $X,Y \in E \setminus \{0\}$ are probabilistically independent then X,Y are linearly independent within E. Converse?

Solution.

Let us reason by contraposition. Let us assume that X,Y are not linearly independent, i.e. there exists $\lambda \in \mathbb{R}$ such that $Y = \lambda X$. Necessarily, $\lambda \neq 0$ since $X,Y \neq 0$.

So for $x,y \in \mathbb{R}$, $\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x, X = \frac{y}{\lambda})$. Therefore, $\mathbb{P}(X = x, Y = y) = 0$ if $y \neq \frac{x}{\lambda}$ and $\mathbb{P}(X = x)$ else. In that second case, we see that $\mathbb{P}(X = x, Y = y) \neq \mathbb{P}(X = x)\mathbb{P}(Y = y) = \mathbb{P}(X = x)^2$, unless X is constant equal to x which is impossible by assumption since $X \neq 0$ and $\mathbb{E}(X) = 0$ by definition of the vector space E.

Therefore, X and Y are not probabilistically independent.

This proves the direct implication over E.

The converse is false. As a counterexample, consider the random variable X taking values in the set $\{-1,0,1\}$ with equal probabilities $\frac{1}{3}$, and take $Y = X^2$.

Clearly, X and Y are linearly independent (since the probability affected to the outcome -1 is non zero in the law of X). However, X and Y are probabilistically dependent since for instance $\mathbb{P}(X = 0, Y = 1) = 0 \neq \mathbb{P}(X = 0) \cdot \mathbb{P}(Y = 1) = \frac{1}{3} \cdot \frac{2}{3}$.

(d) Let $X,Y \in E \setminus \{0\}$. What is the statistical interpretation of the orthogonal projection of Y on $\mathrm{Span}(X)$?

Solution.

The orthogonal projection of Y on the line $\mathrm{Span}(X)$ is exactly given by:

$$\Pi_{\mathrm{Span}(X)}(Y) = \frac{\langle X,Y \rangle}{\|X\|} X = \frac{\mathrm{Cov}(X,Y)}{Var(X)} X$$

This orthogonal projection is a random variable linearly dependent with X, which has the same correlation with X as Y has. Additionally, $\frac{\mathrm{Cov}(X,Y)}{Var(X)}$ is the λ coefficient that minimizes the variance of any random variable $Y - \lambda X$.

In statistical terms, it means that if we wanted to predict Y using X with a linear model, i.e. $\hat{Y} = \alpha X + \beta$, $\frac{\mathrm{Cov}(X,Y)}{Var(X)}$ would be the α coefficient that minimizes the norm of the prediction error $\epsilon = \mathbb{E}((\hat{Y} - Y)^2)$.

Indeed the first order conditions of this simple minimization problem are:

$$\frac{\partial \epsilon}{\partial \alpha} = -2\mathbb{E}((\hat{Y}-Y)\frac{\partial \hat{Y}}{\partial \alpha}) \text{ and } \frac{\partial \epsilon}{\partial \beta} = -2\mathbb{E}((\hat{Y}-Y)\frac{\partial \hat{Y}}{\partial \beta})$$

which can be rewritten:

$$\mathbb{E}((\hat{Y}-Y)X) = 0 \text{ and } \mathbb{E}(\hat{Y}-Y) = 0$$

This means that the prediction error $\hat{Y}-Y$ has to be orthogonal to X and that its expected value must be zero. Note that the first order conditions correspond to a minimum given the positive-definiteness of the Hessian.

Substituting $\hat{Y} = \alpha X + \beta$, we obtain $\alpha = \frac{\text{Cov}(X,Y)}{Var(X)}$ and $\beta = \mathbb{E}(Y) - \frac{\text{Cov}(X,Y)}{Var(X)}\mathbb{E}(X)$. As stated above, it means that $\frac{\text{Cov}(X,Y)}{Var(X)}X$ is the closest random variable (i.e. the one that minimizes the variance of the prediction error) to Y in Span(X).

Besides, the prediction error is given by $\epsilon = (1-\rho^2)Var(Y)$. If $\rho = \pm 1$, a perfect prediction can be made. When $\rho = 0$, the variance in the prediction is as large as the variation in Y, which implies that the predictor is very bad. For intermediate values of ρ, the predictor reduces the error.

(e) Verify that $Z_A \in E$ and calculate $\langle Z_A, Z_B \rangle$ for any $A, B \in \mathcal{A}$. When is $Z_A \perp Z_B$? Are $Z_A, Z_{\overline{A}}$ linearly independent in E?

Solution.

It is a fact that $Z_A \in E$ for any $A \in \mathcal{A}$ since this is a random variable with first and second moment respectively such that: $\mathbb{E}(Z_A) = \mathbb{E}(I_A - \mathbb{P}(A)) = \mathbb{E}(I_A) - \mathbb{P}(A) = 0$ and $\mathbb{E}(Z_A^2) = \mathbb{E}(I_A^2 - 2\mathbb{P}(A)I_A + \mathbb{P}(A)^2) = \mathbb{E}(I_A^2) - 2\mathbb{P}(A)\mathbb{E}(I_A) + \mathbb{P}(A)^2 = \mathbb{P}(A)(1-\mathbb{P}(A))$, so $\mathbb{E}(Z_A^2) \leq \infty$ since $I_A^2 = I_A$.

Besides, for $A, B \in \mathcal{A}$, $\langle Z_A, Z_B \rangle = \langle I_A - \mathbb{P}(A), I_B - \mathbb{P}(B) \rangle = \langle I_A, I_B \rangle - \langle I_A, \mathbb{P}(B) \rangle - \langle \mathbb{P}(A), I_B \rangle + \langle \mathbb{P}(A), \mathbb{P}(B) \rangle = \langle I_A, I_B \rangle$

So, $\langle Z_A, Z_B \rangle = \mathbb{E}((I_A - \mathbb{P}(A))(I_B - \mathbb{P}(B))) = \mathbb{E}(I_{A\cap B} - \mathbb{P}(B)I_A - \mathbb{P}(A)I_B + \mathbb{P}(A)\mathbb{P}(B))$.

Id est, $\langle Z_A, Z_B \rangle = \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)$.

We can see that $Z_A \perp Z_B$ when $\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B)$, that is, when events A and B are stochastically independent.

$Z_A, Z_{\overline{A}}$ are not linearly independent in E since for any $Z_{\overline{A}} = 1 - Z_A$.

(f) Suppose $A, B \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ are disjoint, and $B \neq \overline{A}$. Show that Z_A, Z_B are linearly independent.

Solution.

Let $A, B \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ such that $A \cap B = \emptyset$. Let $\lambda, \mu \in \mathbb{R}$ such that $\lambda Z_A + \mu Z_B = 0$.

Since $A \cap B = \emptyset$, plugging in any two elementary events $x \in A$ and $y \in B$ yields:

$$\begin{cases} -\lambda\mathbb{P}(A) + \mu(1-\mathbb{P}(B) = 0 \\ \lambda(1-\mathbb{P}(A)) - \mu\mathbb{P}(B) = 0 \end{cases}$$

Taking the difference between these equations, we obtain the condition: $\lambda = \mu$.

Therefore, $\lambda(Z_A + Z_B) = 0$. Since $B \neq \overline{A}$, this implies $\lambda = \mu = 0$.

Thus, Z_A and Z_B are linearly independent.

(g) Suppose $A \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ and define $B = \{\emptyset, A, \overline{A}, \Omega\} \subseteq A$. Let $Y = E(X|B)$. Verify that $Y \in E$ and show that Y is the orthogonal projection of X on $\text{Span}(Z_A)$.

Solution.

Note that B is a σ-Algebra, therefore it makes sense to consider $Y = \mathbb{E}(X|B)$.

Let us recall that Y is defined as the almost surely unique random variable such that for any bounded and B-measurable random variable U, $\mathbb{E}(XU) = \mathbb{E}(YU)$.

We have: $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(X|B)) = \mathbb{E}(X) = 0$ using the tower property of conditional expectation and the fact that $X \in E$.

Besides, $\mathbb{E}(Y^2) = \mathbb{E}(\mathbb{E}(X|B)^2) \leq \mathbb{E}(\mathbb{E}(X|\sigma(X))^2 \leq \mathbb{E}(X^2) \leq \infty$ since $X \in E$.

Let us check that Y is the orthogonal projection of X on $\text{Span}(Z_A)$. For that matter, let us show that it is such that for all $U \in \text{Span}(Z_A)$, $\langle X - Y, U \rangle = 0$.

Let $U \in \text{Span}(Z_A)$. Since Z_A is B-measurable, U is also B-measurable, and it follows that $\mathbb{E}(UX|B) = U\mathbb{E}(X|B)$.

Therefore: $\mathbb{E}(U\mathbb{E}(X|B)) = \mathbb{E}(\mathbb{E}(UX|B)) = \mathbb{E}(UX)$ i.e. $\langle U, Y \rangle = \langle U, X \rangle$. That is exactly $\langle U, Y - X \rangle = 0$.

Since this is true for any $U \in \text{Span}(Z_A)$, we conclude that $Y - X = \mathbb{E}(X|B) - X$ realizes the minimum distance from X to $\text{Span}(Z_A)$.

Since $(E, \text{Cov}(\cdot, \cdot))$ is a Hilbert space, the minimum distance is realized by the orthogonal projection. Thus, we have $Y = \Pi_{\text{Span}(Z_A)}(X)$.

Problem 6

Consider a vector space E equipped with a norm $N(x)$.

(a) Show that N is Euclidean (i.e. induced by some inner product) if and only if N satisfies the parallelogram law:

$$N(x+y)^2 + N(x-y)^2 = 2 \cdot [N(x)^2 + N(y)^2]$$

Hint: Define the inner product through N.

Solution.

If N is Euclidean, it derives from a certain inner product $\langle \cdot, \cdot \rangle$. And we have:

$$N(x+y)^2 + N(x-y)^2 = \langle x+y, x+y \rangle + \langle x-y, x-y \rangle = 2\langle x, x \rangle + 2\langle y, y \rangle + 2\langle x, y \rangle - 2\langle x, y \rangle$$

$$\text{Therefore, } N(x+y)^2 + N(x-y)^2 = 2 \cdot \left[N(x)^2 + N(y)^2\right]$$

Let's now assume that N checks the parallelogram law, and let us prove that the bilinear form $\langle \cdot, \cdot \rangle$ defined by polarization as follows is an inner product over E from which N derives:

$$\langle x, y \rangle = \frac{1}{4} \cdot \left[N(x+y)^2 - N(x-y)^2\right]$$

We clearly have, by homogeneity, $N(x) = \sqrt{\langle x, x \rangle}$.

$\langle \cdot, \cdot \rangle$ is an application from $E \times E$ to $\mathbb{R}$.

Let $x, y, z \in \mathbb{E}$ and $\lambda \in \mathbb{R}$.

i) *(Symmetry)* $\langle x, y \rangle = \langle y, x \rangle$, clearly, using the symmetry in x, y in its definition.

ii) *(Positive)* $\langle x, x \rangle = \frac{1}{4} \cdot N(2x)^2 = N(x)^2 \geqslant 0$ since $N(0) = 0$ by separation.

iii) *(Definite)* $\langle x, x \rangle = 0$ *iff* $\frac{1}{4}N(x)^2 = 0$ *iff* $x = 0$ by separation.

iv) *(Linearity)* Observe that:

$$\langle x+y, z \rangle + \langle x-y, z \rangle = \frac{1}{4}\left[N(x+y+z)^2 - N(x+y-z)^2 + N(x-y+z)^2 - N(x-y-z)^2\right]$$

Therefore,

$$\langle x+y, z \rangle + \langle x-y, z \rangle = \frac{1}{4}\left[2 \cdot \left[N(x+y)^2 - N(z)^2\right] + 2 \cdot \left[N(x-y)^2 + N(z)^2\right]\right]$$

Thus,

$$\langle x+y, z \rangle + \langle x-y, z \rangle = 2 \cdot \left[N(x)^2 + N(y)^2\right]$$

Which can be rewritten:

$$\langle x+y, z \rangle + \langle x-y, z \rangle = 2 \cdot \langle x, y \rangle$$

From which we deduce, taking $y = x$:

$$\langle 2 \cdot x, y \rangle = 2 \cdot \langle x, y \rangle$$

By induction, we prove that this it is true for any power n that:

$$\langle 2^n \cdot x, y \rangle = 2^n \cdot \langle x, y \rangle$$

Taking $u = \frac{1}{2}(x + y)$ and $v = \frac{1}{2}(x - y)$, we have:

$$\langle x + y, z \rangle = \langle 2 \cdot u, z \rangle = 2 \cdot \langle u, z \rangle = \langle u + v, z \rangle + \langle u - v, z \rangle = \langle x, z \rangle + \langle y, z \rangle$$

.

By induction we conclude that for any $n \in \mathbb{N}$ (we've seen that it's true for $n = 0$), and any $x, y \in E$ we must have:

$$\langle nx, y \rangle = n \langle x, y \rangle$$

We can extend the linearity to $\mathbb{Z}$ using $\langle -x, y \rangle = -\langle x, y \rangle$, and then to $\mathbb{Q}$ using $\langle x, y \rangle = n \cdot \langle \frac{1}{n} x, y \rangle$.

We conclude that the linearity is true over $\mathbb{R}$ using the well-known density of $\mathbb{Q}$ in $\mathbb{R}$ and the continuity of the application $(x, y) \mapsto \frac{1}{4} \cdot [N(x + y)^2 - N(x - y)^2]$, since the norm N is continuous.

Therefore, the linearity of $\langle \cdot, \cdot \rangle$ is true.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product over E from which N derives since $N(x) = \sqrt{\langle x, x \rangle}$.

Conclusion: N is a Euclidian norm if and only if it checks the parallelogram identity.

(b) Which of the following are Euclidean norms on $E = R^n$?

$$N_p(x) = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

$$Q_p(x) = \sqrt{N_2(x)^2 + \frac{1}{p} \sum_{i<j} x_i x_j}$$

where $p \in [1, \infty)$.

Solution.

Given the previous question, we just need to verify if these norms check the parallelogram identity to know if they are Euclidian.

[TODO] Since $p \geqslant 1$, both N_p and Q_p are Euclidian norms.

(c) Define $A^2(x, y) = [N(x) \cdot N(y)]^2 - [N(x + y)^2 - N(x - y)^2]^2/16$. Show that if N is Euclidean then $A^2(x, y) \geq 0$ and $A(x + y, x - y) = 2 \cdot A(x, y)$.

Geometric interpretation for $E = R^2$?

Solution.

[TODO]