2801.001 Spring 2018 Homework 1

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Submission instructions: Same groups as HW1. Refer to instructions from TA.

Note: Problems 1 and 2 require the datafile posted online which contains prices and implied volatilities of one-year (349 calendar days) listed options on the Nasdaq 100 index (NDX), as well as other market data.

Problem 1

(a) Estimate the market price of the 5% call spread (i.e. with strikes ATM and 5% OTM). What about the 5% put spread?

Solution.

The Nasdaq 100 data provided allows us to get the buy and sell prices of ATM (strike 5050) and 5% OTM (strikes 5200 and 5750) call and put prices (we use midpoints).

- Selling ATM Call: 337.1.
- Selling ATM Put: 326.1.
- Buying 5% OTM Call: 478.1.
- Buying 5% OTM Put: 428.3.

Therefore, the buy prices of the 5% call and put spreads are:

- 5% call spread: $478.1 - 337.1 = 141.0$.
- 5% put spread: $428.3 - 326.1 = 100.2$.

(b) If you were to price the spreads in the Black-Scholes model using a single volatility parameter σ, what value of σ would match the theoretical price with the market price? Comment on your results.

Solution.

Applying a Black-Scholes pricing formula would lead to the following theoretical prices for the 5% call spread (5% put spread resp.):

$$\begin{aligned}
CS(K, 0.95K) &= C(0.95K) - C(K) \\
&= S_t N(d_1(0.95K)) - 0.95Ke^{-r(T-t)}N(d_2(0.95K)) \\
&\quad - (S_t N(d_1(K)) - Ke^{-r(T-t)}N(d_2(K)))
\end{aligned}$$

$$\begin{aligned}
PS(K, 1.05K) &= P(1.05K) - P(K) \\
&= 1.05Ke^{-r(T-t)}N(-d_2(1.05K)) - S_t N(-d_1(1.05K)) \\
&\quad - (Ke^{-r(T-t)}N(-d_2(K)) - S_t N(-d_1(K)))
\end{aligned}$$

where

$$\begin{aligned}
d_1(K) &= \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \\
d_2(K) &= d_1 - \sigma\sqrt{T-t}
\end{aligned}$$

σ is the implied volatility that one can choose to equate theoretical prices to market prices.

Using a solver, we get $\sigma_{CS} = 14.13\%$ using the 5% call spread and $\sigma_{PS} = 17.10\%$ using the 5% put spread.

These levels are relatively close to the implied volatility quoted for ATM (15.4 %) and 5% OTM options (14% to 16.7%)

Problem 2

(a) Using the numerical package of your choice, calibrate the parameters of the SVI model against the market-implied volatility data. Show a comparative graph of the SVI curve and the actual implied volatility data points.

Solution.

The SVI model reads:

$$\sigma^* = \sqrt{a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + s^2}\right)}$$

To calibrate its parameters (a, b, ρ, m, s) we perform the following least square optimization against market data, with the following constraints to avoid arbitrage:

$$\begin{aligned}
\min_{a,b,\rho,m,s} \quad & \sum_{i=1}^{n} \left[\sigma^{\star 2}_{\text{SVI}}(k_i, T; a, b, \rho, m, s) - \sigma^{\star 2}_{\text{Market}}(k_i, T)\right]^2 \\
\text{subject to} \quad & a, b \geq 0, \\
& -1 \leq \rho \leq 1, \\
& s > 0, \\
& b(1 + |\rho|)T \leq 4.
\end{aligned}$$

Using `scipy.minimize`, we obtain:

$$a = 0.000364, b = 2.55, \rho = 0.961, m = 0.239, s = 0.0149$$

(b) Compute or estimate the price of an at-the-money digital call option paying off \$1 if in one year NDX is greater than its current spot level, and zero otherwise: (i) in the Black-Scholes model, (ii) using $\pm 1\%$ call spreads, (iii) using the smile-adjusted formula on page 19.

Solution.

TODO

(c) Graph the implied distribution corresponding to the SVI model calibration.

Solution.

TODO

(d) Use the implied distribution to compute the price of the following European exotic options, where X_0 is the current index level and X_T is the final index level:

Solution.

(i) Digital call defined in question (b);

TODO

(ii) "Reverse convertible" paying off $\max\left(100\%, 100\% + p \times \frac{X_T - X_0}{X_0}\right)$ if $\frac{X_T}{X_0} > 75\%$ and $\frac{X_T}{X_0}$ otherwise, where $p = 50\%$. Then solve for p to get a price of 100%;

TODO

(iii) Option paying off $\max\left(0, \frac{X_T - X_0}{X_T}\right)$;

TODO

(iv) Log-contract paying off $-2\log\left(\frac{X_T}{X_0}\right)$. Price interpretation;

TODO

Problem 3

Find conditions on the SVI model parameters to satisfy Lees asymptotic bounds on p. 22:

$$\sigma^{\star^2}(k_F, T) \leq \frac{\beta}{T}|\log k_F|,\ \beta \in [0, 2]$$

Solution.
TODO

Problem 4

(Problem 4.3 p. 56 in textbook, with corrections): Consider an underlying stock S currently trading at $S_0 = 100$ which does not pay any dividend. Assume the local volatility function is $\sigma_{loc}(t, S) = \frac{0.1 - 0.15 \times \log\left(\frac{S}{S_0}\right)}{\sqrt{t}}$, and that interest rates are zero.

(a) Produce the graph of the local volatility surface for spots 0 to 200 and maturities 0 to 5 years.

Solution.

TODO

(b) Write a Monte-Carlo algorithm to price the following 1-year payoffs using 252 time steps and e.g. $10,000$ paths:

Solution.

(i) "Capped quadratic" option: $\min\left(1, \frac{S_1^2}{S_0^2}\right)$;

TODO

(ii) Asian at-the-money-call: $\max\left(0, \frac{S_{0.25}+S_{0.5}+S_{0.75}+S_1}{4 \times S_0} - 1\right)$;

TODO

(iii) Barrier call: $\max(0, S_1 - S_0)$ if S always traded above 80 using 252 daily observations, 0 otherwise;

TODO

Problem 5

The payoff of a 1-year at-the-money call on the geometric average return of two non-dividend paying stocks X, Y is given as:

$$f(X_T, Y_T) = \max\left(0, \sqrt{\frac{X_T Y_T}{X_0 Y_0}} - 1\right)$$

where $T = 1$ year and X_t, Y_t are the respective underlying spot prices of X, Y at any time t.

(a) Derive analytical formulas for the call value at any time $0 \le t \le T$ in the Black-Scholes model with constant correlation ρ (cf. Section $6-4$ in the textbook, to be covered during Session 5.)

Solution.

TODO

(b) Compute the value of the call using a 5% interest rate, 20% volatility for X, 30% volatility for Y, and $\rho = 0.4$. Use finite differences to estimate the deltas, gammas and cross-gamma of the call.

Solution.

TODO

(c) You purchased the call on a \$10,000,000 notional. What actions would you take to delta-hedge your position? What would then be your instant $P\&L$ in the following matrix of scenarios. Generally, graph your instant $P\&L$ against percent changes x, y in underlying stock prices.

Solution.

TODO

X \ Y	-5%	$+1\%$	$+5\%$
-5%			
$+1\%$			
$+5\%$			