Submission instructions: Groups of three to four, due 14 calendar days after the homework is posted, in electronic format. Submission must contain: (a) master answer sheet (typewritten or handwritten) with header containing your names and NetIDs, and (b) all relevant computer files such as code source, Excel files etc.

HW1-1: You are short 100 at-the-money call option contracts on the S&P 500 index expiring in one year with a contract multiplier of 100. The current index level is 2600, the interest and dividend rates are zero. Simulate the evolution of the index level at periods Δt as a geometric Brownian motion with volatility σ (free parameter) and calculate the corresponding call value, delta, gamma and theta using a fixed 20% implied volatility. Then simulate your actual cumulative P&L when periodically delta-hedging your position (assuming you can trade the index as an asset) and compare it against the proxy formula on slide 11 for the following matrix of parameters:

σ \ Δt	Monthly (12 per year)	Weekly (52 per year)	Daily (252 per year)
$\sigma = 25\%$			
$\sigma = 20\%$			
$\sigma = 15\%$			

Provide a statistical analysis of your results over 10,000 simulations, where one simulation is an entire index path.

HW1-2: Use your knowledge of how the VIX is calculated to show that the VIX is not the price of an investable asset. What about the square of the VIX?

HW1-3: On March 29, 2018 the S&P 500 index (SPX) is 2611.53 and the 12-month VIX is 21.28. The implied volatility smile of SPX options expiring on March 15, 2019 is given as:

$$\sigma^*(x) = \sqrt{a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + s^2}\right)}$$

where $a = 0.009, b = 0.11, \rho = -0.12, m = 0.2, s = 0.05$, $x = \ln\frac{K}{F}$ is log-moneyness and $F = 2625.10$ is the forward price. The continuous interest rate is 2.09% p.a.

(a) Draw the implied volatility smile curve for strikes $500 \leq K \leq 5000$.

(b) Calculate the faire strike K_{var} of a variance swap expiring on March 15, 2019 with the method of your choice. How close is your calculation to the 12-month VIX? Why is it not exactly the same?

HW1-4: Consider a real symmetric matrix A. Show that $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ is symmetric positive-definite. *Hint: Use spectral decomposition*.

HW1-5: Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and E the vector space of random variables with finite second moment and mean zero. Define $\langle X, Y\rangle = \text{Cov}(X, Y)$ and, for any event $A \in \mathcal{A}$, $Z_A = I_A - \mathbb{P}(A)$ where I_A is the indicator variable of A.

(a) Show that $\langle\cdot,\cdot\rangle$ is an inner product on E. What is the induced norm?

(b) Show that $|\rho(X, Y)| \leq 1$ (correlation coefficient).

(c) Show that if $X, Y \in E \setminus \{0\}$ are probabilistically independent then X, Y are linearly independent within E. Converse?

(d) Let $X, Y \in E \setminus \{0\}$. What is the statistical interpretation of the orthogonal projection of Y on Span(X)?

(e) Verify that $Z_A \in E$ and calculate $\langle Z_A, Z_B \rangle$ for any $A, B \in \mathcal{A}$. When is $Z_A \perp Z_B$? Are $Z_A, Z_{\bar{A}}$ linearly independent in E?

(f) Suppose $A, B \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ are disjoint, and $B \neq \bar{A}$. Show that Z_A, Z_B are linearly independent.

(g) Suppose $A \in \mathcal{A} \setminus \{\emptyset, \Omega\}$ and define $\mathcal{B} = \{\emptyset, A, \bar{A}, \Omega\} \subseteq \mathcal{A}$. Let $Y = \mathbb{E}(X|\mathcal{B})$. Verify that $Y \in E$ and show that Y is the orthogonal projection of X on $\mathrm{Span}(Z_A)$.

HW1-6 (optional & not scored): Consider a vector space E equipped with a norm $N(x)$.

(a) Show that N is Euclidean (i.e. induced by some inner product) if and only if N satisfies the parallelogram law:

$$N(x+y)^2 + N(x-y)^2 = 2[N(x)^2 + N(y)^2]$$

Hint: Define the inner product through N.

(b) Which of the following are Euclidean norms on $E = \mathbb{R}^n$? $N_p(x) = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$, $Q_p(x) = \sqrt{N_2(x)^2 + \frac{1}{p}\sum_{i<j} x_i x_j}$, where $p \in [1, \infty)$.

(c) Define $A^2(x,y) = [N(x)N(y)]^2 - [N(x+y)^2 - N(x-y)^2]^2/16$. Show that if N is Euclidean then $A^2(x,y) \geq 0$ and $A(x+y, x-y) = 2A(x,y)$. Geometric interpretation for $E = \mathbb{R}^2$?

