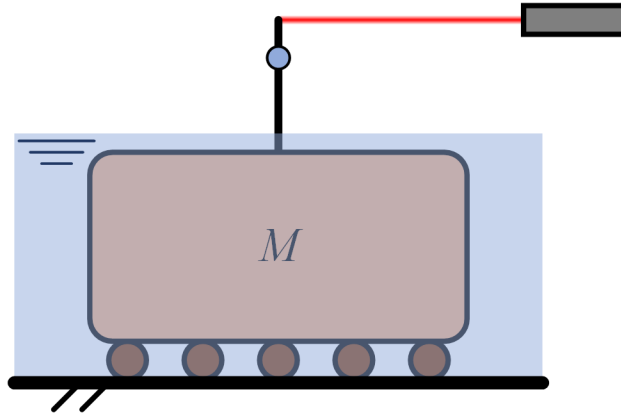


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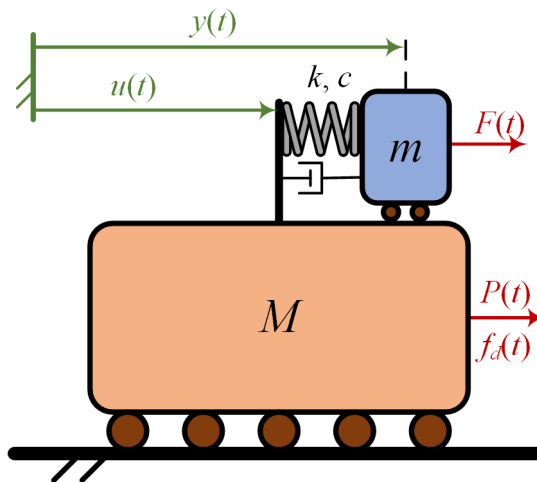
MATLAB[®] PROBLEM SET 3

Consider the following dynamical system below.



A large mass (M) is moving underwater on a track due to external loads, and we want to know its instantaneous position. To resolve the problem, the engineers attached a relatively small cantilever beam onto which an optical displacement sensor shines a laser. A small mass (blue circle) is attached to the cantilever. The figure shown below is the mathematical model of the system where the beam is modeled as a spring (k) and a linear dashpot/damper (c), and the small mass is modeled as a point mass (m). The absolute positions (with respect to the global coordinates) of M and m are $u(t)$ and $y(t)$, respectively. In addition, due to the unique shape of the mass (M) and the fluid, the following drag force acts on the large mass:

$$f_d = -c_{d1}\dot{u} - c_{d2}\dot{u}^2 - c_{d3}\dot{u}^3 \quad (1)$$



The stiffness, mass, and damping of the beam, having a rectangular cross-section, are:

$$k = \frac{6EI}{a^2(3L-a)}, \quad m = \rho(bhL), \quad c = \alpha m + \beta k, \quad I = \frac{bh^3}{12} \quad (2)$$

where E is Young's modulus and I is the beam's second moment of area. The beam has a cross-section with dimensions b and h , its length is L , and the Rayleigh damping coefficient c is proportional to its mass and stiffness (proportional factors α and β are given below). The stiffness, k , depends on the position a of the mass m along the beam.

The engineers want to impress their boss and design the best system (beam + mass) they can, by achieving good tracking of M with minimal error. To do so, they intend to treat the problems in two stages:

- 1) In the first stage, only the step response of the cantilever is studied (i.e., the input is the force acting on m while the large mass M is held, $u = 0$). Therefore, they require the following: (I) minimal overshoot (OS), (II) minimal rise time (t_r), and (III) minimal settling time (t_s).
- 2) In the second stage, they will simulate the whole system and assess the performance of their design. **Notice** that the whole system is nonlinear.

The engineers can choose between 3 different materials for the rod of dimensions:

$$L = 0.2 \text{ m}, a = 0.9L, b = 0.1L, h = 0.2L$$

Steel: $E_S = 200 \times 10^9 \text{ Pa}, \rho_S = 7800 \text{ kg/m}^3, \alpha_S = 0.35 \text{ s}^{-1}, \beta_S = 5.9 \times 10^{-4} \text{ s}$

Aluminum: $E_A = 70 \times 10^9 \text{ Pa}, \rho_A = 2700 \text{ kg/m}^3, \alpha_A = 2 \text{ s}^{-1}, \beta_A = 1 \times 10^{-4} \text{ s}$

Epoxy: $E_E = 5 \times 10^9 \text{ Pa}, \rho_E = 1000 \text{ kg/m}^3, \alpha_E = 4 \text{ s}^{-1}, \beta_E = 8 \times 10^{-4} \text{ s}$

Questions

- 1) Derive the kinetic and potential energies of the system.

The equations of motion of the system are given by:

$$m\ddot{y} + c\dot{y} + ky = c\dot{u} + ku + F$$

$$M\ddot{u} + c\dot{u} + c_{d1}\dot{u} + c_{d2}\dot{u}^2 + c_{d3}\dot{u}^3 + ku = c\dot{y} + ky + P$$

Stage 1 – Consider only the equation of motion of the beam.

- 2) Derive the equation of motion of the beam for fixed mass M .
- 3) Derive the transfer function, where the input is the force F and the output is the target's position (y), where $y(0) = \dot{y}(0) = 0$.
- 4) Derive the standard form of the second-order equation of motion by answering the following:
 - a. Derive the standard form of the transfer function, i.e., use the parameters ω_n (undamped natural frequency) and ζ (damping ratio).
 - b. Express ω_n using m , k , and c .
 - c. Express ζ using m , k , and c .
 - d. Rewrite the equation of motion (in the time domain) using the parameters ω_n , m , and ζ .
- 5) Derive the state-space representation using ω_n , m , and ζ . Use the following state vector:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

Use the following parameters questions 6) and 7):

$$m = 35 \times 10^{-3} \text{ kg}, \quad k = 16 \times 10^5 \text{ N/m}, \quad c = 32 \text{ Ns/m}.$$

- 6) Using the analytical formulas, calculate the undamped natural frequency, damping ratio, maximum percent overshoot, rise time (0%-100%), and settling time (2%).
 - 7) Using both the Transfer Function (TF) and State-Space (SS) models, compute the natural frequencies and damping ratios using `damp`. Compute the overshoot, rise, and settling time using `stepinfo`.
 - a. Present the results in a table and compare them with the analytical results.
 - b. Plot the step response with OS , t_r , and t_s highlighted and comment on the behavior of the system. Is this response suitable for the problem which the engineers are trying to solve.
- Hint:** The results of TF and SS should be identical and close to the ones computed in 5).
- 8) Going back to the problem the engineers wish to optimize, they decided to compare the performance of the three materials proposed. In the end, you should choose the material (steel, aluminum, or epoxy) yielding the best performance.
 - a) Plot the step responses for the three materials with OS , t_r , and t_s highlighted. Comment on the behavior of the three materials and how they relate to the parameters of the dynamical system.
 - b) Conclude on the best material for the task of tracking the cart.

Stage 2 – Consider the whole nonlinear system.

The values of M and c_d are as follows:

$$M = 3 \text{ kg}, \quad c_{d1} = 7 \text{ Ns/m}, \quad c_{d2} = 130 \text{ Ns}^2/\text{m}^2, \quad c_{d3} = 1650 \text{ Ns}^3/\text{m}^3.$$

- 9) Perform order reduction to the governing equations of motion of the whole system, using the following state vector: $\mathbf{z} = (y \quad u \quad \dot{y} \quad \dot{u})^T$
- 10) For the parameters you have chosen in 8), simulate the dynamical response of the system to an impulse acting on M . Model the impulse as follows:

$$P(t) = \begin{cases} A \sin(2\pi t / T) & 0 \leq t < T/2 \\ 0 & t \geq T/2 \end{cases}, \quad A = 4.5 \text{ kN}, \quad T = 10 \mu\text{s}.$$

Complete the `EX_3_NNode_2025` function.

- a. Plot the response for a positive impulse ($P(t)$) and negative impulse ($-P(t)$).
 - b. Comment on the outcome and the difference between the two impulses. What can you conclude about the performance of the selected material when simulating the whole system?
- 11) Estimate the tracking performance of the system.
 - a. Plot the dynamic responses and the relative error in the first two milliseconds as a function of time. What is the maximum relative error for each case? When is it found?
 - b. How long does it take for the relative error to go below 10% of the maximum value? What would happen to the error after a longer period?
 - c. Are these values acceptable? Do you think a better performance can be obtained for different values? How?