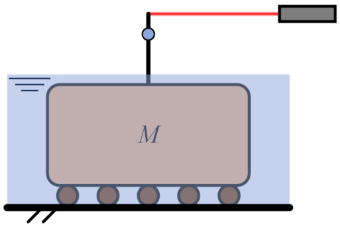
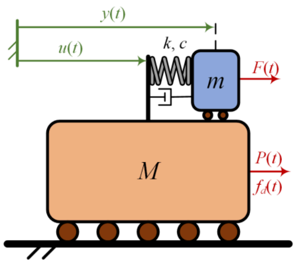
ME – 221
MATLAB® PROBLEM SET 3

Consider the following dynamical system below.



A large mass (M) is moving underwater on a track due to external loads, and we want to know its instantaneous position. To resolve the problem, the engineers attached a relatively small cantilever beam onto which an optical displacement sensor shines a laser. A small mass (blue circle) is attached to the cantilever. The figure shown below is the mathematical model of the system where the beam is modeled as a spring (k) and a linear dashpot/damper (c), and the small mass is modeled as a point mass (m). The absolute positions (with respect to the global coordinates) of M and m are $u(t)$ and $y(t)$, respectively. In addition, due to the unique shape of the mass (M) and the fluid, the following drag force acts on the large mass:

$$f_d = -c_{d1}\dot{u} - c_{d2}\dot{u}^2 - c_{d3}\dot{u}^3 \tag{1}$$



The stiffness, mass, and damping of the beam, having a rectangular cross-section, are:

$$k = \frac{6EI}{a^2(3L-a)}, \quad m = \rho(bhL), \quad c = \alpha m + \beta k, \quad I = \frac{bh^3}{12} \tag{2}$$

where E is Young's modulus and I is the beam's second moment of area. The beam has a cross-section with dimensions b and h, its length is L, and the Rayleigh damping coefficient c is proportional to its mass and stiffness (proportional factors α and β are given below). The stiffness, k, depends on the position a of the mass m along the beam.

The engineers want to impress their boss and design the best system (beam + mass) they can, by achieving good tracking of M with minimal error. To do so, they intend to treat the problems in two stages:

1) In the first stage, only the step response of the cantilever is studied (i.e., the input is the force acting on m while the large mass M is held, u = 0). Therefore, they require the following: (I) minimal overshoot (OS), (II) minimal rise time (t_r), and (III) minimal settling time (t_s).
2) In the second stage, they will simulate the whole system and assess the performance of their design. **Notice** that the whole system is nonlinear.

The engineers can choose between 3 different materials for the rod of dimensions:

L=0.2 m, a = 0.9L, b= 0.1L, h = 0.2L

Steel: $E_S = 200\times10^9$ Pa, $\rho_S = 7800\,\text{kg/m}^3$, $\alpha_S = 0.35\,\text{s}^{-1}$, $\beta_S = 5.9\times10^{-4}$ s

Aluminum: $E_A = 70\times10^9$ Pa, $\rho_A = 2700\,\text{kg/m}^3$ $\alpha_A = 2\,\text{s}^{-1}$, $\beta_A = 1\times10^{-4}$ s

Epoxy: $E_E = 5\times10^9$ Pa, $\rho_E = 1000\,\text{kg/m}^3$. $\alpha_E = 4\,\text{s}^{-1}$, $\beta_E = 8\times10^{-4}$ s

Questions

1) Derive the kinetic and potential energies of the system.

Kinetic NRG :

M : position $u(t)$, velocity $\dot{u}(t)$

m : position $y(t)$, velocity $\dot{y}(t)$

$T = \frac{1}{2}M\dot{u}^2 + \frac{1}{2}m\dot{y}^2$

Potential NRG :

Spring relative displacement : $y(t) - u(t)$

$V_{spring} = \frac{1}{2}k(y-u)^2$

The equations of motion of the system are given by:

$$m\ddot{y} + c\dot{y} + ky = c\dot{u} + ku + F$$
$$M\ddot{u} + c\dot{u} + c_{d1}\dot{u} + c_{d2}\dot{u}^2 + c_{d3}\dot{u}^3 + ku = c\dot{y} + ky + P$$

Stage 1 – Consider only the equation of motion of the beam.

2) Derive the equation of motion of the beam for fixed mass M.

fixed mass M : $\dot{u} = \ddot{u} = 0$ & $u = 0$ (assignment)

$$\begin{cases} m\ddot{y} + c\dot{y} + ky = f \\ 0 = c\dot{y} + ky + P \end{cases}$$

$$\begin{cases} m\ddot{y} + c\dot{y} + ky = f \\ m\ddot{y} = f - P \end{cases}$$

3) Derive the transfer function, where the input is the force F and the output is the target's position (y), where $y(0) = \dot{y}(0) = 0$.

$m\ddot{y} + c\dot{y} + ky = f$ gives us $ms^2 Y(s) + csY(s) + kY(s) = f(s)$

$$Y(s)\,(ms^2 + cs + k) = f(s)$$

Transfer function : $H(s) = \dfrac{Y(s)}{F(s)} = \dfrac{1}{ms^2 + cs + k}$

4) Derive the standard form of the second-order equation of motion by answering the following:
   a. Derive the standard form of the transfer function, i.e., use the parameters ω_n (undamped natural frequency) and ζ (damping ratio).
   b. Express ω_n using m, k, and c.
   c. Express ζ using m, k, and c.
   d. Rewrite the equation of motion (in the time domain) using the parameters ω_n, m, and ζ.

a. Transfer function : $H(s) = \dfrac{1}{ms^2 + cs + k}$

$$H(s) = \frac{1/m}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

$$H(s) = \frac{1/m}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

b. $\omega_0 = \sqrt{\frac{k}{m}}$ natural frequency

c. $\zeta = \frac{c}{2\sqrt{km}}$ damped atio

d. Thus eq. motion : $m\ddot{y} + c\dot{y} + ky = f(t)$

$$\ddot{y} + 2\xi\omega_0\dot{y} + \omega_0^2 y = \frac{1}{m}f(t)$$

5) Derive the state-space representation using ω_n, m, and ζ. Use the following state vector:

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$$

Eq. motion : $\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = \frac{1}{m}f(t)$

$$\begin{cases} z_1 = y \\ z_2 = \dot{y} \end{cases}$$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -2\zeta\omega_0 z_2 - \omega_0^2 z_1 + \frac{1}{m}F(t) \end{cases}$$

$$\dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{pmatrix}}_{A} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}}_{B} f(t)$$

$$y = \underbrace{(1 \quad 0)}_{C} z \qquad D = 0$$

- $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{pmatrix}$
- $\mathbf{B} = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$
- $\mathbf{C} = (1 \quad 0)$
- $\mathbf{D} = 0$

Use the following parameters questions 6) and 7):

$$m = 35\times10^{-3}\text{ kg},\quad k = 16\times10^{5}\text{ N/m},\quad c = 32\text{ Ns/m}.$$

6) Using the analytical formulas, calculate the undamped natural frequency, damping ratio, maximum percent overshoot, rise time (0%-100%), and settling time (2%).

Natural freq undamped :

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16\cdot10^5}{35\cdot10^{-3}}} = \sqrt{4.571\cdot10^7} \approx 6762,6 \text{ rad/s}$$

Damped ratio :

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{32}{2\sqrt{16\cdot10^5\cdot35\cdot10^{-3}}} \approx 0,0675$$

- **Delay time, t_d:** Time required for the response to reach half of the final value the very first time
- **Rise time, t_r:** Time required for the response to rise from 0% to 100% (underdamped system) or from 10% to 90% (overdamped system)
- **Peak time, t_p:** Time required for the response to reach the first peak of the overshoot.
- **Maximum percent overshoot, M_p**
- **Settling time, t_s:** Time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%)

Damped natural freq :

$$\bar{\omega} = \omega_0\sqrt{1-\zeta^2} \approx 6743,2$$

Attenuation : $a = \zeta\omega_0$

Overshoot max (%) :

$$OS = 100\cdot e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \approx 80,8\%$$

Rise time (0% - 100%) :

$$tr = -\frac{\sqrt{1-\zeta^2}}{\zeta}\tan^{-1}(\bar{\omega}) = -\frac{\bar{\omega}}{a}\tan^{-1}(\bar{\omega})$$

Settling time : 2% criterion :

$$\tau = \frac{1}{\zeta\omega_0} = \frac{1}{a}$$

$$t_s = 4\tau = \frac{4}{\zeta\omega_0} \approx 8,76\cdot10^{-5}\text{ s}$$

7) Using both the Transfer Function (TF) and State-Space (SS) models, compute the natural frequencies and damping ratios using `damp`. Compute the overshoot, rise, and settling time using `stepinfo`.

a. Present the results in a table and compare them with the analytical results.

b. Plot the step response with OS, t_r, and t_s highlighted and comment on the behavior of the system. Is this response suitable for the problem which the engineers are trying to solve.

Hint: The results of TF and SS should be identical and close to the ones computed in 5).

a.

```
%% Q7a_tf Define the transfer function
sys_tf = tf(1/m, [1 c/m k/m]); %COMPLETE
```

num (1/m) denum ([1 c/m k/m])

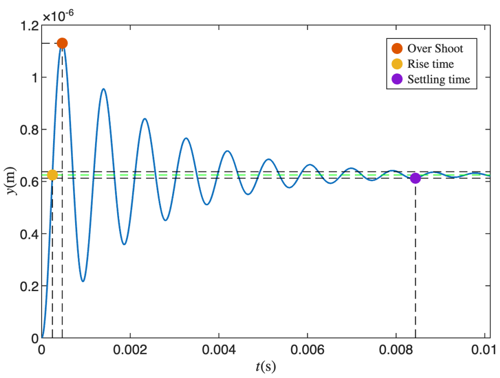
Normalized eq. : $tf = \dfrac{1}{ms^2 + cs + k} = \dfrac{1/m}{s^2 + \frac{c}{m}s + \frac{k}{m}}$

```
%Compute the natural frequency and damping ratio using damp
[wn_tf,zeta_tf] = damp(sys_tf);

%Compute the overshoot, rise and settling time with stepinfo
TF_step_info = stepinfo(sys_tf,'SettlingTimeThreshold',0.02,'RiseTimeLimits',[0 1]);
OS_tf = TF_step_info.Overshoot;
tr_tf = TF_step_info.RiseTime;
ts_tf = TF_step_info.SettlingTime;
```

```
%% Q7a_ss Define the state-space model
% State is z = [y ydot]

A_ss = [0 1; -w0^2 -2*zeta*w0];
B_ss = [0; 1/m];
C_ss = [1 0];
D_ss = 0;

sys_ss = ss(A_ss, B_ss, C_ss, D_ss);

%Compute the natural frequency and damping ratio using damp
[wn_ss,zeta_ss] = damp(sys_ss);

%Compute the overshoot, rise and settling time with stepinfo
SS_step_info = stepinfo(sys_ss,'SettlingTimeThreshold',0.02,'RiseTimeLimits',[0 1]);
OS_ss = SS_step_info.Overshoot;
tr_ss = SS_step_info.RiseTime;
ts_ss = SS_step_info.SettlingTime;
```

b.

```
%% Q7b Plot the step response
tt = linspace(0,1.2*ts_ss,1e4);

y_step = step(sys_ss, tt); %COMPLETE
```

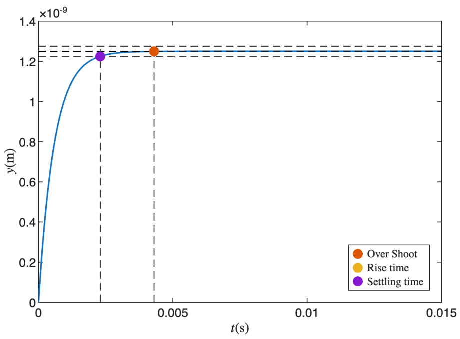


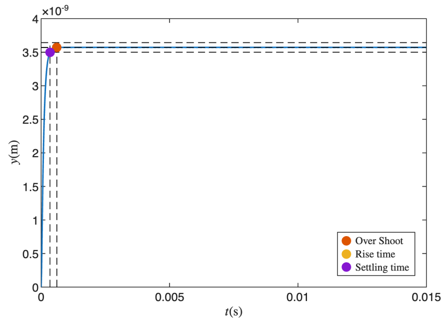
8) Going back to the problem the engineers wish to optimize, they decided to compare the performance of the three materials proposed. In the end, you should choose the material (steel, aluminum, or epoxy) yielding the best performance.

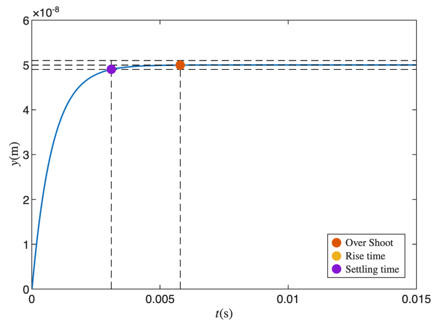
a) Plot the step responses for the three materials with OS, t_r, and t_s highlighted. Comment on the behavior of the three materials and how they relate to the parameters of the dynamical system.

b) Conclude on the best material for the task of tracking the cart.

```
%% Q8 Material choice
E = [200 70 5]*1e9;    % Pa
rho = [7800 2700 1000]; % kg/m^3
alpha = [0.35 2 4];
beta = [5.9 1 8]*10^-4;
L = 0.2; % m
b = 0.1*L; % m
h = 0.2*L; % m
a = 0.9*L; % m

%% Q8a Plot the step response of the best combination of materials.
tt = linspace(0,3e-3,2e3).';

%Compute array with the parameters for the three materials
m = rho .* b .* h .* a;
I = (1/12) * b * h^3;     Inertia
ks = (3 * E * I) ./ (L^3 * (1 - a/L).^2);
c = alpha .* m + beta .* ks;
w0 = sqrt(ks ./ m);
zetas = c ./ (2 .* sqrt(ks .* m));
```







Stage 2 – Consider the whole nonlinear system.

The values of M and c_d are as follows:

$$M = 3\text{ kg},\quad c_{d1} = 7\,\text{Ns/m},\quad c_{d2} = 130\text{Ns}^2/\text{m}^2,\quad c_{d3} = 1650\,\text{Ns}^3/\text{m}^3.$$

9) Perform order reduction to the governing equations of motion of the whole system, using the following state vector: $\mathbf{z} = \begin{pmatrix} y & u & \dot y & \dot u \end{pmatrix}^T$

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} y \\ u \\ \dot y \\ \dot u \end{bmatrix}$$

$$\begin{cases} \dot z_1 = z_3 = \dot y \\ \dot z_2 = z_4 = \dot u \\ \dot z_3 = \ddot y \\ \dot z_4 = \ddot u \end{cases}$$

The equations of motion of the system are given by:

$$m\ddot y + c\dot y + ky = c\dot u + ku + F$$

$$M\ddot u + c\dot u + c_{d1}\dot u + c_{d2}\dot u^2 + c_{d3}\dot u^3 + ku = c\dot y + ky + P$$

$$\begin{cases} m\ddot y + c\dot y + ky = \dot u + ku + F \\ M\ddot u + c\dot u + c_{d1}\dot u + c_{d2}\dot u^2 + c_{d3}\dot u^3 + ku = c\dot y + ky + P \end{cases}$$

gives :

$$\begin{cases} \ddot y = \frac{1}{m}\left[(\dot u - \dot y)c + (u - y)k + F\right] \\ \ddot u = \frac{1}{M}\left[(\dot y - \dot u)c + (y - u)k - c_{d1}\dot u - c_{d2}\dot u^2 - c_{d3}\dot u^3 + P\right] \end{cases}$$

$$\dot z = \begin{pmatrix} \dot z_1 \\ \dot z_2 \\ \dot z_3 \\ \dot z_4 \end{pmatrix} = \begin{pmatrix} z_3 \\ z_4 \\ \frac{1}{m}\left[(z_4 - z_3)c + (z_1 - z_2)k + F\right] \\ \frac{1}{M}\left[(z_3 - z_4)c + (z_1 - z_2)k - c_{d1}z_4 - c_{d2}z_4 - c_{d3}z_4^3 + P\right] \end{pmatrix}$$

10) For the parameters you have chosen in 8), simulate the dynamical response of the system to an impulse acting on *M*. Model the impulse as follows:

$$P(t)=\begin{cases} A\sin(2\pi t/T) & 0\le t<T/2 \\ 0 & t\ge T/2 \end{cases}, \quad A=4.5\,\text{kN}, \quad T=10\,\mu\text{s}.$$

Complete the EX_3_NLode_2025 function.

a. Plot the response for a positive impulse (P(t)) and negative impulse (-P(t)).
b. Comment on the outcome and the difference between the two impulses. What can you conclude about the performance of the selected material when simulating the whole system?