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CALCULATING ELLIPSE OVERLAP AREAS

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ABSTRACT. We present a general algorithm for finding the overlap area between two ellipses. The algorithm is based on finding a segment area (the area between an ellipse and a secant line) given two points on the ellipse. The Gauss-Green formula is used to determine the ellipse sector area between two points, and a triangular area is added or subtracted to give the segment area. For two ellipses, overlap area is calculated by adding the areas of appropriate sectors and polygons. Intersection points for two general ellipses are found using Ferrari's quartic formula to solve the polynomial that results from combining the two ellipse equations. All cases for the number of intersection points (0, 1, 2, 3, 4) are handled. The algorithm is implemented in c-code, and has been tested with a range of input ellipses. The code is efficient enough for use in simulations that require many overlap area calculations.

1. Introduction. Ellipses are useful in many applied scenarios, and in widely disparate fields. In our research, which happens to be in two very different areas, we have encountered a common need for efficiently calculating the overlap area between two ellipses.

In one case, the design for a solar calibrator on-board an orbiting satellite required an efficient algorithm for ellipse overlap area. Imaging systems aboard satellites rely on semi-conductor detectors whose performance changes over time due to many factors. To produce consistent data, some means of calibrating the detectors is required; see, e.g., [1]. Some systems use the sun as a light source for calibration. In a typical solar calibrator, incident sunlight passes through an attenuator grating and impinges on a diffuser plate, which is oriented obliquely to the attenuator grating. The attenuator grating is a pattern of circular openings. When sunlight passes through the circular openings, projections of the circles onto the oblique diffuser plate become small ellipses. The projection of the large circular entrance aperture on the oblique diffuser plate is also an ellipse. The total incident light on the calibrator is proportional to the sum of all the areas of the smaller ellipses that are contained within the larger entrance aperture ellipse. However, as the calibration process proceeds, the satellite is moving through its orbit, and the angle

Key words and phrases. Ellipse Area, Ellipse Sector, Ellipse Segment, Ellipse Overlap, Algorithm, Quartic Formula.

from the sun into the calibrator changes ($\sim 7^\circ$ in 2 minutes). The attenuator grating ellipses thus move across the entrance aperture, and some of the smaller ellipses pass in and out of the entrance aperture ellipse during calibration. Movement of the small ellipses across the aperture creates fluctuations in the total amount of incident sunlight reaching the calibrator in the range of 0.3 to 0.5%. This jitter creates errors in the calibration algorithms. In order to model the jitter, an algorithm is required for determining the overlap area of two ellipses. Monte Carlo integration had been used; however, the method is numerically intensive because it converges very slowly, so it was not an attractive approach for modeling the calibrator due to the large number of ellipses that must be modeled.

In a more down-to-earth setting, populated places such as city streets or building corridors can become quite congested while crowds of people are moving about. Understanding the dynamics of pedestrian movement in these scenarios can be beneficial in many ways. Pedestrian dynamics can provide critical input to the design of buildings or city infrastructure, for example by predicting the effects of specific crowd management strategies, or the behavior of crowds utilizing emergency escape routes. Current research in pedestrian dynamics is making steady progress toward realistic modeling of local movement; see, e.g., [2]. The model presented in [2] is based on the concept of elliptical volume exclusion for individual pedestrians. Each model pedestrian is surrounded by an elliptical footprint area that the model uses to anticipate obstacles and other pedestrians in or near the intended path. The footprint area is influenced by an individual's velocity; for example, the exclusion area in front of a fast-moving pedestrian is elongated when compared to a slower-moving individual, since a pedestrian is generally thinking a few steps ahead. As pedestrians travel through a confined space, their collective exclusion areas become denser, and the areas will eventually begin to overlap. A force-based model will produce a repulsive force between overlapping exclusion areas, causing the pedestrians to slow down or change course when the exclusion force becomes large. Implementing the force-based model with elliptical exclusion areas in a simulation requires calculating the overlap area between many different ellipses in the most general orientations. The ellipse area overlap algorithm must also be efficient, so as not to bog down the simulation.

Simulations for both the satellite solar calibrator and force-based pedestrian dynamic model require efficient calculation of the overlap area between two ellipses. In this paper, we provide an algorithm that has served well for both applications. The core component of the overlap area algorithm is based on determining the area of an *ellipse segment*, which is the area between a secant line and the ellipse boundary. The segment algorithm forms the basis of an application for calculating the overlap area between two general ellipses.

2. Ellipse area, sector area and segment area.

2.1. Ellipse Area. Consider an ellipse that is centered at the origin, with its axes aligned to the coordinate axes. If the semi-axis length along the x -axis is A , and the semi-axis length along the y -axis is B , then the ellipse is defined by a locus of points that satisfy the implicit polynomial equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \quad (1)$$

The same ellipse can be defined parametrically by:

$$\left. \begin{array}{l} x = A \cdot \cos(t) \\ y = B \cdot \sin(t) \end{array} \right\} \quad 0 \leq t \leq 2\pi \quad (2)$$

The area of such an ellipse can be found using the parameterized form with the Gauss-Green formula:

$$\begin{aligned} \text{Area} &= \frac{1}{2} \int_A^B [x(t) \cdot y'(t) - y(t) \cdot x'(t)] dt \\ &= \frac{1}{2} \int_0^{2\pi} A \cdot \cos(t) \cdot B \cdot \cos(t) - B \cdot \sin(t) \cdot (-A) \cdot \sin(t) dt \\ &= \frac{A \cdot B}{2} \int_0^{2\pi} \cos^2(t) + \sin^2(t) dt = \frac{A \cdot B}{2} \int_0^{2\pi} dt \\ &= \pi \cdot A \cdot B \end{aligned} \quad (3)$$

2.2. Ellipse Sector Areas. We define the *ellipse sector* between two points (x_1, y_1) and (x_2, y_2) on the ellipse as the area that is swept out by a vector from the origin to the ellipse, beginning at (x_1, y_1) , as the vector travels along the ellipse in a counter-clockwise direction from (x_1, y_1) to (x_2, y_2) . An example is shown in Fig. 1. The Gauss-Green formula can also be used to determine the area of such an ellipse sector.

$$\begin{aligned} \text{Sector Area} &= \frac{A \cdot B}{2} \int_{\theta_1}^{\theta_2} dt \\ &= \frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} \end{aligned} \quad (4)$$

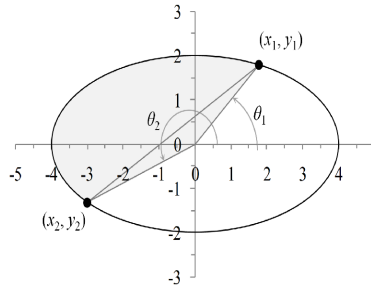


FIGURE 1. The area of an *ellipse sector* between two points on the ellipse is the area swept out by a vector from the origin to the first point as the vector travels along the ellipse in a counter-clockwise direction to the second point. The area of an ellipse sector can be determined with the Gauss-Green formula, using the parametric angles θ_1 and θ_2 .

The parametric angle θ that is formed between the x -axis and a point (x, y) on the ellipse is found from the ellipse parameterizations:

$$\begin{aligned} x &= A \cdot \cos(\theta) \implies \theta = \cos^{-1}(x/A) \\ y &= B \cdot \sin(\theta) \implies \theta = \sin^{-1}(y/B) \end{aligned}$$

For a circle ($A = B$ in the ellipse implicit polynomial form), the parametric angle corresponds to the geometric (visual) angle that a line from the origin to the point (x, y) makes with the x -axis. However, the same cannot be said for an ellipse; that is, the geometric (visual) angle is *not* the same as the parametric angle used in the area calculation. For example, consider the ellipse in Fig. 1; the implicit polynomial form is

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1 \quad (5)$$

Suppose the point (x_1, y_1) is at $(4/\sqrt{5}, 4/\sqrt{5})$. The point is on the ellipse, since

$$\frac{(4/\sqrt{5})^2}{4^2} + \frac{(4/\sqrt{5})^2}{2^2} = \frac{4^2/5}{4^2} + \frac{4^2/5}{2^2} = \frac{1}{5} + \frac{4}{5} = 1$$

A line segment from the origin to $(4/\sqrt{5}, 4/\sqrt{5})$ forms an angle with the x -axis of $\pi/4$ (≈ 0.785398). However, the ellipse parametric angle to the same point is:

$$\theta = \cos^{-1}\left(\frac{4/\sqrt{5}}{4}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 1.10715$$

The same angle can also be found from the parametric equation for y :

$$\theta = \sin^{-1}\left(\frac{4/\sqrt{5}}{2}\right) = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 1.10715$$

The angle found by using the parametric equations does not match the geometric angle to the point that defines the angle.

When determining the parametric angle for a given point (x, y) on the ellipse, the angle must be chosen in the proper quadrant, based on the signs of x and y . For the ellipse in Fig. 1, suppose the point (x_2, y_2) is at $(-3, -\sqrt{7}/2)$. The parametric angle that is determined from the equation for x is:

$$\theta = \cos^{-1}\left(\frac{-3}{4}\right) \approx 2.41886$$

The parametric angle that is determined from the equation for y is:

$$\theta = \sin^{-1}\left(\frac{-\sqrt{7}/2}{2}\right) = \sin^{-1}\left(\frac{-\sqrt{7}}{4}\right) \approx -.722734$$

The apparent discrepancy is resolved by recalling that inverse trigonometric functions are usually implemented to return a ‘principal value’ that is within a conventional range. The typical (principal-valued) $\theta = \arccos(x)$ function returns angles in the range $0 = \theta = \pi$, and the typical (principal-valued) $\theta = \arcsin(x)$ function returns angles in the range $-\pi/2 = \theta = \pi/2$. When the principal-valued inverse trigonometric functions return angles in the typical ranges, the ellipse parametric angles, defined to be from the x -axis, with positive angles in the counter-clockwise direction, can be found with the relations in Table 2.2.

Quadrant II ($x < 0$ and $y \geq 0$) $\theta = \arccos(x/A)$ $= \pi - \arcsin(y/B)$	Quadrant I ($x \geq 0$ and $y \geq 0$) $\theta = \arccos(x/A)$ $= \arcsin(y/B)$
Quadrant III ($x < 0$ and $y < 0$) $\theta = 2\pi - \arccos(x/A)$ $= \pi - \arcsin(y/B)$	Quadrant IV ($x \geq 0$ and $y < 0$) $\theta = 2\pi - \arccos(x/A)$ $= 2\pi + \arcsin(y/B)$

TABLE 1. Relations for finding the parametric angle that corresponds to a given point (x, y) on the ellipse $x^2/A^2 + y^2/B^2 = 1$. The parametric angle is formed between the positive x -axis and a line drawn from the origin to the given point, with counterclockwise being positive. For the standard (principal-valued) inverse trigonometric functions, the resulting angle will be in the range $0 \leq \theta < 2\pi$ for any point on the ellipse.

The point at $(-3, -\sqrt{7}/2)$ on the ellipse of Fig. 1 is in Quadrant III. Using the relations in Table 2.2, the parametric angle that is determined from the equation for x is:

$$\theta = 2\pi - \arccos\left(\frac{-3}{4}\right) \approx 3.86433$$

The parametric angle that is determined from the equation for y is:

$$\theta = \pi - \arcsin\left(\frac{-\sqrt{7}/2}{2}\right) \approx 3.86433$$

With the proper angles, the Gauss-Green formula can be used to determine the area of the sector from the point at $(4/\sqrt{5}, 4/\sqrt{5})$ to the point $(-3, -\sqrt{7}/2)$ in the ellipse of Fig. 1.

$$\begin{aligned} \text{Sector Area} &= \frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} \\ &= \frac{\left[\left(2\pi - \arccos\left(\frac{-3}{4}\right) \right) - \arccos\left(\frac{4/\sqrt{5}}{4}\right) \right] \cdot 4 \cdot 2}{2} \\ &\approx 11.0287 \end{aligned} \tag{6}$$

The Gauss-Green formula is sensitive to the direction of integration. For the larger goal of determining ellipse overlap areas, we define the ellipse sector area to be calculated from the first point (x_1, y_1) to the second point (x_2, y_2) in a *counterclockwise* direction along the ellipse. For example, if the points (x_1, y_1) and (x_2, y_2) of Fig. 1 were to have their labels switched, then the ellipse sector defined by the new points will have an area that is complementary to that of the sector in Fig. 1, as shown in Fig. 2.

Switching the point labels, as shown in Fig. 2, also causes the angle labels to be switched, resulting in the condition that $\theta_1 > \theta_2$. Since using the definitions in Table 2.2 will always produce an angle in the range $0 \leq \theta < 2\pi$ for any point on the ellipse, the first angle can be transformed by subtracting 2π to restore the condition that $\theta_1 < \theta_2$. The sector area formula given above can then be used, with the integration angle from $(\theta_1 - 2\pi)$ through θ_2 . With the angle labels shown

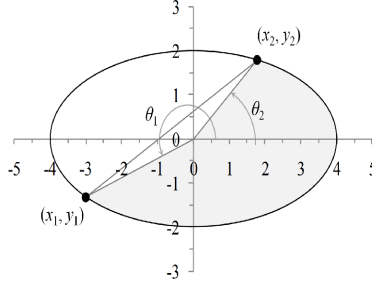


FIGURE 2. We define the ellipse sector area to be calculated from the first point (x_1, y_1) to the second point (x_2, y_2) in a counter-clockwise direction along the ellipse.

in Fig. 2, the area of the sector from the point at $(-3, -\sqrt{7}/2)$ to the point at $(4/\sqrt{5}, 4/\sqrt{5})$ in a counter-clockwise direction is:

$$\begin{aligned} \text{Sector Area} &= \frac{(\theta_2 - (\theta_1 - 2\pi)) \cdot A \cdot B}{2} \\ &= \frac{\left[(2\pi - \arccos(-\frac{3}{4})) - \left(\arccos\left(\frac{4/\sqrt{5}}{4}\right) - 2\pi \right) \right] \cdot 4 \cdot 2}{2} \quad (7) \\ &\approx 14.1040 \end{aligned}$$

The two sector areas shown in Fig. 1 and Fig. 2 are complementary, in that they add to the total ellipse area. Using the angle labels as shown in Fig. 1 for both sector areas:

$$\begin{aligned} \text{Total Area} &= \frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} + \frac{(\theta_1 - (\theta_2 - 2\pi)) \cdot A \cdot B}{2} \\ &= \frac{(2\pi) \cdot A \cdot B}{2} = \pi \cdot A \cdot B \quad (8) \\ &= \pi \cdot 4 \cdot 2 \\ &\approx 25.1327 \end{aligned}$$

2.3. Ellipse Segment Areas. For the overall goal of determining overlap areas between ellipses and other curves, a useful measure is the area of what we will call an *ellipse segment*. A secant line drawn between two points on an ellipse partitions the ellipse area into two fractions, as shown in Fig. 1 and Fig. 2. We define the ellipse segment as the area confined by the secant line and the portion of the ellipse from the first point (x_1, y_1) to the second point (x_2, y_2) *traversed in a counter-clockwise direction*. The segment's complement is the second of the two areas that are demarcated by the secant line. For the ellipse of Fig. 1, the area of the segment defined by the secant line through the points (x_1, y_1) and (x_2, y_2) is the area of the sector *minus* the area of the triangle defined by the two points and the ellipse center. To find the area of the triangle, suppose that the coordinates for the vertices of are known, e.g., as (x_1, y_1) , (x_2, y_2) and (x_3, y_3) . Then the triangle area can be

found by:

$$\begin{aligned} \text{Triangle Area} &= \frac{1}{2} \cdot \left| \det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \right| \\ &= \frac{1}{2} \cdot |x_1 \cdot (y_2 - y_3) - x_2 \cdot (y_1 - y_3) + x_3 \cdot (y_1 - y_2)| \end{aligned} \quad (9)$$

In the case where one vertex, say (x_3, y_3) , is at the origin, then the area formula for the triangle can be simplified to:

$$\text{Triangle Area} = \frac{1}{2} \cdot |x_1 \cdot y_2 - x_2 \cdot y_1| \quad (10)$$

For the case depicted in Fig. 1, subtracting the triangle area from the area of the ellipse sector area gives the area between the secant line and the ellipse, i.e., the area of the ellipse segment counter-clockwise from (x_1, y_1) to (x_2, y_2) :

$$\text{Segment Area} = \frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} - \frac{1}{2} \cdot |x_1 \cdot y_2 - x_2 \cdot y_1| \quad (11)$$

For the ellipse of Fig. 1, with the points at $(4/\sqrt{5}, 4/\sqrt{5})$ and $(-3, -\sqrt{7}/2)$, the area of the segment defined by the secant line is:

$$\begin{aligned} & \frac{\left[(2\pi - \arccos(\frac{-3}{4})) - \arccos(\frac{4/\sqrt{5}}{4}) \right] \cdot 4 \cdot 2}{2} - \frac{1}{2} \cdot \left| \frac{4}{\sqrt{5}} \cdot \frac{-\sqrt{7}}{2} - \frac{4}{\sqrt{5}} \cdot -3 \right| \\ & \approx 9.52865 \end{aligned}$$

For the ellipse of Fig. 2, the area of the segment shown is the sector area *plus* the area of the triangle.

$$\text{Segment Area} = \frac{(\theta_2 - (\theta_1 - 2\pi)) \cdot A \cdot B}{2} + \frac{1}{2} \cdot |x_1 \cdot y_2 - x_2 \cdot y_1| \quad (12)$$

With the points at $(-3, -\sqrt{7}/2)$ and $(4/\sqrt{5}, 4/\sqrt{5})$ the area of the segment is:

$$\begin{aligned} & \frac{\left[(2\pi - \arccos(\frac{-3}{4})) - (\arccos(\frac{4/\sqrt{5}}{4}) - 2\pi) \right] \cdot 4 \cdot 2}{2} + \frac{1}{2} \cdot \left| \frac{4}{\sqrt{5}} \cdot \frac{-\sqrt{7}}{2} - \frac{4}{\sqrt{5}} \cdot -3 \right| \\ & \approx 15.60409411 \end{aligned}$$

For the case shown in Fig. 1 and Fig. 2, the sector areas were shown to be complementary. The segment areas are also complementary, since the triangle area is added to the sector of Fig. 1, but subtracted from the sector of Fig. 2. Using the angle labels as shown in Fig. 1 for both sector areas:

$$\begin{aligned} \text{Total Area} &= \left[\frac{(\theta_2 - \theta_1) \cdot A \cdot B}{2} - \frac{1}{2} \cdot |x_1 \cdot y_2 - x_2 \cdot y_1| \right] \\ &+ \left[\frac{(\theta_1 - (\theta_2 - 2\pi)) \cdot A \cdot B}{2} + \frac{1}{2} \cdot |x_1 \cdot y_2 - x_2 \cdot y_1| \right] \\ &= \pi \cdot A \cdot B = \pi \cdot 4 \cdot 2 \approx 25.1327 \end{aligned} \quad (13)$$

The key difference between the cases in Fig. 1 and Fig. 2 that requires the area of the triangle to be either subtracted from, or added to, the sector area is the size of the *integration angle*. If the integration angle is less than π , then the triangle area

ELLIPSE_SEGMENT Area Algorithm:	
1.	$\theta_1 = \begin{cases} \arccos(x_1/A) & , y_1 \geq 0 \\ 2\pi - \arccos(x_1/A) & , y_1 < 0 \end{cases}$ $\theta_2 = \begin{cases} \arccos(x_2/A) & , y_2 \geq 0 \\ 2\pi - \arccos(x_2/A) & , y_2 < 0 \end{cases}$
2.	$\hat{\theta}_1 = \begin{cases} \theta_1 & , \theta_1 < \theta_2 \\ \theta_1 - 2\pi & , \theta_1 > \theta_2 \end{cases}$
3.	$\text{Area} = \frac{(\theta_2 - \hat{\theta}_1) \cdot A \cdot B}{2} + \frac{\text{sign}(\theta_2 - \hat{\theta}_1 - \pi)}{2} \cdot x_1 \cdot y_2 - x_2 \cdot y_1 $
where: the ellipse implicit polynomial equation is $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ $A > 0$ is the semi-axis length along the x -axis $B > 0$ is the semi-axis length along the y -axis (x_1, y_1) is the first given point on the ellipse (x_2, y_2) is the second given point on the ellipse θ_1 and θ_2 are the parametric angles corresponding to the points (x_1, y_1) and (x_2, y_2)	

TABLE 2. An outline of the ELLIPSE_SEGMENT area algorithm.

must be subtracted from the sector area to give the segment area. If the integration angle is greater than π , the triangle area must be added to the sector area.

2.4. A Core Algorithm for Ellipse Segment Area. A generalization of the cases given in Fig. 1 and Fig. 2 suggests a robust approach for determining the ellipse segment area defined by a secant line drawn between two given points on the ellipse. The ellipse is assumed to be centered at the origin, with its axes parallel to the coordinate axes. We define the segment area to be demarcated by the secant line and the ellipse proceeding counter-clockwise from the first given point (x_1, y_1) to the second given point (x_2, y_2) . The ELLIPSE_SEGMENT algorithm is outlined in Table 2, with pseudo-code presented in List. 1. The ellipse is passed to the algorithm by specifying the semi-axes lengths, $A > 0$ and $B > 0$. The points are passed to the algorithm as (x_1, y_1) and (x_2, y_2) , which must be on the ellipse.

For robustness, the algorithm should avoid divide-by-zero and inverse-trigonometric errors, so data checks should be included. The ellipse parameters A and B must be greater than zero. A check is provided to determine whether the points are on the ellipse, to within some numerical tolerance, ε . Since the points can only be checked as being on the ellipse to within some numerical tolerance, it may still be possible for the x -values to be slightly larger than A , leading to an error when calling the inverse trigonometric functions with the argument x/A . In this case, the algorithm checks whether the x -value close to A or $-A$, that is within a distance that is less than the numerical tolerance. If the closeness condition is met, then the algorithm assumes that the calling function passed a value that is indeed on the ellipse near the point $(A, 0)$ or $(-A, 0)$, so the value of x is nudged back to A or $-A$ to avoid any error when calling the inverse trigonometric functions. The core algorithm, including all data checks, is shown in List. 1.

LISTING 1. The ELLIPSE_SEGMENT algorithm is shown for calculating the area of a segment defined by the secant line drawn between two given points (x_1, y_1) and (x_2, y_2) on the ellipse $x^2/A^2 + y^2/B^2 = 1$. We define the segment area for this algorithm to be demarcated by the secant line and the ellipse proceeding counter-clockwise from the first given point (x_1, y_1) to the second given point (x_2, y_2) .

```

1  ELLIPSE_SEGMENT (A, B, X1, Y1, X2, Y2)
2  do if (A = 0 or B = 0)
3      then return (-1, ERROR_ELLIPSE_PARAMETERS) :DATA CHECK
4      2 2 2 2 2 2
5  do if (|X1|/A + Y1/B > 1 or |X1|/A + Y1/B > 1)
6      then return (-1, ERROR_POINTS_NOT_ON_ELLIPSE) :DATA CHECK
7  do if (|X1|/A > 1)
8      do if |X1| - A > 0
9          then return (-1, ERROR_INVERSE_TRIG) :DATA CHECK
10         else do if X1 < 0
11             then X1 = -X1
12             else X1 = X1
13  do if (|X2|/A > 1)
14      do if |X2| - A > 0
15          then return (-1, ERROR_INVERSE_TRIG) :DATA CHECK
16         else do if X2 < 0
17             then X2 = -X2
18             else X2 = X2
19  do if (Y1 < 0) :ANGLE QUADRANT FORMULA (TABLE 1)
20      then 1 = acos (X1/A)
21      else 1 = acos (X1/A)
22  do if (Y2 < 0) :ANGLE QUADRANT FORMULA (TABLE 1)
23      then 2 = acos (X2/A)
24      else 2 = acos (X2/A)
25  do if (1 > 2) :MUST START WITH 1 < 2
26      then 1 = 1 - 2
27  do if ((2 - 1) > 0) :STORE SIGN OF TRIANGLE AREA
28      then trsgn = +1.0
29      else trsgn = -1.0
30  area = 0.5*(A*B*(2 - 1) + trsgn*|X1*Y2 - X2*Y1|)
31  return (area, NORMAL_TERMINATION)

```

An implementation of the ELLIPSE_SEGMENT algorithm written in c-code is shown in Appendix 4. The code compiles under Cygwin-1.7.7-1, and returns the following values for the two test cases presented in Fig. 1 and Fig. 2:

LISTING 2. Return values for the test cases in Fig. 1 and Fig. 2

```

32  cc call_es.c ellipse_segment.c -o call_es.exe
33  ./call_es
34  Calling ellipse_segment.c
35  Fig. 1: segment area = 9.52864712, return_value = 0
36  Fig. 2: segment area = 15.60409411, return_value = 0
37  sum of ellipse segments = 25.13274123
38  ellipse area by pi*A*B = 25.13274123

```

3. Extending the Core Segment Algorithm to more General Cases.

3.1. Segment Area for a (Directional) Line through a General Ellipse.

The core segment algorithm is based on an ellipse that is centered at the origin with its axes aligned to the coordinate axes. The algorithm can be extended to more general ellipses, such as rotated and/or translated ellipse forms. Start by considering the case for a standard ellipse with semi-major axis lengths of A and B that is centered at the origin and with its axes aligned with the coordinate axes. Suppose that the ellipse is rotated through a counter-clockwise angle φ , and that the ellipse is then translated so that its center is at the point (h, k) . The

rotated+translated ellipse could then be defined by the set of parameters (A, B, h, k, φ) , with the understanding that the rotation through φ is performed before the translation through (h, k) . The approach for extending the core segment area algorithm will be to determine analogs on the standard ellipse corresponding to any points of intersection between a shape of interest and the general rotated and translated ellipse. To identify corresponding points, features of the shape of interest are translated by $(-h, -k)$, and then rotated by $-\varphi$. The translated+rotated features are used to determine any points of intersection with a similar ellipse that is centered at the origin with its axes aligned to the coordinate axes. Then, the core segment algorithm can be called with the translated+rotated intersection points.

Rotation and translation are affine transformations that are also length- and area-preserving. In particular, the semi-axis lengths in the general rotated ellipse are preserved by both transformations, and corresponding points on the two ellipses will demarcate equal partition areas. Fig. 3 illustrates this idea, showing the ellipse of Fig. 1 which has been rotated counter-clockwise through an angle $\varphi = 3\pi/8$, then translated by $(h, k) = (-6, +3)$.

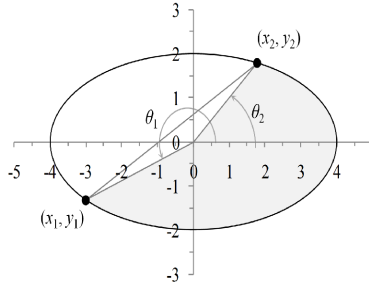


FIGURE 3. Translation and rotation are affine transformations that are also length-and area-preserving. Corresponding points on the two ellipses will demarcate equal partition areas.

Suppose that we desire to find the area of the rotated+translated ellipse sector defined by the line $y = -x$, where the line ‘direction’ travels from lower-right to upper-left, as shown in Fig. 3. We describe an approach for finding a segment in a rotated+translated ellipse, based on the core ellipse segment algorithm.

An ellipse that is centered at the origin, with its axes aligned to the coordinate axes, is defined parametrically by

$$\left. \begin{aligned} x &= A \cdot \cos(t) \\ y &= B \cdot \sin(t) \end{aligned} \right\} \quad 0 \leq t \leq 2\pi$$

Suppose the ellipse is rotated through an angle φ , with counter-clockwise being positive, and that the ellipse is then to be translated to put its center is at the point (h, k) . Any point (x, y) on the standard ellipse can be rotated and translated to end up in a corresponding location on the new ellipse by using the transformation:

$$\begin{bmatrix} x_{TR} \\ y_{TR} \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix} \quad (14)$$

Rotation and translation of the original standard ellipse does not change the ellipse area, or the semi-axis lengths. One important feature of the algorithms presented here is that the semi-axis lengths A and B are in the direction of the x - and y -axes,

respectively, in the *un-rotated* (standard) ellipse. In its rotated orientation, the semi-axis length A will rarely be oriented horizontally (in fact, for $\varphi = \pi/4$, the semi-axis length A will be oriented vertically). Regardless of the orientation of the rotated+translated ellipse, the algorithms presented here assume that the values of A and B passed into the algorithm represent the semi-axis lengths along the x - and y -axes, respectively, for the corresponding un-rotated, un-translated ellipse. The angle φ is the amount of counter-clockwise rotation required to put the ellipse into its desired location. Specifying a negative value for φ will rotate the standard ellipse through a clockwise angle. The angle φ can be specified in anywhere in the range $(-8, +8)$; the working angle in the code will be computed from the given angle, modulo 2π , to avoid any potential errors (?) when calculating trigonometric values. The translation (h, k) is the absolute movement along the coordinate axes of the ellipse center to move a standard ellipse into its desired location. Negative values of h move the standard ellipse to the left; negative values of k move the standard ellipse down.

To find the area between the given line and the rotated+translated ellipse, the two curve equations can be solved simultaneously to find any points of intersection. But instead of searching for the points of intersection with the rotated+translated ellipse, it is more efficient to transform the two given points that define the line back through the translation $(-h, -k)$ then rotation through $-\varphi$. The new line determined by the translated+rotated points will pass through the standard ellipse at points that are analogous to where the original line intersects the rotated+translated ellipse.

The transformations required to move the given points (x_1, y_1) and (x_2, y_2) into an orientation with respect to a standard ellipse that is analogous to their orientation to the given ellipse are the inverse of what it took to rotate+translate the ellipse to its desired position. The translation is performed first, then the rotation:

$$\begin{bmatrix} x_{i_0} \\ y_{i_0} \end{bmatrix} = \begin{bmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{bmatrix} \cdot \begin{bmatrix} x_i - h \\ y_i - k \end{bmatrix} \quad (15)$$

Multiplying the vector by the matrix, and simplifying the negative-angle trig functions gives the following expressions for the translated+rotated points:

$$\begin{aligned} x_{i_0} &= \cos(\varphi) \cdot (x_i - h) + \sin(\varphi) \cdot (y_i - k) \\ y_{i_0} &= -\sin(\varphi) \cdot (x_i - h) + \cos(\varphi) \cdot (y_i - k) \end{aligned}$$

The two new points (x_{1_0}, y_{1_0}) and (x_{2_0}, y_{2_0}) can be used to determine a line, *e.g.*, by the point-slope method:

$$y = y_{1_0} + \frac{y_{2_0} - y_{1_0}}{x_{2_0} - x_{1_0}} (x - x_{1_0}) \quad (16)$$

The equation can also be formulated in an alternative way to accommodate cases where the translated+rotated line is vertical, or nearly so:

$$x = x_{1_0} + \frac{x_{2_0} - x_{1_0}}{y_{2_0} - y_{1_0}} (y - y_{1_0}) \quad (17)$$

Points of intersection are found by substituting the line equations into the standard ellipse equation, and solving for the remaining variable. For each case, define the slope as:

$$m_{yx} = \frac{y_{2_0} - y_{1_0}}{x_{2_0} - x_{1_0}}, \quad m_{xy} = \frac{x_{2_0} - x_{1_0}}{y_{2_0} - y_{1_0}} \quad (18)$$

Then the two substitutions proceed as follows:

$$\begin{aligned}
y = y_{1_0} + m_{yx} \cdot (x - x_{1_0}) & \text{ into } \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \\
\Rightarrow \frac{x^2}{A^2} + \frac{(y_{1_0} + m_{yx} \cdot (x - x_{1_0}))^2}{B^2} &= 1 \\
\Rightarrow \left[\frac{B^2 + A^2 \cdot (m_{yx})^2}{A^2} \right] \cdot x^2 & \\
+ \left[2 \cdot (y_{1_0} \cdot m_{yx} - (m_{yx})^2 \cdot x_{1_0}) \right] \cdot x & \\
+ \left[(y_{1_0})^2 - 2 \cdot m_{yx} \cdot x_{1_0} \cdot y_{1_0} + (m_{yx} \cdot x_{1_0})^2 - B^2 \right] & \\
= 0 &
\end{aligned} \tag{19}$$

$$\begin{aligned}
x = x_{1_0} + m_{xy} \cdot (y - y_{1_0}) & \text{ into } \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \\
\Rightarrow \frac{(x_{1_0} + m_{xy} \cdot (y - y_{1_0}))^2}{A^2} + \frac{y^2}{B^2} &= 1 \\
\Rightarrow \left[\frac{A^2 + B^2 \cdot (m_{xy})^2}{B^2} \right] \cdot y^2 & \\
+ \left[2 \cdot (x_{1_0} \cdot m_{xy} - (m_{xy})^2 \cdot y_{1_0}) \right] \cdot y & \\
+ \left[(x_{1_0})^2 - 2 \cdot m_{xy} \cdot x_{1_0} \cdot y_{1_0} + (m_{xy} \cdot y_{1_0})^2 - A^2 \right] & \\
= 0 &
\end{aligned} \tag{20}$$

If the translated+rotated line is not vertical, then use the first equation to find the x -values for any points of intersection. If the translated+rotated line is close to vertical, then the second equation can be used to find the y -values for any points of intersection. Since points of intersection between the line and the ellipse are determined by solving a quadratic equation $ax^2 + bx + c$, there are three cases to consider:

1. $\Delta = b^2 - 4ac < 0$: Complex Conjugate Roots (no points of intersection)
2. $\Delta = b^2 - 4ac = 0$: One Double Real Root (1 point of intersection; line tangent to ellipse)
3. $\Delta = b^2 - 4ac > 0$: Two Real Roots (2 points of intersection; line crosses ellipse)

For the first two cases, the segment area will be zero. For the third case, the two points of intersection can be sent to the core segment area algorithm. However, to enforce a consistency in area measures returned by the core algorithm, the integration direction is specified to be from the first point to the second point. As such, the ellipse line overlap algorithm should be sensitive to the order that the points are passed to the core segment algorithm. We suggest giving the line a ‘direction’ from the first given point on the line to the second. The line ‘direction’ can then be used to determine which is to be the first point of intersection, i.e., the first intersection point is where the line enters the ellipse based on what ‘direction’ the line is pointing. The segment area that will be returned from ELLIPSE_SEGMENT by passing the line’s entry location as the first intersection point is the area within the ellipse to the right of the line’s path.

The approach outlined above for finding the overlap area between a line and a general ellipse is implemented in the ELLIPSE_LINE_OVERLAP algorithm, with pseudo-code shown in List. 3. The ellipse is passed to the algorithm by specifying the counterclockwise rotation angle φ and the translation (h, k) that takes a standard ellipse and moves it to the desired orientation, along with the semi-axes lengths, $A > 0$ and $B > 0$. The line is passed to the algorithm as two points on the line, (x_1, y_1) and (x_2, y_2) . The ‘direction’ of the line is taken to be from (x_1, y_1) toward (x_2, y_2) . Then, the segment area returned from ELLIPSE_SEGMENT will be the area within the ellipse to the right of the line’s path.

LISTING 3. The ELLIPSE_LINE_OVERLAP algorithm is shown for calculating the area of a segment in a general ellipse that is defined by a given line. The line is considered to have a ‘direction’ that runs from the first given point (x_1, y_1) to the second given point (x_2, y_2) . The line ‘direction’ determines the order in which intersection points are passed to the ELLIPSE_SEGMENT algorithm, which will return the area of the segment that runs along the ellipse from the first point to the second in a counter-clockwise direction. Any routine that calls the algorithm ELLIPSE_LINE_OVERLAP must be sensitive to the order of points that are passed in.

```

39 (Area, Code) ← ELLIPSE\_LINE\_OVERLAP (A, B, H, K,  $\varphi$ , X1, Y1, X2, Y2)
40 do if (A ≤ 0 or B ≤ 0)
41
42     then return (-1, ERROR_ELLIPSE_PARAMETERS) :DATA_CHECK
43
44 do if (| $\varphi$ | > 2 $\pi$ )
45
46     then  $\varphi \leftarrow (\varphi \text{ modulo } 2\pi)$  :BRING  $\varphi$  INTO  $-2\pi \leq \varphi < 2\pi$  (?)
47
48 do if (|X1|/A > 2 $\pi$ )
49
50     then X1 ← -A
51
52 X10 ← cos( $\varphi$ ) * (X1 - H) + sin( $\varphi$ ) * (Y1 - K)
53 Y10 ← -sin( $\varphi$ ) * (X1 - H) + cos( $\varphi$ ) * (Y1 - K)
54
55 X20 ← cos( $\varphi$ ) * (X2 - H) + sin( $\varphi$ ) * (Y2 - K)
56
57 Y20 ← -sin( $\varphi$ ) * (X2 - H) + cos( $\varphi$ ) * (Y2 - K)
58
59 do if (|X20 - X10| >  $\varepsilon$ ) :LINE IS NOT VERTICAL
60
61     then m ← (Y20 - Y10)/(X20 - X10) :STORE QUADRATIC COEFFICIENTS
62
63         a ← (B2 + (A*m)2)/A2
64
65         b ← (2.0*(Y10*m - m2*X10))
66
67         c ← (Y102 - 2.0*m*Y10*X10 + (m*X10)2 - B2)
68
69 else if (|Y20 - Y10| >  $\varepsilon$ ) :LINE IS NOT HORIZONTAL
70
71     then m ← (X20 - X10)/(Y20 - Y10) :STORE QUADRATIC COEFFS
72
73         a ← (A2 + (B*m)2)/B2
74
75         b ← (2.0*(X10*m - m2*Y10))
76
77         c ← (X102 - 2.0*m*Y10*X10 + (m*Y10)2 - A2)
78
79

```

```

80     else return (-1, ERROR_LINE_POINTS)           :LINE POINTS TOO CLOSE
81
82     discrim ← b2 - 4.0*a*c
83
84     do if (discrim < 0.0)                           :LINE DOES NOT CROSS ELLIPSE
85
86         then return (0, NO_INTERSECT)
87
88         else if (discrim > 0.0)                       :TWO INTERSECTION POINTS
89
90         then root1 ← (-b - sqrt (discrim))/(2.0*a)
91
92             root2 ← (-b + sqrt (discrim))/(2.0*a)
93
94         else return (0, TANGENT)                     :LINE TANGENT TO ELLIPSE
95
96     do if (|X20 - X10| > ε)                          :ROOTS ARE X-VALUES
97
98         then do if (X10 < X20)                       :ORDER PTS SAME AS LINE DIRECTION
99
100             then x1 ← root1
101
102                 x2 ← root2
103
104             else x1 ← root2
105
106                 x2 ← root1
107
108         else do if (Y10 < Y20)                       :ROOTS ARE Y-VALUES
109
110             then y1 ← root1                          :ORDER PTS SAME AS LINE DIRECTION
111
112                 y2 ← root2
113
114             else y1 ← root2
115
116                 y2 ← root1
117
118     (Area, Code) ← ELLIPSE_SEGMENT (A, B, x1, y1, x2, y2)
119
120     do if (Code < NORMAL_TERMINATION)
121
122         then return (-1.0, Code)
123
124         else return (Area, TWO_INTERSECTION_POINTS)

```

An implementation of the ELLIPSE_LINE_OVERLAP algorithm in c-code is shown in Appendix 5. The code compiles under Cygwin-1.7.7-1, and returns the following values for the test cases presented above in Fig. 3, with both line ‘directions’:

LISTING 4. Return values for the test cases in Fig. 3.

```

125 cc call_el.c ellipse_line_overlap.c ellipse_segment.c -o call_el.exe
126
127 ./call_el
128
129 Calling ellipse_line_overlap.c
130
131 area =                                4.07186819, return_value = 102
132
133 reverse: area =                        21.06087304, return_value = 102
134
135 sum of ellipse segments =              25.13274123
136
137 total ellipse area by pi*A*B =         25.13274123

```

3.2. Ellipse-Ellipse Overlap Area. The method described above for determining the area between a line and an ellipse can be extended to the task of finding the

overlap area between two general ellipses. Suppose the two ellipses are defined by their semi-axis lengths, center locations and axis rotation angles. Let the two sets of parameters $(A_1, B_1, h_1, k_1, \varphi_1)$ and $(A_2, B_2, h_2, k_2, \varphi_2)$ define the two ellipses for which overlap area is sought. The approach presented here will be to first translate both ellipses by an amount $(-h_1, -k_1)$ that puts the center of the first ellipse at the origin. Then, both translated ellipses are rotated about the origin by an angle $-\varphi_1$ that aligns the axes of the first ellipse with the coordinate axes; see Fig. 4. Intersection points are found for the two translated+rotated ellipses, using Ferrari's quartic formula. Finally, the segment algorithm described above is employed to find all the pieces of the overlap area.

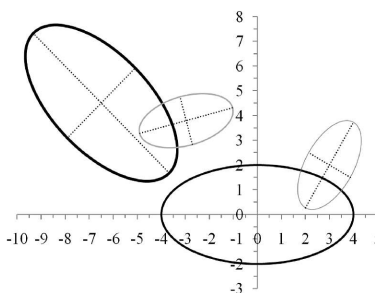


FIGURE 4. Intersection points on each curve are used with the ellipse segment area algorithm to determine overlap area, by calculating the area of appropriate segments, and polygons in certain cases. For the case of two intersection points, as shown above, the overlap area can be found by adding two segments, as shown in Fig. 5.

For example, consider a case of two general ellipses with two (non-tangential) points of intersection, as shown in Fig. 4. The translation+rotation transformations that put the first ellipse at the origin and aligned with the coordinate axes do not alter the overlap area. In the case shown in Fig. 4, the overlap area consists of one segment from the first ellipse and one segment from the second ellipse. The segment algorithm presented above can be used directly for ellipses centered at the origin and aligned with the coordinate axes. As such, the desired segment from the first ellipse can be found immediately with the segment algorithm, based on the points of intersection. To find the desired segment of the second ellipse, the approach presented here further translates and rotates the second ellipse so that the segment algorithm can also be used directly. The overlap area for the case shown in Fig. 4 is equal to the sum of the two segment areas, as shown in Fig. 5. Other cases, e.g. with 3 and 4 points of intersection, can also be handled using the segment algorithm.

The overlap area algorithm presented here finds the area of appropriate sector(s) of each ellipse, which are demarcated by any points of intersection between the two ellipse curves. To find intersection points, the two ellipse equations are solved simultaneously. This step can be accomplished by using the implicit polynomial forms for each ellipse. The first ellipse equation, in its translated+rotated position is written as an implicit polynomial using the appropriate semi-axis lengths:

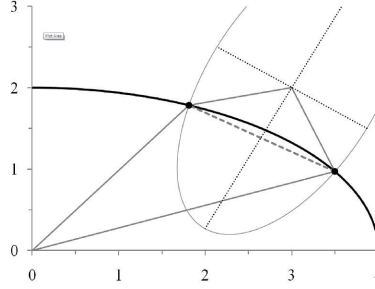


FIGURE 5. The area of overlap between two intersecting ellipses can be found by using the ellipse sector algorithm. In the case of two (non-tangential) intersection points, the overlap area is equal to the sum of two ellipse sectors. The sector in each ellipse is demarcated by the intersection points.

$$\frac{x^2}{A_1^2} + \frac{y^2}{B_1^2} = 1 \quad (21)$$

In a general form of this problem, the translation+rotation that puts the first ellipse centered at the origin and oriented with the coordinate axes will typically leave the second ellipse displaced and rotated. The implicit polynomial form for a more general ellipse that is rotated and/or translated away from the origin is written in the conventional way as:

$$AA \cdot x^2 + BB \cdot x \cdot y + CC \cdot y^2 + DD \cdot x + EE \cdot y + FF = 0 \quad (22)$$

Any points of intersection for the two ellipses will satisfy these two equations simultaneously. An intermediate goal is to find the implicit polynomial coefficients in Ellipse Eq. 22 that describe the second ellipse after the translation+rotation that puts the first ellipse centered at the origin and oriented with the coordinate axes. The parameters that describe the second ellipse after the translation+rotation can be determined from the original parameters for the two ellipses. The first step is to translate the second ellipse center (h_2, k_2) through an amount $(-h_1, -k_1)$, then rotate the center-point through $-\varphi_1$ to give a new center point (h_{2TR}, k_{2TR}) :

$$\begin{aligned} h_{2TR} &= \cos(-\varphi_1) \cdot (h_2 - h_1) - \sin(-\varphi_1) \cdot (k_2 - k_1) \\ k_{2TR} &= \sin(-\varphi_1) \cdot (h_2 - h_1) + \cos(-\varphi_1) \cdot (k_2 - k_1) \end{aligned}$$

The coordinates for a point (x_{TR}, y_{TR}) on the second ellipse in its new translated+rotated position can be found from the following parametric equations, based on an ellipse with semi-axis lengths A_2 and B_2 that is centered at the origin, then rotated and translated to the desired position:

$$\left. \begin{aligned} x_{TR} &= A_2 \cdot \cos(t) \cdot \cos(\varphi_2 - \varphi_1) - B_2 \cdot \sin(t) \cdot \sin(\varphi_2 - \varphi_1) + h_{2TR} \\ y_{TR} &= A_2 \cdot \cos(t) \cdot \sin(\varphi_2 - \varphi_1) + B_2 \cdot \sin(t) \cdot \cos(\varphi_2 - \varphi_1) + k_{2TR} \end{aligned} \right\} 0 \leq t \leq 2\pi$$

To find the implicit polynomial coefficients from the parametric form, further transform the locus of points (x_{TR}, y_{TR}) so that they lie on the ellipse $(A_2, B_2, 0, 0, 0)$, which is accomplished by first translating (x_{TR}, y_{TR}) through $(-h_1 - h_2, -(k_1 - k_2))$ and then rotating the point through the angle $-(\varphi_1 - \varphi_2)$:

$$\begin{aligned} x &= \cos(\varphi_2 - \varphi_1) \cdot (x_{TR} - (h_1 - h_2)) - \sin(\varphi_2 - \varphi_1) \cdot (y_{TR} - (k_1 - k_2)) \\ y &= \sin(\varphi_2 - \varphi_1) \cdot (x_{TR} - (h_1 - h_2)) + \cos(\varphi_2 - \varphi_1) \cdot (y_{TR} - (k_1 - k_2)) \end{aligned}$$

The locus of points (x, y) should satisfy the standard ellipse equation with the appropriate semi-axis lengths:

$$\frac{x^2}{A_2^2} + \frac{y^2}{B_2^2} = 1 \quad (23)$$

Finally, the implicit polynomial coefficients for Ellipse Eq. 22 are found by substituting the expressions for the point (x, y) into the standard ellipse equation, yielding the following ellipse equation:

$$\begin{aligned} & \frac{[\cos(\varphi_2 - \varphi_1) \cdot (x_{TR} - (h_1 - h_2)) - \sin(\varphi_2 - \varphi_1) \cdot (y_{TR} - (k_1 - k_2))]^2}{A_2^2} \\ & + \frac{[\sin(\varphi_2 - \varphi_1) \cdot (x_{TR} - (h_1 - h_2)) + \cos(\varphi_2 - \varphi_1) \cdot (y_{TR} - (k_1 - k_2))]^2}{B_2^2} \\ & = 1 \end{aligned} \quad (24)$$

where (x_{TR}, y_{TR}) are defined as above. Expanding the terms, and then rearranging the order to isolate like terms yields the following expressions for the implicit polynomial coefficients of a general ellipse with the set of parameters $(A_2, B_2, h_{2TR}, k_{2TR}, \varphi_2 - \varphi_1)$:

$$\begin{aligned} AA &= \frac{\cos^2(\varphi_2 - \varphi_1)}{A_2^2} + \frac{\sin^2(\varphi_2 - \varphi_1)}{B_2^2} \\ BB &= \frac{2 \cdot \sin(\varphi_2 - \varphi_1) \cdot \cos(\varphi_2 - \varphi_1)}{A_2^2} - \frac{2 \cdot \sin(\varphi_2 - \varphi_1) \cdot \cos(\varphi_2 - \varphi_1)}{B_2^2} \\ CC &= \frac{\sin^2(\varphi_2 - \varphi_1)}{A_2^2} + \frac{\cos^2(\varphi_2 - \varphi_1)}{B_2^2} \\ DD &= \frac{-2 \cdot \cos(\varphi_2 - \varphi_1) \cdot [h_{2TR} \cdot \cos(\varphi_2 - \varphi_1) + k_{2TR} \cdot \sin(\varphi_2 - \varphi_1)]}{A_2^2} \\ & + \frac{2 \cdot \sin(\varphi_2 - \varphi_1) \cdot [k_{2TR} \cdot \cos(\varphi_2 - \varphi_1) - h_{2TR} \cdot \sin(\varphi_2 - \varphi_1)]}{B_2^2} \\ EE &= \frac{-2 \cdot \sin(\varphi_2 - \varphi_1) \cdot [h_{2TR} \cdot \cos(\varphi_2 - \varphi_1) + k_{2TR} \cdot \sin(\varphi_2 - \varphi_1)]}{A_2^2} \\ & + \frac{2 \cdot \cos(\varphi_2 - \varphi_1) \cdot [h_{2TR} \cdot \sin(\varphi_2 - \varphi_1) - k_{2TR} \cdot \cos(\varphi_2 - \varphi_1)]}{B_2^2} \\ FF &= \frac{[h_{2TR} \cdot \cos(\varphi_2 - \varphi_1) + k_{2TR} \cdot \sin(\varphi_2 - \varphi_1)]^2}{A_2^2} \\ & + \frac{[h_{2TR} \cdot \sin(\varphi_2 - \varphi_1) - k_{2TR} \cdot \cos(\varphi_2 - \varphi_1)]^2}{B_2^2} - 1 \end{aligned} \quad (25)$$

For the area overlap algorithm presented in this paper, the points of intersection between the two general ellipses are found by solving simultaneously the two implicit

polynomials denoted above as Ellipse Eq. 21 and Ellipse Eq. 22. Solving for x in the first equation:

$$\frac{x^2}{A_1^2} + \frac{y^2}{B_1^2} = 1 \implies x = \pm \sqrt{A_1^2 \cdot \left(1 - \frac{y^2}{B_1^2}\right)} \quad (26)$$

Substituting these expressions for x into Ellipse Eq. 22 and then collecting terms yields a quartic polynomial in y . It turns out that substituting either the positive or the negative root gives the same quartic polynomial coefficients, which are:

$$cy[4] \cdot y^4 + cy[3] \cdot y^3 + cy[2] \cdot y^2 + cy[1] \cdot y + cy[0] = 0 \quad (27)$$

where:

$$\begin{aligned} \frac{cy[4]}{B_1} &= A_1^4 \cdot AA^2 + B_1^2 \cdot [A_1^2 \cdot (BB^2 - 2 \cdot AA \cdot CC) + B_1^2 \cdot CC^2] \\ \frac{cy[3]}{B_1} &= 2 \cdot B_1 \cdot [B_1^2 \cdot CC \cdot EE + A_1^2 \cdot (BB \cdot DD - AA \cdot EE)] \\ \frac{cy[2]}{B_1} &= A_1^2 \cdot \{ [B_1^2 \cdot (2 \cdot AA \cdot CC - BB^2) + DD^2 - 2 \cdot AA \cdot FF] - 2 \cdot A_1^2 \cdot AA^2 \} \\ &\quad + B_1^2 \cdot (2 \cdot CC \cdot FF + EE^2) \\ \frac{cy[1]}{B_1} &= 2 \cdot B_1 \cdot [A_1^2 \cdot (AA \cdot EE - BB \cdot DD) + EE \cdot FF] \\ \frac{cy[0]}{B_1} &= [A_1 \cdot (A_1 \cdot AA - DD) + FF] \cdot [A_1 \cdot (A_1 \cdot AA + DD) + FF] \end{aligned} \quad (28)$$

In theory, the quartic polynomial will have real roots if and only if the two curves intersect. If the ellipses do not intersect, then the quartic will have only complex roots. Furthermore, any real roots of the quartic polynomial will represent y -values of intersection points between the two ellipse curves. As with the quadratic equation that arises in the ellipse-line overlap calculation, the ellipse-ellipse overlap algorithm should handle all possible cases for the types of quartic polynomial roots:

1. Four real roots (distinct or not); the ellipse curves intersect.
2. Two real roots (distinct or not) and one complex-conjugate pair; the ellipse curves intersect.
3. No real roots (two complex-conjugate pairs); the ellipse curves do not intersect.

For the method we present here, polynomial roots are found using Ferrari's quartic formula. A numerical implementation of Ferrari's formula is given in [3]. For complex roots are returned, and any roots whose imaginary part is returned as zero is a real root.

When the polynomial coefficients are constructed as shown above, the general case of two distinct ellipses typically results in a quartic polynomial, i.e., the coefficient $cy[4]$ is non-zero. However, certain cases lead to polynomials of lesser degree. Fortunately, the solver in [3] is conveniently modular, providing separate functions BIQUADROOTS, CUBICROOTS and QUADROOTS to handle all the possible polynomial cases that arise when seeking points of intersection for two ellipses.

If the polynomial solver returns no real roots to the polynomial, then the ellipse *curves* do not intersect. It follows that the two ellipse *areas* are either disjoint, or one ellipse area is fully contained inside the other; all three possibilities are shown in Fig. 6. Each sub-case in Fig. 6 requires a different overlap-area calculation, i.e. either the overlap area is zero (Case 0-3), or the overlap is the area of the first ellipse (Case 0-2), or the overlap is the area of the second ellipse (Case 0-1). When the polynomial has no real roots, geometry can be used to determine which specific sub-case of Fig. 6 is represented. An efficient logic starts by determining the relative size of the two ellipses, e.g., by comparing the product of semi-axis lengths for each ellipse. The area of an ellipse is proportional to the product of its two semi-axis lengths, so the relative size of two ellipses can be determined by comparing the product of semi-axis lengths:

$$(\pi \cdot A_1 \cdot B_1) \propto (\pi \cdot A_2 \cdot B_2) \implies (A_1 \cdot B_1) \propto (A_2 \cdot B_2), \quad \alpha \in \{', '>'\} \quad (29)$$

Suppose the first ellipse is larger than the second ellipse, then $A_1 B_1 > A_2 B_2$. In this case, if the second ellipse center (h_{2TR}, k_{2TR}) is inside the first ellipse, then the second ellipse is wholly contained within the first ellipse (Case 0-1); otherwise, the ellipses are disjoint (Case 0-3). The logic relies on the fact that there are no intersection points, which is indicated whenever there are no real solutions to the quartic polynomial. To test whether the second ellipse center (h_{2TR}, k_{2TR}) is inside the first ellipse, evaluate the first ellipse equation at the point $x = h_{2TR}$, and $y = k_{2TR}$; if the result is less than one, then the point (h_{2TR}, k_{2TR}) is inside the first ellipse. The complete logic for determining overlap area when $A_1 B_1 > A_2 B_2$ is:

If the polynomial has no real roots, and $A_1 B_1 > A_2 B_2$, and $\frac{h_{2TR}^2}{A_1^2} + \frac{k_{2TR}^2}{B_1^2} < 1$, then the first ellipse wholly contains the second, otherwise the two ellipses are disjoint.

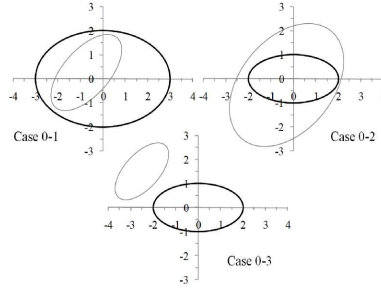


FIGURE 6. When the quartic polynomial has no real roots, the ellipse curves do not intersect. It follows that either one ellipse is fully contained within the other, or the ellipse areas are completely disjoint, resulting in three distinct cases for overlap area.

Alternatively, suppose that the second ellipse is larger than the first ellipse, then $A_1 B_1 < A_2 B_2$. If the first ellipse center $(0, 0)$ is inside the second ellipse, then the first ellipse is wholly contained within the second ellipse (Case 0-2); otherwise the ellipses are disjoint (Case 0-3). Again, the logic relies on the fact that there are no intersection points. To test whether $(0, 0)$ is inside the second ellipse, evaluate the second ellipse equation at the origin; if the result is less than zero, then the origin is inside the second ellipse. The complete logic for determining overlap area when $A_1 B_1 < A_2 B_2$ is:

If the polynomial has no real roots, and $A_1B_1 < A_2B_2$, and $FF < 0$, then the second ellipse wholly contains the first, otherwise the two ellipses are disjoint.

Suppose that the two ellipses are the same size, i.e., $A_1B_1 = A_2B_2$. In this case, when no intersection points exist, the ellipses must be disjoint (Case 0-3). It also turns out that the polynomial solver of [3] will return no real solutions if the ellipses are identical. This special case is also handled in the overlap area algorithm presented below. Pseudo-code for a function NOINTPTS that determines overlap area for the cases depicted in Fig. 6 is shown in Fig. 14.

If the polynomial solver returns either two or four real roots to the quartic equation, then the ellipse curves intersect. For the algorithm presented here, all of the various possibilities for the number and type of real roots are addressed by creating a list of distinct real roots. The first step is to loop through the entire array of complex roots returned by the polynomial solver, and retrieve only real roots, i.e., only those roots whose imaginary component is zero. The algorithm presented here then sorts the real roots, allowing for an efficient check for multiple roots. As the sorted list of real roots is traversed, any root that is ‘identical’ to the previous root can be skipped.

Each distinct real root of the polynomial represents a y -value where the two ellipses intersect. Each y -value can represent either one or two potential points of intersection. In the first case, suppose that the polynomial root is $y = B_1$ (or $y = -B_1$), then the y -value produces a single intersection point, which is at $(0, B_1)$ (or $(0, -B_1)$). In the second case, if the y -value is in the open interval $(-B_1, B_1)$, then there are two potential intersection points where the y -value is on the first ellipse:

$$\left(A_1 \cdot \sqrt{1 - \frac{y^2}{B_1^2}}, y \right) \text{ and } \left(-A_1 \cdot \sqrt{1 - \frac{y^2}{B_1^2}}, y \right)$$

Each potential intersection point (x_i, y_i) is evaluated in the second ellipse equation:

$$AA \cdot x_i^2 + BB \cdot x_i \cdot y_i + CC \cdot y_i^2 + DD \cdot x_i + EE \cdot y_i + FF, \quad i = 1, 2$$

If the expression evaluates to zero, then the point (x, y) is on both ellipses, i.e., it is an intersection point. By checking all points (x, y) for each value of y that is a root of the polynomial, a list of distinct intersection points is generated. The number of distinct intersection points must be either 0, 1, 2, 3 or 4. The case of zero intersection points is described above, with all possible sub-cases illustrated in Fig. 6. If there is only one distinct intersection point, then the two ellipses must be tangent at that point. The three possibilities for a single tangent point are shown in Fig. 7.

For the purpose of determining overlap area, the cases of 0 or 1 intersection points can be handled in the same way. When two intersection points exist, there are three possible sub-cases, shown in Fig. 8. It is possible that both of the intersection points are tangents (Case 2-1 and Case 2-2). In both of these sub-cases, one ellipse must be fully contained within the other. The only other possibility for two intersection points is a partial overlap (Case 2-3).

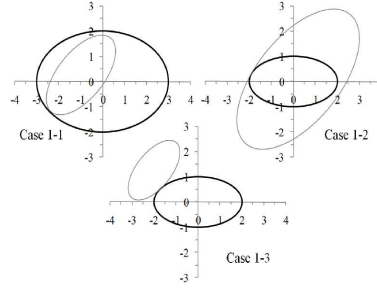


FIGURE 7. When only one intersection point exists, the ellipses must be tangent at the intersection point. As with the case of zero intersection points, either one ellipse is fully contained within the other, or the ellipse areas are disjoint. The algorithm for finding overlap area in the case of zero intersection points can also be used when there is a single intersection point.

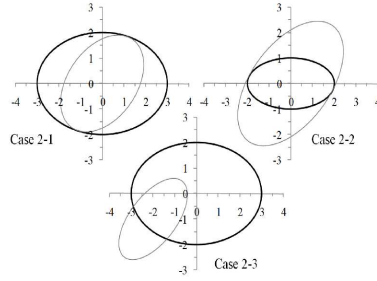


FIGURE 8. When two intersection points exist, either both of the points are tangents, or the ellipse curves cross at both points. For two tangent points, one ellipse must be fully contained within the other. For two crossing points, a partial overlap must exist

Each sub-case in Fig. 8 requires a different overlap-area calculation. When two intersection points exist, either both of the points are tangents, or the ellipse curves cross at both points. Specifically, when there are two intersection points, if one point is a tangent, then both points must be tangents. And, if one point is not a tangent, then neither point is a tangent. So, it suffices to check one of the intersection points for tangency. Suppose the ellipses are tangent at an intersection point; then, points that lie along the first ellipse on either side of the intersection will lie in the same region of the second ellipse (inside or outside). That is, if two points are chosen that lie on the first ellipse, one on each side of the intersection, then both points will either be inside the second ellipse, or they will both be outside the second ellipse. If the ellipse curves cross at the intersection point, then the two chosen points will be in different regions of the second ellipse.

A logic based on testing points that are adjacent to a tangent point can be implemented numerically to test whether an intersection point is a tangent or a cross-point. Starting with an intersection point (x, y) , calculate the parametric angle on the first ellipse, by the rules in Table 2.2:

$$\theta = \begin{cases} \arccos(x/A_1) & y \geq 0 \\ 2\pi - \arccos(x/A_1) & y < 0 \end{cases} \quad (30)$$

A small perturbation angle is then calculated. For the method presented here, we seek to establish an angle that corresponds to a point on the first ellipse that is a given distance, approximately $2EPS$, away from the intersection point:

$$EPS_{\text{Radian}} = \arcsin\left(\frac{2 \cdot EPS}{\sqrt{x^2 + y^2}}\right) \quad (31)$$

The angle EPS_{Radian} is then used with the parametric form of the first ellipse to determine two points adjacent to (x, y) :

$$\begin{aligned} x_1 &= A_1 \cdot \cos(\theta + EPS_{\text{Radian}}) \\ y_1 &= B_1 \cdot \sin(\theta + EPS_{\text{Radian}}) \\ x_2 &= A_1 \cdot \cos(\theta - EPS_{\text{Radian}}) \\ y_2 &= B_1 \cdot \sin(\theta - EPS_{\text{Radian}}) \end{aligned} \quad (32)$$

Each of the points is then evaluated in the second ellipse equation:

$$\text{test}_i = AA \cdot x_i^2 + BB \cdot x_i \cdot y_i + CC \cdot y_i^2 + DD \cdot x_i + EE \cdot y_i + FF, \quad i = 1, 2 \quad (33)$$

If the value of test_i is positive, then the point (x_i, y_i) is outside the second ellipse. It follows that the product of the two test-point evaluations $\text{test}_1\text{test}_2$ will be positive if the intersection point is a tangent, since at a tangent point both test points will be on the same side of the ellipse. The product of the test-point evaluations will be negative if the two ellipse curves cross at the intersection point, since the test points will be on opposite sides of the ellipse. The function `ISTANPT` implements this logic to check whether an intersection point is a tangent or a cross-point; pseudo-code is shown in Fig. 18.

When there are two intersection points, the `ISTANPT` function can be used to differentiate the case 2-3 (Fig. 8) from the cases 2-1 and 2-2. Either of the two known intersection points can be checked with `ISTANPT`. If the intersection point is a tangent, then both of the intersection points must be tangents, so the case is either 2-1 or 2-2, and one ellipse must be fully contained within the other. For cases 2-1 and 2-2, the geometric logic used for 0 or 1 intersection points can also be used, i.e., the function `NOINTPTS` can be used to determine the overlap area for these cases. If the two ellipse curves cross at the tested intersection point, then the case must be 2-3, representing a partial overlap between the two ellipse areas.

For case 2-3, with partial overlap between the two ellipses, the approach for finding overlap area is based on using the two points (x_1, y_1) and (x_2, y_2) with segment the algorithm (Table 2; Fig. 2) to determine the partial overlap area contributed by each ellipse. The total overlap area is the sum of the two segment areas. The two intersection points divide each ellipse into two segment areas (see Fig. 5). Only one sector area from each ellipse contributes to the overlap area. The segment algorithm returns the area between the secant line and the portion of the ellipse from the first point to the second point traversed in a counter-clockwise direction. For the overlap area calculation, the two points must be passed to the segment algorithm in the order that will return the correct segment area. The default order is counter-clockwise from the first point (x_1, y_1) to the second point (x_2, y_2) . A check is made to determine whether this order will return the desired segment area. First,

the parametric angles corresponding to (x_1, y_1) and (x_2, y_2) on the first ellipse are determined, by the rules in Table 2.2:

$$\theta_1 = \begin{cases} \arccos(x_1/A_1) & y_1 \geq 0 \\ 2\pi - \arccos(x_1/A_1) & y_1 < 0 \end{cases} \quad (34)$$

$$\theta_2 = \begin{cases} \arccos(x_2/A_1) & y_2 \geq 0 \\ 2\pi - \arccos(x_2/A_1) & y_2 < 0 \end{cases} \quad (35)$$

Then, a point between (x_1, y_1) and (x_2, y_2) that is on the first ellipse is found:

$$\begin{aligned} x_{\text{mid}} &= A_1 \cdot \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \\ y_{\text{mid}} &= B_1 \cdot \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \end{aligned} \quad (36)$$

The point $(x_{\text{mid}}, y_{\text{mid}})$ is on the first ellipse between (x_1, y_1) and (x_2, y_2) when travelling counter-clockwise from (x_1, y_1) and (x_2, y_2) . If $(x_{\text{mid}}, y_{\text{mid}})$ is inside the second ellipse, then the desired segment of the first ellipse contains the point $(x_{\text{mid}}, y_{\text{mid}})$. In this case, the segment algorithm should integrate in the default order, counterclockwise from (x_1, y_1) to (x_2, y_2) . Otherwise, the order of the points should be reversed before calling the segment algorithm, causing it to integrate counterclockwise from (x_2, y_2) to (x_1, y_1) . The area returned by the segment algorithm is the area contributed by the first ellipse to the partial overlap.

The desired segment from the second ellipse is found in a manner to the first ellipse segment. A slight difference in the approach is required because the segment algorithm is implemented for ellipses that are centered at the origin and oriented with the coordinate axes; but, in the general case the intersection points (x_1, y_1) and (x_2, y_2) lie on the second ellipse that is in a displaced and rotated location. The approach presented here translates and rotates the second ellipse to the origin so that the segment algorithm can be used. It suffices to translate then rotate the two intersection points by amounts that put the second ellipse centered at the origin and oriented with the coordinate axes:

$$\begin{aligned} x_{1\text{TR}} &= (x_1 - h_{2\text{TR}}) \cdot \cos(\varphi_1 - \varphi_2) + (y_1 - k_{2\text{TR}}) \cdot \sin(\varphi_2 - \varphi_1) \\ y_{1\text{TR}} &= (x_1 - h_{2\text{TR}}) \cdot \sin(\varphi_1 - \varphi_2) + (y_1 - k_{2\text{TR}}) \cdot \cos(\varphi_1 - \varphi_2) \\ x_{2\text{TR}} &= (x_2 - h_{2\text{TR}}) \cdot \cos(\varphi_1 - \varphi_2) + (y_2 - k_{2\text{TR}}) \cdot \sin(\varphi_2 - \varphi_1) \\ y_{2\text{TR}} &= (x_2 - h_{2\text{TR}}) \cdot \sin(\varphi_1 - \varphi_2) + (y_2 - k_{2\text{TR}}) \cdot \cos(\varphi_1 - \varphi_2) \end{aligned} \quad (37)$$

The new points $(x_{1\text{TR}}, y_{1\text{TR}})$ and $(x_{2\text{TR}}, y_{2\text{TR}})$ lie on the second ellipse after a translation+rotation that puts the second ellipse at the origin, oriented with the coordinate axes. The new points can be used as inputs to the segment algorithm to determine the overlap area contributed by the second ellipse. As with the first ellipse, the order of the points must be determined so that the segment algorithm returns the appropriate area. The default order is counter-clockwise from the first point $(x_{1\text{TR}}, y_{1\text{TR}})$ to the second point $(x_{2\text{TR}}, y_{2\text{TR}})$. A check is made to determine whether this order will return the desired segment area. First, the parametric angles corresponding to points $(x_{1\text{TR}}, y_{1\text{TR}})$ and $(x_{2\text{TR}}, y_{2\text{TR}})$ on the second ellipse are determined, by the rules in Table 2.2:

$$\theta_1 = \begin{cases} \arccos(x_{1\text{TR}}/A_2) & y_{1\text{TR}} \geq 0 \\ 2\pi - \arccos(x_{1\text{TR}}/A_2) & y_{1\text{TR}} < 0 \end{cases} \quad (38)$$

$$\theta_2 = \begin{cases} \arccos(x_{2\text{TR}}/A_2) & y_{2\text{TR}} \geq 0 \\ 2\pi - \arccos(x_{2\text{TR}}/A_2) & y_{2\text{TR}} < 0 \end{cases} \quad (39)$$

Then, a point on the second ellipse between $(x_{1\text{TR}}, y_{1\text{TR}})$ and $(x_{2\text{TR}}, y_{2\text{TR}})$ is found:

$$\begin{aligned} x_{\text{mid}} &= A_2 \cdot \cos\left(\frac{\theta_1 + \theta_2}{2}\right) \\ y_{\text{mid}} &= B_2 \cdot \sin\left(\frac{\theta_1 + \theta_2}{2}\right) \end{aligned}$$

The point $(x_{\text{mid}}, y_{\text{mid}})$ is on the second ellipse between $(x_{1\text{TR}}, y_{1\text{TR}})$ and $(x_{2\text{TR}}, y_{2\text{TR}})$ when travelling counter-clockwise from $(x_{1\text{TR}}, y_{1\text{TR}})$ and $(x_{2\text{TR}}, y_{2\text{TR}})$. The new point $(x_{\text{mid}}, y_{\text{mid}})$ lies on the centered second ellipse. To determine the desired segment of the second ellipse, the new point $(x_{\text{mid}}, y_{\text{mid}})$ must be rotated then translated back to a corresponding position on the once-translated+rotated second ellipse:

$$\begin{aligned} x_{\text{midRT}} &= x_{\text{mid}} \cdot \cos(\varphi_2 - \varphi_1) + y_{\text{mid}} \cdot \sin(\varphi_1 - \varphi_2) + h_{2\text{TR}} \\ y_{\text{midRT}} &= x_{\text{mid}} \cdot \sin(\varphi_2 - \varphi_1) + y_{\text{mid}} \cdot \cos(\varphi_1 - \varphi_2) + k_{2\text{TR}} \end{aligned}$$

If $(x_{\text{midRT}}, y_{\text{midRT}})$ is inside the first ellipse, then the desired segment of the second ellipse contains the point $(x_{\text{mid}}, y_{\text{mid}})$. In this case, the segment algorithm should integrate in the default order, counterclockwise from $(x_{1\text{TR}}, y_{1\text{TR}})$ to $(x_{2\text{TR}}, y_{2\text{TR}})$. Otherwise, the order of the points should be reversed before calling the segment algorithm, causing it to integrate counterclockwise from $(x_{2\text{TR}}, y_{2\text{TR}})$ to $(x_{1\text{TR}}, y_{1\text{TR}})$. The area returned by the segment algorithm is the area contributed by the second ellipse to the partial overlap. The sum of the segment areas from the two ellipses is then equal to the ellipse overlap area. The TWOINTPTS function calculates the overlap area for partial overlap with two intersection points (Case 2-3); pseudo-code is shown in Fig. 15.

There are two possible sub-cases for three intersection points, shown in Fig. 9. One of the three points must be a tangent point, and the ellipses must cross at the other two points. The cases are distinct only in the sense that the tangent point occurs with ellipse 2 on the interior side of ellipse 1 (Case 3-1), or with ellipse 2 on the exterior side of ellipse 1 (Case 3-2). The overlap area calculation is performed in the same manner for both cases, by calling the TWOINTPTS function with the two cross-point intersections. The ISTANPT function can be used to determine which point is a tangent; the remaining two intersection points are then passed to TWOINTPTS. This logic is implemented in the THREEINTPTS function, with pseudo-code in Fig. 16.

There is only one possible case for four intersection points, shown in Fig. 9. The two ellipse curves must cross at all four of the intersection points, resulting in a partial overlap. The overlap area consists of two segments from each ellipse, and a central convex quadrilateral. For the approach presented here, the four intersection points are sorted ascending in a counter-clockwise order around the first ellipse.

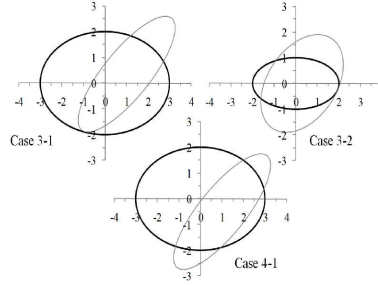


FIGURE 9. When three intersection points exist, one must be a tangent, and the ellipse curves must cross at the other two points, always resulting in a partial overlap. When four intersection points exist, the ellipse curves must cross at all four points, again resulting in a partial overlap

The ordered set of intersection points is (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) . The ordering allows a direct calculation of the quadrilateral area. The standard formula uses the cross-product of the two diagonals:

$$\begin{aligned} \text{area} &= \frac{1}{2} |(x_3 - x_1, y_3 - y_1) \times (x_4 - x_2, y_4 - y_2)| \\ &= \frac{1}{2} |(x_3 - x_1) \cdot (y_4 - y_2) - (x_4 - x_2) \cdot (y_3 - y_1)| \end{aligned} \quad (40)$$

The point ordering also simplifies the search for the appropriate segments of each ellipse that contribute to the overlap area.

Suppose that the first two sorted points (x_1, y_1) and (x_2, y_2) demarcate a segment of the first ellipse that contributes to the overlap area, as shown in Fig. 9 and Fig. 10. It follows that the contributing segments from the first ellipse are between (x_1, y_1) and (x_2, y_2) , and also between (x_3, y_3) and (x_4, y_4) . In this case, the contributing segments from the second ellipse are between (x_2, y_2) and (x_3, y_3) , and between (x_4, y_4) and (x_1, y_1) . To determine which segments contribute to the overlap area, it suffices to test whether a point midway between (x_1, y_1) and (x_2, y_2) is inside or outside the second ellipse. The segment algorithm is used for each of the four areas, and added to the quadrilateral to obtain the total overlap area.

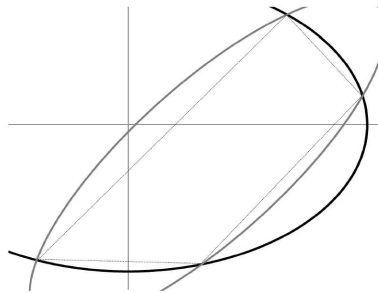


FIGURE 10. Overlap Area with four intersection points (Case 4-1). The overlap area consists of two segments from each ellipse, and a central convex quadrilateral.

An implementation of the ELLIPSE_ELLIPSE_OVERLAP algorithm in c-code is shown in Appendix 6. The code compiles under Cygwin-1.7.7-1, and returns the following values for the test cases presented above in Fig. 6, Fig. 7, Fig. 8 and Fig. 9:

LISTING 5. Return values for the test cases presented above in Fig. 6, Fig. 7, Fig. 8 and Fig. 9.

```

138 cc call\_ee.c ellipse\_ellipse\_overlap.c -o call\_ee.exe
139
140 ./call\_ee
141
142 Calling ellipse\_ellipse\_overlap.c
143
144
145
146 Case 0-1: area =      6.28318531, return_value = 111
147           ellipse 2 area by pi*a2*b2 =      6.28318531
148
149 Case 0-2: area =      6.28318531, return_value = 110
150           ellipse 1 area by pi*a1*b1 =      6.28318531
151
152 Case 0-3: area =      0.00000000, return_value = 103
153           Ellipses are disjoint, overlap area = 0.0
154
155
156
157
158
159 Case 1-1: area =      6.28318531, return_value = 111
160           ellipse 2 area by pi*a2*b2 =      6.28318531
161
162 Case 1-2: area =      6.28318531, return_value = 110
163           ellipse 1 area by pi*a1*b1 =      6.28318531
164
165 Case 1-3: area =     -0.00000000, return_value = 107
166           Ellipses are disjoint, overlap area = 0.0
167
168
169
170
171
172
173 Case 2-1: area =     10.60055478, return_value = 109
174           ellipse 2 area by pi*a2*b2 =     10.60287521
175
176 Case 2-2: area =      6.28318531, return_value = 110
177           ellipse 1 area by pi*a1*b1 =      6.28318531
178
179 Case 2-3: area =      3.82254574, return_value = 107
180
181
182
183
184
185 Case 3-1: area =      7.55370392, return_value = 107
186
187 Case 3-2: area =      5.67996234, return_value = 107
188
189
190
191 Case 4-1: area =     16.93791852, return_value = 109
192

```

LISTING 6. The ELLIPSE_ELLIPSE_OVERLAP algorithm is shown for calculating the overlap area between two general ellipses. The algorithm calls several supporting functions, including the polynomial solvers BIQUADROOTS, CUBICROOTS and QUADROOTS, from CACM Algorithm 326 [2]. The remaining functions are outlined in figures below.

```

193
194 (Area, Code) ← ELLIPSE_ELLIPSE_OVERLAP (A1, B1, H1, K1,  $\varphi 1$ , A2, B2, H2, K2,  $\varphi 2$ )
195
196 do if (A1 = 0 or B1 = 0) OR (A2 = 0 or B2 = 0)
197
198     then return (-1, ERROR_ELLIPSE_PARAMETERS) :DATA CHECK
199
200 do if ( $|\varphi 1| > 2\pi$ )
201
202     then  $\varphi 1 \leftarrow (\varphi 1 \bmod 2\pi)$ 
203
204 do if ( $|\varphi 2| > 2\pi$ )
205
206     then  $\varphi 2 \leftarrow (\varphi 2 \bmod 2\pi)$ 
207
208 H2_TR ← (H2 - H1)*cos( $\varphi 1$ ) + (K2 - K1)*sin( $\varphi 1$ ) :TRANS+ROT ELL2
209
210 K2_TR ← (H1 - H2)*sin( $\varphi 1$ ) + (K2 - K1)*cos( $\varphi 1$ )
211
212  $\varphi 2R \leftarrow \varphi 2 - \varphi 1$ 
213
214 do if ( $|\varphi 2R| > 2\pi$ )
215
216     then  $\varphi 2R \leftarrow (\varphi 2R \bmod 2\pi)$ 
217
218 AA ←  $\cos^2(\varphi 2R)/A2^2 + \sin^2(\varphi 2R)/B2^2$  :BUILD\, IMPLICIT\, COEFFS ELL2TR
219
220 BB ←  $2*\cos(\varphi 2R)*\sin(\varphi 2R)/A2^2 - 2*\cos(\varphi 2R)*\sin(\varphi 2R)/B2^2$ 
221
222 CC ←  $\sin^2(\varphi 2R)/A2^2 + \cos^2(\varphi 2R)/B2^2$ 
223
224 DD ←  $-2*\cos(\varphi 2R)*( \cos(\varphi 2R)*H2\_TR + \sin(\varphi 2R)*K2\_TR)/A2^2$ 
225
226      $- 2*\sin(\varphi 2R)*( \sin(\varphi 2R)*H2\_TR - \cos(\varphi 2R)*K2\_TR)/B2^2$ 
227
228 EE ←  $-2*\sin(\varphi 2R)*( \cos(\varphi 2R)*H2\_TR + \sin(\varphi 2R)*K2\_TR)/A2^2$ 
229
230      $+ 2*\cos(\varphi 2R)*( \sin(\varphi 2R)*H2\_TR - \cos(\varphi 2R)*K2\_TR)/B2^2$ 
231
232 FF ←  $(-\cos(\varphi 2R)*H2\_TR - \sin(\varphi 2R)*K2\_TR)^2/A2^2$ 
233
234      $+ (\sin(\varphi 2R)*H2\_TR - \cos(\varphi 2R)*K2\_TR)^2/B2^2 - 1$ 
235
236 :BUILD QUARTIC POLYNOMIAL COEFFICIENTS FROM THE TWO ELLIPSE EQNS
237
238 cy[4] ←  $A1^4*AA^2 + B1^2*(A1^2*(BB^2 - 2*AA*CC) + B1^2*CC^2)$ 
239
240 cy[3] ←  $2*B1*(B1^2*CC*EE + A1^2*(BB*DD - AA*EE))$ 
241
242 cy[2] ←  $A1^2*((B1^2*(2*AA*CC - BB^2) + DD^2 - 2*AA*FF)$ 
243
244      $- 2*A1^2*AA^2 + B1^2*(2*CC*FF + EE^2))$ 
245
246 cy[1] ←  $2*B1*(A1^2*(AA*EE - BB*DD) + EE*FF)$ 
247
248 cy[0] ←  $(A1*(A1*AA - DD) + FF)*(A1*(A1*AA + DD) + FF)$ 
249
250 py[0] ← 1
251
252 do if ( $|cy[4]| > 0$ ) :SOLVE QUARTIC EQ
253
254     then for i ← 0 to 3 by 1

```

```

255         py[4-i] ← cy[i]/cy[4]
256
257     r [][] ← BIQUADROOTS (py[])
258
259     nroots ← 4
260
261     else if (|cy[3]| > 0)                                :SOLVE CUBIC EQ
262
263     then for i ← 0 to 2 by 1
264
265         py[3-i] ← cy[i]/cy[3]
266
267         r [][] ← CUBICROOTS (py[])
268
269         nroots ← 3
270
271     else if (|cy[2]| > 0)                                :SOLVE QUADRATIC EQ
272
273     then for i ← 0 to 1 by 1
274
275         py[2-i] ← cy[i]/cy[2]
276
277         r [][] ← QUADROOTS (py[])
278
279         nroots ← 2
280
281     else if (|cy[1]| > 0)                                :SOLVE LINEAR EQ
282
283     then r[1][1] ← (-cy[0]/cy[1])
284
285         r[2][1] ← 0
286
287         nroots ← 1
288
289     else                                                  :COMPLETELY DEGENERATE EQ
290
291         nroots ← 0
292
293     nychk ← 0                                            :IDENTIFY REAL ROOTS
294
295     for i ← 1 to nroots by 1
296
297         do if (|r[2][i]| < EPS)
298
299             then nychk ← nychk + 1
300
301                 ychk[nychk] ← r[1][i]*B1
302
303     for j ← 2 to nychk by 1                                :SORT REAL ROOTS
304
305         tmp0 ← ychk[j]
306
307         for k ← (j - 1) to 1 by -1
308
309             do if (ychk[k] = tmp0)
310
311                 then break
312
313                 else ychk[k+1] ← ychk[k]
314
315         ychk[k+1] ← tmp0
316
317     nintpts ← 0                                            :FIND INTERSECTION POINTS
318
319     for i ← 1 to nychk by 1
320
321         do if ((i > 1) and (|ychk[i] - ychk[i-1]| < EPS/2))
322
323             then continue
324
325         do if (|ychk[i]| > -B1)
326
327

```

```

328     then x1 ← 0
329
330     else x1 ← ? A1*sqrt (1.0 - ychk[i]2/B12)
331
332     x2 ← -x1
333
334     do if (|ellipse2tr(x1,ychk[i],AA,BB,CC,DD,EE,FF)| < EPS/2)
335
336         then nintpts ← nintpts + 1
337
338         do if (nintpts > 4)
339
340             then return (-1, ERROR_INTERSECTION_PTS)
341
342             xint[nintpts] ← x1
343
344             yint[nintpts] ← ychk[i]
345
346     do if ((|ellipse2tr(x2,ychk[i],AA,BB,CC,DD,EE,FF)| < EPS/2)
347
348         and (|x2 - x1| > EPS/2))
349
350         then nintpts ← nintpts + 1
351
352         do if (nintpts > 4)
353
354             then return (-1, ERROR_INTERSECTION_PTS)
355
356             xint[nintpts] ← x1
357
358             yint[nintpts] ← ychk[i]
359
360     switch (nintpts) :HANDLE ALL CASES FOR \# OF INTERSECTION PTS
361
362     case 0:
363
364     case 1:
365
366         (OverlapArea,Code) ← NOINTPTS (A1,B1,A2,B2,H1,K1,H2,K2,AA,
367
368             BB,CC,DD,EE,FF)
369
370         return (OverlapArea,Code)
371
372     case 2:
373
374         Code ← istanpt (xint[1],yint[1],A1,B1,AA,BB,CC,DD,EE,FF)
375
376         do if (Code == TANGENT_POINT)
377
378             then (OverlapArea,Code) ← NOINTPTS (A1,B1,A2,B2,H1,K1,
379
380                 H2,K2,AA,BB,CC,DD,EE,FF)
381
382             else (OverlapArea,Code) ← TWOINTPTS (xint[],yint[],A1,
383
384                 PHI_1,A2,B2,H2-TR,K2-TR,PHI_2,AA,BB,CC,DD,EE,FF)
385
386         return (OverlapArea,Code)
387
388     case 3:
389
390         (OverlapArea,Code) ← THREEINTPTS (xint,yint,A1,B1,PHI_1,
391
392             A2,B2,H2-TR,K2-TR,PHI_2,AA,BB,CC,DD,EE,FF)
393
394         return (OverlapArea,Code)
395
396     case 4:
397
398         (OverlapArea,Code) ← FOURINTPTS (xint,yint,A1,B1,PHI_1,
399
400             A2,B2,H2-TR,K2-TR,PHI_2,AA,BB,CC,DD,EE,FF)

```

```

401
402  return (OverlapArea, Code)

```

LISTING 7. The NOINTPTS subroutine. If there are either 0 or 1 intersection points, this function determines whether one ellipse is contained within the other (Cases 0-1, 0-2, 1-1 and 1-2), or if the ellipses are disjoint (Cases 0-3 and 1-3). The function returns the appropriate overlap area, and a code describing which case was encountered.

```

403
404  (OverlapArea, Code) ← NOINTPTS (A1, B1, A2, B2, H1, K1, H2_TR, K2_TR, AA,
405
406                                BB, CC, DD, EE, FF)
407
408  relsize ← A1*B1 - A2*B2
409
410  do if (relsize > 0)
411
412      then do if (((H2_TR*H2_TR)/(A1*A1)+(K2_TR*K2_TR)/(B1*B1)) < 1.0)
413
414          then return (π*A2*B2, ELLIPSE2_INSIDE_ELLIPSE1)
415
416          else return (0, DISJOINT_ELLIPSES)
417
418      else do if (relsize < 0)
419
420          then do if (FF < 0)
421
422              then return (π*A1*B1, ELLIPSE1_INSIDE_ELLIPSE2)
423
424              else return (0, DISJOINT_ELLIPSES)
425
426      else do if ((H1 = H2_TR) AND (K1 = K2_TR))
427
428          then return (π*A1*B1, ELLIPSES_ARE_IDENTICAL)
429
430      else return (-1, ERROR_CALCULATIONS)

```

LISTING 8. The TWOINTPTS subroutine. If there are 2 intersection points where the ellipse curves cross (Case 2-3), this function uses the ellipse sector algorithm to determine the contribution of each ellipse to the total overlap area. The function returns the appropriate overlap area, and a code indicating two intersection points.

```

431  (OverlapArea, Code) ← TWOINTPTS (xint [], yint [], A1, B1, φ1, A2, B2, H2_TR,
432
433                                K2_TR, φ2, AA, BB, CC, DD, EE, FF)
434
435  do if (|x[1]| > A1) :AVOID INVERSE TRIG ERRORS
436
437      then do if (x[1] < 0)
438
439          then x[1] ← -A1
440
441          else x[1] ← A1
442
443  do if (y[1] < 0) :FIND PARAMETRIC ANGLE FOR (x[1], y[1])
444
445      then θ1 ← 2π - arccos (x[1]/A1)
446
447      else θ1 ← arccos (x[1]/A1)
448
449  do if (|x[2]| > A1) :AVOID INVERSE TRIG ERRORS

```

```

450
451   then do if (x[2] < 0)
452
453       then x[2] ← -A1
454
455       else x[2] ← A1
456
457 do if (y[2] < 0)           :FIND PARAMETRIC ANGLE FOR (x[2], y[2])
458
459   then  $\theta_2 \leftarrow 2\pi - \arccos(x[2]/A1)$ 
460
461   else  $\theta_2 \leftarrow \arccos(x[2]/A1)$ 
462
463 do if ( $\theta_1 > \theta_2$ )           :GO CCW FROM  $\theta_1$  TO\,  $\theta_2$ 
464
465   then tmp ←  $\theta_1$ ,  $\theta_1 \leftarrow \theta_2$ ,  $\theta_2 \leftarrow$  tmp
466
467 xmid ←  $A1 \cdot \cos((\theta_1 + \theta_2)/2)$ 
468
469 ymid ←  $B1 \cdot \sin((\theta_1 + \theta_2)/2)$ 
470
471 do if ( $AA \cdot xmid^2 + BB \cdot xmid \cdot ymid + CC \cdot ymid^2 + DD \cdot xmid + EE \cdot ymid + FF > 0$ )
472
473   then tmp ←  $\theta_1$ ,  $\theta_1 \leftarrow \theta_2$ ,  $\theta_2 \leftarrow$  tmp
474
475 do if ( $\theta_1 > \theta_2$ )           :SEGMENT ALGORITHM FOR ELLIPSE 1
476
477   then  $\theta_1 ? \theta_1 - 2\pi$ 
478
479 do if ( $(\theta_2 - \theta_1) > \pi$ )
480
481   then trsign ← 1
482
483   else trsign ← -1
484
485 areal ←  $(A1 \cdot B1 \cdot (\theta_2 - \theta_1) + trsign \cdot \text{textbar } x[1] \cdot y[2] - x[2] \cdot y[1]) \cdot \text{textbar}$ 
486
487    $/2$ 
488
489 x1\_tr ←  $(x[1] - H2\_TR) \cdot \cos(\varphi_1 - \varphi_2) + (y[1] - K2\_TR) \cdot \sin(\varphi_2 - \varphi_1)$ 
490
491 y1\_tr ←  $(x[1] - H2\_TR) \cdot \sin(\varphi_1 - \varphi_2) + (y[1] - K2\_TR) \cdot \cos(\varphi_1 - \varphi_2)$ 
492
493 x2\_tr ←  $(x[2] - H2\_TR) \cdot \cos(\varphi_1 - \varphi_2) + (y[2] - K2\_TR) \cdot \sin(\varphi_2 - \varphi_1)$ 
494
495 y2\_tr ?  $(x[2] - H2\_TR) \cdot \sin(\varphi_1 - \varphi_2) + (y[2] - K2\_TR) \cdot \cos(\varphi_1 - \varphi_2)$ 
496
497 do if ( $|x1\_tr| > A2$ )           :AVOID INVERSE TRIG ERRORS
498
499   then do if ( $x1\_tr < 0$ )
500
501       then  $x1\_tr \leftarrow -A2$ 
502
503       else  $x1\_tr \leftarrow A2$ 
504
505 do if ( $y1\_tr < 0$ )           :FIND PARAMETRIC ANGLE FOR ( $x1\_tr$ ,  $y1\_tr$ )
506
507   then  $\theta_1 \leftarrow 2\pi - \arccos(x1\_tr/A2)$ 
508
509   else  $\theta_1 \leftarrow \arccos(x1\_tr/A2)$ 
510
511 do if ( $|x2\_tr| > A2$ )           :AVOID INVERSE TRIG ERRORS
512
513   then do if ( $x2\_tr < 0$ )
514
515       then  $x2\_tr \leftarrow -A2$ 
516
517       else  $x2\_tr \leftarrow A2$ 
518
519 do if ( $y2\_tr < 0$ )           :FIND PARAMETRIC ANGLE FOR ( $x2\_tr$ ,  $y2\_tr$ )
520
521   then  $\theta_2 \leftarrow 2\pi - \arccos(x2\_tr/A2)$ 
522
523   else  $\theta_2 \leftarrow \arccos(x2\_tr/A2)$ 

```



```

522
523 do if ( $\theta_1 > \theta_2$ )                                :GO CCW FROM  $\theta_1$  TO\,  $\theta_2$ 
524
525     then tmp  $\leftarrow \theta_1$ ,  $\theta_1 \leftarrow \theta_2$ ,  $\theta_2 \leftarrow$  tmp
526
527 xmid  $\leftarrow A_2 \cos((\theta_1 + \theta_2)/2)$ 
528
529 ymid  $\leftarrow B_2 \sin((\theta_1 + \theta_2)/2)$ 
530
531 xmid_rt = xmid*cos( $\varphi_2 - \varphi_1$ ) + ymid*sin( $\varphi_1 - \varphi_2$ ) + H2_TR
532
533 ymid_rt = xmid*sin( $\varphi_2 - \varphi_1$ ) + ymid*cos( $\varphi_2 - \varphi_1$ ) + K2_TR
534
535 do if ( $xmid\_rt^2/A_1^2 + ymid\_rt^2/B_1^2 > 1$ )
536
537     then tmp  $\leftarrow \theta_1$ ,  $\theta_1 \leftarrow \theta_2$ ,  $\theta_2 \leftarrow$  tmp
538
539 do if ( $\theta_1 > \theta_2$ )                                :SEGMENT ALGORITHM FOR ELLIPSE 2
540
541     then  $\theta_1 \leftarrow \theta_1 - 2\pi$ 
542
543 do if ( $(\theta_2 - \theta_1) > \pi$ )
544
545     then trsign  $\leftarrow 1$ 
546
547     else trsign  $\leftarrow -1$ 
548
549 area2  $\leftarrow (A_2*B_2*(\theta_2 - \theta_1)$ 
550
551 + trsign* $|x_{1tr} * y_{2tr} - x_{2tr} * y_{1tr}|$  /2
552
553 return (areal + area2, TWO_INTERSECTION_POINTS)

```

LISTING 9. The THREEINTPTS subroutine. When there are three intersection points, one of the points must be a tangent point, and the ellipses must cross at the other two points. For the purpose of determining overlap area, the TWOINTPTS function can be used with the two cross-point intersections. The ISTANPT function can be used to determine which point is a tangent; the remaining two intersection points are then passed to TWOINTPTS. The function returns the appropriate overlap area, and a code indicating three intersection points.

```

554 OverlapArea, Code)  $\leftarrow$  THREEINTPTS (xint[], yint[], A1, B1,  $\varphi_1$ , A2, B2, H2_TR,
555
556                                     K2_TR,  $\varphi_2$ , AA, BB, CC, DD, EE, FF)
557 tanpts  $\leftarrow 0$ 
558
559 for i  $\leftarrow 1$  to nychk by 1
560
561     code  $\leftarrow$  ISTANPT ISTANPT (x[i], y[i], A1, B1, AA, BB, CC, DD, EE, FF)
562
563     do if (code = TANGENT_POINT)
564
565         then tanpts  $\leftarrow$  tanpts + 1
566
567         tanindex  $\leftarrow i$ 
568
569 do if NOT (tanpts = 1)
570
571     then return (-1, ERROR_INTERSECTION_POINTS)
572
573 switch (tanindex)                                :STORE THE INTERSECTION POINTS
574
575     case 1:                                       :TANGENT POINT IS IN (x[1], y[1])
576
577         xint[1]  $\leftarrow$  xint[3]

```

```

578
579     yint[1] ← yint[3]
580
581     case 2:                                     :TANGENT POINT IS IN (x[2], y[2])
582
583         xint[2] ← xint[3]
584
585         yint[2] ← yint[3]
586
587 (OverlapArea,code) ← TWOINTPTS (xint[], yint[], A1,B1,φ1,A2,B2,H2_TR,
588                                K2_TR,φ2,AA,BB,CC,DD,EE,FF)
589
590
591 return (OverlapArea,THREE_INTERSECTION_POINTS)

```

LISTING 10. The FOURINTPTS subroutine. When there are four intersection points, the ellipse curves must cross at all four points. A partial overlap area exists, consisting of two segments from each ellipse and a central quadrilateral. The function returns the appropriate overlap area, and a code indicating four intersection points.

```

592 verlapArea,Code) ← FOURINTPTS (xint[], yint[], A1,B1,φ1,A2,B2,H2_TR,
593                                K2_TR,φ2,AA,BB,CC,DD,EE,FF)
594
595
596 for i ← 1 to 4 by 1                               :AVOID INVERSE TRIG ERRORS
597
598     do if (|xint[i]| > A1)
599
600         then do if (xint[i] < 0)
601
602             then xint[i] ← -A1
603
604             else xint[i] ← A1
605
606     do if (yint[i] < 0)                             :FIND PARAMETRIC ANGLES
607
608         then θ[i] ← 2π — arccos (xint[i]/A1)
609
610         else θ[i] ← arccos (xint[i]/A1)
611
612 for j ← 2 to 4 by 1                               :PUT POINTS IN CCW ORDER
613
614     tmp0 ← θ[j]
615
616     tmp1 ← xint[j]
617
618     tmp2 ← yint[j]
619
620     for k ← (j-1) to 1 by -1                       :INSERTION SORT BY ANGLE
621
622         do if (θ[k] <= tmp0)
623
624             then break
625
626             else θ[k+1] ← θ[k]
627
628                 xint[k+1] ← xint[k]
629
630                 yint[k+1] ← yint[k]
631
632     area1 ← (|(xint[3] — xint[1])*(yint[4] — yint[2]) —
633
634     xint[4] — xint[2])*(yint[3] — yint[1])| /2)      :QUAD AREA
635
636     for i ← 1 to 4 by 1                               :TRANSLATE+ROTATE ELLIPSE 2
637
638         xint_tr[i] ← (xint[i] — H2_TR)*cos (φ1 — φ2)

```

```

639      + (yint[i] — K2_TR)*sin (φ2 — φ1)
640
641      yint_tr[i] ← (xint[i] — H2_TR)*sin (φ1 — φ2)
642
643      + (yint[i] — K2_TR)*cos (φ1 — φ2)
644
645      do if (|xint_tr[i]| > A2) :AVOID INVERSE TRIG ERRORS
646
647          then do if (xint_tr[i] < 0)
648
649              then xint_tr[i] ← -A2
650
651              else xint_tr[i] ← A2
652
653
654      do if (yint_tr[i] < 0) :FIND PARAM ANGLES FOR (xint_tr, yint_tr)
655
656          then θ_tr[i] ← 2π — arccos (xint_tr[i]/A2)
657
658          else θ_tr[i] ← arccos (xint_tr[i]/A2)
659
660      xmid ← A1*cos ((θ1 + θ2)/2)
661
662      ymid ← B1*sin ((θ1 + θ2)/2)
663
664      do if (AA*xmid2+BB*xmid*ymid+CC*ymid2+DD*xmid+EE*ymid+FF < 0)
665
666          then area2 = (A1*B1*(θ[2] — θ[1]))
667
668          — |(xint[1]*yint[2] — xint[2]*yint[1])|/2
669
670          area3 = (A1*B1*(θ[4] — θ[3]))
671
672          — |(xint[3]*yint[4] — xint[4]*yint[3])|/2
673
674          area4 = (A2*B2*(θ_tr[3] — θ_tr[2]))
675
676          — |(xint_tr[2]*yint_tr[3] — xint_tr[3]*yint_tr[2])|/2
677
678          area5 = (A2*B2*(θ_tr[1] — θ_tr[4] — twopi))
679
680          — |(xint_tr[4]*yint_tr[1] — xint_tr[1]*yint_tr[4])|/2
681
682          else area2 = (A1*B1*(θ[3] — θ[2]))
683
684          — |(xint[2]*yint[3] — xint[3]*yint[2])|/2
685
686          area3 = (A1*B1*(θ[1] — (θ[4] — twopi)))
687
688          — |(xint[4]*yint[1] — xint[1]*yint[4])|/2
689
690          area4 = (A2*B2*(θ_tr[2] — θ_tr[1]))
691
692          — |(xint_tr[1]*yint_tr[2] — xint_tr[2]*yint_tr[1])|/2
693
694          area5 = (A2*B2*(θ_tr[4] — θ_tr[3]))
695
696          — |(xint_tr[3]*yint_tr[4] — xint_tr[4]*yint_tr[3])|/2
697
698      return (area1+area2+area3+area4+area5, FOUR_INTERSECTION_POINTS)

```

LISTING 11. The ISTANPT subroutine. Given an intersection point (x, y) that satisfies both Ellipse Eq. 21 and Ellipse Eq. 22, the function determines whether the two ellipse curves are tangent at (x, y) , or if the ellipse curves cross at (x, y) .

```

699      Code ← ISTANPT (x, y, A1, B1, AA, BB, CC, DD, EE, FF)
700
701      do if (|x| > A1) :AVOID INVERSE TRIG ERRORS
702

```

```

703     then do if x < 0
704
705         then x ← -A1
706
707         else x ← A1
708
709 do if (y < 0)                                :FIND PARAMETRIC ANGLE FOR (x, y)
710
711     then  $\theta \leftarrow 2\pi - \arccos(x/A1)$ 
712
713     else  $\theta \leftarrow \arccos(x/A1)$ 
714
715 branch ←  $v(x^2 + y^2)$                         :DETERMINE PERTURBATION ANGLE
716
717 do if (branch < 100*EPS)
718
719     then eps_radian ← 2*EPS
720
721     else eps_radian ←  $\arcsin(2*EPS/branch)$ 
722
723 x1 ←  $A1 \cos(\theta + \text{eps\_radian})$            :CREATE TEST POINTS ON EACH SIDE
724
725 y1 ←  $B1 \cos(\theta + \text{eps\_radian})$            :OF THE INPUT POINT (x, y)
726
727 x2 ←  $A1 \cos(\theta - \text{eps\_radian})$ 
728
729 y2 ←  $B1 \cos(\theta - \text{eps\_radian})$ 
730
731 test1 ←  $AA*x1^2 + BB*x1*y1 + CC*y1^2 + DD*x1 + EE*y1 + FF$ 
732
733 test2 ←  $AA*x2^2 + BB*x2*y2 + CC*y2^2 + DD*x2 + EE*y2 + FF$ 
734
735 do if (test1*test2 > 0)
736
737     then return TANGENT_POINT
738
739     else return INTERSECTION_POINT

```

LISTING 12. C-SOURCE CODE FOR ELLIPSE_SEGMENT

4. APPENDIX A.

```

740
741
742 /*
743
744 *
745 *
746 * Function: double ellipse_segment
747 *
748 *
749 *
750 * Purpose: Given the parameters of an ellipse and two points that lie on
751 *
752 * the ellipse, this function calculates the ellipse segment
753 *
754 * area
755 *
756 * between the secant line and the ellipse. Points are input as
757 *
758 * (X1, Y1) and (X2, Y2), and the segment area is defined to be
759 *
760 * between the secant line and the ellipse from the first point
761 *
762 * (X1, Y1) to the second point (X2, Y2) in the counter-
763 *
764 * clockwise
765 *
766 * direction.

```

```

766 * Reference: Hughes and Chraibi (2011), Calculating Ellipse Overlap Areas
767 *
768 *
769 *
770 * Dependencies: math.h for calls to trig and absolute value functions
771 *
772 * program_constants.h error message codes and constants
773 *
774 *
775 *
776 * Inputs: 1. double A ellipse semi-axis length in x-direction
777 *
778 * 2. double B ellipse semi-axis length in y-direction
779 *
780 * 3. double X1 x-value of the first point on the ellipse
781 *
782 * 4. double Y1 y-value of the first point on the ellipse
783 *
784 * 5. double X2 x-value of the second point on the ellipse
785 *
786 * 6. double Y2 y-value of the second point on the ellipse
787 *
788 *
789 *
790 * Outputs: 1. int *MessageCode stores diagnostic information
791 *
792 * integer codes in program_constants.h
793 *
794 *
795 *
796 * Return: The value of the ellipse segment area:
797 *
798 * -1.0 is returned in case of an error with input data
799 *
800 *
801 *
802 *****
      */
803
804
805
806 //
=====

807
808 //== INCLUDE ANSI C SYSTEM AND USER-DEFINED HEADER FILES
=====

809
810 //
=====

811
812 #include "program_constants.h"
813
814
815
816 double ellipse_segment (double A, double B, double X1, double Y1, double X2
      ,
817
818                          double Y2, int *MessageCode)
819 {
820 {
821
822     double theta1; //— parametric angle of the first point
823
824     double theta2; //— parametric angle of the second point
825
826     double trsign; //— sign of the triangle area
827
828     double pi = 2.0 * asin \eqref{GrinEQ__1.0_}; //— a maximum-
      precision value of pi
829
830     double twopi = 2.0 * pi; //— a maximum-precision value of 2*pi

```

```

831
832
833
834 //— Check the data first
835
836 //— Each of the ellipse axis lengths must be positive
837
838 if (!(A > 0.0) \textbar \textbar !(B > 0.0))
839 {
840
841     (*MessageCode) = ERROR_ELLIPSE_PARAMETERS;
842
843     return -1.0;
844 }
845
846
847
848
849
850 //— Points must be on the ellipse, within EPS, which is defined
851
852 //— in the header file program_constants.h
853
854 if ( (fabs ((X1*X1)/(A*A) + (Y1*Y1)/(B*B) - 1.0) > EPS) \textbar
      \textbar
855
856         (fabs ((X2*X2)/(A*A) + (Y2*Y2)/(B*B) - 1.0) > EPS) )
857 {
858
859     (*MessageCode) = ERROR_POINTS_NOT_ON_ELLIPSE;
860
861     return -1.0;
862 }
863
864
865
866
867
868 //— Avoid inverse trig calculation errors: there could be an error
869
870 //— if \textbar X1/A\textbar > 1.0 or \textbar X2/A\textbar > 1.0
      when calling acos()
871
872 //— If execution arrives here, then the point is on the ellipse
873
874 //— within EPS. Try to adjust the value of X1 or X2 before giving
875
876 //— up on the area calculation
877
878 if (fabs (X1)/A > 1.0)
879 {
880
881     //— if execution arrives here, already know that \textbar X1\
      \textbar > A
882
883     if ((fabs (X1) - A) > EPS)
884     {
885
886         //— if X1 is not close to A or -A, then give up
887
888         (*MessageCode) = ERROR\_INVERSE\_TRIG;
889
890         return -1.0;
891     }
892
893     else
894     {
895
896         //— nudge X1 back to A or -A, so acos() will work

```

```

901         X1 = (X1 < 0) ? -A : A;
902     }
903 }
904
905
906
907
908
909
910     if (fabs (X2)/A > 1.0)
911     {
912
913         //— if execution arrives here, already know that \textbar X2\
914         textbar > A
915
916         if ((fabs (X2) - A) > EPS)
917         {
918
919             //— if X2 is not close to A or -A, then give up
920
921             (*MessageCode) = ERROR_INVERSE_TRIG;
922
923             return -1.0;
924
925         }
926
927         else
928         {
929
930             //— nudge X2 back to A or -A, so acos() will work
931
932             X2 = (X2 < 0) ? -A : A;
933
934         }
935     }
936 }
937
938
939
940
941
942 //— Calculate the parametric angles on the ellipse
943
944 //— The parametric angles depend on the quadrant where each point
945 //— is located. See Table 1 in the reference.
946
947
948     if (Y1 < 0.0) //— Quadrant III or IV
949
950         theta1 = twopi - acos (X1 / A);
951
952     else //— Quadrant I or II
953
954         theta1 = acos (X1 / A);
955
956
957
958     if (Y2 < 0.0) //— Quadrant III or IV
959
960         theta2 = twopi - acos (X2 / A);
961
962     else //— Quadrant I or II
963
964         theta2 = acos (X2 / A);
965
966
967
968 //— need to start the algorithm with theta1 < theta2
969
970     if (theta1 > theta2)
971
972         theta1 -= twopi;

```

```

973
974
975
976 //— if the integration angle is less than pi, subtract the triangle
977 //— area from the sector, otherwise add the triangle area.
978
979 if ((theta2 - theta1) > pi)
980     trsign = 1.0;
981
982 else
983     trsign = -1.0;
984
985
986 //— The ellipse segment is the area between the line and the ellipse,
987 //— calculated by finding the area of the radial sector minus the
988 //— area
989
990 //— of the triangle created by the center of the ellipse and the two
991 //— points. First term is for the ellipse sector; second term is for
992 //— the triangle between the points and the origin. Area calculation
993 //— is described in the reference.
994
995 (*MessageCode) = NORMAL_TERMINATION;
996
997 return ( 0.5*(A*B*(theta2 - theta1) + trsign*fabs (X1*Y2 - X2*Y1)) );
998
999 }

```

LISTING 13. C-SOURCE CODE FOR ELLIPSE_LINE_OVERLAP

5. APPENDIX B.

```

1010
1011
1012 /*
1013
1014 *
1015 *
1016 * Function: double ellipse_line_overlap
1017 *
1018 *
1019 * Purpose: Given the parameters of an ellipse and two points on a line,
1020 *           this function calculates the area between the two curves. If
1021 *           the line does not cross the ellipse, or if the line is
1022 *           tangent
1023 *           to the ellipse, then this function returns an area of 0.0
1024 *           If the line intersects the ellipse at two points, then the
1025 *           function returns the area between the secant line and the
1026 *           ellipse. The line is considered to have a direction from
1027 *           the first given point (X1,Y1) to the second given point (X2,
1028 *           Y2)
1029
1030
1031
1032
1033
1034

```



```

1035
1036 *      This function determines where the line crosses the ellipse
1037
1038 *      first, and where it crosses second. The area returned is
1039
1040 *      between the secant line and the ellipse traversed counter-
1041
1042 *      clockwise from the first intersection point to the second
1043
1044 *      intersection point.
1045
1046 *
1047
1048 *      Reference: Hughes and Chraibi (2011), Calculating Ellipse Overlap Areas
1049
1050 *
1051
1052 *      Dependencies: math.h for calls to trig and absolute value functions
1053
1054 *      program_constants.h error message codes and constants
1055
1056 *      ellipse_segment.c core algorithm for ellipse segment
1057
1058 *
1059
1060 *      Inputs: 1. double PHI CCW rotation angle of the ellipse, radians
1061
1062 *      2. double A ellipse semi-axis length in x-direction
1063
1064 *      3. double B ellipse semi-axis length in y-direction
1065
1066 *      4. double H horizontal offset of ellipse center
1067
1068 *      5. double K vertical offset of ellipse center
1069
1070 *      6. double X1 x-value of the first point on the line
1071
1072 *      7. double Y1 y-value of the first point on the line
1073
1074 *      8. double X2 x-value of the second point on the line
1075
1076 *      9. double Y2 y-value of the second point on the line
1077
1078 *
1079
1080 *      Outputs: 1. int *MessageCode returns diagnostic information
1081
1082 *      integer codes in program_constants.h
1083
1084 *
1085
1086 *      Return: The value of the ellipse segment area:
1087
1088 *      -1.0 is returned in case of an error with the data or
1089
1090 *      calculation
1091
1092 *      0.0 is returned if the line does not cross the ellipse, or if
1093
1094 *      the line is tangent to the ellipse
1095
1096 *
1097
1098 *****
1099 */
1100
1101
1102 //

```

```

1104 //== DEFINE PROGRAM CONSTANTS
1105 =====
1106 //
1107 =====
1108 #include "program_constants.h" //— error message codes and constants
1109
1110
1111
1112 //
1113 =====
1114 //== DEPENDENT FUNCTIONS
1115 =====
1116 //
1117 =====
1118 double textbf{ellipse_segment} (double A, double B, double X1, double Y1,
1119 double X2,
1120 double Y2, int *MessageCode);
1121
1122
1123
1124 double \textbf{ellipse_line_overlap} (double PHI, double A, double B,
1125 double H,
1126 double K, double X1, double Y1, double X2,
1127 double Y2, int *MessageCode)
1128
1129 \{
1130
1131
1132 //
1133 =====
1134 //== DEFINE LOCAL VARIABLES
1135 =====
1136 //
1137 =====
1138 double X10; //— Translated, Rotated x-value of the first point
1139
1140 double Y10; //— Translated, Rotated y-value of the first point
1141
1142 double X20; //— Translated, Rotated x-value of the second point
1143
1144 double Y20; //— Translated, Rotated y-value of the second point
1145
1146 double cosphi = textbf{cos} (PHI); //— store cos(PHI) to avoid
1147 multiple calcs
1148
1149 double sinphi = \textbf{sin} (PHI); //— store sin(PHI) to avoid
1150 multiple calcs
1151
1152 double m; //— line slope, calculated from input line slope
1153
1154 double a, b, c; //— quadratic equation coefficients  $a*x^2 + b*x + c$ 
1155
1156 double discrim; //— quadratic equation discriminant  $b^2 - 4*a*c$ 
1157
1158 double x1, x2; //— x-values of intersection points
1159
1160 double y1, y2; //— y-values of intersection points

```

```

1159
1160     double mid_X;           //— midpoint of the rotated x-values on the line
1161
1162     double theta1parm;      //— parametric angle of first point
1163
1164     double theta2parm;      //— parametric angle of second point
1165
1166     double xmidpoint;       //— x-value midpoint of secant line
1167
1168     double ymidpoint;       //— y-value midpoint of secant line
1169
1170     double root1, root2;    //— temporary storage variables for roots
1171
1172     double segment_area;    //— stores the ellipse segment area
1173
1174
1175
1176     //— Check the data first
1177
1178     //— Each of the ellipse axis lengths must be positive
1179
1180     if (!(A > 0.0) \textbar (B > 0.0))
1181     {
1182
1183         (*MessageCode) = ERROR_ELLIPSE_PARAMETERS;
1184
1185         return -1.0;
1186     }
1187
1188
1189
1190
1191
1192     //— The rotation angle for the ellipse should be between -2pi and 2pi
1193         (?)
1194
1195     if ( (\textbf{fabs} (PHI) > (2.0*pi)) )
1196
1197         PHI = \textbf{fmod} (PHI, twopi);
1198
1199
1200
1201
1202     //— For this numerical routine, the ellipse will be translated and
1203
1204     //— rotated so that it is centered at the origin and oriented with
1205
1206     //— the coordinate axes.
1207
1208     //— Then, the ellipse will have the implicit (polynomial) form of
1209
1210     //—  $x^2/A^2 + y^2/B^2 = 1$ 
1211
1212
1213
1214     //— For the line, the given points are first translated by the amount
1215
1216     //— required to put the ellipse at the origin, e.g., by (-H, -K).
1217
1218     //— Then, the points are rotated by the amount required to orient
1219
1220     //— the ellipse with the coordinate axes, e.g., through the angle -
1221
1222     PHI.
1223
1224     X10 = cosphi*(X1 - H) + sinphi*(Y1 - K);
1225
1226     Y10 = -sinphi*(X1 - H) + cosphi*(Y1 - K);
1227
1228     X20 = cosphi*(X2 - H) + sinphi*(Y2 - K);
1229
1230     Y20 = -sinphi*(X2 - H) + cosphi*(Y2 - K);

```

```

1230 //— To determine if the line and ellipse intersect, solve the two
1231 //— equations simultaneously, by substituting  $y = Y10 + m(x - X10)$ 
1232 //— and  $x = X10 + mxy(y - Y10)$  into the ellipse equation,
1233 //— which results in two quadratic equations in  $x$ . See the reference
1234 //— for derivations of the quadratic coefficients.
1235
1236
1237
1238
1239
1240
1241 //— If the new line is not close to being vertical, then use the
1242 //— first derivation
1243
1244 if (\textbf{fabs} (X20 - X10) > EPS)
1245 {
1246     //—  $((B^2 + A^2 m^2)/(A^2)) * x^2$ 
1247     //—  $2*(Y10*m - m^2*X10) * x$ 
1248     //—  $(Y10^2 - 2*m*Y10*X10 + m^2*X10^2 - B^2)$ 
1249     m = (Y20 - Y10)/(X20 - X10);
1250     a = (B*B + A*A*m*m)/(A*A);
1251     b = 2.0*(Y10*m - m*m*X10);
1252     c = (Y10*Y10 - 2.0*m*Y10*X10 + m*m*X10*X10 - B*B);
1253 }
1254
1255 //— If the new line is close to being vertical, then use the
1256 //— second derivation
1257
1258 else if (\textbf{fabs} (Y20 - Y10) > EPS)
1259 {
1260     //—  $((A^2 + B^2 m^2)/(B^2)) * y^2$ 
1261     //—  $2*(X10*m - m^2*Y10) * y$ 
1262     //—  $(X10^2 - 2*m*X10*Y10 + m^2*Y10^2 - A^2)$ 
1263     m = (X20 - X10)/(Y20 - Y10);
1264     a = (A*A + B*B*m*m)/(B*B);
1265     b = 2.0*(X10*m - m*m*Y10);
1266     c = (X10*X10 - 2.0*m*Y10*X10 + m*m*Y10*Y10 - A*A);
1267 }
1268
1269 //— If the two given points on the line are very close together in
1270 //— both  $x$  and  $y$  directions, then give up
1271
1272 else
1273 {
1274     (*MessageCode) = ERROR_LINE_POINTS;
1275     return -1.0;
1276 }
1277
1278
1279
1280
1281
1282
1283
1284
1285
1286
1287
1288
1289
1290
1291
1292
1293
1294
1295
1296
1297
1298
1299
1300
1301
1302

```

```

1303
1304
1305
1306 //— Once the coefficients for the Quadratic Equation in  $x$  are
1307 //— known, the roots of the quadratic polynomial will represent
1308 //— the  $x$ - or  $y$ -values of the points of intersection of the line
1309 //— and the ellipse. The discriminant can be used to discern
1310 //— which case has occurred for the given inputs:
1311
1312 //— 1.  $\text{discr} < 0$ 
1313
1314 //— Quadratic has complex conjugate roots.
1315
1316 //— The line and ellipse do not intersect
1317
1318 //— 2.  $\text{discr} = 0$ 
1319
1320 //— Quadratic has one repeated root
1321
1322 //— The line and ellipse intersect at only one point
1323
1324 //— i.e., the line is tangent to the ellipse
1325
1326 //— 3.  $\text{discr} > 0$ 
1327
1328 //— Quadratic has two distinct real roots
1329
1330 //— The line crosses the ellipse at two points
1331
1332 discrim = b*b - 4.0*a*c;
1333
1334 if (discrim < 0.0)
1335 {
1336 //— Line and ellipse do not intersect
1337
1338 (*MessageCode) = NO_INTERSECTION_POINTS;
1339
1340 return 0.0;
1341 }
1342
1343 else if (discrim > 0.0)
1344 {
1345 //— Two real roots exist, so calculate them
1346 //— The larger root is stored in root2
1347
1348 root1 = (-b - \text{sqrt} (discrim)) / (2.0*a);
1349
1350 root2 = (-b + \text{sqrt} (discrim)) / (2.0*a);
1351
1352 }
1353
1354 else
1355 {
1356 //— Line is tangent to the ellipse
1357
1358 (*MessageCode) = LINE_TANGENT_TO_ELLIPSE;
1359
1360 return 0.0;
1361
1362 }
1363
1364 }
1365
1366 }
1367
1368 }
1369
1370
1371
1372
1373
1374
1375

```

```

1376
1377
1378 //— decide which roots go into which x or y values
1379
1380 if (\textbf{fabs} (X20 - X10) > EPS) //— roots are x-values
1381 {
1382
1383     //— order the points in the same direction as X10 -> X20
1384     if (X10 < X20)
1385     {
1386         x1 = root1;
1387         x2 = root2;
1388     }
1389     else
1390     {
1391         x1 = root2;
1392         x2 = root1;
1393     }
1394
1395     //— The y-values can be calculated by substituting the
1396     //— x-values into the line equation  $y = Y10 + m*(x - X10)$ 
1397     y1 = Y10 + m*(x1 - X10);
1398     y2 = Y10 + m*(x2 - X10);
1399 }
1400
1401 else //— roots are y-values
1402 {
1403     //— order the points in the same direction as Y10 -> Y20
1404     if (Y10 < Y20)
1405     {
1406         y1 = root1;
1407         y2 = root2;
1408     }
1409     else
1410     {
1411         y1 = root2;
1412         y2 = root1;
1413     }
1414
1415     //— The x-values can be calculated by substituting the
1416     //— y-values into the line equation  $x = X10 + m*(y - Y10)$ 
1417     x1 = X10 + m*(y1 - Y10);
1418     x2 = X10 + m*(y2 - Y10);
1419 }
1420
1421 //— The area of overlap is the area of the rectangle formed by the
1422 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1423 area = (x2 - x1) * (y2 - y1);
1424
1425 //— The area of overlap is the area of the rectangle formed by the
1426 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1427 area = (x2 - x1) * (y2 - y1);
1428
1429 //— The area of overlap is the area of the rectangle formed by the
1430 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1431 area = (x2 - x1) * (y2 - y1);
1432
1433 //— The area of overlap is the area of the rectangle formed by the
1434 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1435 area = (x2 - x1) * (y2 - y1);
1436
1437 //— The area of overlap is the area of the rectangle formed by the
1438 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1439 area = (x2 - x1) * (y2 - y1);
1440
1441 //— The area of overlap is the area of the rectangle formed by the
1442 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1443 area = (x2 - x1) * (y2 - y1);
1444
1445 //— The area of overlap is the area of the rectangle formed by the
1446 //— points (x1, y1), (x2, y2), (x2, y1), and (x1, y2)
1447 area = (x2 - x1) * (y2 - y1);
1448

```

```

1449         x1 = X10 + m*(y1 - Y10);
1450
1451         x2 = X10 + m*(y2 - Y10);
1452
1453     }
1454
1455
1456
1457     //-- Arriving here means that two points of intersection have been
1458     found. Pass the ellipse parameters and intersection points to
1459     the ellipse_segment() routine.
1460
1461     segment_area = \textbf{ellipse\_segment} (A, B, x1, y1, x2, y2,
1462         MessageCode);
1463
1464
1465
1466
1467     //-- The message code will indicate whether the function encountered
1468     any errors
1469
1470     if ((*MessageCode) < 0)
1471     {
1472         return -1;
1473     }
1474
1475     else
1476     {
1477         (*MessageCode) = TWO_INTERSECTION_POINTS;
1478         return segment_area;
1479     }
1480
1481 }
1482
1483 }
1484
1485 }
1486
1487 }
1488
1489 }
1490

```

LISTING 14. C-SOURCE CODE FOR ELLIPSE_ELLIPSE_OVERLAP

```

1491 6. APPENDIX C.
1492 /*
1493  *
1494  *
1495  * Function: double ellipse_ellipse_overlap
1496  *
1497  *
1498  * Purpose: Given the parameters of two ellipses, this function calculates
1499  *
1500  * the area of overlap between the two curves. If the ellipses
1501  *
1502  * are
1503  *
1504  * disjoint, this function returns 0.0; if one ellipse is
1505  *
1506  * contained
1507  *
1508  * within the other, this function returns the area of the
1509  *
1510  * enclosed
1511  *
1512  * ellipse; if the ellipses intersect, this function returns the
1513  *
1514  *

```

```

1509 *          calculated area of overlap.
1510
1511 *
1512
1513 *   Reference: Hughes and Chraibi (2011), Calculating Ellipse Overlap Areas
1514
1515 *
1516
1517 *   Dependencies: math.h for calls to trig and absolute value functions
1518
1519 *          program_constants.h error message codes and constants
1520
1521 *
1522
1523 *   Inputs:   1. double PHI_1 CCW rotation angle of first ellipse, radians
1524
1525 *             2. double A1    semi-axis length in x-direction first ellipse
1526
1527 *             3. double B1    semi-axis length in y-direction first ellipse
1528
1529 *             4. double H1    horizontal offset of center first ellipse
1530
1531 *             5. double K1    vertical offset of center first ellipse
1532
1533 *             6. double PHI_2 CCW rotation angle of second ellipse, radians
1534
1535 *             7. double A2    semi-axis length in x-direction second
1536 ellipse
1537 *             8. double B2    semi-axis length in y-direction second
1538 ellipse
1539 *             9. double H2    horizontal offset of center second ellipse
1540
1541 *            10. double K2    vertical offset of center second ellipse
1542
1543 *
1544
1545 *   Outputs:  1. int *rtnCode returns diagnostic information integer code
1546
1547 *             integer codes in program_constants.h
1548
1549 *
1550
1551 *   Return:   The calculated value of the overlap area
1552
1553 *            -1 is returned in case of an error with the calculation
1554
1555 *            0 is returned if the ellipses are disjoint
1556
1557 *            pi*A*B of smaller ellipse if one ellipse is contained within
1558 the other ellipse
1559 *
1560
1561 *
1562
1563 *****
1564 */
1565
1566
1567 //
1568
1569 //== DEFINE PROGRAM CONSTANTS
1570
1571 //
1572
1573 #include "program_constants.h" //--- error message codes and constants

```



```

1574
1575
1576
1577 //
=====

1578
1579 //== DEPENDENT FUNCTIONS
=====

1580
1581 //
=====

1582
1583 double nointpts (double A1, double B1, double A2, double B2, double H1,
1584
1585             double K1, double H2_TR, double K2_TR, double AA, double
                BB,
1586
1587             double CC, double DD, double EE, double FF, int *rtnCode);
1588
1589
1590
1591 double twointpts (double xint[], double yint[], double A1, double B1,
1592
1593             double PHI_1, double A2, double B2, double H2_TR,
1594
1595             double K2_TR, double PHI_2, double AA, double BB,
1596
1597             double CC, double DD, double EE, double FF, int *rtnCode)
                ;
1598
1599
1600
1601 double threeintpts (double xint[], double yint[], double A1, double B1,
1602
1603             double PHI_1, double A2, double B2, double H2_TR,
1604
1605             double K2_TR, double PHI_2, double AA, double BB,
1606
1607             double CC, double DD, double EE, double FF,
1608
1609             int *rtnCode);
1610
1611
1612
1613 double fourintpts (double xint[], double yint[], double A1, double B1,
1614
1615             double PHI_1, double A2, double B2, double H2_TR,
1616
1617             double K2_TR, double PHI_2, double AA, double BB,
1618
1619             double CC, double DD, double EE, double FF, int *rtnCode
                );
1620
1621
1622
1623 int istanpt (double x, double y, double A1, double B1, double AA, double BB
                ,
1624
1625             double CC, double DD, double EE, double FF);
1626
1627
1628
1629 double ellipse2tr (double x, double y, double AA, double BB,
1630
1631             double CC, double DD, double EE, double FF);
1632
1633
1634
1635 //-- functions for solving the quartic equation from Netlib/TOMS
1636
1637 void BIQUADROOTS (double p[], double r[][5]);

```

```

1638
1639 void CUBICROOTS (double p[], double r[][5]);
1640
1641 void QUADROOTS (double p[], double r[][5]);
1642
1643
1644
1645 //
=====

1646
1647 //== ELLIPSE-ELLIPSE OVERLAP
=====

1648
1649 //
=====

1650
1651 double ellipse_ellipse_overlap (double PHI_1, double A1, double B1,
1652
1653                                double H1, double K1, double PHI_2,
1654
1655                                double A2, double B2, double H2, double K2,
1656
1657                                int *rtnCode)
1658 {
1659 {
1660
1661 //
=====

1662
1663 //== DEFINE LOCAL VARIABLES
=====

1664
1665 //
=====

1666
1667 int i, j, k, nroots, nychk, nintpts, fnRtnCode;
1668
1669 double AA, BB, CC, DD, EE, FF, H2_TR, K2_TR, A22, B22, PHI_2R;
1670
1671 double cosphi, cosphi2, sinphi, sinphi2, cosphisinphi;
1672
1673 double tmp0, tmp1, tmp2, tmp3;
1674
1675 double cy[5] = {0.0}, py[5] = {0.0}, r[3][5] = {0.0};
1676
1677 double x1, x2, y12, y22;
1678
1679 double ychk[5] = {0.0}, xint[5], yint[5];
1680
1681 double Area1, Area2, OverlapArea;
1682
1683
1684
1685 //
=====

1686
1687 //== DATA CHECK
=====

1688
1689 //
=====

1690
1691 //— Each of the ellipse axis lengths must be positive
1692
1693 if ( (!(A1 > 0.0) \textbar \textbar !(B1 > 0.0)) \textbar \textbar !(
1694       A2 > 0.0) \textbar \textbar !(B2 > 0.0)) )
1695

```

```

1695     {
1696
1697         (*rtncode) = ERROR_ELLIPSE_PARAMETERS;
1698
1699         return -1.0;
1700     }
1701
1702
1703
1704
1705     //— The rotation angles should be between -2pi and 2pi (?)
1706
1707     if ( (fabs (PHI_1) > (twopi)) )
1708
1709         PHI_1 = fmod (PHI_1, twopi);
1710
1711     if ( (fabs (PHI_2) > (twopi)) )
1712
1713         PHI_2 = fmod (PHI_2, twopi);
1714
1715
1716
1717     //
1718
1719     //== DETERMINE THE TWO ELLIPSE EQUATIONS FROM INPUT PARAMETERS
1720
1721     //
1722
1723     //— Finding the points of intersection between two general ellipses
1724
1725     //— requires solving a quartic equation. Before attempting to solve
1726     the
1727
1728     //— quartic, several quick tests can be used to eliminate some cases
1729
1730     //— where the ellipses do not intersect. Optionally, can whittle away
1731     //— at the problem, by addressing the easiest cases first.
1732
1733
1734
1735     //— Working with the translated+rotated ellipses simplifies the
1736
1737     //— calculations. The ellipses are translated then rotated so that
1738     the
1739
1740     //— first ellipse is centered at the origin and oriented with the
1741     //— coordinate axes. Then, the first ellipse will have the implicit
1742     //— (polynomial) form of
1743
1744     //—  $x^2/A1^2 + y^2/B1^2 = 1$ 
1745
1746
1747
1748
1749     //— For the second ellipse, the center is first translated by the
1750     amount
1751
1752     //— required to put the first ellipse at the origin, e.g., by (-H1, -
1753     K1)
1754
1755     //— Then, the center of the second ellipse is rotated by the amount
1756     required to orient the first ellipse with the coordinate axes, e.g
1757     through the angle -PHI_1.

```

```

1758
1759 //— The translated and rotated center point coordinates for the second
1760 //— ellipse are found with the rotation matrix, derivations are
1761 //— described in the reference.
1762
1763 cosphi = cos (PHI_1);
1764
1765 sinphi = sin (PHI_1);
1766
1767 H2_TR = (H2 - H1)*cosphi + (K2 - K1)*sinphi;
1768
1769 K2_TR = (H1 - H2)*sinphi + (K2 - K1)*cosphi;
1770
1771 PHI_2R = PHI_2 - PHI_1;
1772
1773 if ( (fabs (PHI_2R) > (twopi)) )
1774
1775     PHI_2R = fmod (PHI_2R, twopi);
1776
1777
1778 //— Calculate implicit (Polynomial) coefficients for the second
1779 //— ellipse
1780
1781 //— in its translated-by (-H1, -H2) and rotated-by -PHI_1 position
1782
1783 //—  $AA*x^2 + BB*x*y + CC*y^2 + DD*x + EE*y + FF = 0$ 
1784
1785 //— Formulas derived in the reference
1786
1787 //— To speed things up, store multiply-used expressions first
1788
1789 cosphi = cos (PHI_2R);
1790
1791 cosphi2 = cosphi*cosphi;
1792
1793 sinphi = sin (PHI_2R);
1794
1795 sinphi2 = sinphi*sinphi;
1796
1797 cosphisinphi = 2.0*cosphi*sinphi;
1798
1799 A22 = A2*A2;
1800
1801 B22 = B2*B2;
1802
1803 tmp0 = (cosphi*H2_TR + sinphi*K2_TR)/A22;
1804
1805 tmp1 = (sinphi*H2_TR - cosphi*K2_TR)/B22;
1806
1807 tmp2 = cosphi*H2_TR + sinphi*K2_TR;
1808
1809 tmp3 = sinphi*H2_TR - cosphi*K2_TR;
1810
1811
1812 //— implicit polynomial coefficients for the second ellipse
1813
1814 AA = cosphi2/A22 + sinphi2/B22;
1815
1816 BB = cosphisinphi/A22 - cosphisinphi/B22;
1817
1818 CC = sinphi2/A22 + cosphi2/B22;
1819
1820 DD = -2.0*cosphi*tmp0 - 2.0*sinphi*tmp1;
1821
1822 EE = -2.0*sinphi*tmp0 + 2.0*cosphi*tmp1;
1823
1824 FF = tmp2*tmp2/A22 + tmp3*tmp3/B22 - 1.0;
1825
1826
1827
1828
1829

```

```

1830
1831 //
=====

1832
1833 //== CREATE AND SOLVE THE QUARTIC EQUATION TO FIND INTERSECTION POINTS
1834 ==
1835 //
=====

1836
1837 //-- If execution arrives here, the ellipses are at least 'close' to
1838 //-- intersecting.
1839 //-- Coefficients for the Quartic Polynomial in y are calculated from
1840 //-- the two implicit equations.
1841 //-- Formulas for these coefficients are derived in the reference.
1842
1843 cy[4] = pow (A1, 4.0)*AA*AA + B1*B1*(A1*A1*(BB*BB - 2.0*AA*CC)
1844           + B1*B1*CC*CC);
1845
1846 cy[3] = 2.0*B1*(B1*B1*CC*EE + A1*A1*(BB*DD - AA*EE));
1847
1848 cy[2] = A1*A1*((B1*B1*(2.0*AA*CC - BB*BB) + DD*DD - 2.0*AA*FF)
1849           - 2.0*A1*A1*AA*AA) + B1*B1*(2.0*CC*FF + EE*EE);
1850
1851 cy[1] = 2.0*B1*(A1*A1*(AA*EE - BB*DD) + EE*FF);
1852
1853 cy[0] = (A1*(A1*AA - DD) + FF)*(A1*(A1*AA + DD) + FF);
1854
1855
1856
1857 //-- Once the coefficients for the Quartic Equation in y are known, the
1858 //-- roots of the quartic polynomial will represent y-values of the
1859 //-- intersection points of the two ellipse curves.
1860 //-- The quartic sometimes degenerates into a polynomial of lesser
1861 //-- degree, so handle all possible cases.
1862
1863 if (fabs (cy[4]) > 0.0)
1864 {
1865
1866     //== QUARTIC COEFFICIENT NONZERO, USE QUARTIC FORMULA
1867
1868     for (i = 0; i <= 3; i++)
1869         py[4-i] = cy[i]/cy[4];
1870
1871     py[0] = 1.0;
1872
1873     BIQUADROOTS (py, r);
1874
1875     nroots = 4;
1876
1877 }
1878
1879 else if (fabs (cy[3]) > 0.0)
1880 {
1881
1882
1883
1884
1885
1886
1887
1888
1889
1890
1891
1892
1893
1894
1895
1896

```

```

1897      //== QUARTIC DEGENERATES TO CUBIC, USE CUBIC FORMULA
1898
1899      for (i = 0; i <= 2; i++)
1900          py[3-i] = cy[i]/cy[3];
1901
1902      py[0] = 1.0;
1903
1904
1905      CUBICROOTS (py, r);
1906
1907      nroots = 3;
1908
1909  }
1910
1911  else if (fabs (cy[2]) > 0.0)
1912  {
1913      //== QUARTIC DEGENERATES TO QUADRATIC, USE QUADRATIC FORMULA
1914
1915      for (i = 0; i <= 1; i++)
1916          py[2-i] = cy[i]/cy[2];
1917
1918      py[0] = 1.0;
1919
1920
1921      QUADROOTS (py, r);
1922
1923      nroots = 2;
1924
1925  }
1926
1927  else if (fabs (cy[1]) > 0.0)
1928  {
1929      //== QUARTIC DEGENERATES TO LINEAR: SOLVE DIRECTLY
1930
1931      //-- cy[1]*Y + cy[0] = 0
1932
1933      r[1][1] = (-cy[0]/cy[1]);
1934
1935      r[2][1] = 0.0;
1936
1937      nroots = 1;
1938
1939  }
1940
1941  else
1942  {
1943      //== COMPLETELY DEGENERATE QUARTIC: ELLIPSES IDENTICAL???
1944
1945      //-- a completely degenerate quartic, which would seem to
1946      //-- indicate that the ellipses are identical. However, some
1947      //-- configurations lead to a degenerate quartic with no
1948      //-- points of intersection.
1949
1950      nroots = 0;
1951
1952  }
1953
1954
1955
1956
1957
1958
1959
1960
1961
1962
1963
1964
1965

```

```

1966
1967
1968
1969 //
=====

1970
1971 //== CHECK ROOTS OF THE QUARTIC: ARE THEY POINTS OF INTERSECTION?
=====

1972
1973 //
=====

1974
1975 //-- determine which roots are real, discard any complex roots
1976
1977 nychk = 0;
1978
1979 for (i = 1; i <= nroots; i++)
1980 {
1981     if (fabs (r[2][i]) < EPS)
1982     {
1983         nychk++;
1984         ychk[nychk] = r[1][i]*B1;
1985     }
1986 }
1987
1988
1989 //-- sort the real roots by straight insertion
1990
1991 for (j = 2; j <= nychk; j++)
1992 {
1993     tmp0 = ychk[j];
1994
1995     for (k = j - 1; k >= 1; k--)
1996     {
1997         if (ychk[k] <= tmp0)
1998             break;
1999
2000         ychk[k+1] = ychk[k];
2001     }
2002
2003     ychk[k+1] = tmp0;
2004 }
2005
2006
2007 //-- determine whether polynomial roots are points of intersection
2008
2009 //-- for the two ellipses
2010
2011 nintpts = 0;

```

```

2034
2035     for (i = 1; i <= nychk; i++)
2036     {
2037
2038         //— check for multiple roots
2039
2040         if ((i > 1) && (fabs (ychk[i] - ychk[i-1]) < (EPS/2.0)))
2041             continue;
2042
2043
2044
2045
2046         //— check intersection points for ychk[i]
2047
2048         if (fabs (ychk[i]) > B1)
2049             x1 = 0.0;
2050
2051         else
2052             x1 = A1*sqrt (1.0 - (ychk[i]*ychk[i])/(B1*B1));
2053
2054         x2 = -x1;
2055
2056
2057         if (fabs(ellipse2tr(x1, ychk[i], AA, BB, CC, DD, EE, FF)) < EPS
2058             /2.0)
2059         {
2060
2061             nintpts++;
2062
2063             if (nintpts > 4)
2064             {
2065                 (*rtnCode) = ERROR_INTERSECTION_PTS;
2066                 return -1.0;
2067             }
2068
2069             xint[nintpts] = x1;
2070
2071             yint[nintpts] = ychk[i];
2072
2073         }
2074
2075
2076
2077
2078
2079
2080
2081
2082
2083
2084
2085         if ((fabs(ellipse2tr(x2, ychk[i], AA, BB, CC, DD, EE, FF)) < EPS
2086             /2.0)
2087             && (fabs (x2 - x1) > EPS/2.0))
2088         {
2089
2090             nintpts++;
2091
2092             if (nintpts > 4)
2093             {
2094                 (*rtnCode) = ERROR_INTERSECTION_PTS;
2095                 return -1.0;
2096             }
2097
2098             xint[nintpts] = x2;
2099
2100
2101
2102
2103
2104

```



```

2168
2169         return OverlapArea;
2170
2171
2172
2173     case 3:
2174
2175         //— when there are three intersection points, one and only one
2176         //— of the points must be a tangent point.
2177
2178         OverlapArea = threointpts (xint, yint, A1, B1, PHI_1, A2, B2,
2179                                   H2_TR, K2_TR, PHI_2, AA, BB, CC, DD,
2180                                   EE, FF, rtnCode);
2181
2182
2183
2184         return OverlapArea;
2185
2186
2187
2188     case 4:
2189
2190         //— four intersections points has only one case.
2191
2192         OverlapArea = fourintpts (xint, yint, A1, B1, PHI_1, A2, B2,
2193                                   H2_TR, K2_TR, PHI_2, AA, BB, CC, DD,
2194                                   EE, FF, rtnCode);
2195
2196
2197
2198         return OverlapArea;
2199
2200
2201
2202     default:
2203
2204         //— should never get here (but get compiler warning for
2205         missing
2206
2207         //— return value if this line is omitted)
2208
2209         (*rtnCode) = ERROR_INTERSECTIONPTS;
2210
2211         return -1.0;
2212     }
2213 }
2214 }
2215 }
2216
2217
2218
2219 double ellipse2tr (double x, double y, double AA, double BB,
2220                   double CC, double DD, double EE, double FF)
2221 {
2222
2223     return (AA*x*x + BB*x*y + CC*y*y + DD*x + EE*y + FF);
2224 }
2225
2226
2227 }
2228
2229
2230
2231 double nointpts (double A1, double B1, double A2, double B2, double H1,
2232                  double K1, double H2_TR, double K2_TR, double AA, double
2233                  BB,
2234                  double CC, double DD, double EE, double FF, int *rtnCode)
2235 {
2236
2237
2238

```

```

2239  //— The relative size of the two ellipses can be found from the axis
2240
2241  //— lengths
2242
2243  double relsize = (A1*B1) - (A2*B2);
2244
2245
2246
2247  if (relsize > 0.0)
2248  {
2249
2250      //— First Ellipse is larger than second ellipse.
2251
2252      //— If second ellipse center (H2_TR, K2_TR) is inside
2253      //— first ellipse, then ellipse 2 is completely inside
2254      //— ellipse 1. Otherwise, the ellipses are disjoint.
2255
2256      if ( ((H2_TR*H2_TR) / (A1*A1)
2257           + (K2_TR*K2_TR) / (B1*B1)) < 1.0 )
2258      {
2259          (*rtnCode) = ELLIPSE2_INSIDE_ELLIPSE1;
2260
2261          return (pi*A2*B2);
2262      }
2263
2264      else
2265      {
2266          (*rtnCode) = DISJOINT_ELLIPSES;
2267
2268          return 0.0;
2269      }
2270
2271  }
2272
2273  else if (relsize < 0.0)
2274  {
2275
2276      //— Second Ellipse is larger than first ellipse
2277
2278      //— If first ellipse center (0, 0) is inside the
2279      //— second ellipse, then ellipse 1 is completely inside
2280      //— ellipse 2. Otherwise, the ellipses are disjoint
2281
2282      //—  $AA*x^2 + BB*x*y + CC*y^2 + DD*x + EE*y + FF = 0$ 
2283
2284      if (FF < 0.0)
2285      {
2286
2287          (*rtnCode) = ELLIPSE1_INSIDE_ELLIPSE2;
2288
2289          return (pi*A1*B1);
2290      }
2291
2292      else
2293      {
2294
2295          (*rtnCode) = DISJOINT_ELLIPSES;
2296
2297      }
2298
2299  }
2300
2301
2302
2303
2304
2305
2306
2307
2308
2309
2310
2311

```

```

2312
2313         return 0.0;
2314     }
2315 }
2316
2317 }
2318
2319 else
2320 {
2321
2322     /*-- If execution arrives here, the relative sizes are identical.
2323
2324     /*-- Are the ellipses the same? Check the parameters to see.
2325
2326     if ((H1 == H2.TR) && (K1 == K2.TR))
2327     {
2328
2329         (*rtnCode) = ELLIPSES_ARE_IDENTICAL;
2330
2331         return (pi*A1*B1);
2332     }
2333
2334     else
2335     {
2336
2337         /*-- should never get here, so return error
2338
2339         (*rtnCode) = ERROR_CALCULATIONS;
2340
2341         return -1.0;
2342     }
2343 } /*-- end if (relsize > 0.0)
2344 }
2345
2346 /*-- two distinct intersection points (x1, y1) and (x2, y2) find overlap
2347 area
2348
2349 double twointpts (double x[], double y[], double A1, double B1, double
2350     PHI_1,
2351
2352     double A2, double B2, double H2_TR, double K2_TR,
2353
2354     double PHI_2, double AA, double BB, double CC, double DD,
2355
2356     double EE, double FF, int *rtnCode)
2357 {
2358
2359     double area1, area2;
2360
2361     double xmid, ymid, xmid_rt, ymid_rt;
2362
2363     double theta1, theta2;
2364
2365     double tmp, trsign;
2366
2367     double x1_tr, y1_tr, x2_tr, y2_tr;
2368
2369     double discr;
2370
2371     double cosphi, sinphi;
2372
2373
2374
2375
2376
2377
2378
2379
2380
2381
2382

```

```

2383    //-- if execution arrives here, the intersection points are not
2384
2385    //-- tangents.
2386
2387
2388
2389    //-- determine which direction to integrate in the ellipse_segment
2390
2391    //-- routine for each ellipse.
2392
2393
2394
2395    //-- find the parametric angles for each point on ellipse 1
2396
2397    if (fabs (x[1]) > A1)
2398
2399        x[1] = (x[1] < 0) ? -A1 : A1;
2400
2401    if (y[1] < 0.0)      //-- Quadrant III or IV
2402
2403        theta1 = twopi - acos (x[1] / A1);
2404
2405    else                  //-- Quadrant I or II
2406
2407        theta1 = acos (x[1] / A1);
2408
2409
2410
2411    if (fabs (x[2]) > A1)
2412
2413        x[2] = (x[2] < 0) ? -A1 : A1;
2414
2415    if (y[2] < 0.0)      //-- Quadrant III or IV
2416
2417        theta2 = twopi - acos (x[2] / A1);
2418
2419    else                  //-- Quadrant I or II
2420
2421        theta2 = acos (x[2] / A1);
2422
2423
2424
2425    //-- logic is for proceeding counterclockwise from theta1 to theta2
2426
2427    if (theta1 > theta2)
2428
2429    {
2430
2431        tmp = theta1;
2432
2433        theta1 = theta2;
2434
2435        theta2 = tmp;
2436
2437    }
2438
2439
2440
2441    //-- find a point on the first ellipse that is different than the two
2442
2443    //-- intersection points.
2444
2445    xmid = A1*cos ((theta1 + theta2)/2.0);
2446
2447    ymid = B1*sin ((theta1 + theta2)/2.0);
2448
2449
2450
2451    //-- the point (xmid, ymid) is on the first ellipse 'between' the two
2452
2453    //-- intersection points (x[1], y[1]) and (x[2], y[2]) when travelling
2454

```

```

2455  //-- counter- clockwise from (x[1], y[1]) to (x[2], y[2]). If the
2456  point
2457  //-- (xmid, ymid) is inside the second ellipse, then the desired
2458  segment
2459  //-- of ellipse 1 contains the point (xmid, ymid), so integrate
2460  //-- counterclockwise from (x[1], y[1]) to (x[2], y[2]). Otherwise,
2461  //-- integrate counterclockwise from (x[2], y[2]) to (x[1], y[1])
2462
2463  if (ellipse2tr (xmid, ymid, AA, BB, CC, DD, EE, FF) > 0.0)
2464
2465  {
2466
2467      tmp = theta1;
2468
2469      theta1 = theta2;
2470
2471      theta2 = tmp;
2472
2473  }
2474
2475
2476
2477
2478
2479  //-- here is the ellipse segment routine for the first ellipse
2480
2481  if (theta1 > theta2)
2482
2483      theta1 -= twopi;
2484
2485  if ((theta2 - theta1) > pi)
2486
2487      trsign = 1.0;
2488
2489  else
2490
2491      trsign = -1.0;
2492
2493  areal = 0.5*(A1*B1*(theta2 - theta1)
2494
2495      + trsign*fabs (x[1]*y[2] - x[2]*y[1]));
2496
2497
2498
2499  //-- find ellipse 2 segment area. The ellipse segment routine
2500  //-- needs an ellipse that is centered at the origin and oriented
2501  //-- with the coordinate axes. The intersection points (x[1], y[1])
2502  and
2503
2504  //-- (x[2], y[2]) are found with both ellipses translated and rotated
2505  by
2506
2507  //-- (-H1, -K1) and -PHI_1. Further translate and rotate the points
2508
2509  //-- to put the second ellipse at the origin and oriented with the
2510
2511  //-- coordinate axes. The translation is (-H2_TR, -K2_TR), and the
2512
2513  //-- rotation is -(PHI_2 - PHI_1) = PHI_1 - PHI_2
2514
2515  cosphi = cos (PHI_1 - PHI_2);
2516
2517  sinphi = sin (PHI_1 - PHI_2);
2518
2519  x1_tr = (x[1] - H2_TR)*cosphi + (y[1] - K2_TR)*-sinphi;
2520
2521  y1_tr = (x[1] - H2_TR)*sinphi + (y[1] - K2_TR)*cosphi;
2522
2523  x2_tr = (x[2] - H2_TR)*cosphi + (y[2] - K2_TR)*-sinphi;

```

```

2524
2525 y2_tr = (x[2] - H2_TR)*sinphi + (y[2] - K2_TR)*cosphi;
2526
2527
2528
2529 //-- determine which branch of the ellipse to integrate by finding a
2530 //-- point on the second ellipse, and asking whether it is inside the
2531 //-- first ellipse (in their once-translated+rotated positions)
2532 //-- find the parametric angles for each point on ellipse 1
2533
2534
2535 if (fabs (x1_tr) > A2)
2536
2537     x1_tr = (x1_tr < 0) ? -A2 : A2;
2538
2539 if (y1_tr < 0.0) //-- Quadrant III or IV
2540
2541     theta1 = twopi - acos (x1_tr/A2);
2542
2543 else //-- Quadrant I or II
2544
2545     theta1 = acos (x1_tr/A2);
2546
2547
2548
2549
2550 if (fabs (x2_tr) > A2)
2551
2552     x2_tr = (x2_tr < 0) ? -A2 : A2;
2553
2554 if (y2_tr < 0.0) //-- Quadrant III or IV
2555
2556     theta2 = twopi - acos (x2_tr/A2);
2557
2558 else //-- Quadrant I or II
2559
2560     theta2 = acos (x2_tr/A2);
2561
2562
2563
2564
2565 //-- logic is for proceeding counterclockwise from theta1 to theta2
2566
2567 if (theta1 > theta2)
2568 {
2569
2570     tmp = theta1;
2571
2572     theta1 = theta2;
2573
2574     theta2 = tmp;
2575 }
2576
2577
2578
2579
2580
2581 //-- find a point on the second ellipse that is different than the two
2582 //-- intersection points.
2583
2584 xmid = A2*cos ((theta1 + theta2)/2.0);
2585
2586 ymid = B2*sin ((theta1 + theta2)/2.0);
2587
2588
2589
2590
2591 //-- translate the point back to the second ellipse in its once-
2592 //-- translated+rotated position
2593
2594 cosphi = cos (PHI_2 - PHI_1);
2595
2596

```

```

2597     sinphi = sin (PHI_2 - PHI_1);
2598
2599     xmid_rt = xmid*cosphi + ymid*-sinphi + H2_TR;
2600
2601     ymid_rt = xmid*sinphi + ymid*cosphi + K2_TR;
2602
2603
2604
2605     //-- the point (xmid_rt, ymid_rt) is on the second ellipse 'between'
           the
2606
2607     //-- intersection points (x[1], y[1]) and (x[2], y[2]) when travelling
2608
2609     //-- counterclockwise from (x[1], y[1]) to (x[2], y[2]). If the point
2610
2611     //-- (xmid_rt, ymid_rt) is inside the first ellipse, then the desired
2612
2613     //-- segment of ellipse 2 contains the point (xmid_rt, ymid_rt), so
2614
2615     //-- integrate counterclockwise from (x[1], y[1]) to (x[2], y[2]).
2616
2617     //-- Otherwise, integrate counterclockwise from (x[2], y[2]) to
2618
2619     //-- (x[1], y[1])
2620
2621     if (((xmid_rt*xmid_rt)/(A1*A1) + (ymid_rt*ymid_rt)/(B1*B1)) > 1.0)
2622     {
2623
2624
2625         tmp = theta1;
2626
2627         theta1 = theta2;
2628
2629         theta2 = tmp;
2630
2631     }
2632
2633
2634
2635     //-- here is the ellipse segment routine for the second ellipse
2636
2637     if (theta1 > theta2)
2638
2639         theta1 -= twopi;
2640
2641     if ((theta2 - theta1) > pi)
2642
2643         trsign = 1.0;
2644
2645     else
2646
2647         trsign = -1.0;
2648
2649     area2 = 0.5*(A2*B2*(theta2 - theta1)
2650
2651         + trsign*fabs (x1_tr*y2_tr - x2_tr*y1_tr));
2652
2653
2654
2655     (*rtnCode) = TWO_INTERSECTION_POINTS;
2656
2657     return area1 + area2;
2658
2659 }
2660
2661
2662
2663 //-- three distinct intersection points, must have two intersections
2664
2665 //-- and one tangent, which is the only possibility
2666
2667 double threeintpts (double xint[], double yint[], double A1, double B1,
2668

```



```

2669         double PH1_1, double A2, double B2, double H2_TR,
2670         double K2_TR, double PH1_2, double AA, double BB,
2671         double CC, double DD, double EE, double FF,
2672         int *rtnCode)
2673 {
2674     int i, tanpts, tanindex, fnRtn;
2675     double OverlapArea;
2676
2677     //— need to determine which point is a tangent, and which two points
2678     //— are intersections
2679     tanpts = 0;
2680     for (i = 1; i <= 3; i++)
2681     {
2682         fnRtn = istanpt (xint[i], yint[i], A1, B1, AA, BB, CC, DD, EE, FF);
2683
2684         if (fnRtn == TANGENT_POINT)
2685         {
2686             tanpts++;
2687             tanindex = i;
2688         }
2689     }
2690
2691     //— there MUST be 2 intersection points and only one tangent
2692     if (tanpts != 1)
2693     {
2694         //— should never get here unless there is a problem discerning
2695         //— whether or not a point is a tangent or intersection
2696         (*rtnCode) = ERROR_INTERSECTIONPTS;
2697         return -1.0;
2698     }
2699
2700     //— store the two interesection points into (x[1], y[1]) and
2701     //— (x[2], y[2])
2702     switch (tanindex)
2703     {
2704     case 1:
2705         xint[1] = xint[3];

```

```

2742
2743         yint[1] = yint[3];
2744
2745         break;
2746
2747
2748
2749         case 2:
2750
2751             xint[2] = xint[3];
2752
2753             yint[2] = yint[3];
2754
2755             break;
2756
2757
2758
2759         case 3:
2760
2761             /*— intersection points are already in the right places
2762
2763             break;
2764
2765     }
2766
2767
2768     OverlapArea = twointpts (xint, yint, A1, B1, PHI_1, A2, B2, H2_TR,
2769                             K2_TR,
2770
2771                             PHI_2, AA, BB, CC, DD, EE, FF, rtnCode);
2772
2773     (*rtnCode) = THREE_INTERSECTION_POINTS;
2774
2775     return OverlapArea;
2776 }
2777
2778
2779
2780 /*— four intersection points
2781
2782 double fourintpts (double xint[], double yint[], double A1, double B1,
2783                   double PHI_1, double A2, double B2, double H2_TR,
2784                   double K2_TR, double PHI_2, double AA, double BB,
2785                   double CC, double DD, double EE, double FF, int *rtnCode
2786                   )
2787
2788 {
2789     {
2790
2791         int i, j, k;
2792
2793         double xmid, ymid, xint_tr[5], yint_tr[5], OverlapArea;
2794
2795         double theta[5], theta_tr[5], cosphi, sinphi, tmp0, tmp1, tmp2;
2796
2797         double area1, area2, area3, area4, area5;
2798
2799
2800
2801
2802         /*— only one case, which involves two segments from each ellipse, plus
2803
2804         /*— two triangles.
2805
2806         /*— get the parametric angles along the first ellipse for each of the
2807
2808         /*— intersection points
2809
2810         for (i = 1; i <= 4; i++)
2811
2812

```

```

2813 {
2814
2815     if (fabs (xint[i]) > A1)
2816
2817         xint[i] = (xint[i] < 0) ? -A1 : A1;
2818
2819     if (yint[i] < 0.0) //— Quadrant III or IV
2820
2821         theta[i] = twopi - acos (xint[i] / A1);
2822
2823     else //— Quadrant I or II
2824
2825         theta[i] = acos (xint[i] / A1);
2826
2827 }
2828
2829
2830 //— sort the angles by straight insertion, and put the points in
2831 //— counter-clockwise order
2832
2833 for (j = 2; j <= 4; j++)
2834 {
2835     tmp0 = theta[j];
2836
2837     tmp1 = xint[j];
2838
2839     tmp2 = yint[j];
2840
2841
2842     for (k = j - 1; k >= 1; k--)
2843     {
2844         if (theta[k] <= tmp0)
2845             break;
2846
2847         theta[k+1] = theta[k];
2848
2849         xint[k+1] = xint[k];
2850
2851         yint[k+1] = yint[k];
2852     }
2853
2854     theta[k+1] = tmp0;
2855
2856     xint[k+1] = tmp1;
2857
2858     yint[k+1] = tmp2;
2859 }
2860
2861 //— find the area of the interior quadrilateral
2862
2863 area1 = 0.5*fabs ((xint[3] - xint[1])*(yint[4] - yint[2])
2864                  - (xint[4] - xint[2])*(yint[3] - yint[1]));
2865
2866 //— the intersection points lie on the second ellipse in its once

```

```

2886
2887 //— translated+rotated position. The segment algorithm is implemented
2888 //— for an ellipse that is centered at the origin, and oriented with
2889 //— the coordinate axes; so, in order to use the segment algorithm
2890 //— with the second ellipse, the intersection points must be further
2891 //— translated+rotated by amounts that put the second ellipse centered
2892 //— at the origin and oriented with the coordinate axes.
2893
2894 cosphi = cos (PHI.1 - PHI.2);
2895
2896 sinphi = sin (PHI.1 - PHI.2);
2897
2898 for (i = 1; i <= 4; i++)
2899 {
2900     xint_tr[i] = (xint[i] - H2.TR)*cosphi + (yint[i] - K2.TR)*-sinphi;
2901     yint_tr[i] = (xint[i] - H2.TR)*sinphi + (yint[i] - K2.TR)*cosphi;
2902
2903     if (fabs (xint_tr[i]) > A2)
2904         xint_tr[i] = (xint_tr[i] < 0) ? -A2 : A2;
2905     if (yint_tr[i] < 0.0) //— Quadrant III or IV
2906         theta_tr[i] = twopi - acos (xint_tr[i]/A2);
2907     else //— Quadrant I or II
2908         theta_tr[i] = acos (xint_tr[i]/A2);
2909 }
2910
2911 //— get the area of the two segments on ellipse 1
2912
2913 xmid = A1*cos ((theta[1] + theta[2])/2.0);
2914
2915 ymid = B1*sin ((theta[1] + theta[2])/2.0);
2916
2917 //— the point (xmid, ymid) is on the first ellipse 'between' the two
2918 //— sorted intersection points (xint[1], yint[1]) and (xint[2], yint
2919 //— [2])
2920
2921 //— when travelling counter- clockwise from (xint[1], yint[1]) to
2922 //— (xint[2], yint[2]). If the point (xmid, ymid) is inside the
2923 //— second
2924 //— ellipse, then one desired segment of ellipse 1 contains the point
2925 //— (xmid, ymid), so integrate counterclockwise from (xint[1], yint
2926 //— [1])
2927 //— to (xint[2], yint[2]) for the first segment, and from
2928 //— (xint[3], yint[3] to (xint[4], yint[4]) for the second segment.
2929
2930 if (ellipse2tr (xmid, ymid, AA, BB, CC, DD, EE, FF) < 0.0)
2931 {
2932

```

```

2956     area2 = 0.5*(A1*B1*(theta[2] - theta[1])
2957         - fabs (xint[1]*yint[2] - xint[2]*yint[1]));
2958
2959     area3 = 0.5*(A1*B1*(theta[4] - theta[3])
2960         - fabs (xint[3]*yint[4] - xint[4]*yint[3]));
2961
2962     area4 = 0.5*(A2*B2*(theta_tr[3] - theta_tr[2])
2963         - fabs (xint_tr[2]*yint_tr[3] - xint_tr[3]*yint_tr[2]));
2964
2965     area5 = 0.5*(A2*B2*(theta_tr[1] - (theta_tr[4] - twopi))
2966         - fabs (xint_tr[4]*yint_tr[1] - xint_tr[1]*yint_tr[4]));
2967
2968 }
2969
2970 else
2971 {
2972     area2 = 0.5*(A1*B1*(theta[3] - theta[2])
2973         - fabs (xint[2]*yint[3] - xint[3]*yint[2]));
2974
2975     area3 = 0.5*(A1*B1*(theta[1] - (theta[4] - twopi))
2976         - fabs (xint[4]*yint[1] - xint[1]*yint[4]));
2977
2978     area4 = 0.5*(A2*B2*(theta[2] - theta[1])
2979         - fabs (xint_tr[1]*yint_tr[2] - xint_tr[2]*yint_tr[1]));
2980
2981     area5 = 0.5*(A2*B2*(theta[4] - theta[3])
2982         - fabs (xint_tr[3]*yint_tr[4] - xint_tr[4]*yint_tr[3]));
2983
2984 }
2985
2986 OverlapArea = area1 + area2 + area3 + area4 + area5;
2987
2988 (*rtnCode) = FOUR_INTERSECTION_POINTS;
2989
2990 return OverlapArea;
2991 }
2992
2993
2994
2995
2996
2997
2998
2999
3000
3001
3002
3003
3004
3005
3006
3007
3008
3009 ///-- check whether an intersection point is a tangent or a cross-point
3010
3011 int istanpt (double x, double y, double A1, double B1, double AA, double BB
3012             ,
3013             double CC, double DD, double EE, double FF)
3014 {
3015     double x1, y1, x2, y2, theta, test1, test2, branch, eps_radian;
3016
3017
3018
3019
3020
3021 ///-- Avoid inverse trig calculation errors: there could be an error
3022
3023 ///-- if \textbar x1/A\textbar > 1.0 when calling acos(). If execution
3024 arrives here,
3025
3026 ///-- then the point is on the ellipse within EPS.

```

```

3027  if (fabs (x) > A1)
3028
3029      x = (x < 0) ? -A1 : A1;
3030
3031
3032
3033  //— Calculate the parametric angle on the ellipse for (x, y)
3034
3035  //— The parametric angles depend on the quadrant where each point
3036
3037  //— is located. See Table 1 in the reference.
3038
3039  if (y < 0.0)    //— Quadrant III or IV
3040
3041      theta = twopi - acos (x / A1);
3042
3043  else            //— Quadrant I or II
3044
3045      theta = acos (x / A1);
3046
3047
3048
3049  //— determine the distance from the origin to the point (x, y)
3050
3051  branch = sqrt (x*x + y*y);
3052
3053
3054
3055  //— use the distance to find a small angle, such that the distance
3056
3057  //— along ellipse 1 is approximately 2*EPS
3058
3059  if (branch < 100.0*EPS)
3060
3061      eps_radian = 2.0*EPS;
3062
3063  else
3064
3065      eps_radian = asin (2.0*EPS/branch);
3066
3067
3068
3069  //— determine two points that are on each side of (x, y) and lie on
3070
3071  //— the first ellipse
3072
3073  x1 = A1*cos (theta + eps_radian);
3074
3075  y1 = B1*sin (theta + eps_radian);
3076
3077  x2 = A1*cos (theta - eps_radian);
3078
3079  y2 = B1*sin (theta - eps_radian);
3080
3081
3082
3083  //— evaluate the two adjacent points in the second ellipse equation
3084
3085  test1 = ellipse2tr (x1, y1, AA, BB, CC, DD, EE, FF);
3086
3087  test2 = ellipse2tr (x2, y2, AA, BB, CC, DD, EE, FF);
3088
3089
3090
3091  //— if the ellipses are tangent at the intersection point, then
3092
3093  //— points on both sides will either both be inside ellipse 1, or
3094
3095  //— they will both be outside ellipse 1
3096
3097  if ((test1*test2) > 0.0)
3098
3099      return TANGENT_POINT;

```

```

3100
3101     else
3102
3103         return INTERSECTION_POINT;
3104
3105     }
3106
3107
3108
3109 //

```

```

3110
3111 /* CACM Algorithm 326: Roots of low order polynomials.
3112    Nonweiler, Terence R.F., CACM Algorithm 326: Roots of low order
3113    polynomials, Communications of the ACM, vol. 11 no. 4, pages
3114    269-270 (1968). Translated into c and programmed by M. Dow, ANUSF,
3115    Australian National University, Canberra, Australia.
3116    Accessed at http://www.netlib.org/toms/326.
3117    Modified to void functions, integers replaced with floating point
3118    where appropriate, some other slight modifications for readability
3119    and debugging ease.
3120 */
3121 */

```

```

3130
3131 void QUADROOTS (double p[], double r[][5])
3132
3133 {
3134     /*
3135     Array r[3][5] p[5]
3136     Roots of poly p[0]*x^2 + p[1]*x + p[2]=0
3137     x=r[1][k] + i r[2][k] k=1,2
3138     */
3139     double b,c,d;
3140     b=-p[1]/(2.0*p[0]);
3141     c=p[2]/p[0];
3142     d=b*b-c;
3143     if (d>=0.0)
3144     {
3145         if (b>0.0)
3146             b=(r[1][2]=(sqrt(d)+b));
3147         else
3148             b=(r[1][2]=(-sqrt(d)+b));
3149         r[1][1]=c/b;
3150         r[2][1]=(r[2][2]=0.0);
3151     }

```

```

3169     }
3170
3171     else
3172
3173     {
3174         d=(r[2][1]=sqrt(-d));
3175
3176         r[2][2]=-d;
3177
3178         r[1][1]=(r[1][2]=b);
3179
3180     }
3181
3182     return;
3183
3184 }
3185
3186
3187
3188
3189 void CUBICROOTS(double p[], double r[][5])
3190
3191 {
3192     /*
3193     Array r[3][5] p[5]
3194
3195     Roots of poly  $p[0]*x^{\{ \}3} + p[1]*x^{\{ \}2} + p[2]*x + p[3] = 0$ 
3196
3197      $x=r[1][k] + i r[2][k] \quad k=1,\dots,3$ 
3198
3199     Assumes  $0 < \arctan(x) < \pi/2$  for  $x > 0$ 
3200
3201     */
3202
3203     double s,t,b,c,d;
3204
3205     int k;
3206
3207     if(p[0]!=1.0)
3208
3209     {
3210
3211         for(k=1;k<4;k++)
3212
3213             p[k]=p[k]/p[0];
3214
3215         p[0]=1.0;
3216
3217     }
3218
3219     s=p[1]/3.0;
3220
3221     t=s*p[1];
3222
3223     b=0.5*(s*(t/1.5-p[2])+p[3]);
3224
3225     t=(t-p[2])/3.0;
3226
3227     c=t*t*t;
3228
3229     d=b*b-c;
3230
3231     if(d>=0.0)
3232
3233     {
3234
3235         d=pow((sqrt(d)+fabs(b)),1.0/3.0);
3236
3237         if(d!=0.0)
3238
3239         {
3240
3241

```



```

3242
3243         if (b>0.0)
3244             b=-d;
3245
3246         else
3247             b=d;
3248
3249             b=d;
3250
3251             c=t/b;
3252
3253     }
3254
3255     d=r[2][2]=sqrt\eqref{GrindEQ--0-75-}*(b-c);
3256
3257     b=b+c;
3258
3259     c=r[1][2]=-0.5*b-s;
3260
3261     if ((b>0.0 \&\& s<=0.0) \textbar \textbar (b<0.0 \&\& s>0.0))
3262     {
3263
3264         r[1][1]=c;
3265
3266         r[2][1]=-d;
3267
3268         r[1][3]=b-s;
3269
3270         r[2][3]=0.0;
3271
3272     }
3273
3274     else
3275     {
3276
3277         r[1][1]=b-s;
3278
3279         r[2][1]=0.0;
3280
3281         r[1][3]=c;
3282
3283         r[2][3]=-d;
3284
3285     }
3286
3287 } /* end 2 equal or complex roots */
3288
3289 else
3290 {
3291
3292     if (b==0.0)
3293         d=atan\eqref{GrindEQ--1-0-}/1.5;
3294
3295     else
3296         d=atan(sqrt(-d)/fabs(b))/3.0;
3297
3298     if (b<0.0)
3299         b=2.0*sqrt(t);
3300
3301     else
3302         b=-2.0*sqrt(t);
3303
3304     c=cos(d)*b;
3305
3306     t=-sqrt\eqref{GrindEQ--0-75-}*sin(d)*b-0.5*c;
3307
3308
3309
3310
3311
3312
3313
3314

```

```

3315     d=t-c-s;
3316
3317     c=c-s;
3318
3319     t=t-s;
3320
3321     if ( fabs ( c ) > fabs ( t ) )
3322     {
3323
3324         r [ 1 ] [ 3 ] = c ;
3325
3326     }
3327
3328     else
3329     {
3330
3331         r [ 1 ] [ 3 ] = t ;
3332
3333         t=c;
3334
3335     }
3336
3337     if ( fabs ( d ) > fabs ( t ) )
3338     {
3339
3340         r [ 1 ] [ 2 ] = d ;
3341
3342     }
3343
3344     else
3345     {
3346
3347         r [ 1 ] [ 2 ] = t ;
3348
3349         t=d;
3350
3351     }
3352
3353     r [ 1 ] [ 1 ] = t ;
3354
3355     for ( k=1; k<4; k++)
3356         r [ 2 ] [ k ] = 0.0;
3357
3358 }
3359
3360 return ;
3361 }
3362
3363 void BIQUADROOTS ( double p [ ] , double r [ ] [ 5 ] )
3364 {
3365     /*
3366     Array r [ 3 ] [ 5 ] p [ 5 ]
3367     Roots of poly p [ 0 ] * x ^ { 4 } + p [ 1 ] * x ^ { 3 } + p [ 2 ] * x ^ { 2 } + p [ 3 ] * x + p [ 4 ] =
3368     0
3369
3370     x=r [ 1 ] [ k ] + i r [ 2 ] [ k ] k=1, ..., 4
3371
3372     */
3373     double a, b, c, d, e;
3374

```

```

3387     int k, j;
3388
3389     if (p[0] != 1.0)
3390     {
3391
3392         for (k=1; k<5; k++)
3393             p[k]=p[k]/p[0];
3394
3395         p[0]=1.0;
3396     }
3397
3400     e=0.25*p[1];
3401
3402     b=2.0*e;
3403
3404     c=b*b;
3405
3406     d=0.75*c;
3407
3408     b=p[3]+b*(c-p[2]);
3409
3410     a=p[2]-d;
3411
3412     c=p[4]+e*(e*a-p[3]);
3413
3414     a=a-d;
3415
3416     p[1]=0.5*a;
3417
3418     p[2]=(p[1]*p[1]-c)*0.25;
3419
3420     p[3]=b*b/(-64.0);
3421
3422     if (p[3]<0.0)
3423     {
3424
3425         CUBICROOTS(p, r);
3426
3427         for (k=1; k<4; k++)
3428         {
3429
3430             if (r[2][k]==0.0 && r[1][k]>0.0)
3431             {
3432
3433                 d=r[1][k]*4.0;
3434
3435                 a=a+d;
3436
3437                 if (a>=0.0 && b>=0.0)
3438                     p[1]=sqrt(d);
3439
3440                 else if (a<=0.0 && b<=0.0)
3441                     p[1]=sqrt(d);
3442
3443                 else
3444                     p[1]=-sqrt(d);
3445
3446                 b=0.5*(a+b/p[1]);
3447
3448                 goto QUAD;
3449             }
3450         }
3451     }
3452
3453     }
3454
3455     }
3456
3457     }
3458
3459     }

```

```

3460
3461     }
3462
3463     if (p[2]<0.0)
3464     {
3465
3466         b=sqrt ( c ) ;
3467
3468         d=b+b-a ;
3469
3470         p[1]=0.0 ;
3471
3472         if (d>0.0)
3473
3474             p[1]=sqrt ( d ) ;
3475
3476     }
3477
3478     else
3479     {
3480
3481         if (p[1]>0.0)
3482
3483             b=sqrt ( p[2] ) *2.0+p[1] ;
3484
3485         else
3486
3487             b=-sqrt ( p[2] ) *2.0+p[1] ;
3488
3489         if (b!=0.0)
3490
3491         {
3492
3493             p[1]=0.0 ;
3494
3495         }
3496
3497     }
3498
3499     else
3500     {
3501
3502         for (k=1;k<5;k++)
3503
3504         {
3505
3506             r[1][k]=-e ;
3507
3508             r[2][k]=0.0 ;
3509
3510         }
3511
3512         goto END;
3513
3514     }
3515
3516 }
3517
3518
3519 QUAD:
3520
3521     p[2]=c/b ;
3522
3523     QUADROOTS( p , r ) ;
3524
3525     for (k=1;k<3;k++)
3526
3527         for (j=1;j<3;j++)
3528
3529             r[j][k+2]=r[j][k] ;
3530
3531     p[1]=-p[1] ;
3532

```

```

3533     p[2]=b;
3534
3535     QUADROOTS(p,r);
3536
3537     for(k=1;k<5;k++)
3538         r[1][k]=r[1][k]-e;
3539
3540
3541 END:
3542
3543     return;
3544
3545 }

```

LISTING 15. C-SOURCE CODE FOR UTILITY FUNCTIONS

```

7. APPENDIX D.
3547 program\_constants.h:
3548
3549
3550
3551 //

```

```

3552
3553 //== INCLUDE ANSI C SYSTEM HEADER FILES

```

```

3554
3555 //

```

```

3556
3557 #include <math.h> /*-- for calls to trig, sqrt and power functions
3558
3559
3560
3561 //

```

```

3562
3563 //== DEFINE PROGRAM CONSTANTS

```

```

3564
3565 //

```

```

3566
3567 #define NORMAL_TERMINATION 0
3568
3569 #define NO_INTERSECTION_POINTS 100
3570
3571 #define ONE_INTERSECTION_POINT 101
3572
3573 #define LINE_TANGENT_TO_ELLIPSE 102
3574
3575 #define DISJOINT_ELLIPSES 103
3576
3577 #define ELLIPSE2_OUTSIDETANGENT_ELLIPSE1 104
3578
3579 #define ELLIPSE2_INSIDETANGENT_ELLIPSE1 105
3580
3581 #define ELLIPSES_INTERSECT 106
3582
3583 #define TWO_INTERSECTION_POINTS 107
3584
3585 #define THREE_INTERSECTION_POINTS 108
3586
3587 #define FOUR_INTERSECTION_POINTS 109
3588
3589 #define ELLIPSE1_INSIDE_ELLIPSE2 110
3590
3591 #define ELLIPSE2_INSIDE_ELLIPSE1 111

```

```

3592
3593 #define ELLIPSES_ARE_IDENTICAL          112
3594
3595 #define INTERSECTION_POINT              113
3596
3597 #define TANGENT_POINT                    114
3598
3599
3600
3601 #define ERROR_ELLIPSE_PARAMETERS        -100
3602
3603 #define ERROR_DEGENERATE_ELLIPSE        -101
3604
3605 #define ERROR_POINTS_NOT_ON_ELLIPSE     -102
3606
3607 #define ERROR_INVERSE_TRIG              -103
3608
3609 #define ERROR_LINE_POINTS                -104
3610
3611 #define ERROR_QUARTIC_CASE               -105
3612
3613 #define ERROR_POLYNOMIAL_DEGREE         -107
3614
3615 #define ERROR_POLYNOMIAL_ROOTS          -108
3616
3617 #define ERROR_INTERSECTION_PTS          -109
3618
3619 #define ERROR_CALCULATIONS               -112
3620
3621
3622
3623 #define EPS                             +1.0E-07
3624
3625 #define pi      (2.0*asin (1.0)) //— a maximum-precision value of pi
3626
3627 #define twopi   (2.0*pi)          //— a maximum-precision value of 2*pi
3628
3629
3630
3631
3632
3633
3634
3635 call_es.c:
3636
3637
3638
3639 #include <stdio.h>
3640
3641 #include <math.h>
3642
3643 #include "program_constants.h"
3644
3645 double ellipse_segment (double A, double B, double X1, double Y1, double X2
3646 ,
3647                          double Y2, int *MessageCode);
3648
3649
3650
3651 int main (int argc, char ** argv)
3652 {
3653 {
3654     double A, B;
3655     double X1, Y1;
3656     double X2, Y2;
3657     double area1, area2;
3658 }
3659 }
3660
3661
3662

```

```

3663     double pi = 2.0 * asin eqref{GrindEQ--1.0-};    //— a maximum-precision
3664         value of pi
3665     int rtn;
3666
3667     char msg[1024];
3668
3669     printf ("Calling ellipse_segment.ctextbackslash n");
3670
3671
3672
3673     //— case shown in Fig. 1
3674
3675     A = 4.;
3676
3677     B = 2.;
3678
3679     X1 = 4./sqrt (5.);
3680
3681     Y1 = 4./sqrt (5.);
3682
3683     X2 = -3.;
3684
3685     Y2 = -sqrt (7.)/2.;
3686
3687
3688
3689     areal = ellipse_segment (A, B, X1, Y1, X2, Y2, &rtn);
3690
3691     sprintf (msg,"Fig 1: segment area = %15.8f, return_value = %d\
3692         textbackslash n", areal, rtn);
3693
3694     printf (msg);
3695
3696
3697     //— case shown in Fig. 2
3698
3699     A = 4.;
3700
3701     B = 2.;
3702
3703     X1 = -3.;
3704
3705     Y1 = -sqrt (7.)/2.;
3706
3707     X2 = 4./sqrt (5.);
3708
3709     Y2 = 4./sqrt (5.);
3710
3711
3712
3713     area2 = ellipse_segment (A, B, X1, Y1, X2, Y2, &rtn);
3714
3715     sprintf (msg,"Fig 2: segment area = %15.8f, return_value = %
3716         dtextbackslash n", area2, rtn);
3717
3718     printf (msg);
3719
3720
3721     sprintf (msg,"sum of ellipse segments = %15.8ftextbackslash n", areal +
3722         area2);
3723
3724     printf (msg);
3725
3726     sprintf (msg,"total ellipse area by pi*a*b = %15.8ftextbackslash n", pi*
3727         A*B);
3728
3729     printf (msg);
3730

```

```

3731     return rtn;
3732
3733 }
3734
3735
3736
3737
3738
3739 call_el.c:
3740
3741
3742
3743 #include <stdio.h>
3744
3745 #include <math.h>
3746
3747 #include "program_constants.h"
3748
3749 double \textbf{ellipse_segment} (double A, double B, double X1, double Y1,
    double X2,
3750
3751                                double Y2, int *MessageCode);
3752
3753
3754
3755 double \textbf{ellipse_line_overlap} (double PHI, double A, double B,
    double H,
3756
3757                                double K, double X1, double Y1, double X2,
3758
3759                                double Y2, int *MessageCode);
3760
3761
3762
3763 int \textbf{main} (int argc, char ** argv)
3764 {
3765
3766     double A, B;
3767
3768     double H, K, PHI;
3769
3770     double X1, Y1;
3771
3772     double X2, Y2;
3773
3774     double area1, area2;
3775
3776     double pi = 2.0 * \textbf{asin} \eqref{GrindEQ--1.0-}; //— a maximum
    —precision value of pi
3777
3778     int rtn;
3779
3780     char msg[1024];
3781
3782     \textbf{printf} (" Calling ellipse_line_overlap.c\textbackslash n");
3783
3784
3785
3786
3787     //— case shown in Fig. 4
3788
3789     A = 4.;
3790
3791     B = 2.;
3792
3793     H = -6;
3794
3795     K = 3;
3796
3797     PHI = 3.* pi / 8.0;
3798
3799     X1 = -3.;
3800

```



```

3801     Y1 = 3.;
3802
3803     X2 = -7.;
3804
3805     Y2 = 7.;
3806
3807
3808
3809     areal = \textbf{ellipse\_line\_overlap} (PHI, A, B, H, K, X1, Y1, X2, Y2
3810         , \&rtn);
3811
3812
3813     \textbf{sprintf} (msg,"Fig 4: area = \%15.8f, return_value = \%d\
3814         textbackslash n", areal, rtn);
3815
3816
3817     \textbf{printf} (msg);
3818
3819
3820
3821     //— case shown in Fig. 4, points reversed
3822
3823     A = 4.;
3824
3825     B = 2.;
3826
3827     H = -6;
3828
3829     K = 3;
3830
3831     PHI = 3.*pi/8.0;
3832
3833     X1 = -7.;
3834
3835     Y1 = 7.;
3836
3837     X2 = -3.;
3838
3839     Y2 = 3.;
3840
3841
3842
3843     area2 = \textbf{ellipse\_line\_overlap} (PHI, A, B, H, K, X1, Y1, X2, Y2
3844         , \&rtn);
3845
3846
3847     \textbf{sprintf} (msg,"Fig 4 reverse: area = \%15.8f, return_value = \%d\
3848         textbackslash n", area2, rtn);
3849
3850
3851     \textbf{printf} (msg);
3852
3853
3854
3855     \textbf{sprintf} (msg,"sum of ellipse segments = \%15.8ftextbackslash n",
3856         areal + area2);
3857
3858
3859     \textbf{printf} (msg);
3860
3861
3862
3863     \textbf{sprintf} (msg,"total ellipse area by pi*a*b = \%15.8
3864         ftextbackslash n", pi*A*B);
3865
3866
3867     \textbf{printf} (msg);
3868
3869
3870
3871     return rtn;
3872 }
3873
3874 call_ee.c:
3875
3876
3877

```

```

3868
3869 #include <stdio.h>
3870
3871 #include "program_constants.h"
3872
3873 double ellipse_ellipse_overlap (double PHI_1, double A1, double B1,
3874                                double H1, double K1, double PHI_2,
3875                                double A2, double B2, double H2, double K2
3876                                ,
3877                                int *rtnCode);
3878
3879
3880
3881
3882
3883 int main (int argc, char ** argv)
3884 {
3885
3886     double A1, B1, H1, K1, PHI_1;
3887
3888     double A2, B2, H2, K2, PHI_2;
3889
3890     double area;
3891
3892     int rtn;
3893
3894     char msg[1024];
3895
3896     printf ("Calling ellipse_ellipse_overlap.c\\textbackslash n\\
3897             textbackslash n");
3898
3899
3900
3901     //-- case 0-1
3902
3903     A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
3904
3905     A2 = 2.; B2 = 1.; H2 = -.75; K2 = 0.25; PHI_2 = pi/4.;
3906
3907     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
3908                                     PHI_2, A2, B2, H2, K2, &rtn);
3909
3910     sprintf (msg, "Case 0-1: area = %15.8f, return_value = %d\\
3911                 textbackslash n", area, rtn);
3912
3913     printf (msg);
3914
3915     sprintf (msg, "          ellipse 2 area by pi*a2*b2 = %15.8f\\
3916                 textbackslash n", pi*A2*B2);
3917
3918     printf (msg);
3919
3920
3921     //-- case 0-2
3922
3923     A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
3924
3925     A2 = 3.; B2 = 2.; H2 = -.3; K2 = -.25; PHI_2 = pi/4.;
3926
3927     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
3928                                     PHI_2, A2, B2, H2, K2, &rtn);
3929
3930     sprintf (msg, "Case 0-2: area = %15.8f, return\\_value = %d\\
3931                 textbackslash n", area, rtn);
3932
3933     printf (msg);
3934

```

```

3935     sprintf (msg,"           ellipse 1 area by pi*a1*b1 = \%15.8f\
3936             textbackslash n", pi*A1*B1);
3937     printf (msg);
3938
3939
3940
3941     //--- case 0-3
3942     A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
3943
3944     A2 = 1.5; B2 = 0.75; H2 = -2.5; K2 = 1.5; PHI_2 = pi/4.;
3945
3946     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
3947                                     PHI_2, A2, B2, H2, K2, &rtn);
3948
3949
3950     sprintf (msg,"Case 0-3: area = \%15.8f, return_value = \%d\
3951             textbackslash n", area, rtn);
3952
3953     printf (msg);
3954
3955     printf ("           Ellipses are disjoint, overlap area = 0.0\
3956             textbackslash n\textbackslash n");
3957
3958
3959     //--- case 1-1
3960
3961     A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
3962
3963     A2 = 2.; B2 = 1.; H2 = -1.0245209260022; K2 = 0.25; PHI_2 = pi/4.;
3964
3965     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
3966                                     PHI_2, A2, B2, H2, K2, \&rtn);
3967
3968
3969     sprintf (msg,"Case 1-1: area = \%15.8f, return\_value = \%d\
3970             textbackslash n", area, rtn);
3971
3972     printf (msg);
3973
3974     sprintf (msg,"           ellipse 2 area by pi*a2*b2 = \%15.8f\
3975             textbackslash n", pi*A2*B2);
3976
3977     printf (msg);
3978
3979
3980     //--- case 1-2
3981
3982     A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
3983
3984     A2 = 3.5; B2 = 1.8; H2 = .22; K2 = .1; PHI_2 = pi/4.;
3985
3986     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
3987                                     PHI_2, A2, B2, H2, K2, \&rtn);
3988
3989     sprintf (msg,"Case 1-2: area = \%15.8f, return_value = \%d\
3990             textbackslash n", area, rtn);
3991
3992     printf (msg);
3993
3994     sprintf (msg,"           ellipse 1 area by pi*a1*b1 = \%15.8f\
3995             textbackslash n", pi*A1*B1);
3996
3997     printf (msg);
3998
3999     //--- case 1-3
4000

```

```

4001 A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4002
4003 A2 = 1.5; B2 = 0.75; H2 = -2.01796398085; K2 = 1.25; PHI_2 = pi/4.;
4004
4005 area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4006                                  PHI_2, A2, B2, H2, K2, \&rtn);
4007
4008 sprintf (msg,"Case 1-3: area = %15.8f, return\_value = %d\
4009          textbackslash n", area, rtn);
4010
4011 printf (msg);
4012
4013 printf ("          Ellipses are disjoint, overlap area = 0.0\
4014          textbackslash n\textbackslash n");
4015
4016
4017 //--- case 2-1
4018
4019 A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4020
4021 A2 = 2.25; B2 = 1.5; H2 = 0.; K2 = 0.; PHI_2 = pi/4.;
4022
4023 area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4024                                  PHI_2, A2, B2, H2, K2, \&rtn);
4025
4026 sprintf (msg,"Case 2-1: area = %15.8f, return\_value = %d\
4027          textbackslash n", area, rtn);
4028
4029 printf (msg);
4030
4031 sprintf (msg,"          ellipse 2 area by pi*a2*b2 = %15.8f\
4032          textbackslash n", pi*A2*B2);
4033
4034 printf (msg);
4035
4036
4037 //--- case 2-2
4038
4039 A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4040
4041 A2 = 3.; B2 = 1.7; H2 = 0.; K2 = 0.; PHI_2 = pi/4.;
4042
4043 area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4044                                  PHI_2, A2, B2, H2, K2, \&rtn);
4045
4046 sprintf (msg,"Case 2-2: area = %15.8f, return\_value = %d\
4047          textbackslash n", area, rtn);
4048
4049 printf (msg);
4050
4051 sprintf (msg,"          ellipse 1 area by pi*a1*b1 = %15.8f\
4052          textbackslash n", pi*A1*B1);
4053
4054 printf (msg);
4055
4056
4057 //--- case 2-3
4058
4059 A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4060
4061 A2 = 2.; B2 = 1.; H2 = -2.; K2 = -1.; PHI_2 = pi/4.;
4062
4063 area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4064                                  PHI_2, A2, B2, H2, K2, \&rtn);
4065
4066

```

```

4067     sprintf (msg,"Case 2-3: area = \%.15.8f, return\_value = \%.d\
4068               textbackslash n\textbackslash n", area, rtn);
4069     printf (msg);
4070
4071
4072
4073     //--- case 3-1
4074     A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4075     A2 = 3.; B2 = 1.; H2 = 1.; K2 = 0.35; PHI_2 = pi/4.;
4076     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4077                                     PHI_2, A2, B2, H2, K2, \&rtn);
4078
4079     sprintf (msg,"Case 3-1: area = \%.15.8f, return\_value = \%.d\
4080               textbackslash n\textbackslash n", area, rtn);
4081     printf (msg);
4082
4083
4084
4085
4086
4087
4088     //--- case 3-2
4089     A1 = 2.; B1 = 1.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4090     A2 = 2.25; B2 = 1.5; H2 = 0.3; K2 = 0.; PHI_2 = pi/4.;
4091     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4092                                     PHI_2, A2, B2, H2, K2, \&rtn);
4093
4094     sprintf (msg,"Case 3-2: area = \%.15.8f, return\_value = \%.d\
4095               textbackslash n\textbackslash n", area, rtn);
4096     printf (msg);
4097
4098
4099
4100
4101
4102
4103
4104     //--- case 4-1
4105     A1 = 3.; B1 = 2.; H1 = 0.; K1 = 0.; PHI_1 = 0.;
4106     A2 = 3.; B2 = 1.; H2 = 1.; K2 = -0.5; PHI_2 = pi/4.;
4107     area = ellipse_ellipse_overlap (PHI_1, A1, B1, H1, K1,
4108                                     PHI_2, A2, B2, H2, K2, \&rtn);
4109
4110     sprintf (msg,"Case 4-1: area = \%.15.8f, return\_value = \%.d\
4111               textbackslash n\textbackslash n", area, rtn);
4112     printf (msg);
4113
4114
4115
4116
4117
4118
4119
4120
4121     return rtn;
4122
4123 }

```

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