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A new approach to characterizing the relative position of two ellipses depending on one parameter

Fernando Etayo [1,3], Laureano Gonzalez-Vega [*,2,3], Natalia del Rio [3]

Departamento de Matematicas, Estadistica y Computacion, Universidad de Cantabria, Spain

Abstract

By using several tools coming from Real Algebraic Geometry, Computer Algebra and Projective Geometry (Sturm–Habicht sequences and the classification of pencils of conics in $\mathbb{P}_2(\mathbb{R})$), a new approach for characterizing the ten relative positions of two ellipses is introduced.

Each relative position is exclusively characterized by a set of equalities and inequalities depending only on the matrices defining the two considered ellipses and does not require in advance the computation or knowledge of the intersection points between them.

Moreover, this characterization is specially well adapted for computionally treating the case where the considered ellipses depend on one or several parameters.

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Introduction

The problem of detecting the collisions or overlap of two ellipses or ellipsoids is of interest to robotics, CAD/CAM, computer animation, etc., where ellipses or ellipsoids are often used for modelling (or enclosing) the shape of the objects under consideration. The problem to be considered here is obtaining closed formulae characterizing the relative position of two ellipses in the two dimensional real affine space by using several tools coming from Real Algebraic Geometry, Computer Algebra and Projective Geometry. All of them are necessary in order to obtain an algorithm characterizing the relative positions between two ellipses without computing explicitly the intersection points between them. Moreover this characterization should provide easily the manipulation of the formulae for the cases where the considered ellipses depend on one parameter.

Note that the problem considered in this paper is not the computation of the intersection points between the two considered ellipses. This intersection problem can be solved by any numerical nonlinear solver or by "ad-hoc" meth-

ods. Nevertheless, the results later described can be used as a preprocessing step since any intersection problem is highly simplified if the structure of the intersection set is known in advance.

Instead this paper shows that there are exactly ten possible relative positions between two ellipses in the two-dimensional real affine space in such a way that a complete characterization (closed formulae) in terms of the coefficients of the equations defining the two considered ellipses can be easily computed. These positions are listed in Table 1. They are ten if the ellipses are not distinguished and they are fourteen if they are (e.g., if one ellipse is contained in the other one and we want to decide what is the bigger and what is the smaller one). The closed formulae obtained in the paper need to mix two different kinds of information: the algebraic one derived from the coefficients of both ellipses (via the characteristic polynomial, see below) and the geometric one obtained from the pencil of conics generated by the two considered ellipses. None of them is itself enough to obtain the relative position of the ellipses, but mixing them we solve the problem, providing an explicit algorithm (Section 4 of the paper). In this sense, the present paper is a new and complete approach to characterize the relative positions of two ellipses. In some previous papers, as (Wang and Krasauskas, 2003; Wang et al., 2001; Liu and Chen, 2004), the authors obtained the characterization of some concrete positions of ellipses (see Theorem 1.1 in the present work), by using algebraic tools, but there was no a complete classification. On the other hand, there exist algebraic tools (e.g., the index function used in (Wang and Krasauskas, 2003)) allowing to obtain similar information to the algebraic one obtained here (using the Sturm–Habicht sequence). This characterization is specially useful when dealing with ellipses depending on one parameter, since the closed formulae previously mentioned need only to be evaluated in order to study the behaviour of, for example, two moving ellipses.

Let

$$\mathcal{A} = \left\{ (x, y) \in \mathbb{R}^2 \colon a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0 \right\}$$

be the equation of an ellipse. As usual it can be rewritten as $X^T A X = 0$, where $X^T = (x, y, 1)$ and $A = (a_{ij})$ is the symmetric matrix of coefficients. Considering two ellipses $\mathcal{A}$ and $\mathcal{B}$ given by $X^T A X = 0$ and $X^T B X = 0$ and, following the notation in (Wang et al., 2001) and (Wang and Krasauskas, 2003), the degree three polynomial

$$f(\lambda) = \det(\lambda A + B)$$

is called the *characteristic polynomial* of the pencil $\lambda A + B$.

In (Wang et al., 2001) and (Wang and Krasauskas, 2003) the authors give some partial results about the intersection of two ellipsoids, obtaining a complete characterization, in terms of the sign of the real roots of the characteristic polynomial, of the separation case: i.e. when the two ellipsoids can be separated by a plane. More precisely they prove that the two considered ellipsoids are separated if and only if $f(\lambda)$ (a degree four polynomial in this case) has two distinct positive roots. The same result applies to the case of two ellipses $\mathcal{A}$ and $\mathcal{B}$: they are separated by a line if and only if $f(\lambda)$ (a degree three polynomial in this case) has two distinct positive real roots and one negative real root.

The techniques to be presented in this paper will show directly that if

$$f(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$$

(once turned monic) then the following equivalence holds:

The ellipses $\mathcal{A}$ and $\mathcal{B}$ are separated if and only if

$$a \geqslant 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a < 0, \quad -3b + a^2 > 0, \qquad\qquad\qquad\qquad\qquad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \ .$$

It is worth to remark at this point that the obtained characterizations (as the one presented before and covering all the ten possible relative positions) do not require to solve the characteristic polynomial. The equalities and inequalities providing the complete characterization for the ten possible cases for the relative positions of two ellipses in $\mathbb{R}^2$ come from three different sources:

1. the characterization of the sign pattern for the real roots of the characteristic polynomial,
2. the characterization of the kind of degenerate conics in the pencil when the characteristic polynomial has multiple roots (since it has degree three, they can be described in a rational way in terms of the coefficients of the characteristic polynomial), and

3. the characterization of when a pencil contains empty conics under the hypothesis of a characteristic polynomial with three different negative real roots.

The approach presented in this paper is specially well suited for analyzing the relative position of two ellipses depending on a parameter t. For example, let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses that depend on a parameter t in the following way:

$$\mathcal{A}(t): \ X^T A(t) X = 0, \qquad \mathcal{B}(t): \ X^T B(t) X = 0.$$

In this case the characteristic polynomial

$$f(t; \lambda) = \det(\lambda A(t) + B(t))$$

is a degree three polynomial in λ whose coefficients depend on the parameter t. The sign of the real roots of the characteristic polynomial is going to be one of the main arguments in the approach to be presented in this paper. Thus, the study presented in this paper of the behaviour of the real roots of $f(t; \lambda)$ by using algebraic techniques and without requiring the knowledge of any approximation of those roots will provide easy to manipulate formulae in t specially suited in order to obtain the different possibilities (and their transitions) for the relative positions between $\mathcal{A}(t)$ and $\mathcal{B}(t)$ in terms of t.

In the considered case, if

$$f(t; \lambda) = \lambda^3 + a(t)\lambda^2 + b(t)\lambda + c(t)$$

(once turned monic) then the partition of $\mathbb{R}$ generated by the real roots of the equations:

$$a(t) = 0,$$
$$-3b(t) + a(t)^2 = 0,$$
$$3a(t)c(t) + b(t)a(t)^2 - 4b(t)^2 = 0,$$
$$-27c(t)^2 + 18c(t)a(t)b(t) + a(t)^2 b(t)^2 - 4a(t)^3 c(t) - 4b(t)^3 = 0$$

gives the required information in order to know when (in terms of t) the considered ellipses start overlapping.
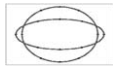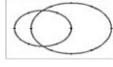
This paper is organized in the following way. In Section 1, we enumerate the ten relative positions of two ellipses and present some properties of the characteristic polynomial which allow us to characterize the nature of the intersection set between two ellipses in some cases (Theorem 1.1). In Section 2, we show how to characterize the sign behaviour of the real roots of a polynomial in terms of its coefficients by using the so called Sturm–Habicht sequence (a generalization of the Sturm sequence specially adapted to this case where parameters are present). In Section 3, we introduce the geometric formalism: the characteristic polynomial characterizes the degenerated conics in the pencil generated by the two considered ellipses. This information, together with the classification of pencils of projective conics (over $\mathbb{C}$ and $\mathbb{R}$), is used in Section 4 to introduce the algorithm which characterizes the relative position of two ellipses. In Section 5, we analyze how to use the described techniques to solve the problem of the determination of the relative position of two ellipses depending on one parameter. Finally, in the last section, we draw several conclusions together with the description of future research related to the problem considered in this paper.

1. The ten different relative positions of two ellipses

Given two ellipses in the two-dimensional real affine plane $\mathbb{R}^2$, we are interested in characterizing all possible relative positions of them, which are presented in Table 1.

By using Bezout Theorem ("Two algebraic curves of degrees p and q in the complex projective plane, which do not have common component, intersect in exactly pq points (counted with multiplicities)"; see (Fulton, 1969)), all the possible relative positions of two ellipses are determined. Thus, two non-degenerate conics of the complex projective plane always intersect in four points, whereas in the real affine plane they intersect in four or less points. In particular, two different ellipses (in the real affine plane) intersect in four or less points. As ellipses do not have points at infinity, the missing points are imaginary. Moreover, for each of them its conjugate appears as well, because the equations of the considered ellipses have real coefficients.

Table 1
The ten possible relative positions of two ellipses in $\mathbb{R}^2$

	Relative positions of two ellipses	
1	Transversal in 4 points	
2	Transversal in 2 points	
3	Separated	
4	One contained in the other one	
5	Transversal in 2 points and tangent in other point	
6	Externally tangent	
7	Internally tangent in one point	
8	Internally tangent in two points	
9	Transversal in one point and tangent in other point	
10	Coincident	

Next, we introduce the following notation for a common point of two ellipses: a point is said to be a *transversal point* (respectively *tangential point*) if the ellipses intersect transversally (respectively tangentially) at that point. It is clear that transversal points have multiplicity one, whereas tangential points have at least multiplicity two. Thus, we have the following relative positions, taking into account the number of intersection points:

- The ellipses are coincident (relative position **10** in Table 1).
- Four points: all of them are transversal (relative position **1** in Table 1).
- Three points: one of them is tangential and the others are transversal (relative position **5** in Table 1).
- Two points: there are three possibilities, both transversal (relative position **2** in Table 1), both tangential (relative position **8** in Table 1), and one of them transversal and the other one tangential (relative position **9** in Table 1).
- One point: it is tangential and the ellipses can be externally or internally tangent (relative positions **6** and **7** in Table 1, respectively). It can not be transversal because there will be three imaginary points.
- No points: the ellipses can be separated by a line (relative position **3** in Table 1), or one is contained in the other one (relative position **4** in Table 1).

This discussion motivates the following definition.

Definition 1.1. Two pairs of ellipses in $\mathbb{R}^2$ are said to have the same relative position if and only if both pairs have the same number of real intersection points and these have the same nature.

Moreover, the positions **4**, **5**, **7** and **8** are not symmetric to both ellipses. Positions **4**, **7** and **8** split into two cases depending on whether the ellipse $\mathcal{A}$ is contained in ellipse $\mathcal{B}$ or vice versa. In order to distinguish these cases we do

the following: we take one point of one of the ellipses and check if this point is an element of the interior or exterior of the other ellipse. This is possible, since we always know the center of any ellipse. Therefore, we take any line passing through the center, intersecting the ellipse in two transversal points. Hence, we take one of these points for testing the relative position with the other ellipse. Furthermore, position **5** splits into two subcases as well, as the set defined by the interior of ellipse $\mathcal{A}$ without the interior of the ellipse $\mathcal{B}$ has one or two connected components. The idea of distinguishing these cases is choosing a line which passes through both ellipses and then studying the behaviour of these points on this line (i.e. if they are in the interior or exterior of both ellipses at the same time or not). In conclusion, even though at the above table we show ten possible positions of two ellipses in $\mathbb{R}^2$, there actually are fourteen. This analysis will be shown in Section 4.

Next similar arguments, as in (Wang et al., 2001), for characterizing the possible positions of two ellipses are used. The equation of any conic $\mathcal{A}$ in $\mathbb{R}^2$ can be written as

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

or in matricial form

$$(x \quad y \quad 1) A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

where A is the symmetric matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}.$$

Moreover, if the conic is an ellipse then a_{11} and a_{22} have the same sign; assuming $a_{11} > 0$ and $a_{22} > 0$, then the interiors of any ellipse $\mathcal{A}$ is defined by $X^T A X < 0$.

Given two ellipses $\mathcal{A}$: $X^T A X = 0$ and $\mathcal{B}$: $X^T B X = 0$, their *characteristic polynomial* is defined as

$$f(\lambda) = \det(\lambda A + B)$$

which is a cubic polynomial in λ with real coefficients.

By using similar techniques as in (Wang et al., 2001) (for example, using the same affine transformation to move the ellipse $\mathcal{A}$ to an ellipse centered at the origin and the ellipse $\mathcal{B}$ to a circle), we can prove the following theorem.

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses with characteristic polynomial $f(\lambda)$. Then*:

(1) *The product of the three roots of the characteristic polynomial is negative.*
(2) *The characteristic polynomial $f(\lambda)$ has always, at least, one negative real root, and $f(0) \neq 0$.*
(3) *The characteristic polynomial $f(\lambda)$ has two distinct positive real roots if and only if the two ellipses are separated by a line* (*case* **3** *in Table* 1).
(4) *The characteristic polynomial $f(\lambda) = 0$ has a positive double root if and only if the two ellipses touch each other externally* (*case* **6** *in Table* 1).
(5) *$\mathcal{A}$ and $\mathcal{B}$ have a common interior point if and only if all the real roots of the characteristic polynomial $f(\lambda)$ are not positive* (*cases* **1**, **2**, **4**, **5**, **7**, **8**, **9** *and* **10** *in Table* 1).

Proof. Items (1), (2), (3), (4) and the "$\Leftarrow$" part of item (5) are derived directly from the corresponding results for ellipsoids presented in (Wang et al., 2001). The "$\Rightarrow$" part is immediate due to the previous items and to the information in Table 1. □

One of the main conclusions to derive from Theorem 1.1 is the fact that some of the possible configurations (for example, case **3**: two ellipses separated by a line) are characterized by the sign behaviour of the real roots of the characteristic polynomial of the two considered ellipses. The next section is devoted to show how to do that without computing explicitly the real roots of the characteristic polynomial (for example, by providing closed formulae depending only on the coefficients of the characteristic polynomial characterizing when this polynomial has exactly three different real roots, two of them positive).

2. Characterization of the sign behaviour of the real roots of the characteristic polynomial in terms of its coefficients

According to Theorem 1.1, in order to classify the relative position of two ellipses, the first step is the study of the sign of the real roots of the characteristic polynomial. The main tools (coming from Computer Algebra and Real Algebraic Geometry) to solve the sign behaviour problem before described will be the Sturm–Habicht sequence and the sign determination scheme.

2.1. Sturm–Habicht sequence

This section is devoted to introduce the definition of the Sturm–Habicht coefficients and their main properties related with the real root counting and sign determination problems. Sturm–Habicht sequence (and coefficients) was introduced in (Gonzalez-Vega et al., 1990); proofs of the results summarized into this section can be found in (Gonzalez-Vega et al., 1990) or (Gonzalez-Vega et al., 1998).

Definition 2.1. Let P, Q be polynomials in $\mathbb{R}[x]$ and $p, q \in \mathbb{N}$ with $\deg(P) \leqslant p$ and $\deg(Q) \leqslant q$:

$$P = \sum_{k=0}^{p} a_k x^k, \qquad Q = \sum_{k=0}^{q} b_k x^k.$$

If $i \in \{0, \ldots, \inf(p, q)\}$ then the polynomial subresultant associated to P, p, Q and q of index i is defined as follows:

$$\mathbf{Sres}_i(P, q, Q, q) = \sum_{j=0}^{i} M_j^i(P, Q) x^j$$

where every $M_j^i(P, Q)$ is the determinant of the matrix built with the columns $1, 2, \ldots, p+q-2i-1$ and $p+q-i-j$ in the matrix:

$$m_i(P, p, Q, q) = \overbrace{\begin{pmatrix} a_p & \ldots & a_0 & & & \\ & \ddots & & \ddots & & \\ & & a_p & \ldots & a_0 \\ b_q & \ldots & b_0 & & & \\ & \ddots & & \ddots & & \\ & & b_q & \ldots & b_0 \end{pmatrix}}^{p+q-i}.$$

The determinant $M_i^i(P, Q)$ will be called ith principal subresultant coefficient and will be denoted by $\mathbf{sres}_i(P, p, Q, q)$.

Next definition introduces Sturm–Habicht sequence associated to P and Q as the subresultant sequence for P and $P'Q$ modulo some well precised sign changes.

Definition 2.2. Let P and Q be polynomials in $\mathbb{R}[x]$ with $p = \deg(P)$ and $q = \deg(Q)$. Writing $v = p + q - 1$ and

$$\delta_k = (-1)^{k(k+1)/2}$$

for every integer k, the Sturm–Habicht sequence associated to P and Q is defined as the list of polynomials $\{\mathbf{StHa}_j(P, Q)\}_{j=0,\ldots,v+1}$ where $\mathbf{StHa}_{v+1}(P, Q) = P$, $\mathbf{StHa}_v(P, Q) = P'Q$ and for every $j \in \{0, \ldots, v-1\}$:

$$\mathbf{StHa}_j(P, Q) = \delta_{v-j} \mathbf{Sres}_j(P, v+1, P'Q, v).$$

For every j in $\{0, \ldots, v+1\}$ the principal jth Sturm–Habicht coefficient is defined as:

$$\mathbf{stha}_j(P, Q) = \mathrm{coef}_j\big(\mathbf{StHa}_j(P, Q)\big).$$

In case $Q = 1$, the notations $\mathbf{StHa}_j(P) = \mathbf{StHa}_j(P, 1)$ and $\mathbf{stha}_j(P) = \mathbf{stha}_j(P, 1)$ are to be used.

It is important to quote that the discriminant of P is equal to the polynomial $\mathbf{stha}_0(P)$ modulo the leading coefficient of P:

$$\mathbf{stha}_0(P) = a_p \, \text{discriminant}(P).$$

Thus, when $\mathbf{stha}_0(P) \neq 0$, the greatest common divisor of P and P' is trivial: in other words P is square free (i.e. without multiple roots). More generally, as these determinants are the formal leading coefficients of the subresultant sequence for P and P' (modulo some sign changes), the greatest common divisor of P and P' is obtained as a by–product thanks to the following equivalence:

$$\mathbf{StHa}_i(P) = \gcd(P, P') \quad \Longleftrightarrow \quad \begin{cases} \mathbf{stha}_0(P) = \cdots = \mathbf{stha}_{i-1}(P) = 0 \\ \mathbf{stha}_i(P) \neq 0. \end{cases} \tag{1}$$

Sign counting on the principal Sturm–Habicht coefficients provides a very useful information about the real roots of the considered polynomial. Next definitions show which are the sign counting functions to be used in the sequel (see (Gonzalez-Vega et al., 1990) or (Gonzalez-Vega et al., 1998)).

Definition 2.3. Let $\mathbb{I} = \{a_0, a_1, \ldots, a_n\}$ be a list of non zero elements in $\mathbb{R}$.

- $\mathbf{V}(\mathbb{I})$ is defined as the number of sign variations in the list $\{a_0, a_1, \ldots, a_n\}$,
- $\mathbf{P}(\mathbb{I})$ is defined as the number of sign permanences in the list $\{a_0, a_1, \ldots, a_n\}$.

Definition 2.4. Let $a_0, a_1, \ldots, a_n$ be elements in $\mathbb{R}$ with $a_0 \neq 0$ and with the following distribution of zeros:

$$\mathbb{I} = \{a_0, a_1, \ldots, a_n\}$$
$$= \{a_0, \ldots, a_{i_1}, \overbrace{0, \ldots, 0}^{k_1}, a_{i_1+k_1+1}, \ldots, a_{i_2}, \overbrace{0, \ldots, 0}^{k_2}, a_{i_2+k_2+1}, a_{i_3}, 0, \ldots\ldots, 0,$$
$$a_{i_{t-1}+k_{t-1}+1}, \ldots, a_{i_t}, \overbrace{0, \ldots, 0}^{k_t}\}$$

where all the a_i's that have been written are not 0. Defining $i_0 + k_0 + 1 = 0$ and:

$$\mathbf{C}(\mathbb{I}) = \sum_{s=1}^{t} \left(\mathbf{P}\left(\{a_{i_{s-1}+k_{s-1}+1}, \ldots, a_{i_s}\}\right) - \mathbf{V}\left(\{a_{i_{s-1}+k_{s-1}+1}, \ldots, a_{i_s}\}\right) \right) + \sum_{s=1}^{t-1} \varepsilon_{i_s}$$

where:

$$\varepsilon_{i_s} = \begin{cases} 0 & \text{if } k_s \text{ is odd,} \\ (-1)^{\frac{k_s}{2}} \text{sign}\left(\frac{a_{i_s+k_s+1}}{a_{i_s}}\right) & \text{if } k_s \text{ is even.} \end{cases}$$

Next the relation between the real zeros of a polynomial $P \in \mathbb{R}[x]$ and the polynomials in the Sturm–Habicht sequence of P is presented. Its proof can be found in (Gonzalez-Vega et al., 1990) or (Gonzalez-Vega et al., 1998).

Definition 2.5. Let $P, Q \in \mathbb{R}[x]$ with $p = \deg(P)$ and $\epsilon \in \{-, 0, +\}$. Then:

$$c_\epsilon(P; Q) = \text{card}\left(\{\alpha \in \mathbb{R}: P(\alpha) = 0, \ \text{sign}(Q(\alpha)) = \epsilon\}\right).$$

With this definition, $c_+(P; 1)$ represents the number of real roots of P and $c_-(P; 1) = 0$.

Theorem 2.1. *If P is a polynomial in $\mathbb{R}[x]$ with $p = \deg(P)$ then*:

$$\mathbf{C}\left(\{\mathbf{stha}_p(P, Q), \ldots, \mathbf{stha}_0(P, Q)\}\right) = c_+(P; Q) - c_-(P; Q).$$

Corollary 2.1. *If P is a polynomial in $\mathbb{R}[x]$ with $p = \deg(P)$ then*:

$$\mathbf{C}\left(\{\mathbf{stha}_p(P), \ldots, \mathbf{stha}_0(P)\}\right) = \#\{\alpha \in \mathbb{R}: P(\alpha) = 0\}.$$

In particular, the number of real roots of P is determined exactly by the signs of the last $p - 1$ determinants $\mathbf{stha}_i(P)$ (the first two ones are $\mathrm{lcof}(P)$ and $p\,\mathrm{lcof}(P)$ with $\mathrm{lcof}(P)$ denoting the leading coefficient of P).

The definition of Sturm–Habicht sequence through determinants allows to perform computations dealing with real roots in a generic way: if P and Q are two polynomials with parametric coefficients whose degrees do not change after specialization then the Sturm–Habicht sequence for P and Q can be computed without specializing the parameters and the result is always good after specialization (modulo the condition over the degrees). This is not true when using Sturm sequences (the computation of the euclidean remainders makes to appear denominators which can vanish after specialization) or negative polynomial remainder sequences (with fixed degree for P the sequence has not always the same number of elements); see (Loos, 1982) and (Gonzalez-Vega et al., 1990) for a more detailed explanation.

For the concrete problem considered here, the using of the results presented in this section allows to characterize when the characteristic polynomial $f(\lambda) = \det(\lambda A + B)$ has a fixed number of real roots. In order to deal with the sign of these real roots, it is needed to use the sign determination scheme (together with Theorem 2.1) which is next presented.

2.2. The sign determination scheme

Let P and Q be polynomials in $\mathbb{R}[x]$. The problem to solve by the so called "sign determination scheme" is the determination of the signs of the evaluation of Q on the real roots of P in a purely formal way without requiring the knowledge of the real roots of P. Denote

$$\mathcal{V}(P, Q) = \mathbf{C}\big(\{\mathbf{stha}_p(P, Q), \ldots, \mathbf{stha}_0(P, Q)\}\big)$$

and according to Theorem 2.1:

$$\mathcal{V}(P, Q) = c_+(P; Q) - c_-(P; Q). \tag{2}$$

Since $\mathcal{V}(P, 1) = c_+(P; 1) - c_-(P; 1) = c_+(P; 1)$ agrees with the number of real roots of P then

$$\mathcal{V}(P, 1) = c_0(P; Q) + c_+(P; Q) + c_-(P; Q) \tag{3}$$

because if α is a real root of P then $Q(\alpha) = 0$ or $Q(\alpha) > 0$ or $Q(\alpha) < 0$. Finally, applying again Theorem 2.1,

$$\mathcal{V}(P, Q^2) = c_+(P; Q^2) - c_-(P; Q^2) = c_+(P; Q^2) - 0 = c_+(P; Q^2) = c_+(P; Q) + c_-(P; Q) \tag{4}$$

because if α is a real root of P such that $Q^2(\alpha) > 0$ then $Q(\alpha) > 0$ or $Q(\alpha) < 0$.

Putting together Eqs. (2), (3) and (4), it is obtained

$$c_0(P; Q) + c_+(P; Q) + c_-(P; Q) = \mathcal{V}(P, 1),$$
$$c_+(P; Q) - c_-(P; Q) = \mathcal{V}(P, Q),$$
$$c_+(P; Q) + c_-(P; Q) = \mathcal{V}(P, Q^2)$$

and the matricial identity

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_0(P; Q) \\ c_+(P; Q) \\ c_-(P; Q) \end{bmatrix} = \begin{bmatrix} \mathcal{V}(P, 1) \\ \mathcal{V}(P, Q) \\ \mathcal{V}(P, Q^2) \end{bmatrix} \tag{5}$$

allowing to compute $c_0(P; Q)$, $c_+(P; Q)$ and $c_-(P; Q)$ once $\mathcal{V}(P, 1)$, $\mathcal{V}(P, Q)$ and $\mathcal{V}(P, Q^2)$ are known. But these integer numbers are directly obtained from the Sturm–Habicht sequences of P and 1, Q and Q^2 by applying the $\mathbf{C}$ function as shown by Theorem 2.1 and Definition 2.4.

When P and Q have no common roots, then $c_0(P; Q) = 0$ and the matricial identity in (5) reduces to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} c_+(P; Q) \\ c_-(P; Q) \end{bmatrix} = \begin{bmatrix} \mathcal{V}(P, 1) \\ \mathcal{V}(P, Q) \end{bmatrix}. \tag{6}$$

More information about the sign determination scheme including historical remarks and the generalization to more than one polynomial can be found in (Basu et al., 2003).

2.3. The study of the signs of the real roots of the characteristic polynomial

The shown techniques presented in Sections 2.1 and 2.2 are going to be applied here to give a complete characterization of the sign behaviour for the real roots of the polynomial

$$P = x^3 + ax^2 + bx + c$$

in terms of the coefficients a, b and c.

First, the Sturm–Habicht sequence associated to P is determined:

$$\mathbf{StHa}_3(P) = x^3 + ax^2 + bx + c,$$
$$\mathbf{StHa}_2(P) = 3x^2 + 2ax + b,$$
$$\mathbf{StHa}_1(P) = 2(-3b + a^2)x - 9c + ab,$$
$$\mathbf{StHa}_0(P) = -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3,$$

together with the principal Sturm–Habicht coefficients associated to P:

$$\mathbf{stha}_3(P) = 1,$$
$$\mathbf{stha}_2(P) = 3,$$
$$\mathbf{stha}_1(P) = 2(-3b + a^2),$$
$$\mathbf{stha}_0(P) = -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3.$$

Next, in order to study the sign of the real roots of the polynomial $P = x^3 + ax^2 + bx + c$, the polynomials P and $Q = x$ are considered. Thus, the Sturm–Habicht sequence associated to P and Q is computed:

$$\mathbf{StHa}_4(P, Q) = x^3 + ax^2 + bx + c,$$
$$\mathbf{StHa}_3(P, Q) = 3x^3 + 2ax^2 + bx,$$
$$\mathbf{StHa}_2(P, Q) = -3ax^2 - 6bx - 9c,$$
$$\mathbf{StHa}_1(P, Q) = 3(3ac + ba^2 - 4b^2)x + 6ca^2 - 18cb,$$
$$\mathbf{StHa}_0(P, Q) = 12c^2a^3 - 3b^2ca^2 - 54c^2ba + 81c^3 + 12cb^3$$

together with the principal Sturm–Habicht coefficients associated to P and Q:

$$\mathbf{stha}_4(P, Q) = 0,$$
$$\mathbf{stha}_3(P, Q) = 3,$$
$$\mathbf{stha}_2(P, Q) = -3a,$$
$$\mathbf{stha}_1(P, Q) = 3(3ac + ba^2 - 4b^2),$$
$$\mathbf{stha}_0(P, Q) = 3c(4b^3 - b^2a^2 - 18bac + 27c^2 + 4ca^3).$$

Since the integer numbers $\mathcal{V}(P, 1)$ and $\mathcal{V}(P, Q)$ depend, only and respectively, on the signs of

1. $\{\mathbf{stha}_3(P, 1), \mathbf{stha}_2(P, 1), \mathbf{stha}_1(P, 1), \mathbf{stha}_0(P, 1)\}$, and
2. $\{\mathbf{stha}_3(P, Q), \mathbf{stha}_2(P, Q), \mathbf{stha}_1(P, Q), \mathbf{stha}_0(P, Q)\}$,

therefore, it is enough to study the five polynomials:

$$a, \quad c, \quad -3b + a^2, \quad 3ac + ba^2 - 4b^2, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3$$

in order to get the values of $\mathcal{V}(P, 1)$ and $\mathcal{V}(P, Q)$.

In the concrete case considered here, the polynomial P represents the characteristic polynomial of the pencil $\lambda A + B$ once it has transformed into a monic polynomial:

$$P(\lambda) = -\frac{f(\lambda)}{k}$$

Table 2
The different possibilities of sign conditions for the polynomials $a, c > 0$, $-3b + a^2$, $3ac + ba^2 - 4b^2$ and $-27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3$

{1}	$[1, 1, 1, 1, 1]$	{2}	$[1, 1, 1, 1, 0]$	{3}	$[1, 1, 1, 1, -1]$
{4}	$[1, 1, 1, 0, 1]$	{5}	$[1, 1, 1, 0, 0]$	{6}	$[1, 1, 1, 0, -1]$
{7}	$[1, 1, 1, -1, 1]$	{8}	$[1, 1, 1, -1, 0]$	{9}	$[1, 1, 1, -1, -1]$
{10}	$[1, 1, 0, 1, 1]$	{11}	$[1, 1, 0, 1, 0]$	{12}	$[1, 1, 0, 1, -1]$
{13}	$[1, 1, 0, 0, 1]$	{14}	$[1, 1, 0, 0, 0]$	{15}	$[1, 1, 0, 0, -1]$
{16}	$[1, 1, 0, -1, 1]$	{17}	$[1, 1, 0, -1, 0]$	{18}	$[1, 1, 0, -1, -1]$
{19}	$[1, 1, -1, 1, 1]$	{20}	$[1, 1, -1, 1, 0]$	{21}	$[1, 1, -1, 1, -1]$
{22}	$[1, 1, -1, 0, 1]$	{23}	$[1, 1, -1, 0, 0]$	{24}	$[1, 1, -1, 0, -1]$
{25}	$[1, 1, -1, -1, 1]$	{26}	$[1, 1, -1, -1, 0]$	{27}	$[1, 1, -1, -1, -1]$
{28}	$[0, 1, 1, 1, 1]$	{29}	$[0, 1, 1, 1, 0]$	{30}	$[0, 1, 1, 1, -1]$
{31}	$[0, 1, 1, 0, 1]$	{32}	$[0, 1, 1, 0, 0]$	{33}	$[0, 1, 1, 0, -1]$
{34}	$[0, 1, 1, -1, 1]$	{35}	$[0, 1, 1, -1, 0]$	{36}	$[0, 1, 1, -1, -1]$
{37}	$[0, 1, 0, 1, 1]$	{38}	$[0, 1, 0, 1, 0]$	{39}	$[0, 1, 0, 1, -1]$
{40}	$[0, 1, 0, 0, 1]$	{41}	$[0, 1, 0, 0, 0]$	{42}	$[0, 1, 0, 0, -1]$
{43}	$[0, 1, 0, -1, 1]$	{44}	$[0, 1, 0, -1, 0]$	{45}	$[0, 1, 0, -1, -1]$
{46}	$[0, 1, -1, 1, 1]$	{47}	$[0, 1, -1, 1, 0]$	{48}	$[0, 1, -1, 1, -1]$
{49}	$[0, 1, -1, 0, 1]$	{50}	$[0, 1, -1, 0, 0]$	{51}	$[0, 1, -1, 0, -1]$
{52}	$[0, 1, -1, -1, 1]$	{53}	$[0, 1, -1, -1, 0]$	{54}	$[0, 1, -1, -1, -1]$
{55}	$[-1, 1, 1, 1, 1]$	{56}	$[-1, 1, 1, 1, 0]$	{57}	$[-1, 1, 1, 1, -1]$
{58}	$[-1, 1, 1, 0, 1]$	{59}	$[-1, 1, 1, 0, 0]$	{60}	$[-1, 1, 1, 0, -1]$
{61}	$[-1, 1, 1, -1, 1]$	{62}	$[-1, 1, 1, -1, 0]$	{63}	$[-1, 1, 1, -1, -1]$
{64}	$[-1, 1, 0, 1, 1]$	{65}	$[-1, 1, 0, 1, 0]$	{66}	$[-1, 1, 0, 1, -1]$
{67}	$[-1, 1, 0, 0, 1]$	{68}	$[-1, 1, 0, 0, 0]$	{69}	$[-1, 1, 0, 0, -1]$
{70}	$[-1, 1, 0, -1, 1]$	{71}	$[-1, 1, 0, -1, 0]$	{72}	$[-1, 1, 0, -1, -1]$
{73}	$[-1, 1, -1, 1, 1]$	{74}	$[-1, 1, -1, 1, 0]$	{75}	$[-1, 1, -1, 1, -1]$
{76}	$[-1, 1, -1, 0, 1]$	{77}	$[-1, 1, -1, 0, 0]$	{78}	$[-1, 1, -1, 0, -1]$
{79}	$[-1, 1, -1, -1, 1]$	{80}	$[-1, 1, -1, -1, 0]$	{81}	$[-1, 1, -1, -1, -1]$

with $k > 0$. According to Theorem 1.1, $f(0) \neq 0$, thus $P(0) \neq 0$ and $P(x)$ and x have no common roots and it is not needed to compute $\mathcal{V}(P, Q^2)$. Therefore the characterization of the sign behaviour of the real roots of P will require the reduced matrix identity in the sign determination scheme as shown in Eq. (6).

Moreover, since $P(0) \neq 0$ then $c \neq 0$. Item (1) in Theorem 1.1 assures that the product of the three roots of $f(\lambda)$ is negative, and the same thing applies to the roots of $P(x)$: since these product agrees with $(-1)^3 c$, it is concluded that in this situation $c > 0$.

Next, the 81 possibilities of sign conditions for the polynomials $a, c > 0$, $-3b + a^2$, $3ac + ba^2 - 4b^2$ and $-27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3$ are generated (see Table 2).

For each one of these 81 possibilities, the sign of the polynomials $\mathbf{stha}_i(P, 1)$ and $\mathbf{stha}_j(P, Q)$ are reconstructed and the application of the function $\mathbf{C}$ (see Definition 2.4) together with the reduced matricial identity in the sign determination scheme (see equation (6)) produce, for each of the 81 possible cases, a list $[a, b]$, where a indicates the number of different real roots of P and b how many of them are positive. This combinatorial way of proceeding produces several impossible cases which are directly discarded: the cases {5} and {10} in Table 2 producing respectively the pairs $[2, 1/2]$ and $[1, -1]$ are removed since a and b, in any produced list $[a, b]$, must be non negative integers.

Moreover, the fact, in the concrete considered case, that P has always one negative real root allows to remove the cases producing the lists $[1, 1]$ and $[2, 2]$. And the fact that the product of the three roots of P is negative implies the removal of the cases producing the list $[3, 1]$.

This process, completely automatized by using the Computer Algebra System `Maple`, produces the following results which completely characterize the sign behaviour of the real roots of the polynomial $x^3 + ax^2 + bx + c$ (assuming that it is the characteristic polynomial of a pencil $\lambda A + B$):

Condition 1: The different possibilities for P having only one real root.
 – $[1, 0]$: One negative real root (one triple real negative root or one negative real root and two conjugated complex).
 The following numbers represent the positions of the signs conditions in Table 2:

{3}, {6}, {9}, {12}, {13}, {14}, {15}, {18}, {21}, {24}, {27}, {33}, {36}, {37}, {42}, {45}, {51}, {54}, {63}, {72}, {81}.

Condition 2: The different possibilities for P having two different real roots.

  – [2, 0]: Two negative real roots (one of them double).
  The following numbers represent the positions of the signs conditions in Table 2:
  {2}.

  – [2, 1]: One negative real root and one positive real root (one of them double).
  The following numbers represent the positions of the signs conditions in Table 2:
  {8}, {29}, {32}, {35}, {62}.

Condition 3: The different possibilities for P having three different real roots.

  – [3, 0]: Three negative real roots.
  The following numbers represent the positions of the signs conditions in Table 2:
  {1}.

  – [3, 2]: One negative real root and two positive real roots.
  The following numbers represent the positions of the signs conditions in Table 2:
  {7}, {31}, {34}, {55}, {58}, {61}.

The obtained results imply directly the following equivalences (assuming $c > 0$, which is what happens in the considered case) after further simplifications by either collapsing cases (by using, for example, $< \cup = \rightarrow \leqslant$) or by direct algebraic manipulations (mostly when $a = 0$):

- The polynomial $x^3 + ax^2 + bx + c$ has three different real roots, all of them negative, if and only if

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0.$$

- The polynomial $x^3 + ax^2 + bx + c$ has two different real roots, all of them negative, if and only if

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 = 0.$$

- The polynomial $x^3 + ax^2 + bx + c$ has two different real roots, one of them positive and the other one negative, if and only if

$$-3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 = 0.$$

- The polynomial $x^3 + ax^2 + bx + c$ has three different real roots, one of them negative and the other two positive, if and only if

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a = 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a < 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a < 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 = 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a < 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0.$$

  In this case, the last three conditions can be replaced by the single one:

$$a < 0, \quad -3b + a^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0$$

  and the first two by the single one:

$$a \geqslant 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0.$$

- The polynomial $x^3 + ax^2 + bx + c$ has one negative real root if and only if:

$$a > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 < 0 \quad \text{or}$$
$$a = 0, \quad 3ac + ba^2 - 4b^2 \leqslant 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 < 0 \quad \text{or}$$
$$a < 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 < 0 \quad \text{or}$$
$$a > 0, \quad -3b + a^2 = 0, \quad 3ac + ba^2 - 4b^2 = 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 = 0.$$

The last condition is the only one corresponding to the case where P has a triple real negative root and can be further simplified to

$$a > 0, \quad -3b + a^2 = 0, \quad 27c - a^3 = 0$$

since if $-3b + a^2 = 0$ then

$$3ac + ba^2 - 4b^2 = \frac{a(27c - a^3)}{9}, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 = -\frac{(-27c + a^3)^2}{27}.$$

Example 2.1. If

$$f(\lambda) = -\lambda^3 - \frac{7}{4}\lambda^2 - \frac{7}{8}\lambda - \frac{1}{8}$$

is the characteristic polynomial of two ellipses $\mathcal{A}$ and $\mathcal{B}$ then (once turned monic):

$$a = \frac{7}{4} > 0, \quad a^2 - 3b = \frac{7}{16} > 0, \quad ba^2 + 3ac - 4b^2 = \frac{35}{128} > 0,$$

$$b^2a^2 - 4b^3 + 18bac - 27c^2 - 4ca^3 = \frac{9}{1024} > 0.$$

This case corresponds to position {1} in Table 2 which implies that $f(\lambda)$ has three different real roots which are all of them negative. As it will be shown later (see Table 5) this corresponds to relative positions **1** (four different intersection points) or **4** (one ellipse contained in the second one) in Table 1.

This last example shows that the sign of the real roots of the characteristic polynomial of the pencil generated by the two ellipses is not enough, in general, to determine their relative position.

Example 2.2. If

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0$$

then $f(\lambda)$ has three real roots, two of them positive. Thus, when the matrices defining $\mathcal{A}$ and $\mathcal{B}$, make the coefficients of $f(\lambda)$ to verify these sign conditions then, according to Theorem 1.1, both ellipses are separated (relative position **3**).

More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are two ellipses, $f(\lambda)$ is the characteristic polynomial of the pencil they define and a, b and c are the coefficients of the polynomial obtained once $f(\lambda)$ is turned monic then the following equivalence holds:
$\mathcal{A}$ and $\mathcal{B}$ are separated if and only if

$$a \geqslant 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 < 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0 \quad \text{or}$$
$$a < 0, \quad -3b + a^2 > 0, \quad \qquad\qquad\qquad\qquad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0.$$

Fig. 1 shows the zero-sets of these five functions (note that in the considered problem $c > 0$): it is worth to remark that according to Theorem 1.1, some of these regions corresponds uniquely to a concrete relative position of the two considered ellipses.

Remark 2.1. There are two cases when it is possible to compute the roots of $P(x) = x^3 + ax^2 + bx + c$ in an exact and rational way. The first one when $P(x)$ has a triple negative root α: then

$$\alpha = -\frac{a}{3}.$$

The second one when $P(x)$ has two different roots which must be real and one of them double, β, and one simple, α. In this case the greatest common divisor of P and P' is of degree 1, has β as the unique root and agrees with

$$\mathbf{StHa}_1(P) = 2(-3b + a^2)x - 9c + ab$$

(see the equivalence in Eq. (1) which can be used also to prove that, under this hypothesis on P, $a^2 - 3b \neq 0$). Thus:
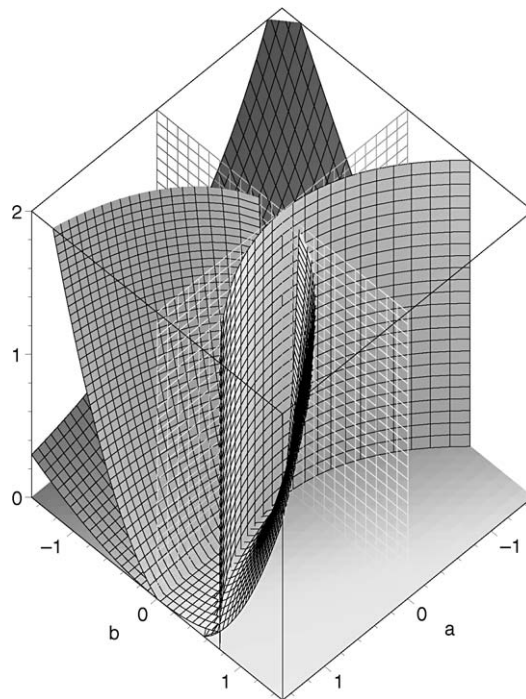
$$\beta = \frac{9c - ab}{2(a^2 - 3b)}$$

Fig. 1. Graphs of $a = 0$, $c = 0$, $-3b + a^2 = 0$, $3ac + ba^2 - 4b^2 = 0$ and $-27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 = 0$.

and

$$\alpha = \frac{4ab - a^3 - 9c}{a^2 - 3b}$$

are the two real roots of $x^3 + ax^2 + bx + c$.

3. Pencils of conics in $\mathbb{P}_2(\mathbb{C})$ and $\mathbb{P}_2(\mathbb{R})$

The problem considered here has been formulated in the real affine plane, $\mathbb{R}^2$. Nevertheless, the most complete model is given by the complex projective plane, since this one contains all the objects that can appear when dealing with conics in $\mathbb{R}^2$: the imaginary ones and those at the infinity. This implies that, for example, the classification of conics in the complex projective plane is simpler than in the real affine plane: there exists only a non-degenerate non-empty conic in $\mathbb{P}_2(\mathbb{C})$ and three, the ellipse, the parabola and the hyperbola, in $\mathbb{R}^2$.

Given two conics $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{P}_2(\mathbb{C})$, the pencil of conics that they define is $\{\lambda A + \mu B\}$ where μ and λ do not vanish simultaneously, being A and B the matrices of the conics $\mathcal{A}$ and $\mathcal{B}$.

The characteristic polynomial associated with two ellipses $\mathcal{A}$ and $\mathcal{B}$ is defined as $f(\lambda) = \det(\lambda A + B)$; therefore $\lambda A + B$ can be regarded as one-parameter family of conics and it can be interpreted as the pencil defined by the ellipses. Furthermore, the roots of the characteristic polynomial $f(\lambda)$ provide the degenerate conics in the pencil.

3.1. Types of pencils of conics in $\mathbb{P}_2(\mathbb{C})$

The different types of pencils of conics can be classified according to the invariant factors, (see e.g. (Abellanas, 1969, pp. 444–448) and (Levy, 1967, pp. 251–267)). There exists six types of pencils of conics in $\mathbb{P}_2(\mathbb{C})$, depending on the intersection of any two conics of the pencil:

- Type I: 4 simple points,
- Type II: 1 double point and 2 simple points,
- Type III: 2 double points,

Table 3
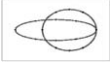Types of pencils of conics in $\mathbb{P}_2(\mathbb{C})$

Type	Intersection	Geometric interpretation
I	4 simple points	
II	1 double point and 2 simple points	
III	2 double points	
IV	1 triple point and 1 simple point	
V	1 quadruple point	
VI	1 conic	

Table 4
Types of pencils of conics in $\mathbb{P}_2(\mathbb{R})$

Type	Degenerate conics	Real roots of $f(\lambda)$	Real base points
I	3 pairs of real lines	α, β, γ	4
Ia	1 pair of real lines and 2 pairs of imaginary lines	α, β, γ	0
Ib	1 pair of real lines	α	2
II	2 pairs of real lines	α, β, β	3
IIa	1 pair of real lines and 1 pair of imaginary lines	α, β, β	1
III	1 pair of real lines and 1 double real line	α, β, β	2
IIIa	1 pair of imaginary lines and 1 double real line	α, β, β	0
IV	1 double real line	α, α, α	2
V	1 double real line	α, α, α	1
VI	no degenerate conics	α, α, α	all of them

- Type IV: 1 triple point and 1 simple point,
- Type V: 1 quadruple point, and
- Type VI: 1 conic, i.e., the pencil generated by two identical conics.

This classification is clearly motivated by Bezout Theorem (see (Fulton, 1969)). The points of intersection of any two conics in the same pencil are called the *base points* of the pencil.

Table 3 summarizes the classification of pencils of conics in $\mathbb{P}_2(\mathbb{C})$. Usually the Type VI is not considered but, for completeness, it has been included.

3.2. Types of pencils of conics in $\mathbb{P}_2(\mathbb{R})$

The classification of the types of pencils of conics in $\mathbb{P}_2(\mathbb{R})$ (see e.g. (Levy, 1967, pp. 251–267)) is a subclassification of the complex classification, according the base points are real or not. The real classification can be also determined according to the degenerate conics and the real base points of the pencil (see Table 4 where the available information about the real roots of the characteristic polynomial has been also included since it can be directly deduced from the nature of the canonical equations for each pencil type).

A very illustrative set pictures corresponding to the different kind of pencils in $\mathbb{P}_2(\mathbb{R})$ (including canonical equations for each pencil type) can be found in http://www.ipfw.edu/math/Coffman/.

3.2.1. Description of the pencils of conics in $\mathbb{P}_2(\mathbb{C})$ of Type I when considered in $\mathbb{P}_2(\mathbb{R})$

The pencils of conics of Type I in $\mathbb{P}_2(\mathbb{C})$ can be of three different types depending on the nature of the base points (or, equivalently, to the nature of the degenerate conics in the considered pencil), when the considered conics are defined over $\mathbb{R}$ and they are regarded in $\mathbb{P}_2(\mathbb{R})$. Thus, three possibilities arise in Table 4 from Type I (in $\mathbb{P}_2(\mathbb{C})$) according to:

1. the four intersection points are all of them real (Type I in Table 4), or
2. two of them are real and two of them are complex non real (Type Ib in Table 4) or
3. all of them are complex non real (Type Ia in Table 4).

Note that according to Theorem 1.1, for Type I in Table 4, the characteristic polynomial of the pencil has three real roots, all of them negative. For Type Ib in Table 4, the characteristic polynomial of the pencil has exactly one real root which is negative and two complex and non real roots (and conjugate) and for Type Ia in Table 4, the characteristic polynomial of the pencil has three real roots: all of them negative or one negative and the other ones positive.

The existence of empty and non-degenerate conics in the pencil will be used to separate in practice pencils from Types I and Ia (in $\mathbb{P}_2(\mathbb{R})$). All the conics of a pencil of Type I (in $\mathbb{P}_2(\mathbb{R})$) are non-empty, because the four base points, which are in $\mathbb{R}^2$, belong to any conic in the pencil. Nevertheless, in the case of a pencil of Type Ia (in $\mathbb{P}_2(\mathbb{R})$) the situation is different since there are no real base points. A pencil of Type Ia has three degenerate conics corresponding to the three different real roots of the characteristic polynomial: $\alpha < \beta < \gamma$. First, it is easy to check that conics in the pencil corresponding to values of λ in $(-\infty, \alpha) \cup (\gamma, \infty)$ are non-empty. Next lemma will show that empty and non-degenerate conics in the pencil come from values of λ belonging to (α, β) or (β, γ).

Lemma 3.1. *Empty and non-degenerate conics in a pencil $\lambda A + B$ of Type Ia (in $\mathbb{P}_2(\mathbb{R})$) come from values of λ belonging to (α, β) or (β, γ).*

Proof. To prove this statement, we shall apply the following results from projective geometry (see (Levy, 1967), for example):

1. There exists a bijection between the conics of projective plane $\mathbb{P}_2(\mathbb{K})$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and the points of projective space $\mathbb{P}_5(\mathbb{K})$. In this bijection each pencil of conics (resp., two generating conics) corresponds to one projective line (resp., two points defining the projective line), i.e., we have the following 1–1 correspondence:
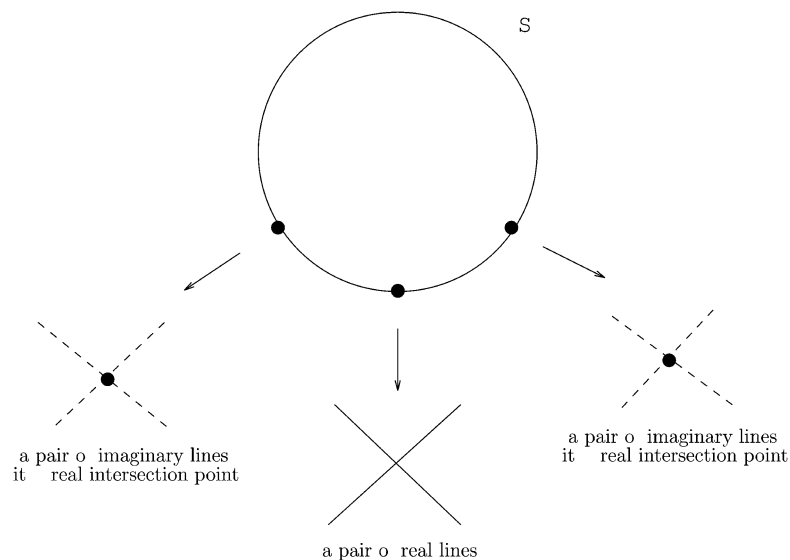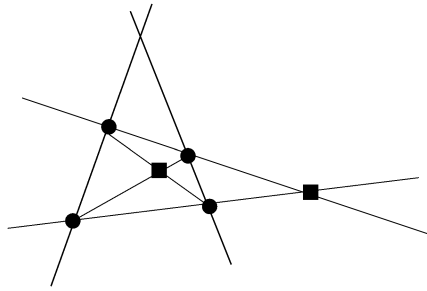

Fig. 2. Root distribution on S^1.

Fig. 3. Harmonic conjugated.

$$\mathbb{P}_2(\mathbb{K}) \longleftrightarrow \mathbb{P}_5(\mathbb{K})$$

{pencil of conics} $\longleftrightarrow$ {projective line}.

2. If the pair of conics intersects in four different points then the conics of the pencil are just those passing through these points. Therefore, five different points determine exactly one conic.
3. If $\mathbb{K} = \mathbb{R}$ then we can parameterize the projective line $\mathbb{P}_1(\mathbb{R})$ as S^1. We recall that for Type Ia the characteristic polynomial $f(\lambda)$ has three different real roots, $\alpha < \beta < \gamma$, which have associated three degenerate conics (see Table 4). This is explained graphically in Fig. 2.
4. Only one conic of the pencil passes through one point of $\mathbb{P}_2(\mathbb{R})$.
5. Two conics of the pencil have always empty intersection in $\mathbb{P}_2(\mathbb{R})$ (because of the previous item).
6. The degenerate conic (a pair of real lines) corresponding to the root β splits $\mathbb{P}_2(\mathbb{R})$ into two connected components.
7. The real points that correspond to the degenerate conics coming from the roots α and γ are in different connected components because they are harmonic conjugated with respect to the degenerate conic coming from β (see Fig. 3). The complex pencil generated by the two conics has four complex non real base points, and the three pairs of lines defined by them are the degenerate conics of the pencil. This configuration of lines is given in Fig. 3, where circles denote the complex base points, whereas squares denote the real points corresponding to the degenerate conics coming from α and γ, which are harmonic conjugate respect to the pair of lines corresponding to the degenerate conic coming from β.
8. Let be p a point in the interior of the ellipse. The conic of the pencil that passes through this point is totally contained in it.
9. The disposal of the pencil is the following: the conics of each connected component are all contained like in a *funnel*, since the singular points are contained in all the conics of their connected component (see Fig. 4).

With all previous items, one we can view the conics of the pencil moving with λ in S^1: if λ in S^1 then one can draw a conic $c(\lambda)$ in $\mathbb{P}_2(\mathbb{R})$. Let us start in α moving counter clockwise towards β on S^1:

- If $\lambda = \alpha$ then the conic $c(\alpha)$ is a point.
- If λ is between α and β then $c(\lambda)$ is a non-degenerate conic in the left-right component in Fig. 4.
- If $\lambda = \beta$ then the conic $c(\beta)$ is a pair of lines.
- If λ is between β and γ then $c(\lambda)$ is a non-degenerate conic in the up-down component in Fig. 4.
- If $\lambda = \gamma$ then the conic $c(\gamma)$ is a point.
- If λ is between γ and α then the conic $c(\lambda)$ is empty.

Other way to prove that in Type Ia there are empty conics would be studying the canonical equation of the pencil (see (Abellanas, 1969)). $\quad\square$

This information will be very useful in order to separate, in practice, pencils of conics in $\mathbb{P}_2(\mathbb{R})$ from Types I and Ia.
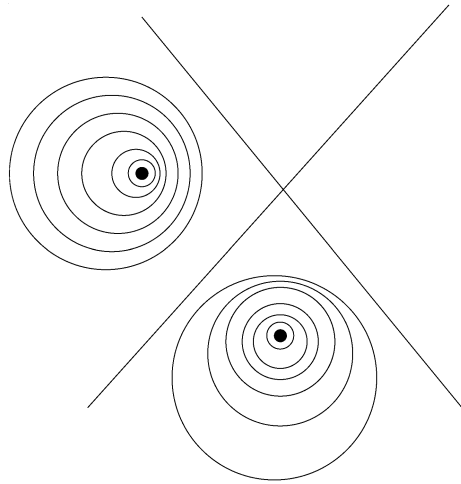
Fig. 4. Disposal of the pencil.

3.2.2. How relative positions in $\mathbb{R}^2$ are linked with the types of pencils in $\mathbb{P}_2(\mathbb{R})$?

Two pairs of ellipses can have the same relative position (according to the classification in Table 1) and generate different pencils in $\mathbb{P}_2(\mathbb{R})$ (according to the classification in Table 4). Next, for each one of the relative positions appearing in Table 1, it is shown which are the possible types of pencils of conics in $\mathbb{P}_2(\mathbb{R})$ (as described in Table 4) corresponding to the considered relative position:

- Relative position **1**, two transversal ellipses intersecting in 4 different points, generates a pencil of Type I.
- Relative position **2**, two transversal ellipses intersecting in 2 points, generates a pencil of Type Ib.
- Relative position **3**, two separated ellipses, generates a pencil of Type Ia.
- Relative position **4**, two contained ellipses, can generate one pencil of Type Ia or one of Type IIIa.
- Relative position **5**, two transversal ellipses in 2 points and tangent in another point, generates a pencil of Type II.
- Relative position **6**, two externally tangent ellipses, generates a pencil of Type IIa.
- Relative position **7**, two internally tangent ellipses in one point, can generate a pencil of Type IIa or one of Type V.
- Relative position **8**, two internally tangent ellipses in 2 points, generates a pencil of Type II.
- Relative position **9**, two transversal ellipses in one point and tangent in another one, generates a pencil of Type IV.
- Relative position **10**, two coincident ellipses, generates a pencil of Type VI.

Also, in the same pencil of conics there can be two different positions of a pair of ellipses, for example:
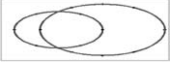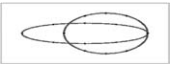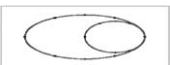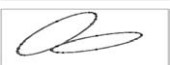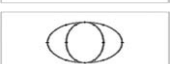
- In a pencil of Type Ia there can be two separated ellipses (the relative position **3**) or two contained ellipses (the relative position **4**).
- In a pencil of Type IIa there can be two externally tangent ellipses (the relative position **6**) or two internally tangent ellipses (the relative position **7**).

4. On the characterization of the relative position of two ellipses

The results shown in Sections 1 and 3 are enough to characterize, from a computational point of view, the ten different relative positions between two ellipse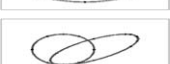s in $\mathbb{R}^2$. Table 5 summarizes the situation by showing that using the information about the positivity of the real roots of the characteristic polynomial of the pencil generated by the two considered ellipses together with the degenerate conics of the considered pencil in $\mathbb{P}_2(\mathbb{R})$, characterizes in an unique way the ten relative positions presented in Table 1.

Table 5 determines uniquely the relative position of two ellipses by, first, analyzing the positivity of the real roots of the characteristic polynomial of the pencil generated by the two ellipses and, second, studying the nature of the

Table 5
Determination of the relative position of two ellipses

Type of pencils ($\mathbb{P}_2(\mathbb{R})$)	Relative position		Degenerate conics	Roots of $f(\lambda)$
I (no empty conics)	**1**		3 pairs of real lines	$\alpha < \beta < \gamma < 0$
Ia (with empty conics)	**3**		1 pair of real lines and 2 pairs of imaginary lines	$\alpha < 0 < \beta < \gamma$
	4		1 pair of real lines and 2 pairs of imaginary lines	$\alpha < \beta < \gamma < 0$
Ib	**2**		1 pair of real lines	$\alpha < 0; \beta, \overline{\beta} \in \mathbb{C}$
II	**5**		2 pairs of real lines	$\alpha < 0; \beta, \beta < 0$
IIa	**7**		1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0; \beta, \beta, < 0$
	6		1 pair of real lines and 1 pair of imaginary lines	$\alpha < 0 < \beta, \beta$
III	**8**		1 pair of real parallel lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IIIa	**4**		1 pair of imaginary lines and 1 double real line	$\alpha < 0; \beta, \beta < 0$
IV	**9**		1 pair of real lines	$\alpha, \alpha, \alpha < 0$
V	**7**		1 double real line	$\alpha, \alpha, \alpha < 0$
VI	**10**		no degenerate conics	$\alpha, \alpha, \alpha < 0$

degenerate conics in the considered pencil. Given two ellipses $\mathcal{A}$ and $\mathcal{B}$, the steps to follow in order to determine their relative position are the following:

(1) Compute the characteristic polynomial $f(\lambda)$ of the pencil $A\lambda + B$.
(2) Characterize the number of real roots of $f(\lambda)$, their multiplicity and their positivity.

This information is already enough to characterize completely the following cases:

- Two separated ellipses (relative position **3** in Table 1):
  – 1 real negative root and 2 real positive roots.
- Two ellipses touching each other externally (relative position **6** in Table 1):
  – 1 real negative root and 1 real double positive root.
- Two ellipses transversal in 2 points (relative position **2** in Table 1):
  – 1 real negative root and 2 complex and non real roots.

If not in any of the three previous cases already characterized (**2**, **3** and **6**), more computations are involved in order characterize the remaining cases. Three are the possibilities according the nature of the roots of the characteristic polynomial $f(\lambda)$ according it has a triple real root, a negative double real root and three negative real roots. Let $\lambda^3 + a\lambda^2 + b\lambda + c$ be the obtained polynomial once $f(\lambda)$ is turned monic.

$f(\lambda)$ *has* 1 *triple real negative root.*

According to Remark 2.1 the real root of $f(\lambda)$ can be computed explicitly in a rational way by the formula:

$$\alpha = -\frac{a}{3}.$$

Relative positions **10**, **9** and **7** are separated according to the nature of the degenerate conics in the pencil

$$\alpha A + B = -\frac{a}{3}A + B$$

in the following way:

- Relative position **10**: if there are no degenerate conics.
- Relative position **9**: if they are a pair of real lines.
- Relative position **7**: if the only degenerate conic in the pencil is a double real line.

$f(\lambda)$ *has* 2 *different real negative roots* (1 *double*).

According to Remark 2.1 the double real root of $f(\lambda)$, β, and the single one, α, can be computed explicitly in a rational way by the formulae:

$$\beta = \frac{9c - ab}{2(a^2 - 3b)}, \qquad \alpha = \frac{4ab - a^3 - 9c}{a^2 - 3b}.$$

Relative positions **4**, **5**, **7** and **8** are separated according to the nature of the degenerate conics in the pencil

$$\beta A + B = \frac{9c - ab}{2(a^2 - 3b)}A + B, \qquad \alpha A + B = \frac{4ab - a^3 - 9c}{a^2 - 3b}A + B \tag{7}$$

in the following way:

- Relative position **4**: a pair of imaginary lines and a double real line.
- Relative position **5**: two pairs of real lines.
- Relative position **7**: a pair of real lines and a pair of imaginary lines.
- Relative position **8**: a pair of real parallel lines and a double real line.

$f(\lambda)$ *has* 3 *different negative real roots.*

Let $\alpha < \beta < \gamma < 0$ be the real roots of $f(\lambda)$. In this situation, there is no rational formulae providing the roots of $f(\lambda)$ in terms of its coefficients as in the previous two possibilities. But, in this case, $f'(\lambda)$ has exactly two real roots u and v:

$$u = \frac{-a - \sqrt{a^2 - 3b}}{3} < v = \frac{-a + \sqrt{a^2 - 3b}}{3}$$

and $\alpha < u < \beta < v < \gamma < 0$. Note that the characterization of the case $\alpha < \beta < \gamma < 0$ contains the conditions $a^2 - 3b > 0$ and $a > 0$ (see Section 2.3).

The separation between relative positions **1** and **4**, under the assumption that the characteristic polynomial has 3 different negative real roots, is made according the existence, or not, of empty conics in the pencil generated by the two considered ellipses (see Section 3.2.1):

- Relative position **1**: if $uA + B$ and $vA + B$ do not represent empty conics.
- Relative position **4**: if $uA + B$ or $vA + B$ represent empty conics.

Thus, relative position **4**, when the characteristic polynomial has 3 different negative real roots, is characterized by

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0,$$
$$m_{22} \det(M) > 0, \quad \det(M_{11}) > 0$$

or

$$a > 0, \quad -3b + a^2 > 0, \quad 3ac + ba^2 - 4b^2 > 0, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3 > 0,$$

$$n_{22} \det(N) > 0, \quad \det(N_{11}) > 0$$

where $M = uA + B = (m_{ij})$, $N = vA + B = (n_{ij})$ and M_{11} and N_{11} are, respectively, the minors of M and N relative to a_{11} and b_{11}.

A slightly more complicated expression, but without involving square roots, can be obtained for characterizing relative positions **1** and **4** by using the same techniques presented in Section 2.3. But, in this case, replacing the characteristic polynomial by its derivative and instead of checking the sign of its roots, analyzing the sign behaviour of the polynomials $m_{22} \det(M)$, $\det(M_{11})$, $n_{22} \det(N)$ and $\det(N_{11})$ (which are polynomials in u or v).

4.1. Formulae summary

Until here the complete characterization (in terms of equalities and inequalities) for the ten possible cases for the relative positions of two ellipses in $\mathbb{R}^2$ has been achieved. These equalities and inequalities come from three different sources:

1. the characterization of the sign pattern for the real roots of the characteristic polynomial,
2. the characterization of the kind of degenerate conics in the pencil provided by the analysis performed when the characteristic polynomial has multiple roots, and
3. the characterization of when a pencil contains empty conics under the hypothesis of a characteristic polynomial with three different negative real roots.

It is worth to remark here that the nature of a conic $\mathcal{A}$ with associated matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

is uniquely determined by the sign of $\det(A)$, the signs of two minors,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{12} & a_{23} \\ a_{13} & a_{33} \end{vmatrix},$$

and the sign of a_{11}.

Next it is summarized how all this information can be used to characterize each of the ten relative positions (presented in Table 1) between two ellipses $\mathcal{A}$ and $\mathcal{B}$, with associated matrices A and B, whose characteristic polynomial $f(\lambda)$ once turned monic is $\lambda^3 + a\lambda^2 + b\lambda + c$:

- *Relative position **1***
  a, b and c satisfy the conditions in order $f(\lambda)$ has three different negative real roots and the conditions in order $uA + B$ and $vA + B$ do not represent empty conics where

$$u = \frac{-a - \sqrt{a^2 - 3b}}{3} < v = \frac{-a + \sqrt{a^2 - 3b}}{3}.$$

- *Relative position **2***
  a, b and c satisfy the conditions in order $f(\lambda)$ has only one real root which is negative.
- *Relative position **3***
  a, b and c satisfy the conditions in order $f(\lambda)$ has three different real roots, one negative and two positive.
- *Relative position **4***
  a, b and c satisfy the conditions in order $f(\lambda)$ has three different negative real roots and the conditions in order $uA + B$ or $vA + B$ represent empty conics where

$$u = \frac{-a - \sqrt{a^2 - 3b}}{3} < v = \frac{-a + \sqrt{a^2 - 3b}}{3}$$

or the conditions in order $f(\lambda)$ has two different real negative roots (one double) and the conditions in order

$$\frac{9c - ab}{2(a^2 - 3b)}A + B \quad \text{and} \quad \frac{4ab - a^3 - 9c}{a^2 - 3b}A + B$$

represent a double real line and a pair of imaginary lines.

- *Relative position* **5**
  a, b and c satisfy the conditions in order $f(\lambda)$ has two different real negative roots (one double) and the conditions in order

  $$\frac{9c - ab}{2(a^2 - 3b)}A + B \quad \text{and} \quad \frac{4ab - a^3 - 9c}{a^2 - 3b}A + B$$

  represent two pair of real lines.

- *Relative position* **6**
  a, b and c satisfy the conditions in order $f(\lambda)$ has two different real roots, one negative and one positive which is double.

- *Relative position* **7**
  a, b and c satisfy the conditions in order $f(\lambda)$ has two different real negative roots (one double) and the conditions in order

  $$\frac{9c - ab}{2(a^2 - 3b)}A + B \quad \text{and} \quad \frac{4ab - a^3 - 9c}{a^2 - 3b}A + B$$

  represent one pair of real lines and one pair of imaginary lines or the conditions in order $f(\lambda)$ has one triple negative real root and the conditions in order

  $$-\frac{a}{3}A + B$$

  represent one double real line.

- *Relative position* **8**
  a, b and c satisfy the conditions in order $f(\lambda)$ has two different real negative roots (one double) and the conditions in order

  $$\frac{9c - ab}{2(a^2 - 3b)}A + B \quad \text{and} \quad \frac{4ab - a^3 - 9c}{a^2 - 3b}A + B$$

  represent one double real line and one pair of real parallel lines.

- *Relative position* **9**
  a, b and c satisfy the conditions in order $f(\lambda)$ has one triple negative real root and the conditions in order

  $$-\frac{a}{3}A + B$$

  represent one pair of real lines.

- *Relative position* **10**
  a, b and c satisfy the conditions in order $\lambda^3 + a\lambda^2 + b\lambda + c = (\lambda + 1)^3$.

4.2. On deciding the internal and external relationships

Next it is shown how the procedure presented in this section is also able to characterize, once in positions **4**, **5**, **7** or **8**, which ellipse is inside and which is outside and when this changes (since this will imply a different relative position of the initial considered ellipses).

In Section 1, we have seen that in some cases the positions of a pair of ellipses $\mathcal{A}$ and $\mathcal{B}$ are not symmetric to both ellipses. If we obtain the positions **4**, **7** or **8** we do the following steps in order to determine whether $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \subset \mathcal{A}$:

1. We take the center of (for example) the ellipse $\mathcal{A}$.
2. We take any line that passes through the center, which intersects the ellipse $\mathcal{A}$ in two points p and q.
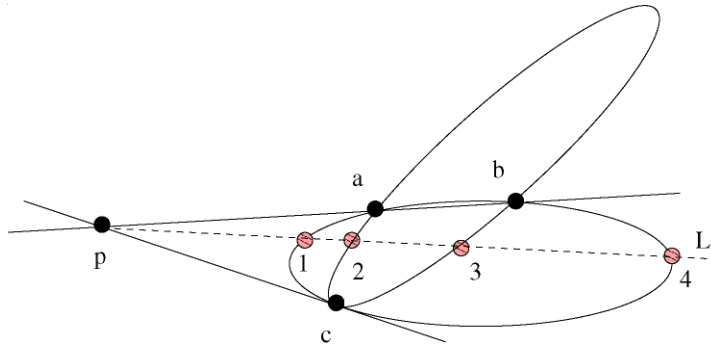
Fig. 5. Position **5**.

3. We use one of these points (for example p) to check if p is in the interior or exterior of ellipse $\mathcal{B}$ (i.e., given p we check if $p^T Bp < 0$ or $p^T Bp > 0$, respectively). Hence, we determine if $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \subset \mathcal{A}$.

If we obtain the position **5**, we do the following steps in order to determine whether the set defined by the interior of the ellipse $\mathcal{A}$ without the interior of ellipse $\mathcal{B}$ has one or two connected components:

1. We take the degenerate conic associated with the single root α (as described in (7)), which is a pair of real lines. In Fig. 5, the degenerate conic is the pair of real lines which intersects in the point p (i.e., they are the lines $\langle p, a, b \rangle$ and $\langle p, c \rangle$, where a, b and c can be calculated explicitly).
2. We choose a line L (see Fig. 5), which passes through p and a point of the segment $\overline{ac}$. Then L intersects the pair of ellipses in four points named 1, 2, 3 and 4.
3. We study the behaviour of the points of the line L (i.e., whether the points are interior or exterior of the ellipses). To do this, we consider the line L as:

$$L \equiv (-\infty, p) \cup \{p\} \cup (p, 1) \cup \{1\} \cup (1, 2) \cup \{2\} \cup (2, 3) \cup \{3\} \cup (3, 4) \cup \{4\} \cup (4, +\infty),$$

where 1, 2, 3 and 4 are the intersection points (see Fig. 5). Then, we study the behaviour of the points in these intervals (for example, we take a point $x \in (-\infty, p)$ and we check $x^T Ax$ and $x^T Bx$, we must do the same with the rest of intervals). To simplify, we need introduce the following notation: $\hat{A} \equiv x^T Ax$ and $\hat{B} \equiv x^T Bx$ for any $x \in L$. If, along the line L, their points verify the following rule:

$$\left\{ \begin{matrix} \hat{A} > 0 \\ \hat{B} > 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} = 0 \\ \hat{B} > 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} < 0 \\ \hat{B} > 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} < 0 \\ \hat{B} = 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} < 0 \\ \hat{B} < 0 \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \hat{A} < 0 \\ \hat{B} = 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} < 0 \\ \hat{B} > 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} = 0 \\ \hat{B} > 0 \end{matrix} \right\} \left\{ \begin{matrix} \hat{A} > 0 \\ \hat{B} > 0 \end{matrix} \right\},$$

the set defined by the interior of ellipse $\mathcal{A}$ without the interior of the ellipse $\mathcal{B}$ has two connected component. Otherwise, it has one connected component.

4.3. Examples

Two examples of the previous discussion are presented.

Example 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses defined by the equations

$$2x^2 + 2y^2 - 5x + 2 = 0, \qquad 2x^2 + 2y^2 - 10x + 2 = 0.$$

The characteristic polynomial of the pencil $\lambda A + B$ is (already turned monic):

$$f(\lambda) = \lambda^3 + 77/9\lambda^2 + 152/9\lambda + 84/9.$$

The evaluation of the polynomials determined in Section 2.3 provides the list of signs $[1, 1, 1, 1, 1]$ together with the information that $f(\lambda)$ has 3 different real negative roots $\alpha < \beta < \gamma < 0$. Since $\alpha < -3 < \beta$ and the conic $-3A + B$, with equation

$$-4x^2 + 5x - 4y^2 - 4 = 0,$$

is empty, then it is concluded that the considered pencil $\lambda A + B$ is of Type Ia and that the relative position of the ellipses $\mathcal{A}$ and $\mathcal{B}$ corresponds to the relative position **4** in Table 1 (two contained ellipses).

Example 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two ellipses defined by the equations

$$3x^2 + 2y^2 - 4x + 2y - 4xy + 1 = 0, \qquad 5x^2 + 2y^2 - 6x + 2y - 6xy + 1 = 0.$$

The characteristic polynomial of the pencil $\lambda A + B$ is (already turned monic):

$$f(\lambda) = \lambda^3 + 5\lambda^2 + 8\lambda + 4.$$

The evaluation of the polynomials determined in Section 2.3 provides the list of signs $[1, 1, 1, 1, 0]$ together with the information that $f(\lambda)$ has 2 different real negative roots, α and β, one of them double, β. In order to decide which of the relative positions **4**, **5**, **7** or **8** is the one under the consideration, it is required to decide the nature of the degenerate conics.

According to the formulae in Remark 2.1, the real roots α and β are rational and given by the formulae:

$$\beta = \frac{9c - ab}{2(a^2 - 3b)} = -2, \qquad \alpha = \frac{4ab - a^3 - 9c}{a^2 - 3b} = -1.$$

Since the conic $\beta A + B = -2A + B$, with equation

$$-x^2 + 2xy + 2x - 2y^2 - 2y - 1 = \bigl(1 - x + (1 + I)y\bigr)\bigl(-1 + x + (-1 + I)y\bigr) = 0,$$

is a pair of imaginary lines, relative positions **5** and **8** are excluded. Since the conic $-1A + B$, with equation

$$2x(x - y - 1) = 0,$$

is a pair of real lines, it is concluded that the considered pencil $\lambda A + B$ is of Type IIa and that the relative position of the ellipses $\mathcal{A}$ and $\mathcal{B}$ corresponds to the relative position **7** in Table 1 (two internally tangent ellipses in one point).

5. Relative position of two ellipses depending on a parameter

It is worth to remark that all the results obtained in the previous section can be applied to study the case of two ellipses depending on one parameter. In this case the goal is to decide when the intersection set moves from a case to another between those appearing in Table 1. More precisely, given two ellipses

$$\mathcal{A}(t)\colon a_{11}(t)x^2 + a_{22}(t)y^2 + 2a_{12}(t)xy + 2a_{13}(t)x + 2a_{23}(t)y + a_{33}(t) = 0,$$
$$\mathcal{B}(t)\colon b_{11}(t)x^2 + b_{22}(t)y^2 + 2b_{12}(t)xy + 2b_{13}(t)x + 2b_{23}(t)y + b_{33}(t) = 0$$

depending on the parameter $t \in \mathbb{R}$ the problem is to compute a partition of $\mathbb{R}$ into a finite number of points and intervals such that the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ is fixed for t belonging to each of the sets in the partition.

If $A(t)$ and $B(t)$ denote the matrices associated to the moving ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ then the characteristic polynomial of the pencil $\lambda A(t) + B(t)$

$$f(t; \lambda) = \det\bigl(\lambda A(t) + B(t)\bigr)$$

is a degree three polynomial in λ with real coefficients depending on the parameter t. In order to determine the relative position of both families of ellipses, the study of the sign behaviour of the roots of the characteristic polynomial for all the possible values of the parameter t is required. This is accomplished by using the techniques presented in Section 4 where the analysis of the possible sign conditions verified by four polynomials in the coefficients of $f(t; \lambda)$ (as polynomial in λ) produces in an automatic manner (and in terms of t) which is the behaviour of the sign of the real roots of $f(t; \lambda)$.

Next subsection presents one detailed example showing how to use the techniques introduced in this paper in order to solve the previously described problem.
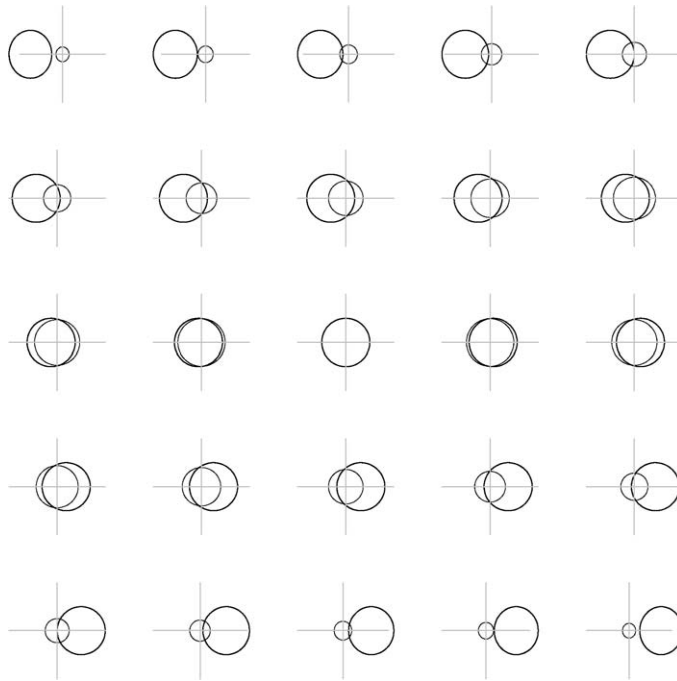
Fig. 6. The two families of moving circles for t from $-3/2$ to $3/2$.

5.1. Example of two ellipses depending on a parameter

Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be two circumferences, depending on $t \in \mathbb{R}$, defined by the equations

$$(t^2 + 1)x^2 + (t^2 + 1)y^2 - 1 = 0, \qquad (x - t)^2 + y^2 - 1 = 0.$$

$\mathcal{A}(t)$ is the set of concentric circumferences of radius less or equal to 1 (circumferences in bold in Fig. 6) and $\mathcal{B}(t)$ is a (radius 1) circumference whose center moves along the axis x (circumferences in gray in Fig. 6).

The matrices associated to $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are in this case:

$$A(t) = \begin{pmatrix} t^2 + 1 & 0 & 0 \\ 0 & t^2 + 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 & 0 & -t \\ 0 & 1 & 0 \\ -t & 0 & t^2 - 1 \end{pmatrix}$$

and the characteristic polynomial of the pencil $\lambda A(t) + B(t)$:

$$f(t; \lambda) = \det\big(\lambda A(t) + B(t)\big) = (-t^4 - 2t^2 - 1)\lambda^3 + (t^6 + t^4 - 3t^2 - 3)\lambda^2 + (t^4 - 3 - t^2)\lambda - 1.$$

Turning $f(t; \lambda)$ into a monic polynomial (with respect to λ) produces the following coefficients

$$a = \frac{-t^6 - t^4 + 3t^2 + 3}{t^4 + 2t^2 + 1}, \qquad b = \frac{-t^4 + 3 + t^2}{t^4 + 2t^2 + 1}, \qquad c = \frac{1}{t^4 + 2t^2 + 1}.$$

According to the results in Section 2.3, the sign behaviour of the real roots of $f(t; \lambda)$ is uniquely determined by the sign conditions verified by the polynomials

$$a, \quad c, \quad -3b + a^2, \quad 3ac + ba^2 - 4b^2, \quad -27c^2 + 18cab + a^2b^2 - 4a^3c - 4b^3.$$

In the concrete problem considered here, once denominators are removed (note that $t^4 + 2t^2 + 1 > 0$ for any $t \in \mathbb{R}$), the following polynomials are obtained

$$-(t^2 + 1)(t^4 - 3), \quad 1, \quad t^2(t^6 - 3t^2 - 3)(t^2 + 1)^2,$$

$$-t^2(t^8 - 2t^6 - 3t^4 + 4t^2 + 6)(t^2 + 1)^3, \quad t^6(t^3 - 2t^2 + 2t - 2)(t^3 + 2t^2 + 2t + 2)(t^2 + 1)^4$$

which can be further simplified by removing those factors without real roots and constant sign ($t^2 + 1$ and $t^8 - 2t^6 - 3t^4 + 4t^2 + 6$):

$$-(t^4 - 3), \quad 1, \quad t^2(t^6 - 3t^2 - 3), \quad -t^2, \quad t^6(t^3 - 2t^2 + 2t - 2)(t^3 + 2t^2 + 2t + 2).$$

Next the real roots of these polynomials are computed producing the following results:

- The real roots of $t^4 - 3$ are $d_1 = -1.316074013$ and $d_2 = 1.316074013$;
- The real roots of $t^6 - 3t^2 - 3$ are $e_1 = -1.450449380$ and $e_2 = 1.450449380$;
- The real root of t is $f = 0$;
- The real root of $t^3 - 2t^2 + 2t - 2$ is $g = 1.543689013$;
- The real root of $t^3 + 2t^2 + 2t + 2$ is $h = -1.543689013$;

and giving the following description, in terms of t, for the sign behaviour of the real roots of $f(t; \lambda)$:

1. If $t \in (-\infty, h)$ then $f(t; \lambda)$ has 2 positive real roots and 1 negative real root.
2. If $t = h$ then $f(t; \lambda)$ has 1 positive (double) real root and 1 negative real root.
3. If $t \in (h, e_1)$ then $f(t; \lambda)$ has 1 negative real root.
4. If $t = e_1$ then $f(t; \lambda)$ has 1 negative real root.
5. If $t \in (e_1, d_1)$ then $f(t; \lambda)$ has 1 negative real root.
6. If $t = d_1$ then $f(t; \lambda)$ has 1 negative real root.
7. If $t \in (d_1, f)$ then $f(t; \lambda)$ has 1 negative real root.
8. If $t = f$ then $f(t; \lambda)$ has 1 negative (triple) real root.
9. If $t \in (f, d_2)$ then $f(t; \lambda)$ has 1 negative real root.
10. If $t = d_2$ then $f(t; \lambda)$ has 1 negative real root.
11. If $t \in (d_2, e_2)$ then $f(t; \lambda)$ has 1 negative real root.
12. If $t = e_2$ then $f(t; \lambda)$ has 1 negative real root.
13. If $t \in (e_2, g)$ then $f(t; \lambda)$ has 1 negative real root.
14. If $t = g$ then $f(t; \lambda)$ has 1 positive(double) real root and 1 negative real root.
15. If $t \in (g, +\infty)$ then $f(t; \lambda)$ has 2 positive real roots and 1 negative real root.

There are only four possible behaviours, in terms of t, for the sign of the real roots of $f(t; \lambda)$:

- If $f(t; \lambda)$ has 1 negative real root and 2 different real positive roots then, by Theorem 1.1, the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are separated.
- If $f(t; \lambda)$ has 1 negative real root and 1 double positive real root then, by Theorem 1.1, the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are externally tangent.
- If $f(t; \lambda)$ has 1 negative real root and 2 conjugate complex roots then, by Table 5, the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ intersect transversally in two points.
- If $f(t; \lambda)$ has 1 triple negative real root then, by Table 5, the relative position of the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ corresponds to relative positions **7**, or **9** or **10**. In this concrete case, $t = 0$, the searched relative position corresponds to the relative position **10**, i.e. two coincident ellipses.

Fig. 7 represents the solution to the problem of characterizing the relative positions, in terms of t, of the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$.
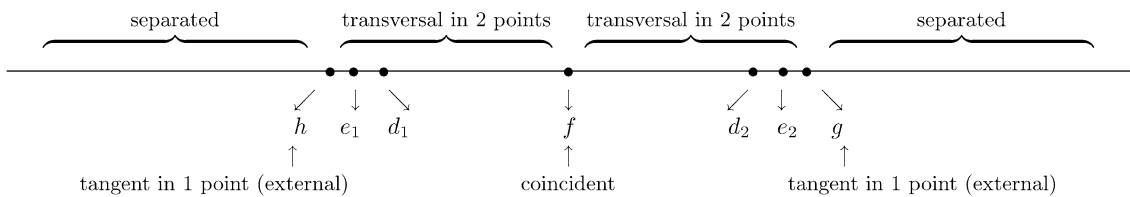

Fig. 7. The relative position of the ellipses $\mathcal{A}(t)$ and $\mathcal{B}(t)$ depending on $t \in \mathbb{R}$.

Note that there are no changes in the relative position of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ in $\mathbb{R}^2$ when t crosses e_1, d_1, d_2 or e_2. This is due to a change of the sign condition in Table 2 but producing the same sign behaviour for the real roots of $f(t; \lambda)$.

6. Conclusions

Previous sections have presented several techniques and results allowing to deal with the problem of determining the relative position of two ellipses without computing their intersection points. The way this problem is solved is specially adapted to deal with the clearly most complicated situation where the considered ellipses depend on one or more parameters.

Two are the main geometrical objects upon the techniques presented here are based. The first one is the characteristic polynomial of the pencil generated by the two considered ellipses together with the properties of its real roots related to the relative position of the two considered ellipses. The second one is the classification of pencils of conics in $\mathbb{P}_2(\mathbb{R})$ (based on the degenerate conics in the pencil and the corresponding base points) which is used to refine the initial discrimination of the relative positions provided by the analysis of the sign of the real roots of the characteristic polynomial.

The main computational tool used comes from Computer Algebra and Real Algebraic Geometry: the Sturm–Habicht sequence. This sequence is used to easily describe the set of univariate polynomials of degree three (in this case) whose real roots have a prescribed sign behaviour.

The techniques here developed can be adapted to the case of ellipsoids which will be the topic of a forthcoming paper. This was the initial problem considered in (Wang et al., 2001) and (Wang and Krasauskas, 2003) where the sign of the real roots of the characteristic polynomial is used to decide when both ellipses or ellipsoids are or are not separated as in Theorem 1.1. In this paper, the use of the Sturm–Habicht sequence together with the classification of pencils of conics in $\mathbb{P}_2(\mathbb{R})$ allows to give a full classification together with a set of precomputed polynomials (four) whose evaluation is the key to determine the relative position of two ellipses under the presence of parameters.

For two ellipsoids $\mathcal{A}$ and $\mathcal{B}$, the characteristic polynomial $f(\lambda)$ of the pencil $\lambda A + B$ has degree four and the fact that this polynomial has two distinct positive real roots is the condition on the real roots of $f(\lambda)$ assuring the separation by a plane of the two considered ellipsoids. The same techniques used in Section 2 can be applied to obtain similar results to those presented in Section 2.3.

It is worth to mention that the results and formulae presented in Section 2.3 apply directly to the characterization of the separation by a plane of an elliptic cylinder $\mathcal{A}$ and an ellipsoid $\mathcal{B}$. According to (Wang and Krasauskas, 2003), $\mathcal{A}$ and $\mathcal{B}$ are separated by a plane if and only if the characteristic polynomial $f(\lambda)$ of the pencil $\lambda A + B$ has two distinct positive real roots. But, in this case, the polynomial $f(\lambda)$ has degree three and Section 2.3 shows how to characterize the set of degree three polynomials with two distinct positive real roots.

The possibility of obtaining similar results for the determination of the relative position of other type of conics like parabolas or hyperbolas (see also (Briand, 2006) and (Liu and Chen, 2004)) or quadrics is also under consideration.

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