FisyMat
Máster en Matemática Aplicada

UNIVERSIDAD DE GRANADA

Física Estelar

Lluís Galbany, Ed. Mecenas (#16)

Inma Domínguez, Ed. Mecenas (#17)

Antonio García, Ed. Mecenas (#16)

5. Energy transport in stellar interiors

Energy Transport

We have seen that in stellar interiors:

LTE is a good approach

Thermal equilibrium

Internal energy sources, nuclear reactions, compensate energy loss from the surface,

$$L = L_{\rm nuc} = -\frac{dE_{\rm nuc}}{dt}$$

The total energy is conserved, and the viral theorem states that the E_{int} and E_{pot} are conserved as well,

$$\dot{E}_{\rm tot} = \dot{E}_{\rm int} = \dot{E}_{\rm pot} = 0$$

This stationary state is known as **Thermal Equilibrium (TE)**

Energy is radiated at the surface at the same rate at which it is produced by nuclear reactions in the interior.

Energy transfer due to a ∇T

Energy Transport

We have seen that in stellar interiors:

LTE is a good approach

Mean free path is extremely small (l<<R)

The time radiation takes to escape from the center of the Sun by the random walk process is the K-H timescale.

Thermal (Kelvin-Helmholtz) timescale, (changes in the thermal structure; *from V. Th.*)

$$\tau_{KH} = \frac{E_{\rm int}}{L} \simeq \frac{|E_{pot}|}{2L} \simeq \frac{GM^2}{2RL} \simeq 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \frac{R_\odot}{R} \frac{L_\odot}{L} yr$$

describes how fast changes in thermal structure of a star can occur

Changes in the Sun luminosity would occur after millions of years, on the timescale for radiative energy transport: K-H timescale for thermal readjustment.

Energy Transport

In this situation, energy can be transported (hot to cold):

DIFFUSION: random thermal motion of particles.

- *Radiative diffusion* in the case of photons
- *Heat conduction* for gas particles

CONVECTION: collective ordered motion of gas particles

Very efficient & rapid mixing

This leads to 2 new equations for stellar structure

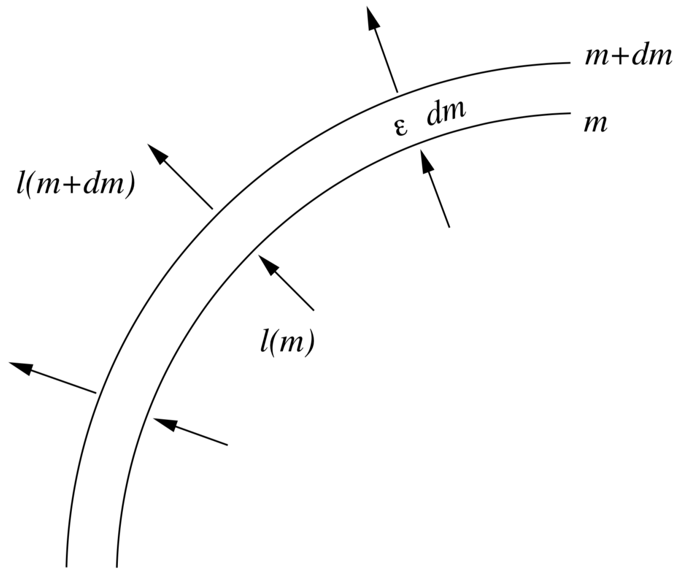
Local energy conservation

Virial Th, regulates the global energy budget.
(conservation)

In local scale the internal energy can be changed by two forms,

$$\underbrace{\delta u = \delta q}_{\text{Heat}} + \underbrace{\frac{P}{\rho^2}\delta\rho}_{\text{Work}}$$

In a spherical shell, changes in the heat content can occur due to:



- Heat is added by the release of nuclear energy ($\epsilon_{\rm nuc}$)

- Heat is removed by the release of neutrinos (ϵ_ν)
  (Not those product of reactions but from weak interaction)

- Heat is absorbed/emitted according to the balance of heat fluxes (*local luminosity*; $l = 4\pi r^2 F$)
  where $l(0) = 0$ and $l(R) = L$

Local energy conservation

The heat content of the shell is: $\delta Q = \delta q \Delta m$

so, $\delta Q = \epsilon_{\rm nuc} \Delta m \delta t - \epsilon_\nu \Delta m \delta t + l(m)\delta t - \underline{l(m+\Delta m)\delta t}$

$l(m+\Delta m)\delta t = l(m) + \frac{\partial l}{\partial m}\Delta m$

rearranging, $$\delta q = \left(\epsilon_{\rm nuc} - \epsilon_\nu - \frac{\partial l}{\partial m} \right) \delta t$$

And combined with the change of internal energy eq., $$\frac{\partial l}{\partial m} = \epsilon_{\rm nuc} - \epsilon_\nu \boxed{- \frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}}$$

$\delta u = \delta q + \frac{P}{\rho^2}\delta\rho$

$= -T\frac{\partial s}{\partial t} = \epsilon_{\rm gr}$

Local energy conservation

LEC eq.

$$\boxed{\frac{\partial l}{\partial m} = \epsilon_{\text{nuc}} - \epsilon_\nu + \epsilon_{\text{gr}}}$$

$\epsilon_{\text{gr}} > 0$ Energy released (contraction)

$\epsilon_{\text{gr}} < 0$ Energy absorbed (expansion)

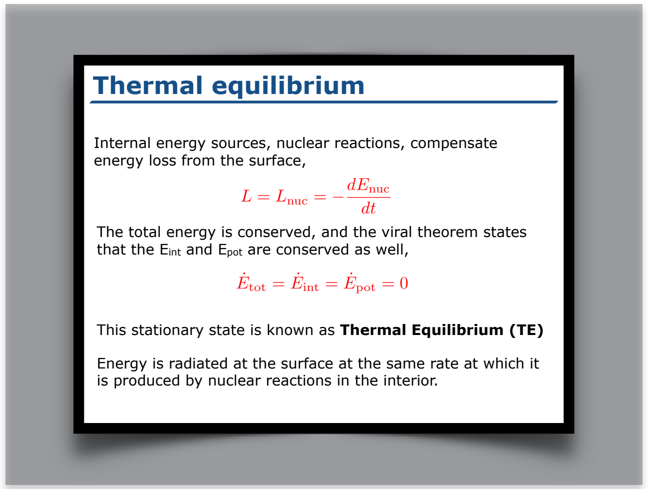
Thermal equilibrium achieved when $\epsilon_{\text{gr}} = 0$, so

$$\frac{\partial l}{\partial m} = \epsilon_{\text{nuc}} - \epsilon_\nu$$

And integrating over the mass (Lagr. coord.)

$$L = \int_0^M \epsilon_{\text{nuc}} dm - \int_0^M \epsilon_\nu dm \equiv L_{\text{nuc}} - L_\nu$$

Neglecting $L_\nu \rightarrow$



Diffusion

Radiative energy transport in stars can be described as a diffusion process

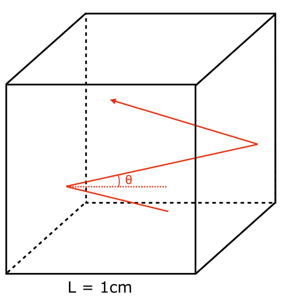
When gradient of particles, diffusion is given by Fick's diffusion law:
Flux equals a constant times the rate it changes in space

Mean free path

$$J = -D\nabla n \quad \text{where} \quad D = \frac{1}{3}\bar{v}l$$

Diffusion coefficient

From Ch. 3:



$$\int_0^{\pi/2} \cos^2\theta \sin\theta d\theta$$

= 1/3

Consider particles crossing a unit surface area

The number of particles crossing a surface

$$\frac{dN}{dt} = \frac{1}{2}\left(\frac{1}{3}\bar{v}\right)n$$

average velocity

half in each direction

if there is a gradient in the particle density, $\frac{\partial n}{\partial z}$, particles moving up have density *n(z-l)*, and particles moving down *n(z+l)*. *So,*

$$J = \frac{1}{6}\bar{v}n(z-l) - \frac{1}{6}\bar{v}n(z+l) = \frac{1}{6}\bar{v}\left[n(z-l) - n(z+l)\right] = \frac{1}{6}\bar{v}\left[n - \frac{dn}{dz}l - \left(n + \frac{dn}{dz}l\right)\right] = -\underbrace{\boxed{\frac{1}{3}\bar{v}l}}_{D}\frac{\partial n}{\partial z}$$

The equivalent, when a gradient of U is present,

$$F = -D\nabla U \quad \text{with} \quad \nabla U = C_V \nabla T$$

$$F = -\frac{1}{3}\bar{v}lC_V\nabla T$$

Eq. for heat conduction

Radiative diffusion (photons)

For photons,

$\bar{v} = c \qquad U = aT^4$

$C_V = \frac{dU}{dT} = 4aT^3$

$$F_{\rm rad} = -\frac{1}{3}\bar{v}l C_V \nabla T = -\boxed{\frac{4}{3}\frac{acT^3}{\kappa\rho}}\nabla T$$

$K_{\rm rad}$

Radiative conductivity

If a source is surrounded by gas, the change in radiation intensity is:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho I_\lambda ds$$

mass *absorption* coefficient mass *emissivity* coefficient

Where *K_λ = opacity (cm^2 g^{-1})*, and can therefore be interpreted as being the **fraction** of radiation absorbed by unit column density of gas.

Or the cross-section for absorbing photons at wavelength λ per g of star stuff

And the mean free path can be defined as the distance over which the intensity decreases by a factor of *e*

$$l_{\rm ph} = \frac{1}{\kappa_\nu \rho}$$

Radiative diffusion (photons)

In spherical symmetry, $F_{\rm rad} \sim \frac{l}{4\pi r^2}$ ← local luminosity so, rearranging,

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho}{16\pi acT^3}\frac{l}{r^2}$$

or (eq. mass continuity)

$$\frac{\partial T}{\partial m} = -\frac{3}{64\pi^2 ac}\frac{\kappa l}{r^4 T^3}$$

which describes the T gradient in stellar interiors when E is transported by radiation

Only valid when $l_{\rm ph} \ll R$, when LTE holds (e.g. in stellar surface $l_{\rm ph} \geq R$)

In HE, we can combine this with eq. Motion $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$

$$\frac{dT}{dm} = \frac{dP}{dm}\frac{dT}{dP} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\frac{d\log T}{d\log P}$$

so then, we can define the radiative T gradient

$$\nabla_{\rm rad} = \left(\frac{d\log T}{d\log P}\right)_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4}$$

Rosseland mean opacity

So far, we have assumed radiative diffusion is independent of frequency

However, in practice, opacity depends on frequency

$$F_\nu = -D_\nu \nabla U_\nu = -D_\nu \frac{\partial U_\nu}{\partial T} \nabla T$$

where $D_\nu = \frac{1}{3} c l_\nu = \frac{c}{3\kappa_\nu \rho}$ and $U_\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}$

Planck function for BB radiation (Ch. 3)

the total flux is integrated over all frequencies

$$F = -\left[\frac{c}{3\rho} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial U}{\partial T} d\nu \right] \nabla T = -\boxed{K_{\mathrm{rad}}} \nabla T$$

From both expressions of conductivity,

$$\frac{1}{\kappa} = \frac{1}{4aT^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial U}{\partial T} d\nu$$

is the Rosseland mean opacity

(the average transparency of the star)

Heat conduction (gas particles)

Collisions between gas particles (e- or ions) can also transport heat

However, their cross sections are typically ~10^{-19} cm^2 and $l_{\rm gas} \ll l_{\rm ph}$

Also $\bar{v} \ll c$, so in general this contribution can be ignored

Only important when particles (e.g. e-) are degenerate

- v increase (momenta approach the Fermi momentum)
- Mean free path $l_{\rm e} \gg l_{\rm ph}$, and e- conduction becomes more efficient

Similarly, $F_{\rm cd} = -K_{\rm cd} \nabla T = -\dfrac{4acT^3}{3\kappa_{\rm cd}\rho} \nabla T$

And in general,

$$F = F_{\rm rad} + F_{\rm cd} = -\left(K_{\rm rad} + K_{\rm cd}\right) \nabla T = -\frac{4acT^3}{3\kappa\rho} \nabla T$$

where $\dfrac{1}{\kappa} = \dfrac{1}{\kappa_{\rm rad}} + \dfrac{1}{\kappa_{\rm cd}}$

Opacity

The opacity coefficient κ determines the flux that can be transported by radiation for a certain temperature gradient

(how large the temperature gradient must be in order to carry a given luminosity *l*)

physical processes that contribute to opacity:

Electron scattering

Free electrons scatter photons. $\mu_e = 2/(1+X)$

Deep inside stars, gas is fully ionized, this is dominant

Basically found dividing cross-section by the unit mass

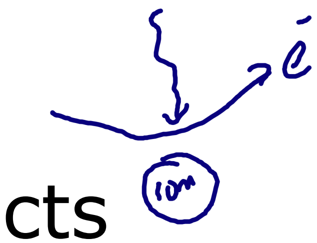
Thomson cross-section

$$\kappa_{\mathrm{es}} = \frac{\overline{\sigma_e}}{\underbrace{\mu_e m_u}_{\rho/n_e}} = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 \frac{1}{\mu_e m_u} = 0.20(1+X)\ \mathrm{cm}^2\mathrm{g}^{-1}$$

Important at high T, not-so-high densities

Opacity

Free-free absorption



Oposite of bremsstrahlung: γ absorbed by e- when interacts with ion

$$\kappa_{\rm ff} \sim Z_i^2 n_i n_e \sim 7.5 \times 10^{22} \left(\frac{1+X}{2}\right) \left\langle \frac{Z_i^2}{A_i} \right\rangle \rho T^{-7/2}\ {\rm cm^2 g^{-1}}$$

ion charge

ion number density

e- number density

(also called Kramer's opacity)

The steep T dependence implies that it is strongest at low T

bound-free absorption

absorption of a photon with sufficient energy by an atom not fully ionized, kicking a bound electron

$$\kappa_{\rm bf} \sim 4.3 \times 10^{25} (1+X) Z \rho T^{-7/2}\ {\rm cm^2 g^{-1}}$$

Not applicable at low T<10^4 K (γ not energetic enough)

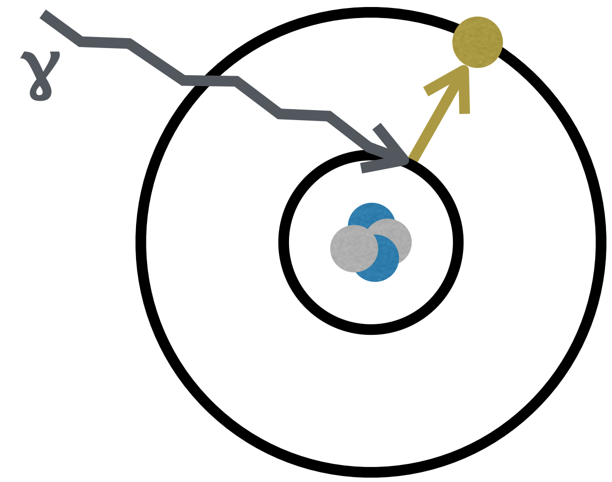
Not applicable at high T (ions fully ionized)

Opacity

Bound-bound absorption κ_{bb}

If gas is not fully ionized, it can absorb γ, resulting in e- transitions from one bound state to another.

Mainly important at $T<10^6$ K



Negative H ion

Important in relatively cool stars (T ~ kK)

Bound-free absorption of H that has been previously ionized with free electrons from metals (Na, K, Ca, or Al)

$$\kappa_{\mathrm{H}-} \simeq 2.5 \times 10^{-31} \left(\frac{Z}{0.02}\right) \rho^{1/2} T^9 \ \mathrm{cm}^2 \mathrm{g}^{-1}$$

Molecules and dust

In cool stars (T< 4000 K) opacity sources from molecules and dust (even lower T~1500 K; dust grains formation) become important

Opacity

+ Conductive opacities

Energy transport by heat conduction can also be described by means of a conductive opacity $\kappa_{\rm cd}$

In general, $\kappa_{\rm cd} \gg \kappa_{\rm rad}$

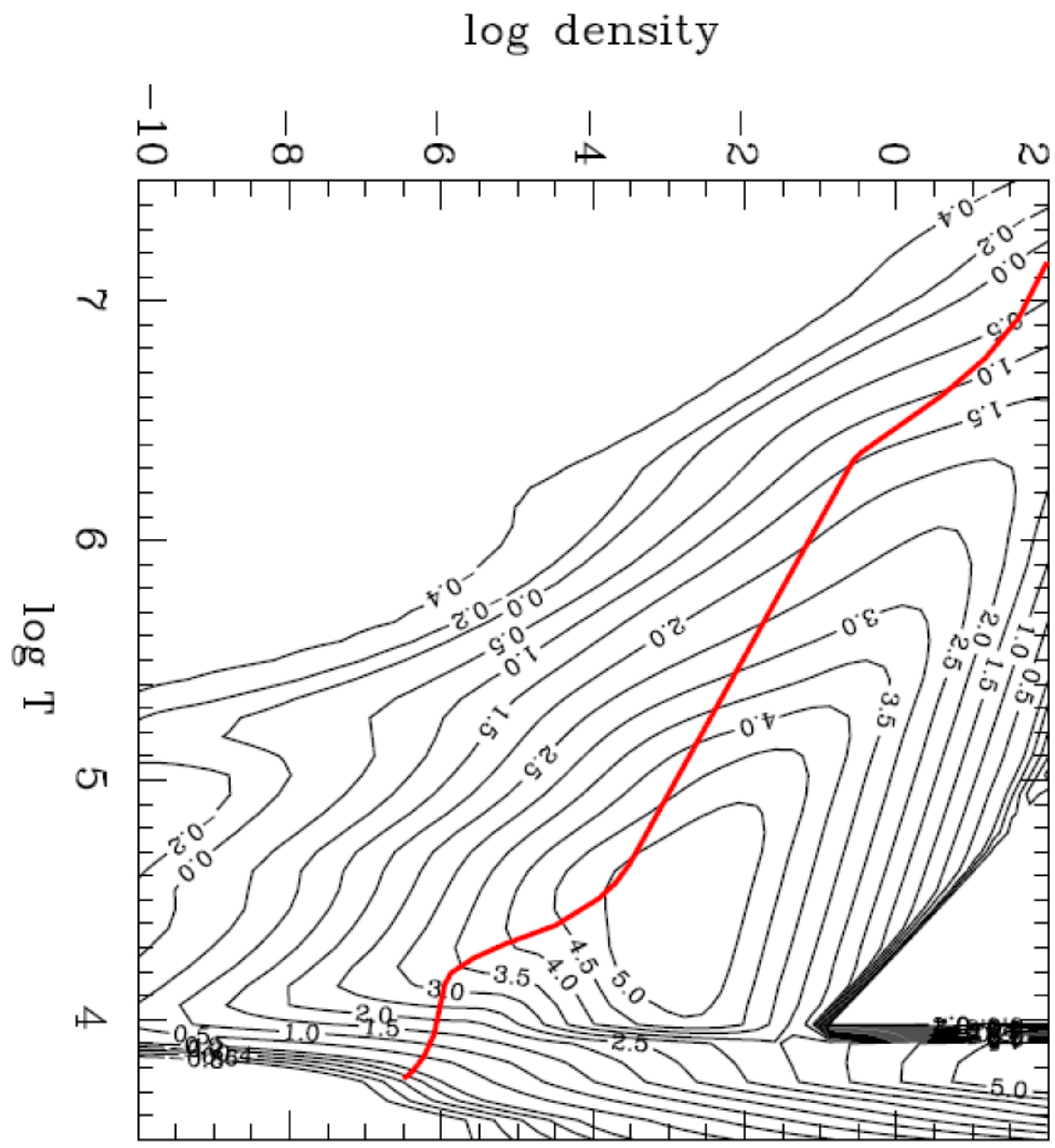
But for a degenerate gas,

$$\kappa_{\rm cd} \simeq 4.4 \times 10^{-3} \frac{\sum_i Z_i^{5/3} X_i/A_i}{(1+X)^2} \frac{(T/10^7 K)^2}{\rho/10^5 \text{ g/cm}^3)^2} \text{ cm}^2\text{g}^{-1}$$

at high ρ and low T, $\kappa_{\rm cd}$ becomes very small (large l_e)

In general, $\kappa = \kappa(\rho, T, X_i)$ is a complicated function (because it is not additive). But it has been calculated in a few cases…

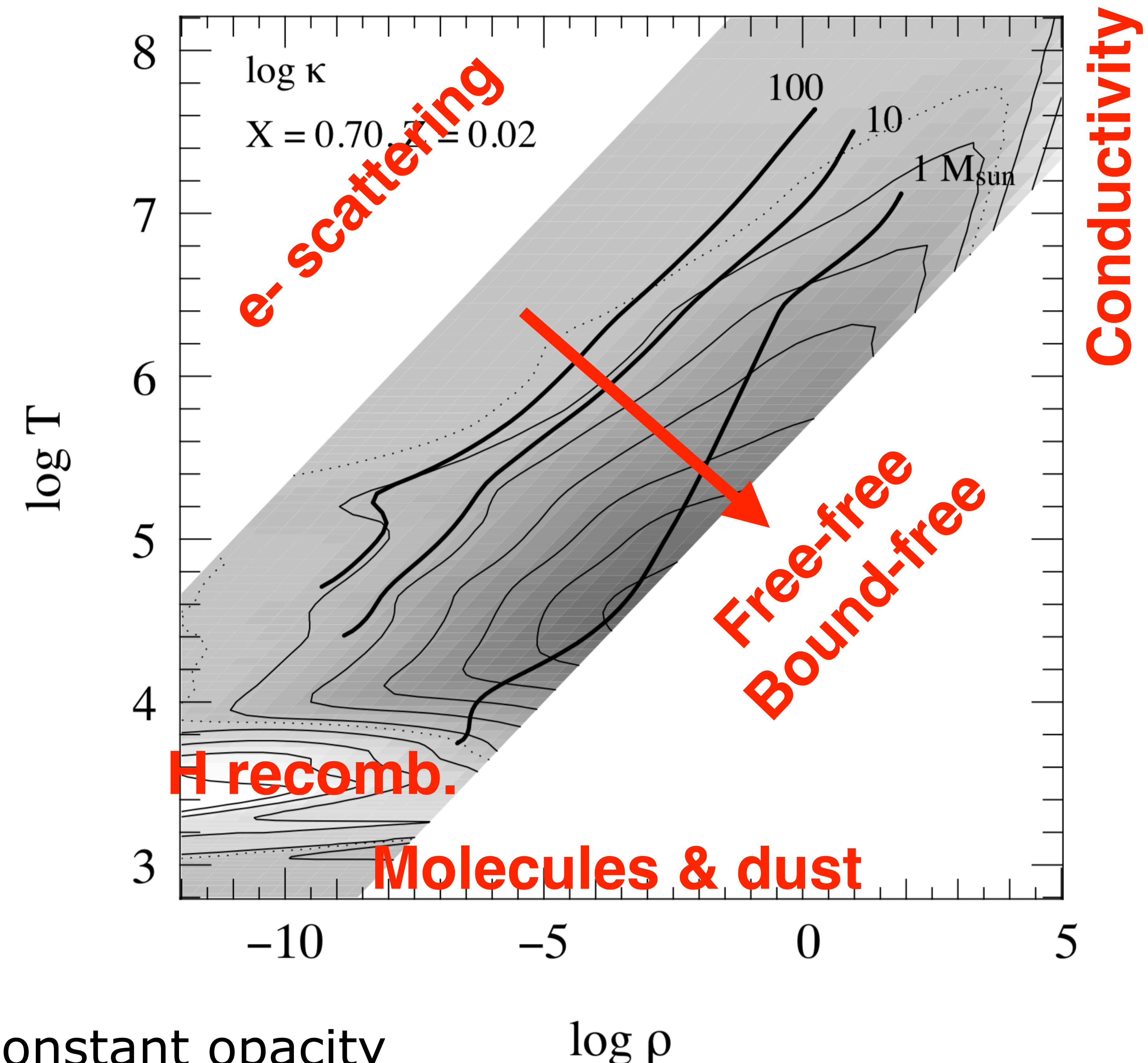
Opacity



Opacities measured by the OPAL project (1990s)

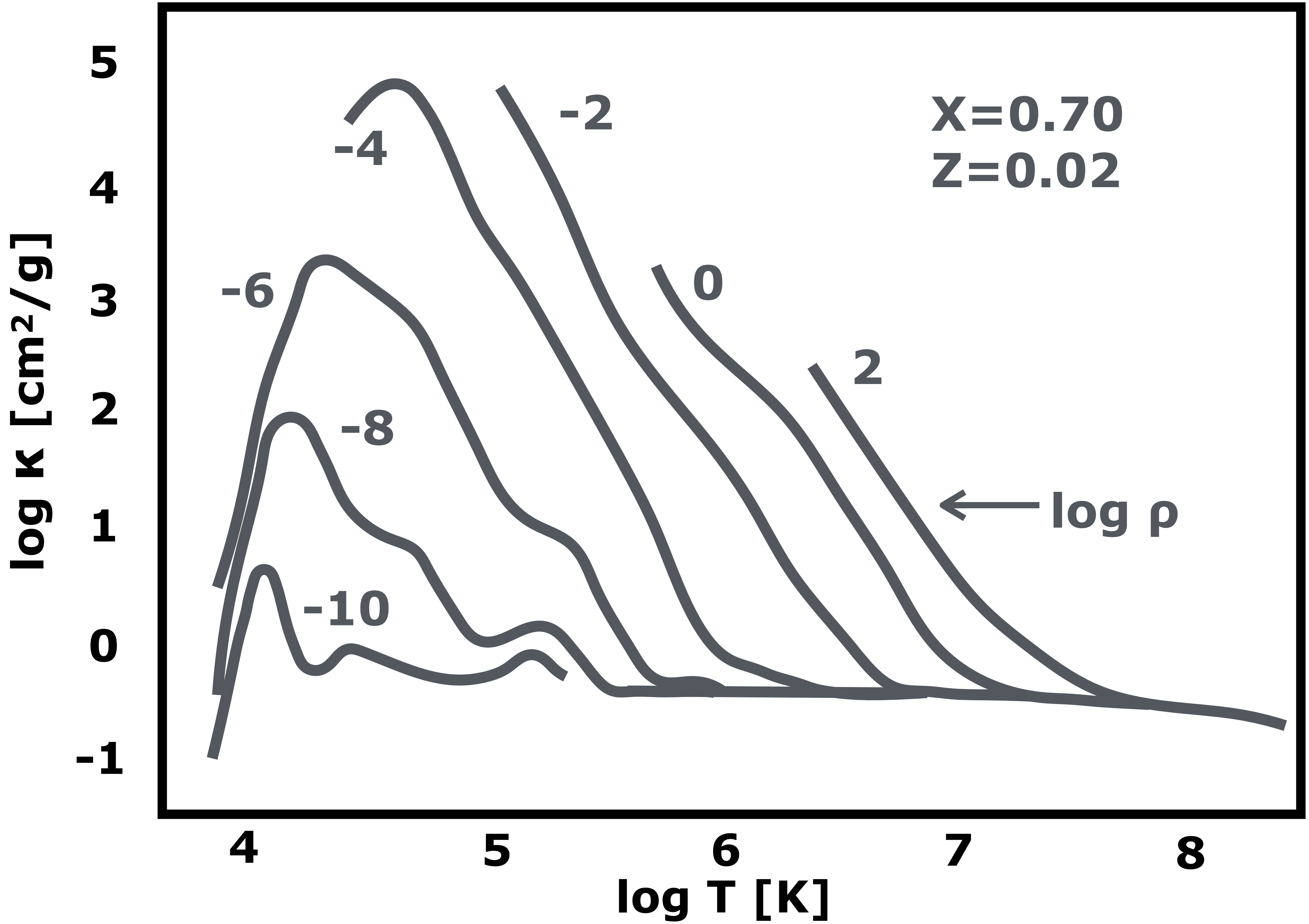
https://opalopacity.llnl.gov/

Opacity

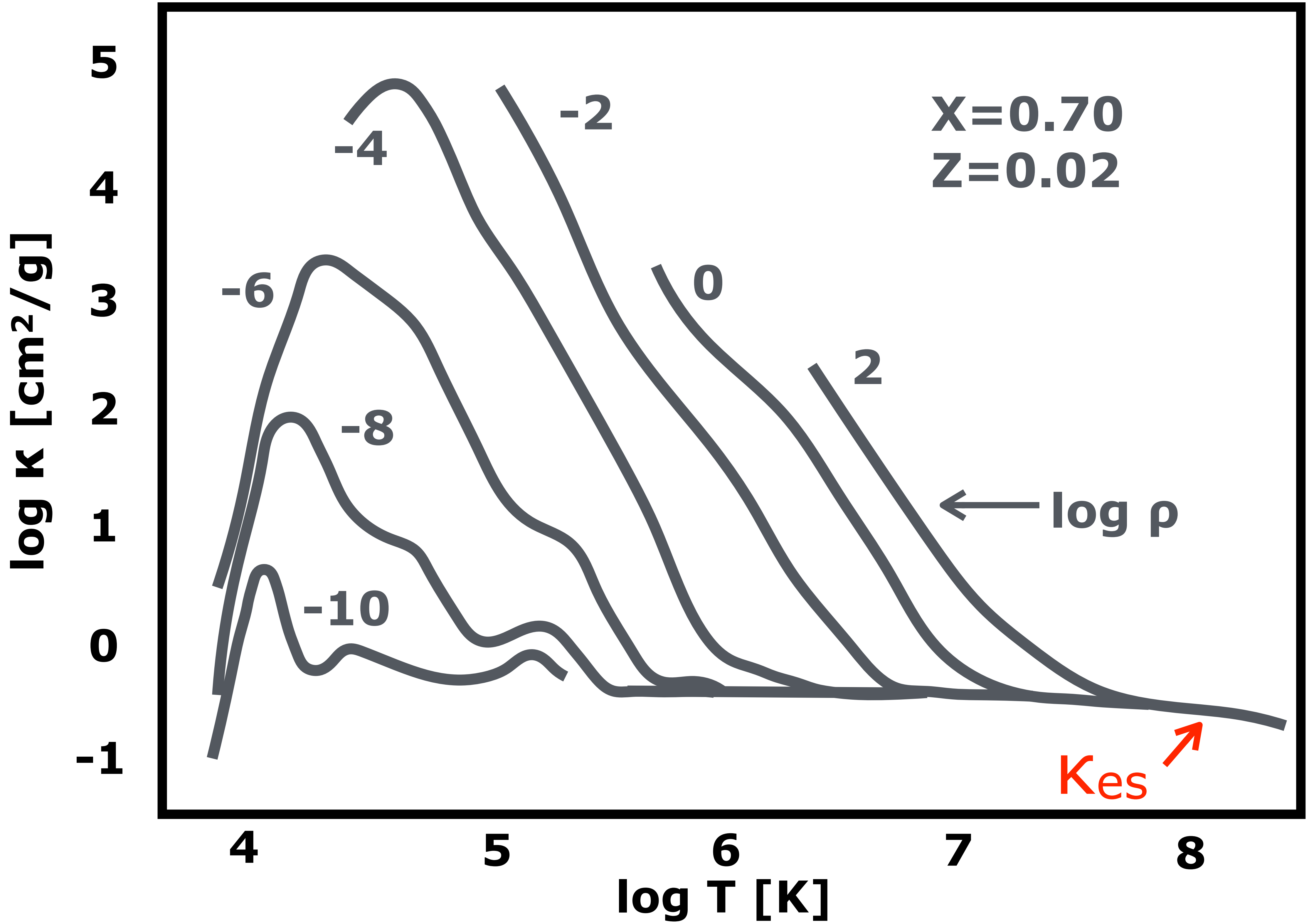


lines of constant opacity

Sources of Opacity

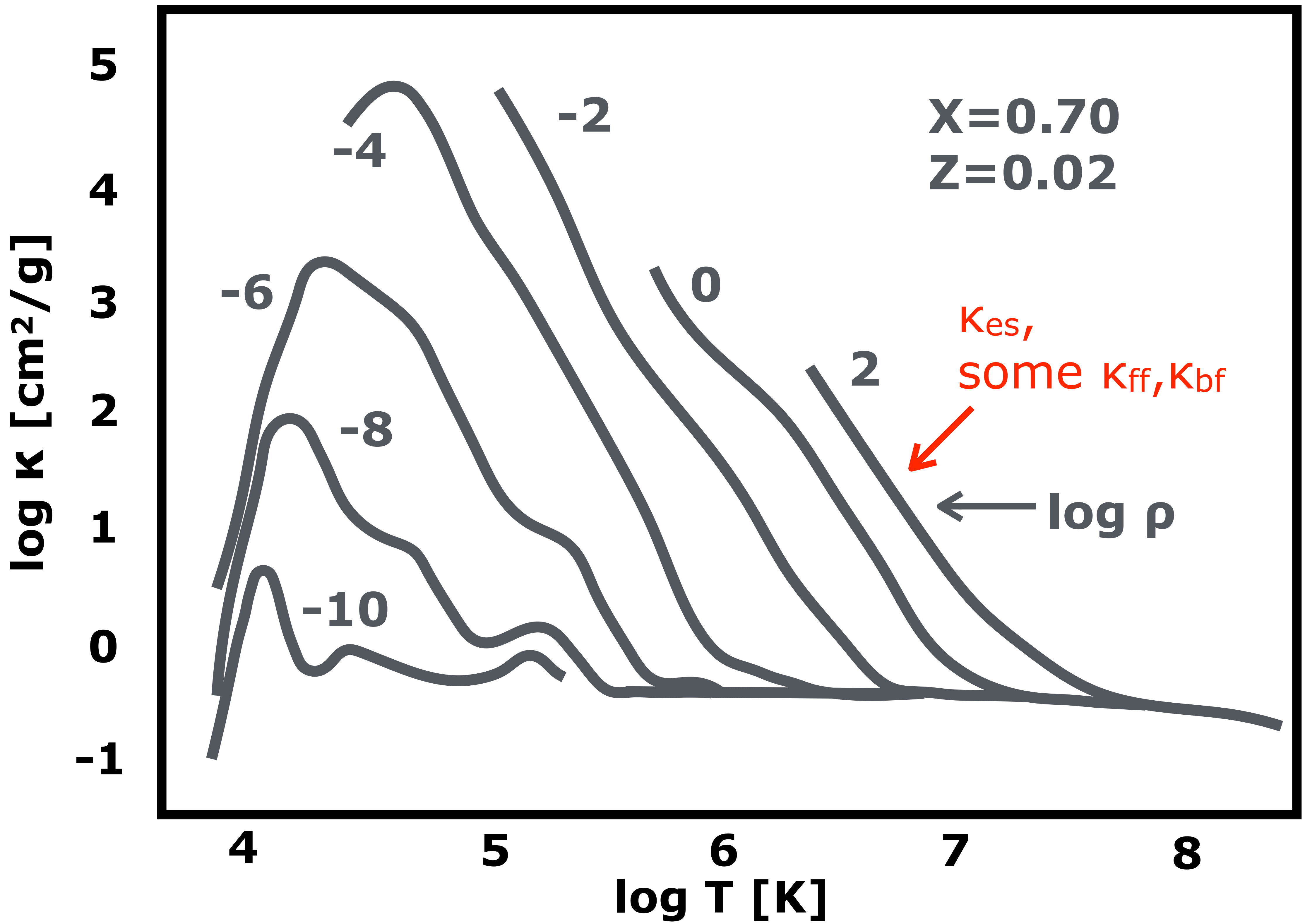


Sources of Opacity



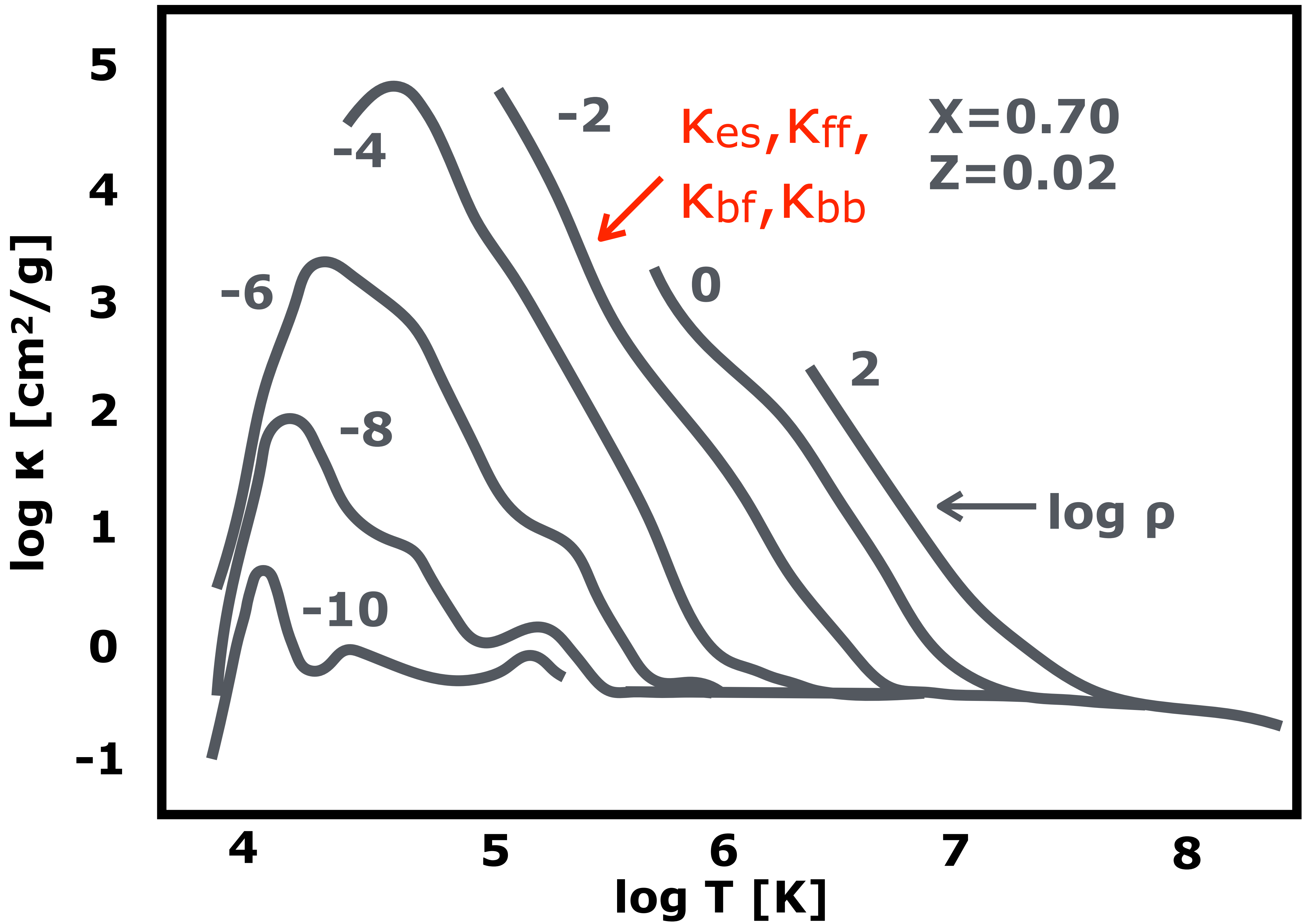
At high T and low ρ, all matter is ionized, $\kappa=\sigma_e$

Sources of Opacity



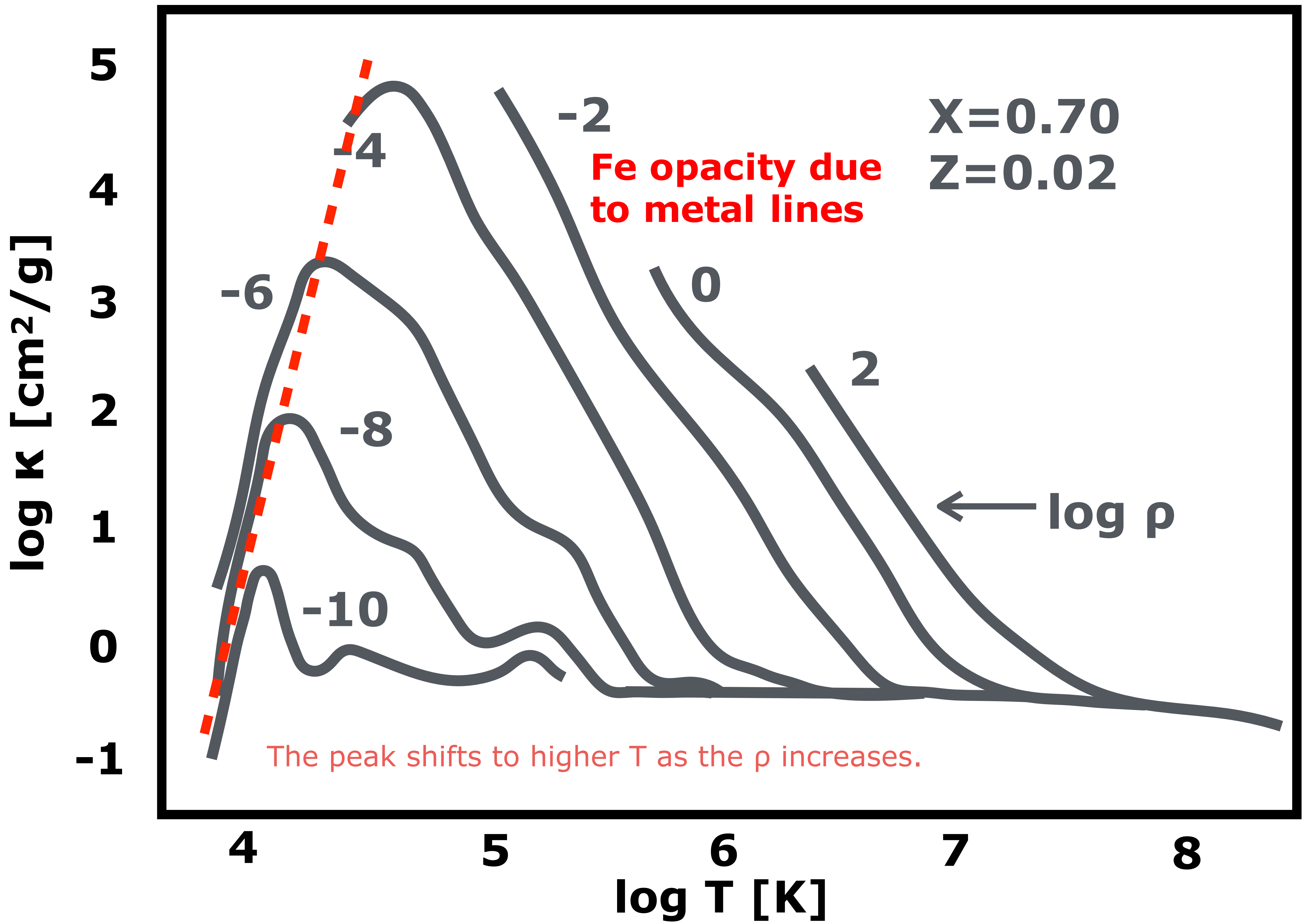
As T decreases, the bound–free and free–free absorption coefficient increases as $\kappa \sim \rho T^{-7/2}$

Sources of Opacity



The gas is partly ionized, resulting in many more e- transitions and huge κ.

Sources of Opacity

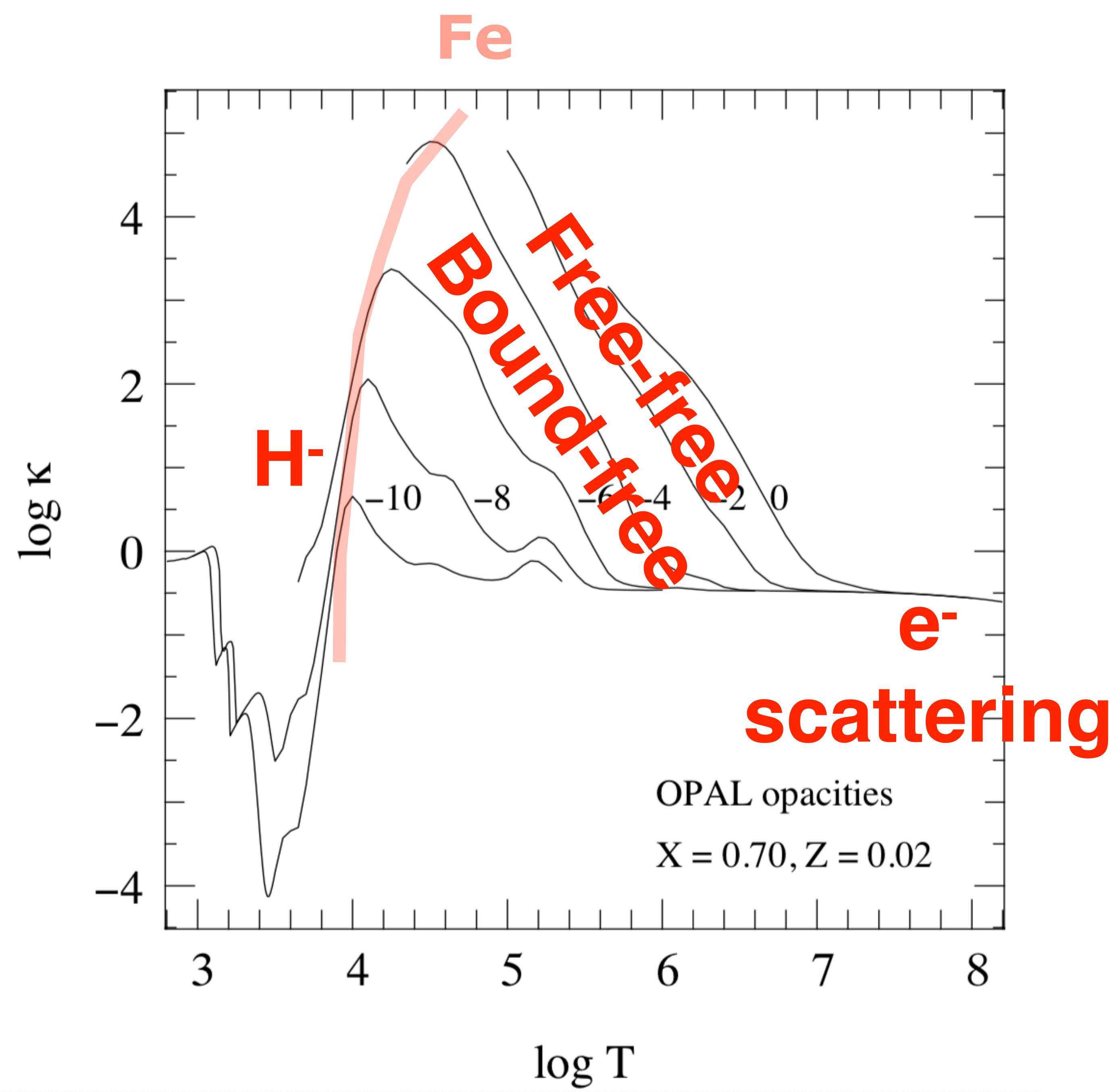


This produces the so-called Fe-opacity peak around 10^5 K for densities 10^{-6} to 10^{-4} g cm^{-3}

Opacity

- The peak at $10^4<T<10^5$ K at very low ρ (~10^{-10} to 10^{-8} g cm^{-3}) is due to H.

- At very low $T<10^4$ K, κ due to H− decreases steeply toward lower T as $\kappa(H^-)\sim T^9$.



The Eddington luminosity

Radiative transport requires a T gradient

$$\frac{\partial T}{\partial r} = -\frac{3\kappa\rho}{16\pi a c T^3}\frac{l}{r^2}$$

Since $P_{\rm rad} = \frac{1}{3}aT^4$, this also implies a Pressure gradient

$$\frac{dP_{\rm rad}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr} = -\frac{\kappa\rho}{4\pi c}\frac{l}{r^2}$$

this is an outward force that goes against gravitation in HE

$$\left(\frac{dP_{\rm rad}}{dr}\right) < \underbrace{\left(\frac{dP}{dr}\right)_{\rm HE}}_{\left(\frac{dP_{\rm rad}}{dr}\right)+\left(\frac{dP_{\rm gas}}{dr}\right)} = -\frac{Gm\rho}{r^2} \quad \rightarrow \quad \frac{\kappa\rho}{4\pi c}\frac{l}{r^2} < \frac{Gm\rho}{r^2} \quad \rightarrow \quad l < \frac{4\pi c G m}{\kappa} \equiv l_{\rm Edd}$$

Which is the maximum luminosity that can be carried out by radiation

Sometimes $l > l_{\rm Edd}$, which may result from intense nuclear burning or large κ (e.g. at very low T)

At that point energy cannot be transported by radiation, and to maintain HE the star needs another mechanism: *convection*

(collective motion of gas bubbles that carry heat)

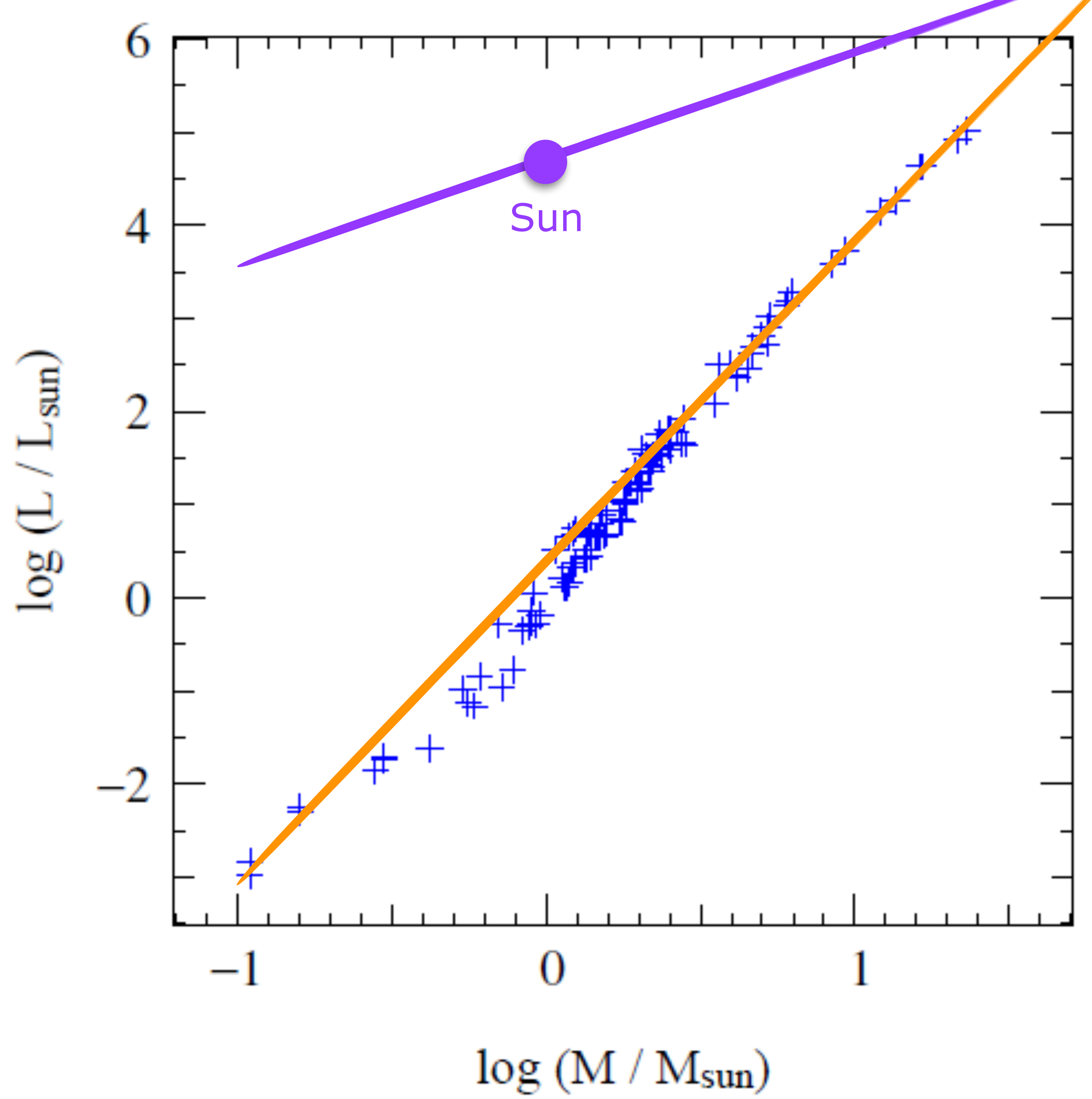
The Eddington luminosity

At the surface (m -> M) energy scapes, so the surface layer is always radiative. Assuming κ constant,

(so, breaking Ledd limit, means breaking HE)

e- scatter opacity for X=0.7 ~ solar

$$L_{\rm Edd} = \frac{4\pi c G M}{\kappa} = 3.8\times10^4\left(\frac{M}{M_\odot}\right)\left(\frac{0.34\ {\rm cm\ g^{-1}}}{\kappa}\right)L_\odot$$



For the most massive stars, only e- scatter contributes to κ

Observationally, $L \sim M^{3.8}$, so L at some point will exceed the Eddington limit

$$\log_{10}\left(\frac{L_{\rm Edd}}{L_\odot}\right) \approx 4.5 + \log_{10}\left(\frac{M}{M_\odot}\right)$$

$$M_{\rm max} \sim 100 M_\odot$$

Convective energy transfer

For diffusion to happen, a certain T gradient is needed

The larger the L, the larger the T gradient needed

If the limit T gradient is exceeded, it leads to an instability in the gas, that produces cyclical motions of the gas

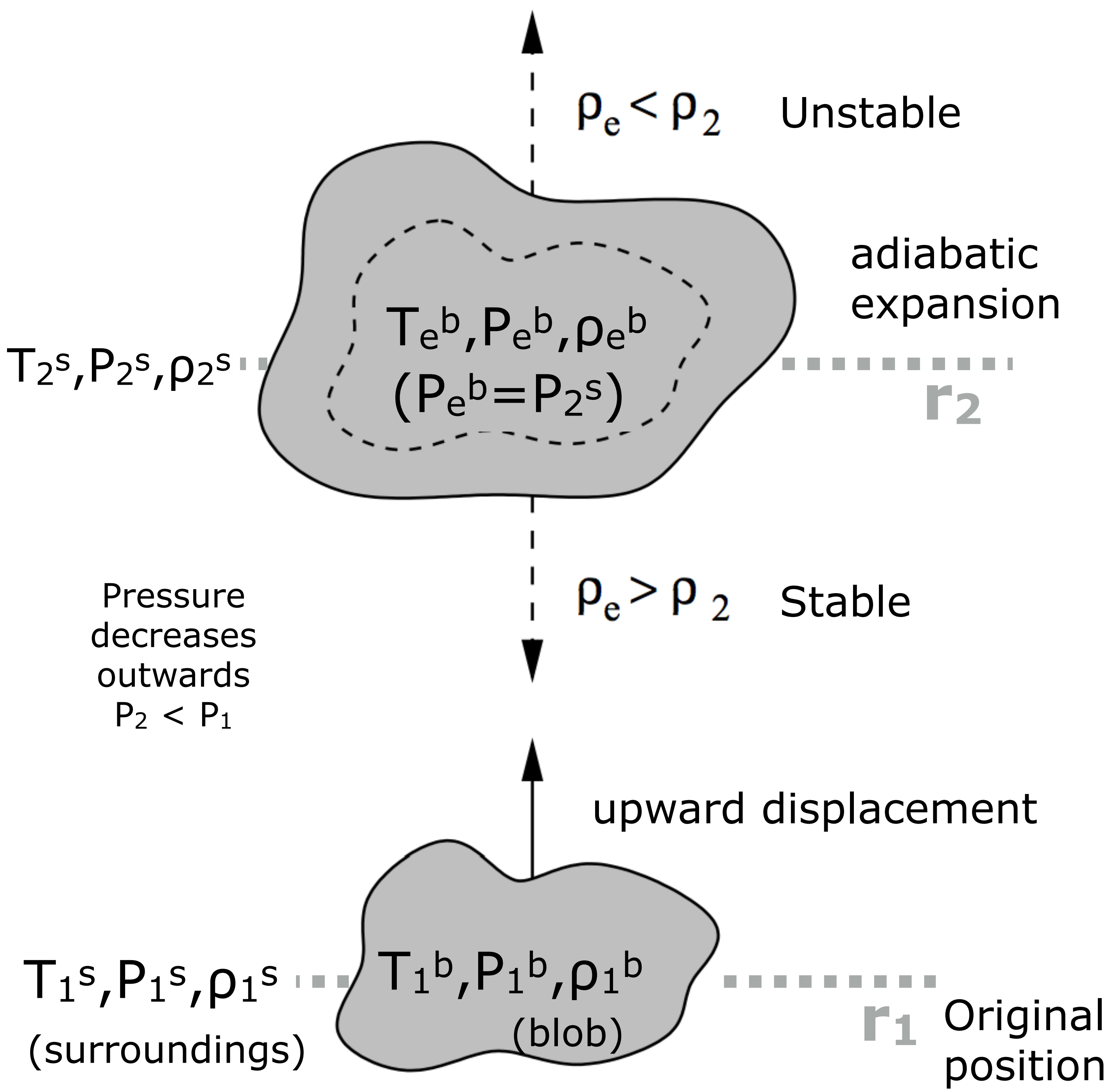
Convection is a dynamical instability that do not break HE.

(no disruptive consequences)

Convection affects the structure of a star only as an efficient means of ***heat transport*** and as an efficient ***mixing mechanism***.

Criteria for stability against convection

Consider a blob of gas in a star that due to small perturbations starts moving upwards. (by thermal motions of gas particles)



At r_2, and at lower pressure P_2, the blob will expand: $P_e = P_2$, but not necessarily $\rho_e = \rho_2$

If $\rho_e > \rho_2$, we have a stable situation. It will go back down

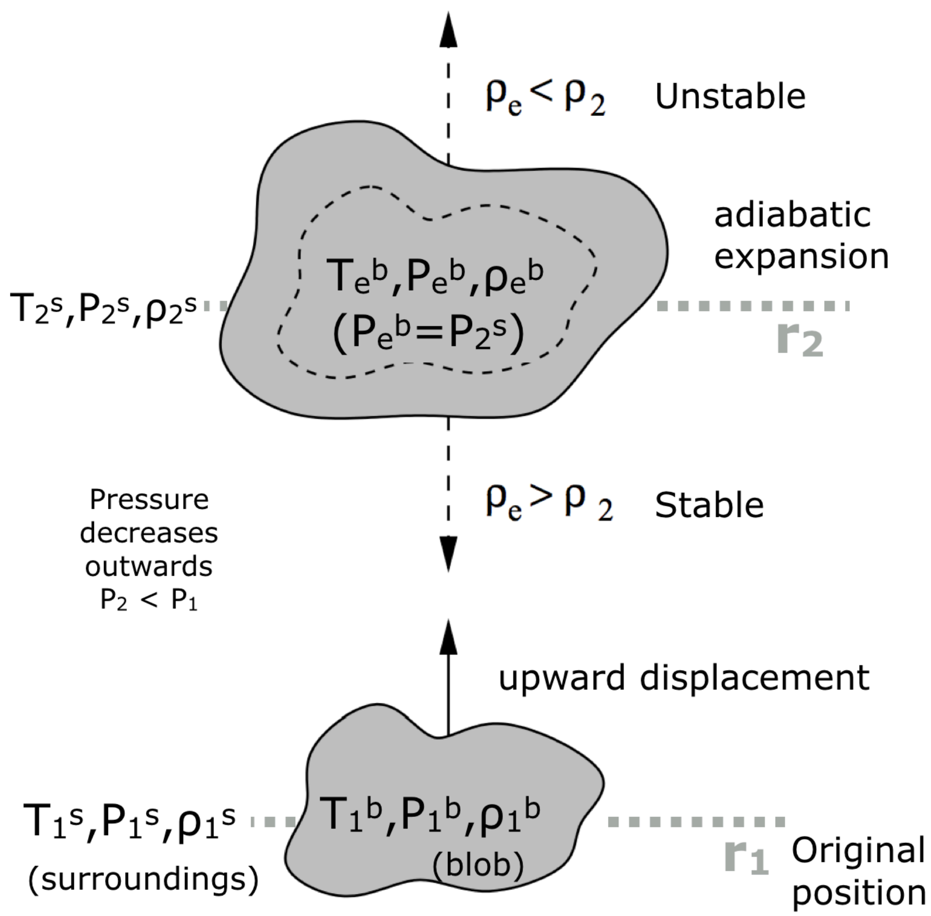
If $\rho_e < \rho_2$, it will keep going up. We have an unstable situation that leads to convection.

The gas expansion occurs on the local $\tau_{dyn} \ll \tau_{KH}$ (heat exchange), so it is an adiabatic process

$$\gamma_{\rm ad} = \frac{d \log P}{d \log \rho} \quad \longrightarrow \quad \frac{\delta P_e}{P_e} = \gamma_{\rm ad} \frac{\delta \rho_e}{\rho_e}$$

Criteria for stability against convection

where the difference in pressure is described by the pressure gradient



$$\frac{\delta P_e}{P_e} = \gamma_{\rm ad}\frac{\delta \rho_e}{\rho_e}$$

$$P_e = P_2$$

$$\delta P_e = P_2 - P_1 = \frac{dP}{dr}\Delta r$$

$$\longrightarrow \quad \delta\rho_e = \frac{\rho_e}{P_e}\frac{1}{\gamma_{\rm ad}}\frac{dP}{dr}\Delta r$$

writing $\rho_e = \rho_1 + \delta\rho_e \qquad \rho_2 = \rho_1 + (d\rho/dr)\Delta r$

we can express the stability criterion

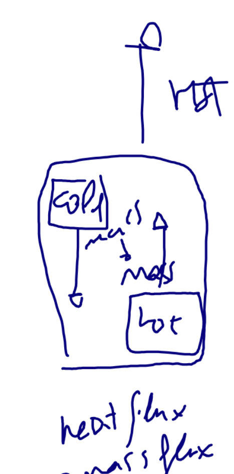
$\rho_e > \rho_2$ as, $\delta\rho_e > \dfrac{d\rho}{dr}\Delta r$

which combined with above expression yields a limit to the density gradient for stability

$$\frac{1}{\rho}\frac{d\rho}{dr} < \frac{1}{P}\frac{dP}{dr}\frac{1}{\gamma_{\rm ad}} \quad \rightarrow \quad \frac{d\log\rho}{d\log P} > \frac{1}{\gamma_{\rm ad}}$$

The general criterion for stability against convection

If violated, convection motions will develop $\rightarrow$



Criteria for stability against convection

(not part of the stellar structure equations)
The stability criterion involves the calculation of a density gradient

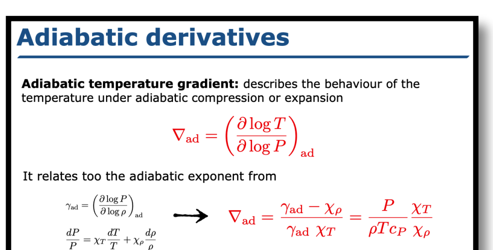
(Because it is also in the eq radiative transfer)
We can rewrite the criterion using EOS to get a T gradient

$$\frac{dP}{P} = \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho} + \chi_\mu \frac{d\mu}{\mu} \qquad \text{where} \qquad \chi_i = \left(\frac{\partial \log P}{\partial \log i}\right)_{j,k}$$

In general, we can rearrange

$$\frac{\partial \log \rho}{\partial \log P} = \frac{1}{\chi_\rho}\left(1 - \chi_T \frac{\partial \log T}{\partial \log P} - \chi_\mu \frac{\partial \log \mu}{\partial \log P}\right) = \frac{1}{\chi_\rho}\left(1 - \chi_T \nabla - \chi_\mu \nabla_\mu\right)$$

In the displaced gas element, composition does not change, so from adiabatic derivatives,



Adiabatic derivatives

Adiabatic temperature gradient: describes the behaviour of the temperature under adiabatic compression or expansion

$$\nabla_{\rm ad} = \left(\frac{\partial \log T}{\partial \log P}\right)_{\rm ad}$$

It relates too the adiabatic exponent from

$$\gamma_{\rm ad} = \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\rm ad} \qquad \frac{dP}{P} = \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho} \longrightarrow \nabla_{\rm ad} = \frac{\gamma_{\rm ad} - \chi_\rho}{\gamma_{\rm ad}\ \chi_T} = \frac{P}{\rho T c_P}\frac{\chi_T}{\chi_\rho}$$

$$\nabla_{\rm ad} = \frac{\gamma_{\rm ad} - \chi_\rho}{\gamma_{\rm ad}\chi_T} \rightarrow \frac{1}{\gamma_{\rm ad}} = \frac{1}{\chi_\rho}(1 - \chi_T \nabla_{\rm ad})$$

And the stability criterion $\frac{d\log\rho}{d\log P} > \frac{1}{\gamma_{\rm ad}}$ becomes

$$\nabla < \nabla_{\rm ad} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

Criteria for stability against convection

If all energy is transported by radiation $\nabla = \nabla_{\rm rad}$

$$\nabla_{\rm rad} < \nabla_{\rm ad} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$$

Ledoux criterion
(a layer is stable against convection if…)

For an ideal gas χ_μ = -1 and χ_T = 1, so $\nabla_{\rm rad} < \nabla_{\rm ad} + \nabla_\mu$

$\nabla_\mu > 0$, so it contributes to stability

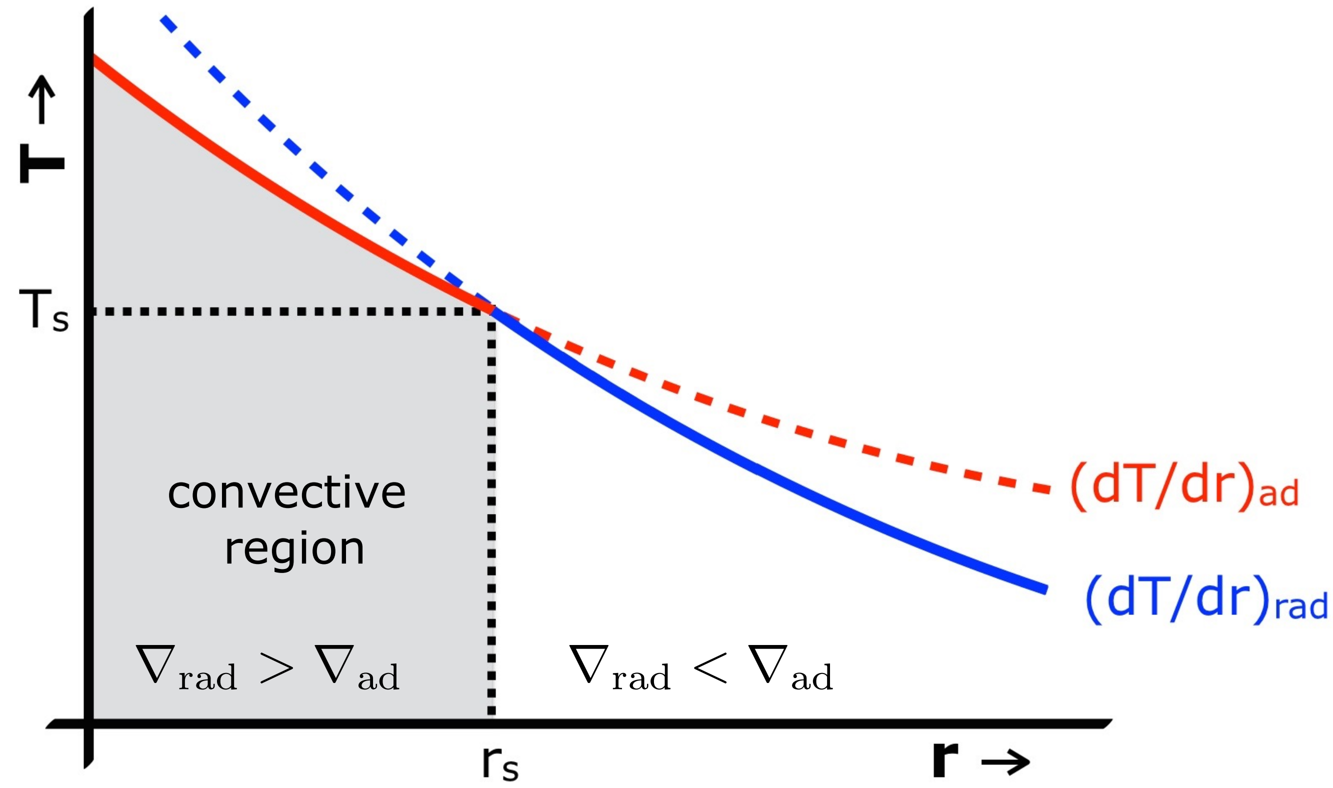
In chemically homogenous layers $\nabla_\mu = 0$

$$\nabla_{\rm rad} < \nabla_{\rm ad}$$

Schwarzschild criterion

So we expect convection to occur when

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$



Schwarzschild criterion

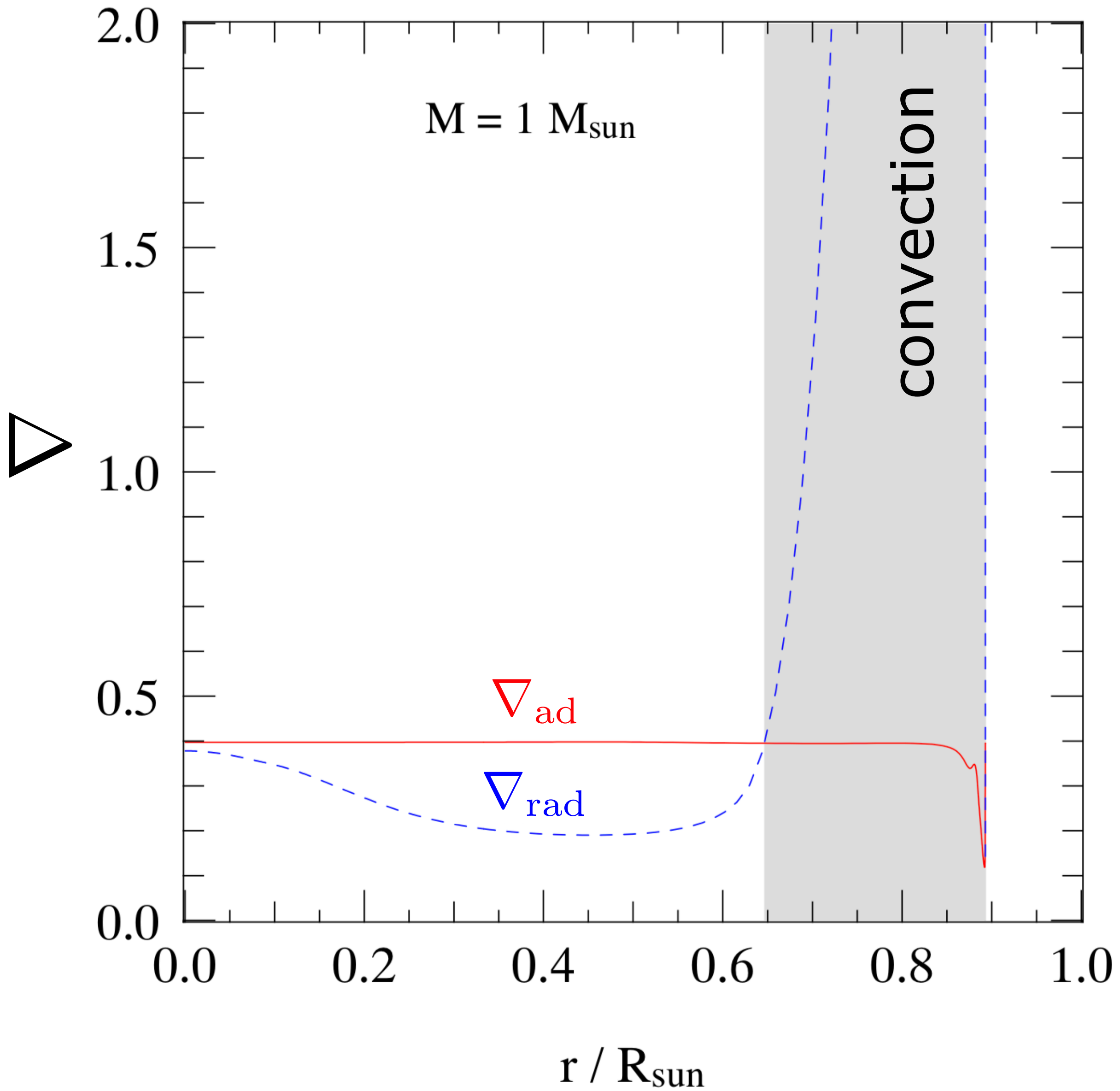
convection requires:

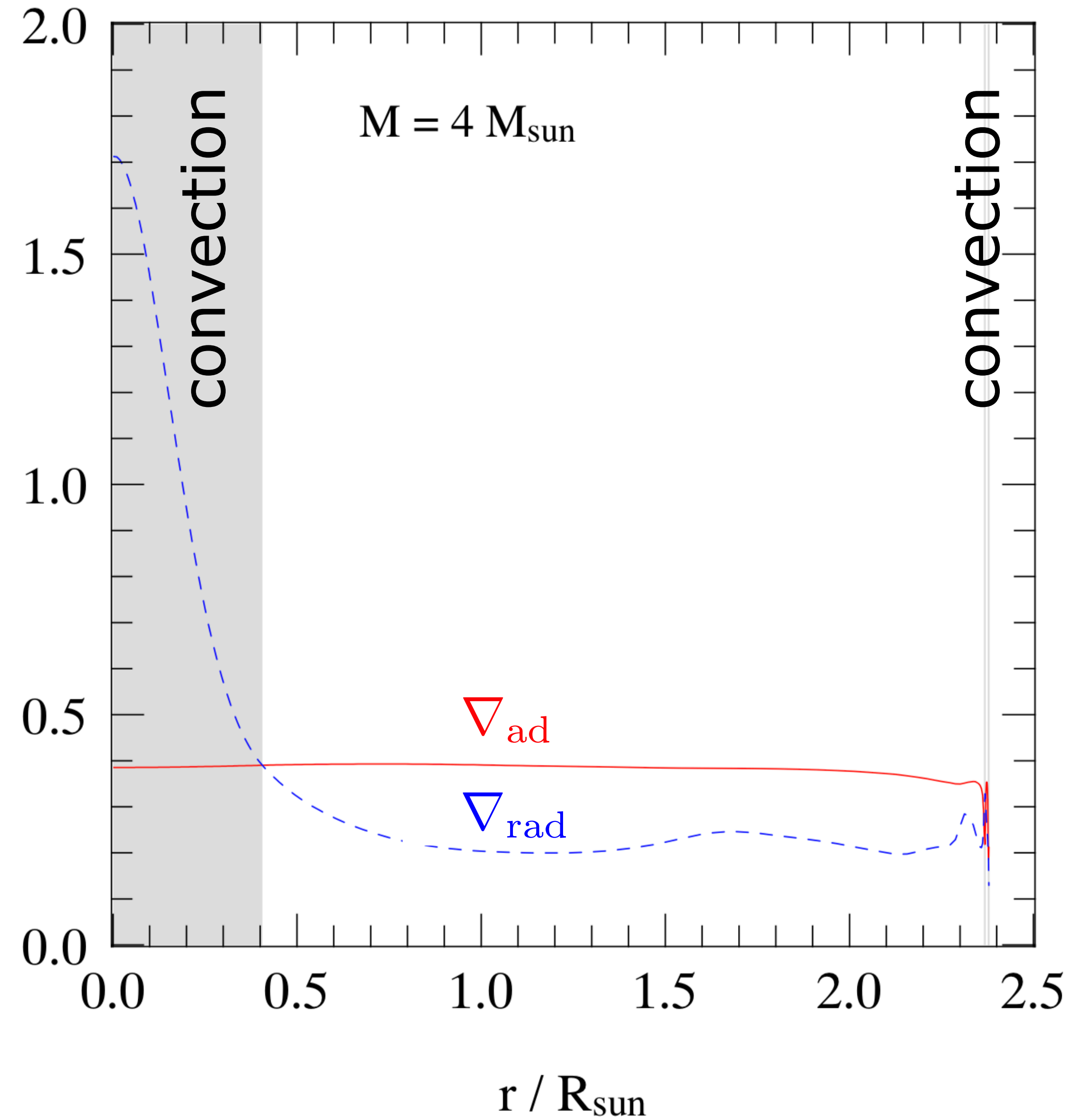
$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

Large κ opacity increases with decreasing T. low-mass stars to have convective envelopes

Large $\frac{l}{m}$ towards the centre of a star $l/m \approx \epsilon_{\rm nuc}$ massive stars will have convective cores.

Small $\nabla_{\rm ad}$ occurs in not fully ionized zones at low T. Even if $\boldsymbol{\kappa}$ is not large, surface layers may be unstable to convection.



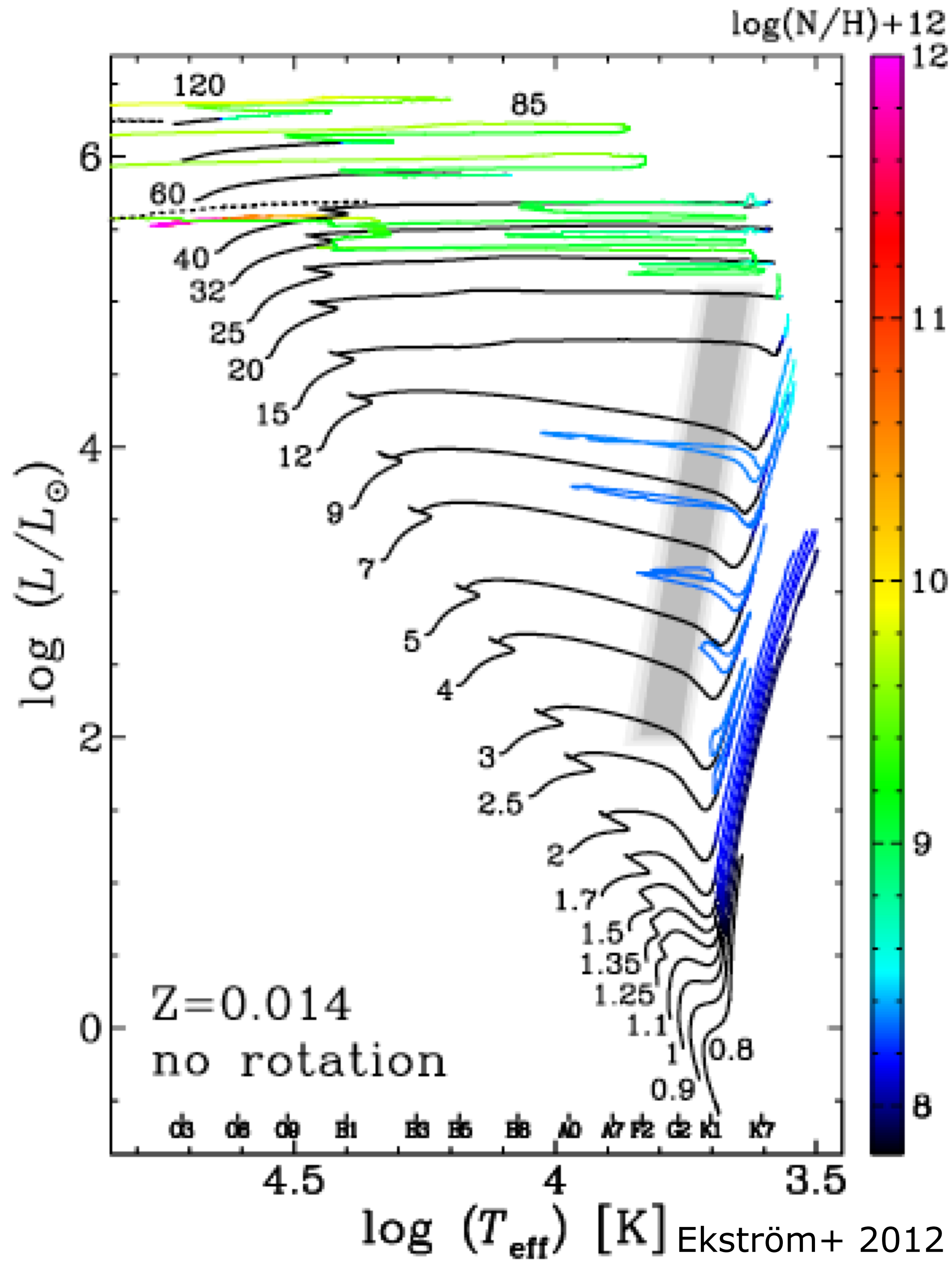


Convection in Stars

<u>Where does convection actually happen?</u>

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

1. Large κ and low T



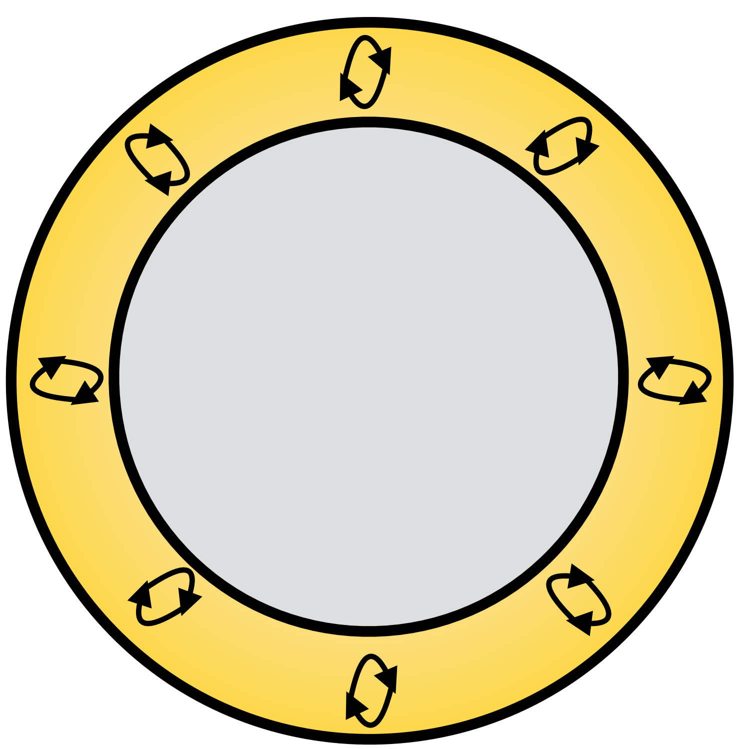
Ekström+ 2012

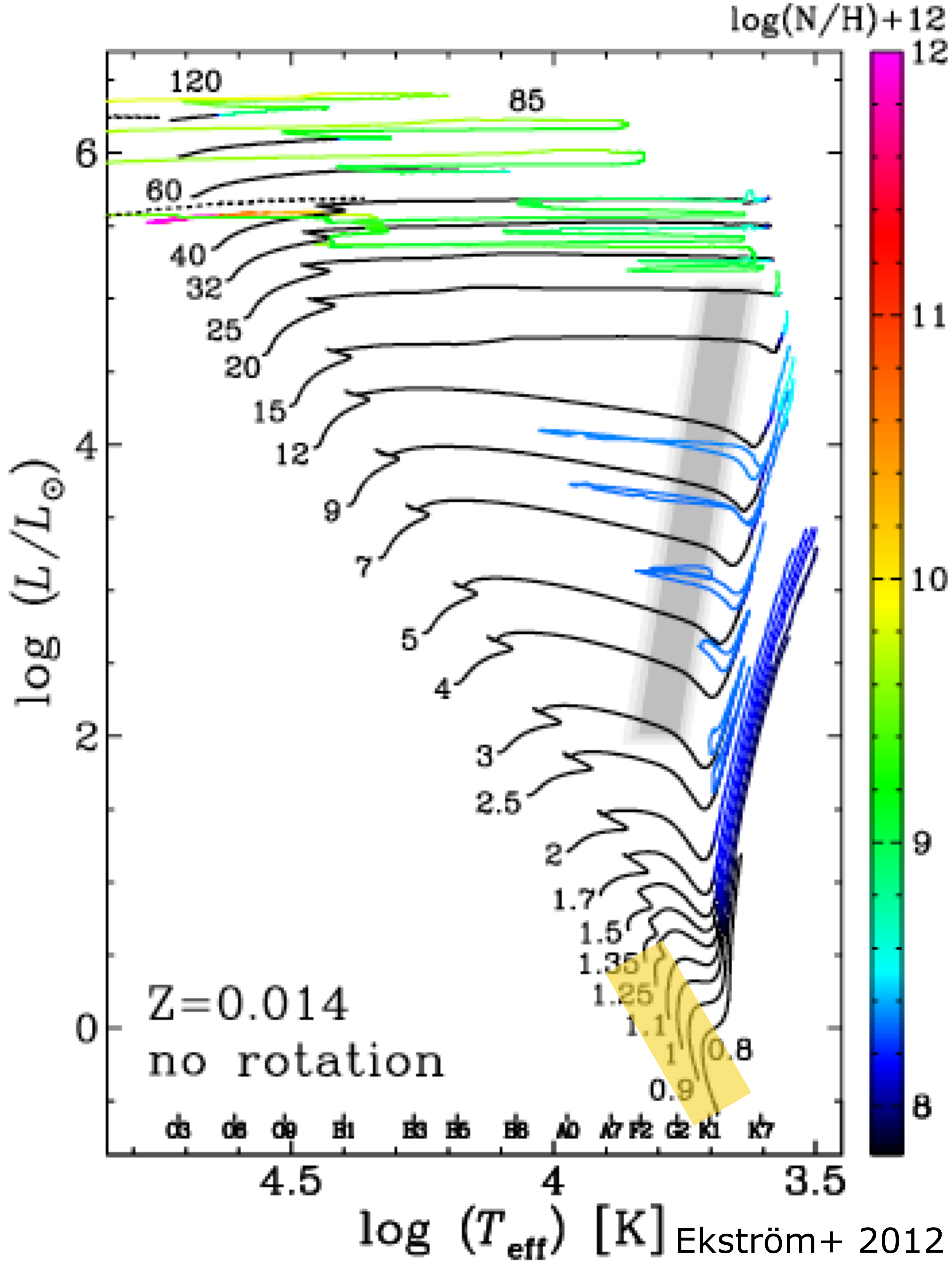
Convection in Stars

<u>Where does convection actually happen?</u>

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

1. Large κ and low T
   —outer layers of cool stars





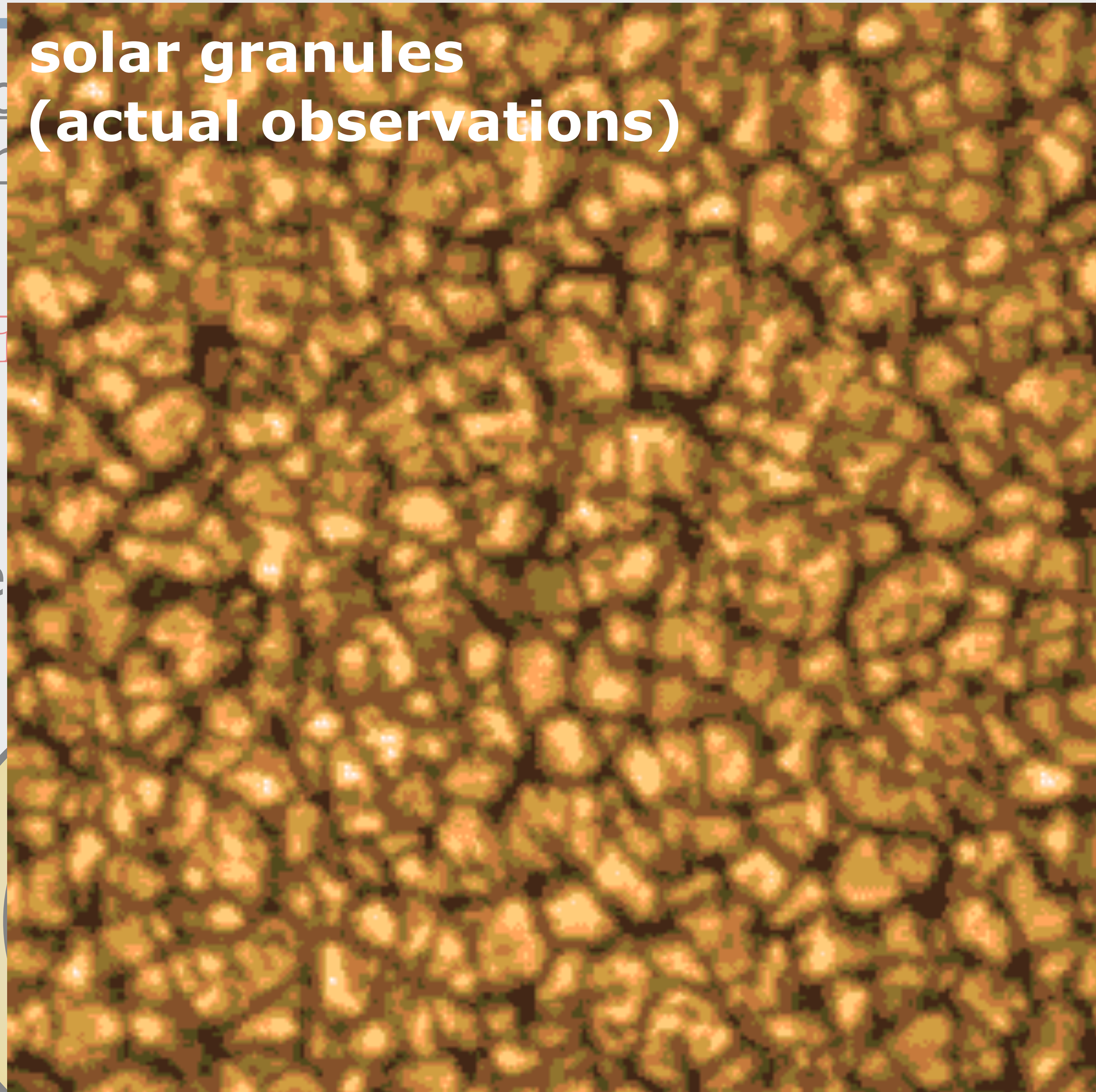
Ekström+ 2012

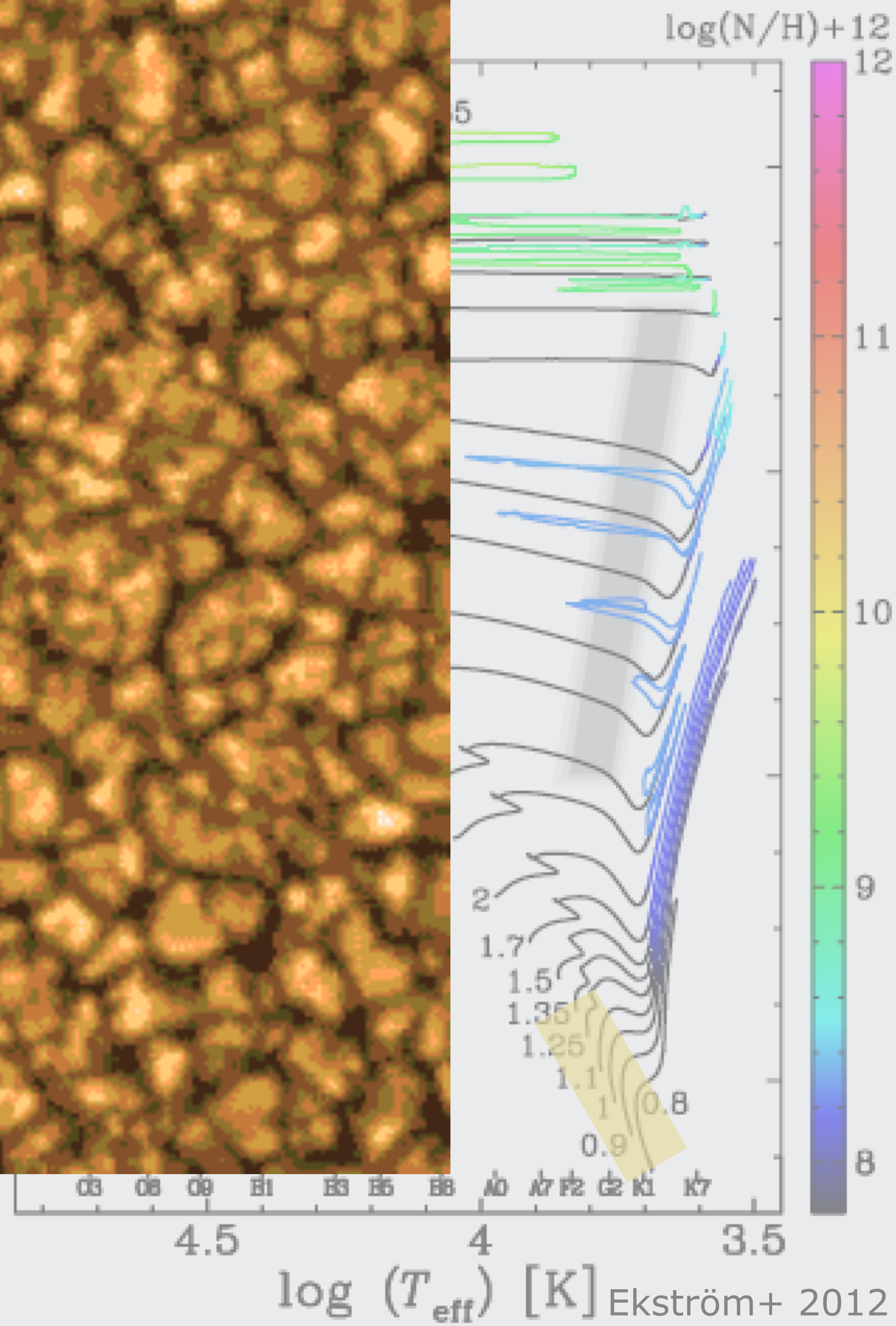
Convection in Stars

Where do
actually h

$\nabla_{\mathrm{rad}} =$

1. Large
—oute





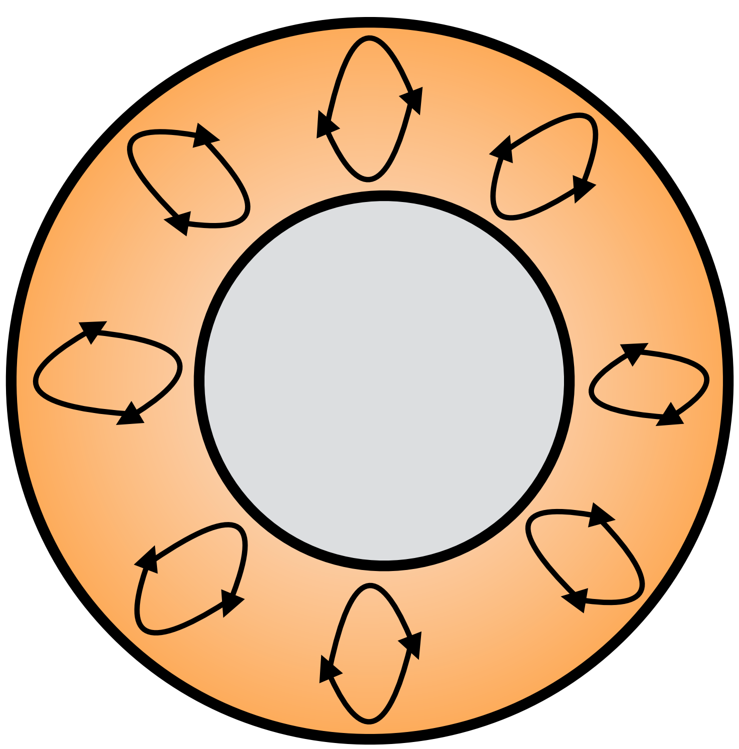
Ekström+ 2012

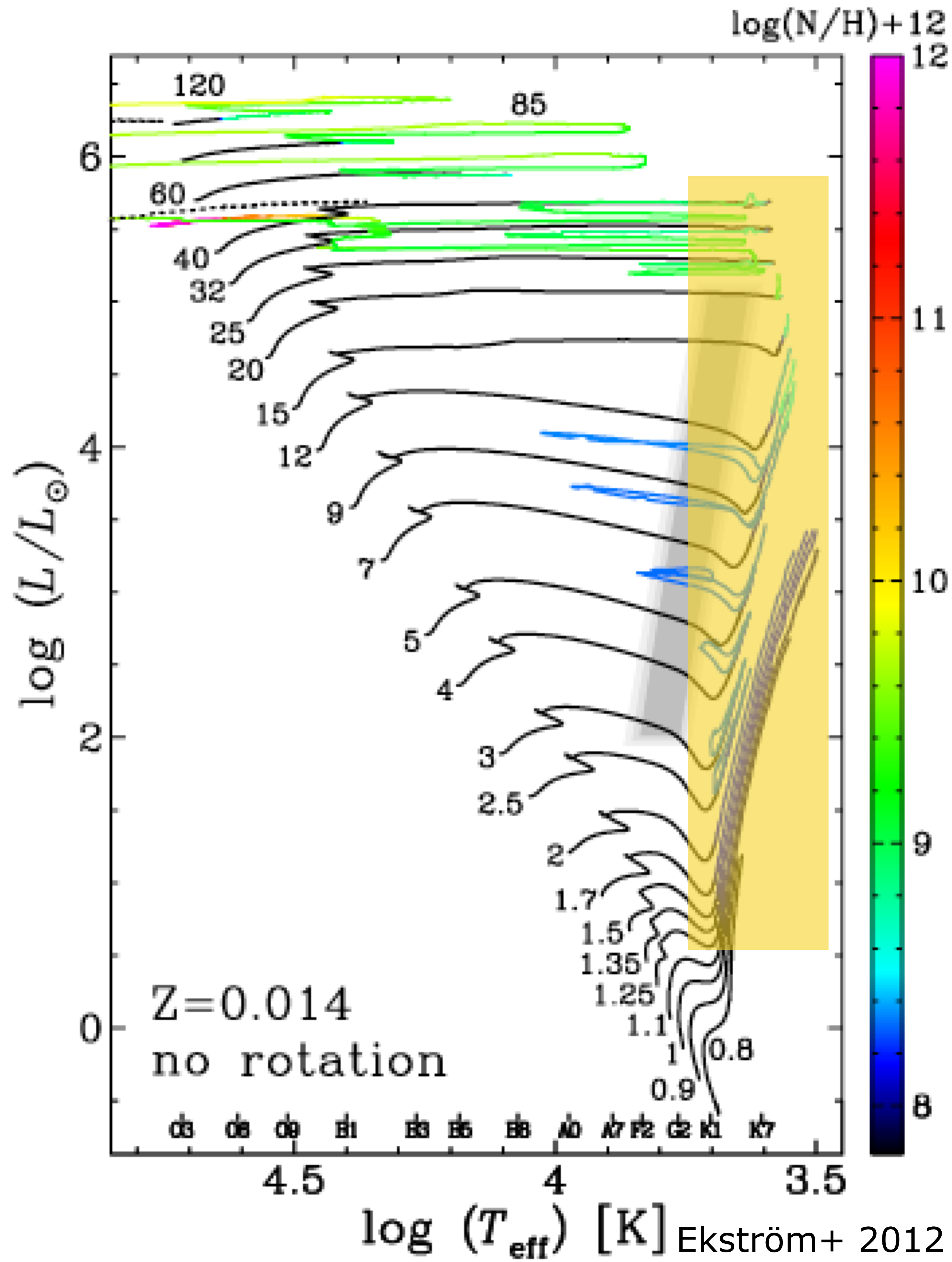
Convection in Stars

<u>Where does convection actually happen?</u>

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

1. Large κ and low T
   —outer layers of cool stars





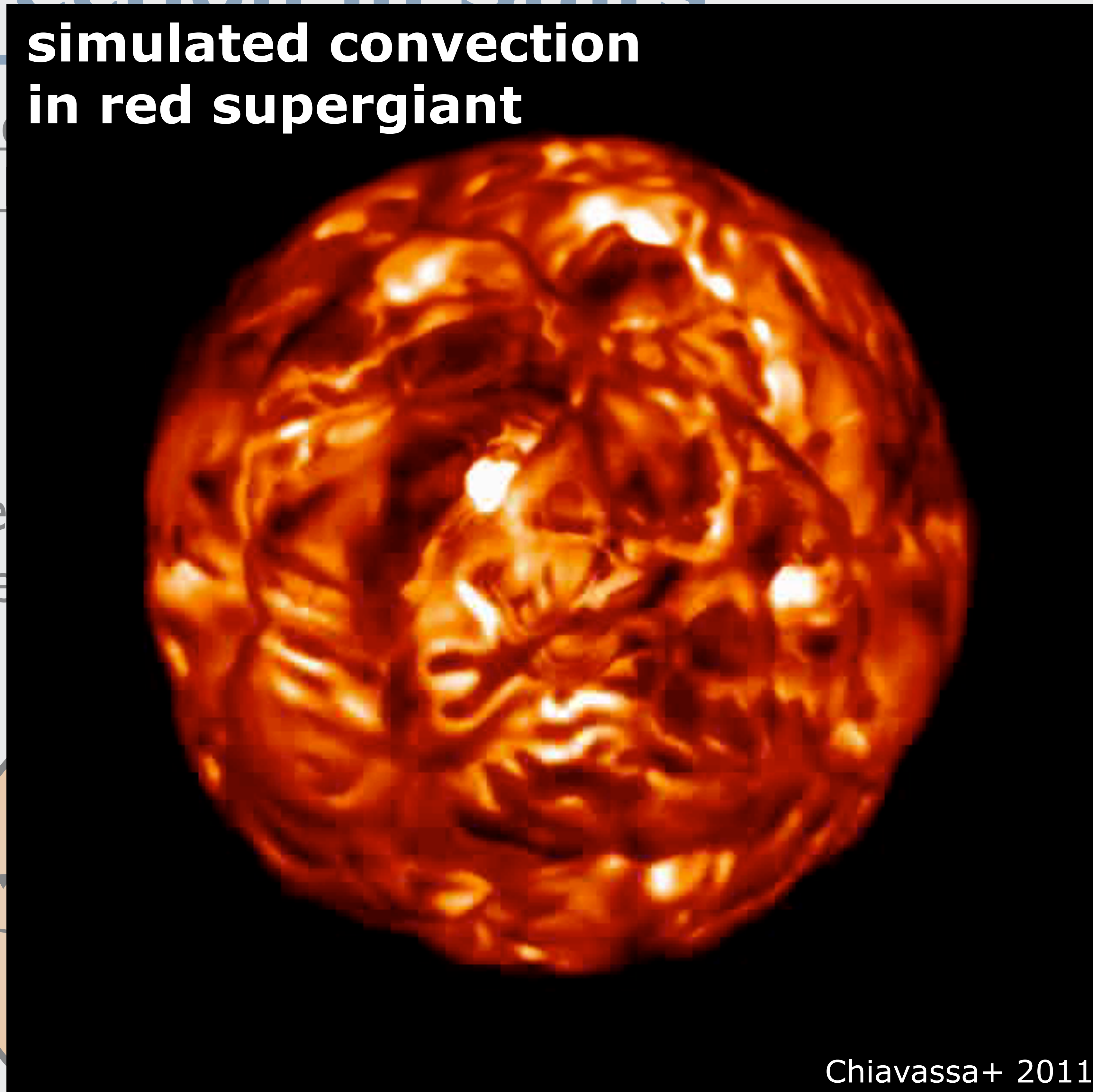
Ekström+ 2012

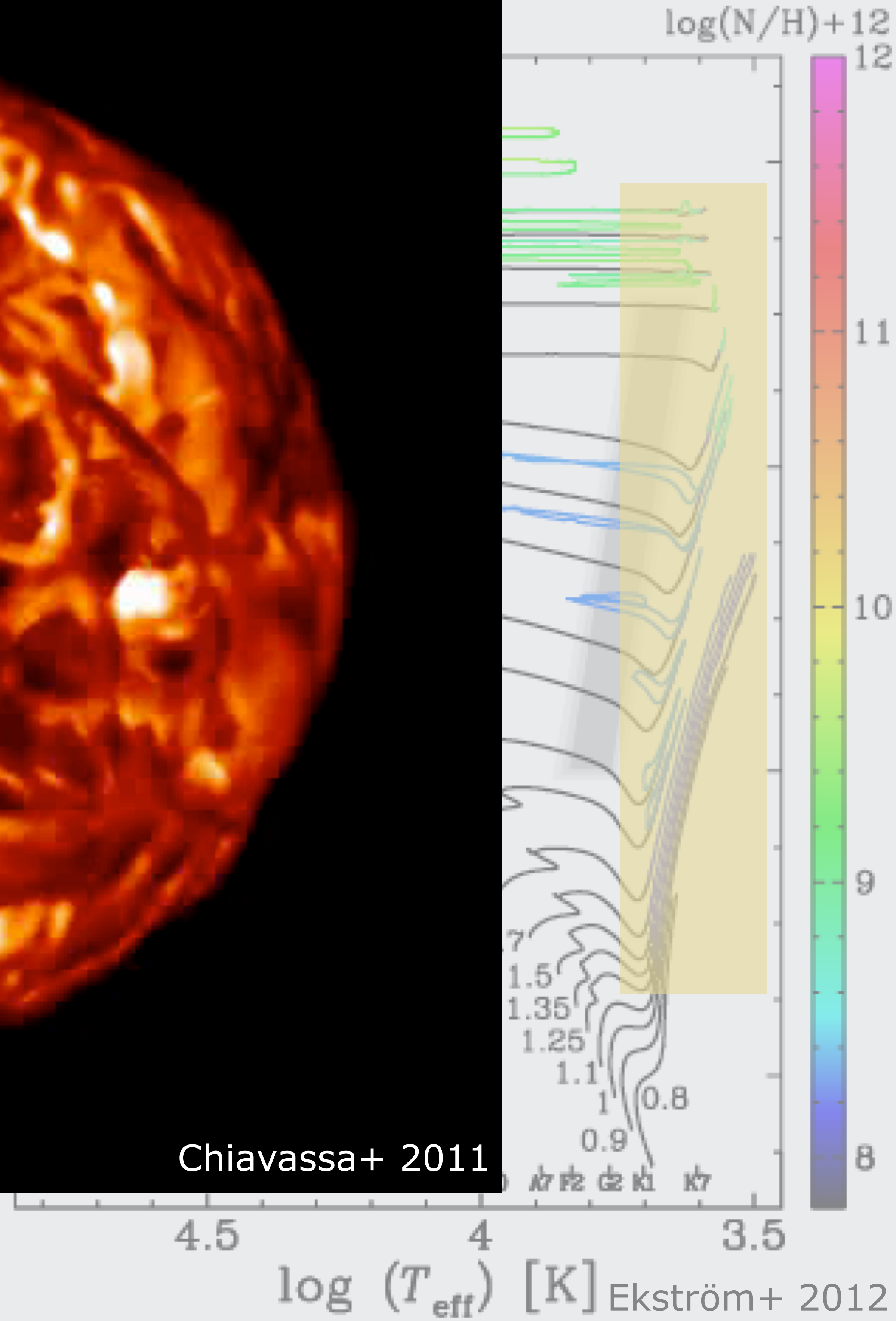
Convection in Stars

Where d
actually

$\nabla_{\rm rad} =$

1. Large
—oute





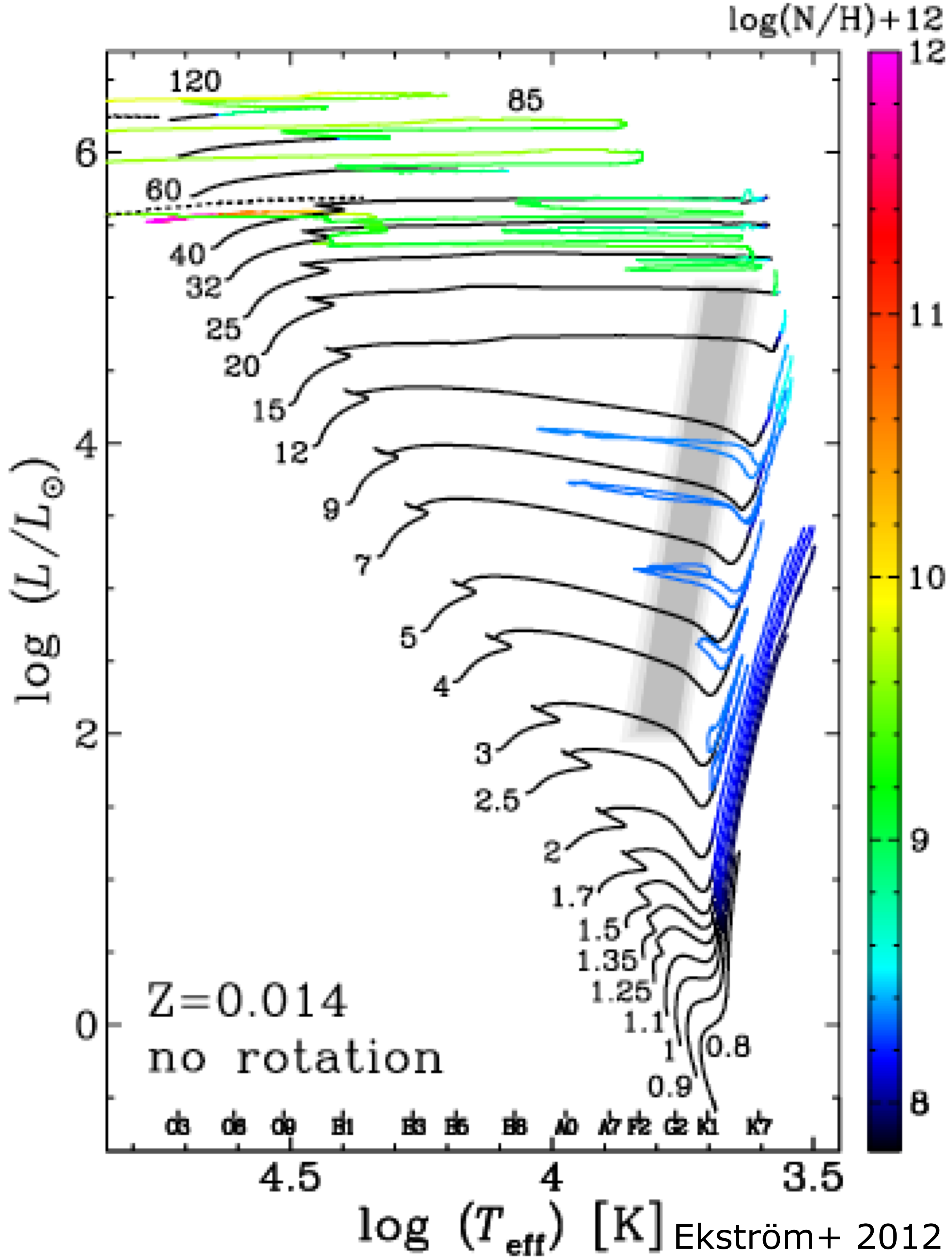
Ekström+ 2012

Convection in Stars

Where does convection actually happen?

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

1. Large κ and low T
   —outer layers of cool stars
2. Large $L_r/4\pi r^2$



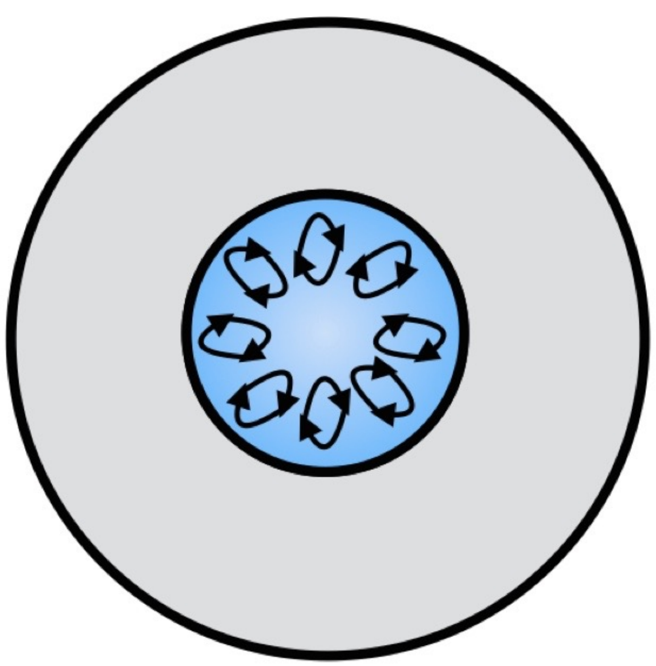
Ekström+ 2012

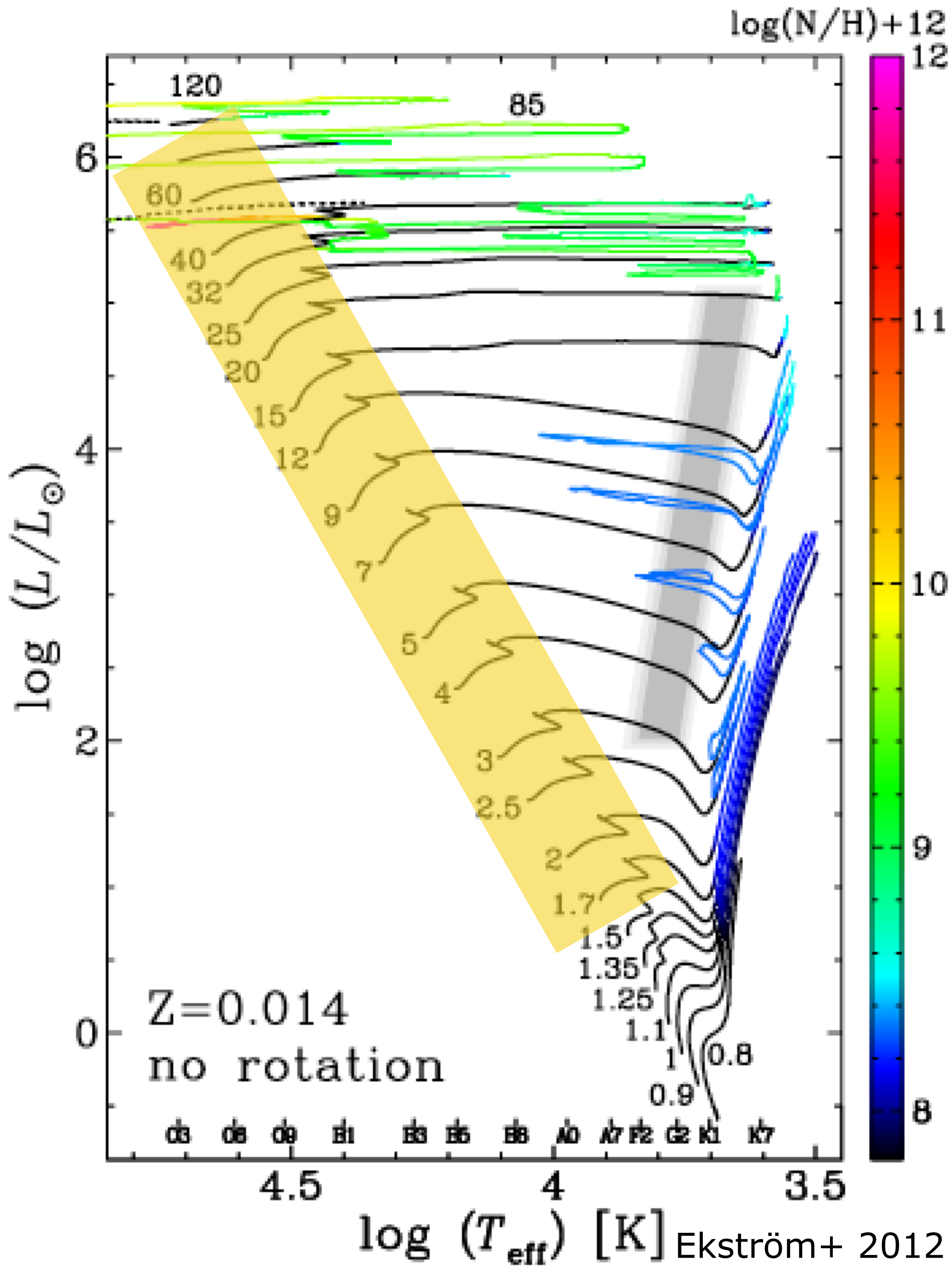
Convection in Stars

<u>Where does convection actually happen?</u>

$$\nabla_{\rm rad} = \frac{3}{16\pi a c G}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}$$

1. Large κ and low T
   —outer layers of cool stars
2. Large $L_r/4\pi r^2$
   —cores of massive stars





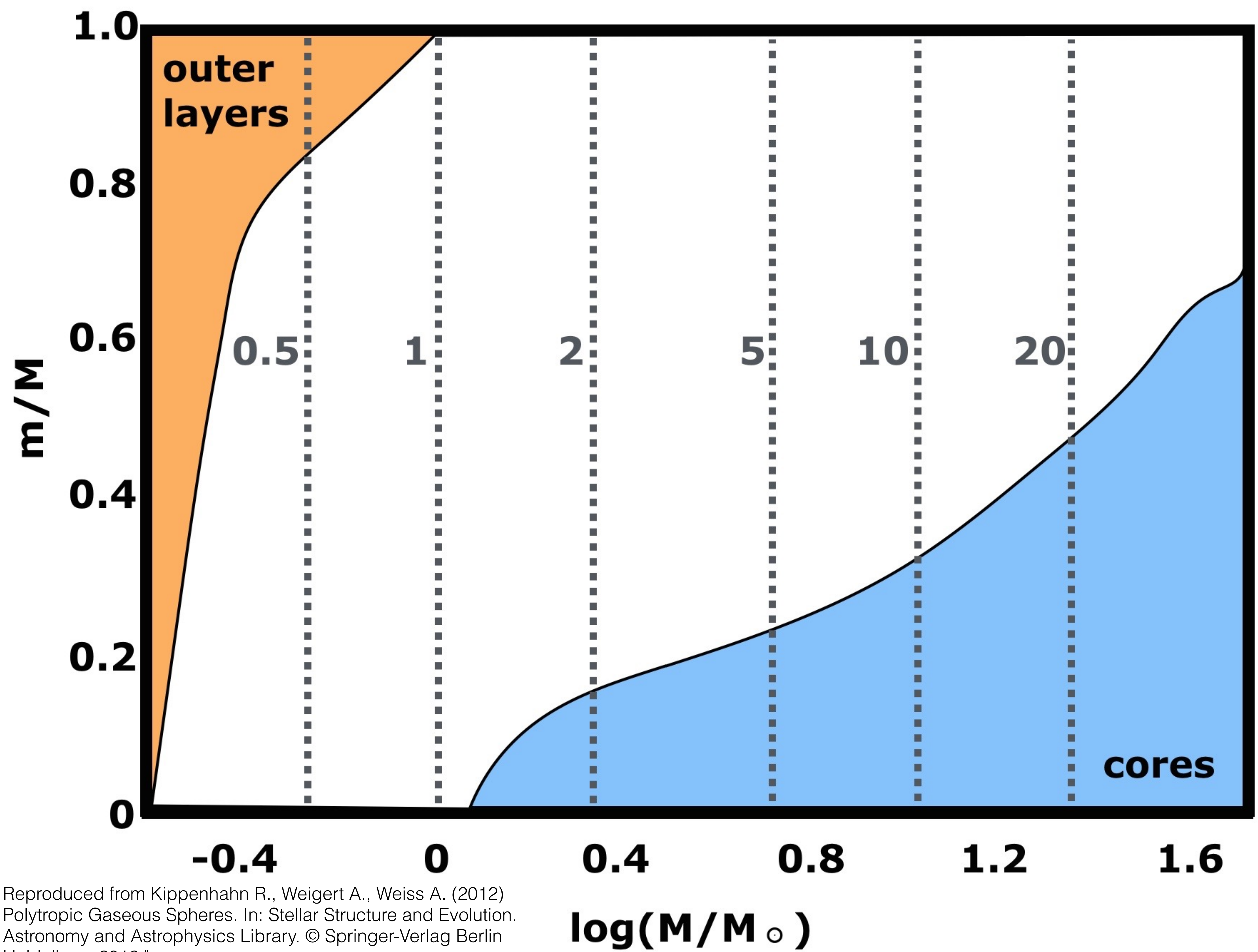
Ekström+ 2012

Convection in Stars

Whe
actu

1. L

2. L



Reproduced from Kippenhahn R., Weigert A., Weiss A. (2012) Polytropic Gaseous Spheres. In: Stellar Structure and Evolution. Astronomy and Astrophysics Library. © Springer-Verlag Berlin Heidelberg 2012."

log (T_{eff}) [K] Ekström+ 2012

Convective Energy Transport

(how much can be transferred, T gradient needed,...)

To address in detail how convective E transport works ($F_{\rm conv}$) we need detailed and complicated 3D models

However, we can get an approximate results with the 1D *mixing length theory (MLT)*

In *MLT* convective motions are approximated to blobs that move a distance l_m, before they dissolve and transfer heat

l_m can be approx. to

$$l_m \sim H_P = \frac{\Re T}{\mu}\frac{R^2}{GM} = \frac{\Re T}{\mu g} = \frac{P}{\rho g}$$

(local P scale height: radial distance over which P changes by an *e*-folding factor)

In HE, $P(r+h) \sim P(r)e^{-h/H_P}$; $\frac{dP}{dr} = -\rho\frac{GM}{r^2}$; $\frac{d\ln P}{dr} = -\frac{GM}{r^2}\frac{\mu}{\Re T}$

After rising l_m, the T difference between blob and media is

$$\Delta T = T_e - T_s = \left[\left(\frac{dT}{dr}\right)_e - \frac{dT}{dr}\right] l_m$$

e: gas element s: surroundings

Convective Energy Flux

e: gas element s: surroundings

$$\Delta T = T_e - T_s = \left[\left(\frac{dT}{dr}\right)_e - \frac{dT}{dr}\right] l_m$$

T gradient in the surroundings

Due to the T variation the gas experiences as it raises and expands adiabatically

We can write ΔT in terms of ∇ and $\nabla_{\rm ad}$

$$\frac{dT}{dr} = T\frac{d\ln T}{dr} = T\underbrace{\frac{d\ln T}{d\ln P}}_{\nabla}\underbrace{\frac{d\ln P}{dr}}_{-1/H_P} = -\frac{T}{H_P}\nabla$$

$$\left(\frac{dT}{dr}\right)_e = -\frac{T}{H_P}\nabla_{\rm ad}$$

$$\Delta T = T\frac{l_m}{H_P}\left(\nabla - \nabla_{\rm ad}\right)$$

And the energy flux carried by the convection gas

$$F_{\rm conv} = v_c \rho \Delta u = v_c \rho c_P \Delta T$$

we therefore need a way to calculate the convective velocity

buoyancy force by a $\Delta\rho$

$$a = -g\frac{\Delta\rho}{\rho} \sim g\frac{\Delta T}{T}$$

$$P \propto \rho T$$

$$\Delta P = 0$$

for an ideal gas

Accelerated over a distance $l_m = \frac{1}{2}at^2$

$$v_c \approx l_m/t = \sqrt{\frac{1}{2}l_m a} \approx \sqrt{\frac{1}{2}l_m g\frac{\Delta T}{T}} \approx \sqrt{\frac{l_m^2 g}{2H_P}\left(\nabla - \nabla_{\rm ad}\right)}$$

Convective Energy Flux

Combined with the previous expression

$$F_{\rm conv} = \rho c_P T \left(\frac{l_m}{H_P}\right)^2 \sqrt{\frac{1}{2} g H_P} \underline{(\nabla - \nabla_{\rm ad})}^{3/2}$$

Where is *superadiabaticity*

the degree to which the actual temperature gradient ∇ exceeds the adiabatic

The superadiabaticity needed to carry the whole Flux by convection

$F_{\rm conv} = \frac{l}{4\pi r^2} \sim \frac{L}{R^2}$ for an ideal gas $\rho \approx \bar{\rho} = \frac{3M}{4\pi R^3}$ $T \approx \bar{T} \sim \frac{\mu}{\Re} \frac{GM}{R}$ $c_P = \frac{5}{2} \frac{\Re}{\mu}$ $\sqrt{g H_P} = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{\Re}{\mu} T} \sim \sqrt{\frac{GM}{R}}$

$$F_{\rm conv} \sim \frac{M}{R^3} \left(\frac{GM}{R}\right)^{3/2} (\nabla - \nabla_{\rm ad})^{3/2} \quad \rightarrow \quad \nabla - \nabla_{\rm ad} \sim \left(\frac{LR}{M}\right)^{2/3} \frac{R}{GM}$$

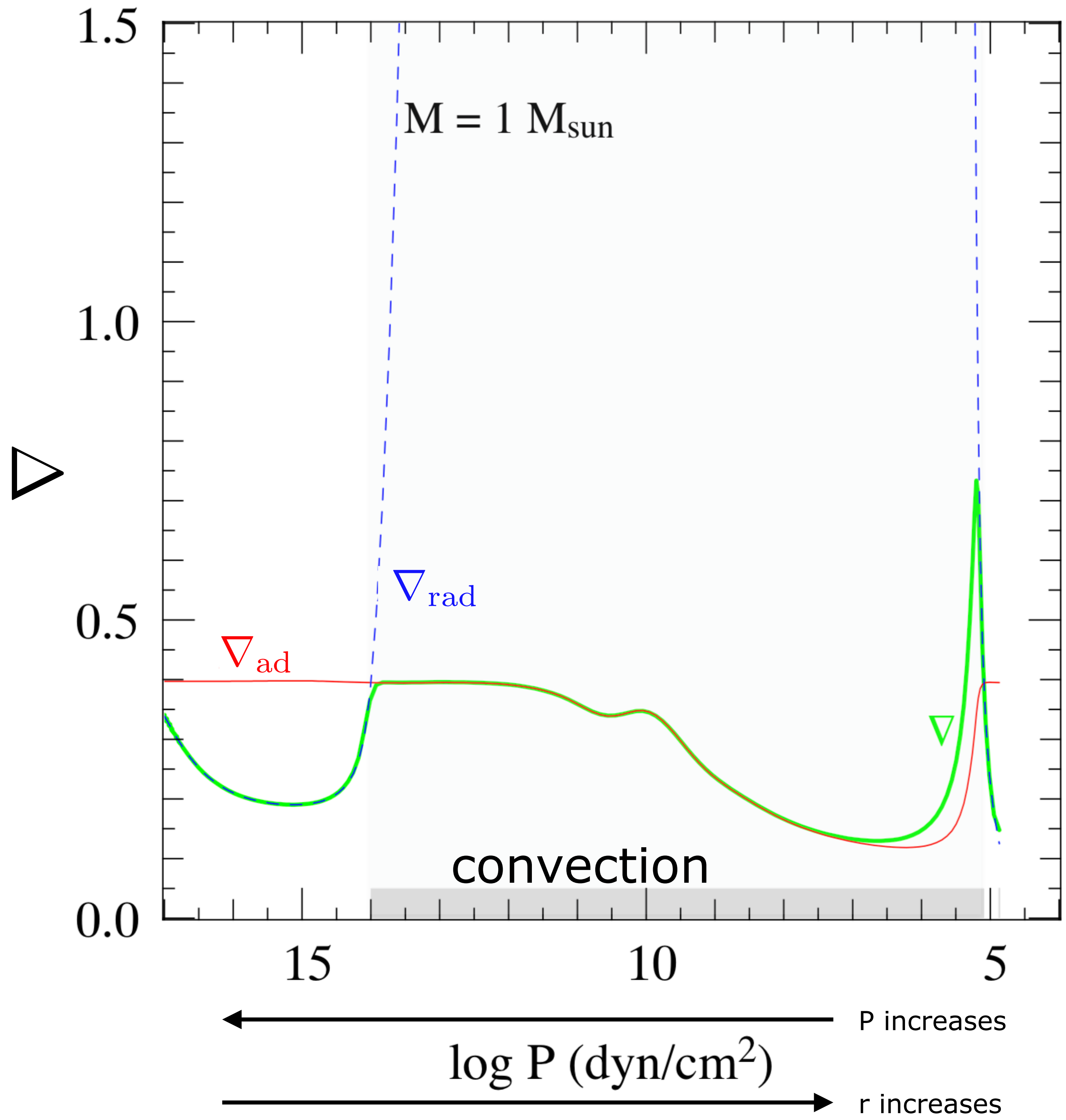
For the Sun, $\nabla - \nabla_{\rm ad} \sim 10^{-8}$, a small difference translates in a huge amount of energy transfer. *Convection is highly efficient.*

$F_{\rm conv} \gg F_{\rm rad}$ in convective regions

Convective Energy Flux

In a convective zone $\nabla \sim \nabla_{\rm ad}$ therefore $$\frac{dT}{dm} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla$$

eq. of structure in convection



However, at the surface

$$\rho \ll \bar{\rho} \qquad T \ll \bar{T}$$

And $\nabla > \nabla_{\rm ad}$, so convection becomes inefficient, so

$$F_{\rm conv} \ll F_{\rm rad}$$

And $\nabla \approx \nabla_{\rm rad}$

Convection mixing

Convection is also an efficient mixing mechanism

From the expression of v_c and approximating

$$v_c \approx \sqrt{\frac{l_m^2 g}{2H_P}(\nabla - \nabla_{\rm ad})} \approx v_s\sqrt{(\nabla - \nabla_{\rm ad})}$$

$$\sqrt{gH_P} \approx v_s$$
$$l_m \sim H_P$$

So in general, $v_c \ll v_s$

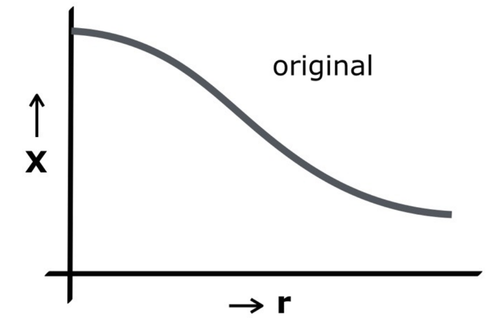
(e.g. for the Sun $v_c \sim 5\ 10^3$ cm/s)

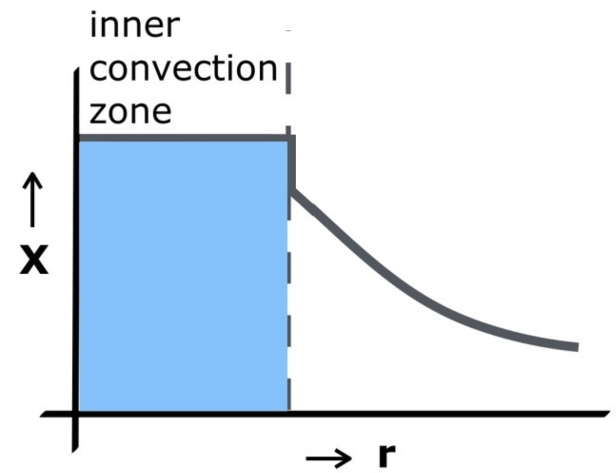
but high enough to mix different layers in a timescale

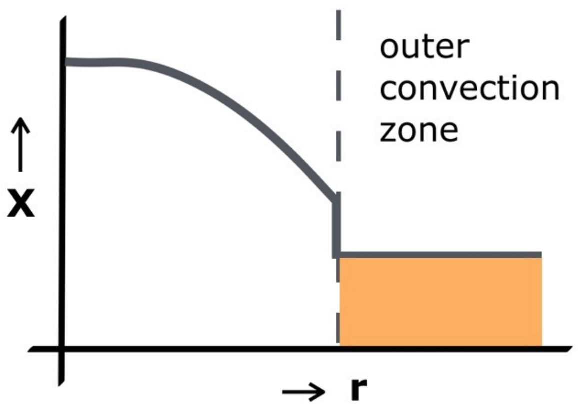
$$\tau_{\rm mix} \ll \tau_{\rm KH} \ll \tau_{\rm nuc} \ll \tau_{\rm life}$$

$$\tau_{\rm mix} \approx \frac{d}{v_c} \sim \frac{qR}{v_c} = q\left(\frac{R^2 M}{L}\right)^{1/3} \sim q\ 10^7 s$$

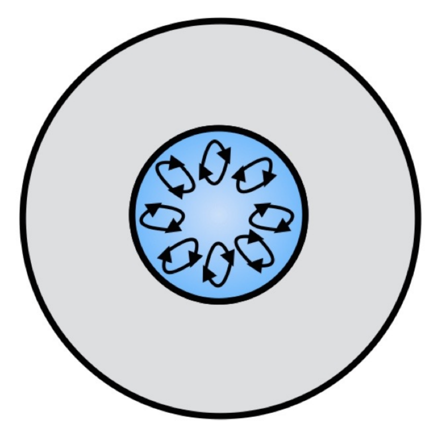
consequences:



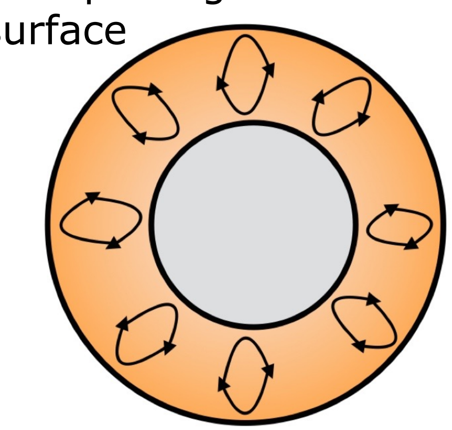




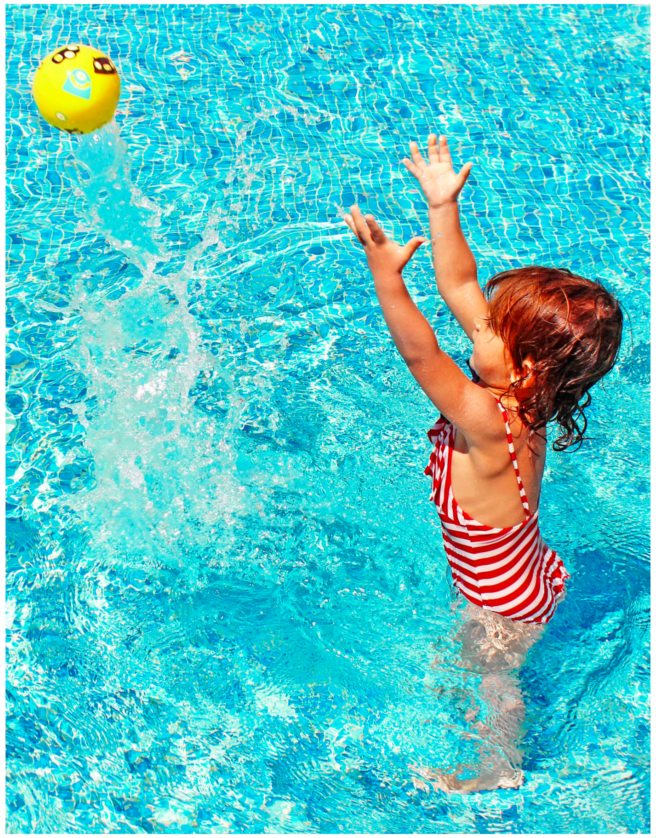
Mixing in massive star cores extends their MS lifetimes by transporting ashes upwards and fuel downwards



Mixing in stars with large convective *outer* layers causes "dredge-up" transporting ashes to the surface



Convective overshooting



At the boundary between radiative and convective zones (Sc. criterion, $\nabla_{\rm rad} = \nabla_{\rm ad}$) the acceleration due to buoyancy vanishes.

However, the blob has certain inercia and will *overshoot* by some distance

In principle this distance is $l_{\rm ov} << H_P$, so Schwarzschild criterion would still stand

But as it also carry heat and mix, it makes $|\nabla - \nabla_{\rm ad}|$ to decrease, and creates a positive feedback loop

As a consequence, overshooting introduces a large *uncertainty in the extent of mixed regions*

Chapter 9

Usually it is parametrized as $d_{\rm ov} = \alpha_{\rm ov} H_P$ where $\alpha_{\rm ov}$ is calibrated against observations

Exercises (1)

5.1 Radiation transport

The most important way to transport energy form the interior of the star to the surface is by radiation, i.e. photons traveling from the center to the surface.

(a) How long does it typically take for a photon to travel from the center of the Sun to the surface? Hint: use the mean free path of a photon in the central regions of the Sun, 1 cm. How it would take if the mea free path is 0.5 cm? And 2cm? How does this relate to the thermal timescale of the Sun?

(b) Estimate a typical value for the temperature gradient dT/dr. Use it to show that the difference in temperature ΔT between to surfaces in the solar interior one photon mean free path l_{ph} apart is $\Delta = l_{ph}\frac{dT}{dr} \approx 2 \times 10^{-4} K$

In other words the anisotropy of radiation in the stellar interior is very small. This is why radiation in the solar interior is close to that of a black body.

(c) Verify that a gas element in the solar interior, which radiates as a black body, emits ≈ 6×10^{23} erg cm^{-2} s^{-1}. If the radiation field would be exactly isotropic, then the same amount of energy would radiated into this gas element by the surroundings and so there would be no net flux.

(d) Show that the minute deviation from isotropy between two surfaces in the solar interior one photon mean free path apart at $r \sim R_{\odot}/10$ and $T \sim 10^7$ K, is sufficient for the transfer of energy that results in the luminosity of the Sun.

(e) Why does the diffusion approximation for radiation transport break down when the surface (photosphere) of a star is approached?

5.2 Opacity

(a) Identify the various processes contributing to the opacity as shown in Fig. 5.2, and the T and ρ ranges where they are important.

(b) Compare the opacity curve for $\log\rho = -6$ in the left panel of Fig.5.2 to the approximations given in Sec. 5.3.1 for (1) electron scattering, (2) free-free absorption, (3) bound-free absorption and (4) the H^- ion. How well do these approximations fit the realistic opacity curve?

(c) Calculate (up to an order of magnitude) the photon mean free path in a star of 1 $M_{\odot}$ at radii where the temperature is 10^7 K, 10^5 K and 10^4 K, using the right panel of Fig. 5.2.

(d) Suppose that the frequency-dependent opacity coefficient has the form $\kappa_\nu = \kappa_0 \nu^{-a}$. Show that the Rosseland mean opacity depends on the temperature as $\kappa \propto T^{-a}$.

Exercises (2)

5.3 Mass-luminosity relation for stars in radiative equilibrium

Without solving the stellar structure equations, we can already derive useful scaling relations. In this question you will use the equation for radiative energy transport with the equation for hydrostatic equilibrium to derive a scaling relation between the mass and the luminosity of a star.

(a) Derive how the central temperature, T_c, scales with the mass, M, radius, R, and luminosity, L, for a star in which the energy transport is by radiation. To do this, use the stellar structure equation (5.16) for the temperature gradient in radiative equilibrium. Assume that $r \sim R$ and that the temperature is proportional to T_c, $l \sim L$ and estimating $dT/dr \sim -T_c/R$.

(b) Derive how T_c scales with M and R, using the hydrostatic equilibrium equation, and assuming that the ideal gas law holds.

(c) Combine the results obtained in (a) and (b), to derive how L scales with M and R for a star whose energy transport is radiative.

You have arrived at a mass-luminosity relation without assuming anything about how the energy is *produced*, only about how it is *transported* (by radiation). It shows that the luminosity of a star is *not* determined by the rate of energy production in the centre, but by how fast it can be transported to the surface!

(d) Compare your answer to the relation between M and L which you derived from observations (Exercise 1.3). Why does the derived power-law relation start to deviate from observations for low mass stars? Why does it deviate for high mass stars?

Exercises (3)

5.4 Conceptual questions: convection

(a) Why does convection lead to a net heat flux upwards, even though there is no net mass flux (upwards and downwards bubbles carry equal amounts of mass)?

(b) Explain the Schwarzschild criterion

$$\left(\frac{d\ln T}{d\ln P}\right)_{\rm rad} > \left(\frac{d\ln T}{d\ln P}\right)_{\rm ad}$$

in simple physical terms (using Archimedes law) by drawing a schematic picture . Consider both cases $\nabla_{\rm rad} > \nabla_{\rm ad}$ and $\nabla_{\rm rad} < \nabla_{\rm ad}$. Which case leads to convection?

(c) What is meant by the *superadiabaticity* of a convective region? How is it related to the convective energy flux (qualitatively)? Why is it very small in the interior of a star, but can be large near the surface?

5.5 Applying Schwarzschild's criterion

(a) Low-mass stars, like the Sun, have convective envelopes. The fraction of the mass that is convective increases with decreasing mass. A 0.1 $M_\odot$ star is completely convective. Can you qualitatively explain why?

(b) In contrast higher-mass stars have radiative envelopes and convective cores, for reasons we will discuss in the coming lectures. Determine if the energy transport is convective or radiative at two different locations ($r = 0.242R_\odot$ and $r = 0.670R_\odot$) in a $5M_\odot$ main sequence star. Use the data of a 5 $M_\odot$ model in the table below. You may neglect the radiation pressure and assume that the mean molecular weight $\mu = 0.7$.

$r/R_\odot$	$m/M_\odot$	$L_r/L_\odot$	T [K]	ρ [g cm^{-3}]	κ [g^{-1} cm^2]
0.242	0.199	3.40×10^2	2.52×10^7	18.77	0.435
0.670	2.487	5.28×10^2	1.45×10^7	6.91	0.585

Exercises (4)

5.6 The Eddington luminosity

The Eddington luminosity is the maximum luminosity a star (with radiative energy transport) can have, where radiation force equals gravity.

(a) Show that

$$l_{\rm max} = \frac{4\pi c G m}{\kappa}.$$

(b) Consider a star with a uniform opacity κ and of uniform parameter $1 - \beta = P_{\rm rad}/P$. Show that $L/L_{\rm Edd} = 1 - \beta$ for such a star.

(c) Show that the Schwarzschild criterion for stability against convection $\nabla_{\rm rad} < \nabla_{\rm ad}$ can be rewritten as:

$$\frac{l}{l_{\rm max}} < 4\frac{P_{\rm rad}}{P}\nabla_{\rm ad}$$

(d) Consider again the star of question (b). By assuming that it has a convective core, and no nuclear energy generation outside the core, show that the mass fraction of this core is given by

$$\frac{M_{\rm core}}{M} = \frac{1}{4\nabla_{\rm ad}}.$$

Exercises (5)

5.7 The predicted mass–luminosity relation implies that $L \sim \mu^4/\kappa$.

(a) Compare this with the results of the stellar models for zero-age main-sequence stars with $Z = 0.014$ and $Z = 0.002$,

(b) Explain the trend that the luminosity of massive stars is about the same for both compositions, but that the luminosity of the metal-poor low-mass stars is higher than those of $Z = 0.014$ stars.

5.8 The Eddington limit was derived by extrapolating the empirical mass–luminosity relation of $L/L_\odot \approx 12(M/M_\odot)^{2.9}$ to higher mass. This can be improved by adopting the predicted M-L relation for zero-age main-sequence stars,

(a) Derive the approximate M-L relation in the mass range of 30 to $60M_\odot$.

(b) Extrapolate this relation and derive the Eddington limit, in mass and luminosity.

5.9 (a) Identify the ranges in T and ρ, where electron scattering dominates, and where the bound–free and free–free opacity dominates.

(b) Check the dependence of κ on T and ρ in these regions.

5.10 The average radial distance traveled by a photon in a random walk is $r \sim l\, N$, where N is the number of random steps and l is the step length (i.e., the mean-free path length).

- (a) Use $l \approx 1$ cm and $r = R_\odot$ to estimate the number of scatterings a photon will undergo before reaching the surface of the star.
- (b) Calculate the total path length L and the time $\tau_L = L/c$ it takes a photon created in the center to leave the Sun.
- (c) Is it still the same photon? (Same λ, ν?)

Exercises (6)

5.11 Calculate the properties of the convective envelope of the Sun.

(a) Calculate the thermal energy content of a cm^3 of gas in the Sun at $r = 0.9R_\odot$, using data from Appendix C.

(b) Calculate the speed of sound at that location.

(c) Calculate g and the pressure scale height

(d) Assume that the temperature difference between the ascending and descending bubbles is δ_T of the mean local value, and that the velocity of the bubbles is δ_V of the local sound speed. Calculate the luminosity that would be transported by convection in this case.

(e) Compare this with the true luminosity, L_r, at $0.9R_\odot$.

(f) Derive the values of δ_V and δ_T, and of ΔT and v_c.

5.12 Consider the Ledoux criterion.

(a) Sketch a diagram of outward decreasing μ as a function of r/R. Adopt some lower limit and some upper limit for a zone that would be convective according to the Schwarzschild criterion.

(b) Show schematically where the convection zone would be accord- ing to the Ledoux criterion for convection in a medium with a μ-gradient.

(c) Show how the μ-profile would be changed by convection in these two cases.

Exercises (7)

5.13 Estimate the duration of the main-sequence phase for stars with convective cores. During the main-sequence phase of the Sun, which lasts 8 $\times$ 10^9 yr, the inner 20% of the solar mass takes part in the H-fusion.

(a) Use the mass of the convective cores to estimate the duration of the main-sequence phase of stars of 4, 12, 20, and 40$M_\odot$.

(b) Compare the results with the lifetime computed from evolutionary models.

5.14 Sketch the chemical profile of He in a massive star in which the mass of the convective core decreases during the H-fusion phase. Sketch it at three epochs: at the beginning of the main-sequence phase (ZAMS), halfway through the main-sequence phase, and at the end of the main-sequence phase (TAMS).

Questions

Q (5.1) Free–free absorption and bound–free absorption are real absorption: photons disappear and are reemitted at another wavelength. But electron scattering does not “absorb” photons; it just sends them in another direction (with a very small mean-free path). With this in mind, why does electron scattering play a role at all in radiative transfer and in the structure of stars?

Q (5.2) Why is this relation not valid for lower main-sequence stars, red giants, and red supergiants?

Q (5.3) Is the relation valid for degenerate stars? Why?

Q (5.4) Why is κ_{ff} in $cm^2\ g^{!1}$ proportional to ρ?

Q (5.5) Show that for a low-metallicity composition of $Z << 1$ the factor $<Z_i^2/A_i> \approx 1$.

Q (5.6) Why is κ_{bf} in $cm^2\ g^{!1}$ proportional to ρ?

Q (5.7) Why does the Fe opacity peak shift to higher T with increasing ρ?

Q (5.8) If radiation inside a star was isotropic, how could there be a radiative outward moving flux?

Q (5.9) What is the reason for the absolute value signs?

Q (5.10) Show that a descending blob will keep descending under the same condition.

Q (5.11) Why would the mean particle mass decrease with the distance from the center?

Q (5.12) Estimate the pressure scale height in the Earth’s atmosphere. Do you need an oxygen mask when you climb Mount Everest (~10 km)?

Q (5.13) Explain in words the physical reason for $\Delta T \cdot v_c$ being very small. Hint: think in terms of gas-energy content and compare it with a typical stellar luminosity.

Q (5.14) Explain why this is required.

Q (5.15) What sorts of observations can be used to estimate the value of α_{os}?

Q (5.16) The predicted vertical jumps of the μ-profile shown in figure 7.8 will in reality be slightly smoothed. Which physical effects could contribute to this smoothing?