

Física Estelar

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4. Polytropic stellar models

Polytropes

Stellar evolution equations (HE) can be solved if EOS is $P=P(\rho)$

In particular, we call *polytropes* when the EOS is

$$P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}$$

n: polytrope index
K: polytrope ctnt

(Hint: this simple P/ρ relation has no T-dependence)

Examples seen already:

$$\frac{dP}{P} = \frac{\phi+1}{\phi} \frac{d\rho}{\rho} = \gamma_{\text{ad}} \frac{d\rho}{\rho}$$

Adiabatic processes ($dq=0$)

$$P_e = \frac{1}{3} \int_0^{p_F} \frac{8\pi p^4}{h^3 m_e} dp = \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} m_u^{-5/3} \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

completely e-degenerate stars
fully convective stars

$$P_e = \frac{1}{3} \int_0^{p_F} \frac{8\pi p^3}{h^3} dp = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} m_u^{-4/3} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

relativistic degenerate stars
stars dominated by P_{rad}
stars with a constant $P_{\text{gas}}/P_{\text{tot}}$

Structure of polytropes

In polytropes, combining *mass continuity* eq. (dm/dr) and the *eq. of motion* in HE (dP/dr) ,

$$\left. \begin{aligned} \frac{r^2}{\rho} \frac{dP}{dr} = -Gm &\rightarrow \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dm}{dr} \\ \frac{dm}{dr} = 4\pi r^2 \rho \end{aligned} \right\} \quad \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho$$

and with the EOS, $\frac{dP}{dr} = K\gamma \rho^{\gamma-1} \frac{d\rho}{dr}$

$$\frac{K\gamma}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \rho^{\gamma-2} \frac{d\rho}{dr} \right) = -\rho$$

With 2 b.c. in
the center ($r=0$)

$$\rho(0) = \rho_c$$

$$\left(\frac{d\rho}{dr} \right)_{r=0} = 0$$

And defining,

$$\rho = \rho_c w^n$$

$$r = \alpha z$$

$$w = \left(\frac{\rho}{\rho_c} \right)^{1/n}$$

$$\alpha = \left(\frac{n+1}{4\pi G} K \rho_c^{1/n-1} \right)^{1/2}$$

Structure of polytropes

We end up getting the *Lane-Emden equation*

$$\frac{1}{z^2} \frac{d}{dz} \left(z^2 \frac{dw}{dz} \right) + w^n = 0$$

From which a polytropic stellar model can be constructed by integrating it outwards from the center $\int_0^\infty dz$.

The equation describes the density structure w with only one parameter n

There are only 3 analytical solutions ($n=0, 1, 5$).

For $n < 5$ $w(z)$ decrease monotonically and reach 0 at $z=z_n$

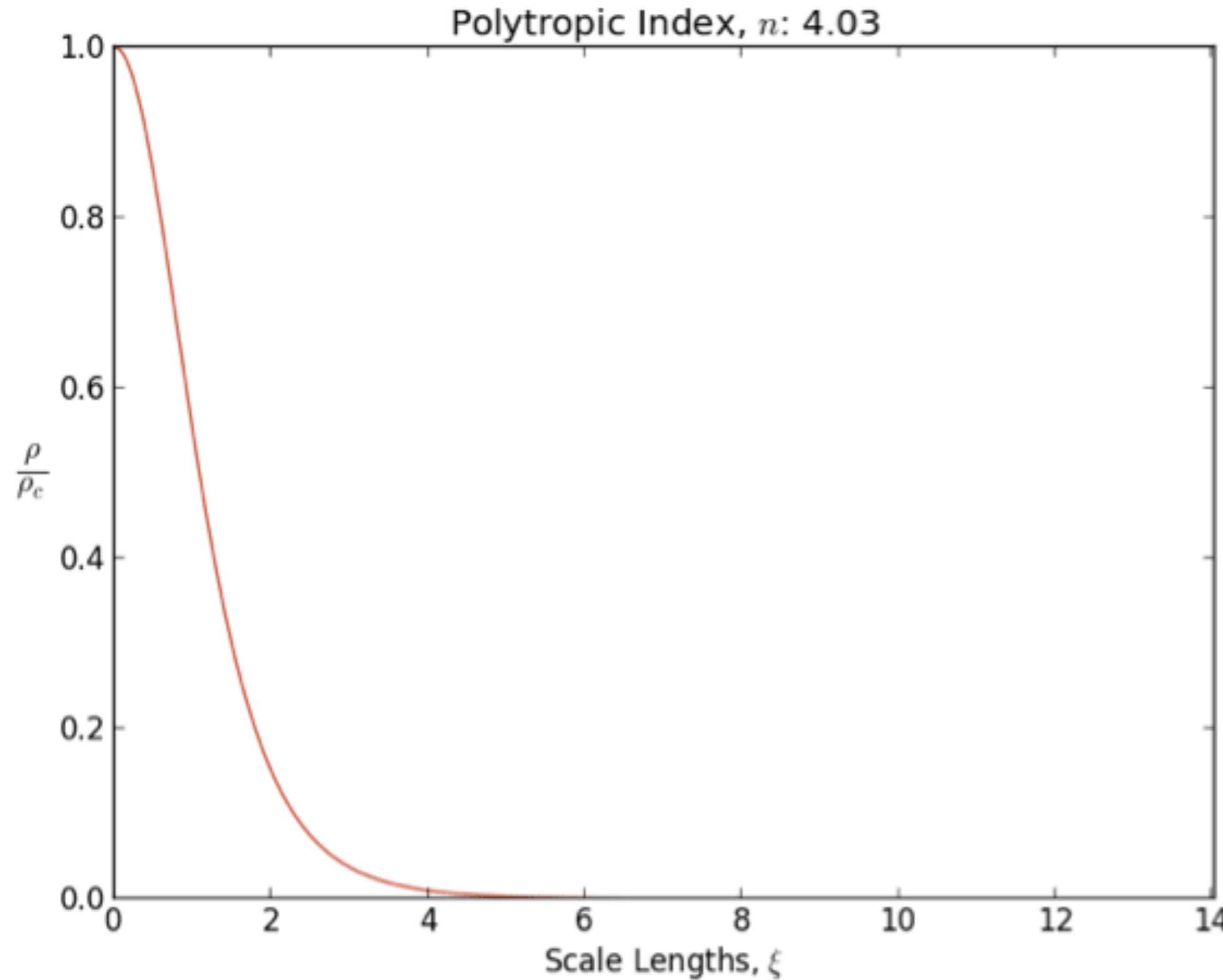
Structure of polytropes

Table 4.1. Numerical values for polytropic models with index n .

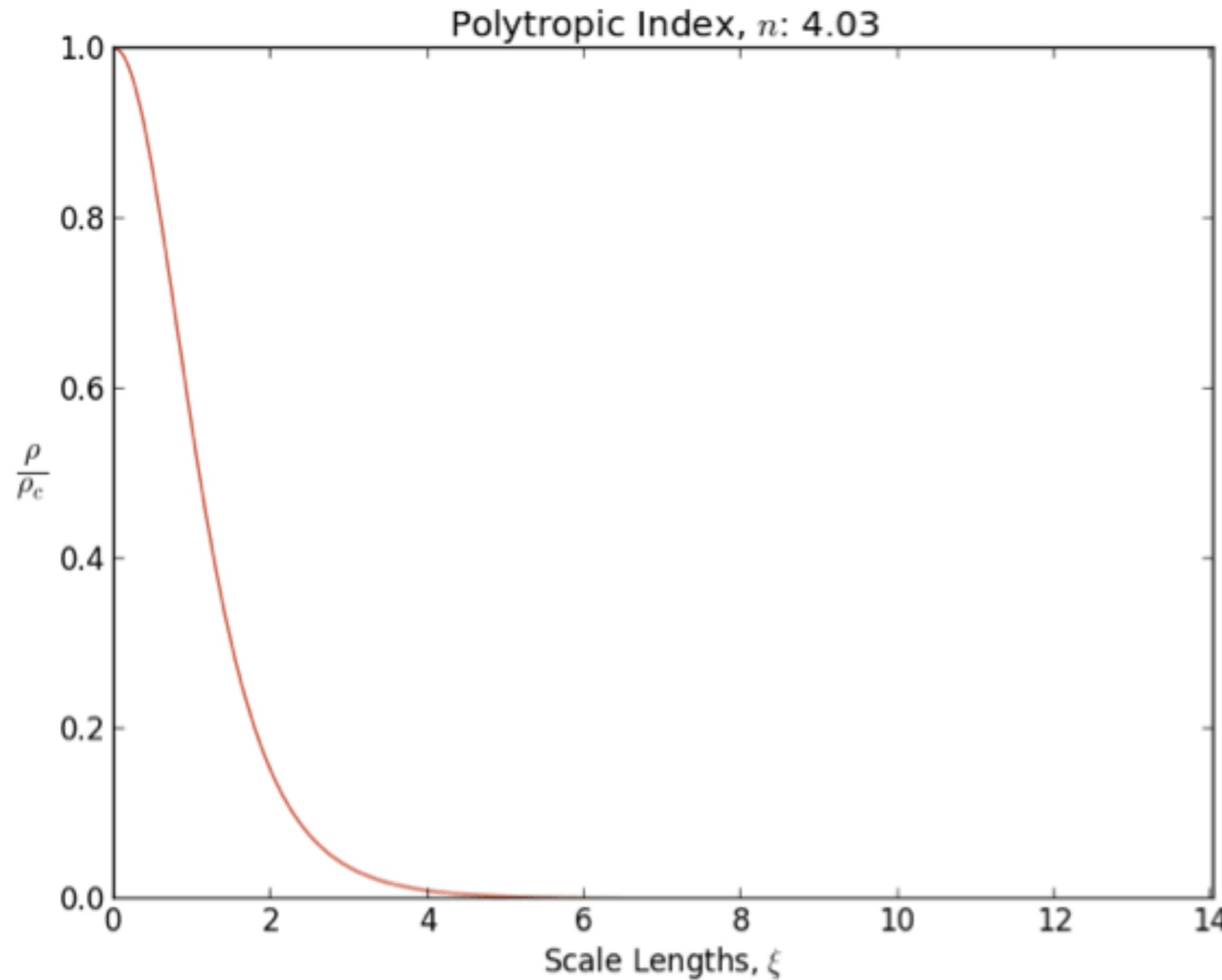
n	z_n	Θ_n	$\rho_c/\bar{\rho}$	N_n	W_n
0	2.44949	4.89898	1.00000	...	0.119366
1	3.14159	3.14159	3.28987	0.63662	0.392699
1.5	3.65375	2.71406	5.99071	0.42422	0.770140
2	4.35287	2.41105	11.40254	0.36475	1.638183
3	6.89685	2.01824	54.1825	0.36394	11.05068
4	14.97155	1.79723	622.408	0.47720	247.559
4.5	31.8365	1.73780	6189.47	0.65798	4921.84
5	∞	1.73205	∞	∞	∞

$n = 0; \gamma = \infty$	$w(z) = 1 - \frac{z^2}{6}$	$z_0 = \sqrt{(6)}$	$\text{Polytrope with constant } \rho$
$n = 1; \gamma = 2$	$w(z) = 1 - \frac{\sin z}{z}$	$z_1 = \pi$	(neutron stars $0.5 < n < 1.0$)
			(WD NR $n \sim 1.5$)
			(WD ER $n \sim 3.0$)
$n = 5; \gamma = 6/5$	$w(z) = 1 - \left(1 + \frac{z^2}{3}\right)^{-1/2}$	$z_5 = \infty$	Infinite R
$n = \infty; \gamma = 1$	$w(z) = 1$	$z_5 = \infty$	(Also for $n > 5$)

Structure of polytropes



Structure of polytropes



Physical properties of the solutions

Once we have solved for w , the density distribution is determined by n .
The physical properties of a polytropic stellar model are:

Radius, $R(K, \rho_c)$

$$R = \alpha z_n = \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/2n} z_n$$

Mass interior to z ,

$$m(z) = \int_0^{\alpha z} 4\pi r^2 \rho dr = -4\pi \alpha^3 \rho_c z^2 \frac{dw}{dz}$$

Total mass, $M(K, \rho_c)$

$$\frac{M = 4\pi \alpha^3 \rho_c \left(-z^2 \frac{dw}{dz} \right)_{z=z_n}}{\Theta_n} = 4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \Theta_n$$

Physical properties of the solutions

Eliminating ρ_c from R and M equations:

R-M relation,

$$R = (4\pi)^{\frac{1}{n-3}} \left[\frac{(n+1)K}{G} \right]^{\frac{n}{3-n}} \left[-z^2 \frac{dw}{dz} \right]_{z=z_n}^{\frac{n-1}{3-n}} M^{\frac{1-n}{3-n}}$$

K-R-M relation,

$$K = \frac{(4\pi)^{1/n}}{n+1} \left(-z^2 \frac{dw}{dz} \right)_{z=z_n}^{(1-n)/n} z_n^{(n-3)/n} \Theta_n GM^{(n-1)/n} R^{(3-n)/n}$$

From the above, one sees that $n=1$ and $n=3$ are special cases

For $n=1$, R is independent of M, and only depends on K: $R(K)$

For $n=3$, M is independent of R, and only depends on K: $M(K)$

Physical properties of the solutions

Central density,

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \left(-\frac{3}{z} \frac{dw}{dz} \right)_{z=z_n} \quad \rho_c = \frac{3\Theta_n}{z_n^3} \rho_c$$

ρ_c/ρ
Concentration index
of the polytope,
only depends on n

Central pressure,

$$P_c = K\rho_c^{(n+1)/n} = \boxed{\frac{1}{4\pi(n+1)} \left(-\frac{dw}{dz} \right)_{z=z_n}^{-2}} \frac{GM^2}{R^4}$$

$$= \boxed{\frac{(4\pi)^{1/3}}{(n+1)} \left(-z_n^2 \frac{dw}{dz} \right)_{z=z_n}^{-2/3}} \frac{GM^{2/3}}{\rho_c^{4/3}}$$

Central temperature,

$$T_c = \frac{\mu m_u}{K\rho_c} P_c = \frac{\mu m_u}{(n+1)K} \left(-z^2 \frac{dw}{dz} \right)^{-1} \frac{GM}{R}$$

Application to stars

For white dwarfs, where P only depends of ρ (indep. of T), is well described by polytropic models

Non-relativistic

$$P_e = \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} m_u^{-5/3} \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

Polytrope
 $n=1.5$
 $\gamma = 1 + \frac{1}{n} = \frac{5}{3}$

$$R \propto M^{-\frac{1}{3}} \quad \text{So when } M \uparrow \Leftrightarrow R \downarrow \Leftrightarrow \rho \uparrow \quad \dots \text{which goes towards ER}$$

Extremely relativistic

$$P_e = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} m_u^{-4/3} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

Polytrope
 $n=3.0$
 $\gamma = 1 + \frac{1}{n} = \frac{4}{3}$

M does not depend on R , and has an unique M

$$M = 4\pi \left(-z_n^2 \frac{dw}{dz} \right)_{z=z_n} \left(\frac{K}{\pi G} \right)^{3/2} = 5.836 \mu_e^{-2} M_\odot = 1.46 M_\odot$$

which is a upper limit of a sphere in HE supported by e^- deg.
so then, the maximum mass of a white dwarf (Ch. limit)

Application to stars

Eddington' standard model:

We consider a star where $P_{\text{gas}} = \beta P$, $P_{\text{rad}} = (1-\beta)P$, where β is fixed

$$\beta = \frac{P_{\text{gas}}}{P} = \frac{\mathfrak{R}}{\mu} \frac{\rho T}{P} \quad 1 - \beta = \frac{P_{\text{rad}}}{P} = \frac{aT^4}{3P}$$

and $P = \left(\frac{3\mathfrak{R}^4}{a\mu^4} \frac{1-\beta}{\beta^4} \right)^{1/3} \rho^{4/3}$

Polytrope n=3.0
With constant β

Since we are free to choose $0 < \beta < 1$, the constant $K(\beta)$

This is Eddington's standard model equation,
which explains well most MS stars

Exercises (1)

4.1 The Lane-Emden equation

- (a) Derive eq. (4.2) from the stellar structure equations for mass continuity and hydrostatic equilibrium. (Hint: multiply the hydrostatic equation by r^2/p and take the derivative with respect to r).
- (b) What determines the second boundary condition of eq. (4.4), i.e., why does the density gradient have to vanish at the center?
- (c) By making the substitutions (4.3), (4.5) and (4.6), derive the Lane-Emden equation (4.7).
- (d) Solve the Lane-Emden equation analytically for the cases $n = 0$ and $n = 1$.

4.2 Polytropic models

- (a) Derive K and γ for the EOS of an ideal gas at a fixed T , of a non-relativistic degenerate gas and of a relativistic degenerate gas.
- (b) Using the Lane-Emden equation, show that the mass distribution in a polytropic star is given by eq. (4.12), and show that this yields eq. (4.13) for the total mass of a polytrope.
- (c) Derive the expressions for the central density p_c and the central pressure P_c as function of mass and radius, eqs. (4.16) and (4.17).
- (d) Derive eq. (4.18) and compute the constant C_n for several values of n .

4.3 White dwarfs

To understand some of the properties of white dwarfs (WDs) we start by considering the equation of state for a degenerate, non-relativistic electron gas.

- (a) What is the value of K for such a star? Remember to consider an appropriate value of the mean molecular weight per free electron μ_e .
- (b) Derive how the central density p_c depends on the mass of a non-relativistic WD. Using this with the result of Exercise 4.2(b), derive a radius-mass relation $R = R(M)$. Interpret this physically.
- (c) Use the result of (b) to estimate for which WD masses the relativistic effects would become important.
- (d) Show that the derivation of a $R = R(M)$ relation for the extreme relativistic case leads to a unique mass, the so-called *Chandrasekhar mass*. Calculate its value, i.e. derive eq. (4.22).

Exercises (2)

4.4 Eddington's standard model

(a) Show that for constant β the virial theorem leads to $E_{\text{tot}} = \beta/2 E_{\text{gr}} = -\beta/(2-\beta)E_{\text{int}}$, for a classical, non-relativistic gas. What happens in the limits $\beta \rightarrow 1$ and $\beta \rightarrow 0$?

(b) Verify eq. (4.25), and show that the corresponding constant K depends on β and the mean molecular weight μ as

$$K = \frac{2.67 \times 10^{15}}{\mu^{4/3}} \left(\frac{1 - \beta}{\beta^4} \right)^{1/3}$$

(c) Use the results from above and the fact that the mass of an $n=3$ polytrope is uniquely determined by K , to derive the relation $M = M(\beta, \mu)$. This is useful for numerically solving the amount of radiation pressure for a star with a given mass.

4.5 Explain in simple physical terms why a star with $\gamma = 4/3$ has a more concentrated density structure than a star with $\gamma = 5/3$.

4.6 Suppose that zero-age main-sequence stars are described by the Eddington standard model with electron scattering as the dominant opacity.

(a) What would be the mean values of β and $P_{\text{rad}}/P_{\text{gas}}$ for stars of 1 and $60M_\odot$?

(b) What is the predicted luminosity of these stars?

(c) Compare this with the data in Appendix D and comment on the comparison.

Questions

Q (4.1) Although θ is dimensionless, can you think of what it may describe in physical terms? Why was the symbol θ chosen?

(Hint: suppose the gas has a polytropic EoS and obeys the ideal gas law,
What is the range of θ ?)

Q (4.2) What does a model with $n = 0 \rightarrow \gamma = \infty$ describe in physical terms?

Q (4.3) What does a model with $n = \infty \rightarrow \gamma = 1$ describe in physical terms?