FisyMat
Máster en Matemática Aplicada

UNIVERSIDAD DE GRANADA

# Física Estelar


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# 2. Mechanical and thermal equilibrium

# Summary

We will derive 2 fundamental stellar structure equations by applying mass and momentum conservation:

- Mass continunity equation

- Equation of motion (hydro. eq.)

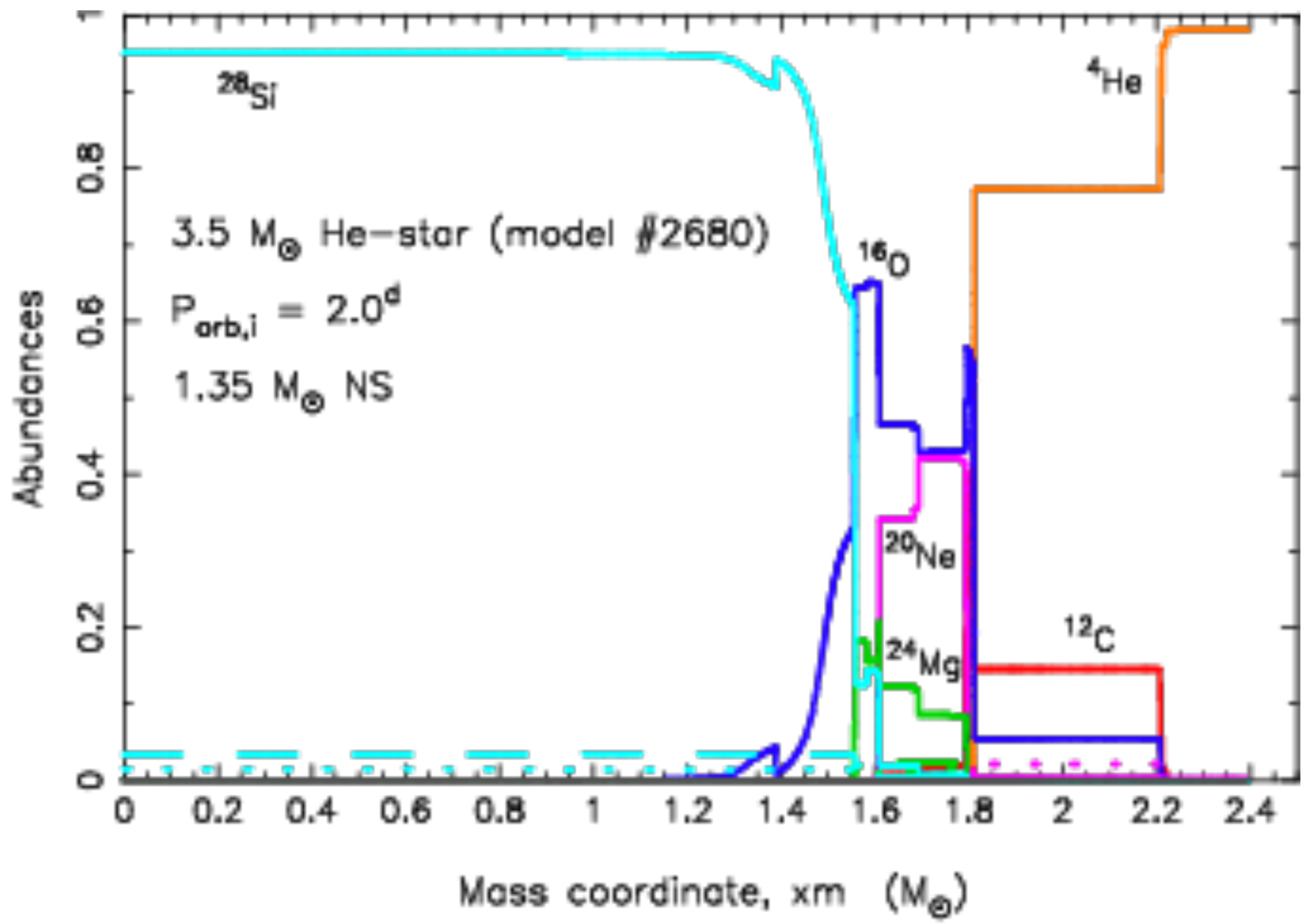
Stars are in a state of almost complete *mechanical equilibrium*, which will allow us to derive and apply the *virial theorem*.

We will consider the basic *stellar timescales* and see that most stars are also in a state of *thermal equilibrium*.

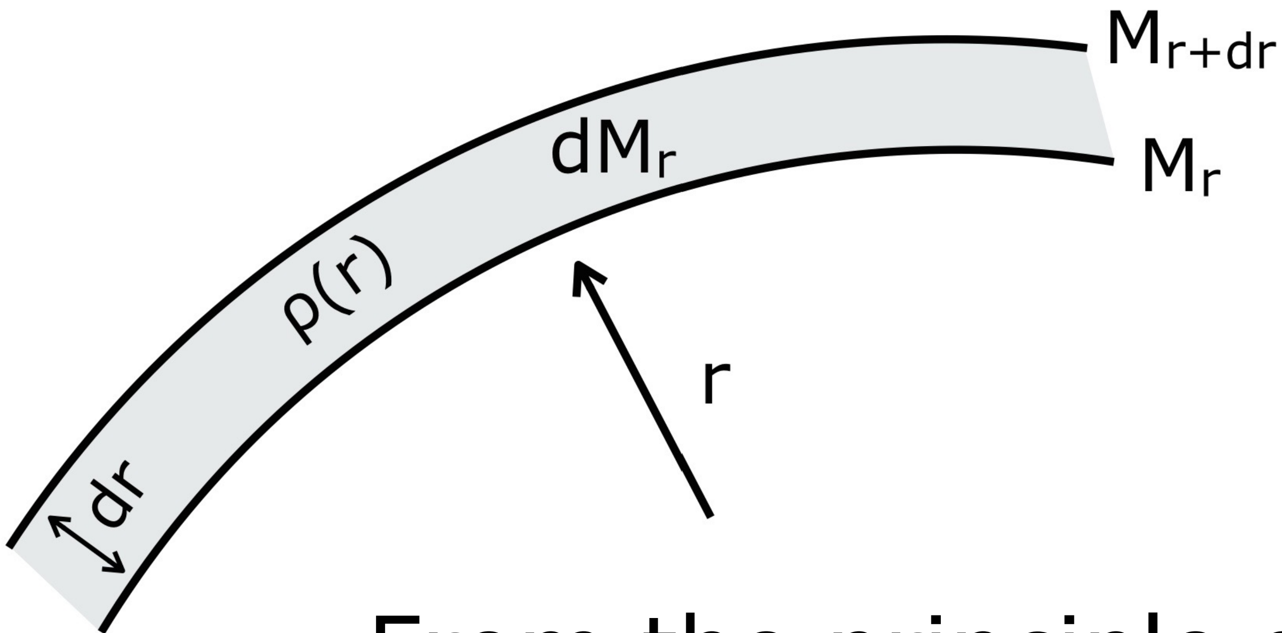
# Coordinate system

Spherical symmetry $\longrightarrow$ $X(r,t)$

Eulerian and Lagrangian coordinates



# Mass distribution




Spherical shell

Volume: $4\pi r^2 dr$

Mass: $4\pi r^2 \rho(r) dr$

From the principle of mass conservation:

$$\underbrace{dm = 4\pi r^2 \rho(r) dr}_{\frac{\partial m}{\partial r}} - \underbrace{4\pi r^2 \rho v dt}_{\frac{\partial m}{\partial t}}$$

At the surface,
this is the mass-loss rate

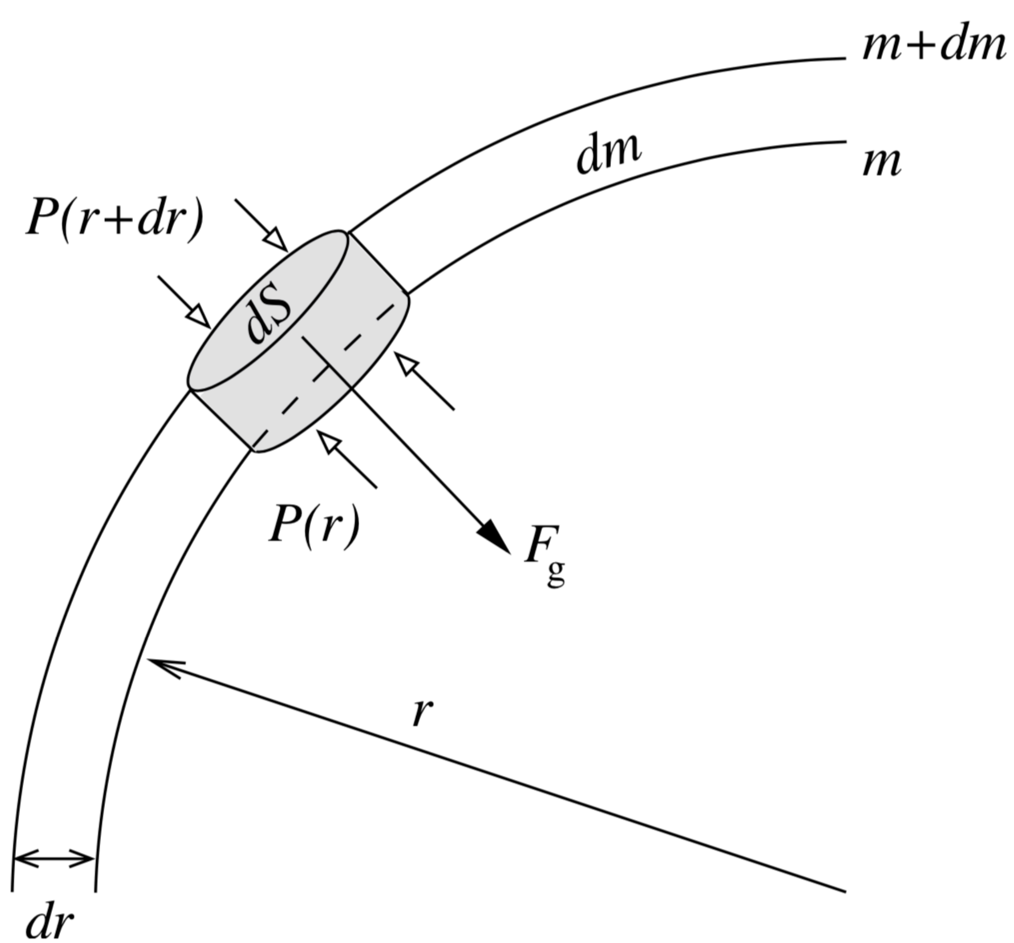
In a static situation, v=0

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) \qquad m(r) = \int_0^r 4\pi r^2 \rho(r) dr$$

Mass continuity equation: $\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho(r)}}$

In Lagrangian *m* coordinate

# Equation of motion




Gravity (inward)

$$F_g = -\frac{G\, m\, dm}{r^2} = -\frac{G\, m\, \rho\, dr\, dS}{r^2}$$

Pressure (net force due to difference in pressure between upper and lower faces)

$$\begin{aligned} F_p &= P(r)dS - P(r+dr)dS \\ &= P(r)dS - \left[P(r) + \frac{dP}{dr}dr\right] dS \\ &= -\frac{dP}{dr} dr dS \end{aligned}$$

$$\sum F \quad \rightarrow \quad \rho\, dr\, dS\, \frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2}\rho\, dr\, dS - \frac{\partial P}{\partial r} dr\, dS$$

**radial acceleration** **force of gravity** **pressure gradient**

# Hydrostatic Equilibrium

The equation of motion
of one cm³ of gas, density ρ in shell dr

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

or

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}$$

In **hydrostatic equilibrium** (HE), $\frac{d^2 r}{dt^2} = 0$

$$\boxed{\frac{dP}{dr} = -\frac{Gm}{r^2}\rho}$$

(pressure decreases outwards) or

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

These 2 eq. together determine the mechanical structure of a star in HE.

2 eq. With 3 (P, $\rho$, r) unknowns. We need eq. state P($\rho$,T)

$\uparrow$

Only solvable for polytropes where P($\rho$) (e.g. WD)

# Central pressure

Estimate of central pressure: $$\left.\frac{dP}{dm}\right|_{\mathrm{cent}}^{\mathrm{surf}} = \frac{P_{\mathrm{surf}} - P_{\mathrm{cent}}}{M - 0} = -\frac{P_c}{M}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \quad \rightarrow \quad (\ldots) \quad \rightarrow \quad P_c > \frac{GM^2}{8\pi R^4}$$

or from $$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \quad \rightarrow \quad (\ldots) \quad \rightarrow \quad P_c = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

For the sun, $P_c^{\odot} \simeq \frac{7\times10^{-8}(2\times10^{33})^2}{(7\times10^{10})^4} \simeq 1\times10^{16}\,\mathrm{dyne/cm^2}$ (~ $10^{10}$ atm)

# Dynamical timescale

eq. motion

What happens if HE is violated?

$$\frac{\partial^2 r}{\partial t^2} = -\frac{Gm}{r^2} - \cancel{\frac{1}{\rho}\frac{\partial P}{\partial r}}$$

e.g. suppose that the pressure gradient drops. All shells will be accelerated inwards, and the star collapse in free fall.

From $x = x_0 - v_o t - \frac{1}{2}at^2$

The resulting acceleration can be approx. as

$$\ddot{r} \simeq \frac{R}{\tau_{\rm ff}^2}$$

$\rho = 3M/4\pi R^3$

Where,

$$\tau_{\rm dyn} \simeq \sqrt{\frac{R}{g}} \simeq \sqrt{\frac{R^3}{GM}} \simeq \frac{1}{2}(G\rho)^{-1/2}$$

(can also be estimated assuming no gravitational force)

averaged over all cells.

The time that a star will survive until collapse (Sun -> 30m)

# Dynamical timescale

Consequences:

- Any departure from HE should very quickly lead to observable phenomena. If the star do not recover HE, it should lead to collapse or explosion.

- A perturbation of the HE may lead to small-scale oscillations on the dynamical timescale.

- Stars are extremely close to HE, since any disturbance is immediately quenched in $t \lesssim \tau_{\rm dyn}$ . Stars evolve quasi-statically, throuough a series of nnear-perfect HE states

# Virial Theorem

Important consequence of hydrostatic equilibrium

Th. V. connects two E reservoirs of the star (Pot/Int)

Taking hydrostatic equilibrium:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

we can multiply both sides by $V=4/3\ \pi r^3$ and integrate from 0 -> M
(similarly, from dP/dr, we can integrate from 0 -> R)

$$\int_0^M \frac{4\pi}{3} r^3 \frac{dP}{dm} dm \qquad -\frac{1}{3}\int_0^M \left(\frac{Gm}{r}\right) dm$$

Integrated by parts $\int_{P_c}^{P_s} V dP = [VP]_c^s - \int_0^{V_s} P dV$ gives 1/3 gravitational $E_{pot}$

$$\cancel{\left[\frac{4\pi}{3} R^3 P\right]_c^s} - \int_0^{V_s} P dV \quad \xrightarrow[dV = dm/\rho]{} \quad \boxed{E_{\rm pot} = -3\int_0^M \frac{P}{\rho} dm}$$

$R_C \sim 0$, $P_S \sim 0$

**General form of Virial Th.**

P needed to support the star: $P = -\frac{1}{3} E_{\rm Pot}/V$

# Virial Theorem

For an ideal gas, $P = nkT = \frac{N}{V}kT = \frac{m}{\mu m_u}\frac{1}{V}kT = \frac{\rho}{\mu m_u}kT$

number of particles

number density

particle mass in $m_u$

*weighted average of all isotopes of that element*

*μ factor: average number of amu per particle*

Kinetic energy per particle of ideal gas $\epsilon_{\rm kin} = \frac{3}{2}kT$

and the internal energy per unit mass $u = \frac{3}{2}\frac{kT}{\mu m_u} = \frac{3}{2}\frac{P}{\rho}$

so, $$\int_0^M \frac{P}{\rho}dm = \frac{2}{3}\int_0^M u dm = \frac{2}{3}E_{\rm int}$$

And the viral th., for an ideal gas becomes $\boxed{E_{\rm int} = -\frac{1}{2}E_{\rm pot}}$

Ideal gas, but holds for MS stars

# Central temperature

Using the virial th., we can obtain an estimate of the *average* temperature inside a star of ideal gas

$$E_{\rm pot} = -\frac{GM^2}{R}$$

Average T

$$u = \frac{3}{2}\frac{kT}{\mu m_u} \quad \rightarrow \quad E_{\rm int} = \frac{3}{2}\frac{k}{\mu m_u}\int T dm \approx \frac{3}{2}\frac{k\overline{T}M}{\mu m_u}$$

$$E_{\rm int} = -\frac{1}{2}E_{\rm pot} \quad \rightarrow \quad \overline{T} = \frac{1}{3}\frac{\mu m_u}{k}\frac{1}{M}\frac{GM^2}{R} = \frac{1}{3}\frac{\mu m_u}{k}\frac{GM}{R}$$

Taking $\mu$=0.5 for Hydrogen, $T_c \simeq 4\times 10^6 \mathrm{K}$ for the Sun

This is an *average*. Given the gradient of T, at the center $T_c \simeq 10^7 \mathrm{K}$

# Virial theorem in general

For an equation of state other than an ideal gas $\left(u = \frac{3}{2}\frac{P}{\rho}\right)$:

$$u = \phi \frac{P}{\rho}$$

where $\phi = \frac{3}{2}$ for an ideal gas [and all non-relativistic particles]

For relativistic particles $\phi = 3$. So, in general the V. Th.:

$$E_{\rm int} = -\frac{1}{3}\phi E_{\rm pot}$$

# Virial Theorem implications

Due to bulk motions, not thermal motions =0 in HE

For an ideal gas

$$E_{\rm tot} = \cancel{E_{\rm kin}} + E_{\rm int} + E_{\rm pot} = \left(1 - \frac{1}{3}\phi\right) E_{\rm pot} \approx \frac{1}{2} E_{\rm pot} < 0$$

so the total energy of a star is negative!

Consequences:

- Gravitationally bound gas spheres must be hot to maintain HE. Heat provides the pressure needed to balance gravity

- A hot sphere of gas radiates into surrounding space, so a star must lose energy from its surface. The rate of energy loss is the luminosity. $L = -dE_{tot}/dt > 0$

- Taking the derivative of $E_{tot}$:
  $dE_{pot}/dt = -2L < 0$ meaning that the star *contracts*, and
  $dE_{int}/dt = L > 0$ meaning that the star *gets hotter*

# Thermal equilibrium

*(So far, we have not assumed nuclear reactions, just $E_{int}$. So, stars would contract by radiating E)*

Internal energy sources, nuclear reactions, compensate energy loss from the surface,

$$L = L_{\rm nuc} = -\frac{dE_{\rm nuc}}{dt}$$

The total energy is conserved, and the viral theorem states that the $E_{int}$ and $E_{pot}$ are conserved as well,

$$\dot{E}_{\rm tot} = \dot{E}_{\rm int} = \dot{E}_{\rm pot} = 0$$

This stationary state is known as **Thermal Equilibrium (TE)**

*Energy is radiated at the surface at the same rate at which it is produced by nuclear reactions in the interior.*

# Timescales of stellar evolution

## Dynamical timescale, (changes in the mechanical structure; *from eq. motion*)

$$\tau_{\rm dyn} \simeq \sqrt{\frac{R^3}{GM}} \simeq 0.02 \left(\frac{R}{R_\odot}\right)^{3/2} \left(\frac{M_\odot}{M}\right)^{1/2} days$$

on which the star reacts to a perturbation of HE

## Thermal (Kelvin-Helmholtz) timescale, (changes in the thermal structure; *from V. Th.*)

Energy sources of a star: - gravitational contraction
- nuclear reactions

Assuming no nuclear reactions, and an ideal gas: $L = \frac{dE_{\rm int}}{dt} = -\frac{1}{2}\frac{dE_{\rm pot}}{dt}$

$$\tau_{KH} = \frac{E_{\rm int}}{L} \simeq \frac{|E_{pot}|}{2L} \simeq \frac{GM^2}{2RL} \simeq 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \frac{R_\odot}{R}\frac{L_\odot}{L} yr$$

describes how fast changes in thermal structure of a star can occur

Basically, how much time it would take to a star to contract and lose all its internal energy by radiation.

# Timescales of stellar evolution

## Nuclear timescale, (changes in the composition; *see Ch. 6*)

$$L = \frac{dE_{\rm nuc}}{dt}$$

For H

$$\tau_{\rm nuc} = \frac{E_{\rm nuc}}{L} = \phi f_{nuc} \frac{Mc^2}{L} \simeq 10^{10} \frac{M}{M_\odot} \frac{L_\odot}{L} yr$$

$f_{\rm nuc}$: fraction of M serving as fuel

$\phi$: fraction of M converted to E

is the time a star remains in thermal equilibrium by burning nuclear supply

## Convection timescale, (Rising and descending cells in convection zones; *see Ch. 5*)

The longest time is reached in AGB stars. In general, $\tau_{conv} \approx \text{days}$

$$\boxed{\tau_{\rm nuc} \gg \tau_{KH} \gg \tau_{\rm conv} \gg \tau_{\rm dyn}}$$

*Nuclear reactions determine the pace of stellar evolution*

*The dynamical (thermal) timescale is so short that stars can be assumed to remain in HE (TE).*

# Exercises

## 2.1 Density profile

In a star with mass $M$, assume that the density decreases from the center to the surface as a function of radial distance $r$, according to

$$\rho = \rho_c \left[1 - \left(\frac{r}{R}\right)^2\right],$$

where $\rho_C$ is a given constant and $R$ is the radius of the star.

(a) Find $m(r)$.

(b) Derive the relation between $M$ and $R$.

(c) Show that the average density of the star is $0.4\rho_C$.

## 2.2 Hydrostatic equilibrium

(a) Consider an infinitesimal mass element d$m$ inside a star (Fig. in slide 6). What forces act on this mass element?

(b) Newton's second law of mechanics, or the equation of motion, states that the net force acting on a body is equal to its acceleration times it mass. Write down the equation of motion for the gas element.

(c) In hydrostatic equilibrium the net force is zero and the gas element is not accelerated. Find an expression of the pressure gradient in hydrostatic equilibrium.

(d) Find an expression for the central pressure $P_C$ by integrating the pressure gradient. Use this to derive the lower limit on the central pressure of a star in hydrostatic equilibrium (slide 7).

(e) Verify the validity of this lower limit for the case of a star with the density profile of Ex. 2.1.

# Exercises (2)

**2.3 The virial theorem**

An important consequence of hydrostatic equilibrium is that it links the gravitational potential energy $E_{gr}$ and the internal thermal energy $E_{int}$.

(a) Estimate the gravitational energy $E_{gr}$ for a star with mass $M$ and radius $R$, assuming (1) a constant density distribution and (2) the density distribution of exercise 2.1.

(b) Assume that a star is made of an ideal gas. What is the kinetic internal energy per particle for an ideal gas? Show that the total internal energy, $E_{int}$ is given by:

$$E_{int} = \int_0^R \left(\frac{3}{2}\frac{k}{\mu m_u}\rho(r)\,T(r)\right) 4\pi r^2 dr$$

(c) Estimate the internal energy of the Sun by assuming constant density and $T(r) \simeq \langle T \rangle \simeq \frac{1}{2}T_c \simeq 5 \times 10^6 K$

and compare your answer to your answer for a). What is the total energy of the Sun? Is the Sun bound according to your estimates?

It is no coincidence that the order of magnitude for $E_{gr}$ and $E_{int}$ are the same. This follows from hydrostatic equilibrium and the relation is known as the virial theorem. In the next steps we will derive the virial theorem starting from the pressure gradient in the form of eq. of HR (slide 7).

(d) Multiply by both sides of that equation by $4\pi r^3$ and integrate over the whole star. Use integration by parts to show that

$$\int_0^R 3P4\pi r^2 dr = \int_0^R \frac{Gm(r)}{r}\rho 4\pi r^2 dr$$

(e) Now derive a relation between $E_{gr}$ and $E_{int}$, the virial theorem for an ideal gas.

(f) Also show that for the average pressure of the star

$$\langle P \rangle = \frac{1}{V}\int_0^R P4\pi r^2 dr = -\frac{1}{3}\frac{E_{gr}}{V}$$

where V is the volume of the star.

As the Sun evolved towards the main sequence, it contracted under gravity while remaining close to hydrostatic equilibrium. Its internal temperature changed from about 30000 K to about 6 × $10^6$K.

(g) Find the total energy radiated during away this contraction. Assume that the luminosity during this contraction is comparable to $L_\odot$ and estimate the time taken to reach the main sequence.

# Exercises (3)

**2.4 Conceptual questions**

(a) Use the virial theorem to explain why stars are hot, i.e. have a high internal temperature and therefore radiate energy.

(b) What are the consequences of energy loss for the star, especially for its temperature?

(c) Most stars are in thermal equilibrium. What is compensating for the energy loss?

(d) What happens to a star in thermal equilibrium (and in hydrostatic equilibrium) if the energy production by nuclear reactions in a star drops (slowly enough to maintain hydrostatic equilibrium)?

(e) Why does this have a stabilizing effect? On what time scale does the change take place?

(f) What happens if hydrostatic equilibrium is violated, e.g. by a sudden increase of the pressure.

(g) On which timescale does the change take place? Can you give examples of processes in stars that take place on this timescale.

# Exercises (4)

**2.5 Three important timescales in stellar evolution**

(a) The nuclear timescale $\tau_{nuc}$.

i. Calculate the total mass of hydrogen available for fusion over the lifetime of the Sun, if 70% of its mass was hydrogen when the Sun was formed, and only 13% of all hydrogen is in the layers where the temperature is high enough for fusion.

ii. Calculate the fractional amount of mass converted into energy by hydrogen fusion.

iii. Derive an expression for the nuclear timescale in solar units, i.e. expressed in terms of $R/R_\odot$, $M/M_\odot$ and $L/L_\odot$.

iv. Use the mass-radius and mass-luminosity relations for main-sequence stars to express the nuclear timescale of main-sequence stars as a function of the mass of the star only.

v. Describe in your own words the meaning of the nuclear timescale.

(b) The thermal timescale $\tau_{KH}$.

i-iii. Answer question (a) iii, iv and v for the thermal timescale and calculate the age of the Sun according to Kelvin.

iv. Why are most stars observed to be main-sequence stars and why is the Hertzsprung-gap called a gap?

(c) The dynamical timescale $\tau_{dyn}$.

i-iii. Answer question (a) iii, iv and v for the dynamical timescale.

iv. In stellar evolution models one often assumes that stars evolve *quasi-statically*, i.e. that the star remains in hydrostatic equilibrium throughout. Why can we make this assumption?

v. Rapid changes that are sometimes observed in stars may indicate that dynamical processes are taking place. From the timescales of such changes - usually oscillations with a characteristic period - we may roughly estimate the average density of the Star. The sun has been observed to oscillate with a period of minutes, white dwarfs with periods of a few tens of seconds. Estimate the average density for the Sun and for white dwarfs.

(d) Comparison.

i. Summarize your results for the questions above by computing the nuclear, thermal and dynamical timescales for a 1, 10 and 25 $M_\odot$ main-sequence star. Put your answers in tabular form.

ii. For each of the following evolutionary stages indicate on which timescale they occur: *pre- main sequence contraction, supernova explosion, core hydrogen burning, core helium burning*.

iii. When the Sun becomes a red giant (RG), its radius will increase to $200R_\odot$ and its luminosity to $3000L_\odot$. Estimate $\tau_{dyn}$ and $\tau_{KH}$ for such a RG.

iv. How large would such a RG have to become for $\tau_{dyn} > \tau_{KH}$? Assume both R and L increase at constant effective temperature.

# Exercises (5)

**2.6** We have derived the VT mathematically, using some tricks such as multiplication of the integral and substitutions of variables for integration in parts. Can you also explain in physical terms that a relation between $E_{pot}$ and $E_{kin}$ is to be expected?

**2.7** Why is the estimate of $T_c$ (Equation 3.9) better than that of $P_c$ (Equation 3.8) in comparison with the solar value? Hint: consider how the density is used in both estimates.

**2.8** We will show later that for non-relativistic degenerate gas P = 1/3u. What does that imply for the VT for degenerate stars?

**2.9** If $\bar{T}$ and $\bar{\rho}$ are the mean temperature and density of a star, then show by using the VT that $\bar{T}$ can be described by a relation that is similar to Equation (3.9).

**2.10** The VT also applies to the orbits of the individual planets around the Sun. Show that Kepler's third law follows directly from the VT.

**2.11** The estimates of $P_c$ and $T_c$, derived from the HE condition in Equations (3.7a) and (3.9), were obtained by assuming a constant mean density. These estimates can be improved by adopting a more realistic density distribution. It turns out that a density distribution of the type $\rho/\rho \approx 3e^{-10x}$ with x = r/R $c$ is a reasonable approximation for main-sequence stars in the range of 0.11 < x < 0.98. Adopt this, plus a constant density core with $\rho/\rho = 1$ at x < 0.11.

(a) Derive an expression for the mean density by computer.

(b) Use the HE condition to derive an expression for the central pressure.

(c) Derive an expression for the central temperature, using the ideal gas law.

(d) Apply these estimates to a zero-age main-sequence star of $1M_\odot$ and compare the results with the values from stellar models: $\varrho_c = 78$ g cm$^3$, $P_c = 1.2\times10^{17}$ dyne cm$^2$, and $T_c = 1.36\times10^7$ K.

(e) Do the same for a zero-age main-sequence star of $60M_\odot$: $\varrho_c = 2.2$ gcm$^3$, $P_c = 1.2\times10^{16}$ dynecm$^2$, and $T_c = 8.31\times10^7$ K.

(f) Comment on the differences.

# Exercises (6)

**2.12** Calculate dynamical, thermal, and nuclear timescales and their ratios for different stars.

– MS star of 1 $M_{sun}$.

– MS star of 60 $M_{sun}$.

– Red supergiant of 15 $M_{sun}$.

– White dwarf of 0.6 $M_{sun}$.

Comment on the consequences of these results.

**2.13** If the nuclear fusion in the Sun were to suddenly stop, how long would it take before an observer would notice? Photon travel time $\tau_L$? Dynamical time? Thermal timescale? Nuclear timescale? Give arguments.

# Questions

**Q1:** Argue that, for any realistic density distribution of the star, this will also be a lower limit.

**Q2:** Could you have guessed that the central pressure is of the order of $GM^2/R^4$? Hint: consider the weight of a column of gas on the center.

**Q3:** Does the VT apply to massive main-sequence stars, where HE is partly supported by radiation pressure? Why?

**Q4:** Does the Virial Theorem apply to degenerate stars? Hints: (a) Did we use the ideal gas law in deriving it? (b) How did we use it?