

# Física Estelar

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### **3. Equation of state of stellar interiors**

# Local Thermodynamic equilibrium (LTE)

In an isolated space, matter and radiation tend towards a state of TE, achieved by interaction (collision, scattering, absorption)

Stars are not isolated systems, they emit radiation and generate energy

(e.g. in the Sun,  $T_{\text{surf}} \sim 6000 \text{ K}$  while  $T_c \sim 10^{17} \text{ K}$ )

Stars are not in TE, but locally ( $d \ll R$ ) there is a well-defined local temperature:

$$l_{ph} = 1/\kappa\rho$$

$l_{ph}$ : mean free path for photons  
k: opacity, effective cross-section per unit mass

In the Sun,  $\sim 1 \text{ cm}$

$$\Delta T \simeq \frac{dT}{dr} l_{ph} \simeq \frac{T_c}{R} l_{ph} \simeq \frac{10^7}{10^{11}} \simeq 10^{-4} K$$

11 orders of magnitude  
smaller than  $T_c$

We can therefore talk about ***local thermodynamical equilibrium (LTE)***, which enables the calculation of all thermodynamic properties  $\rho$ ,  $T$ ,  $\mu$ ,  $P$ , etc...

# Equation of State (EOS)

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The equation of state describes the relationship between  $P$ ,  $T$ ,  $X$ , and  $\rho$ .

$$P = P(\rho, T, X_i)$$

Using laws of thermodynamics and a similar equation for the internal energy  $U = U(\rho, T, X_i)$ , we can derive the properties needed to describe the structure of a star:

- specific heats  $c_V, c_P$
- the adiabatic exponent  $\gamma_{\text{ad}}$
- The adiabatic temperature gradient  $\nabla_{\text{ad}}$

# EOS for an gas of free particles

EOS for a perfect gas provides a description of ions, electrons, and photons (**individually!**) in the stellar interior

**Number density**  
(number of particles per unit volume)

$$n = \int_0^{\infty} \frac{n(p)dp}{\text{Number of particles per unit volume With momentum between } p, p+dp}$$

Distribution of momenta of gas particles

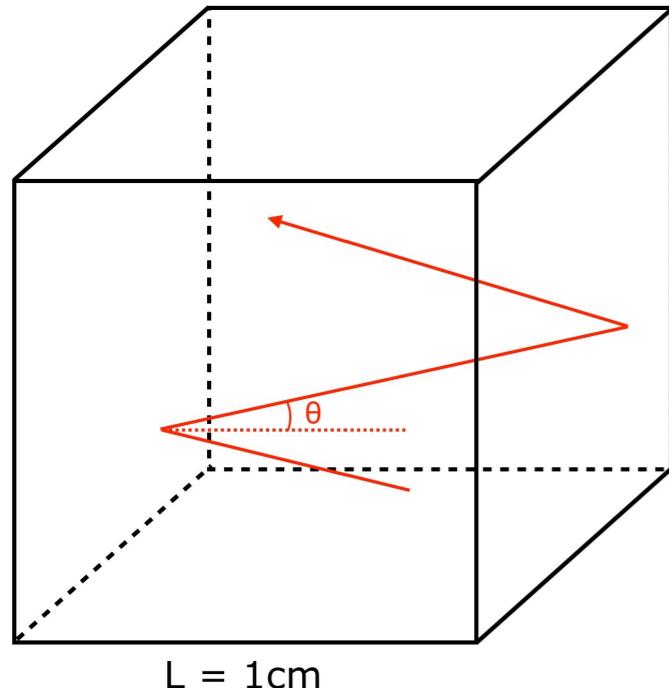
**Internal energy density**  
(internal energy per unit volume)

$$U = \int_0^{\infty} \frac{\epsilon_p n(p)dp}{\text{Kinetic energy of a particle with momentum } p} = n \langle \epsilon_p \rangle$$

Pressure

$$P = \frac{1}{3} \int_0^{\infty} p v_p n(p)dp = \frac{1}{3} n \langle p v_p \rangle$$

# EOS for an gas of free particles



To estimate the pressure, we can calculate the force on one side ( $1\text{cm}^2$ ) by the collisions (p exchange) of all particles per sec.

$\Delta t$ : time between 2 collisions by the same particle

$$\Delta t = \frac{2L}{v \cos \theta}$$

Collisions are elastic, so

$$\Delta p = 2p \cos \theta$$

and the momentum transferred

$$\frac{\Delta p}{\Delta t} = vp \cos^2 \theta$$

the number of particles in the box is

$$n(\theta, p)d\theta dp$$

so, the contribution to the pressure is

$$dP = vp \cos^2 \theta n(\theta, p)d\theta dp$$

Integrating over all directions  $n(\theta, p)d\theta = n(p) \sin \theta d\theta$

$$P = \int_0^\infty p v_p n(p) dp \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta = \frac{1}{3} n \langle p v_p \rangle$$

# EOS for an gas of free particles

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Relation between pressure and internal energy

$$\epsilon^2 = p^2 c^2 + m^2 c^4 \quad \epsilon_p = \epsilon - mc^2 \quad v_p = \frac{\partial \epsilon}{\partial p} = \frac{pc^2}{\epsilon}$$

Non-relativistic limit ( $p \ll mc$ ):

$$\epsilon_p = \frac{1}{2} \frac{p^2}{m} \quad v_p = \frac{p}{m} \quad \langle pv \rangle = \frac{p^2}{m} = 2\langle \epsilon_p \rangle$$

$$P = \frac{2}{3} U$$

Extremely relativistic limit ( $p \gg mc$ ):

$$\epsilon_p = pc \quad v = c \quad \langle pv \rangle = \langle pc \rangle = \langle \epsilon_p \rangle$$

$$P = \frac{1}{3} U$$

# EOS for a classical ideal gas

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For an ideal gas, the momentum distribution is given by the Maxwell-Boltzmann distribution

Normalization of total number density                                    volume in momentum space

$$n(p)dp = \frac{n}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT} 4\pi p^2 dp$$

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Equilibrium distribution of kinetic energies

Using  $v = p/m$ , this yields the ideal gas law,

$$P = \frac{1}{3} \frac{n}{(2\pi mkT)^{3/2}} \int_0^\infty \frac{p^2}{m} e^{-p^2/2mkT} 4\pi p^2 dp = nkT$$

# EOS for a mixture of ideal gases

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For a mixture of gases, (e.g. ions and electrons)

$$P_{\text{gas}} = P_{\text{ion}} + P_{\text{e}} = \sum_i P_i + P_{\text{e}} = \left( \sum_i n_i + n_{\text{e}} \right) kT = nkT$$

where,  $n_i = \frac{X_i \rho}{A_i m_u}$

$$n_{\text{ion}} = \sum_i \frac{X_i}{A_i} \frac{\rho}{m_u} \equiv \frac{1}{\mu_{\text{ion}}} \frac{\rho}{m_u}$$

$$P_{\text{ion}} = \frac{1}{\mu_{\text{ion}}} \frac{\rho}{m_u} kT$$

$$n_{\text{e}} = \sum_i Z_i n_i = \sum_i \frac{Z_i X_i}{A_i} \frac{\rho}{m_u} \equiv \frac{1}{\mu_{\text{e}}} \frac{\rho}{m_u}$$

$$P_{\text{e}} = \frac{1}{\mu_{\text{e}}} \frac{\rho}{m_u} kT$$

$$P_{\text{gas}} = \left( \frac{1}{\mu_{\text{ion}}} + \frac{1}{\mu_{\text{e}}} \right) \frac{\rho kT}{m_u}$$

# Quantum mechanical description of the gas

At high  $\rho$  (or low  $T$ !) quantum mechanical effects become important and change the EOS

According to Heisenberg uncertainty pple. in 3D

$$\Delta V \Delta^3 p \geq h^3$$

and the number of quantum states is

$$g(p)dp = g_s \frac{V}{h^3} 4\pi p^2 dp$$

- Fermions obey the Pauli exclusion pple. (two particles cannot occupy the same quantum state)  
e.g. electrons and nucleons

the fraction of states is

$$f_{FD}(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} + 1} \leq 1$$

$\mu$  — chemical potential. Normalization ctnt

- Bosons have no restriction

e.g. photons

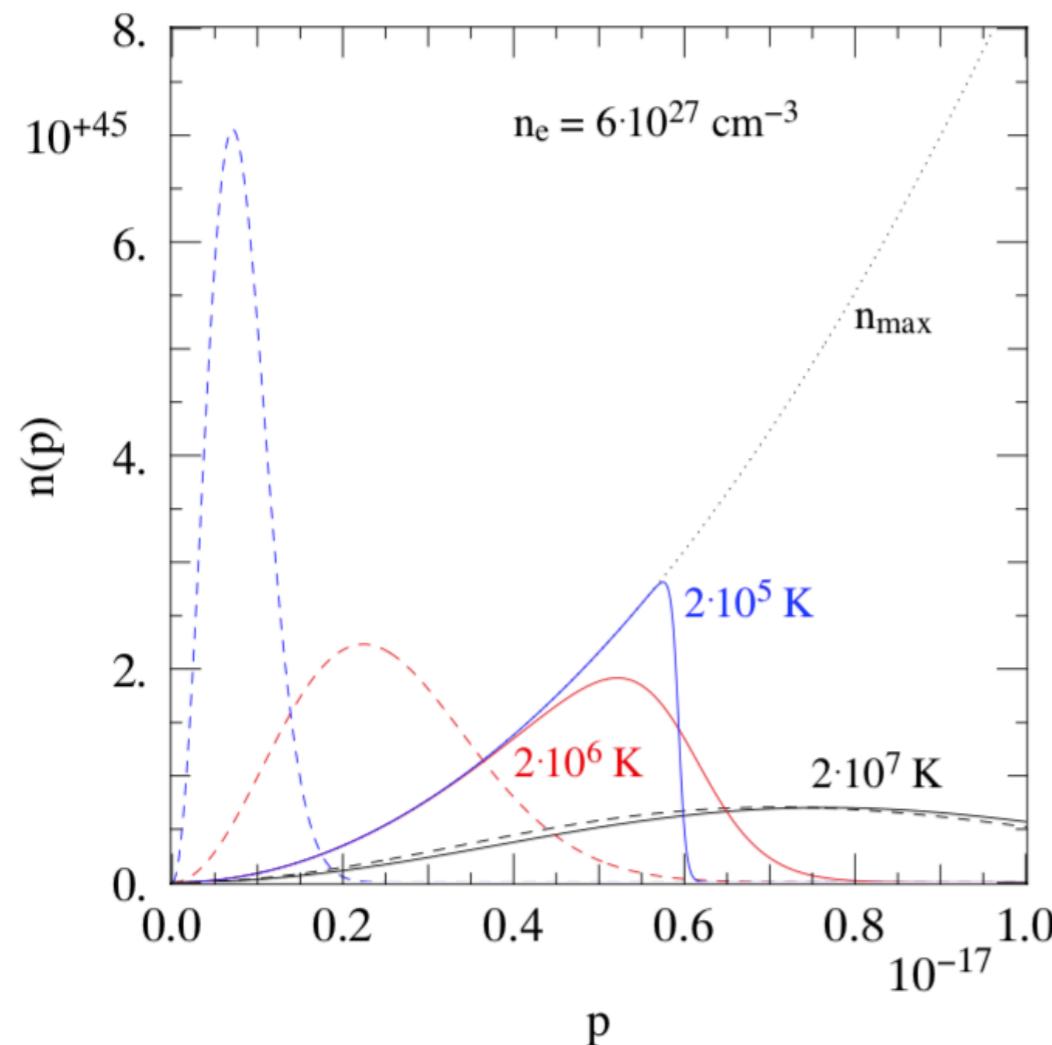
the fraction of states is

$$f_{BE}(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} - 1} > 1$$

The distribution of momenta in LTE id given by  $f(\epsilon_p)$  times  $g(p) dp$

# Electron degeneracy

Electrons are fermions with 2 spin states,  $g_e = 2$ .



dashed lines: classic mech.  
solid olines: quantum mech.

the maximum number density of electrons is,

$$n_{\max}(p)dp = \frac{g_e}{h^3} 4\pi p^2 dp = \frac{8\pi}{h^3} p^2 dp$$

the momentum distribution of electrons

$$n_e(p)dp = \frac{2}{h^3} \frac{1}{e^{(p^2/2m_e kT) - \Psi} + 1} 4\pi p^2 dp$$

where  $\Psi = \mu/kT$  is the degeneracy parameter  
the limitation imposed by the Pauli exc. ppl.  
Means that e- can exert a higher pressure  
compared to the predicted by the MB eq.

$$n_{\text{MB}}(p)dp = \frac{n_e}{(2\pi m_e kT)^{3/2}} e^{-p^2/2m_e kT} 4\pi p^2 dp$$

this extra pressure is the *degeneracy pressure*

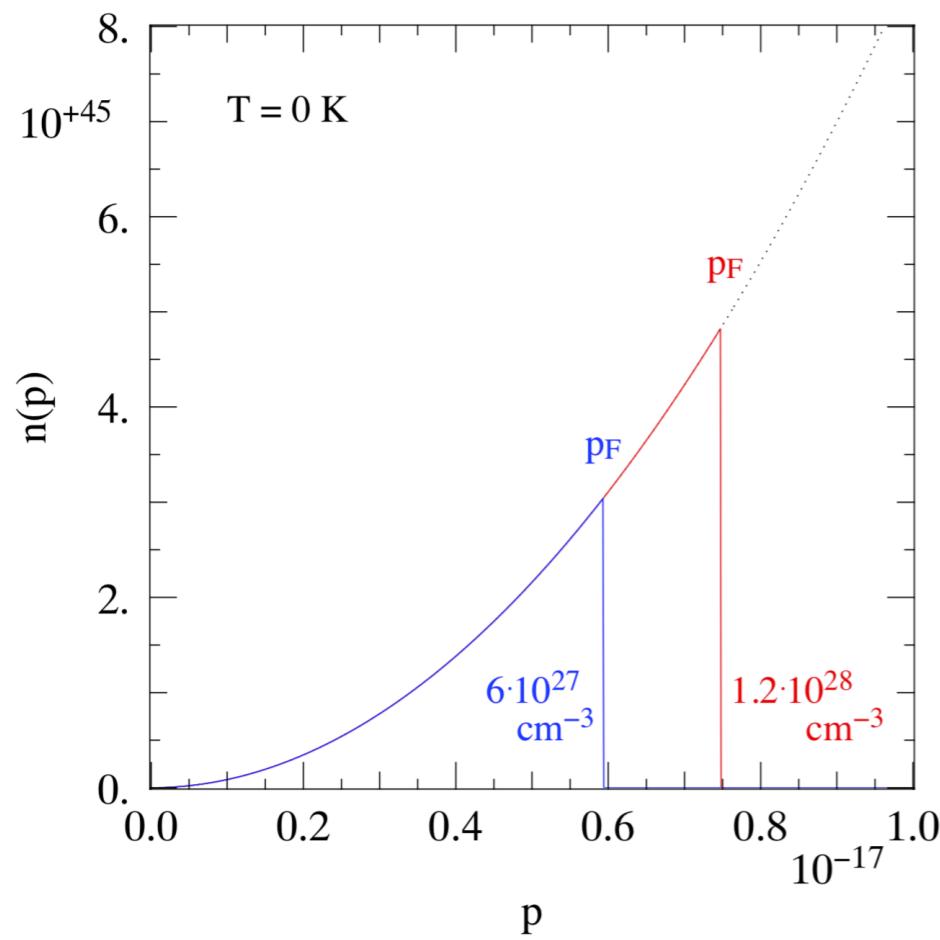
# Electron degeneracy

$\Psi = \mu/kT$  is the degeneracy parameter

$\Psi \gg 0$	complete degeneracy ( $T \sim 0$ ; $f(\epsilon_p) \sim 1$ )
$\Psi \sim 0$	partial degeneracy
$\Psi \ll 0$	no degeneracy

*(Maxwell-Boltzmann)*

## complete degeneracy



$$n_e(p) = \frac{8\pi p^2}{h^3} \quad \text{for } p \leq p_F$$

$$n_e(p) = 0 \quad \text{for } p > p_F$$

Depends on  $n_e$

Non-relativistic ( $v=p/m$ )       $3 \times 10^7 < \rho < 6 \times 10^{15} \text{ g/cm}^3$

$$P_e = \frac{1}{3} \int_0^{p_F} \frac{8\pi p^4}{h^3 m_e} dp = \frac{h^2}{20m_e} \left( \frac{3}{\pi} \right)^{2/3} m_u^{-5/3} \left( \frac{\rho}{\mu_e} \right)^{5/3}$$

Extremelly relativistic ( $v=c$ )       $\rho > 6 \times 10^{15} \text{ g/cm}^3$

$$P_e = \frac{1}{3} \int_0^{p_F} \frac{8\pi p^3}{h^3} dp = \frac{hc}{8} \left( \frac{3}{\pi} \right)^{1/3} m_u^{-4/3} \left( \frac{\rho}{\mu_e} \right)^{4/3}$$

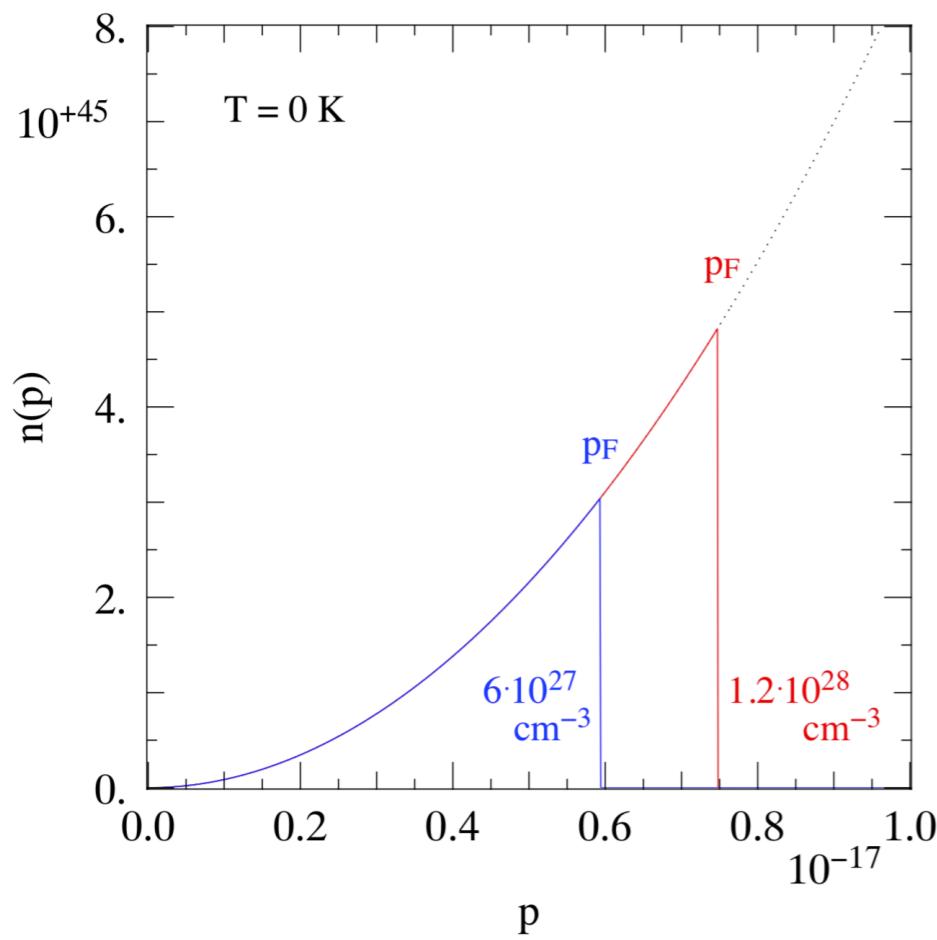
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## complete degeneracy



$$n_e(p) = \frac{8\pi p^2}{h^3}$$

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for  $p \leq p_F$

for  $p > p_F$

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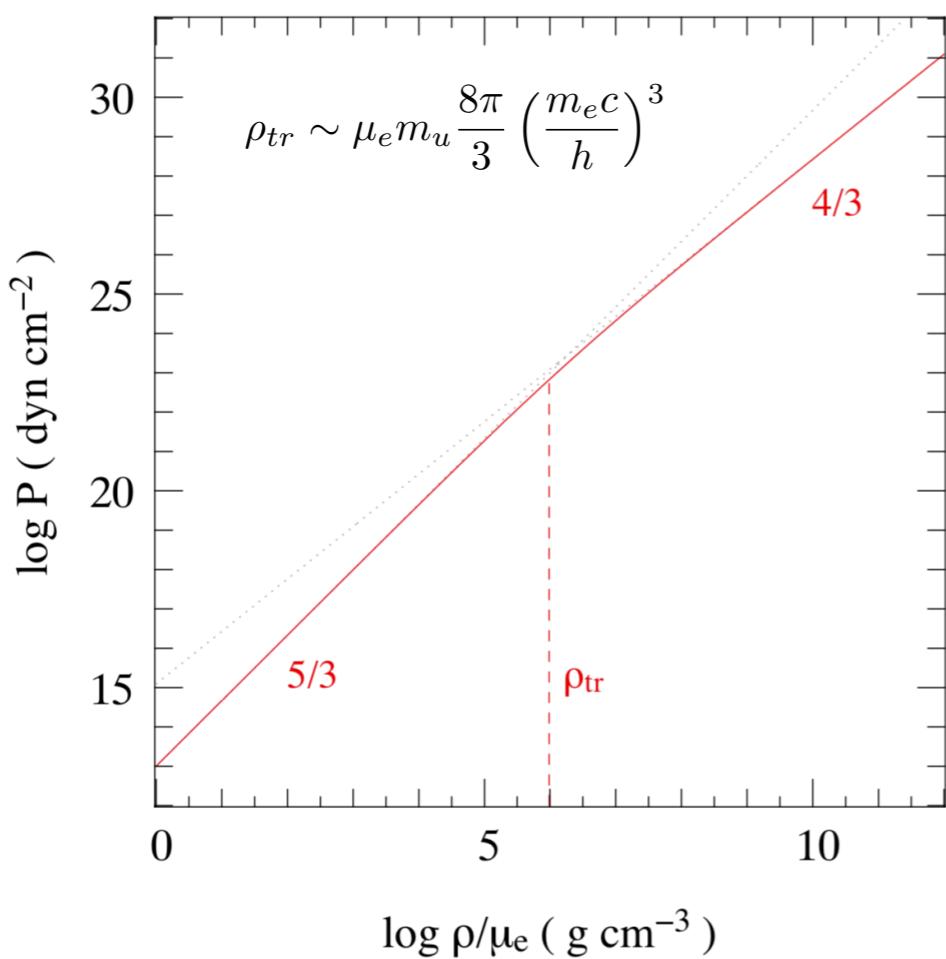
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*(Maxwell-Boltzmann)*

## complete degeneracy



$$n_e(p) = \begin{cases} \frac{8\pi p^2}{h^3} & \text{for } p \leq p_F \\ 0 & \text{for } p > p_F \end{cases}$$

Depends on  $n_e$

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# Radiation pressure

Photons can be treated as quantum-mechanical particles that carry momentum and exert pressure

Photons are bosons with  $g_s = 2$ , so they are described by the Bose-Einstein statistics,  $f_{BE}(\epsilon_p)$ , with  $\mu=0$  b/c they can be destroyed and created

$$n(p)dp = \frac{2}{h^3} \frac{1}{e^{\epsilon_p/kT} - 1} 4\pi p^2 dp \quad \xrightarrow{\epsilon_p = pc = h\nu} \quad n(\nu)d\nu = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1}$$

Planck function for BB radiation

Photon number density

$$n_\gamma = \int_0^\infty n(p)dp = bT^3 \quad b = 20.3 \text{ cm}^{-3}\text{K}^{-3}$$

Energy density of radiation

$$U_\gamma = \int_0^\infty pc n(p)dp = aT^4 \quad a = \frac{8\pi k^4}{15h^3 c^3} = 7.56 \times 10^{-15} \text{ erg cm}^{-3}\text{K}^{-4}$$

Since they are always extremely-relativistic

$$P = \frac{1}{3}U = \frac{1}{3}aT^4$$

# Radiation pressure

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Photons are bosons with  $g_s = 2$ , so they are described by the Bose-Einstein statistics,  $f_{BE}(\epsilon_p)$ , with  $\mu=0$  b/c they can be destroyed and created

Total pressure:

$$P = P_{\text{rad}} + P_{\text{ion}} + P_e = \frac{1}{3}aT^4 + \frac{\rho kT}{\mu_{\text{ion}} m_u} + P_e$$

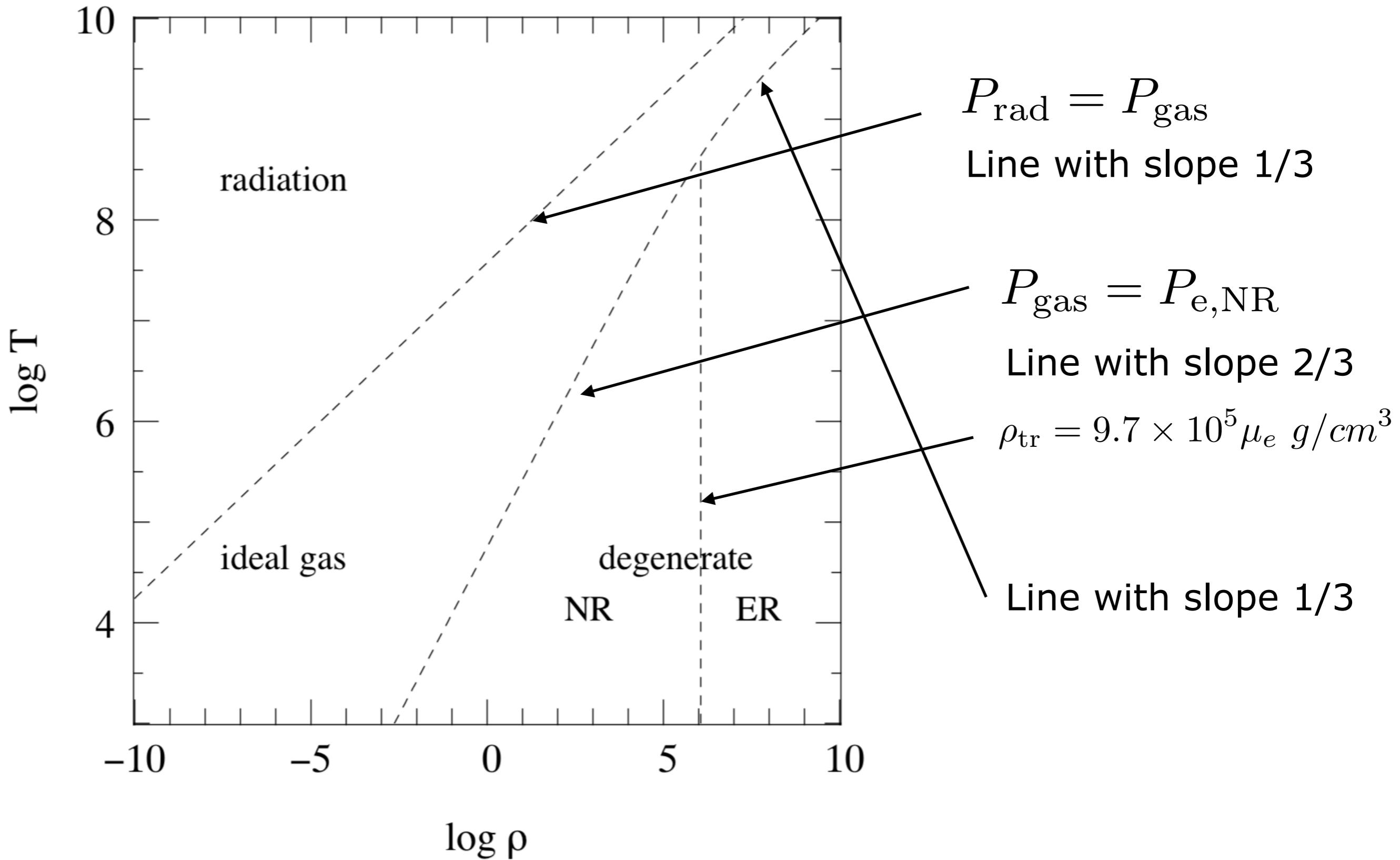
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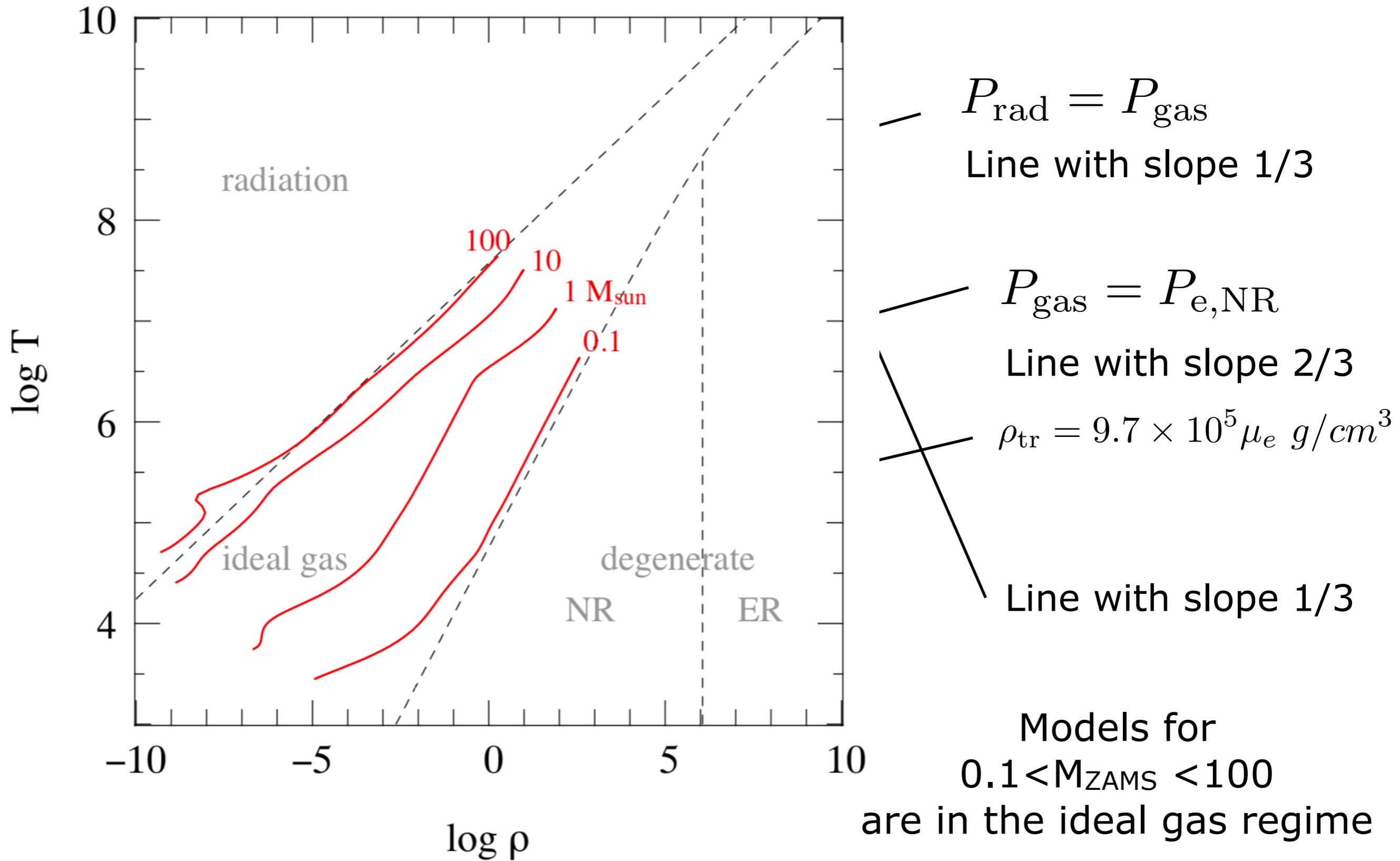
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$$P = \frac{1}{3}U = \frac{1}{3}aT^4$$

# EOS regimes



# EOS regimes



# Adiabatic processes

Process that occur on  $t \sim \tau_{\text{dyn}}$ , so there is no heat exchange

$$\frac{dq}{\text{Change of heat}} = T \frac{\overline{ds}}{\text{Specific entropy}} = du + P \frac{\overline{dv}}{\text{Volume per unit mass}} = \frac{du}{\text{Change of internal energy}} - \frac{P}{\rho^2} d\rho$$

1st and 2nd  
thermodynamic laws

to calculate specific heats, we need EOS in differential form

$$\frac{dP}{P} = \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho}$$

where

$$\chi_T = \left( \frac{\partial \log P}{\partial \log T} \right)_{\rho, X_i} = \frac{T}{P} \left( \frac{\partial P}{\partial T} \right)_{\rho, X_i}$$

$$\chi_\rho = \left( \frac{\partial \log P}{\partial \log \rho} \right)_{T, X_i} = \frac{\rho}{P} \left( \frac{\partial P}{\partial \rho} \right)_{T, X_i}$$

For instance, for a ideal gas  $\chi_\rho \sim \chi_T \sim 1$

for a radiation-dominated gas  $\chi_T = 4 ; \chi_\rho = 0$

# Adiabatic processes

The specific heats at constant volume and pressure for a unit mass

$$c_V = \left( \frac{dq}{dT} \right)_V = \left( \frac{\partial u}{\partial T} \right)_V$$

$$c_P = \left( \frac{dq}{dT} \right)_P = \left( \frac{\partial u}{\partial T} \right)_P - \frac{P}{\rho^2} \left( \frac{\partial \rho}{\partial T} \right)_P$$

For an ideal gas,  $u = U/\rho = \frac{3}{2}P/\rho = \frac{3}{2}\frac{PkT}{\mu m_u \rho}$

$$c_V = \frac{3}{2} \frac{k}{\mu m_u} = \frac{3 \mathfrak{R}}{2 \mu}$$

For a radiation-dominated gas,  $u = U/\rho = \frac{aT^4}{\rho}$

$$c_V = \frac{4aT^3}{\rho}$$

Generally,  $c_P - c_V = \frac{P}{\rho T} \frac{\chi_T^2}{\chi_P}$

So,  $c_P = \frac{5 \mathfrak{R}}{2 \mu}$  for an ideal gas, and  $c_P \rightarrow \infty$  for a rad-dom. gas

$P_{\text{rad}}$  only depends on T, so a change in T cannot be performed at const P

The ratio of specific heats,  $\gamma = \frac{c_P}{c_V} = 1 + \frac{P}{\rho T c_V} \frac{\chi_T^2}{\chi_\rho}$  = 5/3 for an ideal gas

# Adiabatic derivatives

The response of a system to adiabatic changes is measured by the *adiabatic derivatives*

**Adiabatic exponent:** measures the response of the pressure to adiabatic compression or expansion (change in density)

for an adiabatic process  $dq=0$ , so

$$du = \frac{P}{\rho^2} d\rho$$

For a simple, perfect gas

$$u = \phi \frac{P}{\rho} \quad \rightarrow \quad \frac{dP}{P} = \frac{\phi + 1}{\phi} \frac{d\rho}{\rho} = \gamma_{ad} \frac{d\rho}{\rho}$$

$\phi$  a constant between 3/2 and 3

In general,

$$\gamma_{ad} = \left( \frac{\partial \log P}{\partial \log \rho} \right)_{ad} = \chi_\rho + \frac{P}{\rho T c_V} \chi_T^2 = \gamma \chi_\rho$$

For ideal gas and non-relativistic particles

$$\phi = \frac{3}{2} \quad \rightarrow \quad \gamma_{ad} = \frac{5}{3}$$

For extremely relativistic particles (photons & deg. e-)

$$\phi = 3 \quad \rightarrow \quad \gamma_{ad} = \frac{4}{3}$$

For a mixture of gas and radiation

$$\frac{4}{3} \leq \gamma_{ad} \leq \frac{5}{3}$$

# Adiabatic derivatives

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**Adiabatic temperature gradient:** describes the behaviour of the temperature under adiabatic compression or expansion

$$\nabla_{\text{ad}} = \left( \frac{\partial \log T}{\partial \log P} \right)_{\text{ad}}$$

It relates too the adiabatic exponent as,

$$\nabla_{\text{ad}} = \frac{\gamma_{\text{ad}} - \chi_{\rho}}{\gamma_{\text{ad}} \chi_T} = \frac{P}{\rho T c_P} \frac{\chi_T}{\chi_{\rho}}$$

For ideal gas and non-relativistic particles       $\chi_T = \chi_{\rho} = 1$        $\gamma_{\text{ad}} = \frac{5}{3}$        $\nabla_{\text{ad}} = \frac{2}{5} = 0.4$

For extremely relativistic particles (photons & deg. e-)       $\chi_T = 4 ; \chi_{\rho} = 0$        $\nabla_{\text{ad}} = \frac{1}{4} = 0.25$

For a mixture of gas and radiation       $0.25 \leq \nabla_{\text{ad}} \leq 0.4$

# Ionization

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So far, we assumed complete ionization of the gas. This is a good approx. at  $T > 10^6$  K, when typical energies  $kT \gg$  ionization of the atom

For lower T, we need to consider *partial ionization*, and that changes in  $\rho, T, P$  could change the degree of ionization.

In LTE the number densities of ionized and neutral species are determined by the *Saha equation*

$$\frac{n_{r+1}}{n_r} = \frac{\text{partition functions}}{\frac{u_{r+1}}{u_r} \frac{2(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}}$$

Ionization potential

r: times ionized

# Ionization

For only hydrogen  $u_H = u_0 = 2$  and  $u_{H+} = u_1 = 1$ , so

$$\frac{n_+}{n_0} n_e = \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_H/kT} \quad \text{and } \chi_H = 13.6 \text{ eV}$$

the gas pressure is  $P_{\text{gas}} = (n_0 + n_+ + n_e)kT$  and density  $\rho = (n_0 + n_+)m_u$

the degree of ionization is defined as  $x = \frac{n_+}{n_0 + n_+}$  so  $P_{\text{gas}} = (1 + x)\mathfrak{R}\rho T = \frac{kT\rho}{m_u \mu}$

and, we can rewrite the Saha equation as,

$$\frac{x}{1 - x^2} = \frac{(2\pi m_e)^{3/2}}{h^3} \frac{(kT)^{5/2}}{P_{\text{gas}}} e^{-\chi_H/kT} \quad \begin{aligned} &x \uparrow \text{ when } T \uparrow \\ &x \downarrow \text{ when } P \uparrow \text{ (with } T \text{ ctnt)} \\ &x \uparrow \text{ when } \mu = 1/(1 + x) \downarrow \end{aligned}$$

In the case of partial ionization, internal energy has an additional contribution from the available potential energy of recombination

$$n_+ \chi_H \text{ per unit volume}$$
$$n_+ \chi_H / \rho = x \chi_H / m_u \text{ per unit mass}$$

$$u = \frac{3}{2} \frac{P_{\text{gas}}}{\rho} + x \frac{\chi_H}{m_u} = \frac{3}{2} (1 + x) \mathfrak{R}T + x \frac{\chi_H}{m_u}$$

# Coulomb interaction

We have so far ignored the effect of electrostatic interaction because normally kinetic energy is much more important

Is this a good approximation? It is measured by the Coulomb parameter  $\Gamma_C$  the ratio of Coulomb energy to kinetic energy

$$\Gamma_C = \frac{Z^2 e^2}{dkT} = \frac{Z^2 e^2}{kT} \left( \frac{4\pi\rho}{3Am_u} \right)^{1/3} = 2.275 \times 10^5 \frac{Z^2}{A^{1/3}} \frac{\rho^{1/3}}{T}$$

Where,  $d \sim \left( \frac{4\pi}{3} n \right)^{-1/3}$  ;  $n = \rho/(Am_u)$  ;  $\epsilon_C \sim Z^2 e^2/d$  ;  $\epsilon_{\text{kin}} = \frac{3}{2} kT$

$\Gamma_C$  becomes important at high densities or low temperatures (i.e.  $\Gamma_C \gtrapprox 1$ )

At stellar interiors, we approximate

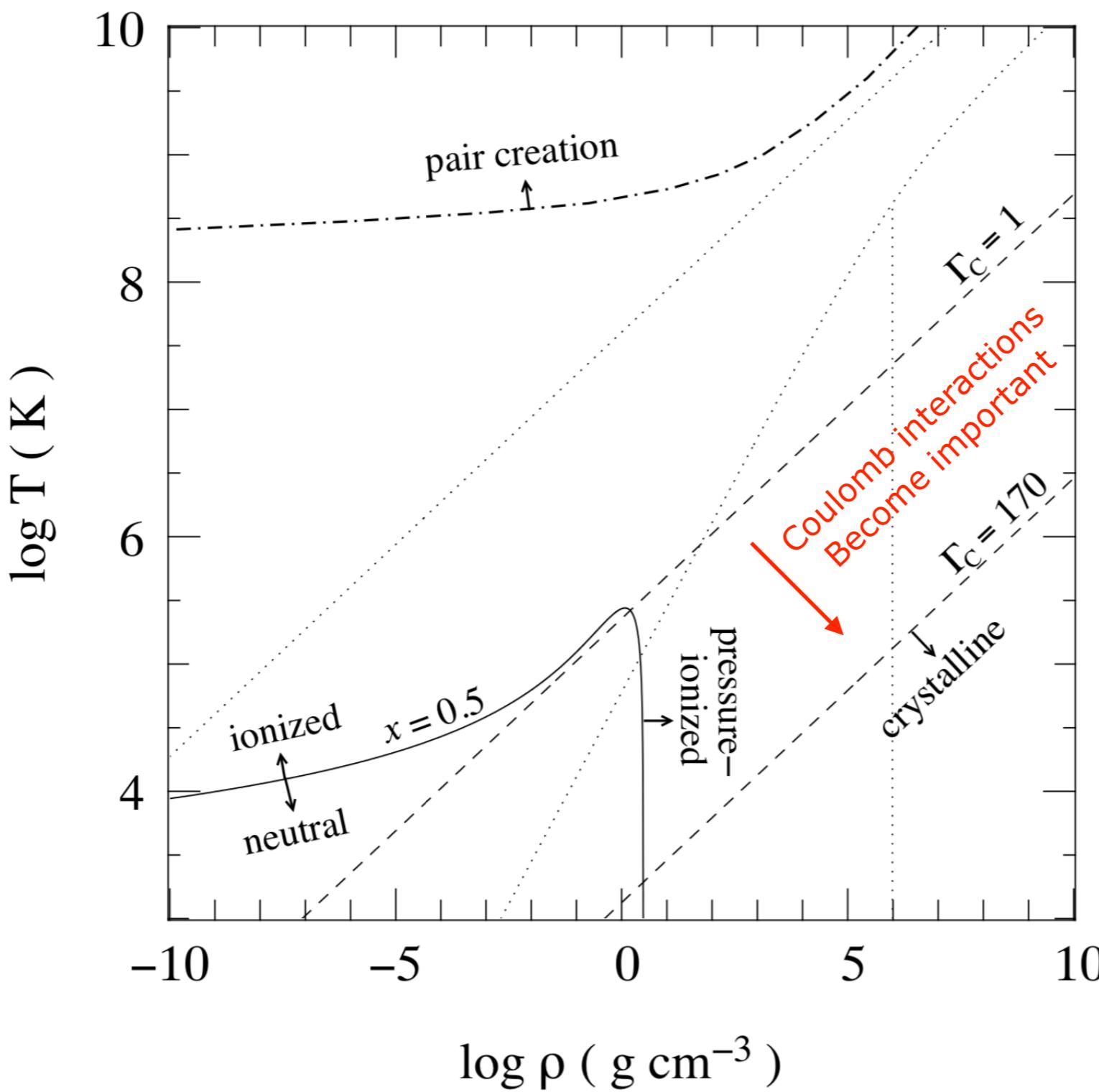
$$\rho \sim M / \left( \frac{4\pi}{3} R^3 \right)$$

$$T \sim \frac{1}{3} \frac{Am_u}{k} \frac{GM}{R}$$

$$\Gamma_C \sim 0.01 \frac{Z^2}{A^{4/3}} \left( \frac{M}{M_\odot} \right)^{-2/3}$$

It is very important for  $M \ll M_\odot$   
(planets)

# Crystallization and pair production



For  $\Gamma_C \gg 1$ , ions will tend to settle down and form a crystalline lattice, where e- move freely

This could happen in cooling white dwarfs interiors, but rarely in H stars

For  $\Gamma_C \ll 1$ , high T and low  $\rho$ , photons may turn into  $e^-/e^+$  pairs if  $h\nu > 2m_e c^2$

Usually at  $T > 1.2 \times 10^{10} \text{ K}$

# Exercises (1)

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## 3.1 Conceptual questions

These questions are intended to test your understanding of the lectures. Try to answer them without referring to the lecture notes.

- (a) What do we mean by *local thermodynamic equilibrium* (LTE)? Why is this a good assumption for stellar interiors? What is the difference between LTE and *thermal equilibrium* (as treated in Ch. 2)?
- (b) In what type of stars does degeneracy become important? Is it important in main-sequence stars? Is it more important in high mass or low mass MS stars?
- (c) Explain qualitatively why for degenerate matter, the pressure increases with the density.
- (d) Why do electrons become relativistic when they are compressed into a smaller volume? Why does the pressure increase less steeply with the density in this case?
- (e) In the central region of a star we find free electrons and ions. Why do the electrons become degenerate first? Why do the ions never become degenerate in practice?

## 3.2 Mean molecular weight

Derive a general expression for the mean molecular weight of an ionized gas, as a function of composition  $X, Y, Z$ . Assume that, for elements heavier than H, nuclei are composed of equal number of protons and neutrons, so that the nuclear charge  $Z_i$  is half of the mass number  $A_i$ .

## 3.3 The $\rho - T$ plane

Consider a gas of ionized hydrogen. In the  $\rho-T$  plane compute the approximate boundary lines between the regions where:

- (a) radiation pressure dominates,
- (b) the electrons behave like a classical ideal gas,
- (c) the electrons behave like a degenerate gas,
- (d) the electrons are relativistically degenerate.

# Exercises (2)

## 3.4 The pressure of a gas of free particles

In this exercise you will derive some important relations from this chapter for yourself.

- (a) Suppose that the particles in a gas have momenta distributed as  $n(p)dp$ . Show that the pressure can be expressed by eq. Pressure (slide 5).
- (b) For classical particles in LTE, the momentum distribution is given by the Maxwell-Boltzmann distribution (slide 8). Calculate the pressure using eq. Pressure (slide 5). Does the result look familiar?
- (c) Show that for a gas of free, non-relativistic particles  $P = 2/3 U$ , where  $U$  is the internal energy density. Show that in the extremely relativistic limit  $P = 1/3 U$ ;
- (d) Electrons are fermions with 2 spin states. Explain why the maximum number of electrons per volume with momentum  $p$  can be written as eq. In (slide 11).
- (e) In the extreme case of complete degeneracy,  $T \rightarrow 0$ , the electrons fill up all available quantum states up to a maximum  $p_F$ , the Fermi momentum. Show that
$$p_F = h \left( \frac{3n_e}{8\pi} \right)^{\frac{1}{3}}$$
- (f) Show that the pressure as a function of the density for a non-relativistic degenerate electron gas can be written as
$$P = K_{NR} \left( \frac{\rho}{\mu_e} \right)^x$$
and derive an expression for  $K_{NR}$  and  $x$ .
- (g) Show that the pressure as function of the density for an extremely relativistic degenerate electron gas can be written as
$$P = K_{ER} \left( \frac{\rho}{\mu_e} \right)^y$$
and derive an expression for  $K_{ER}$  and  $y$ .
- (h) Photons are bosons, and the distribution of their momenta is given by the Planck function (slide 10). Show that in this case  $U \propto T^4$   
(Hint: to derive an expression for the proportionality constant  $a$ , you might want to use Mathematica or a list of standard integrals.)
- (i) Now use (c) to show that the radiation pressure is given by  $P_{rad} = 1/3 aT^4$ .

# Exercises (3)

## 3.5 Adiabatic derivatives

- (a) Use the first law of thermodynamics to show that, for an ideal gas in an adiabatic process,  $P \propto \rho^{\gamma_{ad}}$  and give a value for the adiabatic exponent  $\gamma_{ad}$ .
- (b) Use the ideal gas law in combination with eq. in (a) to show that  $\nabla_{ad} = \left( \frac{d \ln T}{d \ln P} \right)_{ad,id} = 0.4$
- (c) The quantity  $\nabla_{ad}$  is referred to as the *adiabatic temperature gradient*. Normally you would use the term 'gradient of a quantity  $A$ ' for  $dA/dr$ , or if you use mass coordinates instead of radius coordinates,  $dX/dm$ . Do you understand why  $\nabla_{ad}$  can be referred to as a temperature 'gradient'?
- (d) Show that for a mixture of an ideal gas plus radiation, the adiabatic exponent is given by  
where  $\beta = P_{\text{gas}}/P$  
$$\gamma_{ad} = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}$$

(Hints: write down the equation of state for the mixture in differential form (slide 15), and express  $xT$  and  $x\rho$  in terms of  $\beta$ . Then apply the first law of thermodynamics for an adiabatic process.)

- (e) What is the value of  $\gamma_{ad}$  in the limit where radiation dominates and where pressure dominates? Does this look familiar?

## 3.6 Ionization effects

- (a) The particles in an ionized gas are charged and therefore undergo electrostatic (Coulomb) interactions. Why can we nevertheless make the ideal-gas assumption in most stars (i.e. that the internal energy of the gas is just the sum of the kinetic energies of the particles)? For which stars do Coulomb interactions have a significant effect?
- (b) Why does the gas in the interior of a star become pressure-ionized at high densities?
- (c) Explain qualitatively why partial ionization leads to  $\nabla_{ad} < \nabla_{ad,\text{ideal}} = 0.4$ , in other words: why does adiabatic compression lead to a smaller temperature increase when the gas is partly ionized, compared to a completely ionized (or unionized) gas?

# Exercises (4)

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- 3.7** Show that for a fully ionized gas the number of electrons per gram is  $(1+X)/2m_H$  and that  $\mu_e = 2/(1+X)$
- 3.8** Plot the run of temperature versus density in the Sun in Figure (slide 14). What do you conclude about the equation of state?  
(Data: [https://www.ap.smu.ca/~guenther/evolution/ssmyz47\\_0200.txt](https://www.ap.smu.ca/~guenther/evolution/ssmyz47_0200.txt))
- 3.9** Plot the values of the central temperatures versus densities of zero-age main-sequence stars of 1, 4, 12, 60, and  $120M_\odot$  in Figure (slide 14). What do you conclude?
- 3.10** Compare the densities at which neutrons and electrons become degenerate at the same temperature. Hint: consider two effects.
- Energy exchange results in  $\langle m_e v_e^2 \rangle = 3/2 kT = \langle m_n v_n^2 \rangle$ . Calculate the ratio of the momenta  $p_n/p_e$  of neutrons and electrons.
  - Degeneracy occurs when  $d^3p \sim h^3/\Delta Vol$ , with  $\Delta Vol \sim 1/n_n$ . Calculate the ratio  $n_n/n_e$  when the values of  $(d^3p \times \Delta Vol)$  for electrons and neutrons are equal.
  - Calculate the corresponding density ratio  $\rho_n/\rho_e$ .
- 3.11** Compare the expected radius of a non-relativistic complete electron degenerate star with that of a non-relativistic complete neutron degenerate star of the same mass.  
Comment on the result.

# Questions

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**Q (3.1)** Calculate  $P_{\text{gas}}$  and  $P_{\text{rad}}$  in the center of the Sun and show that  $P_{\text{rad}} \ll P_{\text{gas}}$  (use data in Appendix C).

**Q (3.2)** What is the role of temperature in the transition from the Maxwellian to the partial degenerate distributions?

Hint: consider the width of the distributions for different temperatures.

**Q (3.3)** What is the role of the particle mass?

**Q (3.4)** Why does this require a density limit and not a temperature limit?

**Q (3.5)** Can a fully convective star be completely adiabatic?

**Q (3.6)** Explain why  $\gamma = 5/3$  at both the high- and low-temperature ends.

**Q (3.7)** What is the physical reason that  $\gamma$  drops in a partially ionized region?