FisyMat
Máster en Matemática Aplicada

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4. Polytropic stellar models

Polytropes

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$$

n: polytrope index
K: polytrope ctnt

(Hint: this simple P/ρ relation has no T-dependence)

Examples seen already:

$$\frac{dP}{P} = \frac{\phi+1}{\phi}\frac{d\rho}{\rho} = \gamma_{\mathrm{ad}}\frac{d\rho}{\rho}$$

Adiabatic processes (dq=0)

$$P_e = \frac{1}{3}\int_0^{p_F} \frac{8\pi p^4}{h^3 m_e} dp = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} m_u^{-5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

completely e^--degenerate stars

fully convective stars

$$P_e = \frac{1}{3}\int_0^{p_F} \frac{8\pi p^3}{h^3} dp = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} m_u^{-4/3}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

relativistic degenerate stars

stars dominated by P_{rad}

stars with a constant P_{gas}/P_{tot}

Polytropes

Stellar evolution equations (HE) can be solved if EOS is P=P(ρ)

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relativistic degenerate stars

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Polytropes

Stellar evolution equations (HE) can be solved if EOS is P=P(ρ)

In particular, we call *polytropes* when the EOS is

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$$

n: polytrope index
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(Hint: this simple P/ρ relation has no T-dependence)

Examples seen already:

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completely e$^-$-degenerate stars
fully convective stars

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relativistic degenerate stars
stars dominated by P_{rad}
stars with a constant P_{gas}/P_{tot}

Structure of polytropes

In polytropes, combining *mass continuity* eq. (dm/dr) and the *eq. of motion* in HE (dP/dr),

$$\left.\begin{array}{r} \frac{r^2}{\rho}\frac{dP}{dr} = -Gm \rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \\ \frac{dm}{dr} = 4\pi r^2 \rho \end{array}\right\} \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

and with the EOS, $\frac{dP}{dr} = K\gamma\rho^{\gamma-1}\frac{d\rho}{dr}$

$$\frac{K\gamma}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -\rho$$

With 2 b.c. in the center (r=0)

$$\rho(0) = \rho_c$$

$$\left(\frac{d\rho}{dr}\right)_{r=0} = 0$$

And defining,

$$\rho = \rho_c w^n \qquad w = \left(\frac{\rho}{\rho_c}\right)^{1/n}$$

$$r = \alpha z \qquad \alpha = \left(\frac{n+1}{4\pi G}K\rho_c^{1/n-1}\right)^{1/2}$$

Structure of polytropes

We end up getting the *Lane-Emden equation*

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right)+w^n=0$$

From which a polytropic stellar model can be constructed by integrating it outwards from the center $\int_0^\infty dz$.

The equation describes the density structure *w* with only one parameter *n*

There are only 3 analytical solutions (*n*=0, 1, 5).

For *n*<5 *w(z)* decrease monotonically and reach 0 at $z=z_n$

Structure of polytropes

Table 4.1. Numerical values for polytropic models with index n.

n	z_n	Θ_n	$\rho_c/\bar{\rho}$	N_n	W_n
0	2.44949	4.89898	1.00000	...	0.119366
1	3.14159	3.14159	3.28987	0.63662	0.392699
1.5	3.65375	2.71406	5.99071	0.42422	0.770140
2	4.35287	2.41105	11.40254	0.36475	1.638183
3	6.89685	2.01824	54.1825	0.36394	11.05068
4	14.97155	1.79723	622.408	0.47720	247.559
4.5	31.8365	1.73780	6189.47	0.65798	4921.84
5	∞	1.73205	∞	∞	∞

$$r = \alpha z$$

$$\gamma = 1 + \frac{1}{n}$$

$$\rho = \rho_c w^n$$

$n = 0; \gamma = \infty$ $\quad$ $w(z) = 1 - \frac{z^2}{6}$ $\quad$ $z_0 = \sqrt{(6)}$ $\quad$ Polytrope with constant ρ

(neutron stars 0.5<n<1.0)

$n = 1; \gamma = 2$ $\quad$ $w(z) = 1 - \frac{\sin z}{z}$ $\quad$ $z_1 = \pi$

(WD NR n~1.5)

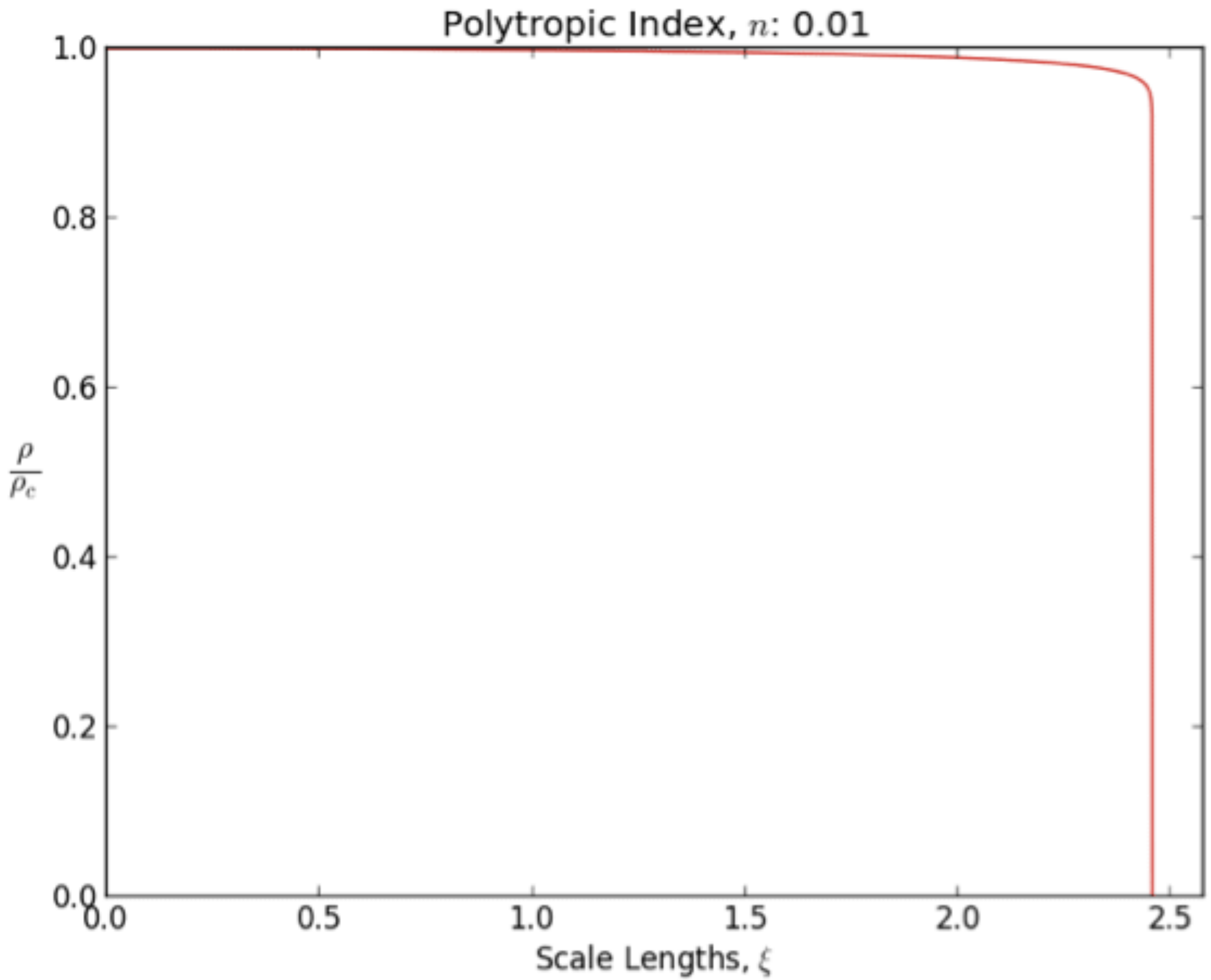
(WD ER n~3.0)

$n = 5; \gamma = 6/5$ $\quad$ $w(z) = 1 - \left(1 + \frac{z^2}{3}\right)^{-1/2}$ $\quad$ $z_5 = \infty$ $\quad$ Infinite R

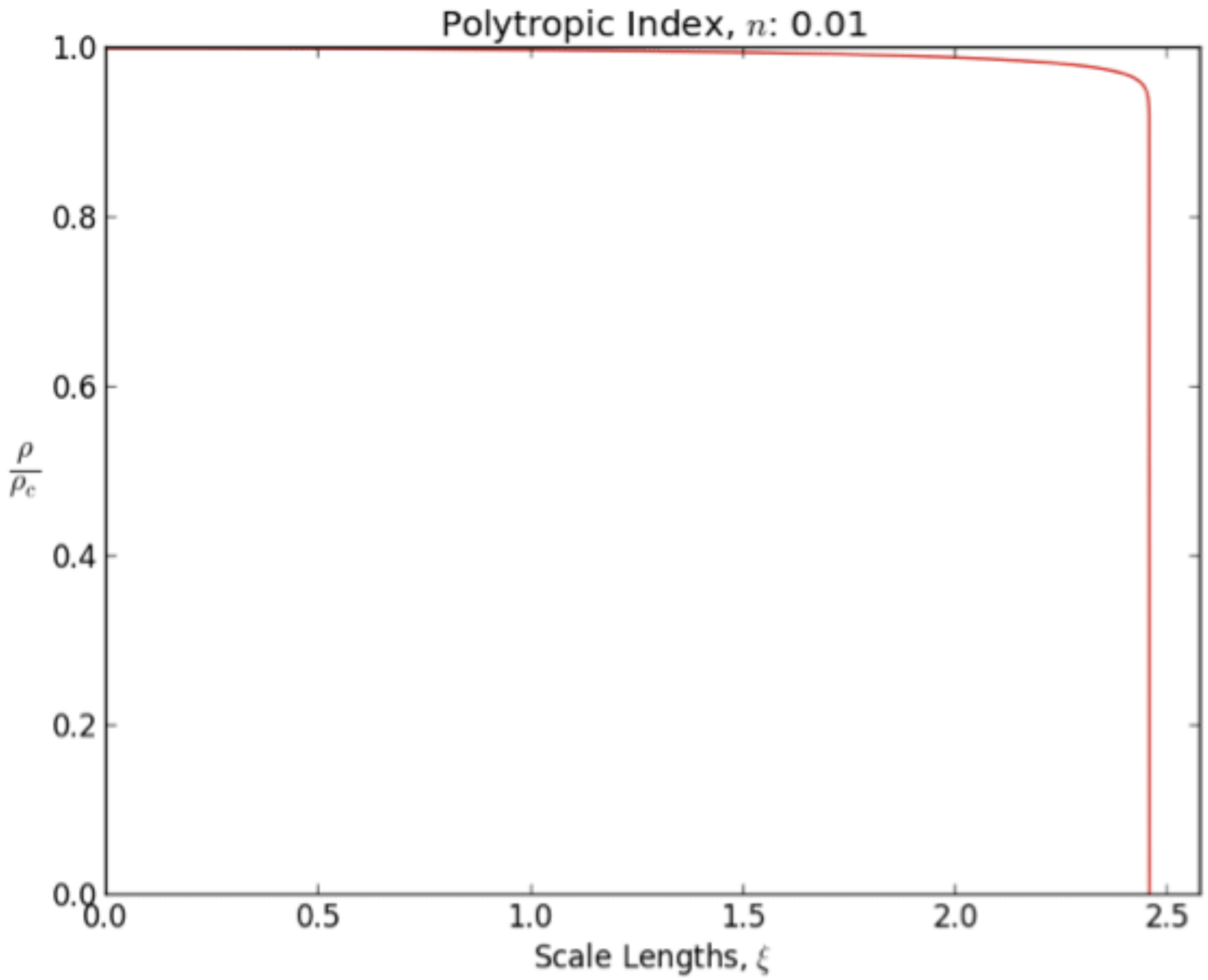
(Also for n>5)

$n = \infty; \gamma = 1$ $\quad$ $w(z) = 1$ $\quad$ $z_5 = \infty$

Structure of polytropes



Structure of polytropes



Physical properties of the solutions

Once we have solved for *w*, the density distribution is determined by *n*

The physical properties of a polytropic stellar model are:

Radius, $R(K, \rho_c)$

$$R = \alpha z_n = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n} z_n$$

Mass interior to z,

$$m(z) = \int_0^{\alpha z} 4\pi r^2 \rho dr = -4\pi\alpha^3 \rho_c z^2 \frac{dw}{dz}$$

Total mass, $M(K, \rho_c)$

$$M = 4\pi\alpha^3\rho_c \underbrace{\left(-z^2\frac{dw}{dz}\right)_{z=z_n}}_{\Theta_n} = 4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-)/2n} \left(-z^2\frac{dw}{dz}\right)_{z=z_n}$$

Physical properties of the solutions

Eliminating ρ_c from R and M equations:

R-M relation,

$$R = (4\pi)^{\frac{1}{n-3}} \left[\frac{(n+1)K}{G}\right]^{\frac{n}{3-n}} \underbrace{\left[-z^2 \frac{dw}{dz}\right]_{z=z_n}^{\frac{n-1}{3-n}}}_{\Theta_n} M^{\frac{1-n}{3-n}}$$

K-R-M relation,

$$K = \underbrace{\frac{(4\pi)^{1/n}}{n+1} \left(-z^2 \frac{dw}{dz}\right)_{z=z_n}^{(1-n)/n} z_n^{(n-3)/n}}_{N_n} G M^{(n-1)/n} R^{(3-n)/n}$$

From the above, one sees that n=1 and n=3 are special cases

For n=1, R is independent of M, and only depends on K: R(K)

For n=3, M is independent of R, and only depends on K: M(K)

Physical properties of the solutions

Central density,

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \left(-\frac{3}{z}\frac{dw}{dz}\right)_{z=z_n} \rho_c = \frac{3\Theta_n}{z_n^3}\rho_c$$

ρ_c/ρ

Concentration index of the polytope, only depends on n

Central pressure,

$$P_c = K\rho_c^{(n+1)/n} = \overbrace{\boxed{\frac{1}{4\pi(n+1)}\left(-\frac{dw}{dz}\right)^{-2}_{z=z_n}}}^{W_n} \frac{GM^2}{R^4}$$

$$= \underbrace{\boxed{\frac{(4\pi)^{1/3}}{(n+1)}\left(-z_n^2\frac{dw}{dz}\right)^{-2/3}_{z=z_n}}}_{C_n} \frac{GM^{2/3}}{\rho_c^{4/3}}$$

Central temperature,

$$T_c = \frac{\mu m_u}{K\rho_c}P_c = \frac{\mu m_u}{(n+1)K}\left(-z^2\frac{dw}{dz}\right)^{-1}\frac{GM}{R}$$

Application to stars

For white dwarfs, where P only depends of ρ (indep. of T), is well described by polytropic models

Non-relativistic $$P_e = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3} m_u^{-5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$ Polytrope n=1.5 $\gamma = 1 + \frac{1}{n} = \frac{5}{3}$

$R \propto M^{-\frac{1}{3}}$ So when M↑ ⇨ R↓ ⇨ ρ↑ ...which goes towards ER

Extremely relativistic $$P_e = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} m_u^{-4/3}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$ Polytrope n=3.0 $\gamma = 1 + \frac{1}{n} = \frac{4}{3}$

M does not depend on R, and has an unique M

$$M = 4\pi\left(-z_n^2\frac{dw}{dz}\right)_{z=z_n}\left(\frac{K}{\pi G}\right)^{3/2} = 5.836\mu_e^{-2}M_\odot = 1.46M_\odot$$

which is a upper limit of a sphere in HE supported by e$^-$ deg.
so then, the maximum mass of a white dwarf (Ch. limit)

Application to stars

Eddington' standard model:

We consider a star where $P_{gas} = \beta P$, $P_{rad} = (1-\beta)P$, where β is fixed

$$\beta = \frac{P_{\rm gas}}{P} = \frac{\Re}{\mu}\frac{\rho T}{P} \qquad 1-\beta = \frac{P_{\rm rad}}{P} = \frac{aT^4}{3P}$$

and

$$P = \left(\frac{3\Re^4}{a\mu^4}\frac{1-\beta}{\beta^4}\right)^{1/3}\rho^{4/3}$$

Polytrope n=3.0

With constant β

Since we are free to choose $0 < \beta < 1$, the constant $K(\beta)$

This is Eddington's standard model equation, which explains well most MS stars

Exercices (1)

4.1 The Lane-Emden equation

(a) Derive eq. (4.2) from the stellar structure equations for mass continuity and hydrostatic equilib- rium. (Hint: multiply the hydrostatic equation by r^2/ρ and take the derivative with respect to r).

(b) What determines the second boundary condition of eq. (4.4), i.e., why does the density gradient have to vanish at the center?

(c) By making the substitutions (4.3), (4.5) and (4.6), derive the Lane-Emden equation (4.7).

(d) Solve the Lane-Emden equation analytically for the cases $n = 0$ and $n = 1$.

4.2 Polytropic models

(a) Derive K and γ for the EOS of an ideal gas at a fixed T, of a non-relativistic degenerate gas and of a relativistic degenerate gas.

(b) Using the Lane-Emden equation, show that the mass distribution in a polytropic star is given by eq. (4.12), and show that this yields eq. (4.13) for the total mass of a polytrope.

(c) Derive the expressions for the central density ρ_c and the central pressure P_c as function of mass and radius, eqs. (4.16) and (4.17).

(d) Derive eq. (4.18) and compute the constant C_n for several values of n.

4.3 White dwarfs

To understand some of the properties of white dwarfs (WDs) we start by considering the equation of state for a degenerate, non-relativistic electron gas.

(a) What is the value of K for such a star? Remember to consider an appropriate value of the mean molecular weight per free electron μ_e.

(b) Derive how the central density ρ_c depends on the mass of a non-relativistic WD. Using this with the result of Exercise 4.2(b), derive a radius-mass relation $R = R(M)$. Interpret this physically.

(c) Use the result of (b) to estimate for which WD masses the relativistic effects would become important.

(d) Show that the derivation of a $R = R(M)$ relation for the extreme relativistic case leads to a unique mass, the so-called *Chandrasekhar mass*. Calculate its value, i.e. derive eq. (4.22).

Exercices (2)

4.4 Eddington's standard model

(a) Show that for constant β the virial theorem leads to $E_{tot}=\beta/2\ E_{gr}=-\ \beta/(2-\beta)E_{int}$, for a classical, non-relativistic gas. What happens in the limits $\beta \rightarrow 1$ and $\beta \rightarrow 0$?

(b) Verify eq. (4.25), and show that the corresponding constant K depends on β and the mean molecular weight μ as

$$K = \frac{2.67 \times 10^{15}}{\mu^{4/3}} \left(\frac{1-\beta}{\beta^4} \right)^{1/3}$$

(c) Use the results from above and the fact that the mass of an n=3 polytrope is uniquely determined by K, to derive the relation $M = M(\beta,\mu)$. This is useful for numerically solving the amount of radiation pressure for a star with a given mass.

4.5 Explain in simple physical terms why a star with $\gamma = 4/3$ has a more concentrated density structure than a star with $\gamma = 5/3$.

4.6 Suppose that zero-age main-sequence stars are described by the Eddington standard model with electron scattering as the dominant opacity.

(a) What would be the mean values of β and P_{rad}/P_{gas} for stars of 1 and $60M_{\odot}$?

(b) What is the predicted luminosity of these stars?

(c) Compare this with the data in Appendix D and comment on the comparison.

Questions

Q (4.1) Although θ is dimensionless, can you think of what it may describe in physical terms? Why was the symbol θ chosen?

(Hint: suppose the gas has a polytropic EoS and obeys the ideal gas law, What is the range of θ?)

Q (4.2) What does a model with $n = 0 \rightarrow \gamma = \infty$ describe in physical terms?

Q (4.3) What does a model with $n = \infty \rightarrow \gamma = 1$ describe in physical terms?