Distribución Gaussiana Condicional para una Gaussiana Multivariada.

Distribución normal Multivariada.

Suponiendo $X = \begin{bmatrix} X_a \\ X_b \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_a \\ \mu_b \end{bmatrix}$; $\Sigma = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}$

Donde $N(X|\mu, \Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(X-\mu)^T \Sigma (X-\mu)\right)$

Con $\Sigma_{ab} = \Sigma_{ba}^T$

A partir de la matriz de precisión:

$\Delta = \Sigma^{-1}$; $\Delta = \begin{bmatrix} \Delta_{aa} & \Delta_{ab} \\ \Delta_{ba} & \Delta_{bb} \end{bmatrix}$

Para $P(X_b | X_a)$, con $p(X) = p(X_a, X_b)$

$-\frac{1}{2}(X-\mu)^T \Sigma (X-\mu) = -\frac{1}{2}([X_a, X_b], [\mu_a, \mu_b])^T \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}^{-1} ([X_a, X_b] - [\mu_a, \mu_b])$

$-\frac{1}{2}(X-\mu)^T \Sigma (X-\mu) = -\frac{1}{2}[[X_a-\mu_a, X_b-\mu_b]^T \begin{bmatrix} \Delta_{aa} & \Delta_{ab} \\ \Delta_{ba} & \Delta_{bb} \end{bmatrix} [X_a-\mu_a, X_b-\mu_b]$

$-\frac{1}{2}[X^T\Sigma^{-1}X - X^T\Sigma^{-1}\mu - \mu^T\Sigma^{-1}X + \mu^T\Sigma^{-1}\mu] = -\frac{1}{2}X^T\Sigma^{-1}X + \frac{2}{2}X^T\Sigma^{-1}\mu - \frac{1}{2}\mu^T\Sigma^{-1}\mu$

$X^T\Sigma^{-1}\mu = \mu^T\Sigma^{-1}X = \langle \mu, X\rangle \Sigma^{-1}$, con $\Sigma > 0 \rightarrow$ Definida positiva.

Por lo tanto:

$\underbrace{-\frac{1}{2}X^T\Sigma^{-1}X}_{\text{Cuadrático } X} + \underbrace{X^T\Sigma^{-1}\mu}_{\text{lineal } X} \underbrace{-\frac{1}{2}\mu^T\Sigma^{-1}\mu}_{\text{Constante}} = -\frac{1}{2}\left[[X_a-\mu_a, X_b-\mu_b]^T \begin{bmatrix} \Delta_{aa} & \Delta_{ab} \\ \Delta_{ba} & \Delta_{bb} \end{bmatrix} [X_a-\mu_a, X_b-\mu_b]\right]$

$-\frac{1}{2}X^T\Sigma^{-1}X + X^T\Sigma^{-1}\mu + cte = -\frac{1}{2}[(X_a-\mu_a)^T\Delta_{aa} + (X_b-\mu_b)^T\Delta_{ba}, (X_a-\mu_a)^T\Delta_{ab}$
$+ (X_b-\mu_b)^T\Delta_{bb}][X_a-\mu_a, X_b-\mu_b]]$

$= -\frac{1}{2}[(X_a-\mu_a)^T\Delta_{aa}(X_a-\mu_a) + (X_b-\mu_b)^T\Delta_{ba}(X_a-\mu_a)$
$+ (X_a-\mu_a)^T\Delta_{ab}(X_b-\mu_b) + (X_b-\mu_b)^T\Delta_{bb}(X_b-\mu_b)]$

$\mu_{b|a}$ y $\Sigma_{b|a}$ Completando cuadrados.

$p(x) = p([x_a, x_b])$, $p(x_b|x_a) = N(x_b|\mu_{b|a}, \Sigma_{b|a})$

Reescribiendo exponencial

$A_{ab}^T = A_{ba}$

$$-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu) = -\frac{1}{2} x_a^T \Delta_{aa} x_a + x_a^T \Delta_{aa}\mu_a - \frac{1}{2}\mu_a^T \Delta_{aa}\mu_a$$

$$\cdots = -\frac{1}{2} x_b^T \Delta_{ba} x_a + \frac{1}{2} x_b^T \Delta_{ba}\mu_a + \frac{1}{2}\mu_b^T \Delta_{ba} x_a - \frac{1}{2}\mu_b^T \Delta_{ba}\mu_a$$

$$-\frac{1}{2} x_a^T \Delta_{ab} x_b + \frac{1}{2} x_a^T A_{ab}\mu_b + \frac{1}{2}\mu_a^T A_{ab} x_b - \frac{1}{2}\mu_a^T \Delta_{ab}\mu_b.$$

$$-\frac{1}{2} x_b^T \Delta_{bb} x_b + x_b^T A_{bb}\mu_b - \frac{1}{2}\mu_b^T A_{bb}\mu_b.$$

Para determinar $p(x_b|x_a)$ se encuentra la dependencia de x_b con x_a asumiendo x_a constante.

Término Cuadrático en x_b : $-\frac{1}{2} x_b^T \Delta_{bb} x_b$.

Del término cuadrático $\underline{\Sigma_{b|a} = \Delta^{-1}_{bb}}$.

Términos lineales en x_b:

$$-\frac{1}{2} x_b^T \Delta_{ba} x_a + \frac{1}{2} x_b^T \Delta_{ba}\mu_a - \frac{1}{2} x_a^T A_{ab} x_b + \frac{1}{2}\mu_a^T A_{ab} x_b$$

$$+ x_b^T A_{bb}\mu_b.$$

Se tiene que

- $x_b^T A_{bb}\mu_b - \frac{1}{2} x_b^T \Delta_{ba} x_a + \frac{1}{2} x_b^T A_{ba}\mu_a - \frac{1}{2} x_a^T A_{ab} x_b + \frac{1}{2}\mu_a^T \Delta_{ab} x_b$

- $x_b^T A_{bb}\mu_b + (-x_b^T A_{ba} x_a) + x_b^T A_{ba}\mu_a$

- $x_b^T (A_{bb}\mu_b + \Delta_{ba}(\mu_a - x_a))$

Donde $\Delta_{ba} = \Delta_{ab}$

Se busca despejar término donde

$$\cancel{x_b^T} \Sigma_{b|a}^{-1} \mu_{b|a} = \cancel{x_b^T} (\Delta_{bb} \mu_b + (\Delta_{ba} (\mu_a - x_a)))$$

$$\Sigma_{b|a}^{-1} = \Delta_{bb}.$$

$$\Sigma_{b|a}^{-1} \mu_{b|a} = \Sigma_{b|a}^{-1} \mu_b + (\Delta_{ba} (\mu_a - x_a))$$

$$\cancel{\Sigma_{b|a}^{-1} \Sigma_{b|a}} \mu_{b|a} = \Sigma_{b|a}^{-1} \Sigma_{b|a} \mu_b + (\Delta_{ba} (\mu_a - x_a))$$

$$\mu_{b|a} = \cancel{\Sigma_{b|a}^{-1} \Sigma_{b|a}} \mu_b + \Sigma_{ba}^{-1} (\mu_a - x_a) \Sigma_{b|a}$$

$$\Delta_{ba} = \Sigma_{ba}^{-1}$$

$$\mu_{b|a} = \mu_b + \Sigma_{b|a} \Sigma_{ba}^{-1} (\mu_a - x_a)$$

$$\Sigma_{b|a} = \Delta_{bb}^{-1}$$

$$\mu_{b|a} = \mu_b + \Delta_{bb}^{-1} \Delta_{ba} (\mu_a - x_a)$$

$$P(x_b | x_a) = N(x_b | \mu_{b|a}, \Sigma_{b|a})$$

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{bmatrix}^{-1} = \Delta = \begin{bmatrix} \Delta_{aa} & \Delta_{ab} \\ \Delta_{ba} & \Delta_{bb} \end{bmatrix}$$

Matriz^{-1} por partes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} M & -MBD^{-1} \\ -D^{-1}CM & D^{-1} + D^{-1}CMBD^{-1} \end{bmatrix}$$

$$M = (A - BD^{-1}C)^{-1}$$

$$\Sigma_{ab} = \Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}$$

$$\Sigma_{b|a} = \Sigma_{bb}^{-1} + \Sigma_{bb}^{-1} \Sigma_{ba} (\Sigma_{aa} - \Sigma_{ab} \Sigma_{bb}^{-1} \Sigma_{ba}) \Sigma_{ab} \Sigma_{bb}^{-1}$$