The Exponentiated Generalized G Geometric family of distributions: Theory, Properties and Applications

Abstract

In this article,We propose a new class of distributions called the exponentiated generalized G geometric family motivated mainly by lifetime issues which can generate several lifetime models discussed in the literature. Some mathematical properties of the new family including asymptotes and shapes, moments, quantile and generating functions, extreme values and order statistics are fully investigated....

1 Introduction

Let N be a geometric random variable with failure probability parameter $\mathbf{p} \in (0,1)$ and probability mass function (for $n \geq 1$) given by $\Pr(N = n) = (1 - \mathbf{p})\,\mathbf{p}^{n-1}$. We motivate the new family following the mechanism pioneered by Adamidis and Loukas (1998) Suppose that a system has N subsystems functioning independently at a given time, and that each subsystem is made of b parallel units, so that the system will fail if all of the subsystems fail. We consider that the failure times of the components $Z_{i,1}, Z_{i,2}, ..., Z_{i,b}$ for the i th subsystem are independent and identically distributed random variables with common cumulative distribution function (cdf) $H\left(x;\boldsymbol{\phi}\right)$ depending on a parameter vector $\boldsymbol{\phi}$. Let Y_i and X denote the failure time of the i th subsystem and the time to failure of the first out of the N functioning subsystems, respectively. If we define $X = min\{Y_i, ..., Y_N\}$, the unconditional cdf of X becomes

$$\begin{aligned} F\left(x\right) &= \sum_{n=1}^{\infty}\left(1-\mathbf{p}\right)\mathbf{p}^{n-1}\left\{1-\left[1-H\left(x;\boldsymbol{\phi}\right)^{b}\right]^{n}\right\} \\ &= \frac{H\left(x;\boldsymbol{\phi}\right)^{b}}{1-\mathbf{p}\left[1-H\left(x;\boldsymbol{\phi}\right)^{b}\right]}. \end{aligned} \tag{1}$$

The class of distributions (1) is called the exponentiated H geometric ('EHGc' for short) family,where $b > 0$ and $0 < \mathbf{p} < 1$ are two additional shape parameters.

Suppose that the failure time of each subsystem has the Generalized-G ("G-G(a)" for short) defined by the cumulative distribution function (cdf) and (pdf)

$$H\left(x;\boldsymbol{\phi}\right) = 1-\overline{G}\left(x;\boldsymbol{\phi}\right)^{a}, \tag{2}$$

$$h(x;\boldsymbol{\phi}) = ag\left(x;\boldsymbol{\phi}\right)\overline{G}\left(x;\boldsymbol{\phi}\right)^{a-1}. \tag{3}$$

respectively.Where $\overline{G}\left(x_i;\boldsymbol{\phi}\right) = 1-G\left(x;\boldsymbol{\phi}\right)$ is the reliability function (rf) and $a > 0$ additional shape parameter.

let Y_i denote the failure time of the ith subsystem and X denote the time to failure of the first out of the N functioning subsystems.

Then

$$F\left(x;a,b,\mathbf{p},\boldsymbol{\phi}\right) = \frac{\left\{1-\overline{G}\left(x;\boldsymbol{\phi}\right)^{a}\right\}^{b}}{1-\mathbf{p}\left[1-\left\{1-\overline{G}\left(x;\boldsymbol{\phi}\right)^{a}\right\}^{b}\right]}. \tag{4}$$

Clearly when $a = 1$ we get (1)

The cdf in (4) is called the exponentiated generalized G geometric ("EGGGc") family of Distributions.

The corresponding probability density function (pdf)

$$f(x;a,b,\mathbf{p},\boldsymbol{\phi}) = \frac{ab(1-\mathbf{p})\,g(x;\boldsymbol{\phi})\,\overline{G}(x;\boldsymbol{\phi})^{a-1}\left\{1-\overline{G}(x;\boldsymbol{\phi})^{a}\right\}^{b-1}}{\left\{1-\mathbf{p}\left[1-\left\{1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right\}^{b}\right]\right\}^{2}}. \tag{5}$$

Based on the Taylor series expansion for $z\in(0,1)$

$$(1-\mathbf{p}+\mathbf{p}z)^{-a} = (1-\mathbf{p})^{-a}\sum_{i=0}^{\infty}\binom{-a}{i}\frac{(\mathbf{p}z)^{i}}{(1-\mathbf{p})^{i}},$$

the pdf in (5) can be expressed as

$$\begin{aligned} f(x) &= ab(1-\mathbf{p})\,g(x;\boldsymbol{\phi})\,\overline{G}(x;\boldsymbol{\phi})^{a-1}\left\{1-\overline{G}(x;\boldsymbol{\phi})^{a}\right\}^{b-1} \\ &\quad\times\left\{1-\mathbf{p}+\mathbf{p}\left\{1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right\}^{b}\right\}^{-2}, \end{aligned}$$

$$\begin{aligned} f(x) &= abg(x;\boldsymbol{\phi})\,\overline{G}(x;\boldsymbol{\phi})^{a-1} \\ &\quad\times\sum_{i=0}^{\infty}\binom{-2}{i}\frac{\mathbf{p}^{i}\left\{1-\overline{G}(x;\boldsymbol{\phi})^{a}\right\}^{b(i+1)-1}}{(1-\mathbf{p})^{i+1}}. \end{aligned}$$

Applying the series expantion to $\left\{1-\overline{G}(x;\boldsymbol{\phi})^{a}\right\}^{b(i+1)-1}$ we get

$$f(x) = abg(x;\boldsymbol{\phi})\sum_{i,j=0}^{\infty}(-1)^{j}\binom{-2}{i}\binom{b(i+1)-1}{j}\frac{\mathbf{p}^{i}\overline{G}(x;\boldsymbol{\phi})^{a(j+1)-1}}{(1-\mathbf{p})^{i+1}}.$$

Finally, applying the series expantion to $\overline{G}(x;\boldsymbol{\phi})^{a(j+1)-1}$ we get

$$f(x) = \sum_{i,j,k=0}^{\infty}\frac{ab\mathbf{p}^{i}(-1)^{j+k}\binom{-2}{i}\binom{b(i+1)-1}{j}\binom{a(j+1)-1}{k}}{(1-\mathbf{p})^{i+1}}g(x;\boldsymbol{\phi})\,G(x;\boldsymbol{\phi})^{k}.$$

The pdf can be expressed as a mixture of E-G densities

$$f(x) = \sum_{k=0}^{\infty}t_k\pi_{k+1}(x), \tag{6}$$

where

$$t_k = \frac{ab(-1)^{k}}{k+1}\sum_{i,j=0}^{\infty}\frac{\mathbf{p}^{i}(-1)^{j}\binom{-2}{i}\binom{b(i+1)-1}{j}\binom{a(j+1)-1}{k}}{(1-\mathbf{p})^{i+1}},$$

and is $\pi_{k+1}(x) = (k+1)\,g(x;\boldsymbol{\phi})\,G(x;\boldsymbol{\phi})^{k}$ the E-G pdf with power parameter $(k+1)$.

The cdf of the TEG-G family can also be expressed as a mixture of E-G densities. By integrating (6), we obtain the same mixture representation

$$F(x) = \sum_{k=0}^{\infty}t_k\,\Pi_{k+1}(x), \tag{7}$$

where $\Pi_{k+1}(x)$ is the cdf of the E-G family with power parameter $(k+1)$.

The rest of the paper is outlined as follows. In Section 2, ...

2 Special EGGGc Models...

...

3 Some Mathematical Properties

Here, we investigate mathematical properties of the EGGGc family of distributions including Asymptotes and shapes, Quantile Measure, moments,central moments,kumulants, incomplete moments, generating function , and Lorenz, Bonferroni and Zenga curves. Established algebraic expansions to determine some structural properties of the EGGGc family of distributions can be more efficient than computing those directly by numerical integration of its density function.

3.1 Moments, Cumulants and Generating Function

The rth ordinary moment of X is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r\, f(x)\, dx.$$

Then we obtain

$$\mu_r' = \sum_{k=0}^{\infty} t_k E(Y_{k+1}^r). \tag{8}$$

Henceforth, Y_{k+1} denotes the exp-G distribution with power parameter$(k+1)$ .Setting $r=1$ in (8), we have the mean of X. The last integration can be computed numerically for most parent distributions.The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

The nth central moment of X, say M_n, follows as

$$M_n = E(X-\mu)^n = \sum_{h=0}^{n} (-1)^h \binom{n}{h} (\mu_1')^n\, \mu_{n-h}'.$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r\, \mu_{n-r}',$$

where $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc. The skewness and kurtosis measures also can be calculated from the ordinary moments using well-known relationships.

Here, we provide two formulae for the mgf $M_X(t) = E\left(\mathrm{e}^{t\,X}\right)$ of X. Clearly, the first one can be derived from equation (6) as

$$M_X(t) = \sum_{k=0}^{\infty} t_k\, M_{k+1}(t),$$

where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M_X(t)$ can be determined from the exp-G generating function. A second formula for $M_X(t)$ follows from (6) as

$$M_X(t) = \sum_{k=0}^{\infty} t_k\, \tau(t,k),$$

where $\tau(t,k) = \int_0^1 \exp\left[t\, Q_G(u)\right] u^k du$ and $Q_G(u)$ is the qf corresponding to $G(x;\boldsymbol{\phi})$, i.e., $Q_G(u) = G^{-1}(u;\boldsymbol{\phi})$.

3.2 Incomplete Moments and Mean Deviations

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The sth incomplete moment, say $\varphi_s(t)$, of X can be expressed from (9) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx = \sum_{k=0}^{\infty} t_k \int_{-\infty}^{t} x^s\, \pi_{k+1}(x)\, dx. \tag{9}$$

The mean deviations about the mean $\delta_1 = E(|X - \mu_1'|)$ and about the median $\delta_2 = E(|X - M|)$ of X are given by $\delta_1 = 2\mu_1' F(\mu_1') - 2\varphi_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\varphi_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q(0.5)$ is the median, $F(\mu_1')$ is easily calculated from (4) and $\varphi_1(t)$ is the first incomplete moment given by (9) with $s = 1$.

Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $\varphi_1(t)$ can be derived from (9) as

$$\varphi_1(t) = \sum_{k=0}^{\infty} t_k\, J_{k+1}(x),$$

where $J_{k+1}(x) = \int_{-\infty}^{t} x\, \pi_{k+1}(x)\, dx$ is the first incomplete moment of the exp-G distribution. A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k=0}^{\infty} t_k\, V_k(t),$$

where $v_k(t) = (k+1) \int_0^{G(t)} Q_G(u)\, u^k du$ can be computed numerically. These equations for $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q) / (\pi \mu_1')$ and $L(\pi) = \varphi_1(q) / \mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

3.3 Probability Weighted Moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.

The (s,r)th PWM of X following the EGGGc family, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\{X^s\, F(X)^r\} = \int_{-\infty}^{\infty} x^s\, F(x)^r\, f(x)\; dx.$$

Using equations (4) and (5), we can write

$$\begin{aligned}
f(x)\; F(x)^r &= \frac{ab\,(1-\mathbf{p})\, g(x;\boldsymbol{\phi})\, \overline{G}(x;\boldsymbol{\phi})^{a-1} \left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^{b(r+1)-1}}{\left\{1 - \mathbf{p}\left[1 - \left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^b\right]\right\}^{r+2}} \\
&= \frac{ab\,(1-\mathbf{p})\, g(x;\boldsymbol{\phi})\, \overline{G}(x;\boldsymbol{\phi})^{a-1} \left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^{b(r+1)-1}}{\left\{1 - \mathbf{p} + \mathbf{p}\left\{1 - \overline{G}(x_i;\boldsymbol{\phi})^a\right\}^b\right\}^{r+2}} \\
&= ab\,(1-\mathbf{p})\, g(x;\boldsymbol{\phi})\, \overline{G}(x;\boldsymbol{\phi})^{a-1} \left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^{b(r+1)-1} \\
&\quad \times \left\{1 - \mathbf{p} + \mathbf{p}\left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^b\right\}^{-r-2}.
\end{aligned}$$

Using the Taylor series expansion we get

$$f(x)\; F(x)^r = ab g(x;\boldsymbol{\phi})\, \overline{G}(x;\boldsymbol{\phi})^{a-1} \sum_{i=0}^{\infty} \frac{\mathbf{p}^i \binom{-r-2}{i}}{(1-\mathbf{p})^{r+i+1}} \left\{1 - \overline{G}(x;\boldsymbol{\phi})^a\right\}^{b(r+i+1)-1}.$$

Applying the series expantion to $\left\{1-\overline{G}\left(x_i;\boldsymbol{\phi}\right)^a\right\}^{b(r+i+1)-1}$ we get

$$f\left(x\right)\ F(x)^r = abg\left(x;\boldsymbol{\phi}\right)\sum_{i,j=0}^{\infty}\frac{\mathbf{p}^i\left(-1\right)^j\binom{-r-2}{i}\binom{b(r+i+1)-1}{j}}{\left(1-\mathbf{p}\right)^{r+i+1}}\overline{G}\left(x;\boldsymbol{\phi}\right)^{a(j+1)-1}.$$

Applying the series expantion to $\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{a(j+1)-1}$ we get

$$f\left(x\right)\ F(x)^r = \sum_{i,j,k=0}^{\infty}\frac{ab\mathbf{p}^i\left(-1\right)^{j+k}\binom{-r-2}{i}\binom{b(r+i+1)-1}{j}\binom{a(j+1)-1}{k}}{\left(1-\mathbf{p}\right)^{r+i+1}}g\left(x;\boldsymbol{\phi}\right)G\left(x;\boldsymbol{\phi}\right)^k.$$

$$f\left(x\right)\ F(x)^r = \sum_{k=0}^{\infty}\boldsymbol{\Upsilon}_k\pi_{k+1}\left(x\right).$$

where

$$\boldsymbol{\Upsilon}_k = \frac{ab\left(-1\right)^k}{\left(k+1\right)}\sum_{i,j=0}^{\infty}\frac{\mathbf{p}^i\left(-1\right)^j}{\left(1-\mathbf{p}\right)^{r+i+1}}\binom{-r-2}{i}\binom{b\left(r+i+1\right)-1}{j}\binom{a\left(j+1\right)-1}{k}.$$

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Then, the (s,r)th PWM of X can be expressed as

$$\rho_{s,r} = \sum_{k=0}^{\infty}\boldsymbol{\Upsilon}_k E\left(Y_{k+1}^s\right)\ dx.$$

3.4 Entropy Measures

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\theta\left(X\right) = \frac{1}{1-\theta}\ \log\int_{-\infty}^{\infty}f\left(x\right)^\theta dx,\ \ \theta>0 \text{ and } \theta\neq 1.$$

Using the pdf (6), we can write

$$\begin{aligned}
f\left(x\right)^\theta &= \frac{a^\theta b^\theta\left(1-\mathbf{p}\right)^\theta g\left(x;\boldsymbol{\phi}\right)^\theta\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{a\theta-\theta}\left\{1-\overline{G}\left(x;\boldsymbol{\phi}\right)^a\right\}^{\theta b-\theta}}{\left\{1-\mathbf{p}+\mathbf{p}\left\{1-\overline{G}\left(x;\boldsymbol{\phi}\right)^a\right\}^b\right\}^{2\theta}}\\
&= a^\theta b^\theta g\left(x;\boldsymbol{\phi}\right)^\theta\overline{G}\left(x;\boldsymbol{\phi}\right)^{a\theta-\theta}\sum_{i=0}^{\infty}\frac{\mathbf{p}^i\binom{-2\theta}{i}}{\left(1-\mathbf{p}\right)^{\theta+i}}\left\{1-\overline{G}\left(x;\boldsymbol{\phi}\right)^a\right\}^{b(i+\theta)-\theta}\\
&= a^\theta b^\theta g\left(x;\boldsymbol{\phi}\right)^\theta\sum_{i,j=0}^{\infty}\frac{\mathbf{p}^i\left(-1\right)^j\binom{-2\theta}{i}\binom{b(i+\theta)-\theta}{j}}{\left(1-\mathbf{p}\right)^{\theta+i}}\overline{G}\left(x;\boldsymbol{\phi}\right)^{a(j+\theta)-\theta}\\
&= a^\theta b^\theta\sum_{i,j,k=0}^{\infty}\frac{\mathbf{p}^i\left(-1\right)^{j+k}\binom{-2\theta}{i}\binom{b(i+\theta)-\theta}{j}\binom{a(j+\theta)-\theta}{k}}{\left(1-\mathbf{p}\right)^{\theta+i}}g\left(x;\boldsymbol{\phi}\right)^\theta G\left(x;\boldsymbol{\phi}\right)^k
\end{aligned}$$

$$f\left(x\right)^\theta = \sum_{k=0}^{\infty}\Phi_k g\left(x;\boldsymbol{\phi}\right)^\theta G\left(x;\boldsymbol{\phi}\right)^k.$$

where

$$\Phi_k = a^\theta b^\theta\left(-1\right)^k\sum_{i,j=0}^{\infty}\frac{\mathbf{p}^i\left(-1\right)^j\binom{-2\theta}{i}\binom{b(i+\theta)-\theta}{j}\binom{a(j+\theta)-\theta}{k}}{\left(1-\mathbf{p}\right)^{\theta+i}}.$$

Then, the Rényi entropy of the EGGGc family is given by

$$I_{\theta}(X)=\frac{1}{1-\theta}\log\left\{\sum_{k=0}^{\infty}\Phi_{k}\int_{-\infty}^{\infty}g(x)^{\theta}\,G(x)^{k}\,dx\right\},$$

The q-entropy, say $H_q(X)$, can be obtained as

$$H_{q}(X)=\frac{1}{q-1}\log\left\{1-\left[\sum_{k=0}^{\infty}\Phi_{k}^{*}\int_{-\infty}^{\infty}g(x;\phi)^{q}\,G(x;\phi)^{k}\,dx\right]\right\},$$

where $q>0$, $q\neq 1$.

The Shannon entropy of a random variable X, say SI, is defined by

$$SI=E\left\{-\left[\log f(X)\right]\right\}.$$

It is the special case of the Rényi entropy when $\theta\uparrow 1$. So, using equation (9), we can write

$$SI=-E\left\{\log\left[\sum_{k=0}^{\infty}t_{k}\,\pi_{k+1}(x)\right]\right\}=-\left\{\log\left[\sum_{k=0}^{\infty}t_{k}\,J_{k+1}\right]\right\},$$

where $J_{\gamma}(x)=\gamma\int_{0}^{\infty}x\,g(x)\,G^{\gamma-1}(x)\,dx$.

The last equation can be determined numerically for any exp-G distribution.

3.5 Residual Life Function and Mean Residual Life

The nth moment of the residual life, say $m_n(t)=E[(X-t)^n\mid X>t]$, $n=1,2,\ldots$, uniquely determine $F(x)$ (see Navarro et al., 1998). The nth moment of the residual life of X is given by

$$m_{n}(t)=\frac{1}{R(t)}\int_{t}^{\infty}(x-t)^{n}dF(x).$$

Therefore,

$$m_{n}(t)=\frac{1}{R(t)}\sum_{k=0}^{\infty}t_{k}^{*}\int_{t}^{\infty}x^{r}\pi_{k+1}(x),$$

where $t_{k}^{*}=t_{k}\sum_{r=0}^{n}\binom{n}{r}(-t)^{n-r}$. Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t)=E\left[(X-t)\mid X>t\right]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting $n=1$ in the last equation.

3.6 Reversed Residual Life and Mean Waiting Time

The nth moment of the reversed residual life, say $M_n(t)=E\left[(t-X)^n\mid X\leq t\right]$ for $t>0$ and $n=1,2,\ldots$ uniquely determines $F(x)$ (Navarro et al., 1998). We obtain

$$M_{n}(t)=\frac{1}{F(t)}\int_{0}^{t}(t-x)^{n}dF(x).$$

Therefore, the nth moment of the reversed residual life of X becomes

$$M_{n}(t)=\frac{1}{F(t)}\sum_{k=0}^{\infty}t_{k}^{**}\int_{0}^{t}x^{r}\pi_{k+1}(x),$$

where $t_k^{**} = t_k \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$.The mean waiting time (MWT) or mean inactivity time (MIT) also called the mean reversed residual life function is given by $M_1(t) = E[(t-X) \mid X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.The MIT of the EGGGc family of distributions can be obtained easily by setting $n = 1$ in the above equation. For further information about the properties of the MIT, we refer to Kayid and Ahmad (2004) and Ahmad et al. (2005).

4 Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the TEG family of distributions and let $X_{(1)}, \ldots, X_{(n)}$ be the corresponding order statistics. The pdf of ith order statistic, say $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \tag{13}$$

where $B(\cdot, \cdot)$ is the beta function

Using (4) and (5),we can get

$$f(x)\ F(x)^{j+i-1} = \sum_{k=0}^{\infty} \mathbf{\Psi}_k \pi_{k+1}(x),$$

where

$$\mathbf{\Psi}_k = \frac{ab(-1)^k}{(k+1)} \sum_{m,w=0}^{\infty} \frac{\mathbf{p}^m (-1)^w}{(1-\mathbf{p})^{j+i+m}} \binom{-j-i-1}{m} \binom{b(j+i+m)-1}{w} \binom{a(w+1)-1}{k},$$

the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{\infty} \mathbf{\Psi}_k^* \pi_{k+1}.$$

wher$\mathbf{\Psi}_k^* = \mathbf{\Psi}_k \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{\mathrm{B}(i,n-i+1)}$

Then, the density function of theTEG order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those properties of Y_{k+1}. For example, the moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^q\right) = \sum_{k=0}^{\infty} \mathbf{\Psi}_k^* E\left(Y_{k+1}^q\right). \tag{14}$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are relatively robust to the effects of outliers. Based upon the moments in equation (14), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable Kw-TG order statistics. They are linear functions of expected order statistics defined by

$$\boldsymbol{\lambda}_r = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E\left(X_{r-d:r}\right),\ r \geq 1.$$

The first four L-moments are given by:

$$\boldsymbol{\lambda}_1 = E\left(X_{1:1}\right), \ \boldsymbol{\lambda}_2 = \frac{1}{2}E\left(X_{2:2} - X_{1:2}\right),$$

$$\boldsymbol{\lambda}_3 = \frac{1}{3}E\left(X_{3:3} - 2X_{2:3} + X_{1:3}\right)$$

and

$$\boldsymbol{\lambda}_4 = \frac{1}{4}E\left(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}\right).$$

One simply can obtain the $\boldsymbol{\lambda}$'s for X from (14) with $q = 1$.

5 Stress-Strength Model

Stress-strength model is the most widely approach used for reliability estimation. This model is used in many applications of physics and engineering such as strength failure and system collapse. In stress-strength modeling, $R = \Pr(X_2 < X_1)$ is a measure of reliability of the system when it is subjected to random stress X_2 and has strength X_1.

The system fails if and only if the applied stress is greater than its strength and the component will function satisfactorily whenever $X_1 > X_2$. R can be considered as a measure of system performance and naturally arise in electrical and electronic systems. Other interpretation can be that, the reliability, say R, of the system is the probability that the system is strong enough to overcome the stress imposed on it. Let X_1 and X_2 be two independent random variables have EGGGc$(a_1, b_1, \mathbf{p}_1, \boldsymbol{\phi})$ and EGGGc$(a_2, b_2, \mathbf{p}_2, \boldsymbol{\phi})$ distributions .The *pdf* of X_1 and the *cdf* of X_2 can be written from Equations (6) and (7), respectively as

$$f_1\left(a_1, b_1, \mathbf{p}_1, \boldsymbol{\phi}\right) = \sum_{k=0}^{\infty} a_1 b_1 \left(-1\right)^k \sum_{i,j=0}^{\infty} \frac{\mathbf{p}_1^i \left(-1\right)^j \binom{-2}{i}\binom{b_1(i+1)-1}{j}\binom{a_1(j+1)-1}{k}}{\left(1-\mathbf{p}_1\right)^{i+1}} g\left(x;\boldsymbol{\phi}\right) G\left(x;\boldsymbol{\phi}\right)^k$$

and

$$F_2\left(a_2, b_2, \mathbf{p}_2, \boldsymbol{\phi}\right) = \sum_{h=0}^{\infty} \frac{a_2 b_2 \left(-1\right)^h}{h+1} \sum_{m,w=0}^{\infty} \frac{\mathbf{p}_2^m \left(-1\right)^w \binom{-2}{m}\binom{b_2(m+1)-1}{j}\binom{a_2(w+1)-1}{h}}{\left(1-\mathbf{p}_2\right)^{m+1}} G\left(x;\boldsymbol{\phi}\right)^{h+1}.$$

Then, the reliability is given by

$$R = \int_0^\infty f_1\left(\delta_1, a_1, \mathbf{p}_1, \boldsymbol{\phi}\right) F_2\left(\delta_2, a_2, \mathbf{p}_2, \boldsymbol{\phi}\right) dx.$$

Then

$$R = \sum_{k,h=0}^{\infty} \boldsymbol{\Omega}_{k,h} \int_0^\infty \pi_{k+h+2}\left(x\right) dx.$$

Where

$$\begin{aligned}\boldsymbol{\Omega}_{k,h} \quad = \quad & \frac{a_1 a_2 b_1 b_2}{\left(h+1\right)\left(k+h+2\right)} \sum_{i,j=0}^{\infty} \mathbf{p}_1^i \mathbf{p}_2^m \left(-1\right)^{j+k+w+h} \binom{-2}{i}\binom{-2}{m} \\ & \times \frac{\binom{b_1(i+1)-1}{j}\binom{b_2(m+1)-1}{j}\binom{a_1(j+1)-1}{k}\binom{a_2(w+1)-1}{h}}{\left(1-\mathbf{p}_1\right)^{i+1}\left(1-\mathbf{p}_2\right)^{m+1}}.\end{aligned}$$

then

$$R = \sum_{k,h=0}^{\infty} \boldsymbol{\Omega}_{k,h} E\left(Y_{k+h+2}^r\right).$$

6 Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood.

Let $x_1, \ldots, x_n$ be a random sample from the EGGGc distribution with parameters δ, a and $\boldsymbol{\phi}$. Let $\Theta = (a, b, \mathbf{p}, \boldsymbol{\phi}^\intercal)^\intercal$ be the $p \times 1$ parameter vector. For determining the MLE of Θ, we have the log-likelihood function

$$\begin{aligned} \ell &= \ell(\Theta) = n \log a + n \log b + n \log(1-\mathbf{p}) + \sum_{i=1}^{n} \log g(x_i; \boldsymbol{\phi}) \\ &+ (a-1) \sum_{i=1}^{n} \log \overline{G}(x_i; \boldsymbol{\phi}) + (b-1) \sum_{i=1}^{n} \log s_i - 2 \sum_{i=1}^{n} \log z_i, \end{aligned}$$

where $s_i = 1 - \overline{G}(x_i; \boldsymbol{\phi})^a$ and $z_i = 1 - \mathbf{p}\left(1 - s_i^b\right)$.

The components of the score vector, $\mathbf{U}(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \mathbf{p}}, \frac{\partial \ell}{\partial \boldsymbol{\phi}}\right)^\intercal$, are

$$\begin{aligned} U_a &= \frac{n}{a} + \sum_{i=1}^{n} \log \overline{G}(x_i; \boldsymbol{\phi}) + (b-1) \sum_{i=1}^{n} \frac{u_i}{s_i} - 2 \sum_{i=1}^{n} \frac{m_i}{z_i}, \\ U_b &= \frac{n}{b} + \sum_{i=1}^{n} \log s_i - 2 \sum_{i=1}^{n} \frac{w_i}{z_i}, U_{\mathbf{p}} = \frac{n}{(1-\mathbf{p})} - 2 \sum_{i=1}^{n} \frac{q_i}{z_i}, \end{aligned}$$

and

$$U_{\boldsymbol{\phi}} = \sum_{i=1}^{n} \frac{g'(x_i; \boldsymbol{\phi})}{g(x_i; \boldsymbol{\phi})} - (a-1) \sum_{i=1}^{n} \frac{G'(x_i; \boldsymbol{\phi})}{\overline{G}(x_i; \boldsymbol{\phi})} + (b-1) \sum_{i=1}^{n} \frac{d_i}{s_i} - 2 \sum_{i=1}^{n} \frac{t_i}{z_i},$$

where

$$\begin{aligned} q_i &= -\left(1 - s_i^b\right), \ u_i = -\overline{G}(x_i; \boldsymbol{\phi})^a \log \overline{G}(x_i; \boldsymbol{\phi}), G'(x_i; \boldsymbol{\phi}) = \partial G(x_i; \boldsymbol{\phi}) / \partial \boldsymbol{\phi}, m_i = \mathbf{p} b u_i s_i^{b-1}, \\ t_i &= \mathbf{p} b d_i s_i^{b-1}, g'(x_i; \boldsymbol{\phi}) = \partial g(x_i; \boldsymbol{\phi}) / \partial \boldsymbol{\phi}, w_i = \mathbf{p} s_i^b \log s_i \ \text{ and } d_i = a G'(x_i; \boldsymbol{\phi}) \overline{G}(x_i; \boldsymbol{\phi})^{a-1}. \end{aligned}$$

Setting the nonlinear system of equations $U_\delta = U_a = 0$ and $U_{\boldsymbol{\phi}} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{a}, \widehat{b}, \widehat{\mathbf{p}}, , \widehat{\boldsymbol{\phi}}^\intercal)^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\Theta) = \{\frac{\partial^2 \ell}{\partial r \, \partial s}\}$ (for $r, s = a, b, \mathbf{p}, \boldsymbol{\phi}$), whose elements can be computed numerically.

Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. The elements of $J(\Theta)$ are given in **Appendix A.**

7 Applications...

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8 Simulation Study...

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9 Conclusions...

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Appendix A

$$
\begin{aligned}
U_{aa} &= \frac{-n}{a^2} + (b-1)\sum_{i=1}^{n}\frac{s_i\left(\partial u_i/\partial a\right) - u_i^2}{s_i^2} - 2\sum_{i=1}^{n}\frac{z_i\left(\partial m_i/\partial a\right) - m_i^2}{z_i^2}, \\
U_{ab} &= \sum_{i=1}^{n}\frac{u_i}{s_i} - 2\sum_{i=1}^{n}\frac{\mathbf{p}u_i z_i s_i^{b-1}\left(1+\log s_i\right) - w_i m_i}{z_i^2}, U_{a\mathbf{p}} = -2\sum_{i=1}^{n}\frac{b u_i z_i s_i^{b-1} - m_i q_i}{z_i^2}, \\
U_{a\boldsymbol{\phi}} &= (b-1)\sum_{i=1}^{n}\frac{G'\left(x_i;\boldsymbol{\phi}\right)}{s_i\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{-a+1}} + (b-1)\sum_{i=1}^{n}\frac{-aG'\left(x_i;\boldsymbol{\phi}\right)\log\overline{G}\left(x_i;\boldsymbol{\phi}\right)}{s_i\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{-a+1}} \\
&\quad -\sum_{i=1}^{n}\frac{G'\left(x_i;\boldsymbol{\phi}\right)}{\overline{G}\left(x_i;\boldsymbol{\phi}\right)} + (b-1)\sum_{i=1}^{n}\frac{-u_i d_i}{s_i^2} - 2\sum_{i=1}^{n}\frac{\left(\partial m_i/\partial\boldsymbol{\phi}\right)}{z_i} - 2\sum_{i=1}^{n}\frac{-t_i m_i}{z_i^2}, \\
U_{bb} &= \frac{-n}{b^2} - 2\sum_{i=1}^{n}\frac{\mathbf{p}z_i s_i^{b}\left(\log s_i\right)^2 - w_i^2}{z_i^2}, U_{b\mathbf{p}} = -2\sum_{i=1}^{n}\frac{z_i s_i^{b}\log s_i - q_i w_i}{z_i^2}, \\
U_{b\boldsymbol{\phi}} &= \sum_{i=1}^{n}\frac{d_i}{s_i} - 2\sum_{i=1}^{n}\frac{z_i\left(\partial w_i/\partial\boldsymbol{\phi}\right) - t_i w_i}{z_i^2}, U_{\mathbf{pp}} = \frac{n}{\left(1-\mathbf{p}\right)^2} + 2\sum_{i=1}^{n}\frac{q_i^2}{z_i^2}, U_{\mathbf{p}\boldsymbol{\phi}} = -2\sum_{i=1}^{n}\frac{b d_i z_i s_i^{b-1} - t_i q_i}{z_i^2},
\end{aligned}
$$

$$
\begin{aligned}
U_{\boldsymbol{\phi\phi}} &= +\sum_{i=1}^{n}\frac{g\left(x_i;\boldsymbol{\phi}\right)g''\left(x_i;\boldsymbol{\phi}\right) - \left[g'\left(x_i;\boldsymbol{\phi}\right)\right]^2}{g\left(x_i;\boldsymbol{\phi}\right)^2} - (a-1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\boldsymbol{\phi})G''\left(x_i;\boldsymbol{\phi}\right) + \left[G'\left(x_i;\boldsymbol{\phi}\right)\right]^2}{\overline{G}(x_i;\boldsymbol{\phi})^2} \\
&\quad + (b-1)\sum_{i=1}^{n}\frac{a\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{a-1}\left[G''\left(x_i;\boldsymbol{\phi}\right) - (a-1)\left[G'\left(x_i;\boldsymbol{\phi}\right)\right]^2\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{-1}\right]}{s_i} \\
&\quad -2\sum_{i=1}^{n}\frac{\mathbf{p}bz_i\left[(a-1)d_i^2 s_i^{b-2} + v_i s_i^{b-1}\right]}{z_i^2} + (b-1)\sum_{i=1}^{n}\frac{-d_i^2}{s_i^2} - 2\sum_{i=1}^{n}\frac{-t_i^2}{z_i^2},
\end{aligned}
$$

where

$$
\begin{aligned}
g''\left(x_i;\boldsymbol{\phi}\right) &= \left[\partial^2 g\left(x_i;\boldsymbol{\phi}\right)/\partial\boldsymbol{\phi}^2\right], G''\left(x_i;\boldsymbol{\phi}\right) = \left[\partial^2 G\left(x_i;\boldsymbol{\phi}\right)/\partial\boldsymbol{\phi}^2\right], \\
v_i &= a\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{a-1}\left[G''\left(x_i;\boldsymbol{\phi}\right) - (a-1)\left[G'\left(x_i;\boldsymbol{\phi}\right)\right]^2\overline{G}\left(x_i;\boldsymbol{\phi}\right)^{-1}\right]
\end{aligned}
$$