From tree matching to sparse graph alignment.

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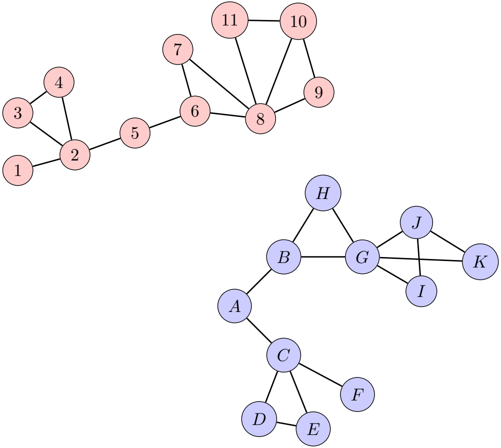
INRIA, Paris

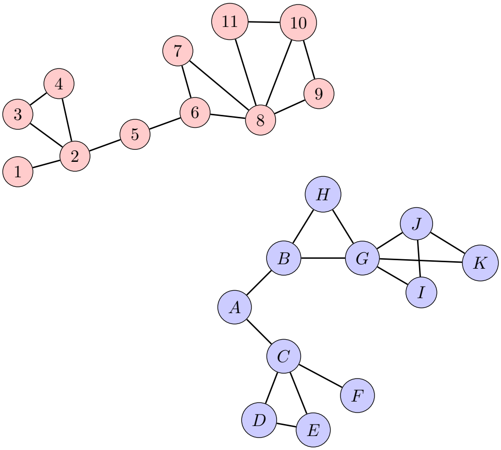
Conference on Learning Theory, July 2020

Introduction: the graph isomorphism problem



Introduction: the graph isomorphism problem

Question: Given two graphs $G = (V, E)$ and $G' = (V', E')$, is there a *graph isomorphism*, i.e. a bijection $f : V \to V'$ such that $(i, j) \in E \iff (f(i), f(j)) \in E'$?



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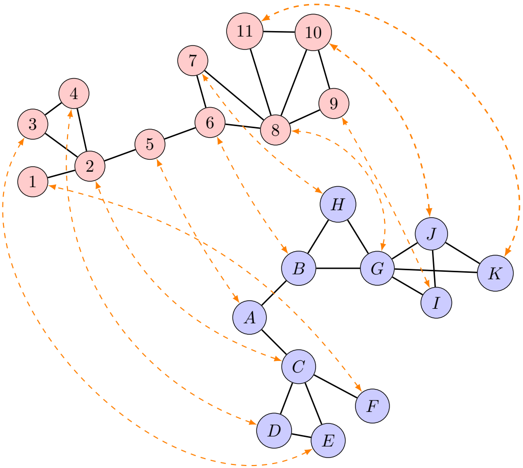
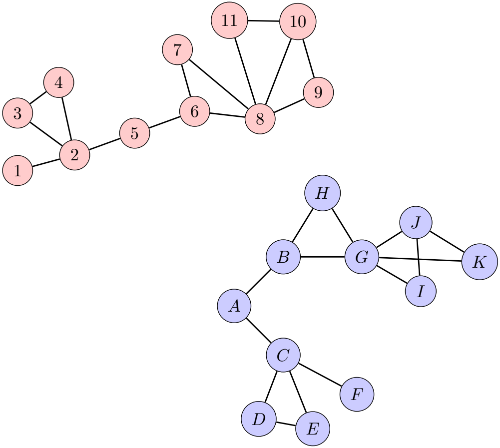


Problem in NP, thought to be neither in P nor NP-complete.

Introduction: graph alignment



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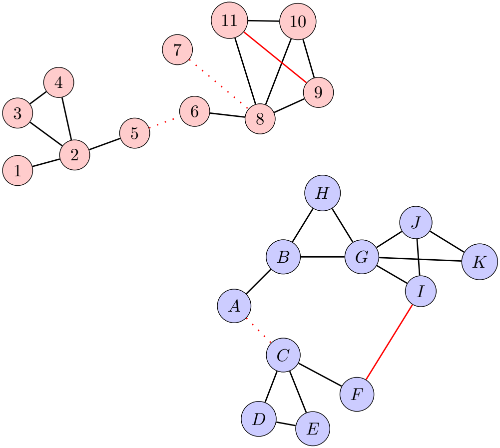
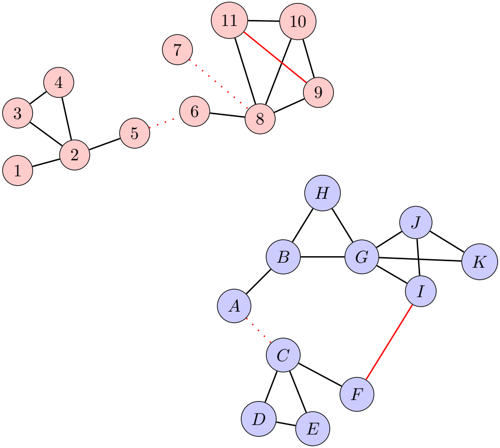


Introduction: graph alignment

Relaxed version: Is there a bijection $f : V \to V'$ that *preserves most edges*?



Introduction: graph alignment

Formally: f minimizes

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$\longrightarrow$ instance of the NP-hard quadratic assigment problem (QAP):

$$\max_{\Pi} \mathrm{Tr}\left(G\Pi G'\Pi^\top\right),$$

where Π runs over all permutation matrices.

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- Form G_2' as an other independent $s-$sub-sampling of G_0.
- Shuffle labels of G_2' uniformly at random to form G_2. Formally, $G_2 = \Pi^\top G_2' \Pi$, where $\Pi = \Pi_\sigma$ is the matrix of a uniform permutation σ.

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Exact recovery of σ:

- Information-theoretically feasible iff $nps = \log n + \omega(1)$ [Cullina-Kiyavash'16].
- Polynomial time feasible if $np \geq (\log n)^{\alpha}$ and $1 - s \leq (\log n)^{-\beta}$[Ding et al.'18].
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This work: partial recovery of σ

- Polynomial-time recovery, in sparse regime $np = O(1)$.
- Relaxed objective: find a one-to-one $\hat{\sigma}$ from G_1, G_2, such that

$$\mathrm{overlap}(\hat{\sigma}) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\hat{\sigma}(i) = \sigma(i)} = \Omega(1),$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{\hat{\sigma}(i) \neq \sigma(i)} = o(1).$$

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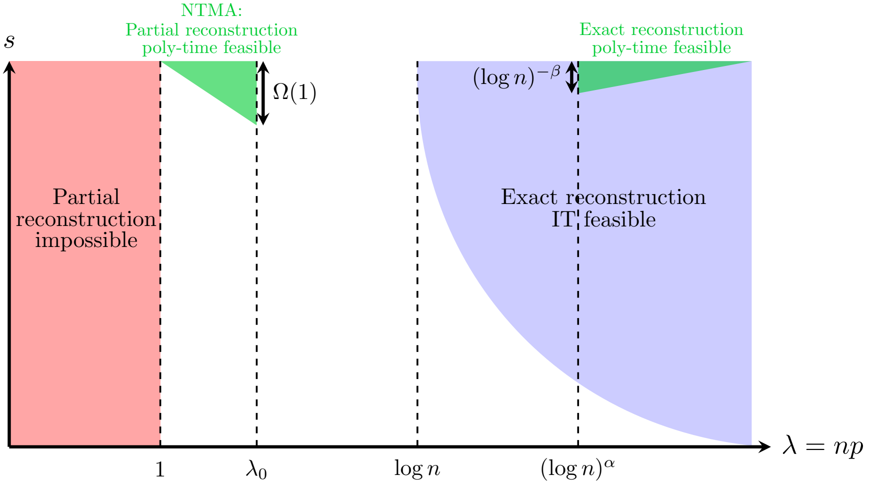

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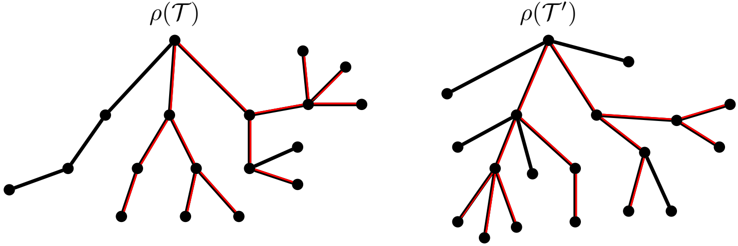

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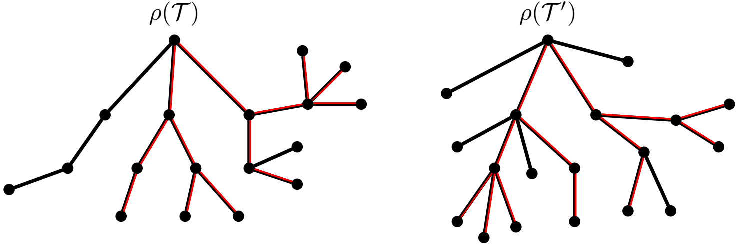

Recursive computation:

$$\mathcal{W}_d(\mathcal{T}, \mathcal{T}') = \sup_{\mathfrak{m}} \sum_{(i,u)\in\mathfrak{m}} \mathcal{W}_{d-1}(\mathcal{T}_i, \mathcal{T}'_u).$$

Intuition for main result

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- if $u = \sigma(i)$, the neighborhoods $\mathcal{N}_i$ of i in G_1 and $\mathcal{N}_u$ in $G_2 \simeq$ GW trees of offspring $\mathcal{P}(\lambda)$, with intersection of offspring $\mathcal{P}(\lambda s)$. Thus

$$\mathcal{W}_d(\mathcal{N}_i, \mathcal{N}_u) \geq \sharp\text{leaves at depth } d \text{ in the intersection} \simeq (\lambda s)^d$$

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Theorem

For $\lambda \in (1, \lambda_0]$ and $s \in (s^*(\lambda), 1]$, then there exists $\gamma < \lambda s$ such that

$$\mathcal{W}_d(\mathcal{T}, \mathcal{T}') \ll \gamma^d \text{ as } d \to \infty.$$

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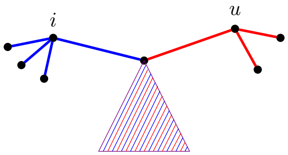

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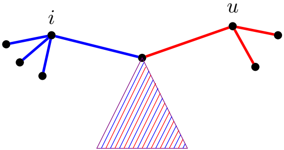
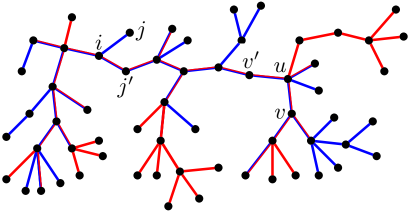


'Dangling trees trick': look at both $\mathcal{W}_{d-1}(j \leftarrow i, v \leftarrow u)$ and $\mathcal{W}_{d-1}(j' \leftarrow i, v' \leftarrow u)$.



Neighborhood tree matching algorithm, main result

NTMA algorithm: $\mathcal{S} = \varnothing$. For all pairs $(i, u) \in V(G_1) \times V(G_2)$ whose $d-$neighborhoods $\mathcal{N}_i$ and $\mathcal{N}_u$ are trees:
If there exists $j \neq j' \overset{G_1}{\sim} i$, $v \neq v' \overset{G_2}{\sim} u$ such that $\mathcal{W}_{d-1}(j \leftarrow i, v \leftarrow u) > \tau$ and $\mathcal{W}_{d-1}(j' \leftarrow i, v' \leftarrow u) > \tau$, then add (i, u) to $\mathcal{S}$.

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Theorem

Assume $\lambda s > 1$, $\lambda \in (1, \lambda_0]$ and $s \in (s^*(\lambda), 1]$. Then for $d = \Theta(\log n)$ and $\tau = \Theta(\gamma^{d-1})$, with high probability:

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(i,\sigma(i))\in\mathcal{S}} = \Omega(1) \quad \textit{and} \quad \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\exists u\neq\sigma(i),(i,u)\in\mathcal{S}} = o(1).$$

Thank you!