

1 Semi-groupoid Algebra

Let M be a semi-group, R a ring, $f : M \rightarrow R$, we write $f = \sum f_m x^m$ formally and call $f_m \in R$ the coefficient of f at x^m , where f_m is the image of m in R under f . This kind of symblic representation can be used in mapping between any sets.

Let

$$R[[M]] = \{f | f : M \rightarrow R\}$$

the algebra of formal series, and

$$R[M] = \{f | f : M \rightarrow R, \#\{m | f(m) \neq 0\} < +\infty\}$$

then algebra of polynomials. Notice that $R[[M]]$ is the free R -module with basis M . Define a binary operation on $R[[M]]$

$$fg = \sum (\sum_{hk=m} f_h g_k) x^m$$

Now take $G = (\mathbb{N}, +)$, the semi-ring of natural numbers, with addition operation, then $R[G]$ is right the polynomial ring of one variable, we usually denoted by $R[X_1]$. The formal sum $\sum f_m x^m$ now becomes the generating function of the coefficient sequence.

When $G = \mathbb{Z}$, $R[G]$ becomes the Laurrent series.

When $G = (\mathbb{N}, *)$, $R[G]$ becomes the algrbra of arithmetic functions, and the product is the Dirichlet convolution.

But the incidence algebra can not be defined in this way, which needs further generalization. Recall the definition of incidence algebra: Let P be a poset, $Q = \{(g, h) | g \leq h\} \subset P \times P$, $\alpha, \beta : Q \rightarrow R$,

$$\alpha\beta = \sum (\sum_{g \leq k \leq h} \alpha_{g,k} \beta_{k,h}) x^{g,h}$$

We can define an associative partial binary operation \circ on Q ,

$$\circ : Q \times Q \rightarrow Q$$

$$(x, y) \circ (y, z) \rightarrow (x, z)$$

but it doesn't make Q into a semi-group. This kind of algebra strutcutr is called a semi-groupoid. However a semi-group is the special case of a semi-groupoid, therefore we may generatlize the definition of semi-group algebra to semi-groupoid algebra and still use $R[Q]$ for the incidence algebra of P . For $f : P \rightarrow R$, we have another kind of product

$$(\alpha f) = \sum (\sum_{g \leq h} \alpha_{g,h} f_h) x^g$$

$$(f \alpha) = \sum (\sum_{h \leq g} f_h \alpha_{h,g}) x^g$$

2 Representation

Now consider the representation of $R[[M]]$. Let V be a R -module, and $GL(V)$ the general linear algebra over V , the representation of $R[[M]]$ is a homomorphism

$$R[[M]] \rightarrow GL(V)$$

this homomorphism is determined by

$$M \rightarrow GL(V)$$

3 Linear Transform

Since $R[[M]]$ is a free monoid, therefore each $f = \sum f_m x^m \in R[[M]]$ can be viewed as a vector in coordinate representation with basis $\{x^m\}$. Let g be a fixed element in $R[[M]]$, then

$$\circ g : f \rightarrow f \circ g$$

$$g \circ : f \rightarrow g \circ f$$

are linear transforms over $R[[M]]$ and admits a matrix form with given basis $\{x^m\}$ when M is finite. Let $M = \mathbb{N}$ and $g = 1 + x + x^2 + \dots = \frac{1}{1-x}$, then $\circ g$ sends each sequence into its partial sum sequence.

Each semi-group restrict to an arbitrary subset becomes a semi-groupoid. Let $[0, n)$ be the restriction of $(\mathbb{N}, +)$, then $\sum_{k < n} x^k$ is the partial sum operator for finite sequence.