$$\iff \quad C^0\hat{U}^1 = |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{U}$$
$$= \hat{I} \otimes \hat{I} + |1\rangle\langle 1| \otimes (\hat{U} - \hat{I})$$