### Problem Set 7 (Last one!)

Total 15 points

Due Monday, 25 April 2022 before 11.59pm

*Instructions. This is an individual assignment. Handwritten solutions are acceptable here, please either scan them to submit.. Of course, you can also typeset your solutions and upload them electronically.*

*(a) The file name should be: PSn.FAMILYNAME1Initial1.xls, for example my solution to Problem set 1 would have the file name PS1.WIDDICKSM.xls.*

*(b) Please make sure that your solutions are well-organized and clear, with appropriate text, explanations, and formatting. If I cannot figure out what you did, then what you did is wrong.*

**1**. (3 points) Suppose that the U.S. dollar price of a British pound (GBP) follows the process

where *s, m*  are constants. The U.S. dollar and Pound interest rates are constants *r* and, respectively. Thus, the prices of dollar and euro bonds (or equivalently, money-market accounts) *B* and *BP* follow the processes

with initial conditions *B*0 = 1 and *BP0* = 1, respectively.

The U.S. dollar price of the euro bond is *Yt* = *StBPt*.

**(a)** (1 point) Write down the dynamics of the U.S. dollar price of a GBP bond, *Yt,* under the risk-neutral probability (with the US bond as numeraire)? (Take the perspective of a U.S. dollar investor.)

*Hint*: Under the risk-neutral probability, what is the expected return on an asset when using the dollar bond as numeraire? Assume that the euro-denominated bond is an asset that is available to a U.S. investor. Then, what is the expected return on a euro-denominated bond for a US investor under the risk-neutral probability?

***Solution*. The euro-denominated bond/money-market account is an asset that is available to a U.S. investor. (The U.S. investor can convert USD to Pound, and then invest in the GBP-denominated money-market account.) Thus, from the perspective of the U.S. investor, the expected return on the GBP-denominated money-market account must be equal to the USD riskless rate *r*. The, the dynamics of the USD value of a GBP bond or money-market account are**

**d*Y*(*t*) = *rY*(*t*)d*t* + *sY*(*t*)d*XQ*(*t*),**

**where *XQ* is a Brownian motion under the risk-neutral probability**

**(b)** (1 point) Using your answer to a) determine the dynamics of the U.S. dollar price of a British Pound, *St*, (not a euro bond) under the risk-neutral probability? (Continue to take the perspective of a U.S. dollar investor.)

***Solution*: In the real world measure**

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**This via the Girsanov theorem dXQ = dX – (m + *r*P – *r*)/*s* and so**

**d*S*(*t*) = (*r – rP*)*S*(*t*)d*t* + *sS*(*t*)d*XQ*(*t*).**

**(c)** (1 point) Use the probabilistic solution to obtain an explicit formula giving the value of a European call option giving the owner the right to buy one euro for an exercise price of *K* dollars on date *T*.

***Solution.* For an ordinary European call option on a common stock that pays dividends at the rate *d*, the value at time *t* is:**

****

**with**

**.**

**Then, by analogy,**

****

**with**

**.**

**2.** (4 points) Assume that we are in the Black-Scholes world where

,

*dBt = rBtdt*

and choose the **stock as the numeraire asset**.

1. (1/2 point) If the stock is the numeraire asset, what is true about the process followed by the ratio of the stock price to the bond price, *Bt/St*?

**It must be a martingale under the appropriate measure.**

**(b)** (1 point) By using your answer to part (a), Ito’s lemma, and Girsanov’s theorem, determine the stochastic process followed by *St* when *St* is the numeraire (this is not as trivial as it sounds).

**Let Y = B/S, We have ∂*Y*/∂*t* = 0, ∂*Y*/∂*S* = -B/*S*2, ∂2*Y*/∂*S*2 = -2*B*/*S*3, ∂*Y*/∂*B* = 1/S. From Itô’s formula under the real world probability,**

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**So, introduce a new probability measure, Q, such that X = – gt is a Brownian motion. Then**

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**if g = -/s in this case, then**

***dS = (r +s 2)Sdt + sSd***

1. (1/2 point) Now consider a European call option, *Vt*. Write down the value of the option in terms of an expectation when *St* is the numeraire.

**According to the probabilistic solution, then under the measure associated with St as the numeraire asset, V/S is a martingale and so V0 = S0EQ0[VT/ST].**

**(d)** (2 points) Using your answers to parts (b) and (c) write *Vt*as the sum of two integrals and **solve the integral with S in it** (this is the d1 one! Leave the other integral as it is more difficult!). Was this easier than evaluating this term (i.e d1) when *Bt* was numeraire?

**Now we have that**

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**Where**

****

**And so the first term is simply *S0N(d1)* with the usual definition of d1. This is very straightforward!**

**3.** (8 Points)

Let *B* be a $-money market account, *BP* be a Korean Won (KRW)-denominated money market account and *S* be the price of Samsung stock in KRW, the exchange rate (or the value of KRW in dollars) is *e*. These prices follow the processes

where X1t and X2t are Brownian motions under the (real world) ***P*** measure and where the correlation, *r*, the risk-free rates, *r* and *r*K, the drifts, m, me and standard deviation, *s* and se,terms are constants.

1. What is the process followed by the dollar value of the Samsung stock price SDt = etSt (I would not combine dX1t and dX2t here)

Text, letter

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1. What is the process followed by the dollar value of the KRW money market account BKDt = etBKt

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1. Define Yt = BKDt/Bt. What are the dynamics of *Yt*under the original probability, ***P***? (That is, what is *dYt*?).

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1. Define Zt = SDt/Bt. What are the dynamics of *Zt* under the original probability, ***P***? (That is, what is *dZt*?).

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1. By the fundamental theorem of finance *Yt* must be a martingale under a unique probability measure ***Q***. By applying Giranov’s theorem, find a new probability measure, ***Q*** and a new Brownian motion (under ***Q***) so that *Yt* is a martingale under the new measure.

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1. By the fundamental theorem of finance *Zt* must also be a martingale under the unique probability measure ***Q***. By applying Giranov’s theorem, find the new Brownian motion (under ***Q***) so that *Zt* is also a martingale under the new measure. (*You will need your result from d) and e)*)

**Text

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Finally, Consider a Quanto digital call option with the following payoff **in dollars**:

where ST is the value of the Samsung stock (in KRW) at time T. The current date is *t* = 0.

1. By using the results from f) above, or otherwise, determine a formula for the value of this digital call option.

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