Computational Statistics Homework 1

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1 Question 1

The function $p(x) = sin(x)$ is a density for $x \in (0, \pi/2)$.

1.1 Describe an inverse CDF transform method to sample random variables with this density. Plot the histogram and the true density for visual verification.

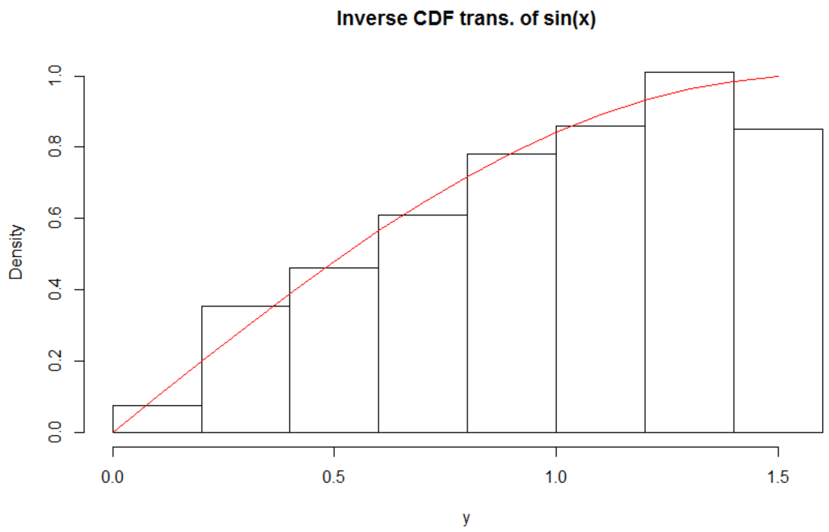
if the PDF is $p(x) = sin(x)$ from $(0, \pi/2)$, then we write the CDF as:

$$\int_0^{\pi/2} sin(x)dx$$

so the result of the integral should be $-cos(x)$. Now we need the inverse of $cos(x)$, which is $arccos(x)$. so our inverse CDF transform should be: $-arccos(x)$, but it resulted in the incorrect output. We can see the code for my inverse transform, and how I drew the graph below:

```
isin_rvgen = function(n=1000) {
  u = runif(n)
  x = acos(1-u)
  return(x)
}

x = seq(0, pi/2, 0.1)
fx = sin(x)
y = isin_rvgen()
hist(y, freq = F, main = "Inverse CDF trans. of sin(x)")
lines(x, fx, col="red")
```

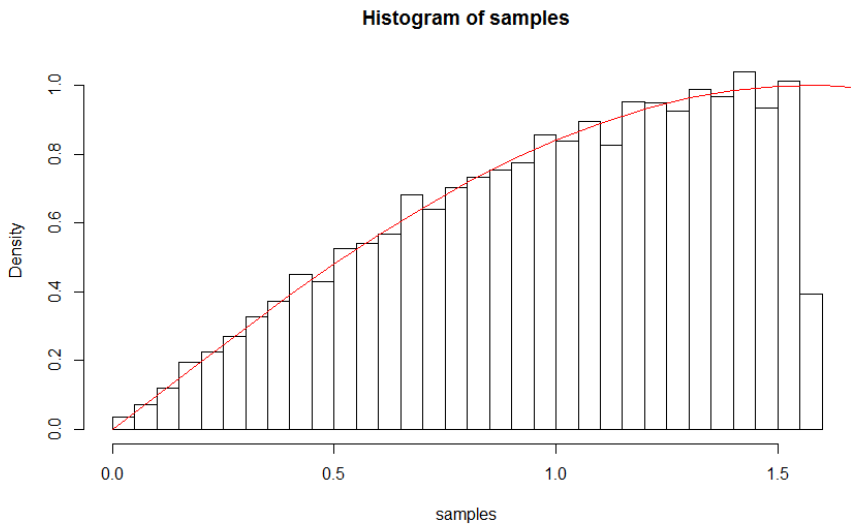


The red line is the graph of $sin(x)$ from $(0, \pi/2)$. and the histogram is contains the samples from our inverse CDF, $arccos(x)$.

1.2 Set up a rejection sampling method to sample from $p(x)$ using a proposal density $g(x)$. Plot the histogram and the true density for visual verification

Hint: You can take $g(x)$ as the uniform density on the interval $(0, \pi/2)$.

```
arfunc = function(n=1000) {
  rv = rep(0,n)
  a = 0
  for(i in 1:n) {
    x = runif(1)
    if(x > -acos(x)) {
      rv[i] = x
      a = a + 1
    }
  }
  cat("Efficiency is ", a/n)
  plot(rv)
  title("accept reject plot")
}
arfunc()
```



2 Question 2

Determine a method to draw samples from the distribution with PDF: $f(x) \propto exp(-x^4/12)$, for $x \in R$. Turn in derivation and code. Plot histogram and true density for visual verification.

We can relate this density to its general form, where $f(x) \propto exp(-kx^n)$. In our case, $k = 1/12$, and $n = 4$. Let $Y = X^4$. Note $x > 0$ so $y > 0$. We want to perform a change of variables so we can have our density in terms of y.

$\frac{dy}{dx} = 4x^3$. From the change of variables theorem, $f_y(y) = f_x(x) \mid \frac{dy}{dx} \mid$. Thus, $f_y(y) = exp(-x^4/12)\frac{1}{4}x^{-3}$. Now we need to swap out x for y. Observe the relationship: $y = x^4$ Which is change to $x = y^{1/4}$. By subsitution, we have

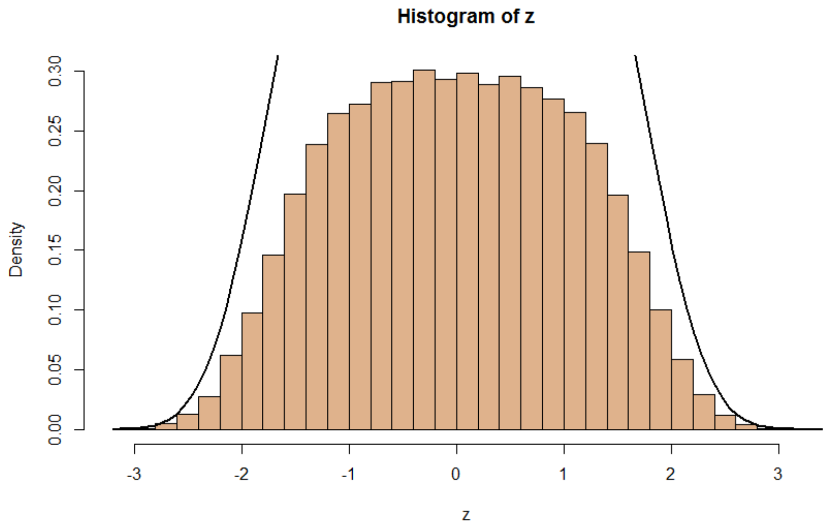
$$f_y(y) = exp(-y/12)\frac{1}{4}y^{-3/4}$$

Note that this is similar to the Gamma density:

$$f(u) = \frac{\lambda^\alpha e^{-\lambda u} u^{\alpha-1}}{\Gamma(\alpha)}, u \geq 0$$

However, we are sampling over all of R, so we need to do a coin-flip on every sample, and decide whether the sample must be negative or positive.

```r
z = rgamma(1e5, shape = (1/4), rate = (1/12))
z = z^(1/4)
f = runif(1e5)
for(i in 1:1e5) {
  if(f[i] < 0.5) {
    z[i] = z[i] * -1
  }
}
hist(z, breaks = 30, freq = F, col = rgb(0.75,0.4,0.1,0.5)) # z is your sample
lambda = 1/12; alpha = 1/4
target <- function(x){exp(-lambda*x^(1/alpha))/
    integrate(function(x) exp(-lambda*x^(1/alpha)),0,Inf)$value}
curve(target,lwd=2,add=T)
```



3 Question 3

Suppose that $X \in R^{nxn}$ is a random matrix with independent $\mathcal{N}(0,1)$ entries. Let ℓ_1 be the smallest Eigenvalue of $S_n = X^T X/n$ and let $Y = n\ell_1$. Edelman showed that as $n \to \infty$, the PDF of Y approaches

$$f(y) = \frac{1+\sqrt{y}}{2\sqrt{y}} e^{-(y/2+\sqrt{y})},\ 0 < y < \infty$$

3.1 Develop a method to sample Y from the density $f(y)$ given in the equation above. Show derivation and code. Plot histogram and true density for visual verification.

3.2 Test method by estimating $E(log(Y))$ by simple Monte Carlo, and giving a 99% confidence interval. Edelman found that the answer was roughly -1.68788.