Senior Paper: The Mandelbrot Set

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March 8, 2019

1 Introduction

The Mandelbrot Set is a set of numbers in C that satisfy behaviors under the constrains of a given function. While the numbers within the set, and the numbers clearly not in the set seem straight-forward, the popular interest in this set comes from the bizarre behavior of what occurs on the boundary of the set. The behavior of what occurs on the boundary of the set seems to go on with inifinitely minute detail, and there are many videos on the internet of people exploring certain areas of this boundary to incredible precision. What we want to do is explore some of the concepts behind this behavior, in hopes to have a better understanding of what is occuring when we look at this set.

2 Iteration of a Function

To *iterate* a function f means to take the value of the function for a given input, and pipe that value back in as an input, i.e.

$$f(x_0) = x_1, f(x_1) = x_2, f(x_2) = x_3...$$

Where x_0 is some sort of initial input or condition. We can label these iterations like this: $f^{(0)}(x_0) = x_1, f^{(1)}(x_1) = x_2, ...$ Where $f^{(i)}(x_i) = x_{i+1}$. It is worth noting that when we graph the iteration of a function, the coordinates look like this:

$$(\ x_i, f^{(i)}(x_i)\ ) = (x_i, x_{i+1})$$

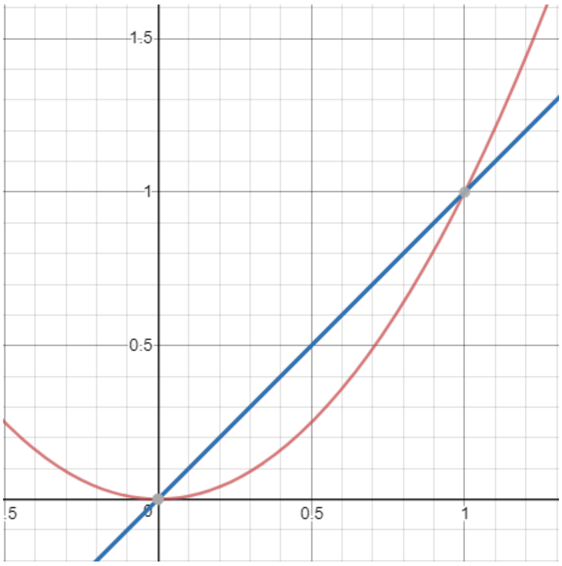
The reason it is worth noting this is there are cases when $x_i = x_{i+1}$ for some point (x_i, x_{i+1}) that are important in understanding the construction of the Mandelbrot Set.

3 Orbits

In the previous section we introduced the concept of *iterating* a function f over an initial input x_0. The *orbit* of x_0 under f is the sequence of points that result from iterating f over its initial input x_0. This initial input is called the *seed* of the orbit.

There are many different kinds of orbits, but the most important one, which we will look at first, is called a *fixed point*, where $f(x) = x$. If you recall from the previous section where we mentioned the point (x_i, x_{i+1}), where $x_i = x_{i+1}$ for all i, this is a fixed point under our function. An example of fixed points under a function f is the function $f(x) = x^3$, where the fixed points are $-1, 0, 1$, as$f^n(-1) = -1$, $f^n(0) = 0$, and $f^n(1) = 1$ for any n. Remember than the n "power" does not actually mean raise the function to the nth power, but rather *iterate* the function by composing it n times, i.e. $f^n(f^{n-1}(...f^1(f^0(x_0))..))$.

Fixed points may also be found geometrically by graphing the function, as well as the identity function, $y = x$. Wherever the graph of our function intersects that line, we have a fixed point. This is true because of what we mentioned earlier about $x_i = x_{i+1}$ for all i. For example, given the function $f(x) = x^2$, and super-imposing the identity function $y = x$ over it, we can quickly spot the fixed orbits.



(a) fixed point orbits $\{0, 1\}$ of $f(x) = x^2$

Another type of orbit is called the *eventually fixed* orbit. the orbit of some point x_0 is *eventually fixed* if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic. For example, given the function:

$$f^n(x_0) = \frac{x_0}{2^n}$$

Given any $x_0 \neq 0$, as $n \to \infty$, the sequence $\{\frac{1}{2^n}\}$ tends towards 0. So we can say, the orbit of x_0 converges to the fixed point 0, and thus is *eventually fixed*.

Now, returning to our function $f(x) = x^2$, we can graphically analyze the orbits of a given point by tracing how the output of one iteration pipes into the input of the next iteration. For example, let's choose a point x_0 such that $0 < x_0 < 1$, That is, between the two fixed points.