Senior Paper: The Mandelbrot Set

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1 Introduction

The Mandelbrot Set is a set of numbers in $\mathbb{C}$ that satisfy behaviors under the constrains of a given function. While the numbers within the set, and the numbers clearly not in the set seem straight-forward, the popular interest in this set comes from the bizarre behavior of what occurs on the boundary of the set. The behavior of what occurs on the boundary of the set seems to go on with inifinitely minute detail, and there are many videos on the internet of people exploring certain areas of this boundary to incredible precision. What we want to do is explore some of the concepts behind this behavior, in hopes to have a better understanding of what is occuring when we look at this set.

2 Biography: Benoit Mandelbrot

Benoit began his life at 1924 in Poland, where he was introduced to mathematics by his two uncles. His family, with an academic background, took responsibility for his education. Mandelbrot's contributions to the field of mathematics are largely responsible for the present interest in fractal geometry, which have been used to describe diverse behavior in economics, finance, the stock market, astronomy, and computer science *(Cunningham, n.d.)*.

His education was considered unconventional, due to it being interrupted by the onset of World War II. He stated that the nature of his education forced him to think in unconventional ways, rather than someone with a conventional education would be encouraged to think in standard ways *(na, 1999)*. His highly successful book, *The Fractal Geometry of Nature*, his work is a stimulating mixture of conjecture and observation, both into mathematical processes and their occurrance in nature and economics *(Cunningham, n.d.)*.

in 1980, he proposed that a certain set governs the behavior of some *iterative* processes in mathematics that are easy to define but have remarkably subtle properties. His evidence about this set helped to generate a substantial and continuing interest in the subject *(Cunningham, n.d.)*. The set, *The Mandelbrot Set*, which this paper revolves around, reflects the statements about being easy to define, yet has very remarkable and subtle behavior. Benoit has won numerous awards and honory degrees, and became a fellow of the *American Academy or Arts and Sciences* and *National Academy of Sciences (Cunningham, n.d.)*.

3 Iteration of a Function

To *iterate* a function f means to take the value of the function for a given input, and pipe that value back in as an input, i.e.

$$f(x_0) = x_1, f(x_1) = x_2, f(x_2) = x_3...$$

Where x_0 is some sort of initial input or condition. We can label these iterations like this: $f^{(0)}(x_0) = x_1, f^{(1)}(x_1) = x_2, ...$ Where $f^{(i)}(x_i) = x_{i+1}$. It is worth noting that when we graph the iteration of a function, the coordinates look like this:

$$(\ x_i, f^{(i)}(x_i) \ ) = (x_i, x_{i+1})$$

Which we notate as:

$$f^{(i)}(x_0) = f^{(i-1)} \circ f^{(i-2)} \circ ...f^{(1)} \circ f^{(0)}(x_0) \text{ where: } f^{(i)}(x_0) = x_i$$

The reason it is worth noting this is there are cases when $x_i = x_{i+1}$ for some point (x_i, x_{i+1}) that are important in understanding the construction of the Mandelbrot Set.

4 Orbits

In the previous section we introduced the concept of *iterating* a function f over an initial input x_0. The *orbit* of x_0 under f is the sequence of points that result from iterating f over its initial input x_0. This initial input is called the *seed* of the orbit.

There are many different kinds of orbits, but the most important one, which we will look at first, is called a *fixed point*, where $f(x) = x$. If you recall from the previous section where we mentioned the point (x_i, x_{i+1}), where $x_i = x_{i+1}$ for all i, this is a fixed point under our function. An example of fixed points under a function f is the function $f(x) = x^3$, where the fixed points are $-1, 0, 1$, as$f^n(-1) = -1$, $f^n(0) = 0$, and $f^n(1) = 1$ for any n. Remember than the n "power" does not actually mean raise the function to the nth power, but rather *iterate* the function by composing it n times, i.e. $f^n(f^{n-1}(...f^1(f^0(x_0))..))$.

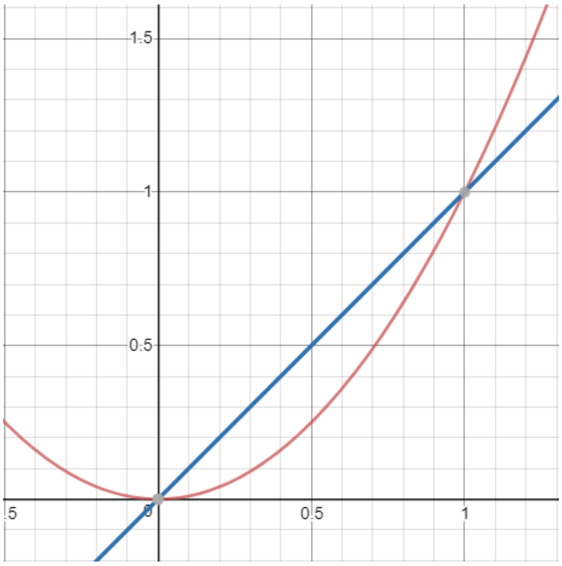
Fixed points may also be found geometrically by graphing the function, as well as the identity function, $y = x$. Wherever the graph of our function intersects that line, we have a fixed point. This is true because of what we mentioned earlier about $x_i = x_{i+1}$ for all i. For example, given the function $f(x) = x^2$, and super-imposing the identity function $y = x$ over it, we can quickly spot the fixed orbits.

Another type of orbit is called the *eventually fixed* orbit. the orbit of some point x_0 is *eventually fixed* if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic. For example, given the function:

$$f^n(x_0) = \frac{x_0}{2^n}$$

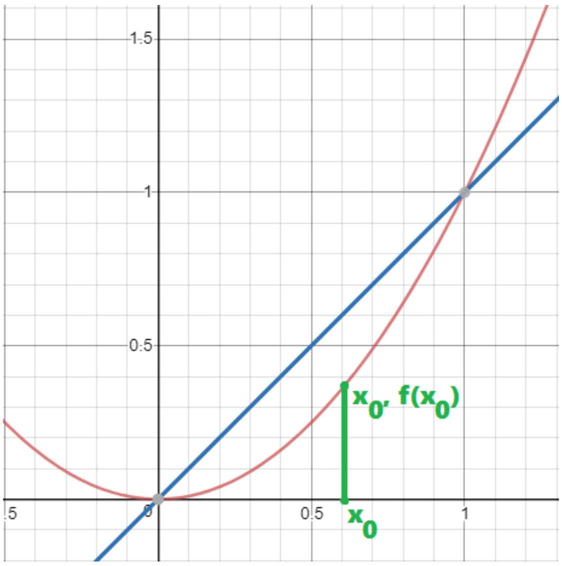
Given any $x_0 \neq 0$, as $n \to \infty$, the sequence $\{\frac{1}{2^n}\}$ tends towards 0. So we can say, the orbit of x_0 converges to the fixed point 0, and thus is *eventually fixed.*

Now, returning to our function $f(x) = x^2$, we can graphically analyze the orbits of a given point by tracing how the output of one iteration pipes into the input of the next iteration. For example, let's choose a point x_0 such that $0 < x_0 < 1$, That is, between the two fixed points.



(a) fixed point orbits $\{0, 1\}$ of $f(x) = x^2$

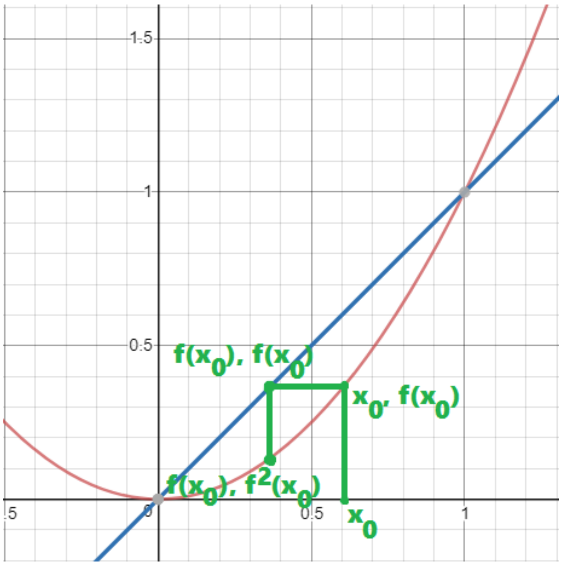
We plot our point on the x-axis, and apply the function $f(x) = x^2$ to our *seed* x_0, which takes us up vertically to the curve.



(a) PLACE HOLDER

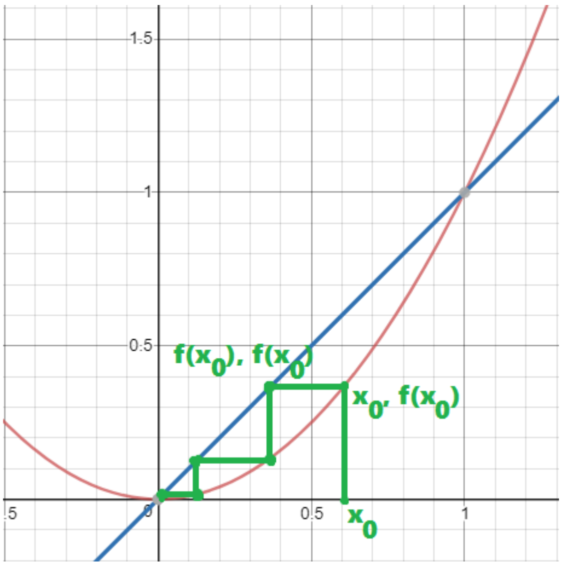
Now recall that we are iterating this function, so the output, i.e. $f(x_0)$ becomes our input for our second iteration. This is equivalent to drawing a horizontal line from the point $(x_0, f(x_0))$ to the $y = x$ line, such that we are at the point $(f(x_0), f(x_0))$. Now we want to compute $f^2(x_0)$, so we need to draw a vertical line from the point $(f(x_0), f(x_0))$, to the point $(f(x_0), f(f(x_0))) = (f(x_0), f^2(x_0))$.

By continuing this process of calcuating the point $(x_i, f(x_i))$, then drawing to the identity



(a) PLACE HOLDER

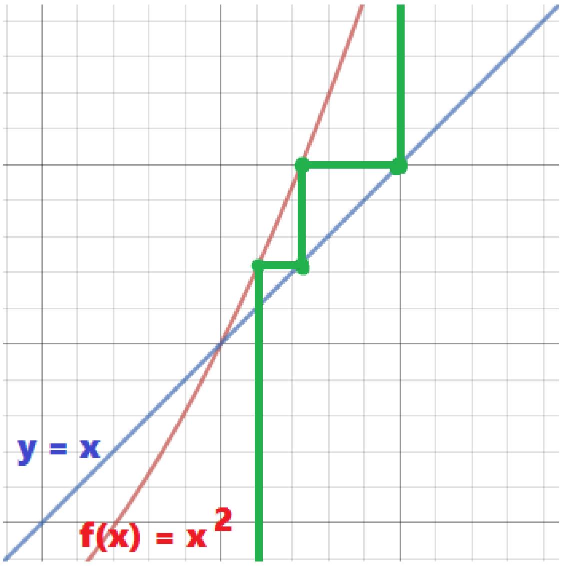
line to transform the output to the new input, we can find the orbit of our given *seed* under our function f.



(a) PLACE HOLDER

So we can infer from this example, that for all points $0 < x_0 < 1$, the orbit is *eventually fixed*, that it converges to 0. This concept it critical to understanding how values become accepted into the Mandelbrot Set.

Let's look at the same graph again, but in a different window. Here we can see again our fixed point of 1, but this time let's choose an x_0 such that $x_0 > 1$. If we play the same game, plotting first $(x_0, f(x_0))$, then drawing to the $y = x$ line, and so on, we can see we have much different behavior.



(a) PLACE HOLDER

The orbit explodes, or as $n \to \infty$, the orbit diverges to infinity. So we can see that orbits are sensitive to the initial seed, that, given a certain function, the initial condition can determine how the function behaves under iteration.

We can change this problem, such as, if instead of varying our seed x_0, we instead vary the y-intercept, c. We can then find for what values of c does the orbit given the seed $x_0 = 0$ explode. Then our equation will look more like $f(x_0) = x_0^2 + c$. So for $n = 0$, we have $f(0)^0 = 0^2 + c$. Then our first point will be $(0, c)$, our second will be $(c, c^2 + c)$, and our third $(c^2 + c, (c^2 + c)^2 + c)$ and so on.

So the obvious next question to ask is, for what values of c do we have orbits that explode? Or rather, for what values of c does the orbit converge? Let's try values of c such that $0 < c < 1$, or rather, $c = 1/x$, where $x > 1$. We could constrain x such that $|x| > 1$, but let's just stick to positive numbers for now. So lets begin with our seed 0, and crunch the numbers.

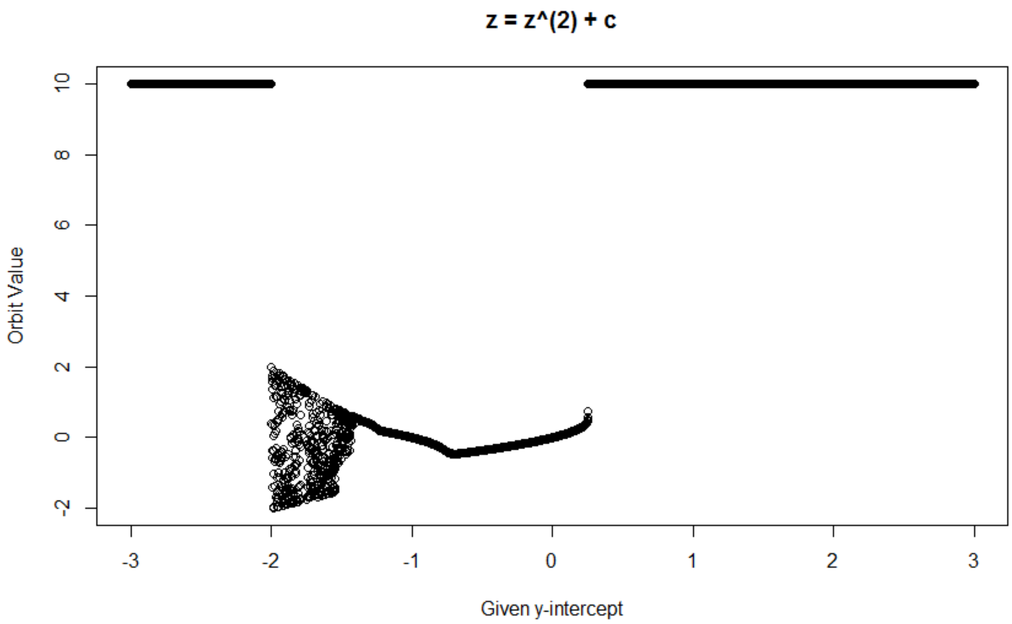
$$f^0(0) = (0)^2 + 1/x$$
$$f^1(1/x) = (1/x)^2 + 1/x = \frac{1}{x^2} + \frac{1}{x} = \frac{x^2 + x}{x^3}$$

.

$$f^2(\frac{x^2 + x}{x^3}) = (\frac{x^2 + x}{x^3})^2 + \frac{1}{x} = \frac{x^4 + 2x^3 + x^2}{x^6} + \frac{1}{x} = \frac{x^6 + x^5 + 2x^4 + x^3}{x^7}$$

After this the algebra becomes quite bothersome to do by hand, but it appears that the equation will converge to 0, as the equation appears to always be dominated by the x-term in the denominator. But how do we prove this? Also, is it simply values of c between 0 and 1? Or is the criteria something more complicated?

To check my intuition, I did an empirical test. I used the R language, which, if you are unfamiliar with, is a computer programming language commonly used in statistical computing. I wrote some code to generate some evenly distributed values of c and compute the behavior of their orbits for $n = 50$ iterations. I tested values of c such that $-3 \leq c \leq 3$, and changed c by a value of 0.001 each time. After running, I plotted the points and the corresponding value after 50 iterations.



(a) PLACE HOLDER

I put a condition on the graph, such that if the absolute value of the orbit was ≥ 10, positive, or negative infinity, to set the value automatically to 10. On the right side, the values seem to explode pretty quickly after what appears to be 0.25, but on the left the behavior is more interesting. The values are scattered all around the place, only exploding around $c < -2$. I'm assuming this is because the negative value of c is doing its best to push down the increasing value of z^2, where z is the output of the equation from the previous iteration.

So my initial assumption that values of c such that $|c| < 1$ gave us a non-exploding orbit was false. So what causes a value of c to have a divergent orbit? If you recall from earlier, given a seed x_0, one of the types of orbits we can have are *eventually fixed*. But there must be a fixed point, where $y = x$ for the orbit to converge to. If there are no fixed points, then there cannot be any eventually fixed points. So one thing we may try is to find the fixed points of the equation. We can use the quadratic equation, but instead of simply solving for when $y = 0$, we solve for $y = x$, thus, solve for $z^2 + c = z$. This gives us two points, $\frac{1}{2}(1 + \sqrt{1 - 4c})$ and $\frac{1}{2}(1 - \sqrt{1 - 4c})$. Note that the points are real if and only if $c \leq 1/4$. This explains why the orbits suddenly diverge on the right side of our plot, because there is no real fixed points, and thus nothing to be eventually fixed to.

Now what about the case when $c < -2$? it also seems to diverge, but it has real fixed points. At $c = -2$, the function actually converges exactly to a fixed point 2 in a few iterations. So why does $c < -2$ diverge? There are actually a few different kinds of fixed points, *attracting*, *repelling*, and *neutral*. If we pause for a moment, at look back at our first example, $f(x) = x^2$, we have two fixed, points, 0 and 1. if we select a *seed* x_0 such that $0 \leq x_0 < 1$, the orbit is attracted towards 0. However, if $|x_0| > 1$, then the point is repelled and diverges. You can re-read and look at the graphical analysis performed on the equation previously in this paper for understanding.

So how do we tell, in general, if a point is *attracting* or *repelling*? We simply take the derivative of our function, and plug in the fixed point. for some fixed point x_0, if $|F'(x_0)| > 1$, then it is a *repelling fixed point*. Otherwise, if $|F'(x_0)| < 1$, then it an *attracting fixed point*. if $|F'(x_0)| = 1$, then it is *neutral*. So in our case, $|F'(0)| = 2(0) < 1$, and $|F'(1)| = 2(1) > 1$. So since $f(z) = z^2 - 2$ has a *repelling fixed point* at 2, then for any $c > -2$ will diverge. And thus explains why c-values in the range $2 \leq c \leq 1/4$ don't diverge towards infinity.

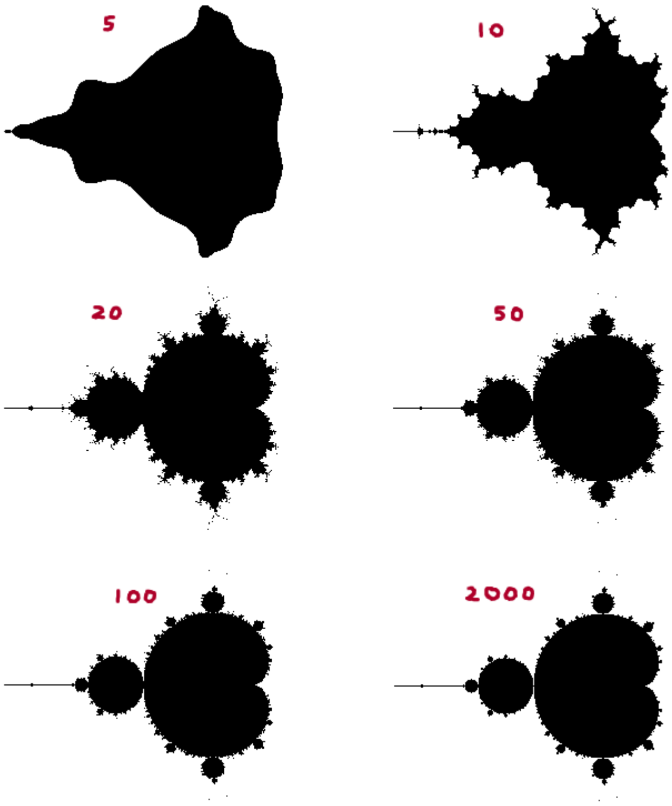
5 Mandelbrot Set

Now that we have finally described all the necessary prerequisite knowledge, we can discuss the point of this paper. The Mandelbrot Set $\mathcal{M}$ is a subset of the complex plane where $\mathcal{M} = \{c \in C \mid |F_c^n(0)| \nrightarrow \infty\}$. for each parameter c, we will test if the orbit of 0 tends to infinity. If it does not, then the point lies within $\mathcal{M}$.

So what orbits explode, and which ones don't? Recall when we were exploring the version of $F_c^n(z) = z^2 + c$ where $z, c \in R$. If you remember, for any value of $c < -2$, the orbit of 0 exploded. But when $c = -2$, the orbit is *eventually fixed* and actually equals its fixed point positive 2 after a few iterations. Similarly, if we select any seed x_0 such that $-2 \leq x_0 \leq 2$, it remains *bounded*, where $|F_{-2}^n(x_0)| \leq 2$ for all n. Taking this back to our function, If at any point z becomes greater than some bound K, then we say the orbit of z is unbounded. This is interesting to think about in the case of the example we just underlined, $F_{-2}^n(z) = z^2 - 2$, because, if at any point $|z|$ steps over its bound of 2, then the orbit has no hope of staying bounded, and is guaranteed to go towards infinity. This can be also seen in $F_0^n(z) = z^2$, where the bound is 1. If $|F_0^n(z)| > 1$ for any n, then the orbit will certainly goto infinity.

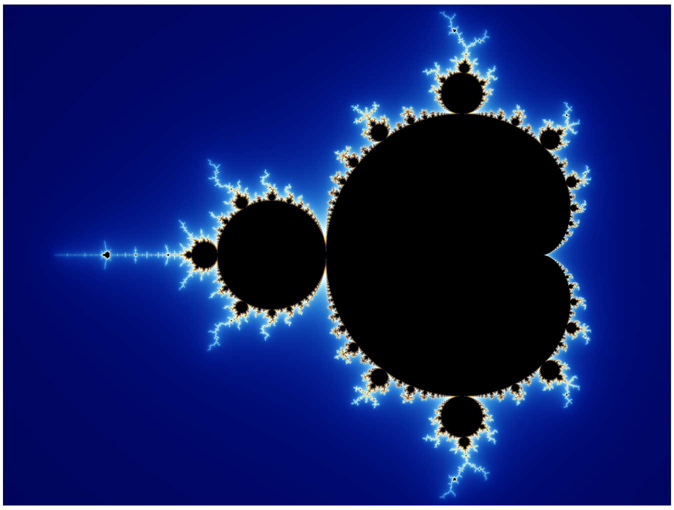
So we can safely say that if $|c| > 2$, then the orbit spirals to infinity. But not all values of c under 2 stay bounded either, we just know that there is no hope for any numbers greater than it. This is where computing comes back into play. recall that the n in our function F_c^n measures the number of iterations. To compute the Mandelbrot Set, we choose a finite maximum number of iterations, N. Then, for each complex number c, we compute $F_c^N(z)$, with orbit seed 0. Then, if at any point, $|F_c^i(z)| > 2$ for any i, we can stop immediately, as we are guaranteed to have a point diverge, and it is not accepted into the set. if $|F_c^i(z)| > 2$ for all N, then it is in the Mandelbrot Set.

It is important to note that we will always have some margin of error, as there may be some points which take significantly longer iterations of N than we selected to grow past 2. However, by increasing N, we can reduce that error. The exciting part of the Mandelbrot Set comes from its visualization. If a point is contained in $\mathcal{M}$, then we color it black. Otherwise, we color it white. And by mapping real and imaginary parts to (x, y) coordinates of an image, we can plot the Mandelbrot Set itself. Note that we only graph points in the range $[-2, 2]$ in both the real and imaginary axis.

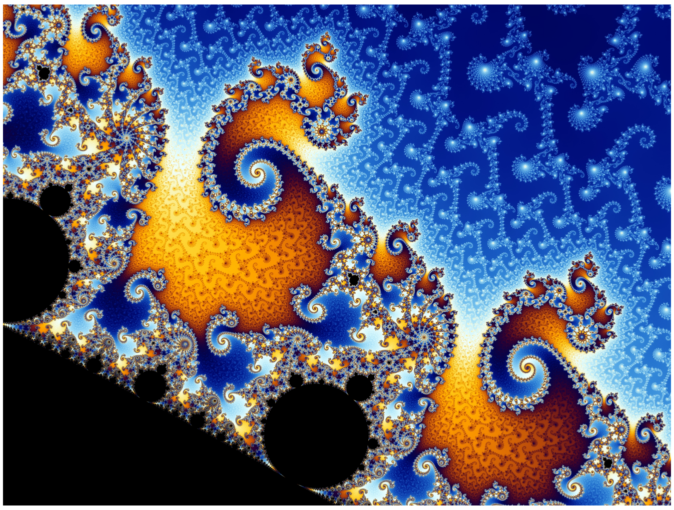


(a) Increasing the value of N improves margin of error (Fonesca, 2018)

In addition, reducing the space between calculated points will improve the resolution of the image. This is especially important if one wishes to visualize a smaller section of the set. There are additional aspects of information that we can use to improve upon the visualization of the Mandelbrot Set. For example, we can color the points based on the speed at which $|z|$ grows past 2. For example, if $|z|$ grows past 2 in just a few iterations, then we color it blue. If it does so in the last few iterations up to N, then we color it red. And we proportionally color all the points between in their respective gradient color. Recall that a point is only black if it is actually in the set.



(a) Mandelbrot Set. (Beyer, Wikipedia, 2013)



(a) Mandelbrot set zoom. (Beyer, Wikipedia, 2005)

If we observe the first picture, we can see that the farthest left point of the graph is the point -2 on the real axis, and the point that exists on the farthest right "heart shape" (known as the main cardioid) is the point 1/4 on the real axis. This shows that the Mandelbrot Set intersects the real axis at the closed interval $[-2, 1/4]$, precisely as we had shown earlier. I've also included some R code that you can use to construct your own Mandelbrot Set.

```
in.mandelbrot.set <- function(c, iterations = 20, bound = 2) {
  z <- 0
  for (i in 1:iterations) {
    z <- z ** 2 + c
    if (Mod(z) > bound) {
      return(FALSE)
    }
  }
  return(TRUE)
}

resolution <- 0.001
sequence <- seq(-1, 1, by = resolution)
m <- matrix(nrow = length(sequence), ncol = length(sequence))
for (x in sequence) {
  for (y in sequence) {
    mandelbrot <- in.mandelbrot.set(complex(real = x, imaginary = y))
    m[round((x + resolution + 1) / resolution), round((y + resolution + 1) / resoluti
  }
}

jpeg('mandelbrot.jpg')
image(m)
dev.off()

# (Unknown author, n.d.)
```

This will save an output image of the mandelbrot set at the given "zoom" to your home directory ("" on Unix systems, or "*My Documents*" on Windows).

6 References

1. Devaney, R. (1992). `AFirstCourseinChaoticDynamicalSystems.` Reading, MA: Addison-Wesley.

2. Benoit Mandelbrot. (1999, July). Retrieved April 13, 2019, from `http://www-history.mcs.st-and.ac.uk/Biographies/Mandelbrot.html`

3. Cunningham, J. (n.d.). Benoit Mandelbrot. Retrieved April 14, 2019, from `https://www.britannica.com/biography/Benoit-Mandelbrot`

6.1 Images

1. Fonesca, R. 2018. MandelbrotIterationshttp://renatofonseca.net/mandelbrotset.php
2. Beyer, W. 2013 https://en.wikipedia.org/wiki/Mandelbrot_set=#/media/File:Mandel_zoom_00_mandelbrot_set.jpg
3. Beyer, W. 2005 https://commons.wikimedia.org/wiki/User:Wolfgangbeyer#/media/File:Mandel_zoom_11_satellite_double_spiral.jpg