



**The 3D Interaction Tool:
A Pointing Device for Virtual Reality Applications**

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Contents

List of Figures	3
I. Nomenclature	3
II. Introduction	4
A. Motivation	4
B. Background	4
B.1 Commercial Solutions	4
B.2 3D Interaction Techniques	5
B.3 Tracking System Design	5
C. Design Objectives	5
C.1 Usability	5
C.2 Performance	5
III. Design	6
A. Requirements	6
B. Geometry	7
B.1 Option 1: World-referenced geometry	7
B.2 Option 2: Display-referenced geometry	7
B.3 Geometry Selection: Comparing portability and tracking volume	8
C. Technology	8
D. Final Design	9
D.1 Tracking measurements	9
D.2 Inferring the tool position	9
IV. Implementation	10
V. Orientation Tracking	12
A. Naive Measurement Model: TRIAD	12
A.1 Error Analysis of TRIAD Measurements	13
B. Mathematical Representation of Orientation	15
C. Dynamic Measurement Model	15
C.1 Filter Design	17
C.2 Filter Initialization	18
C.3 Filter Tuning	19
C.4 Filter Performance	20
VI. Position Tracking	21
A. Measurement Model	21
A.1 Camera Calibration	21
A.2 Object Tracking	22
A.3 Tool Maneuvers	23
A.4 Error Analysis of Motion in the Image Plane	24
A.5 Error Analysis of Motion in the Depth Dimension	25
B. Dynamic Measurement Model	26
VII. Performance Analysis	26
A. Estimation Error	26
B. Covariance Transformations	26
C. Estimation Error for \hat{p}_l : Finding position error due to orientation error	27
VIII. Conclusion	28
IX. Future Work	28
References	28

List of Figures

1	Usability design objectives	5
2	Performance design objectives	6
3	Position and orientation tracking requirements	6
4	Position tracking with a world-referenced geometry	7
5	Position tracking with a display-referenced geometry	8
6	Comparison of five tracking technologies	9
7	Final tracking system design: a display-referenced tracking topology	9
8	Geometrically equivalent tracking system	11
9	Full system diagram of implementation	12
10	Pitch maneuver reference versus estimates	13
11	Pitch error TRIAD estimation	14
12	TRIAD orientation tracking errors	14
13	Tuning the measurement noise parameter of the Kalman filter	20
14	Kalman filter orientation tracking errors	20
15	Kalman filter and TRIAD root-mean-squared error comparison	20
16	Computer vision tracking model	21
17	Camera calibration routine	22
18	Vertical and horizontal motion in the image plane	23
19	Motion of the interaction tool in the depth dimension	24
20	Tool tracking errors in the yz -plane	24
21	Distribution of depth-tracking errors	25

I. Abstract

The 3D Interaction Tool is a hand-held device that enables a user to interact with virtual reality applications. Since both the position and orientation of the tool are tracked, user input with the tool accomplishes tasks such as selecting, translating, and rotating three-dimensional objects, or free-hand drawing in space. The tools position is tracked by a software application that filters observations from a camera attached to the virtual reality display, and the orientation is found by means of observations from an inertial measurement unit attached to the tool itself. Simple computer-aided design and drawing applications demonstrate the capabilities of the 3D Interaction Tool.

II. Nomenclature

Position vectors definitions in the inertial coordinate frame:

\vec{p}_s \equiv position of the spherical tracking target on tool rear

\vec{p}_r \equiv position of the tool rear

\vec{p}_f \equiv position of the tool front

\vec{p}_c \equiv position of the tracking camera

\vec{p}_d \equiv position of the virtual reality display

Rotated position vectors:

\vec{p}'_s \equiv position of the spherical tracking target on tool rear, camera coordinates

\vec{p}'_r \equiv position of the tool rear in display coordinates

\vec{p}'_f \equiv position of the tool front in display coordinates

Attitude matrices:

\mathbf{A}_c \equiv attitude of the tracking camera

\mathbf{A}_d \equiv attitude of the virtual reality display

\mathbf{A}_t \equiv attitude of the interaction tool

\mathbf{A}'_t \equiv attitude of the interaction tool in display coordinates

Body-referenced quaternion representations of attitude:

$\bar{\mathbf{q}}_c$ \equiv attitude of the tracking camera

$\bar{\mathbf{q}}_d$ \equiv attitude of the virtual reality display

$\bar{\mathbf{q}}_t$ \equiv attitude of the interaction tool

Position error covariances in inertial coordinate frame:
 $\Sigma_{ps} \equiv$ covariance of spherical tracking target position

 $\Sigma_{pr} \equiv$ covariance of the position of the rear of the interaction tool

 $\Sigma_{pf} \equiv$ covariance of the position of the front of the interaction tool

 $\Sigma_{pc} \equiv$ covariance of the position of the tracking camera

 $\Sigma_{pd} \equiv$ covariance of the position of the virtual reality display
Attitude error covariances in the inertial coordinate frame:
 $\Sigma_{qc} \equiv$ covariance of the tracking camera attitude quaternion error

 $\Sigma_{qd} \equiv$ covariance of the display attitude quaternion error

 $\Sigma_{qt} \equiv$ covariance of the interaction tool attitude quaternion error
Position error covariances in rotated body coordinates:
 $\Sigma'_{ps} \equiv$ covariance of the spherical tracking target position

 $\Sigma'_{pr} \equiv$ covariance of the position of the rear of the interaction tool

 $\Sigma'_{pf} \equiv$ covariance of the position of the front of the interaction tool
Attitude error covariances in the rotated body coordinates:
 $\Sigma'_{ac} \equiv$ covariance of the tracking camera attitude angular error

 $\Sigma'_{ad} \equiv$ covariance of the display attitude angular error

 $\Sigma'_{at} \equiv$ covariance of the interaction tool attitude angular error

III. Introduction

A. Motivation

As much as \$25 billion could be spent on virtual reality (VR) and augmented reality (AR) software applications by 2025 [1]. There are two primary modes of user-input that are possible for these applications: indirect and direct. Indirect input devices translate the user's physical movements into the motion of a virtual cursor by means of a fixed transformation. Common examples of indirect input devices are a mouse or joystick. Direct input devices also translate physical to virtual movements, but this transformation is one-to-one. For example, a touch screen is a direct input device because the physical location of the user's press is colocated with the virtual click.

With the advent of virtual reality applications the following question must be answered: which of these input paradigms – indirect or direct – will prove most useful for interacting with virtual reality applications? While indirect input devices have achieved widespread popularity for two-dimensional displays, the direct input paradigm for VR applications embraces the immersiveness of VR experiences by enabling intuitive interactions for users. Direct input devices allow users to reach out and interact with the objects of their virtual space, just as they would with objects in their physical space.

B. Background

B.1 Commercial Solutions

Recent market trends suggest that direct input devices are on the rise for consumers. Two of these direct input devices, the "Oculus Touch" controller and the HTC Valve controller, are achieving widespread publicity surrounding the commercial realease of virtual reality headsets [2] [3] [4]. While originally scheduled for release alongside the Oculus VR headset at the end of March 2016, the Oculus Touch controller will be available in the second half of 2016 [5]. Meanwhile the Valve controllers have been reviewed in preparation for the release of the HTC Vive headset at the beginning of April, 2016, and their utility has been demonstrated with a three-dimensional sketching program by Google called TiltBrush [?]. These input devices track both their position and orientation, making them degree-of-freedom (6DOF) controllers. While the 6DOF controllers are reported to work well with their respective headsets, the headsets are expensive – both retail for more than \$600.

B.2 3D Interaction Techniques

Three-dimensional interaction techniques can be separated into several categories. First, there are interactions that manipulate objects. These interactions accomplish tasks such as rotating, translating, or scaling three dimensional objects. Second, there are interactions that manipulate 3D

viewpoint, allowing the user to adjust their field of view. Lastly, there are interactions that control an application. Application control interactions include navigating menus or widgets associated with the application logic. The first set of tasks, those that manipulate objects, have largely been the focus of three-dimensional input device design [6].

B.3 Tracking System Design

C. Design Objectives

There are two broad classes of objectives in 3D input device design: usability and performance.

C.1 Usability

The usability requirements presented in Figure 1 must be met for the tool to succeed as an *input* device for VR software applications.

Usability		
Objective	Justification	Target
Good portability	Target scenarios are seated applications for home and workplace.	The interaction tool setup should be easily translatable between uses.
Low configuration overhead	----	Limited setup time.
Natural form factor	The interaction tool should not require the user to learn a new interaction paradigm.	Form factor similar to a pen or stylus.
Large tracking volume	The user should be able to use the interaction tool over their full range of movement.	Tracking volume of about 1m ³ .
Low cost	----	Less than \$500.

Figure 1: Usability design objectives

C.2 Performance

The performance requirements presented in Figure 2 must be met for the tool to succeed as a *direct* input device for VR software applications.

Performance		
Objective	Justification	Target
High Accuracy	Tracking apparatus must translate user movements into direct input signals.	10cm or less.
Very High Precision	High precision ensures that cursor movements are repeatable.	1-2cm.
Low Latency	Too much discrepancy between physical and virtual motion of the tool induces simulator sickness.	Less than 60ms.
High Robustness	The system should be usable in non-ideal environments and use-cases.	Robust to tool velocity.

Figure 2: Performance design objectives

IV. Design

A. Requirements

In order to enable direct input with the interaction tool, the tool's tracking system must estimate both the position and orientation of the interaction tool. Additionally, rendering a virtual representation of the tool requires measuring its position and orientation relative to the virtual reality display orientation, given by the attitude matrix \mathbf{A}_d . To track the tool's absolute trajectory, its position is determined relative to the world – that is, relative to a non-accelerating, inertial coordinate system in the user's physical environment. The position and orientation of the tool in the world coordinates of

the inertial frame are \vec{p}_f and \mathbf{A}_t , respectively. The position of the tool in the coordinate frame of the virtual reality display is denoted by a 'prime' sign: the tip or front of the tool in display coordinates is \vec{p}'_f . Thus, there are four tracking targets: \vec{p}_f , \mathbf{A}_t , \vec{p}'_f , and \mathbf{A}_d ; these are shown in Figure 3.

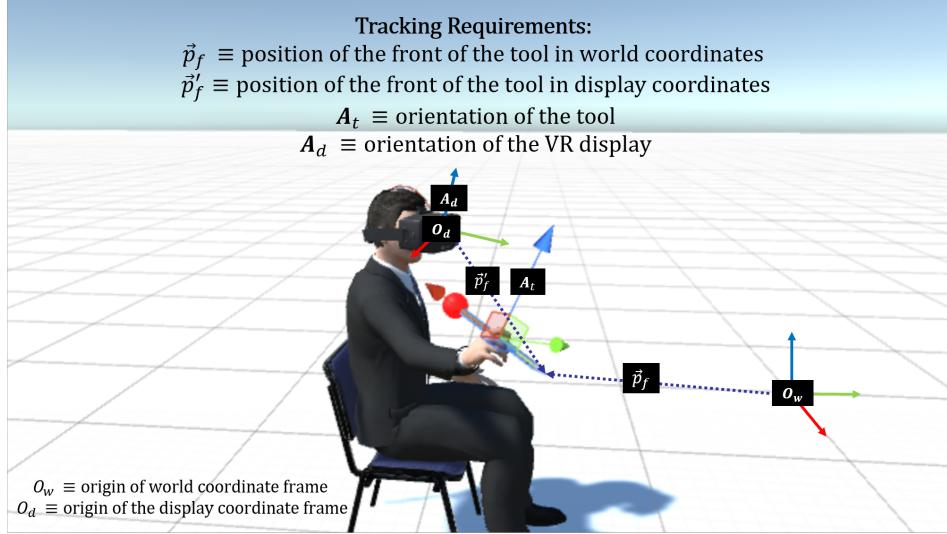


Figure 3: Position and orientation tracking requirements

B. Geometry

The geometry of the tool tracking system affects both the usability and the performance of the interaction tool. The display-referenced geometry is chosen because it eliminates the burden on the user to pre-calibrate their environment before providing direct input to VR applications. At the same time, this geometry is compatible with VR applications with which a user interacts while seated.

B.1 Option 1: World-referenced geometry

The world-referenced tracking geometry is shown in Figure 4.

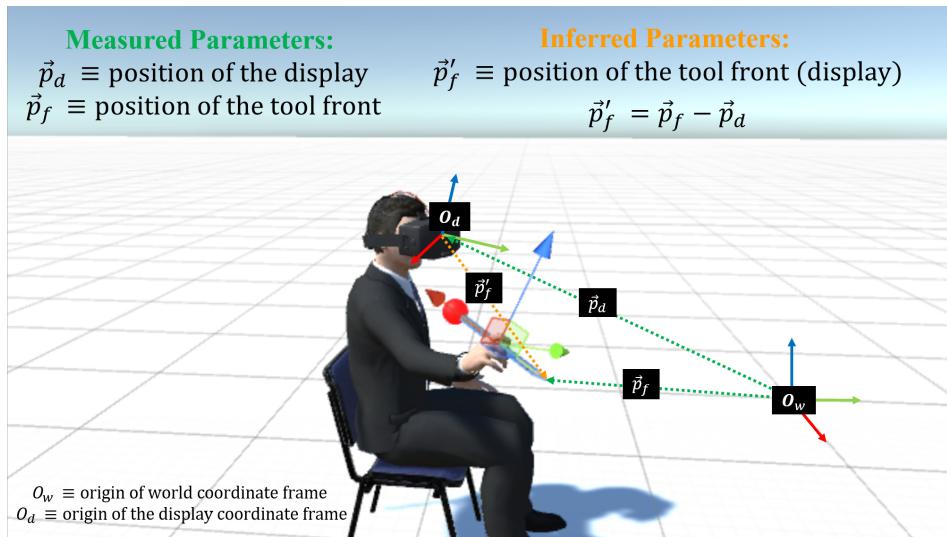


Figure 4: Position tracking with a world-referenced geometry

B.2 Option 2: Display-referenced geometry

The display-referenced tracking geometry is shown in Figure 5.

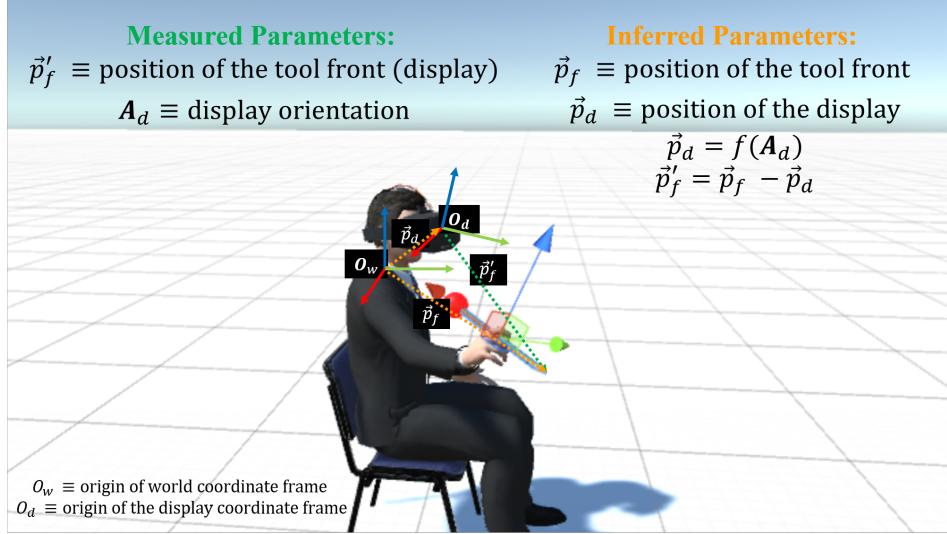


Figure 5: Position tracking with a display-referenced geometry

B.3 Geometry Selection: Comparing portability and tracking volume

Tracking geometries 1 and 2 have advantages and disadvantages when evaluated using the usability design objectives. Relative to the usability criteria, the two geometries differ greatly in their portability and tracking volume.

The world-referenced tracking geometry requires a fixed coordinate system in the user's physical environment, decreasing portability by increasing the configuration overhead for using the direct input device. However, once this coordinate system is established, the user can provide direct input to a VR application over a very large tracking volume, up to the size of the room in which the tracking system is located. The display-referenced geometry, on the other hand, has the advantage of enabling a user to provide direct input to a VR application without pre-calibrating the physical environment in which they are using the VR display. The tradeoff is that the tracking volume is small. Since the origin of the world coordinate system in the display-referenced geometry is located beneath the user's head, the user must remain seated while using a direct input device with a display-referenced tracking geometry.

C. Technology

A combination between optical and inertial sensors was chosen to track the 3D Interaction Tool's position and orientation. The chart in Figure 6 summarizes the comparison between various alternatives for the choice of the tracking technology.

Tracking Technology	Advantages	Disadvantages
Mechanical	Very high accuracy and precision . Good portability .	Small tracking volume .
Magnetic	Very high accuracy and precision . Low latency . Good robustness .	Very high cost .
Ultrasound	High accuracy and precision .	Poor portability . Incompatible with slender form factor . Poor robustness to acoustic noise.
Optical	High accuracy and precision . Low configuration overhead . Low cost .	Poor robustness to occlusions and tool velocity. High latency . Narrow field of view .
Inertial	High accuracy and precision of orientation. Very low latency . Low cost .	Low accuracy and precision of position over large time scales.

Figure 6: Comparison of five tracking technologies

D. Final Design

The final system layout is shown in figure 7.

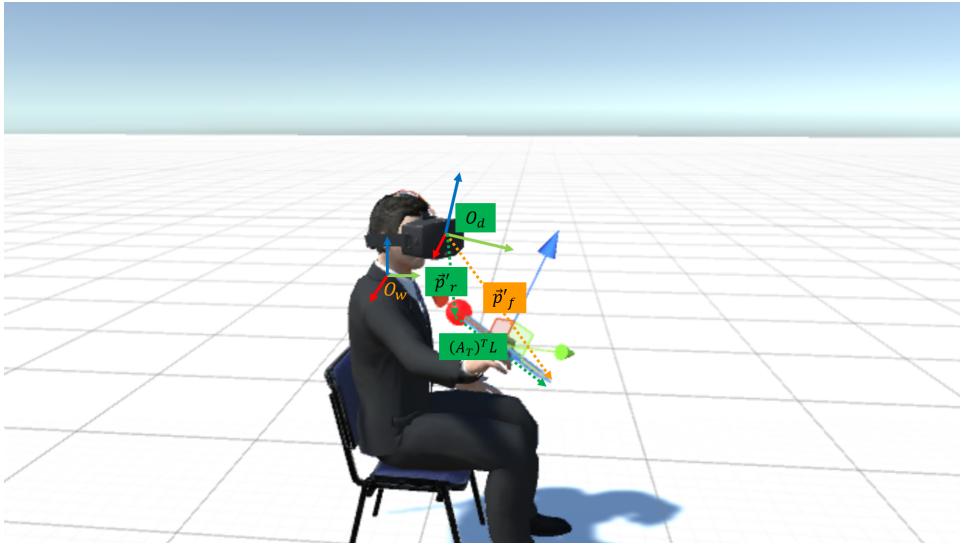


Figure 7: Final tracking system design: a display-referenced tracking topology

D.1 Tracking measurements

Not all of these three tracking targets are measured directly. The tracking targets are inferred from the measurements of the position of the rear of the interaction tool relative to the camera, the attitude of the camera, and the attitude of the tool. The position of the tool relative to the camera is denoted by \vec{p}'_s , the attitude of the camera relative to the inertial coordinate system by \mathbf{A}_c and the attitude of the tool relative to the inertial coordinate system by \mathbf{A}_t . The vector \vec{p}'_s is measured by finding the .

D.2 Inferring the tool position

The position of the front of the tool in world coordinates is given by \vec{p}_f . This position is inferred through measurements of the position of the rear of the tool and the tool's orientation. The equation for the position of the front of the tool is

$$\vec{p}_f = \vec{p}_r + \mathbf{A}_t^{-1} \vec{p}_l$$

where the position of the rear of the tool, \vec{p}_r is given by

$$\vec{p}_r = \mathbf{A}_c^{-1} \vec{p}'_s - \vec{p}_c$$

The attitude matrix, \mathbf{A}_t^{-1} , is the rotation from the body coordinates of the tool to the world coordinate system.

In order to render the tool on the display for the user, the position of the tool must be known to the VR application in the coordinate system of the VR display. The position of the front of the tool in display coordinates is \vec{p}'_f and it is found by multiplying the tool's position in the world coordinate system by the attitude of the display. Additionally, the position of the display must be added to the tool position before multiplying by the display attitude matrix:

$$\vec{p}'_f = \mathbf{A}_d(\vec{p}_f + \vec{p}_d)$$

The tool attitude in the coordinate system of the display is denoted by \mathbf{A}'_t and it is found by composing the rotation from the tool's body coordinates to world coordinates with the display attitude matrix:

$$\mathbf{A}'_t = \mathbf{A}_d \mathbf{A}_t^{-1} \vec{p}_l$$

This is the tool orientation visible to the user.

V. Implementation

The system's physical layout is shown in ???. For simplicity, the tracking camera is assumed to be fixed. The model for this system is geometrically equivalent to the one presented in Figure 7. The webcam of the computer is used as the tracking camera. The inertial measurement unit, the Adafruit BNO055, is attached to the breadboard. A colored foam sphere is attached to the rear of the interaction tool to serve as the tracking target.

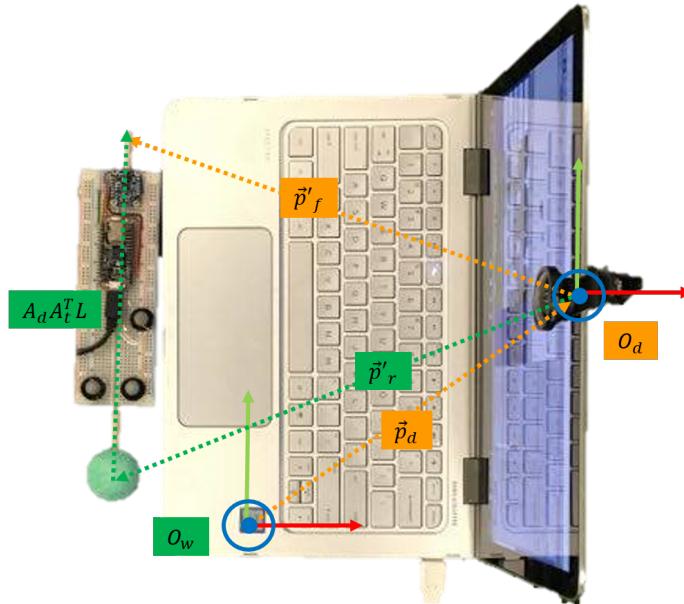


Figure 8: Geometrically equivalent tracking system

The virtual reality application runs on the computer shown in Figure ???. While this virtual reality application is not immersive, it creates a virtual 3D environment displayed to the user on the computer screen, where they can interact with virtual objects. Thus, this geometry is sufficient to prove the usefulness of the 3D Interaction Tool for virtual reality applications.

The spherical tracking target is chosen because its size in each image does not depend on its orientation. Additionally, if the angle subtended by the tracking target in the field of view is small, then the midpoint of the tracking target in world coordinates is on the same plane as the vanishing points on the edges of the target.

The block diagram of the implemented tracking system is shown in Figure 9.

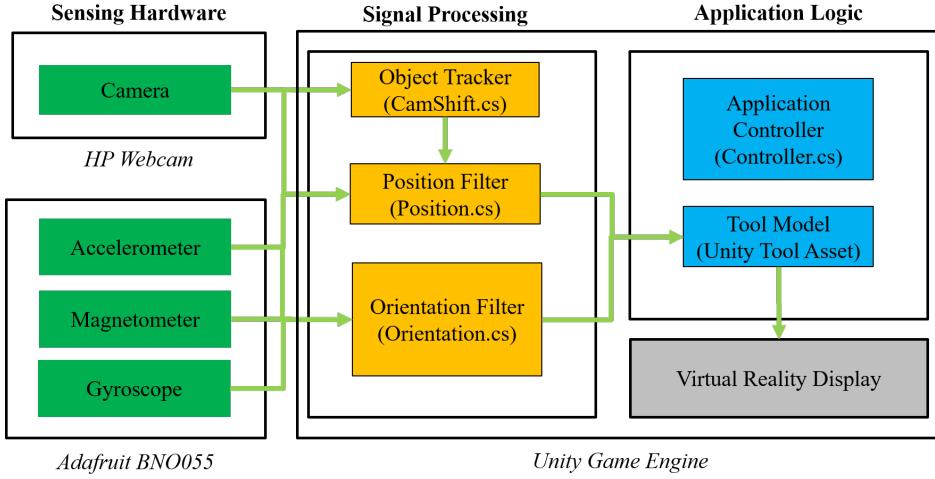


Figure 9: Full system diagram of implementation

VI. Orientation Tracking

The orientation of the interaction tool must be estimated to find the span vector, $\vec{p}_l = \vec{p}_f - \vec{p}_r = \mathbf{A}_t \vec{p}'_l$. The naive approach for estimating the orientation of the tool is to start with two known vectors in the reference coordinate system – the gravity vector and the earth’s magnetic field vector are good candidates. Then we can find the attitude matrix directly by sensing these two reference vectors in the body coordinates of the device and finding the rotation matrix that maps them back to the reference coordinate frame. While this method is computationally and conceptually simple, the estimation is very sensitive to noise from both the accelerometer and the magnetometer that sense the gravity and magnetic field vectors, respectively. We would like some way of filtering these estimates in real-time.

A. Naive Measurement Model: TRIAD

The TRIAD estimation method finds the attitude matrix A that maps three reference frame vectors \vec{r}_k to their corresponding body frame vectors \vec{b}_k for $k = 1, 2, 3$ [7]. Two observations of reference frame vectors provide sufficient information to compute a right-handed orthonormal triad of vectors in the reference frame. If \vec{v}_1 and \vec{v}_2 are reference frame vectors for the accelerometer and the magnetometer, respectively, then

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 9.81 \end{bmatrix} \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 0.3 \\ -1 \\ 0 \end{bmatrix}$$

so that the gravity vector points down and the magnetometer vector points in the direction of magnetic North in Boston, relative to the world reference frame. The orthonormal vector triad representing the reference frame is given by,

$$\vec{r}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}, \quad \vec{r}_2 = \frac{\vec{v}_1 \times \vec{v}_2}{\|\vec{v}_1 \times \vec{v}_2\|}, \quad \vec{r}_3 = \vec{r}_1 \times \vec{r}_2$$

Let \vec{w}_1 be the measurement of the normalized unit vector from the accelerometer and \vec{w}_2 be the measurement of the normalized unit vector from the magnetometer, then the orthonormal vector triads for the reference vectors are given by,

$$\vec{b}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|}, \quad \vec{b}_2 = \frac{\vec{w}_1 \times \vec{w}_2}{\|\vec{w}_1 \times \vec{w}_2\|}, \quad \vec{b}_3 = \vec{w}_1 \times \vec{w}_2$$

If there was no noise in the system, then it would be the case that $\vec{b}_k = A\vec{r}_k$ for $k = 1, 2, 3$. An equivalent way of writing this in matrix form is the following:

$$M_b = AM_r$$

where

$$M_b = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix}$$

$$M_r = \begin{bmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \end{bmatrix}$$

Since M_r is an orthogonal matrix, its inverse is equal to its transpose: $M_r^{-1} = M_r^T$. Thus, the solution for the attitude matrix is,

$$A = M_b M_r^T$$

A.1 Error Analysis of TRIAD Measurements

There are three independent maneuvers that are possible with the interaction tool: roll maneuvers, pitch maneuvers, and yaw maneuvers. The error analysis focuses on the pitch maneuver but similar results are obtained for a rotation of the interaction tool in any dimension. Using the TRIAD algorithm, the roll, pitch, and yaw angles are estimated, and these are seen side-by-side with the absolute orientation reference in Figure 10.

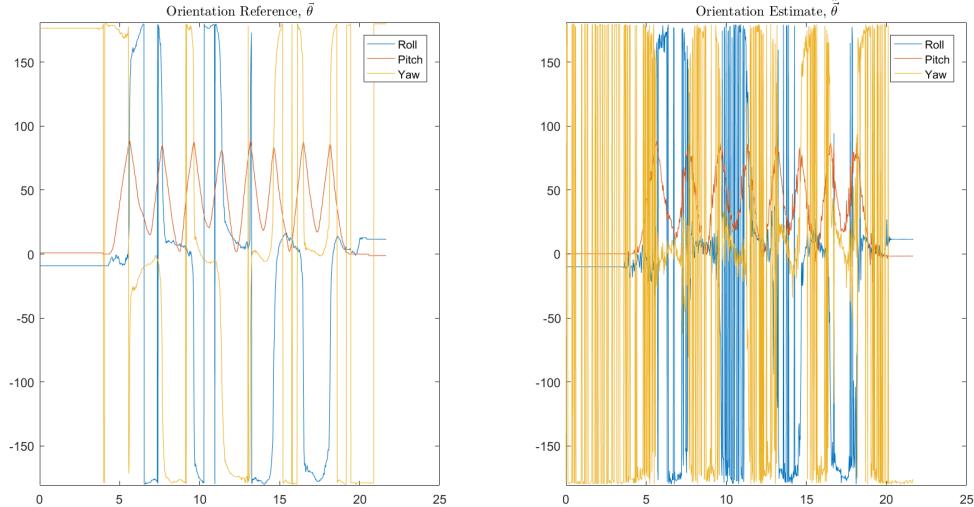


Figure 10: Pitch maneuver reference versus estimates

The error associated with the estimation of roll, pitch, and yaw is shown in ??.

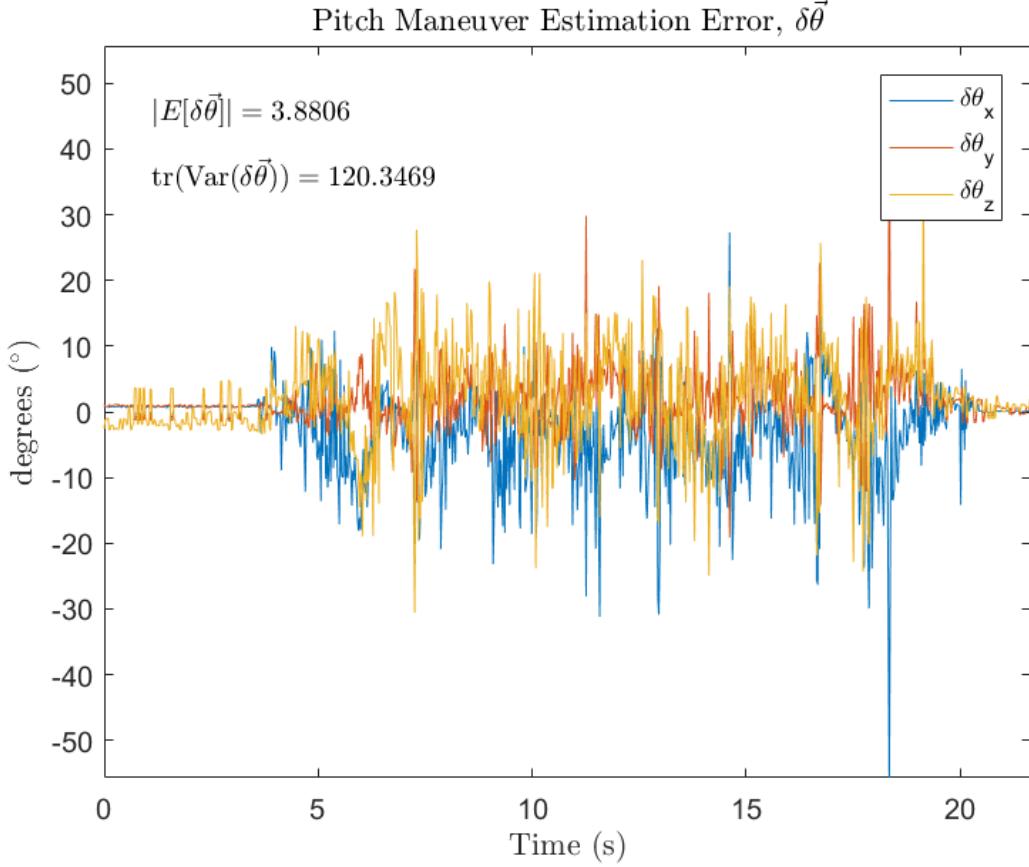


Figure 11: Pitch error TRIAD estimation

The bias of the measurements is just 4.3° but the variance is more than 125° , corresponding to a root-mean-squared error of 11.2° . If a user notices an angular change of 5° , a deviation of more than 10° is likely to give them a headache. Similar results are obtained when analyzing maneuvers in the roll and yaw dimensions of the rotation. These errors and root-mean-squared errors are summarized in the table of Figure 12.

Bias of angular error, $\delta\vec{\theta}$ (degrees °)			
	Yaw Maneuver	Pitch Maneuver	Roll Maneuver
$E[\delta\vec{\theta}_x]$	-1.3	-2.8	-0.2
$E[\delta\vec{\theta}_y]$	-0.0	1.7	-1.0
$E[\delta\vec{\theta}_z]$	-0.3	2.0	1.3
$ E[\delta\vec{\theta}] $	-1.4	3.9	1.6

Variances and RMSE of angular error, $\delta\vec{\theta}$ (degrees °)			
	Yaw Maneuver	Pitch Maneuver	Roll Maneuver
$\text{var}(\delta\vec{\theta}_x)$	$(9.3)^2$	$(7.0)^2$	$(7.9)^2$
$\text{var}(\delta\vec{\theta}_y)$	$(6.6)^2$	$(4.6)^2$	$(9.1)^2$
$\text{var}(\delta\vec{\theta}_z)$	$(14.2)^2$	$(7.1)^2$	$(7.4)^2$
$\sqrt{\text{trace}(\text{cov}(\delta\vec{\theta}))}$	18.2	11.0	14.1

Figure 12: TRIAD orientation tracking errors

B. Mathematical Representation of Orientation

The orientation of the interaction tool relative to the world coordinate frame can be represented in a number of ways. The two representations that will be important here are the attitude matrix and the

attitude quaternion. First, the attitude of the orientation tool can be described by the attitude matrix A . The attitude matrix typically maps a vector in the reference frame to a vector in the body frame [8]. This mapping is given by the relation,

$$\mathbf{b} = \mathbf{A}\mathbf{r}$$

Second, the attitude of the interaction tool can be represented as a quaternion. A rotation in three-dimensions is equivalent to an axis of rotation and an angle of rotation. If \hat{n} is the axis of a rotation and θ is the angle of the rotation, then the quaternion \bar{q} represents the attitude:

$$\bar{q} = \begin{bmatrix} \vec{q} \\ q_4 \end{bmatrix}$$

with $\vec{q} = \hat{n} \sin \frac{\theta}{2}$ and $q_4 = \cos \frac{\theta}{2}$. The quaternion must have length 1: $\bar{q}^T \bar{q} = 1$.

The attitude matrix and the quaternion representing attitude are related by the following equation, where A is a function of \bar{q} [9]:

$$A(\bar{q}) = (q_4^2 - \|\vec{q}\|^2)I_{3x3} + 2\vec{q}\vec{q}^T - 2q_4[\vec{q}\times]$$

where $[\vec{q}\times]$ is the skew-symmetric cross product matrix,

$$[\vec{q}\times] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}$$

The relationship between A and \bar{q} can also be expressed as the product of two matrices [8]:

$$A(\bar{q}) = \Xi(\bar{q})^T \Psi(\bar{q})$$

with

$$\Xi(\bar{q}) = \begin{bmatrix} q_4 \mathbf{I}_{3x3} + [\vec{q}\times] \\ -\vec{q}^T \end{bmatrix}$$

$$\Psi(\bar{q}) = \begin{bmatrix} q_4 \mathbf{I}_{3x3} - [\vec{q}\times] \\ -\vec{q}^T \end{bmatrix}$$

C. Dynamic Measurement Model

The rate of change of the attitude can be written in the form of a general state space model:

$$\dot{\vec{x}} = A(\vec{x}, t)\vec{x} + B(\vec{x}, t)\vec{u}$$

The kinematic equation for the change in attitude as a function of the angular rate can be written as a rate of change of the attitude matrix or a rate of change in the quaternion [9]. The angular velocity vector is given by $\vec{\omega}$. The rate of change of the attitude matrix is,

$$\dot{A}(t) = [\vec{\omega}\times]A(t)$$

In this equation, $[\vec{\omega}\times]$ is an instance of the state transition matrix $A(\vec{x}, t)$ from the general schema for a state space model. The kinematic relation for the rate of change of the quaternion can be written in the following two ways:

$$\dot{\bar{q}} = \frac{1}{2}\Omega(\vec{\omega})\bar{q}$$

$$\dot{\bar{q}} = \frac{1}{2} \Xi(\bar{q}) \vec{\omega}$$

where the first relation is a state space model with state transition matrix $A(\vec{\omega}) = \Omega(\omega)$, where the state transition matrix is a function of the angular rate vector $\vec{\omega}$:

$$\Omega(\vec{\omega}) = \begin{bmatrix} -[\vec{\omega} \times] & \vec{w} \\ -\vec{\omega}^T & 0 \end{bmatrix}$$

The second kinematic equation for the rate of change of the quaternion is an instance of the schema for the state space model where the angular rate $\vec{\omega}$ is treated as the input \vec{u} to the system and the input matrix $B(\bar{q}) = \Xi(\bar{q})$ is a function of the current attitude \bar{q} .

There are two good reasons to use the quaternion representation of attitude: first, the rate of change of the quaternion is linearly related to the state of the quaternion; and second, successive rotations are computed by taking the product of quaternions [8]. The following identity holds of the composition of quaternions [9]:

$$A(\bar{q}_1)A(\bar{q}_2) = A(\bar{q}_1 \otimes \bar{q}_2)$$

with the operation $A(\bar{q})$ finding the attitude matrix A that corresponds with the quaternion \bar{q} . Following the notation in [8], the product of the quaternion is linear in each of the quaternions \bar{q}_1 and \bar{q}_2 :

$$\bar{q}_1 \otimes \bar{q}_2 = \begin{bmatrix} \Psi(\bar{q}_1) & \bar{q}_1 \end{bmatrix} \bar{q}_2 = \begin{bmatrix} \Xi(\bar{q}_1) & \bar{q}_1 \end{bmatrix} \bar{q}_2$$

The inverse of the quaternion is found by negating the imaginary component of the quaternion:

$$\bar{q}^{-1} = \begin{bmatrix} -\bar{q} \\ q_4 \end{bmatrix}$$

C.1 Filter Design

How does the orientation of the interaction tool change from one time step to another? To discretize the dynamic model of rotations, we must know how the orientation of the tool will change between time steps. The change in orientation of the hand-held tool over time must be dependent on the dynamics of wrist movement. The peak velocity for human wrist movements is on the order of $10^2 \frac{\circ}{s}$ for fast and accurate wrist rotations [10]. We can assume that users of the interaction tool will pursue accurate movements from one orientation to another and not move the tool wildly. Therefore, the change in the orientation of the tool between time steps for maximum angular velocity movements will be on the order of 2° , given a sample rate of 50 Hz. This rotation is so small as to be imperceptible to most users [11]. Given these assumptions, we can assume that the orientation of the tool is more or less fixed between samples and we can treat changes to the orientation as solely due to small deviations – essentially noise.

The small deviations in the angular velocity between samples can be approximated by integrating the angular velocity over the time step. The integrated angular velocity can be treated as a small deviation to the quaternion at time step k :

$$\delta \bar{q}_t = \begin{bmatrix} \frac{1}{2} \vec{\omega} \Delta t \\ 1 \end{bmatrix}$$

Thus, given this small deviation we can predict the orientation at time step k given the orientation at step $k-1$:

$$\bar{q}_k^- = \delta \bar{q}_t \otimes \bar{q}_{k-1} = \begin{bmatrix} \Xi(\bar{q}_{k-1}) & \bar{q}_{k-1} \end{bmatrix} \delta \bar{q}_t = \bar{q}_{k-1} + \Xi(\bar{q}_k)(\frac{1}{2} \vec{\omega} \Delta t)$$

Now this is our discrete-time dynamic model! The state transition matrix $F = I_{3 \times 3}$ and the input matrix G is parameterized by $\bar{\mathbf{q}}_k$ so that $G = \Xi(\bar{\mathbf{q}}_k)$.

In order to be able to integrate this orientation prediction with our measurements, we must also approximate the evolution of the covariance between time steps. Here the assumption that the orientation is fixed comes in handy for the covariance of the estimation does not change if the body is in a fixed configuration. However, the input from the gyroscope adds uncertainty. Let Q be the process noise in the gyroscope input, then the predicted covariance at time k given the covariance at step $k-1$ is

$$\mathbf{P}_k^- = \mathbf{P}_{k-1} + Q$$

How do these predicted quantities relate to our measurements? Ultimately we want to be able to compute the innovation, the difference between the predicted measurements and the actual measurements. Once we multiply the innovation by the Kalman gain, we will know how to filter our prediction to produce a better estimate of the orientation. So how do we predict what will be measured?

We expect that the measured gravity vector and magnetic field vector will simply be the gravity reference vector rotated into the current body coordinates of the interaction tool. Similarly, the expected measurement for the magnetic field vector depends on the tool's current orientation. Thus, we estimate that what we will measure is

$$\hat{z}_k = \begin{bmatrix} A(\bar{\mathbf{q}}_k) \vec{g} \\ A(\bar{\mathbf{q}}_k) \vec{b} \end{bmatrix}$$

Now we can compute the innovation μ as the difference between what we predicted we would measure and what we actually measure:

$$\mu = z_k - \hat{z}_k$$

Multiplying the innovation by the gain of the filter, K , we compute the update residual

$$\delta\bar{\mathbf{q}}_t^+ = K\mu$$

This update residual is given as an error in the rotation of the interaction tool. Therefore instead of adding it to the predicted quaternion directly, we compose the two, as we do for rotations

$$\bar{\mathbf{q}}_k^+ = \delta\bar{\mathbf{q}}_t^+ \otimes \bar{\mathbf{q}}_k^-$$

This is the estimated orientation for time step k .

There is some subtlety to finding the filter gain K . The Kalman filter equations for the extended Kalman Filter give K as

$$K = P^- H_k (H_k P^- H_k^T + R)^{-1}$$

but this requires that we know both the measurement noise, R , and the measurement sensitivity matrix H_k . While the measurement noise is initialized with reasonable values based on the expected noise of the sensors, the measurement sensitivity matrix changes on each time step because the errors in the measurements depend on the current orientation. It turns out that if $\hat{z}_k = h(\bar{\mathbf{q}}_k)$ then the measurement sensitivity is

$$H_k = \frac{\partial h}{\partial \delta\bar{\mathbf{q}}_t} = \begin{bmatrix} 2A(\bar{\mathbf{q}}_k) \vec{g}^\times \\ 2A(\bar{\mathbf{q}}_k) \vec{b}^\times \end{bmatrix}$$

where \vec{g}^\times is the cross product matrix. This measurement sensitivity matrix is the linearization of the rotation around the current orientation. This result will be derived in the later section on the overall performance analysis of the tracking system.

C.2 Filter Initialization

The initial state of the orientation is estimated using the TRIAD approach. The rotation matrix output of the TRIAD approach is converted to a quaternion to initialize the dynamic model. In order to initialize the covariance, an initial uncertainty in the orientation of 10 degrees is assumed.

The Kalman filter assumes process noise and measurement noise in the dynamics. The process noise for the dynamic model of rotations assumes that the body is fixed at each time step, so the noise in the dynamics must come from noise on the gyroscope input. The root-mean squared noise density of gyroscope on the Adafruit BNO055 given in the data sheet as $0.014 \frac{\text{°}}{\sqrt{\text{Hz}}}$ [12]. To find the mean squared noise, this value is squared and multiplied by bandwidth of the signal, taken here to be the sample rate, 47 Hz. This yields a value of approximately $(0.1)^2 \frac{\text{°}}{\text{s}}$. Converting to radians, the mean squared noise of the gyroscope is $2.8\text{ee}-6 \frac{\text{rad}^2}{\text{s}}$. Thus, the process noise Q is initialized as

$$Q = \begin{bmatrix} 2.8\text{ee}-6 & 0 & 0 \\ 0 & 2.8\text{ee}-6 & 0 \\ 0 & 0 & 2.8\text{ee}-6 \end{bmatrix}$$

Similarly, the measurement noise is initialized with noise values for the accelerometer and magnetometer listed in the datasheet [12]. Again, squaring the noise densities and multiplying by the bandwidths yields mean squared errors of $1.1\text{ee}-6 g^2$ and $17\mu T^2$ for the accelerometer and the magnetometer, respectively. The six dimensional measurement noise matrix, R , is initialized as

$$R = \begin{bmatrix} 1.1\text{ee}-6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.1\text{ee}-6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.1\text{ee}-6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 17 & 0 & 0 \\ 0 & 0 & 0 & 0 & 17 & 0 \\ 0 & 0 & 0 & 0 & 0 & 17 \end{bmatrix}$$

C.3 Filter Tuning

The Kalman filter assumes that the measurement noise and process noise are Gaussian. However, this assumption rarely holds, so the noise parameters of the filter must be tuned to achieve better performance of the filter. The measurement noise parameters are particularly important. Increasing the measurement noise parameter, represented by R in the Kalman filter equations, decreases the Kalman gain.

Intuitively, a larger value for the measurement noise means that the variability of the measurements is high, so the filter is less responsive to measurements and more responsive to the model. Conversely, if the measurement noise parameter decreases, the Kalman gain increases. The graph in figure ?? shows how the Kalman filter responds to a step input in the roll for different values of R , the measurement gain.

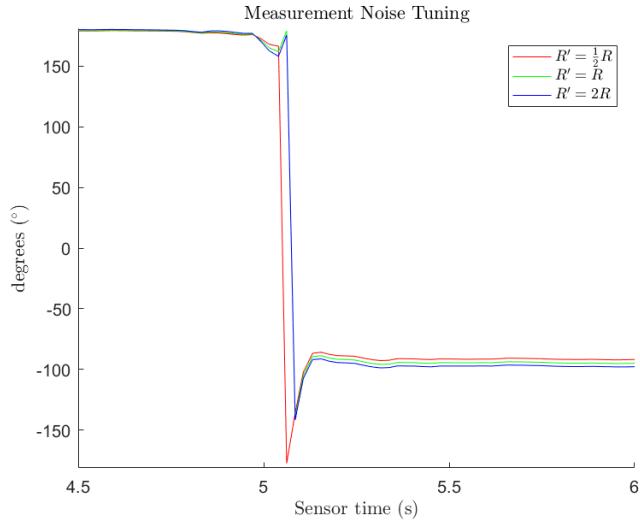


Figure 13: Tuning the measurement noise parameter of the Kalman filter

Lower values of R lead to a faster step response but more overshoot because the Kalman gain is high. Higher values of R have a slower step response but less overshoot because the Kalman gain is lower.

C.4 Filter Performance

The performance of the extended Kalman Filter model for the orientation is summarized in the charts in 15.

Bias of angular error, $\delta\vec{\theta}$ (degrees °)			
	Yaw Maneuver	Pitch Maneuver	Roll Maneuver
$E[\delta\vec{\theta}_x]$	1.0	0.2	-1.8
$E[\delta\vec{\theta}_y]$	2.7	0.9	-0.4
$E[\delta\vec{\theta}_z]$	-0.1	1.4	0.7
$ E[\delta\vec{\theta}] $	-2.9	1.6	2.0

Variances and RMSE of angular error, $\delta\vec{\theta}$ (degrees °)			
	Yaw Maneuver	Pitch Maneuver	Roll Maneuver
$\text{var}(\delta\vec{\theta}_x)$	$(3.6)^2$	$(1.2)^2$	$(3.5)^2$
$\text{var}(\delta\vec{\theta}_y)$	$(2.9)^2$	$(1.9)^2$	$(1.9)^2$
$\text{var}(\delta\vec{\theta}_z)$	$(8.8)^2$	$(2.5)^2$	$(2.2)^2$
$\sqrt{\text{trace}(\text{cov}(\delta\vec{\theta}))}$	9.9	3.3	4.5

Figure 14: Kalman filter orientation tracking errors

The comparison in the precision of these results to the TRIAD estimation algorithm is shown in ??.

Comparison of RMSE for TRIAD and EKF (degrees °)			
	Yaw Maneuver	Pitch Maneuver	Roll Maneuver
TRIAD	18.2	11.2	14.1
EKF	.9	3.3	4.5

Figure 15: Kalman filter and TRIAD root-mean-squared error comparison

VII. Position Tracking

A. Measurement Model

The following is a model of a single frame from the camera. In this frame, shown in image Figure 16 the red circular object represents the rear of the interaction tool. The spherical object of the rear of the interaction tool is called the tracking target. The width of the target along the x dimension of the image is s_u .

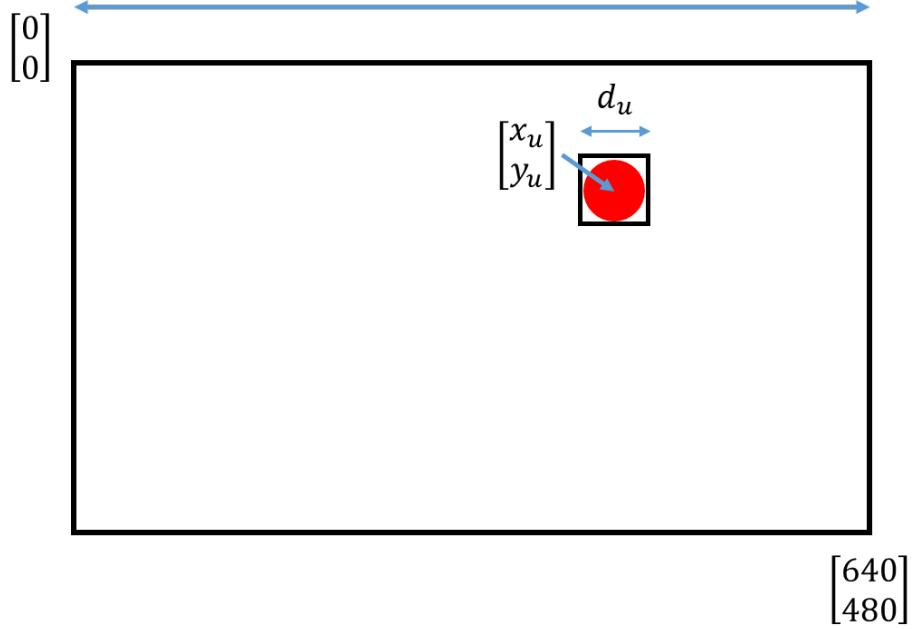


Figure 16: Computer vision tracking model

The midpoint of the target in the ideal undistorted image is $\vec{p}_u = \begin{bmatrix} x_u & y_u \end{bmatrix}^T$. The center of the undistorted image is c_x in the x direction and c_y in the y direction. The focal lengths in the x and y directions are f_x and f_y , respectively. These parameters are found through the camera calibration routine, explained in detail below. The measurements of the center of the target and its width are related to the hidden state $\vec{p}'_s = \begin{bmatrix} x_s & y_s & z_s \end{bmatrix}^T$ in the following way:

$$\vec{y} = h(\vec{x}) = \begin{bmatrix} x_u \\ y_u \\ s_u \end{bmatrix} = \begin{bmatrix} \frac{f_x x_s}{z_s} + c_x \\ \frac{f_y y_s}{z_s} + c_y \\ \frac{s_c}{z_s \sqrt{\frac{1}{f_x^2} + \frac{1}{f_y^2}}} \end{bmatrix}$$

These relationships are found through using a simple pin-hole camera model [13].

A.1 Camera Calibration

The measurement model requires knowledge of the intrinsic camera parameters, f_x, f_y, c_x, c_y , and the radial and tangential distortion factors. There are two steps to translating a point from camera coordinates to image coordinates [13]. First, the pin-hole camera model shows how to find the ideal image coordinates of an object, $(\mathbf{x}_u, \mathbf{y}_u)$ based on the position of the object in camera coordinates, $\vec{p}'_s = (x, y, z)$:

$$\mathbf{x}_u = f_x \frac{x}{z}$$

$$\mathbf{y}_u = f_y \frac{\mathbf{y}}{z}$$

Second, the radial and tangential distortion relations show how the actual image coordinates, $(\mathbf{x}_u, \mathbf{y}_u)$, relate to the image coordinates on the actual or distorted image plane, $(\mathbf{x}_d, \mathbf{y}_d)$. The radial distortion correction equation is as follows:

$$\begin{aligned}\mathbf{x}_u &= x_d(1 + k_1 r^2 + k_2 r^4 + k_3 r^6) \\ \mathbf{y}_u &= y_d(1 + k_1 r^2 + k_2 r^4 + k_3 r^6)\end{aligned}$$

and the tangential distortion equation is given by,

$$\begin{aligned}\mathbf{x}_u &= x_d + (2p_1 xy + p_2(r^2 + 2x^2)) \\ \mathbf{y}_u &= y_d + (p_1(r^2 + 2y^2) + 2p_2xy)\end{aligned}$$

Thus, the five distortion parameters are summarized in a vector D :

$$D = \begin{bmatrix} k_1 & k_2 & p_1 p_2 & k_3 \end{bmatrix}$$

The calibration parameters are discovered by using an open source implementation provided by OpenCV. The camera collects a series of images that contain a checkerboard pattern. The calibration computes the transformation from the image coordinates of the checkerboard to its real world coordinates. The calibration images look like the ones in figure 1 below.

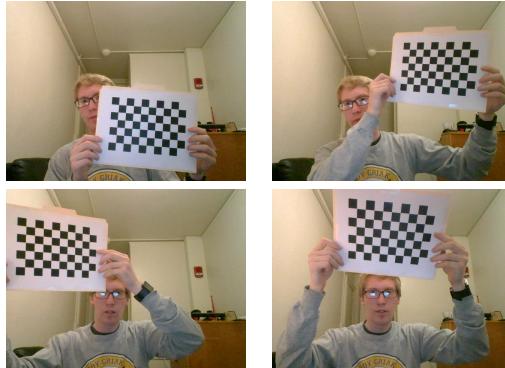


Figure 17: Camera calibration routine

A.2 Object Tracking

The signal processing layer contains an object tracker to find the rear of the interaction tool in each frame. Since a robust estimate of the object size in each image is required, its edges must be estimated as exactly as possible. One simple way to estimate the edges of the spherical tracking target is to segment the tracking target from the background by using color. The color of the object is used determine which pixels belong to it and which to not. Additionally, the color-based object tracking must happen in real time. To meet the demands of real-time, color-based object tracking, the CamShift algorithm was chosen [14].

A.3 Tool Maneuvers

The possible maneuvers of the interaction tool are classified into two categories: maneuvers in the image plane and those in the depth dimension. For a user of the 3D Interaction Tool, side-to-side or up-and-down movements of the interaction tool map to motions in the image plane of the tracking camera, while movements of the tool towards and away from their body map to motions in the depth dimension. More precisely, the maneuvers in the image plane are movements of the interaction tool that occur perpendicular to the viewing angle of the camera.

The convention established for the purposes of testing is that the image plane is the yz -plane of the coordinate system and the depth dimension maps to the negative x -dimension. Maneuvers in the

x-dimension encompass the movements of the tool towards and away from the tracking camera. Figure 18 compares the actual motion of the interaction tool to the estimates made by the tracking system.

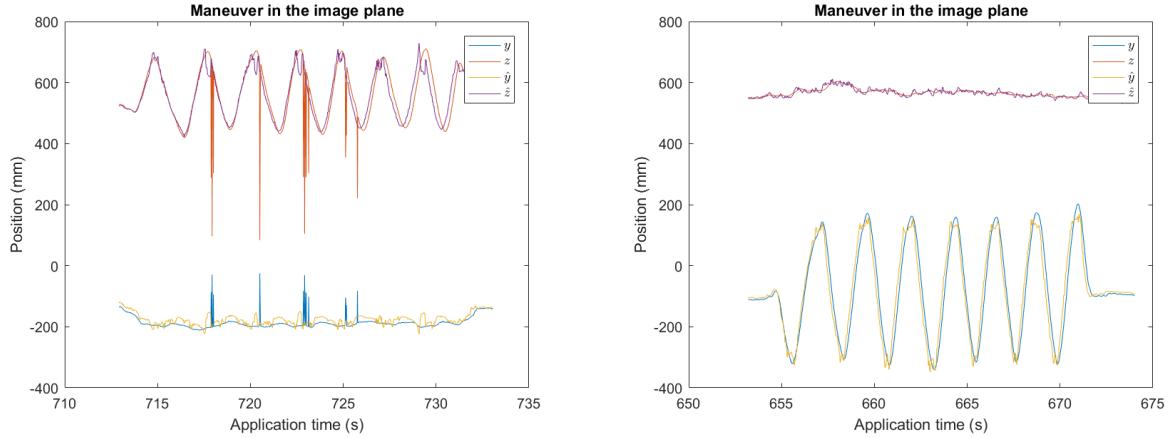


Figure 18: Vertical and horizontal motion in the image plane

In 18, the traces on the left correspond with vertical maneuvers of the tool and the traces on the right correspond to horizontal movements of the tool. In both graphs it is apparent that the tracking system is very accurate with low bias. The estimates of \hat{y} and \hat{z} map almost exactly to the true values.

The second set of maneuvers are those in the depth dimension. Figure 19 shows a trace of the actual depth x and the estimated depth \hat{x} .

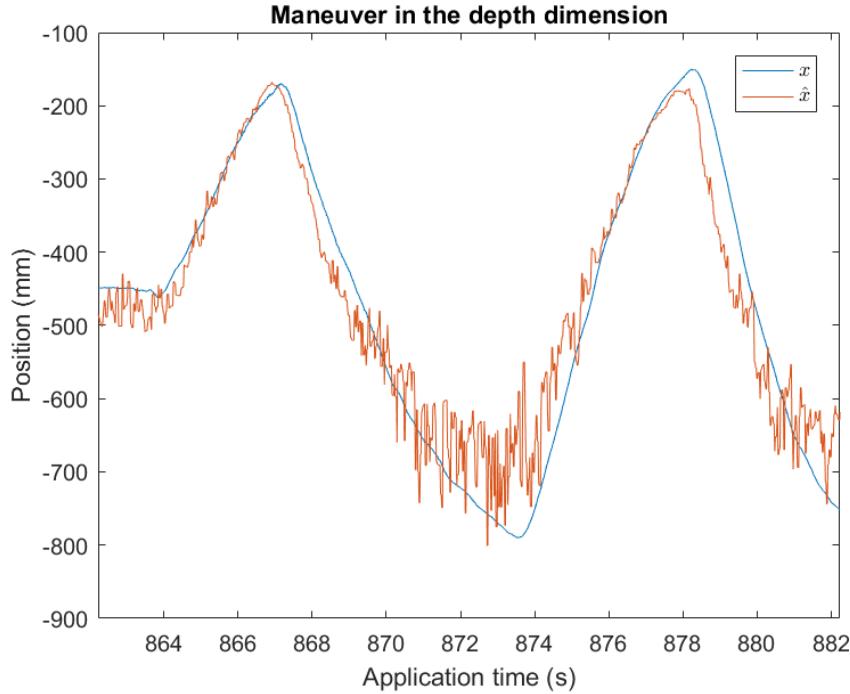


Figure 19: Motion of the interaction tool in the depth dimension

The natural conclusion from the traces in 19 is that the depth tracking accuracy and precision depend on how far away the interaction tool is from the tracking camera.

A.4 Error Analysis of Motion in the Image Plane

The accuracy and precision of the tracking system depends on the direction of the tool maneuvers. This is evident from analyzing the covariance of the error in the y and z directions for vertical and horizontal maneuvers. Figure 20 shows the error ellipse associated with motion in the image plane. On the left is the error ellipse associated with vertical motion and on the right is the error associated with horizontal motion.

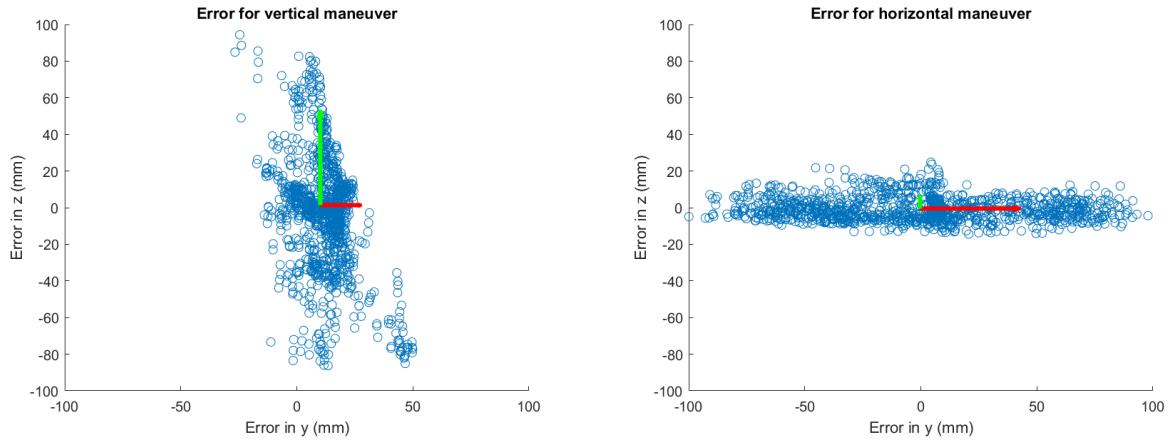


Figure 20: Tool tracking errors in the yz -plane

In 20 two arrows are placed with their origins at the mean of the yz -error distribution and their length corresponding to the square root of the variance along the direction they point. It is clear from the magnitudes of these arrows that there is a breakdown of precision of the tracking system along the direction of motion for the 3D Interaction Tool.

What explains the higher variance along the direction of motion for the interaction tool? There are essentially two sources of error for tracking the target relative to the camera. First, there is error that comes from the camera calibration routine in the form of error of intrinsic parameters of the camera – errors in fx , fy , for example. Second, there is error that comes from finding the centroid of the object in each image frame. Broadly, we could classify the calibration errors as systemic and the image processing errors as dynamic. Since the variance of the tool tracking increases in the direction of motion of the object, it is not likely that this error comes from a systemic problem with the calibration routine. Therefore, we can attribute the decrease in precision to error in the image segmentation routine.

Why does motion along a particular direction increase the error in the image segmentation? When the object is moving, its boundaries become blurred along the direction of motion in each frame. These blurred edges increase the uncertainty about the location of the object along the direction of motion. Therefore, when the object is moving along a particular direction, the uncertainty of its location in the direction of motion increases, but quickly drops again when the motion of the object halts.

A.5 Error Analysis of Motion in the Depth Dimension

The distribution of errors is larger for depth-tracking than tracking in the image plane. This result is displayed graphically in Figure 21.

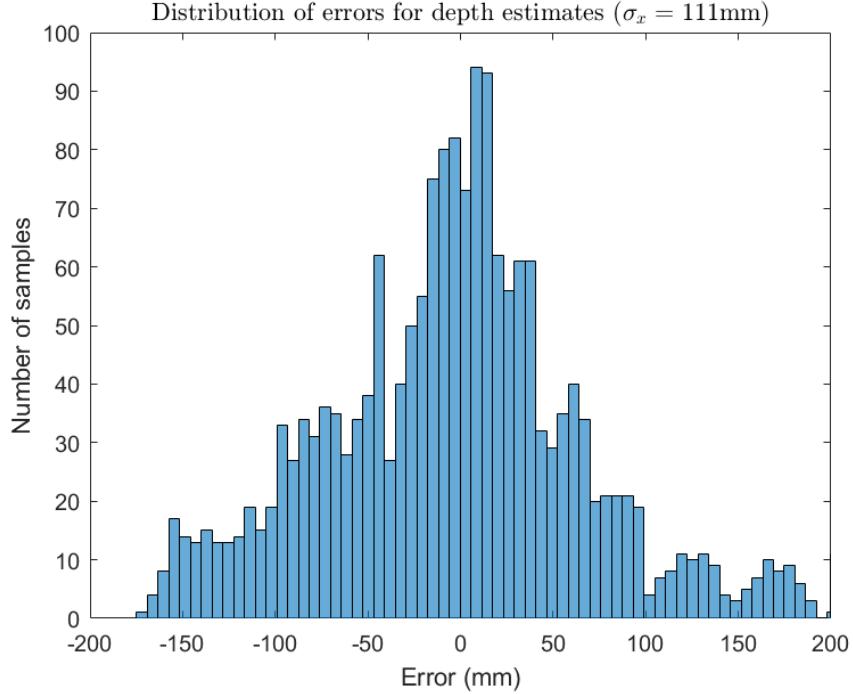


Figure 21: Distribution of depth-tracking errors

While the bias of the tracking errors tends to zero, the variance is much larger than for the depth-tracking in the image plane.

B. Dynamic Measurement Model

Since the user's movement of the interaction tool is voluntary, no assumptions can be made about its overall trajectory. However, human-motion obeys minimum-jerk constraints [15]. Since the jerk of the tool's motion is small, it can be modelled as a white-noise process. The dynamical model for the tool's motion along the x axis is estimated by the discrete time state-space model

$$\vec{x}_{k+1} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} \ddot{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$

where w_k is the jerk. Additionally, the motion of the tool is assumed to be coordinate uncoupled, since the user is free to move the tool in any direction. Therefore this same model can be extended to motion along the y and z axes as well.

VIII. Performance Analysis

A. Estimation Error

Given the tool model $\hat{\vec{p}}_f = \hat{\vec{p}}_r + \hat{\vec{p}}_l$, the overall error covariance Σ_{pf} is a function of the error due to the position estimation and the error to the orientation estimation. Since the estimation of \vec{p}_r and \vec{p}_l occur independently, it is reasonable to assume that the errors in the estimation of these vectors is not correlated. Thus, the covariances of the errors for these two vectors simply add to yield the covariance of the error for the tool front:

$$\Sigma_{pf} = \Sigma_{pr} + \Sigma_{pl} \quad (1)$$

B. Covariance Transformations

Here are three properties of covariance transformations, useful for analysing the performance of the tracking system as a whole:

(1) Given two independent random variables X, Y , and their sum $Z = X + Y$ how does the covariance of $Z \Sigma_Z$ relate to the covariances Σ_X and Σ_Y of X and Y , respectively? In this case the covariances simply sum so that

$$\Sigma_Z = \Sigma_X + \Sigma_Y$$

(2) Given two random variables X and Y related by the linear operator T so that $Y = TX$ how does the covariance of Y relate to the covariance of X ? If Σ_X is the covariance of X and Σ_Y is the covariance of Y , then

$$\Sigma_Y = T\Sigma_XT^T$$

(3) Given two random variables X and Y , and the non-linear relationship that $Y = f(X)$, how does the covariance of Y change with the covariance of X ? As a first-order approximation, the randomness of Y can be approximated as a linear function of X [16]:

$$Y - \mu_Y = T(X - \mu_X) \quad (2)$$

Now, given the linear relationship, $Y - \mu_Y = T(X - \mu_X)$, the covariance of Y is related to the covariance of X by

$$\Sigma_Y = T\Sigma_XT^T$$

using property (2) of the covariance transformation under a linear operator.

C. Estimation Error for $\hat{\vec{p}}_l$: Finding position error due to orientation error

How do errors in the orientation estimation lead to errors the tool position estimation? In the tool model, the position of the front of the tool is estimated from three pieces of information: the position of the rear of the tool \vec{p}_r , the orientation of the tool \mathbf{A}_t , and the tool length in body coordinates \vec{p}'_l . This relationship is described by the equation

$$\vec{p}_f = \vec{p}_r + \vec{p}_l = \vec{p}_r + \mathbf{A}_t^T \vec{p}'_l$$

There is no error in the measurement of \vec{p}'_l ; that is a well known parameter of the interaction tool. The error in \vec{p}_l comes from the estimation of the rotation matrix from body coordinates to world coordinates.

The attitude matrix \mathbf{A}_t is parameterized by the quaternion $\bar{\mathbf{q}}_t$ estimated from the EKF so that $\vec{p}_l = \mathbf{A}_t(\bar{\mathbf{q}}_t)^T \vec{p}'_l$. This expression is helpful because it describes the tool span vector \vec{p}_l as a function of $\bar{\mathbf{q}}_t$. The relationship between \vec{p}_l and $\bar{\mathbf{q}}_t$ must be linearized in order to understand how errors in the estimation of $\bar{\mathbf{q}}_t$ translate to errors in \vec{p}_l .

Let the small deviations in the orientation be represented by the error quaternion $\delta\bar{\mathbf{q}}_t$ so that $\vec{p}_l = \mathbf{A}(\delta\bar{\mathbf{q}}_t)\mathbf{A}_t(\bar{\mathbf{q}}_t)^T \vec{p}'_l$, where $\mathbf{A}(\delta\bar{\mathbf{q}}_t)$ is the attitude matrix parameterized by the orientation error. Then, the goal here is to linearize the relationship between \vec{p}_l and $\delta\bar{\mathbf{q}}_t$, and to accomplish we find the matrix first-order derivative of \vec{p}_l with respect to the three dimensional representation of the error quaternion $\delta\vec{q}$:

$$\frac{\partial \vec{p}_l}{\partial \delta\vec{q}} = \frac{\partial}{\partial \delta\vec{q}} \mathbf{A}(\delta\bar{\mathbf{q}}_t)\mathbf{A}_t(\bar{\mathbf{q}}_t)^T \vec{p}'_l$$

Now simplify $\mathbf{A}(\delta\bar{\mathbf{q}}_t)$ using the assumptions that $\delta q_4 \approx 1$ and $|\delta\vec{q}| \approx 0$, both of which will be true for small errors:

$$\mathbf{A}(\delta\bar{\mathbf{q}}_t) = (|\delta q_4|^2 - |\delta\vec{q}|^2)I_{3x3} + 2\delta\vec{q}\delta\vec{q}^T - 2\delta q_4 \left[\begin{array}{c} \delta\vec{q}^\times \\ \vdots \end{array} \right] = I_{3x3} - 2 \left[\begin{array}{c} \delta\vec{q}^\times \\ \vdots \end{array} \right]$$

Pplugging in this expression for $\mathbf{A}(\delta\bar{\mathbf{q}}_t)$:

$$\frac{\partial \vec{p}_l}{\partial \delta \vec{q}} = \frac{\partial}{\partial \delta \vec{q}} (I_{3x3} - 2 \begin{bmatrix} \delta \vec{q}^\times \end{bmatrix}) \mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l$$

Note that $\mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l$ is a constant vector – in fact it is simply the estimate for \vec{p}_l !

$$\frac{\partial \vec{p}_l}{\partial \delta \vec{q}} = \frac{\partial}{\partial \delta \vec{q}} (I_{3x3} \vec{p}_l - 2 \begin{bmatrix} \delta \vec{q}^\times \end{bmatrix} \vec{p}_l) = -2 \frac{\partial}{\partial \delta \vec{q}} \begin{bmatrix} \delta \vec{q}^\times \end{bmatrix} \vec{p}_l$$

The quantity $\begin{bmatrix} \delta \vec{q}^\times \end{bmatrix} \vec{p}_l$ is simply $\delta \vec{q} \times \vec{p}_l = -\vec{p}_l \times \delta \vec{q}$. Therefore,

$$\frac{\partial \vec{p}_l}{\partial \delta \vec{q}} = -2 \frac{\partial}{\partial \delta \vec{q}} (-\vec{p}_l \times \delta \vec{q}) = 2 \begin{bmatrix} \vec{p}_l^\times \end{bmatrix} \frac{\partial}{\partial \delta \vec{q}} \delta \vec{q} = 2 \begin{bmatrix} \vec{p}_l^\times \end{bmatrix}$$

Now we can write deviations in \vec{p}_l as a linear function of the quaternion error or angular error ($\delta \vec{q} = \frac{1}{2} \delta \vec{\theta}$):

$$\delta \vec{p}_l = 2 \begin{bmatrix} \mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l^\times \end{bmatrix} \delta \vec{q} = \begin{bmatrix} \mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l^\times \end{bmatrix} \delta \vec{\theta}$$

This is written in the form of Equation 2. Thus, if $\Sigma_{\theta t}$ is the covariance of the estimation of the tool's orientation, then the covariance Σ_l of \vec{p}_l is

$$\Sigma_l = \left[\mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l^\times \right] \Sigma_{\theta t} \left[\mathbf{A}_t (\bar{\mathbf{q}}_t)^T \vec{p}'_l^\times \right]^T = \mathbf{A}_t (\bar{\mathbf{q}}_t)^T \left[\vec{p}'_l^\times \right] \Sigma_{\theta t} \left[\vec{p}'_l^\times \right]^T \mathbf{A}_t (\bar{\mathbf{q}}_t) \quad (3)$$

IX. Conclusion

X. Future Work

References

- [1] J. Verhage, “Goldman Sachs Has Four Charts Showing the Huge Potential in Virtual and Augmented Reality.” <http://www.bloomberg.com/news/articles/2016-01-13/goldman-sachs-has-four-charts-showing-the-huge-potential-in-virtual-and-augmented-reality>, 2016. Accessed: 2016-03-19.
- [2] A. Robertson, “Valve’s virtual reality headset is great, but its controllers are the real story.” <http://www.theverge.com/2015/3/4/8150653/valve-steam-controller-vive-vr-gdc-2015>, 2015. Accessed: 2016-03-01.
- [3] E. Frederiksen, “Oculus Touch will be your virtual hands.” <http://www.technobuffalo.com/2015/06/11/oculus-touch-will-be-your-virtual-hands/>, 2015. Accessed: 2016-03-01.
- [4] J. Constine, “Oculus Previews Oculus Touch Handheld Motion-Tracking Haptic Controllers.” <http://techcrunch.com/2015/06/11/oculus-touch/>, 2015. Accessed: 2016-03-01.
- [5] B. Russell, “Oculus Touch delayed into second half of 2016.” <http://www.technobuffalo.com/2015/12/31/oculus-touch-delayed-2016/>, 2015. Accessed: 2016-03-01.
- [6] C. Hand, “A survey of 3d interaction techniques,” in *Computer graphics forum*, vol. 16, pp. 269–281, Wiley Online Library, 1997.
- [7] M. D. Shuster, “Deterministic three-axis attitude determination,” *Journal of Astronautical Sciences*, vol. 52, no. 3, pp. 405–419, 2004.
- [8] J. L. Crassidis, F. L. Markley, and Y. Cheng, “Survey of nonlinear attitude estimation methods,” *Journal of guidance, control, and dynamics*, vol. 30, no. 1, pp. 12–28, 2007.
- [9] E. J. Lefferts, F. L. Markley, and M. D. Shuster, “Kalman filtering for spacecraft attitude estimation,” *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, 1982.

- [10] D. S. Hoffman and P. L. Strick, “Step-tracking movements of the wrist in humans. i. kinematic analysis,” *The Journal of neuroscience*, vol. 6, no. 11, pp. 3309–3318, 1986.
- [11] K. Hinckley, J. Tullio, R. Pausch, D. Proffitt, and N. Kassell, “Usability analysis of 3d rotation techniques,” in *Proceedings of the 10th annual ACM symposium on User interface software and technology*, pp. 1–10, ACM, 1997.
- [12] B. Sensortec, “BNO055: Intelligent 9-axis absolute orientation sensor.” https://www.adafruit.com/datasheets/BST_BN0055_DS000_12.pdf, 2014. Accessed: 2016-01-28.
- [13] R. Y. Tsai, “A versatile camera calibration technique for high-accuracy 3d machine vision metrology using off-the-shelf tv cameras and lenses,” *Robotics and Automation, IEEE Journal of*, vol. 3, no. 4, pp. 323–344, 1987.
- [14] G. R. Bradski, “Real time face and object tracking as a component of a perceptual user interface,” in *Applications of Computer Vision, 1998. WACV'98. Proceedings., Fourth IEEE Workshop on*, pp. 214–219, IEEE, 1998.
- [15] T. Flash and N. Hogan, “The coordination of arm movements: an experimentally confirmed mathematical model,” *The journal of Neuroscience*, vol. 5, no. 7, pp. 1688–1703, 1985.
- [16] R. M. Haralick, “Propagating covariance in computer vision,” *International journal of pattern recognition and artificial intelligence*, vol. 10, no. 05, pp. 561–572, 1996.