Laplace transform

Tuesday, December 14, 2021 7:30 AM

What is Laplace transform ?

- integral transform :
  a function of some variable (s)
  to a function of some other variable (s),
  by means of an improper integral
  e.g. integral depending on a parameter
- tool to study ODEs

Suppose $f(t)$ is a function on $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{>0}$

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$= \lim_{T \to \infty} \int_0^T f(t) \underbrace{e^{-st}}_{\text{kernel}} dt$$

$f(t)$ original function

$F(s)$ Laplace transform

$$F(s) = \mathcal{L}[f(t)](s) = \mathcal{L}[f(t)]$$

$$= L[f(t)](s) = L[f(t)]$$

$$= L[f(t)](s) = L[f(t)]$$

Starting examples

Tuesday, December 14, 2021 7:49 AM

<u>Example 1</u> $f(t) = 1$ on $\mathbb{R}_{\geq 0}$.

Compute $L[f]$

$$L[f] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 1 \cdot e^{-st}\, dt = \left[\frac{e^{-st}}{-s} \right]_{t=0}^{t=\infty}$$

$$= \lim_{T \to \infty} \frac{e^{-s \cdot T}}{-s} + \frac{1}{s} \overset{(s>0)}{=} \frac{1}{s}$$

$$L[f](s) = \frac{1}{s} \qquad (s > 0)$$

<u>Example 2</u> $f(t) = e^{kt}$ $(k \in \mathbb{R})$

$L[f] = ?$

$$L[e^{kt}] = \int_0^\infty e^{kt}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{(k-s)t}\, dt = \left[\frac{e^{(k-s)t}}{k-s} \right]_{t=0}^{t=\infty} \qquad (s > k)$$

$$= \frac{1}{s-k}$$

$$L[e^{kt}] = \frac{1}{s-k} \quad \text{if } s > k$$

We are tabulating the formulas

eq 3 $\qquad f(t) = t^r \qquad (r > -1)$

$L[f] = ?$

$$\Gamma(k) = \int_0^\infty e^{-u} u^k \frac{du}{u}$$

$$L[t^r] = \int_0^\infty t^r e^{-st}\, dt \qquad u = st, \quad du = s\,dt$$

$$= \frac{1}{s} \int_0^\infty \left(\frac{u}{s}\right)^r e^{-u}\, du$$

$$= s^{-r-1} \int_0^\infty u^{r+1} e^{-u} \frac{du}{u} \overset{(r>-1)}{=} \frac{\Gamma(r+1)}{s^{r+1}} \quad \text{if } s > 0$$

$$L[t^r] = \frac{\Gamma(r+1)}{s^{r+1}} \quad \text{if } \begin{array}{l} r > -1 \\ s > 0 \end{array}$$

Linearity

Tuesday, December 14, 2021 8:02 AM

$$L\left[k_1 f_1(t) + k_2 f_2(t) \right] = k_1 L\left[f_1(t)\right] + k_2 L\left[f_2(t)\right]$$

What are the conditions for $f(t)$ so that $L[f]$ is defined ?

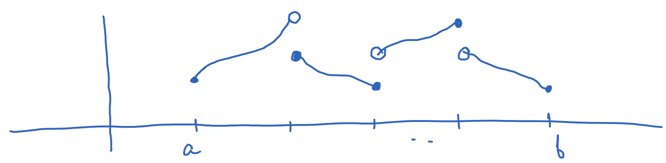
<u>Definition:</u> Let $f(t)$ be a function on $[a, b]$

We say $f(t)$ is <u>piecewise continuous</u> if

(1) $f(t)$ is continuous on $[a, b]$ except for a finite number of discontinuities

(2) $f(t)$ only has discontinuity of Type 1

$\exists f(t_0^-)$, $f(t_0^+)$ if t_0 is a discontinuity

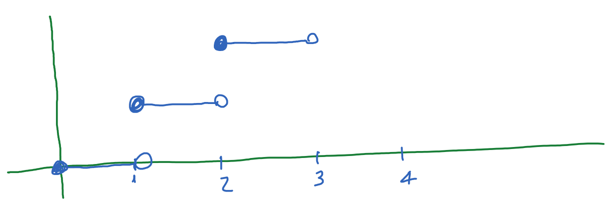


<u>Definition:</u> Let $f(t)$ be a function on $\mathbb{R}_{\geq 0}$

$f(t)$ is <u>piecewise continuous</u> if

$\forall T > 0$, $f(t)$ is piecewise continuous on $[0, T]$

<u>Example 1</u>



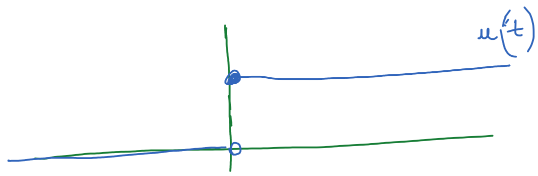
$t \in \mathbb{R}_{\geq 0}$: $\lfloor t \rfloor$ = biggest integer not exceeding t (floor of t)

$\lfloor e \rfloor = \lfloor 2.71 \ldots \rfloor = 2$

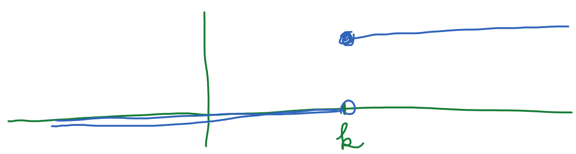
$\lfloor \pi \rfloor = \lfloor 3.14159265358979 3 \ldots \rfloor = 3$

Example 2 Heaviside function

$$u(t) = \begin{cases} 0 & (t < 0) \\ 1 & (t \geq 0) \end{cases}$$



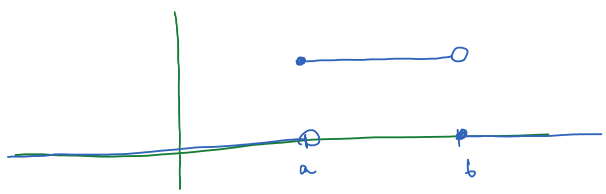
Example 3 $u(t-k) = \begin{cases} 0 & \text{if } t < k \\ 1 & \text{if } t \geq k \end{cases}$



Example 4 $[a, b]$ is an interval

$$u_{ab}(t) = \begin{cases} 0 & \text{if } t \notin [a, b) \\ 1 & \text{if } t \in [a, b) \end{cases}$$

$$= u(t-a) - u(t-b)$$



Laplace transform of Heaviside.

$$L[u(t-k)] = ? \qquad (k \geq 0)$$

$$\int_0^\infty u(t-k)\, e^{-st}\, dt$$

$$= \int_k^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s} \right]_k^\infty = \frac{e^{-sk}}{s} \qquad (s > 0)$$

$$L[u(t-k)] = \frac{e^{-sk}}{s} \qquad (s > 0)$$

$$L\left[u(t-k)\right] = \frac{-}{s} \quad (s>0)$$

- $L\left[u_{ab}(t)\right] = ?$

$$L\left[u_{ab}(t)\right] = L\left[u(t-a) - u(t-b)\right]$$

$$= \frac{e^{-sa} - e^{-sb}}{s} \quad (s>0)$$

$$L\left[u_{ab}(t)\right] = \frac{e^{-sa} - e^{-sb}}{s} \quad (s>0)$$

<u>Definition</u>: . $f(t)$ on $\mathbb{R}_{\geq 0}$ has <u>exponential order at most R</u> $(R \in \mathbb{R})$

if $\sup_{t \in \mathbb{R}_{\geq 0}} \frac{|f(t)|}{e^{Rt}} < +\infty$.

. $f(t)$ on $\mathbb{R}_{\geq 0}$ has <u>exponential order</u>

if it has exponential order at most R for some $R \in \mathbb{R}$

<u>Examples</u>. . $f(t) = t + 5e^{kt}$ $(k \in \mathbb{R}_{\geq 0})$ is of exp. ord. at most k

$$\sup_{t \geq 0} \frac{|t+5e^{kt}|}{e^{kt}} \leq \underbrace{\sup_{t \geq 0} \frac{t}{e^{kt}}}_{<+\infty} + 5 < +\infty$$

. $f(t) = e^{t^2}$ is not of exp. ord.

$$\sup_{t \geq 0} \frac{e^{t^2}}{e^{kt}} = \sup_{t \geq 0} \underbrace{e^{t^2 - kt}}_{\longrightarrow +\infty} = +\infty$$

Theorem. Suppose $f(t)$ is piecewise continuous on $\mathbb{R}_{\geq 0}$ and of exponential order at most $k \in \mathbb{R}$. Then $L[f(t)]$ exists for $s > k$

Proof

$$\sup_{t \geq 0} \frac{|f(t)|}{e^{kt}} < M < +\infty$$

$$|f(t)| < M \cdot e^{kt}$$

$$\int_0^\infty \left| f(t) \, e^{-st} \right| dt < M \int_0^\infty e^{(k-s)t} \, dt$$

$$< +\infty \quad \text{if} \quad s > k$$

So $L[f(t)]$ converges when $s > k$ $\square$

Lemma If $f(t)$ is of exp. ord. at most k, then $f(t)$ is of exp. ord at most $k + r$ for any $r \geq 0$

Proof use definition

$$\sup_{t \geq 0} \frac{|f(t)|}{e^{kt}} < +\infty$$

Then

$$\sup_{t \geq 0} \frac{|f(t)|}{e^{(k+r)t}} \leq \sup_{t \geq 0} \frac{|f(t)|}{e^{kt}} < +\infty$$

$\Rightarrow f(t)$ is of exp ord. at most $(k+r)$ $\square$

exp. ord. at most k (standard terminology)

Further examples

Tuesday, December 14, 2021 8:44 AM

A good way to remember these examples is to actually compute these formulas

$$L[\sin kt] = \frac{k}{s^2+k^2} \quad (s>0)$$

$$L[\cos kt] = \frac{s}{s^2+k^2} \quad (s>0)$$

$$L[\sinh kt] = \frac{k}{s^2-k^2} \quad (s>|k|)$$

$$L[\cosh kt] = \frac{s}{s^2-k^2} \quad (s>|k|)$$

$$L[t^n] = \frac{n!}{s^{n+1}} \quad (n \in \mathbb{N})$$

For other Laplace transforms, compute directly.

There are tables of Laplace transforms

<u>Example</u>. $L[\cos kt] = ?$

$$\int_0^\infty \cos(kt)\, e^{-st}\, dt \quad (*)$$

$$\int \cos(kt)\, e^{rt}\, dt = \frac{e^{rt}}{r^2+k^2}\left(r\cos kt + k \sin rt\right)$$

<u>$t=0$</u> $\text{RHS} = \dfrac{1}{r^2+k^2}\left(r + k\, 0\right) = \dfrac{r}{r^2+k^2}$

$\underline{t=0}$ $\quad RHS = \dfrac{1}{r^2+k^2}\left(r + k\, 0\right) = \dfrac{r}{r^2+k^2}$

$\underline{t=+\infty}$ $\quad RHS = 0 \quad$ if $r<0$.

$$\circledast = \left[RHS \right]_{t=0}^{t=+\infty} \underset{=}{\overset{(s>0)}{=}} 0 - \frac{-s}{s^2+k^2} = \frac{s}{s^2+k^2}$$

$$L\left[\cos kt\right] = \frac{s}{s^2+k^2} \qquad (s>0)$$

$$L\left[\sin kt\right] = \frac{k}{s^2+k^2} \quad (s>0)$$

$$\int \sin(kt)\, e^{rt}\, dt = \frac{e^{rt}}{r^2+k^2}\left(r \sin kt - k \sin kt\right)$$

$\underline{t=0}$ $\qquad RHS = \ldots \qquad\qquad$ Do this exercise!

$\underline{t=+\infty}$ $\qquad RHS = \ldots$

$$L\left[\sin kt\right] = \left[RHS\right]_{t=0}^{t=+\infty} = \frac{k}{s^2+k^2}$$

$L\left[\cos kt\right] = ? \qquad\qquad L\left[\sin kt\right] = ?$

$$e^{ikt} = \cos kt + i \sin kt$$

$$\int_0^\infty \cos kt \; e^{-st}\, dt = ? \qquad\qquad \int_0^\infty \sin kt \; e^{-st}\, dt = ?$$

$$\int^\infty e^{ikt}\; e^{-st}\, dt = ? = \int^{+\infty} \cos kt \; e^{-st}\, dt + i \int_0^{+\infty} \sin kt \; e^{-st}\, dt$$

$$\underbrace{\int_0^\infty e^{ikt}\, e^{-st}\, dt}_{} = ? = \int_0 \cos kt\, e^{-st}\, dt + i \int_0 \sin kt\, e^{-st}\, dt$$

$$= \int_0^\infty e^{(ik-s)t}\, dt = \left[\frac{e^{(ik-s)t}}{ik - s} \right]_{t=0}^{t=+\infty}$$

$$= \begin{cases} 0 - \dfrac{1}{ik - s} & \text{if } s > 0 \\ \text{diverges} & \text{if } s < 0 \end{cases}$$

$$\overset{(s>0)}{=} \frac{1}{s - ik} = \frac{s + ik}{s^2 + k^2} = \underbrace{\frac{s}{s^2 + k^2}}_{L[\cos kt]} + i \underbrace{\frac{k}{s^2 + k^2}}_{L[\sin kt]}$$

$$L[\cos kt] = \frac{s}{s^2 + k^2} \quad (s > 0)$$

$$L[\sin kt] = \frac{k}{s^2 + k^2} \quad (s > 0)$$

$$\int_0^{+\infty} f(t)\, e^{-st}\, dt = L[f(t)]$$

In fact, $L[f(t)]$ can be considered as a function of a <u>complex variable</u> s.

Inverse Laplace transform

Tuesday, December 14, 2021 8:53 AM

Theorem. $f(t)$, $g(t)$ piecewise continuous on $\mathbb{R}_{\geq 0}$, exp ord.

. $F = L[f]$, $G = L[g]$

. $F(s) = G(s)$ for all s sufficiently large

Then at every point of continuity t, $f(t) = g(t)$

Given $F(s)$, what is $f(t)$ such that $L[f(t)] = F(s)$?

inverse Laplace transform

$$f(t) = L^{-1}[F(s)]$$

Example: . $L[e^{kt}] = \dfrac{1}{s-k}$

$$L^{-1}\left[\frac{1}{s-k}\right] = e^{kt}$$

. $L[\cos kt] = \dfrac{s}{s^2+k^2}$

$$L^{-1}\left[\frac{s}{s^2+k^2}\right] = \cos kt$$

. $L[\sin kt] = \dfrac{k}{s^2+k^2}$

$$L^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin kt$$

. $L[\cosh kt] = \dfrac{s}{s^2-k^2}$

$$L[\cosh kt] = \frac{s}{s^2 - k^2}$$

$$L^{-1}\left[\frac{s}{s^2 - k^2}\right] = \cosh kt$$

$$L[\sinh kt] = \frac{k}{s^2 - k^2}$$

$$L^{-1}\left[\frac{k}{s^2 - k^2}\right] = \sinh kt$$

Laplace correspondence

Tuesday, December 28, 2021 7:20 AM

Laplace transform :

$$f : [0, +\infty) \longrightarrow \mathbb{R}$$

$$L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$F(s) = L[f(t)]$$ in domain of convergence s

$$f(t) \rightsquigarrow F(s)$$

$f(t)$	$\longrightarrow$	$F(s)$
1		$\frac{1}{s}$ $(s > 0)$
e^{kt}		$\frac{1}{s-k}$ $(s > 0)$
t		$\frac{1}{s^2}$ $(s > 0)$
te^{kt}		$\frac{1}{(s-k)^2}$ $(s > 0)$

$\vdots$

$$f_1(t),\ f_2(t) \longrightarrow F_1(s),\ F_2(s)$$

If $F_1(s) = F_2(s)$, then $f_1(t) = f_2(t)$

This means we can reverse the process

$$f(t) \longleftarrow F(s)$$

we don't know the reverse formula yet

$f(t) = L^{-1}[F(s)]$ inverse Laplace transform

(find by tables and elementary operations of $F(s)$)

$$f(t) \longrightarrow F(s) = L[f(t)]$$

$$f(t) \rightsquigarrow F(s) = L[f(t)]$$

$$f(t) = L^{-1}[F(s)] \longleftarrow F(s)$$

$$\boxed{\begin{array}{c} f(t) \longleftrightarrow F(s) \\ \text{Laplace correspondence} \end{array}}$$

We have a correspondence between a function $f(t)$

and a function $F(s)$,

from $f(t)$ to $F(s)$. have Laplace transform

from $F(s)$ to $f(t)$. use tables

Laplace correspondence

$$f(t) \longleftrightarrow F(s)$$

<u>Questions</u>: . if we change $f(t)$, how does $F(s)$ change ?

. if we change $F(s)$, how does $f(t)$ change ?

$$f(t) \longleftrightarrow F(s)$$

$$f(t-k) \longleftrightarrow ?$$

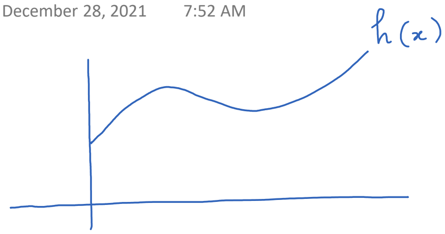
$$f'(t) \longleftrightarrow ?$$

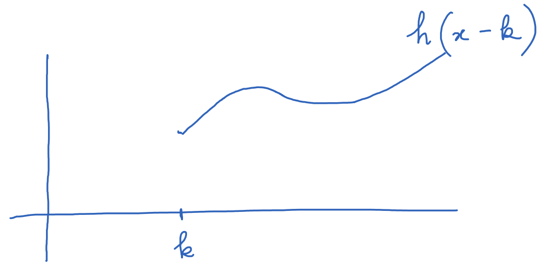
$$? \longleftrightarrow F(s-k)$$

$$? \longleftrightarrow F'(s)$$

Translations

Tuesday, December 28, 2021 7:52 AM





$$f(t) \longleftrightarrow F(s)$$

$$? \longleftrightarrow F(s-k)$$

Suppose $F(s) = L\left[f(t)\right]$ for $s > k'$

Then $F(s-k) = L\left[e^{kt} \ f(t)\right]$ for $s > k + k'$

$$\boxed{\begin{array}{lcll} f(t) & \longleftrightarrow & F(s) & (s > k') \\ e^{kt} \cdot f(t) & \longleftrightarrow & F(s-k) & (s > k + k') \end{array}}$$

Proof $\quad F(s) = \int_0^\infty e^{-st} f(t)\, dt$ for $s > k'$

$$\Rightarrow \quad L\left[e^{kt} f(t)\right] = \int_0^\infty e^{-st} \cdot e^{kt} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-k)t} f(t)\, dt$$

$$= F(s-k) \text{ if } s - k > k' \ (\Leftrightarrow s > k + k') \quad \square$$

Example: Find $L^{-1}\left(\frac{1}{(s-k)^2}\right)$.

$$L[t] = \frac{1}{s^2} = F(s) \quad \text{for } s > 0$$

$$L\left[e^{kt} \cdot t\right] = \frac{1}{(s-k)^2} = F(s-k) \quad \text{for } s > k$$

$$L^{-1}\left[\frac{1}{(s-k)^2}\right] = e^{kt}\, t$$

$$L^{-1}\left[\frac{1}{(s-k)^2}\right] = e^{kt}\, t$$

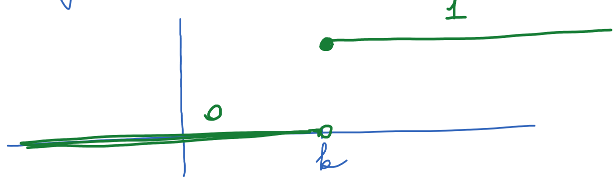
$$f(t) \longleftrightarrow F(s)$$

$$e^{kt} f(t) \longrightarrow F(s-k)$$

$$e^{kt} f(t) \longleftarrow F(s-k)$$

(translation on s-domain)

<u>Heaviside function</u> $u(t-k)$

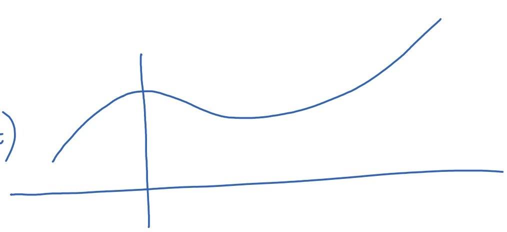


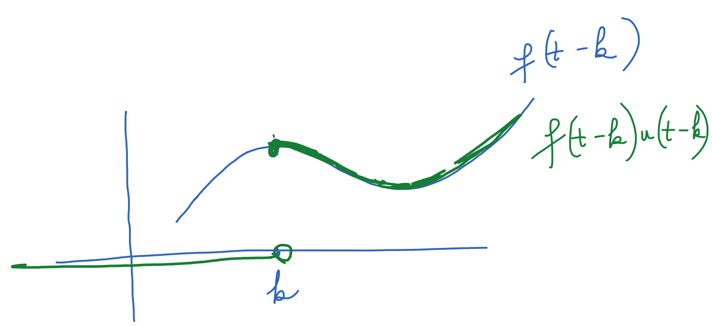
<u>Translation on t domain</u>

$$f(t) \longleftrightarrow F(s)$$

$$f(t-k)\, u(t-k) \longleftrightarrow \ ?$$

$f(t)$





$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$ Suppose $k \geq 0$

$$L\left[f(t-k)\, u(t-k)\right] = \int_{0\ k}^{\infty} e^{-st} f(t-k)\, u(t-k)\, dt$$

$-\infty$

$$L\left[f(t-k)\,u(t-k)\right] = \int_0^{\infty} e^{-st} f(t-k)\,u(t-k)\,dt$$

$$= \underbrace{\int_0^{k}}_{0} + \int_k^{\infty}$$

$$= \int_k^{\infty} e^{-st} f(t-k)\,dt$$

$$= \int_k^{\infty} e^{-s(t-k)} f(t-k)\,dt \cdot e^{-sk}$$

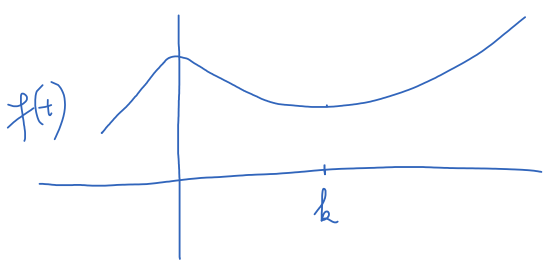
$$= \int_0^{\infty} e^{-st'} f(t')\,dt' \quad e^{-sk} = F(s)\,e^{-sk}$$

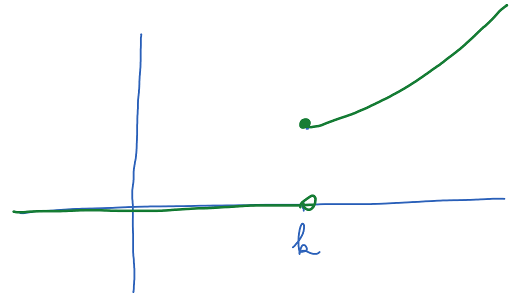
$\square$

$$f(t) \longleftrightarrow F(s) \qquad (s > k')$$

$$f(t-k)\,u(t-k) \xleftrightarrow{(k \geq 0)} F(s)\,e^{-sk} \qquad (s > k')$$

$$f(t)\quad u(t-k) \longleftrightarrow \ ?$$





$$f(t)\quad u(t-k) \longleftrightarrow e^{-sk}\ L\left[f(t+k)\right]$$

Summary on translations

$$f(t) \longleftrightarrow F(s) \qquad (s > k')$$

$$e^{kt}\ f(t) \xleftrightarrow{(k \in \mathbb{R})} F(s-k) \qquad (s > k + k')$$

(on s- domain)

$$f(t-k)\ u(t-k) \xleftrightarrow{(k \geq 0)} F(s)\ e^{-sk}$$

$$f(t-k)\, u(t-k) \xleftrightarrow{(k \geq 0)} F(s)\, e^{-sk}$$

$$f(t)\, u(t-k) \xleftrightarrow{(k \geq 0)} L\left[f(t+k)\right] e^{-sk}$$

(on t - domain)

<u>Example.</u> Find $L\left[\sin t \;\; u(t-\pi)\right]$

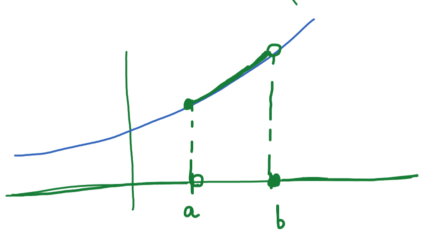
$$L\left[\sin t \;\; u(t-\pi)\right] = e^{-s\pi} \; L\Big[\underbrace{\sin\,(t+\pi)}_{-\sin t}\Big]$$

$$= - \; e^{-s\pi} \; L\left[\sin t\right]$$

$$= \frac{-\,e^{s\pi}}{s^2+1}$$

<u>Example.</u>

$$f(t) = \begin{cases} 0 & \text{if } t < a \\ e^t & \text{if } a \leq t < b \\ 0 & \text{if } t \geq b \end{cases}$$



Find $L\left[f(t)\right]$

We have $f(t) = e^t \left(u(t-a) - u(t-b) \right)$

We find $L\left[e^t \;\; u(t-a)\right]$ and $L\left[e^t \;\; u(t-b)\right]$

$$L\left[e^t \;\; u(t-a)\right] = L\left[e^{t-a} \;\; u(t-a)\right] \cdot e^a$$

$$= e^{-as} \cdot \frac{1}{s-1} \cdot e^a$$

$$= e^{-as} \cdot \frac{1}{s-1} \cdot e^{a}$$

$$= \frac{e^{a(1-s)}}{s-1}.$$

So $L[f(t)] = L\left[e^{t}\left(u(t-a) - u(t-b)\right)\right]$

$$= \frac{e^{a(1-s)} - e^{b(1-s)}}{s-1}$$

Derivatives

Tuesday, December 28, 2021 8:29 AM

$$f(t) \longleftrightarrow F(s)$$

$$f'(t) \longleftrightarrow ?$$

$$? \longleftrightarrow F'(s)$$

$$f^{(k)}(t) \longleftrightarrow ?$$

$$? \longleftrightarrow F^{(k)}(s)$$

We know how Laplace transform changes if we take $f'(t)$, $f^{(k)}(t)$

$$f'(t) \longleftrightarrow L[f'(t)] = sF(s) - f(0)$$

$$f^{(k)}(t) \longleftrightarrow L[f^{(k)}(t)] = s^k F(s) - \left(s^{k-1} f(0) + s^{k-2} f'(0) + \dots + f^{(k-1)}(0)\right)$$

(on t - domain)

On s - domain:

$$f(t) \longleftrightarrow F(s)$$

$$? \longleftrightarrow F'(s)$$

We have $F'(s) = \dfrac{d}{ds} \displaystyle\int_0^\infty e^{-st} f(t)\, dt$

$$\overset{\text{(suppose ok)}}{=} \int_0^\infty \frac{d}{ds} e^{-st} f(t)\, dt$$

$$= -\int_0^\infty e^{-st}\, t f(t)\, dt$$

$$= -L[t f(t)] = L[-t f(t)]$$

$$F^{(k)}(s) = \frac{d^k}{} \int^\infty e^{-st} f(t)\, dt$$

$$F^{(k)}(s) = \frac{d^k}{ds^k} \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty \frac{d^k}{ds^k} e^{-st} f(t)\,dt$$

$$= \int_0^\infty (-1)^k t^k f(t) e^{-st}\,dt$$

$$= (-1)^k L\left[t^k f(t)\right] = L\left[(-1)^k t^k f(t)\right]$$

$$f(t) \longleftrightarrow F(s) \qquad (s > k')$$

$$-t f(t) \longleftrightarrow F'(s) \qquad (s > k')$$

$$(-1)^k t^k f(t) \longleftrightarrow F^{(k)}(s) \qquad (s > k')$$

on s- domain

<u>Example:</u> find $L[t \cos t]$

$$L[t \cos t] = -\frac{d}{ds} L[\cos t]$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{s^2-1}{(s^2+1)^2}$$

<u>Example.</u> find $L[e^{kt}\, t \cos t]$

$$L[e^{kt}\, t \cos t] = \frac{(s-k)^2 - 1}{\left((s-k)^2+1\right)^2}$$

$$f(t) \longleftrightarrow F(s)$$

translation on t - domain $\longleftrightarrow$ ✓

/ translation on s- domain

translation on t - domain ⟷ ✓

✓ ⟷ translation on s - domain

derivatives on t - domain ⟷ ✓

✓ ⟷ derivatives on s - domain

How about integration ?

(next time)

Inverse Laplace transform: Post's inversion formula

Tuesday, December 28, 2021 8:51 AM

Let $f : [0, +\infty) \longrightarrow \mathbb{R}$ be continuous

Say f is of <u>exponential order k</u> if

$$\sup_{t \in \mathbb{R}_{\geq 0}} \frac{|f(t)|}{e^{kt}} < +\infty$$

$$F(s) = \int_0^\infty f(t) e^{-st} \, dt \qquad (s > k)$$

<u>Theorem</u>: (Post's inversion formula)

$$\lim_{k \to \infty} \frac{(-1)^k}{k!} \cdot \left(\frac{k}{t}\right)^{k+1} \cdot F^{(k)}\left(\frac{k}{t}\right) = f(t)$$

<u>Example</u>. find $L^{-1}\left[\frac{1}{s}\right]$

$$F(s) = \frac{1}{s}, \qquad F^{(k)}(s) = \frac{(-1)^k \, k!}{s^{k+1}}$$

$$\lim_{k \to \infty} \frac{\cancel{(-1)^k}}{\cancel{k!}} \cdot \left(\frac{\cancel{k}}{t}\right)^{k+1} \cdot \frac{\cancel{(-1)^k} \, \cancel{k!}}{(\cancel{k}/t)^{k+1}} = L^{-1}\left[\frac{1}{s}\right]$$

$$L^{-1}\left[\frac{1}{s}\right] = \lim_{k \to \infty} 1 = 1$$

<u>Example</u>: find $L^{-1}\left[\frac{1}{s+h}\right]$

$$F(s) = \frac{1}{s+h}, \qquad F^{(k)}(s) = \frac{(-1)^k \, k!}{(s + \cancel{k}h)^{k+1}}$$

$$\cancel{(-1)^k} \quad \left(\frac{k}{}\right)^{k+1} \quad \frac{\cancel{(-1)^k} \, \cancel{k!}}{}$$

$$L^{-1}\left[\frac{1}{s+h}\right] = \lim_{k\to\infty} \frac{\cancel{(-1)^k}}{\cancel{k!}} \left(\frac{k}{t}\right)^{k+1} \cdot \frac{\cancel{(-1)^k}\ \cancel{k!}}{\left(\frac{k}{t} + h\right)^{k+1}}$$

$$= \lim_{k\to\infty} \frac{k^{k+1}}{\cancel{t^{k+1}}} \frac{\cancel{t^{k+1}}}{(k + th)^{k+1}}$$

$$= \lim_{k\to\infty} \frac{1}{\left(1 + \frac{th}{k}\right)^{k+1}}$$

$$= \lim_{k\to\infty} \exp\left[-(k+1) \underbrace{\log\left(1 + \frac{th}{k}\right)}_{\sim \frac{th}{k}}\right]$$

$$= \lim_{k\to\infty} e^{-\frac{k+1}{k} \cdot th} = e^{-th}$$

$$L^{-1}\left[\frac{1}{s+h}\right] = e^{-h \cdot t}$$

Brief review on Laplace transforms

Tuesday, January 4, 2022 7:37 AM

Laplace transforms are useful tools to solve differential equations, integral equations.

$f : [0, +\infty) \longrightarrow \mathbb{R}$, $f(t)$

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$\left.\begin{matrix} f_1(t) \\ f_2(t) \end{matrix}\right\} \to F(s) \Rightarrow f_1(t) = f_2(t)$$

$F(s) = L[f(t)]$, $f(t) = L^{-1}[F(s)]$

$$f(t) \longleftrightarrow F(s)$$

Laplace correspondence

Laplace correspondences :

$f(t)$	$\longleftrightarrow$ $F(s)$	
$f'(t)$	$sF(s) - f(0)$	Derivatives
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$	
$f^{(n)}(t)$	$s^n F(s) - \left(s^{n-1} f(0) + s^{n-2} f'(0) + \dots + f^{(n-1)}(0)\right)$	
$-t f(t)$	$F'(s)$	
$(-1)^n t^n f(t)$	$F^{(n)}(s)$	
$f(t) \cdot e^{kt}$	$F(s-k)$	Translations

$f(t) \cdot e^{kt}$	$F(s-k)$
$u(t-k)$	$\dfrac{e^{-ks}}{s}$
$f(t-k)\, u(t-k)$	$e^{-ks}\, F(s)$
$f(t)\, u(t-k)$	$e^{-ks}\, L\left[f(t+k)\right]$
antiderivative of $f(t)$	?
periodic function $f(t)$	?
convolution $f(t) * g(t)$	?

Laplace transform of an antiderivative

Tuesday, January 4, 2022 7:53 AM

$$F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$$

$$F(s) = L\left[f(t) \right]$$

$$f(t) \longleftrightarrow F(s)$$

$$g(t) = \int_0^t f(x)\, dx \qquad ?$$

By the fundamental theorem of calculus. $g'(t) = f(t)$

$$L\left[g(t) \right] = \ ?$$

Theorem: Suppose $f(t)$ is piecewise continuous

$f(t)$ has exponential order k $(k \geqslant 0)$

$$\sup \left\{ \frac{f(t)}{e^{kt}} : t \geqslant 0 \right\} < M$$

constant

$$F(s) = L\left[f(t) \right]$$

$$g(t) = \int_0^t f(x)\, dx$$

Then $L\left[g(t) \right] = \dfrac{F(s)}{s}$ $(s > k)$

Proof First we show that $g(t)$ has exponential order k

$$g(t) = \int_0^t f(x)\, dx \qquad \left(\left| \frac{f(t)}{e^{kt}} \right| \leq M \right)$$

$$\Rightarrow \left| g(t) \right| \leq \int_0^t \left| f(x) \right| dx$$

$$\int^t \ldots kx \qquad M \left[\ldots kx \right]^{x=t}$$

$$\leq \int_0^t H e^{kx}\,dx = \frac{H}{k}\left[e^{kx}\right]_{x=0}^{x=t}$$

$$\leq \frac{H}{k}\left(e^{kt}-1\right) \leq \frac{H}{k} e^{kt} = H' e^{kt}$$

with $H' = \frac{H}{k}$

$\Rightarrow$ $g(t)$ has exponential order k

$$\Rightarrow L[f(t)] = L[g'(t)]$$

$$= s\,L[g(t)] - g(0)$$

$$F(s) = s\,L[g(t)] - 0 \quad \left(g(t) = \int_0^t f(x)\,dx \Rightarrow g(0) = 0 \right)$$

$$\Rightarrow L[g(t)] = \frac{1}{s} F(s).$$

$\square$

Example: compute $L\left[\int_0^t x\cdot e^{ax}\,dx\right]$

$$L\left[\int_0^t x\, e^{ax}\,dx\right] = \frac{1}{s}\cdot L\left[t\, e^{at}\right]$$

$$= \frac{1}{s}\cdot L[t](s-a)$$

$$= \frac{1}{s}\cdot \frac{1}{(s-a)^2}$$

Example: compute $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

$$\frac{1}{s(s^2+a^2)} = \frac{1}{s}\cdot\frac{1}{s^2+a^2}$$

$$f(t) \longleftrightarrow \frac{1}{s^2 + a^2}$$

$$\int_0^t f(x)\,dx \xleftrightarrow{?} \frac{1}{s} \frac{1}{s^2 + a^2}$$

$$f(t) = \frac{1}{a} L^{-1}\left[\frac{a}{s^2+a^2}\right] = \frac{1}{a} \sin(at)$$

$$\int_0^t f(x)\,dx = \int_0^t \frac{1}{a} \sin(ax)\,dx$$

$$= \frac{1}{a}\left(-\frac{\cos(ax)}{a}\right)_{x=0}^{x=t} = \frac{1}{a^2}\left(1 - \cos(at)\right)$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}\left(1 - \cos(at)\right)$$

Example. find a function $f(t)$ satisfying

$$f(t) - \int_0^t f(x)\,dx = t^2 e^t$$

'integral equation'

Write $F(s) = \int_0^t f(t) e^{-st}\,dt = L[f(t)]$

$$L[t^2 e^t] = L[t^2](s-1) = \frac{2}{(s-1)^3}.$$

Taking Laplace transform, we have $F(s) - \frac{1}{s} F(s) = \frac{2}{(s-1)^3}$

$$F(s) \frac{s-1}{s} = \frac{2}{(s-1)^3}$$

$$F(s) = \frac{2s}{(s-1)^4} = \frac{2(s-1+1)}{(s-1)^4}$$

$$F(s) = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^4}$$

$$F(s) = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^4}$$

$$f(t) = L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{2}{(s-1)^3} + \frac{2}{(s-1)^4}\right]$$

$$= e^t \cdot L^{-1}\left[\frac{2}{s^3} + \frac{2}{s^4}\right]$$

$$= e^t \cdot \left(t^2 + \frac{t^3}{3}\right)$$

$\square$

Laplace transform can be used to solve many integral equations!

Laplace transform of a periodic function

Tuesday, January 4, 2022 8:22 AM

Definition: Suppose $f : [0, +\infty) \longrightarrow \mathbb{R}$

Say f is periodic if there exists $P > 0$ such that $f(t+P) = f(t) \ \forall\, t \geqslant 0$

The smallest such P is called the period of f

Suppose f is periodic with period P

$L[f(t)] = \ ?$

- since f is periodic, f has exponential order 0

$$\sup_{t \geqslant 0} \left| \frac{f(t)}{e^{0 \cdot t}} \right| < +\infty$$

$\Rightarrow \exists\, L[f(t)]$

- $F(s) = \int_0^{+\infty} f(t)\, e^{-st}\, dt$

$= \int_0^P + \int_P^{+\infty}$

$$F(s) = \int_0^P f(t)\, e^{-st}\, dt + \underbrace{\int_P^{+\infty} f(t)\, e^{-st}\, dt}_{t = x + P}$$

$$= \int_P^{+\infty} f(x+P)\, e^{-s(x+P)}\, d(x+P)$$

$$= \int_0^{+\infty} f(x)\, e^{-sx}\, dx \cdot e^{-sP}$$

$$= F(s)\, e^{-sP}$$

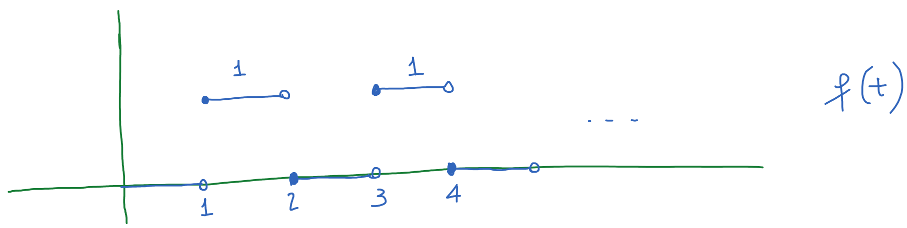
$$F(s)\left(1 - e^{-sP}\right) = \int_0^P f(t)\, e^{-st}\, dt.$$

Theorem. Suppose f is periodic with period P.

Then $F(s) = L\left[f(t)\right]$ exists

and $$F(s)\left(1 - e^{-sP}\right) = \int_0^P f(t)\, e^{-st}\, dt$$

Example.



$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 2 \end{cases} \qquad (\text{period } 2)$$

Compute $L\left[f(t)\right]$.

$$F(s)\left(1 - e^{-2s}\right) = \int_0^2 f(t)\, e^{-st}\, dt$$

$$= \int_1^2 e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_1^2$$

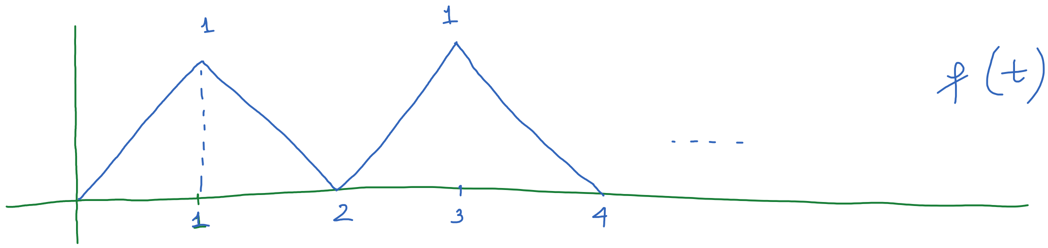
$$= \frac{e^{-s} - e^{-2s}}{s}$$

$$F(s) = \frac{e^{-s}\left(1 - e^{-s}\right)}{s\left(1 - e^{-s}\right)\left(1 + e^{-s}\right)}$$

$$F(s) = \frac{e^{-s}}{s\left(1 + e^{-s}\right)} \qquad \square$$

Example:

Example :



$$f(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 2 - t & \text{if } 1 \leq t \leq 2 \end{cases} \qquad (\text{period } 2)$$

$$F(s) = L\left[f(t) \right]$$

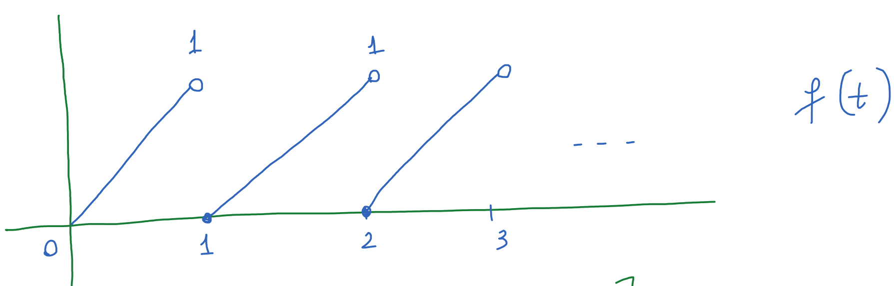
$$\left(1 - e^{-2s}\right) F(s) = \int_0^2 f(t)\, e^{-st}\, dt$$

$$= \int_0^1 t\, e^{-st}\, dt + \int_1^2 (2 - t)\, e^{-st}\, dt$$

Integrate by parts

$$F(s) = \frac{1}{s} \frac{1 - e^{-s}}{1 + e^{s}} \qquad \square$$

Example (sawtooth function)



Compute $F(s) = L\left[f(t) \right]$

$$f(t) = t \quad \text{if} \quad 0 \leq t < 1 \qquad (\text{period } 1)$$

$$F(s)\left(1 - e^{-s}\right) = \int^1 f(t)\, e^{-st}\, dt$$

$$F(s)\left(1-e^{-s}\right) = \int_0^1 f(t)\, e^{-st}\, dt$$

$$= \int_0^1 t\, e^{-st}\, dt = \int_0^1 t \; d\left(e^{-st} \cdot \frac{1}{-s}\right)$$

$$= \left[\frac{e^{-st}}{-s}\, t\right]_{t=0}^{t=1} - \int_0^1 \frac{e^{-st}}{-s}\, dt$$

$$= \frac{e^{-s}}{-s} + \frac{1}{s} \cdot \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=1}$$

$$F(s)\left(1-e^{-s}\right) = \frac{e^{-s}}{-s} + \frac{1}{s} \cdot \frac{e^{-s}}{-s} = \frac{e^{-s}}{-s}\left(1+\frac{1}{s}\right)$$

$$F(s) = -\frac{e^{-s}}{1-e^{-s}} \cdot \frac{s+1}{s^2} . \qquad \square$$

Convolution product

Tuesday, January 4, 2022 8:50 AM

Suppose $f, g: [0, +\infty) \longrightarrow \mathbb{R}$ are piecewise continuous.

$$h(t) = f(t) \cdot g(t) \quad \text{pointwise product}$$

$$\int_0^t f(x)\, g(t-x)\, dx =: f(t) * g(t)$$
$$=: (f * g)(t)$$

convolution product / convolution 'tích chập'

<u>Example:</u> $f(t) = 1$

$g(t) = 1$

Compute $(f * g)(t)$.

Know $(f\,g)(t) = 1$ (pointwise product)

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx$$
$$= \int_0^t 1 \cdot 1\, dx = t \Rightarrow (f * g)(t) = t.$$

(convolution product)

<u>Example.</u> $f(t) = t^p$

$g(t) = t^q$

Compute $(f * g)(t)$. convolution product

$(f\,g)(t) = t^{p+q}$ pointwise product

t

$(f\ g)(t) = t^{..}$ pointwise product

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx$$

$$= \int_0^t x^p (t-x)^q\, dx$$

If $p \in \mathbb{N}$ or $q \in \mathbb{N}$, we can use binomial expansion to compute $f * g$

$$f(t) = t^p, \quad g(t) = t \quad (q = 1)$$

$$(f*g)(t) = \int_0^t x^p (t-x)\, dx$$

$$= t \int_0^t x^p\, dx - \int_0^t x^{p+1}\, dx$$

$$= t \cdot \left[\frac{x^{p+1}}{p+1}\right]_0^t - \left[\frac{x^{p+2}}{p+2}\right]_0^t$$

$$= \frac{t^{p+2}}{p+1} - \frac{t^{p+2}}{p+2}$$

$$= \frac{t^{p+2}}{(p+1)(p+2)}$$

$$\Rightarrow t^p * t = \frac{t^{p+2}}{(p+1)(p+2)}$$ convolution product

$$t^p\ t = t^{p+1}$$ pointwise product

$$f(t) \longleftrightarrow F(s)$$

$$g(t) \longleftrightarrow G(s)$$

$$(f * g)(t) \longleftrightarrow F(s) \cdot G(s)$$

Theorem: If $F(s) = L[f(t)]$

$G(s) = L[g(t)]$

then $F(s)G(s) = L[f(t) * g(t)]$

$$L^{-1}\left[\frac{1}{(s^2+k^2)^2}\right] = L^{-1}\left[\frac{1}{s^2+k^2}\right] * L^{-1}\left[\frac{1}{s^2+k^2}\right]$$

$$F(s) = \frac{s^5 + 4s^4 + 3s^3 + 2s^2 + 1}{(s^2+2s+2)(s^2-1)(s^2+5)(s^2+10)}$$

$$L^{-1}[F(s)] = ?$$

Use convolution product

Convolution - review

Tuesday, January 11, 2022 7:33 AM

$f, g : [0, +\infty) \longrightarrow \mathbb{R}$

$$(f * g)(t) = \int_0^t f(x)\, g(t-x)\, dx$$

Example. $f(t) = e^{-t}$, $g(t) = \sin t$

$f * g = ?$

$$\int_0^t f(x)\, g(t-x)\, dx = \int_0^t e^{-x} \underbrace{\sin(t-x)\, dx}_{d\cos(t-x)}$$

$$= \underbrace{\Big[\cos(t-x)\, e^{-x}\Big]_{x=0}^{x=t}}_{e^{-t} - \cos t} - \int_0^t \cos(t-x)\, \underbrace{d(e^{-x})}_{-e^{-x}dx}$$

$$= e^{-t} - \cos t + \int_0^t e^{-x} \underbrace{\cos(t-x)\, dx}_{-d\sin(t-x)}$$

$$= e^{-t} - \cos t - \underbrace{\Big[e^{-x}\sin(t-x)\Big]_{x=0}^{x=t}}_{-\sin t} + \int_0^t \sin(t-x)\, d(e^{-x})$$

$$\Rightarrow 2\underbrace{\int_0^t \sin(t-x)\, e^{-x}\, dx}_{e^{-t} * \sin t} = e^{-t} + \sin t - \cos t$$

$$\Rightarrow e^{-t} * \sin t = \frac{1}{2}\left(e^{-t} + \sin t - \cos t\right) \quad \square$$

Properties of convolutions

Tuesday, January 11, 2022 7:43 AM

Proposition: Let f, g, h be piecewise-continuous functions. Then:

(i) $f * g = g * f$ (commutativity)

(ii) $(f * g) * h = f * (g * h)$ (associativity)

(iii) $f * (g + h) = f * g + f * h$ (distributivity)

(iv) $f * 0 = 0$

Convolution is commutative:

$$(f * g)(t) = \int_0^t f(x)\, g(t - x)\, dx \quad (1) \qquad \text{asymmetric formula}$$

$$f * g \overset{?}{=} g * f$$

Change variables: put $u = t - x$

$$(1) = \int_t^0 f(t-u)\, g(u)\, (-du)$$

$$= \int_0^t g(u)\, f(t-u)\, du = (g * f)(t)$$

$$\Rightarrow f * g = g * f$$

Convolution is associative $(f * g) * h \overset{?}{=} f * (g * h)$

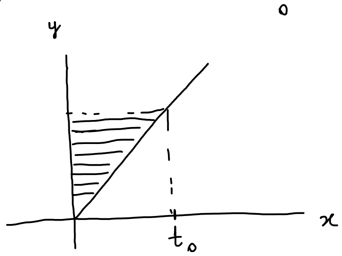
we want to show

$$\left[(f * g) * h\right](t_0) = \left[f * (g * h)\right](t_0)$$

$$\left[(f * g) * h\right](t_0) = \int_0^{t_0} \underbrace{(f * g)(x)}_{\text{unpack}}\, h(t_0 - x)\, \underline{dx}$$

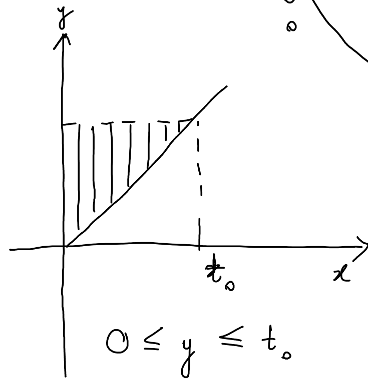
$$\underbrace{\qquad}_{\text{unpack}} \quad \int_0^x f(y)\,g(x-y)\,dy$$

$$= \int_0^{t_0} h(t_0 - x)\,dx \int_0^x f(y)\,g(x-y)\,dy = \int_0^{t_0} dy \int_y^{t_0} dx \; h(t_0 - x)\,\underline{\underline{f(y)}}\,g(x-y)$$



$0 \le x \le t_0$

$0 \le y \le x$



$0 \le y \le t_0$

$y \le x \le t_0$

$$= \int_0^{t_0} f(y)\,dy \underbrace{\int_y^{t_0} h(t_0 - x)\,g(x-y)\,dx}_{x' = x - y}$$

$$= \int_0^{t_0} f(y)\,dy \underbrace{\int_0^{t_0 - y} g(x')\,h(t_0 - y - x')\,dx'}_{g * h \; (t_0 - y)}$$

$$= \int_0^{t_0} f(y) \; (g * h)(t_0 - y)\,dy$$

$$= \left[f * (g * h) \right] (t_0)$$

$$\Rightarrow (f * g) * h = f * (g * h)$$

group / ring / field

What is a ring ?

The set of piecewise-continuous functions with addition, convolution forms a commutative ring without unit

(the unit is the 'delta function', not a piecewise - continuous function)

Laplace transform of convolutions

Tuesday, January 11, 2022 8:08 AM

Proposition. Let f, g be piecewise - continuous functions of exponential order

Then $L[f * g] = L[f] \; L[g]$

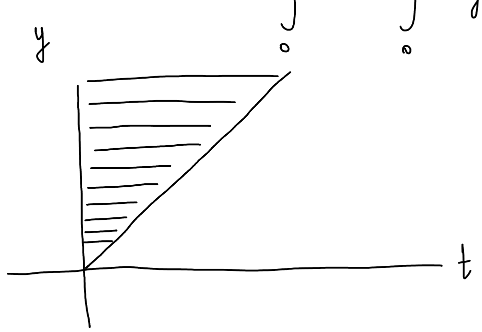
Laplace transform takes convolution to pointwise - product

Laplace transform of convolution.

$$L[f*g](s) = \int_0^\infty e^{-st} \underbrace{(f*g)(t)}_{\text{unpack}} \, dt$$

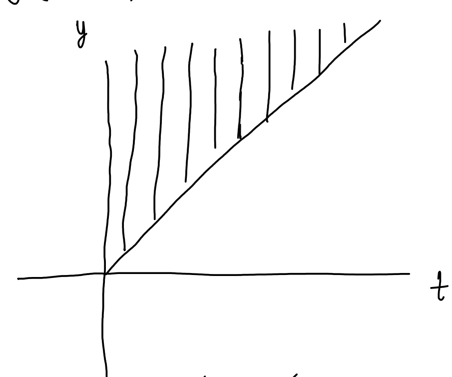
$$\int_0^t f(y)\, g(t-y)\, dy$$

$$= \int_0^\infty dt \int_0^t dy \;\; e^{-st} f(y)\, g(t-y)$$



$0 \le t < \infty$

$0 \le y \le t$



$0 \le y < \infty$

$y \le t < \infty$

$$= \int_0^\infty dy \int_y^\infty dt \;\; e^{-st} f(y)\, g(t-y)$$

put $t' = t - y$

$$= \int_0^\infty dy \int_0^\infty dt' \;\; e^{-s(t'+y)} f(y)\, g(t')$$

$$= \int^\infty f(y)\, e^{-sy}\, dy \quad \int^\infty g(t')\, e^{-st'}\, dt'$$

$$= \underbrace{\int_0 f(y)\, e^{-sy}\, dy}_{L[f](s)} \cdot \underbrace{\int_0 g(t')\, e^{-st'}\, dt'}_{L[g](s)}$$

$$\Rightarrow \quad L[f * g](s) = L[f](s) \quad L[g](s)$$

$$\Rightarrow \quad L[f * g] = L[f] \quad L[g]$$

$\square$

Convolution: examples

Tuesday, January 11, 2022 8:21 AM

$$L[f * g] = L[f] \cdot L[g]$$

Laplace correspondence

$$f \longleftrightarrow L[f] = F$$

$$g \longleftrightarrow L[g] = G$$

$$f * g \longleftrightarrow L[f * g] = F \cdot G$$

$$L^{-1}[F \cdot G] = L^{-1}[F] * L^{-1}[G]$$

This identity allows us to compute the inverse Laplace transform of any rational function!

Example. compute $L^{-1}\left[\frac{1}{s^2(s-1)}\right]$

$$L^{-1}\left[\frac{1}{s^2(s-1)}\right] = L^{-1}\left[\frac{1}{s^2} \cdot \frac{1}{s-1}\right]$$

$$= L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s-1}\right]$$

$$= t * e^t$$

$$= \int_0^t dx \, (t-x) e^x = \int_0^t (t-x) \, de^x$$

$$= \underbrace{\Big[e^x (t-x)\Big]_{x=0}^{x=t}} - \int_0^t e^x \underbrace{d(t-x)}_{-dx}$$

$$= -t + \int_0^t e^x \, dx = e^t - 1 - t$$

$$L^{-1}\left[\frac{1}{s^2(s-1)}\right] = t * e^t = e^t - t - 1.$$

$$L^{-1}\left[\frac{1}{s^2(s-1)}\right] = t * e^t = e^t - t - 1 .$$

<u>Example</u>: compute $L^{-1}\left[\dfrac{1}{(s^2+k^2)^2}\right]$ ①

$$① = \frac{1}{k^2} L^{-1}\left[\frac{k}{s^2+k^2} \quad \frac{k}{s^2+k^2}\right]$$

$$= \frac{1}{k^2} \underbrace{L^{-1}\left[\frac{k}{s^2+k^2}\right]}_{\sin(kt)} * \underbrace{L^{-1}\left[\frac{k}{s^2+k^2}\right]}_{\sin(kt)}$$

$$= \frac{1}{k^2} \sin(kt) * \sin(kt)$$

$$= \frac{1}{2k^2} \int_0^t dx \cdot \underbrace{2 \sin k(t-x) \quad \sin kx}$$

$$= \frac{1}{2k^2} \int_0^t dx \cdot \left(\cos k(2x-t) - \cos kt\right)$$

$$= \frac{1}{2k^2} \left[\frac{\sin k(2x-t)}{2k}\right]_{x=0}^{x=t} - \frac{1}{2k^2} \cdot t \cos kt$$

$$= \frac{1}{2k^2}\left(\sin kt - kt \cos kt\right)$$

Laplace transform can solve some integral equation

<u>Example</u> find $y(t)$ satisfying t $(t > 0)$

<u>Example</u> find $y(t)$ satisfying

$$y(t) = t + \int_0^t y(x)\sin(t-x)\,dx \qquad (t>0)$$

This is an integral equation

$$y(t) = t + y(t) * \sin t$$

$\Big\downarrow$ Laplace transform $\quad Y(s) = L[y(t)]$

$$Y = L[t] + Y\, L[\sin t]$$

$$Y = \frac{1}{s^2} + Y \frac{1}{s^2+1}$$

$$Y \frac{s^2}{s^2+1} = \frac{1}{s^2} \Rightarrow Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$y = L^{-1}[Y] = L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s^4}\right]$$

$$= t + \frac{t^3}{6}$$

Convolution allows us to 'Laplace inverse transform' rational functions