ĐẠI HỌC BÁCH KHOA

25 YEARS ANNIVERSARY SOICT

HA NOI UNIVERSITY OF SCIENCE AND TECHNOLOGY
SCHOOL OF INFORMATION AND COMMUNICATION TECHNOLOGY

FUNDAMENTALS OF OPTIMIZATION

Modelling

CONTENT

- Optimization problems
- Modelling overview
- Examples

Optimization problems

- Maximize or minimize some function relative to some set (range of choices)
- The function represents the quality of the choice, indicating which is the "best"

Modelling overview

prob. text ⇒ math model

M_1 M_2 ... M_k

- Modelling consists of specifying
  - Decision variables
  - Constraints
  - Objective functions

ILP

P ⇐ CP

$x_1 \dots x_n$

C: $f(x_1 \dots x_k)$

- A problem can be modelled in different ways (how to define variables)

Ortools

- Take into account the modelling languages of software tools
  - Constraint Programming solvers: constraints can be stated in flexible ways

Constraint linearization

C: $f(x_1, \dots x_n)$

Linear

Simplex method: LP

(I) LP $\Leftarrow$ Linear Solver $\Leftarrow$ P

Linear form

- Motivation
  - Linear Programming solvers are very efficient

$l \le a_1 x_1 + a_2 x_2 + \dots + a_n x_n \le u$

- Examples
  - How to model $\boxed{X = \min\{x_1, x_2\}}$?

not linear.

Constraint linearization

- Motivation
  - Linear Programming solvers are very efficient
- Examples
  - How to model $X = \min\{x_1,x_2\}$?

$X \le x_1$
$X \le x_2$

$X = x_1$
$X \ge x_1$
$X \le x_1$

$\{0, 1\}$

Solution: define an auxiliary binary variable y, use big constant M
- $x_1 \ge X$
- $x_2 \ge X$
- $X \ge x_1 - M(1-y)$
- $X \ge x_2 - My$

if $y = 0$

$X \ge x_1 - M(1-0)$ ✓
$X \ge x_2 - M.0 = x_2$

$X = x_2.$

$X \ge x_1 - M.(1-1)$
$X \ge x_2 - M.1$

$y = 1 \Leftrightarrow \begin{cases} X \ge x_1 \\ X \ge x_2 - M. \end{cases}$ $X = x_1$

Constraint linearization

Linear form. $\boxed{x = 1}$

$\{0, 1\}$

- Examples
  - How to model $\boxed{(x = 1) \Rightarrow (z \geq y)}$ where x is binary variable, y and z are real variables ?

Constraint linearization

each values of (x,y,z)

- Examples
  - How to model $(x = 1) \Rightarrow (z \geq y)$ where x is binary variable, y and z are real variables ?

if $x = 0$

Solution: use big constant M

- $\boxed{M(x\text{-}1) + y \leq z}$

$M(0-1) + y \leq z$ ✓ True

if $x = 1$, $\boxed{z \geq y}$ must be True.

$M(1-1) + y \leq z$

$M \cdot 0 + y \leq z$

$\boxed{y \leq z}$

each value (x,y,z)

Constraint linearization

- Example
  - $(x > 0) \Rightarrow (z \geq y)$ in which x, y and z are real variables (and x >=0) ?

or-tools

$\left(\frac{x}{|x|} - 1\right)M + y \leq z$

$(x = 0, y = 10^6, z = 1)$

$l \leq a_1 x_1 + a x_2 - \dots a_n x_n \leq u$

$z \geq y + Mx$

$x * (y - z) \geq 0$

$z \geq y - Mx$

$z \geq y - M.x$

$x = 1$
$y = 10, z = 2$

$C_1 \Leftrightarrow C_2$

$(x_1 \dots x_n)$

$z \geq y + M(1 - x)$

$\cancel{z \geq y + Mx + N}$

$Mx\cancel{x}y - z \geq 0$

$x = 0,5$
$y = 2, z = 3$

$M(x + x_2) + y \leq z$, $x_2 < 0$

$M(1 + (0.5)) + 2 \leq 3$

$z \geq y - Mx - N$

$2 \geq 3 - M1 - N$

$z \geq y - N$

$1 \geq 10^6 - N$

Constraint linearization

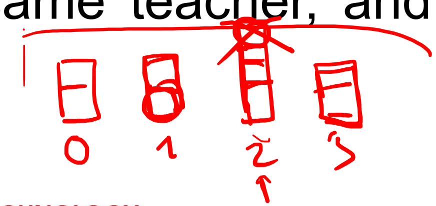
- Example
  - ($x > 0$) $\Rightarrow$ ($z \geq y$) in which x, y and z are real variables (and x >=0) ?

Solution:

- Let M be a very big constant,
- Introduce a binary variable $t \in \{0,1\}$:
  - $t = 1$ indicates that $x > 0$, and $t = 0$ indicates that $x = 0$
- Equivalent linear constraints
  - $x \leq M.t$
  - $z + (1-t)M \geq y$
  - $x \geq t/M$

Balanced Course Assignment Problem

- At the beginning of the semester, the head of a computer science department *D* have to assign courses to teachers in a balanced way. The department *D* has *m* teachers $T=\{0, 2, ..., m\text{-}1\}$ and *n* courses $C=\{0, 2,..., n\text{-}1\}$.
  - Each teacher $t \in T$ has a preference list which is a list of courses he/she can teach depending on his/her specialization. The preference information is represented by a 0-1 matrix $A_{m \times n}$ in which $\boxed{A(t,c) = 1}$ indicates that teacher t can teach the course c and $A(t,c) = 0$, otherwise
  - We know a set *B* of pairs of conflicting two courses that cannot be assigned to the same teacher as these courses have been already scheduled in the same slot of the timetable. (i, j)
  - The load of a teacher is the number of courses assigned to her/him. How to assign *n* courses to *m* teacher such that each course assigned to a teacher is in his/her preference list, no two conflicting courses are assigned to the same teacher, and the maximal load among teachers is minimal.



Balanced Course Assignment Problem

- Example

Course	0	1	2	3	4	5	6	7	8	9	10	11	12
credits	3	3	4	3	4	3	3	3	4	3	3	4	4

Teachers	Preference Courses
0	0, 2, 3, 4, 8, 10
1	0, 1, 3, 5, 6, 7, 8
2	1, 2, 3, 7, 9, 11, 12

Conflicting courses

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

Balanced Course Assignment Problem

- Example

Course	0	1	2	3	4	5	6	7	8	9	10	11	12
credits	3	3	4	3	4	3	3	3	4	3	3	4	4

Teachers	Preference Courses
0	0, 2, 3, 4, 8, 10
1	0, 1, 3, 5, 6, 7, 8
2	1, 2, 3, 7, 9, 11, 12

Teacher	Assigned courses	Load
0	2, 4, 8, 10	15
1	0, 1, 3, 5, 6	15
2	7, 9, 11, 12	14

Conflicting courses

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

Balanced Course Assignment: CP model

- Decision variables
  - $X(i)$: teacher assigned to course i, $\forall i \in C$, domain $D(X(i)) = \{t \in T \mid A(t,i) = 1\}$
  - $Y(i)$: load of teacher i, domain $D(Y(i)) = \{0,1,\ldots,n-1\}$
  - Z: maximum load among teachers
- Constraints
  - $X(i) \neq X(j)$, $\forall (i,j) \in B$
  - $Y(i) = \sum_{j \in C}(X(j) = i)$, $\forall i \in T$
  - $Z \geq Y(i)$, $\forall i \in T$
- Objective function to be minimized: Z

CP model.

Balanced Course Assignment: ILP model

- Decision variables
  - $X(i,j) = 1$: teacher i is assigned to course j, and $X(i,j) = 0$, otherwise, $\forall\ i \in T, j \in C$, domain $D(X(i,j)) = \{0,1\}$
  - $Y(i)$: load of teacher i, domain $D(Y(i)) = \{0,1,\dots,n\}$
  - Z: maximum load among teachers
- Constraints
  - $\sum_{i\in T} X(i,j) = 1,\ \forall j \in C$
  - $X(t,i) + X(t,j) \le 1,\ \forall (i,j) \in B,\ t \in T$
  - $Y(i) = \sum_{j\in C} X(i,j),\ \forall i \in T$
  - $Z \ge \boxed{Y(i)},\ \forall i \in T$
- Objective function to be minimized: Z

$f(X,Y,Z) = Z \to \min$

$$X:\quad \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ i \to 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$i = 1$

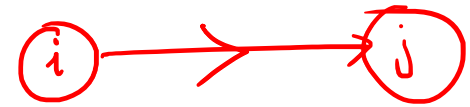
3

Travelling Salesman Problem (TSP)

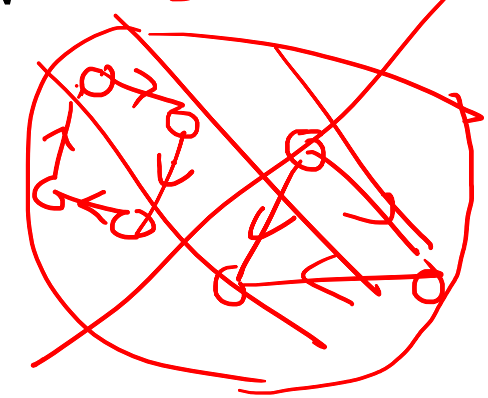
- A salesman departs from point 1, visiting $N-1$ points 2, …, N and comes back to the point 1. The travelling distance from point i and point j is $d(i,j)$, $i,j = 1,\dots,N$. Compute the route of minimal total travelling distance

Travelling Salesman Problem (TSP)

- Decision variables
  - Binary variable $X(i,j) = 1$ if the route traverses from point i to point j, and $X(i,j) = 0$, otherwise.



- Constraints
  - $\sum_{j=1}^{N} X(i,j) = \sum_{j=1}^{N} X(j,i) = 1, \forall i \in \{1,2,\ldots,N\}$
  - $\sum_{(i,j)\in S} X(i,j) \leq |S| - 1, \forall S \subseteq \{1,2,\ldots,N\}$ and $|S| < N$ SEC



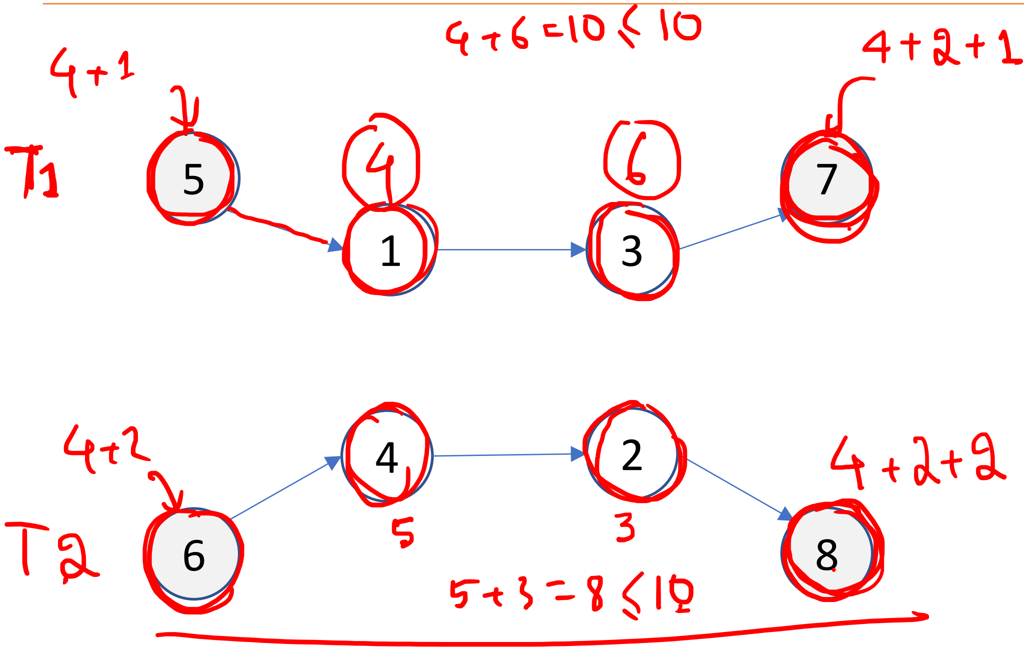
- Objective function to be minimized

$$f(X) = \sum_{j=1}^{N} \sum_{i=1}^{N} d(i,j)X(i,j)$$

Capacitated Vehicle Routing

- A fleet of K trucks 1, 2, …, K must be scheduled to visit N customers 1, 2, …, N for collecting items
  - Customer i located at point i and requests to be collected $r(i)$ items, $i = 1, 2, …, N$
  - Truck k (k = 1,…,K)
    - Departs from point $N+k$ and terminates at point $N + K + k$ ($N+k$ and $N+K+k$ might refer to the central depot)
    - has capacity $c(k)$ which is the maximum number of items it can carry at a time → tải trọng
  - Travel distance from point i to point j is $d(i,j)$, $i,j = 1,…,N + 2K$
- Compute the delivery solution such that the total travelling distance is minimal
  - Satisfy capacity constraints
  - Each customer is visited exactly once by exactly one truck

Capacitated Vehicle Routing Problem



K=2, N=4

feasible

d

	1	2	3	4	5	6	7	8
1	0	2	3	4	3	3	3	3
2	4	0	2	6	1	1	1	1
3	2	4	0	2	1	1	1	1
4	5	7	7	0	4	4	4	4
5	3	1	5	7	0	0	0	0
6	3	1	5	7	0	0	0	0
7	3	1	5	7	0	0	0	0
8	3	1	5	7	0	0	0	0

	1	2	3	4
r	4	3	6	5

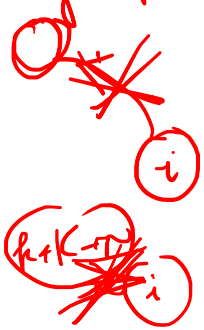
	1	2
c	10	10

Capacitated Vehicle Routing Problem

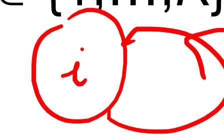
TSP $\boxed{K}$ N, K, d
c, r

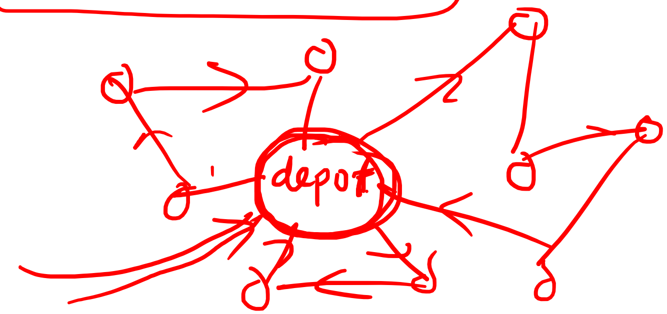
- Notations

graph



  - $B = \{1,\dots, N+2K\}$
  - $F_1 = \{(i, k+N) \mid i \in B, k \in \{1,\dots,K\}\}$
  - $F_2 = \{(k+K+N, i) \mid i \in B, k \in \{1,\dots,K\}\}$
  - $F_3 = \{(i, i) \mid i \in B\}$
  - $\boxed{A} = B^2 \setminus F_1 \setminus F_2 \setminus F_3$
  - $A^+(i) = \{ j \mid (i, j) \in A\}$, $A^-(i) = \{ j \mid (j, i) \in A\}$





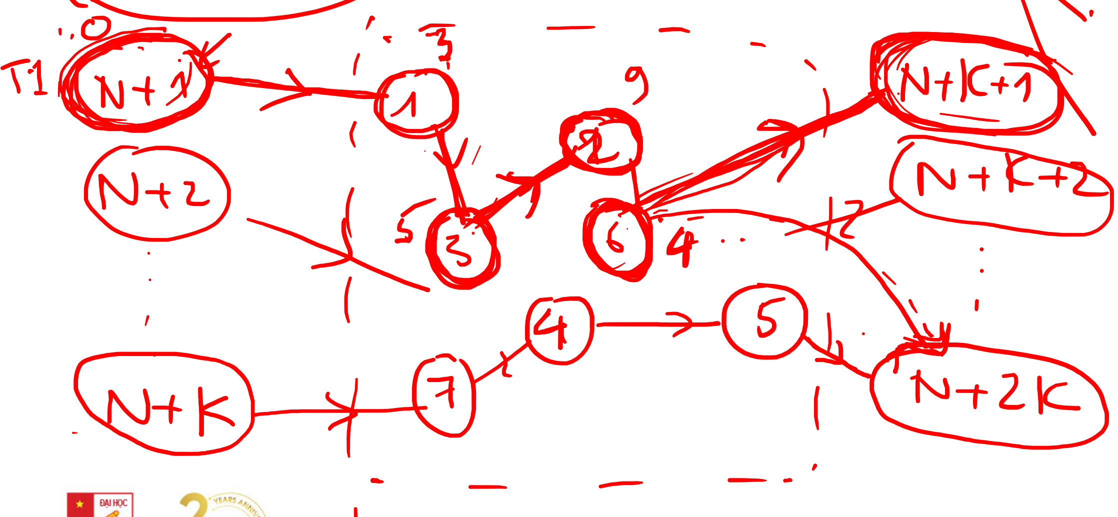
- Model ?

ILP model.

decision variables (i, j)

constraints

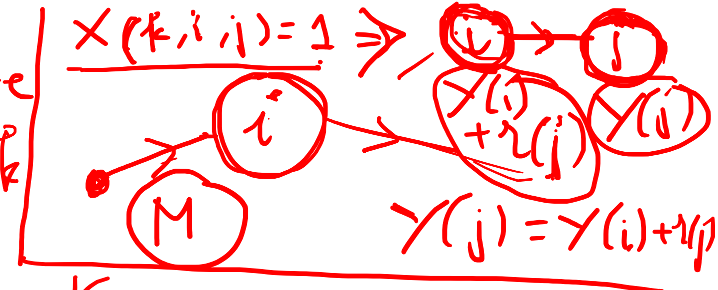
objective function.



Truck k $i \to j$

$X(k,i,j) = 1$: truck k traverses $i \to j$
$= 0$, otherwise

Variables:
- $X(k,i,j) \in \{0,1\}$
- $Y(i)$: số items tích lũy trên route từ đầu đến điểm i
- $Z(i)$: chỉ số của xe đi qua điểm i $(1,\dots,k)$



Constraints:
- Balance flow constraints

$$\sum_{k=1}^{K} \sum_{j \in A^+(i)} X(k,i,j) = \sum_{k=1}^{K} \sum_{j \in A^-(i)} X(k,i,j) = 1 \quad \forall i \in \{1,\dots,N\}$$

$$\sum_{j \in A^+(i)} X(k,i,j) = \sum_{j \in A^-(i)} X(k,i,j), \quad \forall i \in \{1,\dots,N\}, \ \forall k \in \{1,\dots,K\}$$

$$\sum_{j=1}^{N} X(k,k+N,j) = \sum_{j=1}^{N} X(k,j,k+K+N) = 1, \quad \forall k = \{1,\dots,K\}$$

$$Y(k+K+N) \leq c(k), \quad \forall k \in \{1,\dots,K\}$$

$$Z(k+N) = Z(k+K+N) = k, \quad \forall k = \{1,\dots,K\}$$

$$\left.\begin{aligned} & M(1 - X(k,i,j)) + Y(j) \geq Y(i) + r(j) \\ & M(X(k,i,j) - 1) + Y(j) \leq Y(i) + r(j) \end{aligned}\right\} \forall k,i,j$$

$$M(1 - X(k,i,j)) + Z(j) \geq Z(i) \quad \forall k,i,j$$

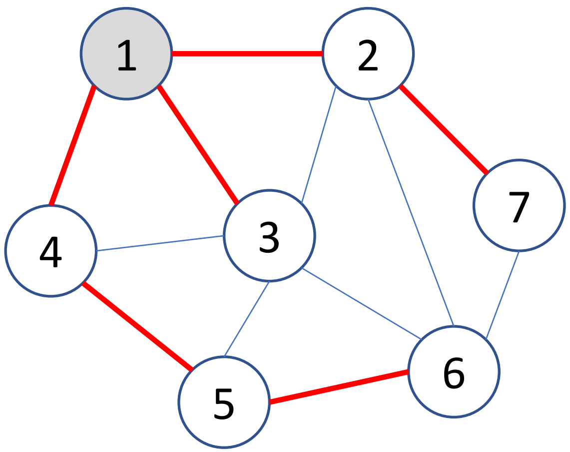
$$M(X(k,i,j) - 1) + Z(j) \leq Z(i)$$

Objective

$$\sum_{k=1}^{K} \sum_{(i,j) \in A} d(i,j) * X(k,i,j) \to \min$$

MultiCast Routing Problem

- Given a network $V = \{1,\ldots,N\}$ is the set of nodes, $E \subseteq V^2$ is the set of links between nodes. A node $s \in V$ is the source node which will transmit a package to others nodes. A node receiving the package can continue transmit this package to adjacent nodes.
  - $t(i,j)$ and $c(i,j)$ are transmission time and transmission cost when transmitting the package from node i to node j
- Compute the set of links used for broadcasting the package from the source node to all other nodes such that
  - Total transmission time from s to any node cannot exceed a given value L
  - Total transmission cost is minimal



Facility Location

- There are M sites 1, 2, …, M that can be used to open facility for servicing N customers 1, 2, …, N.
  - $f(i)$ is the cost for opening the site i
  - $Q(i)$ is the capacity of site i (maximum amount of good it can serve customers)
  - $c(i,j)$ is the cost for transporting unit of good from site i to customer j
  - $d(j)$ is the total demand (amount of goods) of customer j
- Compute a planning (which site to be opened and amount of good each opened site serves a customer) such that
  - Capacity constraint is satisfied
  - Total cost is minimal

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Thank you for your attentions!