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Calculus2 notebook.
File 1: BT_GT2_CTTT

Chapter 3: Multiple integrals

3.1 Double integrals

53

a) $I = \iint\limits_{[0,\frac{\pi}{2}]^2} x\sin(x+y)\,dxdy = \int_0^{\pi/2} -x\cos(x+y)\Big|_{y=0}^{y=\pi/2} dx$

$= \int_0^{\pi/2} x\cos(x) + x\sin(x)\,dx$

$= \int_0^{\pi/2} x\,d\sin(x) - \int_0^{\pi/2} x\,d\cos(x)$

$= x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(x)\,dx + \int_0^{\pi/2}\cos(x)\,dx - x\cos x\Big|_0^{\pi/2}$

$= \left(\frac{\pi}{2} - 0\right) - 1 + 1 - 0 = \frac{\pi}{2}$

b) $I = \iint\limits_{[0,2]\times[1,2]} x - 3y^2\,dxdy = \int_0^2 dx \int_1^2 x - 3y^2\,dy = \int_0^2 xy - y^3\Big|_{y=1}^{y=2} dx$

$= \int_0^2 x - 7\,dx = \frac{x^2}{2} - 7x\Big|_0^2 = -12$

c) $I = \iint\limits_{[1,2]\times[0,\pi]} y\sin(xy)\,dxdy = \int_0^{\pi} dy \int_1^2 y\sin(xy)\,dx$

$= \int_0^{\pi} -\cos(xy)\Big|_{x=1}^{x=2} dy = \int_0^{\pi} \cos y - \cos(2y)\,dy = 0$

d) $I = \iint\limits_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \sin(x-y)\,dxdy = \int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x-y)\,dy$

$= \int_0^{\pi/2} \cos(x-y)\Big|_{y=0}^{y=\pi/2} dx = \int_0^{\pi/2} \sin(x) - \cos(x)\,dx = 0$

e) $\iint\limits_{[0,2]\times[1,2]} (y+xy^{-2})\,dx\,dy = \int_0^2 \frac{y^2}{2} + \frac{-x}{y}\Big|_{y=1}^{y=2} dx$

$= \int_0^2 \frac{3}{2} + \frac{x}{2}\,dx = \frac{3}{2}x + \frac{x^2}{4}\Big|_{x=0}^{x=2} = 5$

f) $\iint\limits_{[0,1]\times[-3,2]} xy^2/(x^2+1)\,dx\,dy = \int_0^1 \frac{x}{x^2+1}\,dx \cdot \int_{-3}^{2} y^2\,dy = \frac{\ln(2)}{2}\cdot\frac{35}{3}$

$= \frac{35\ln 2}{6}$

g) $\iint\limits_{[0,1]\times[0,1]} \frac{1+x^2}{1+y^2}\,dx\,dy = \left(x+\frac{x^3}{3}\right)\Big|_0^1 \cdot \arctan(y)\Big|_0^1 = \frac{4}{3}\cdot\frac{\pi}{4} = \frac{\pi}{3}$

h) $\iint\limits_{[0,\frac{\pi}{6}]\times[0,\frac{\pi}{3}]} x\sin(x+y)\,dx\,dy \left(= \int_0^{\pi/3} dy \int_0^{\pi/6} x\sin(x+y)\,dx\right)$

$= \int_0^{\pi/6} dx \int_0^{\pi/3} x\sin(x+y)\,dy$

$= \int_0^{\pi/6} -x\cos(x+y)\Big|_{y=0}^{y=\pi/3} dx$

$= \int_0^{\pi/6} x\cos(x) - x\cos\left(x+\frac{\pi}{3}\right) dx$

$= \int_0^{\pi/6} x\cos(x)\,dx - \int_0^{\pi/6} \left(x+\frac{\pi}{3}\right)\cos\left(x+\frac{\pi}{3}\right) d\left(x+\frac{\pi}{3}\right) + \frac{\pi}{3}\int_0^{\pi/6} \cos\left(x+\frac{\pi}{3}\right) d\left(x+\frac{\pi}{3}\right)$

$= x\sin(x)\Big|_0^{\pi/6} - \int_0^{\pi/6} \sin x\,dx - \left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{\pi}{3}\right)\Big|_0^{\pi/6} + \int_0^{\pi/6} \sin\left(x+\frac{\pi}{3}\right) d\left(x+\frac{\pi}{3}\right)$

$+ \frac{\pi}{3}\sin\left(x+\frac{\pi}{3}\right)\Big|_0^{\pi/6}$

$= \left(\frac{\pi}{6}\cdot\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2} - 1\right) - \left(\frac{\pi}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} - 0\right) + \frac{\pi}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$

$= \pi\left(\frac{1}{12} - \frac{1}{2} + \frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{6} + \frac{1}{3}\right) + \frac{\sqrt{3}}{4}$

$= \frac{-\pi}{12} + \frac{\sqrt{3}}{2} - \frac{1}{2}$

i) $I = \iint\limits_{[0,1]\times[0,1]} x/(1+xy)\,dx\,dy = \int_0^1 dx \int_0^1 \frac{x}{1+xy}\,dy$

$= \int_0^1 \ln(1+xy)\Big|_{y=0}^{y=1} dx = \int_0^1 \ln(1+x)\,dx = x\ln(1+x) + \ln(1+x) - x\Big|_0^1 = 2\ln(2) - 1$

j) $I = \iint\limits_{[0,2]\times[0,3]} ye^{-xy}\,dx\,dy = \int_0^3 dy \int_0^2 ye^{-xy}\,dx = \int_0^3 -e^{-xy}\Big|_{x=0}^{x=2} dy$

$= \int_0^3 1 - e^{-2y}\,dy = y + \frac{e^{-2y}}{2}\Big|_0^3 = 3 + \frac{e^{-6}}{2} - \frac{1}{2} = \frac{5}{2} + \frac{e^{-6}}{2}$

k) $I = \iint\limits_{[1,3]\times[1,2]} \frac{1}{1+x+y}\,dx\,dy$

$= \int_1^3 \ln(1+x+y)\Big|_{y=1}^{y=2} dx$

$= \int_1^3 \ln(x+3) - \ln(x+2)\,dx$

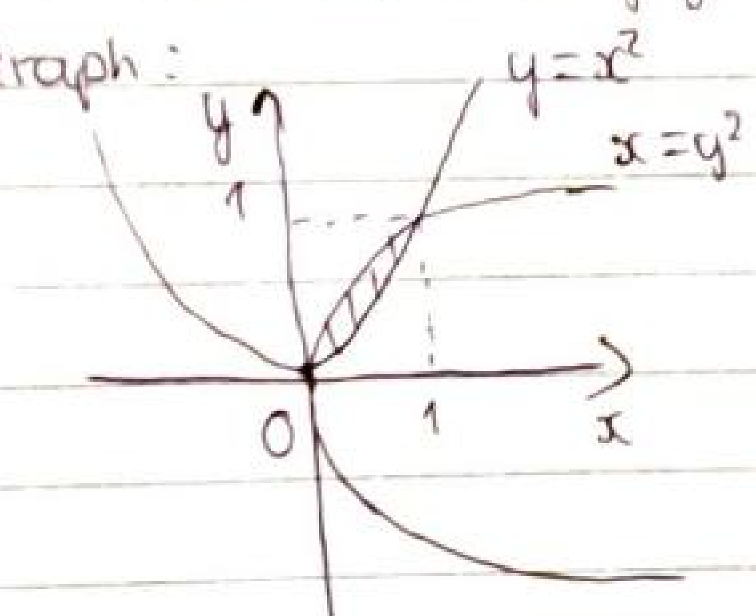
$= \left((x+3)\ln(x+3) - (x+2)\ln(x+2)\right)\Big|_1^3$

$= 6\ln 6 - 5\ln 5 - 4\ln(4) + 3\ln 3$

54 a) $\iint x^2(y-x)\,dx\,dy = I$

D: bounded by $y=x^2$ and $x=y^2$

+) Graph:



+) $I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} x^2(y-x)\,dy = \int_0^1 \frac{x^2y^2}{2} - x^3y\Big|_{y=x^2}^{y=\sqrt{x}} dx = \int_0^1 \frac{x^2(x-x^4)}{2} - x^3(\sqrt{x} - x^2)\,dx$

$= \frac{3}{56} - \frac{1}{18} = -\frac{1}{504}$

b) $I = \iint_D |x+y| \, dx dy$, $D: \{(x,y) \in \mathbb{R}^2, |x| \leq 1, |y| \leq 1\}$

$= \int_{-1}^{1} dx \int_{-x}^{1} x+y \, dy + \int_{-1}^{1} dx \int_{-1}^{-x} -x-y \, dy$

$= \int_{-1}^{1} x(1+x) + \frac{1-x^2}{2} \, dx + \int_{-1}^{1} -x(-x+1) - \frac{x^2-1}{2} \, dx$

$= \frac{8}{3}$

c) $\iint_D \sqrt{|y-x^2|} \, dx dy$

$D: |x| \leq 1$

$0 \leq y \leq 1$

$= \int_{-1}^{1} dx \int_{x^2}^{1} \sqrt{y-x^2} \, dy + \int_{-1}^{1} dx \int_{0}^{x^2} \sqrt{x^2-y} \, dy$

$= \int_{-1}^{1} (y-x^2)^{\frac{3}{2}} \cdot \frac{2}{3} \Big|_{y=x^2}^{y=1} + -(x^2-y)^{\frac{3}{2}} \cdot \frac{2}{3} \Big|_{y=0}^{y=x^2} dx$

$= \int_{-1}^{1} \frac{2}{3} (1-x^2)^{\frac{3}{2}} + \frac{2}{3} (x^2)^{\frac{3}{2}} dx$

$= 2 \cdot \frac{2}{3} \int_{0}^{1} (1-x^2)^{\frac{3}{2}} + (x^2)^{\frac{3}{2}} dx$

$= \frac{4}{3} \cdot \left(\frac{3\pi}{16} + \frac{1}{4} \right) = \frac{\pi}{4} + \frac{1}{3}$

d) $\iint_{[0,1] \times [0,1]} \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}} \, dx dy$

$= \int_{0}^{1} \frac{1}{(1+x^2+y^2)^{\frac{1}{2}}} \cdot (-2) \cdot \frac{1}{2} \Big|_{y=0}^{y=1} dx$

$= \int_{0}^{1} \frac{1}{(1+x^2)^{\frac{1}{2}}} - \frac{1}{(2+x^2)^{\frac{1}{2}}} dx = \ln|x+\sqrt{1+x^2}| - \ln|x+\sqrt{2+x^2}| \Big|_0^1$

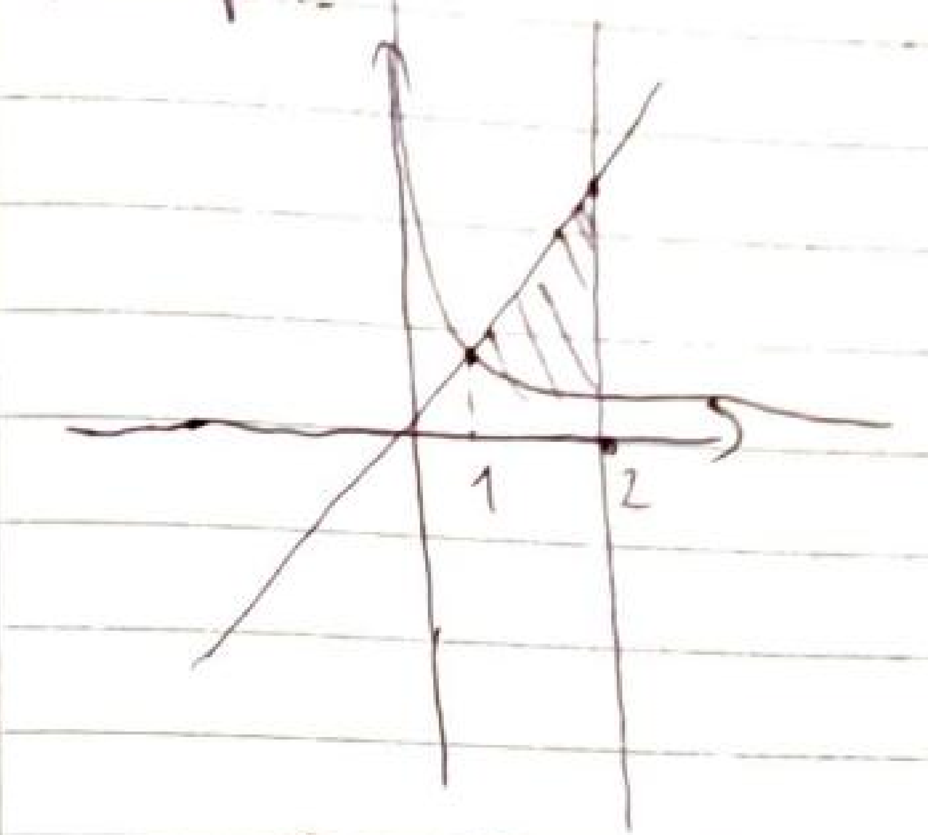
$= (\ln(1+\sqrt{2}) - \ln 1) - (\ln(1+\sqrt{3}) - \ln\sqrt{2})$

$= \ln \frac{(1+\sqrt{2})\sqrt{2}}{(1+\sqrt{3})} = \ln \frac{2+\sqrt{2}}{1+\sqrt{3}}$

e) $\iint_D \frac{x^2}{y^2} \, dx dy = I$

$D: x=2, y=x, xy=1$

+) Graph:



+) $I = \int_{1}^{2} dx \int_{1/x}^{x} \frac{x^2}{y^2} \, dy$

$= \int_{1}^{2} \frac{-x^2}{y} \Big|_{y=1/x}^{y=x} dx$

$= \int_{1}^{2} x^3 - x \, dx = \frac{9}{4}$

f) $\iint_D \frac{y}{1+x^5} \, dx dy = I$

$D: 0 \leq x \leq 1, 0 \leq y \leq x^2$

+) $I = \int_{0}^{1} dx \int_{0}^{x^2} \frac{y}{1+x^5} \, dy$

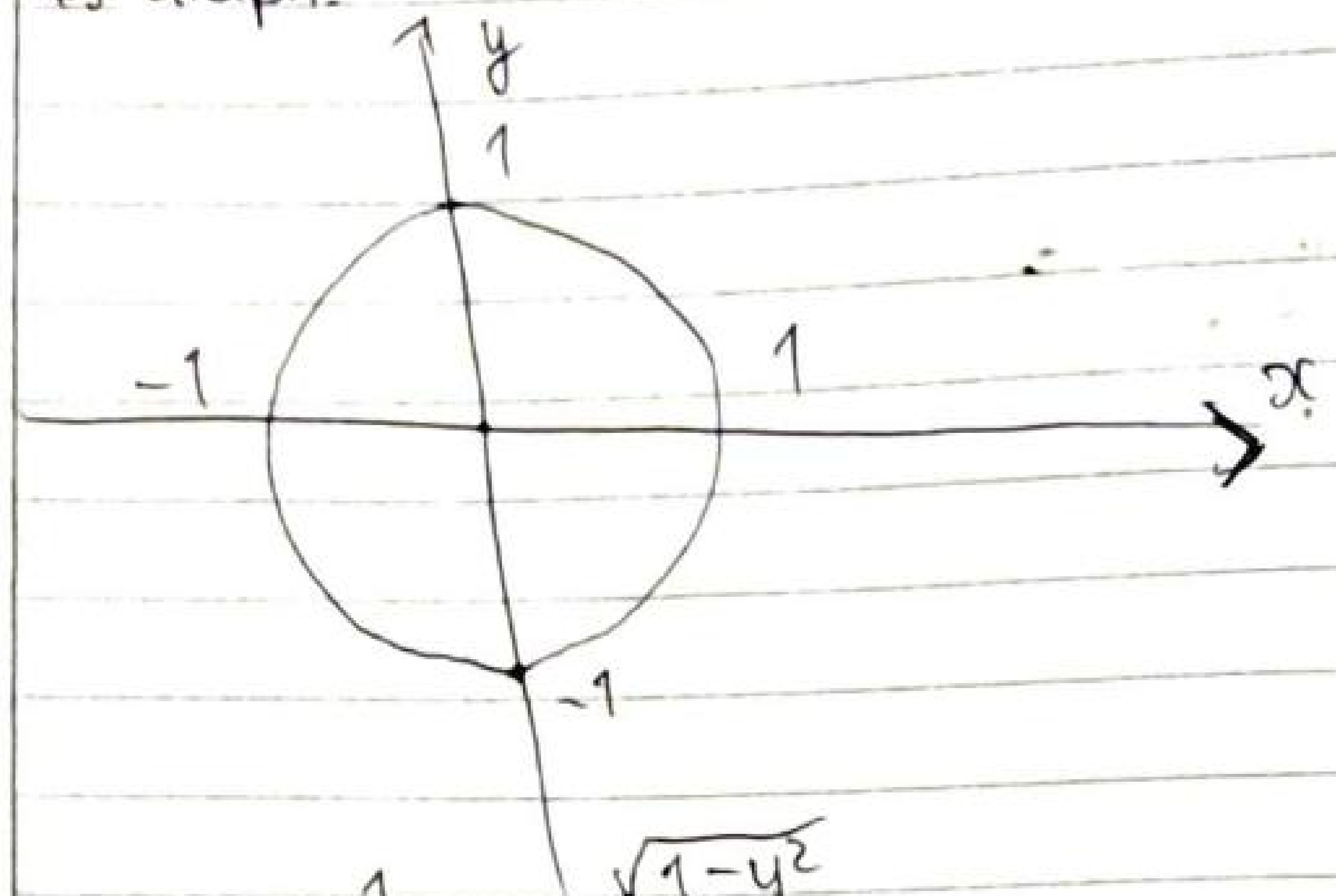
$= \int_{0}^{1} \frac{y^2}{2(x^5+1)} \Big|_{y=0}^{y=x^2} dx$

$= \int_{0}^{1} \frac{x^4}{2(x^5+1)} \, dx$

$= \frac{1}{2} \cdot \ln(x^5+1) \cdot \frac{1}{5} \Big|_0^1 = \frac{1}{10} \ln 2$

55) a) $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dxdy$

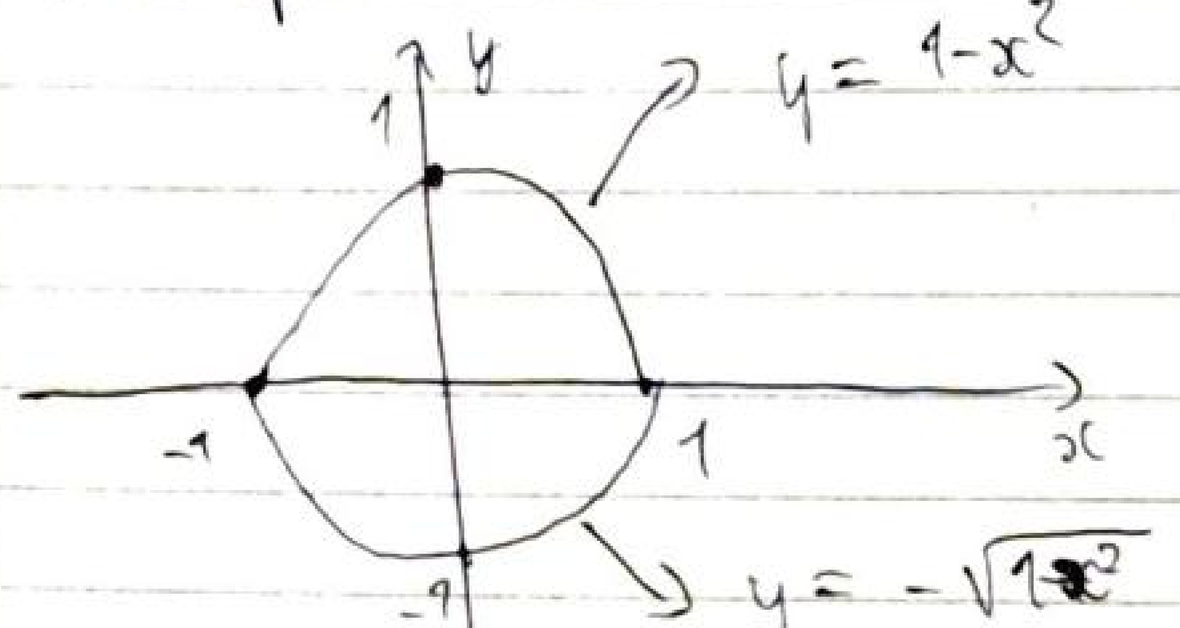
+) Graph:



+) $I = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$

a) $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)dy$

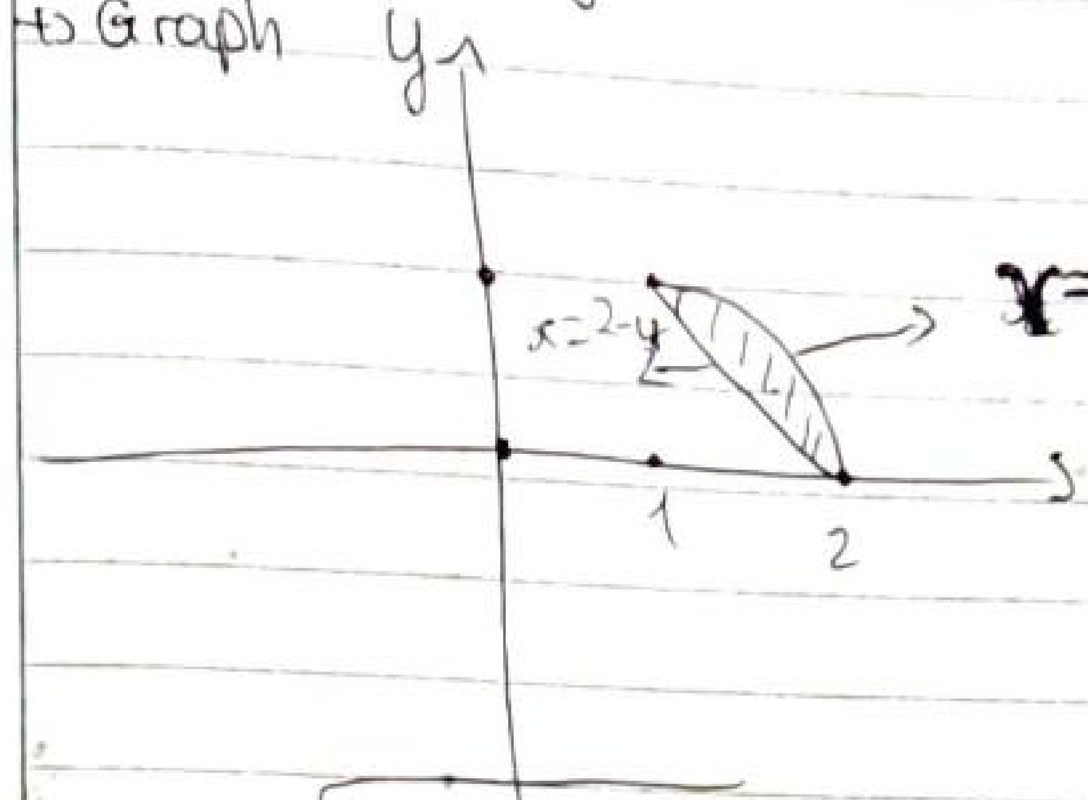
+) Graph:



+) $I = \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx + \int_{0}^{1} dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dy$

b) $\int_{0}^{1} dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx$

+) Graph

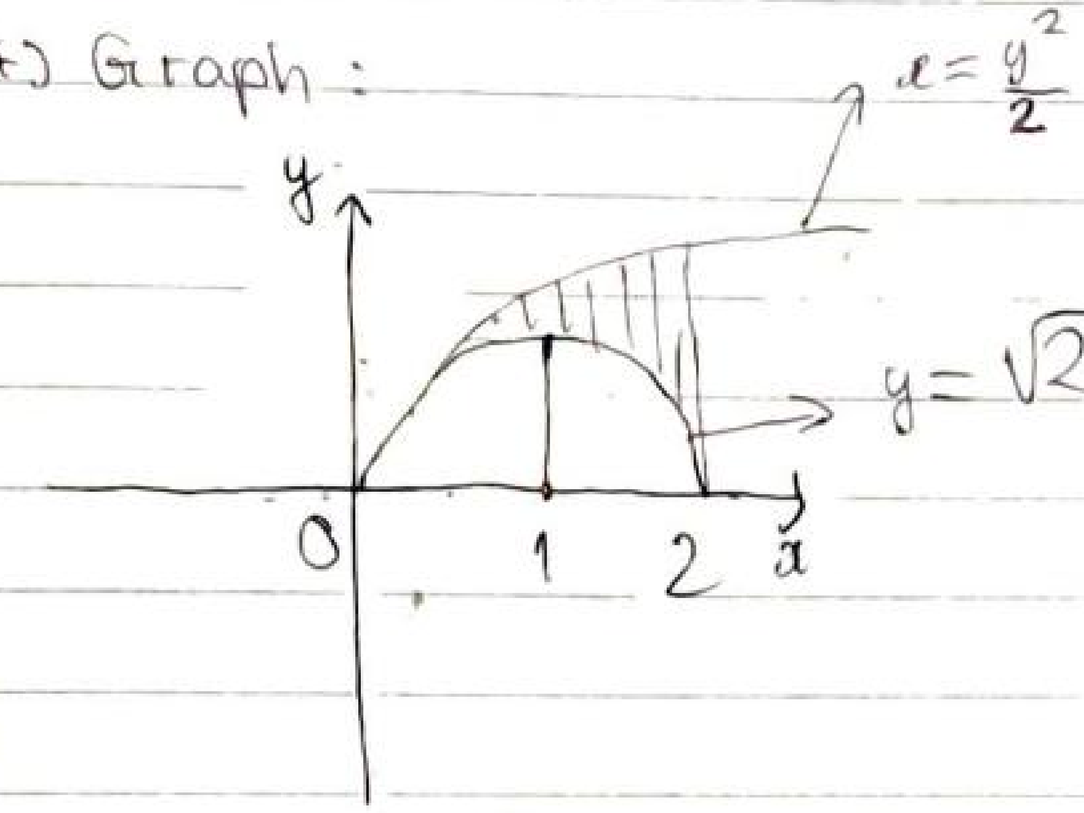


$x = 1+\sqrt{1-y^2} \Rightarrow y = \sqrt{1-(x-1)^2}$

+) $I = \int_{1}^{2} dx \int_{2-x}^{\sqrt{1-(-1+x)^2}} f(x,y)\,dy$

c) $\int_{0}^{2} dx \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} f(x,y)\,dy$

+) Graph:



$y = \sqrt{2x-x^2} \Rightarrow y^2 = 1-(1-x)^2$

$\Rightarrow 1-y^2 = (x-1)^2$

$x \leq 1: \sqrt{1-y^2} = 1-x$

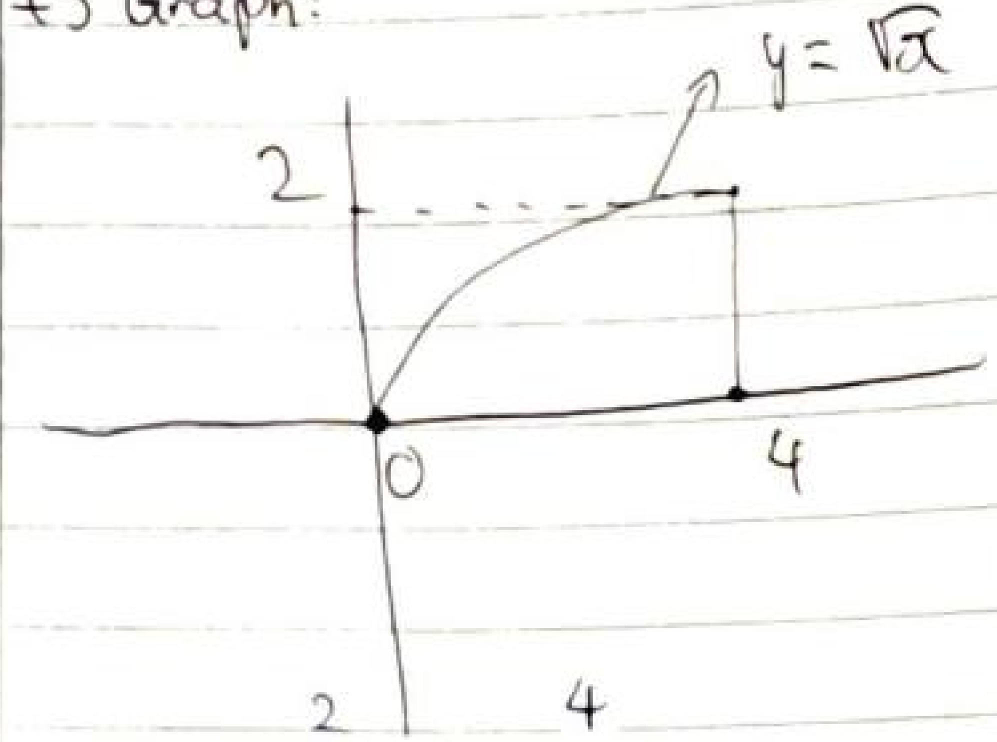
$\Rightarrow x = 1-\sqrt{1-y^2}$

$x \geq 1: x = 1+\sqrt{1-y^2}$

+) $I = \int_{0}^{1} dy \int_{\frac{y^2}{2}}^{1-\sqrt{1-y^2}} f(x,y)dx + \int_{0}^{1} dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)dx + \int_{1}^{2} dy \int_{\frac{y^2}{2}}^{2} f(x,y)dx$

d) $\int_0^4 dx \int_0^{\sqrt{x}} f(x,y)dy$

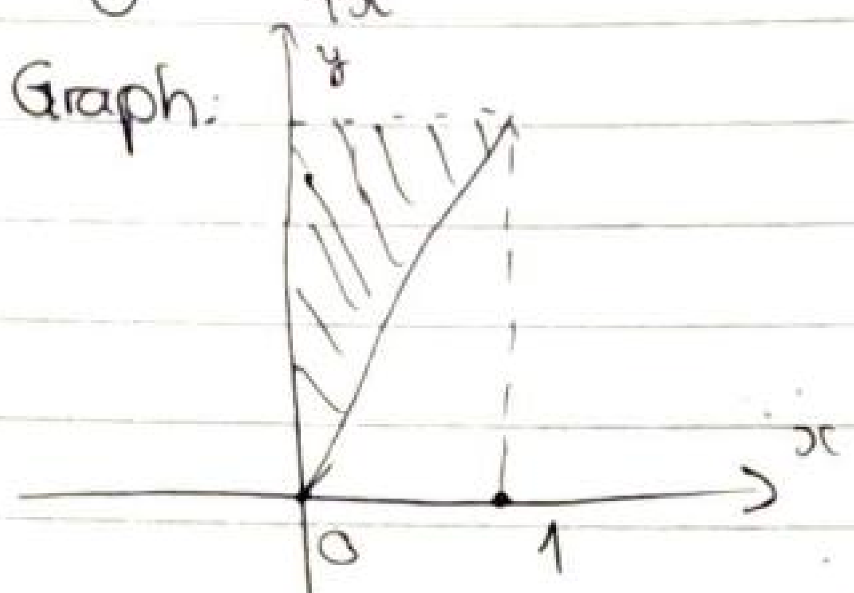
+) Graph:



+) $I = \int_0^2 dy \int_{y^2}^4 f(x,y)dx$

e) $\int_0^1 dx \int_{4x}^4 f(x,y)dy$

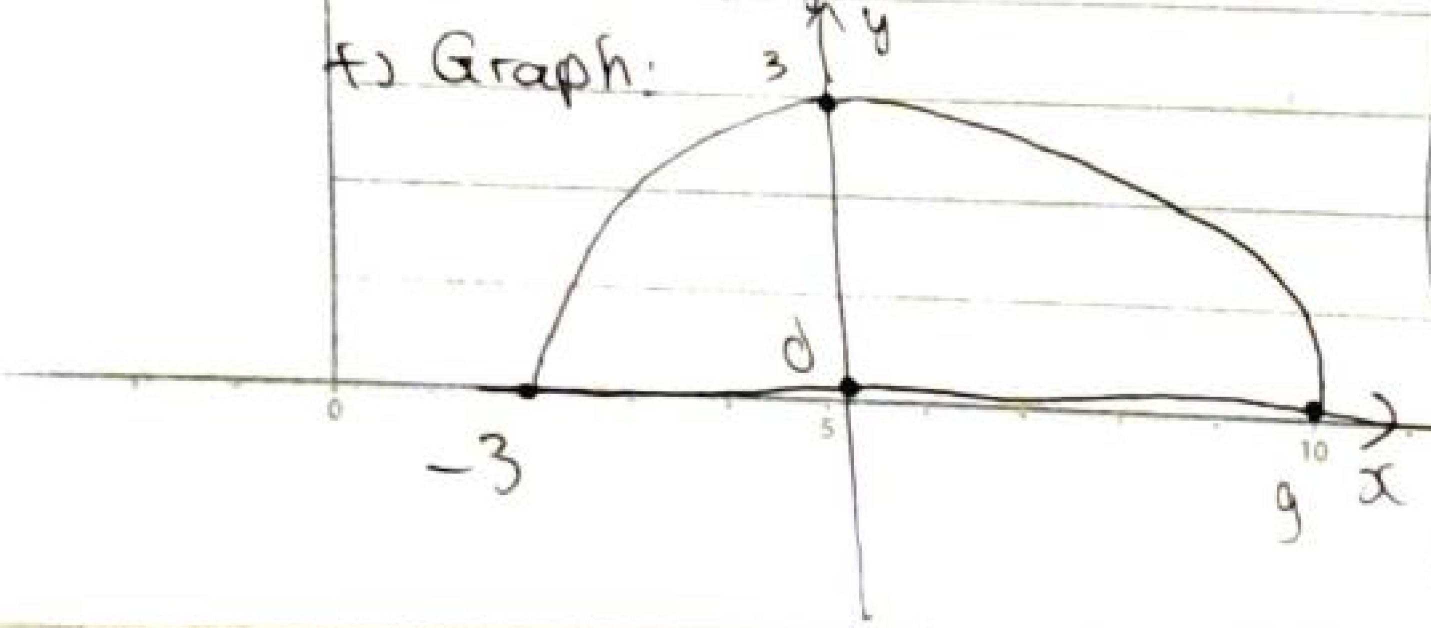
+) Graph:



+) $I = \int_0^4 dy \int_0^{y/4} f(x,y)dx$

f) $\int_0^3 dy \int_{-\sqrt{9-y^2}}^{9-y^2} f(x,y)\,dx$

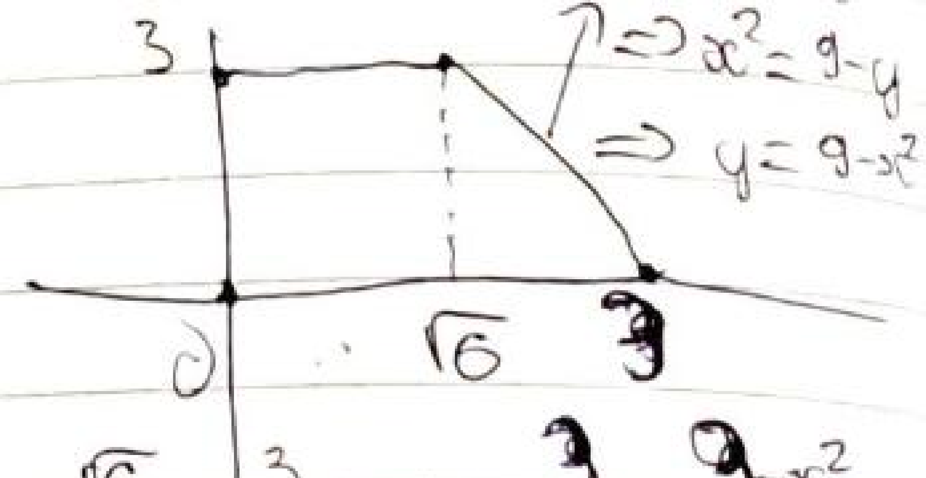
+) Graph:



+) $I = \int_{-3}^0 dx \int_0^{\sqrt{9-x^2}} f(x,y)dy + \int_0^9 dx \int_0^{\sqrt{9-x}} f(x,y)\,dy$

g) $I = \int_0^3 dy \int_0^{\sqrt{9-y}} f(x,y)dx$

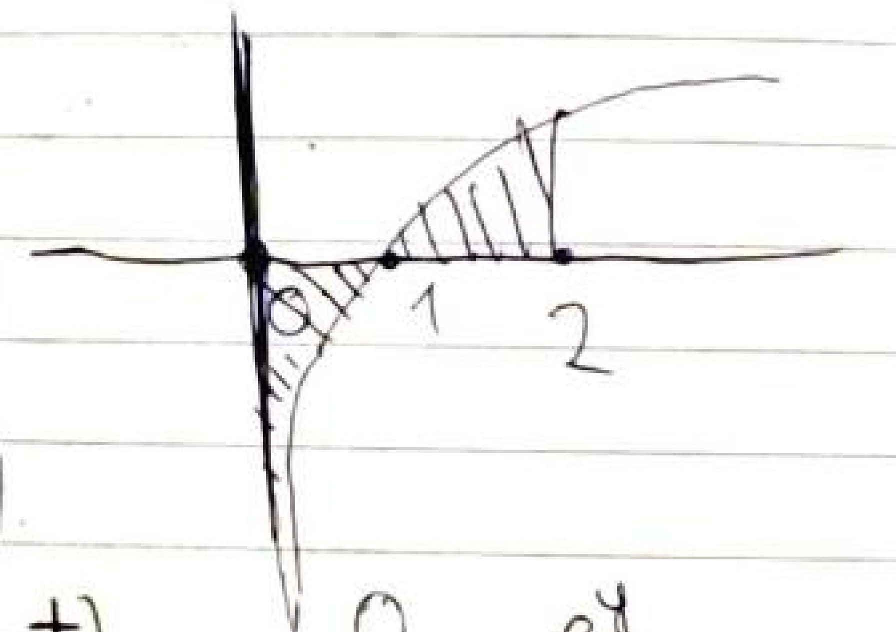
+) Graph:



$I = \int_0^{\sqrt{6}} dx \int_0^{3} dy + \int_{\sqrt{6}}^{3} dx \int_0^{9-x^2} f(x,y)dy$

h) $\int_0^2 dx \int_0^{\ln x} f(x,y)\,dy$

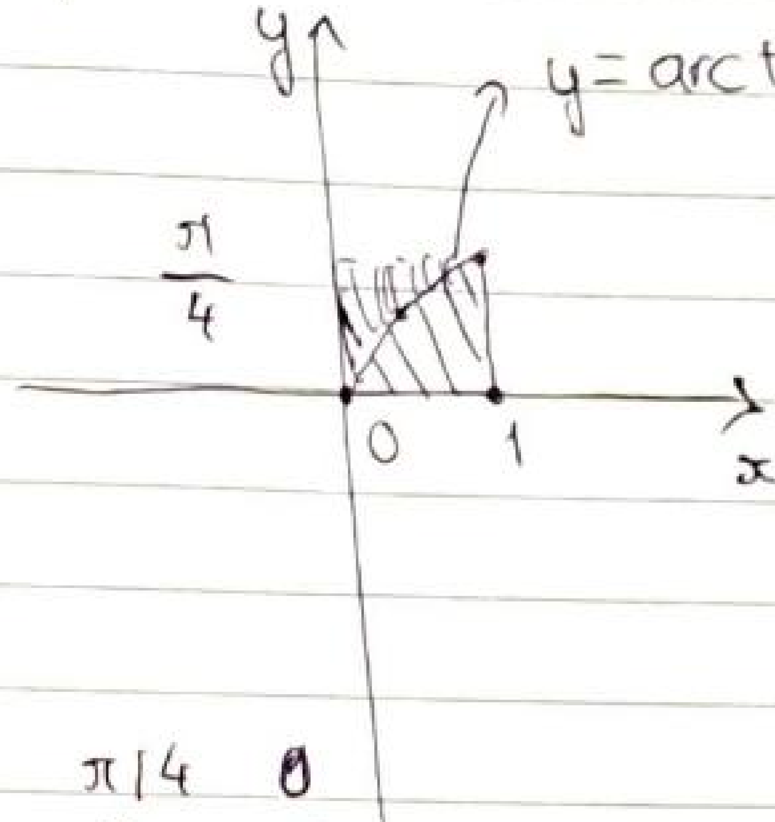
+) Graph:



+) $I = -\int_{-\infty}^{0} dy \int_{0}^{e^y} f(x,y)\,dx + \int_0^{\ln(2)} dy \int_{e^y} f(x,y)\,dx$

i) $I = \int_0^1 dx \int_{\arctan(x)}^{\pi/4} f(x,y)\,dy$

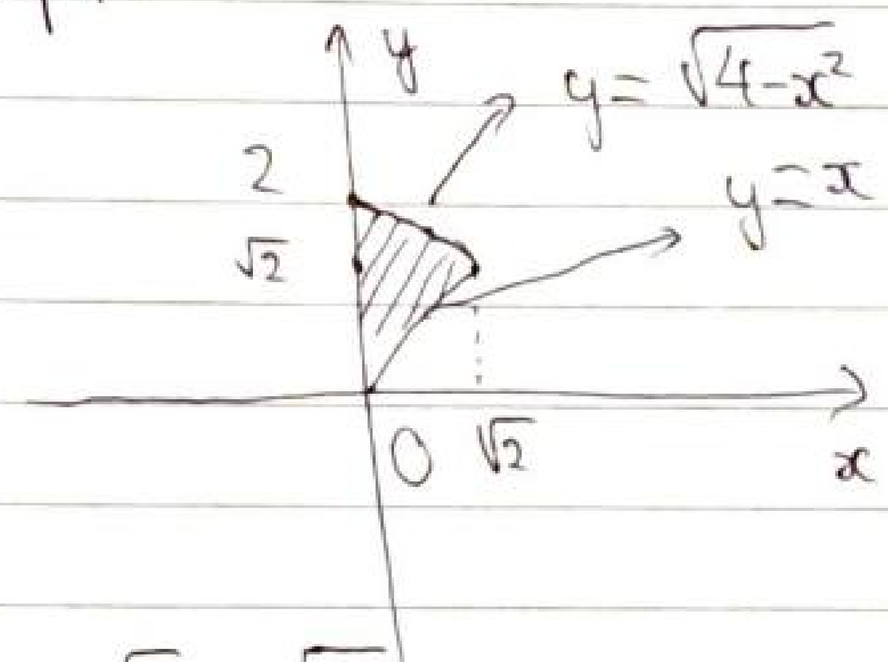
• Graph:



• $I = -\int_0^{\pi/4} dy \int_{\tan y}^{0} f(x,y)\,dx$

j) $I = \int_0^{\sqrt{2}} dy \int_0^{y} f(x,y)\,dx + \int_{\sqrt{2}}^{2} dy \int_0^{\sqrt{4-y^2}} f(x,y)\,dx$

• Graph:



• $I = \int_0^{\sqrt{2}} dx \int_x^{\sqrt{4-x^2}} f(x,y)\,dy$

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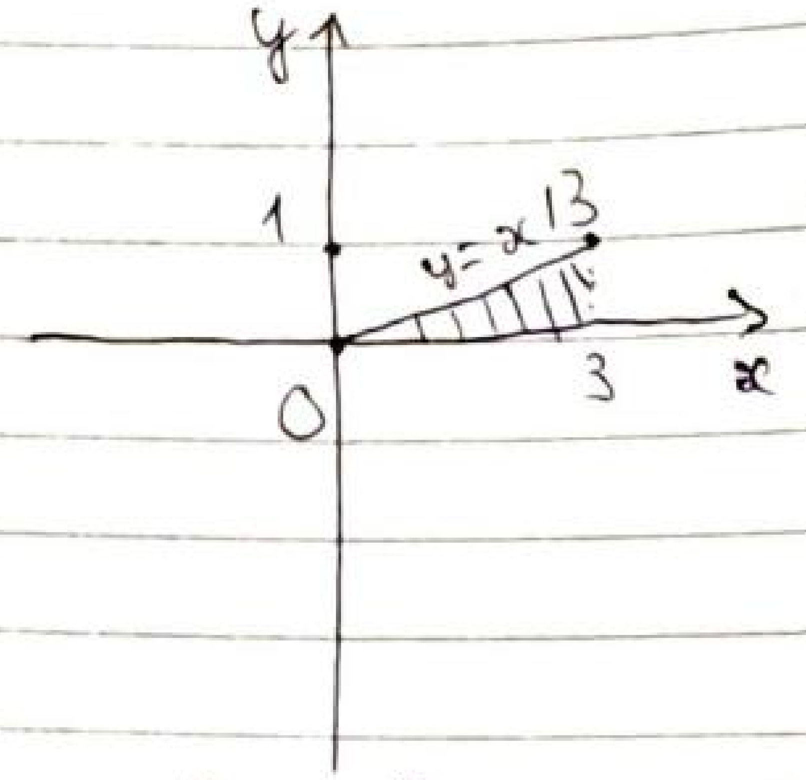
(56)

a) $\int_0^1 dy \int_{3y}^3 e^{x^2} dx$

• Graph:

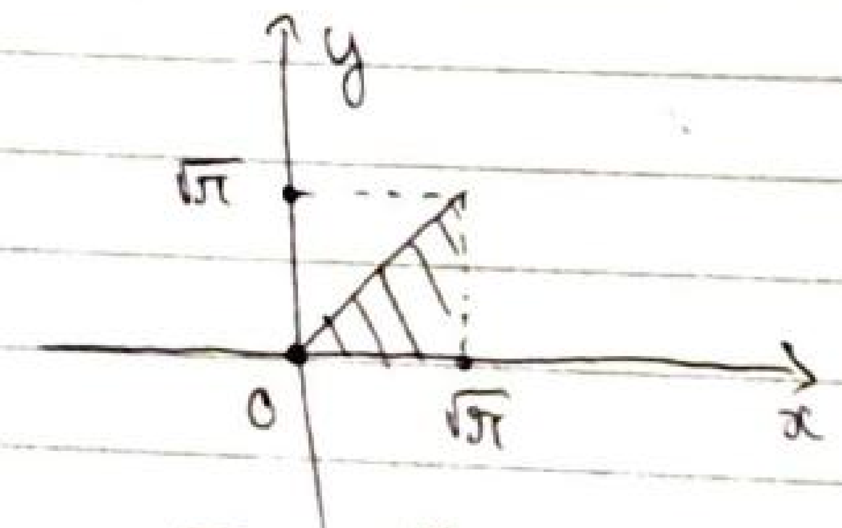


• $I = \int_0^3 dx \int_0^{x/3} e^{x^2} dy = \int_0^3 \frac{x}{3} e^{x^2} dx = \int_0^3 \frac{1}{6} e^{x^2} dx^2 = \frac{1}{6} e^{x^2} \Big|_0^3$

$= \frac{1}{6}(e^9 - 1)$

b) $\int_0^{\sqrt{\pi}} dy \int_y^{\sqrt{\pi}} \cos(x^2) dx$

• Graph:



• $I = \int_0^{\sqrt{\pi}} dx \int_0^x \cos(x^2) dy = \int_0^{\sqrt{\pi}} x\cos(x^2) dx = \frac{1}{2} \int_0^{\sqrt{\pi}} \cos(x^2) dx^2$

$= \frac{1}{2} \sin(x^2) \Big|_0^{\sqrt{\pi}} = 0$

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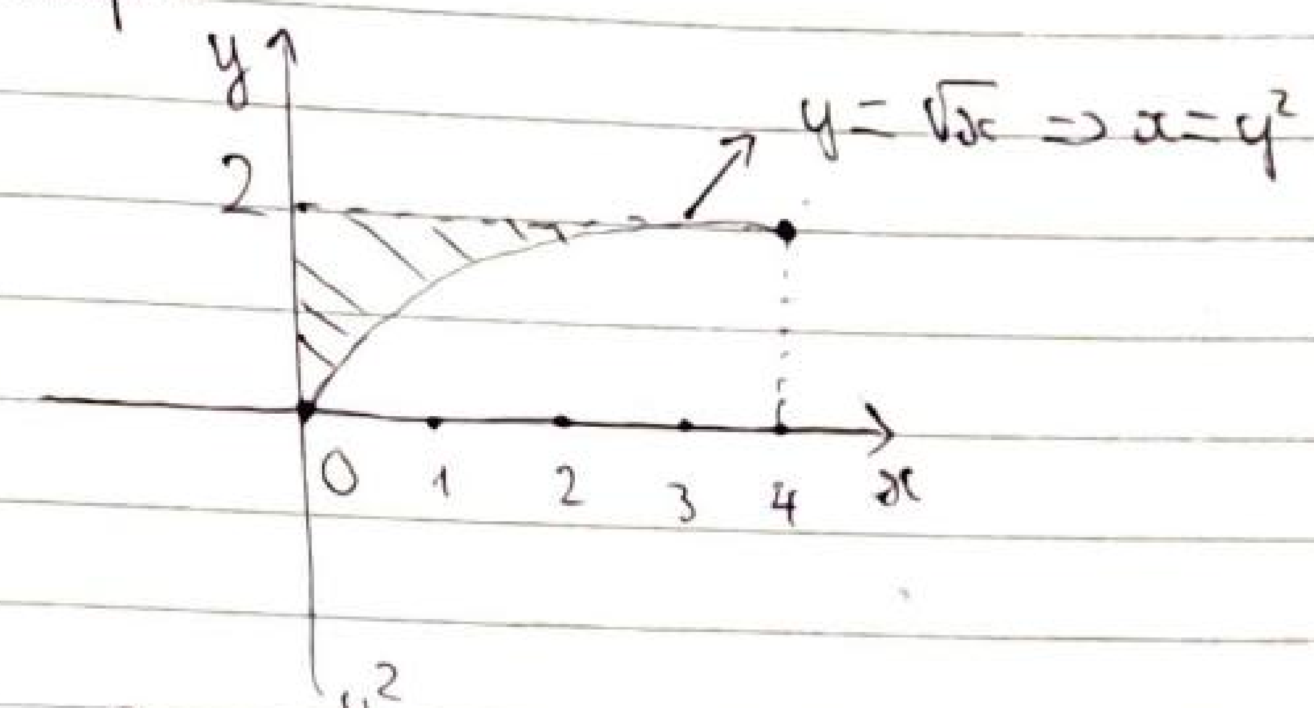
c) $\int_0^4 dx \int_{\sqrt{x}}^2 \frac{1}{y^3+1} dy$

• Graph:

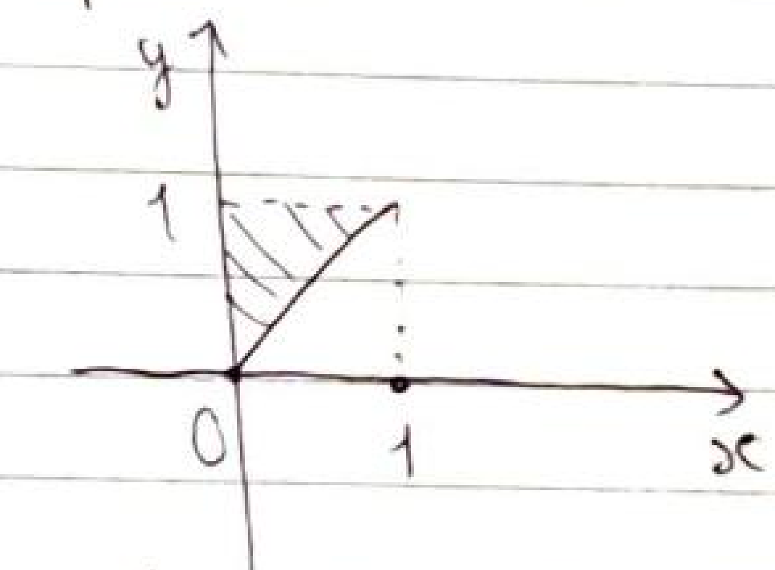


• $I = \int_0^2 dy \int_0^{y^2} \frac{1}{y^3+1} dx$

$= \int_0^2 \frac{y^2}{y^3+1} dy = \frac{1}{3} \ln(y^3+1) \Big|_0^2 = \frac{1}{3} \ln(9) = \frac{2}{3} \ln(3)$

d) $\int_0^1 dx \int_x^1 e^{x/y} dy$

• Graph:



• $I = \int_0^1 dy \int_0^y e^{x/y} dx = \int_0^1 y(e-1) dy = \frac{e-1}{2}$

e) $\int_0^1 dy \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx$

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$$+)\ I = \iint_{D_{uv}} \frac{(u^2v)^{2/3}\sin(uv)}{(uv^2)^{1/3}} \cdot \frac{1}{3}\ du\,dv$$

$$= \iint_{D_{uv}} \frac{1}{3} u \sin(uv)\,.$$

$$= \frac{1}{3}\int_a^b du \int_p^q u\sin(uv)\,dv$$

$$= \frac{1}{3}\int_a^b -\cos(uv)\Big|_{v=p}^{v=q} du$$

$$= \frac{1}{3}\left(-\sin(qu) + \sin(pu)\Big|_{u=a}^{u=b}\right)$$

$$= \frac{1}{3}(\sin(pb) - \sin(pa) - \sin(qb) + \sin(qa))$$

(59) $I = \iint_D xy\,dx\,dy$, D is bounded by $y = ax^3$, $y = bx^3$, $y^2 = px$, $y^2 = qx$ $(0 < b < a), (0 < p < q)$

+) Put $u = \frac{x^3}{y}$, $v = \frac{y^2}{x}$

- $D_{uv}: \begin{cases} 1/a \leq u \leq 1/b \\ p \leq v \leq q \end{cases}$
- $x = (u^2 v)^{1/5} = u^{2/5} v^{1/5}$

  $y = (uv^3)^{1/5} = u^{1/5} v^{3/5}$
- $J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{2}{5}u^{-3/5}v^{1/5} & \frac{1}{5}u^{2/5}v^{-4/5} \\ \frac{1}{5}u^{-4/5}v^{3/5} & \frac{3}{5}u^{1/5}v^{-2/5} \end{vmatrix}$

$$= \frac{6}{25}u^{-2/5}v^{-1/5} - \frac{1}{25}u^{-2/5}v^{-1/5} = \frac{1}{5}u^{-2/5}v^{-1/5}$$

$$+)\ I = \iint_{D_{uv}} u^{3/5}v^{4/5}\frac{1}{5}u^{-2/5}v^{-1/5}\,du\,dv = \frac{1}{5}\int_{1/a}^{1/b} du \int_p^q u^{1/5}v^{3/5}\,dv$$

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$$= \frac{1}{5}\cdot\frac{5}{6}\left(\frac{1}{b^{6/5}} - \frac{1}{a^{6/5}}\right)\cdot\frac{5}{8}\left(q^{8/5} - p^{8/5}\right)$$

$$= \frac{5}{48}\left(b^{-6/5} - a^{-6/5}\right)\left(q^{8/5} - p^{8/5}\right)$$

(60) $\int_0^1 dx \int_0^{1-x} e^{y/(x+y)}\,dy = \frac{e-1}{2}$?

+) Put $u = x+y$, $v = y$

- $D_{uv}: v \leq u \leq 1$

  $0 \leq v \leq 1$
- $J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

$$+)\ I = \int_0^1 dv \int_v^1 e^{\frac{v}{u}}\,du = \int_0^1 du \int_0^u e^{v/u}\,dv$$

$$= \int_0^1 u(e-1)\,du = \frac{e-1}{2}$$

(61) Find area bounded by $xy = 4$, $xy = 8$, $xy^3 = 5$, $xy^3 = 15$

+) Put $u = xy$

  $v = xy^3$

- $D_{uv}: \begin{cases} 4 \leq u \leq 8 \\ 5 \leq v \leq 15 \end{cases}$
- $J^{-1} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} y & x \\ y^3 & 3xy^2 \end{vmatrix} = 2xy^3 = 2v$

  $\Rightarrow J = \frac{1}{2v}$

$$+)\ A = \iint_{D_{xy}} dx\,dy = \iint_{D_{uv}} \frac{1}{2v}\,du\,dv = \int_4^8 du \int_5^{15} \frac{1}{2v}\,dv$$

$$= \frac{1}{2}(8-4)(\ln(15) - \ln(5))$$

$$= 2(\ln(15) - \ln(5)) = 2\ln 3$$

KOKUYO

62) Area of domain bounded by: $y^2=x$, $y^2=8x$, $x^2=y$, $x^2=8y$

+) Put $u=\frac{y^2}{x}$, $v=\frac{x^2}{y}$

• $D_{uv}: \begin{cases} 1\le u\le 8 \\ 1\le v\le 8 \end{cases}$

• $J^{-1}=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix}=-3$

$\Rightarrow J=\frac{-1}{3}$

+) $A=\iint_{D_{xy}} dxdy=\iint_{D_{uv}}\left|\frac{-1}{3}\right| dudv=\left|\frac{-1}{3}\right|(8-1)(8-1)=\frac{49}{3}$

63) $y=x^3$, $y=4x^3$, $x=y^3$, $x=4y^3$

+) Put $u=\frac{y}{x^3}$, $v=\frac{x}{y^3}$

• $D_{uv}\begin{cases} 1\le u\le 4 \\ 1\le v\le 4 \end{cases}$

• $J^{-1}=\frac{D(u,v)}{D(x,y)}=\begin{vmatrix} \frac{-3y}{x^4} & \frac{1}{x^3} \\ \frac{1}{y^3} & \frac{-3x}{y^4} \end{vmatrix}=\frac{8}{x^3y^3}$

$\Rightarrow J=\frac{x^3y^3}{8}=\frac{1}{8}(uv)^{-3/2}$

+) $A=\iint_{D_{xy}} dxdy=\iint_{D_{uv}}\frac{1}{8}u^{-3/2}v^{-3/2}dudv=\frac{1}{8}.(-2).(4^{-1/2}-1^{-1/2})(-2)(4^{-1/2}-1^{-1/2})$

$=\frac{1}{8}$

64) Prove $\iint_{x+y\le 1, x\ge 0, y\ge 0}\cos\left(\frac{x-y}{x+y}\right)dxdy=\frac{\sin 1}{2}$

+) Put $u=x-y$
$v=x+y$

• $D_{uv}: \begin{cases} -v\le u\le v \\ 0\le v\le 1 \end{cases}$

• $J^{-1}=\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}=2 \Rightarrow J=\frac{1}{2}$

+) $I=\iint_{D_{uv}}\frac{1}{2}\cos\left(\frac{u}{v}\right)dudv$

$=\int_0^1 dv\int_{-v}^{v}\frac{1}{2}\cos\left(\frac{u}{v}\right)du$

$=\frac{1}{2}\int_0^1 v\sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v}dv$

$=\frac{1}{2}\int_0^1 v(\sin(1)-\sin(-1))dv$

$=\frac{1}{2}.2\sin(1).\frac{1}{2}=\frac{\sin(1)}{2}$

65) $I=\iint_D\left(\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}\right)dxdy$

D bounded by $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1$

+) Put $u=\sqrt{\frac{x}{a}}$, $v=\sqrt{\frac{y}{b}}$

• $D_{uv}\begin{cases} u\ge 0, v\ge 0 \\ u+v\le 1 \end{cases}$

• $J^{-1}=\begin{vmatrix} \frac{1}{\sqrt{a}}\cdot\frac{1}{2\sqrt{x}} & 0 \\ 0 & \frac{1}{\sqrt{b}}\cdot\frac{1}{2\sqrt{y}} \end{vmatrix}=\frac{1}{4\sqrt{abxy}}$

$\Rightarrow J=4abuv$

+) $I=\int_0^1 du\int_0^{1-u}4abuv(u+v)dv=4ab\int_0^1 u\frac{(1-u)^3}{3}+u^2\frac{(1-u)^2}{2}du$

$=\frac{2ab}{15}$

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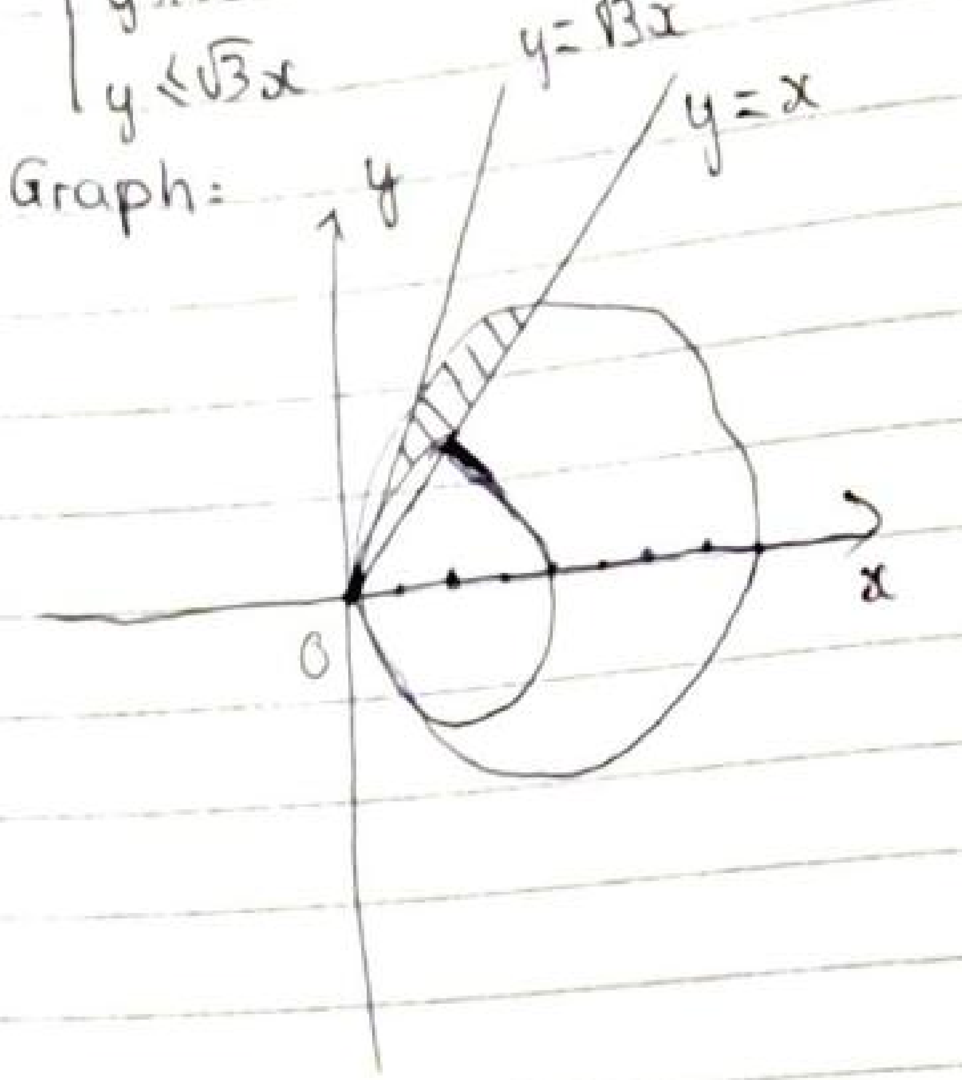
(66) $I = \iint_D f(x,y)\,dx\,dy$

$D: \begin{cases} x^2+y^2 \geq 4x \Rightarrow (x-2)^2+y^2 \geq 4 \\ x^2+y^2 \leq 8x \Rightarrow (x-4)^2+y^2 \leq 16 \\ y \geq x \\ y \leq \sqrt{3}x \end{cases}$

+) Graph:



+) Put $x = r\cos\alpha$
$y = r\sin\alpha$

$\cdot\ D_{r\alpha}: \begin{cases} 0 \leq \alpha \leq 2\pi,\ r \geq 0 \\ r^2 \geq 4r\cos\alpha \Rightarrow r \geq 4\cos\alpha \\ r^2 \leq 8r\cos\alpha \Rightarrow r \leq 8\cos\alpha \\ r\sin\alpha \geq r\cos\alpha \Rightarrow \alpha \geq \frac{\pi}{4} \\ r\sin\alpha \leq \sqrt{3}r\cos\alpha \Rightarrow \alpha \leq \frac{\pi}{3} \end{cases}$

$\cdot\ J = r$

+) $I = \int_{\pi/4}^{\pi/3} d\alpha \int_{4\cos\alpha}^{8\cos\alpha} r f(r\cos\alpha, r\sin\alpha)\,dr$

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(67) $\iint_D xy^2\,dx\,dy$, D bounded by $\begin{cases} x^2+(y-1)^2 = 1 \\ x^2+y^2-4y = 0 \end{cases}$

+) $D \begin{cases} x^2+(y-1)^2 = 1 \\ x^2+(y-2)^2 = 4 \end{cases}$

+) Put $x = r\cos\alpha$
$y = r\sin\alpha$

$\cdot\ J = r$

$\cdot\ D_{r\alpha}: \begin{cases} 0 \leq \alpha \leq \pi \\ 2\sin\alpha \leq r \leq 4\sin\alpha \end{cases}$

+) $I = \int_0^{\pi} d\alpha \int_{2\sin\alpha}^{4\sin\alpha} r^4\cos\alpha\sin^2\alpha\,dr$

$= \int_0^{\pi} \cos\alpha\sin^2\alpha\,[(4\sin\alpha)^5 - (2\sin\alpha)^5]\cdot\frac{1}{5}\,d\alpha$

$= 0$

(68) $\iint_D |x+y|\,dx\,dy$; $D: x^2+y^2 \leq 1$

+) Put $x = r\cos\alpha$
$y = r\sin\alpha$

$\cdot\ J = r$

$\cdot\ D_{r\alpha} \begin{cases} 0 \leq \alpha \leq 2\pi \\ 0 \leq r \leq 1 \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^1 |r(\sin\alpha+\cos\alpha)|\cdot r\,dr$

$= \int_{-\frac{\pi}{4}}^{3\pi/4} d\alpha \int_0^1 r^2(\sin\alpha+\cos\alpha)\,dr + \int_{3\pi/4}^{7\pi/4} d\alpha \int_0^1 -r^2(\sin\alpha+\cos\alpha)\,dr$

$= \frac{1}{3}\cdot 2\sqrt{2} - \frac{1}{3}\cdot(-2\sqrt{2}) = \frac{4\sqrt{2}}{3}$

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b) $\iint |x-y|\,dxdy$
$D: x^2+y^2 \le 1$

$I = \int_0^{2\pi} d\alpha \int_0^1 r\,|r(-\sin\alpha+\cos\alpha)|\,dr$

$= \int_{\pi/4}^{5\pi/4} d\alpha \int_0^1 -r^2(-\sin\alpha+\cos\alpha)\,dr + \int_{5\pi/4}^{9\pi/4} d\alpha \int_0^1 -r^2(\sin\alpha - \cos\alpha)\ldots$

$= \frac{1}{3}2\sqrt{2} + \frac{1}{3}2\sqrt{2} = \frac{4\sqrt{2}}{3}$

69) $\iint_D \frac{dxdy}{(x^2+y^2)^2}$; $D: \begin{cases} 4y \le x^2+y^2 \le 8y \\ x \le y \le x\sqrt{3} \end{cases}$

+) Put $x = r\cos\alpha$
$y = r\sin\alpha$

$.D_{r\alpha}: \begin{cases} 4\sin\alpha \le r \le 8\sin\alpha \\ \frac{\pi}{4} \le \alpha \le \frac{\pi}{3} \end{cases}$ $.\ J = r$

+) $I = \int_{\pi/4}^{\pi/3} d\alpha \int_{4\sin\alpha}^{8\sin\alpha} \frac{1}{r^3}\,dr = \int_{\pi/4}^{\pi/3} \frac{-1}{2}\cdot\left(\frac{1}{(8\sin\alpha)^2} - \frac{1}{(4\sin\alpha)^2}\right) d\alpha$

$= \frac{3}{128}\int_{\pi/4}^{\pi/3} \frac{1}{\sin^2(\alpha)}\,d\alpha$

$= \frac{3}{128}\left(\frac{\sqrt{3}}{3}+1\right) = \frac{3-\sqrt{3}}{128}$

70) $\iint_D \frac{xy}{x^2+y^2}\,dxdy$; $D: \begin{cases} x^2+y^2 \le 12,\ x^2+y^2 \ge 2x \\ x^2+y^2 \ge 2\sqrt{3}y,\ x \ge 0, y \ge 0 \end{cases}$

+) Put $x = r\cos\alpha$ $.\ J = r$
$y = r\sin\alpha$

• $D_{r\alpha}: \begin{cases} r^2 \le 12,\ r^2 \ge 2r\cos\alpha \\ r^2 \ge 2\sqrt{3}\,r\sin\alpha \\ 0 \le \alpha \le \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} r \le 2\sqrt{3} \\ r \ge 2\cos\alpha \\ r \ge 2\sqrt{3}\sin\alpha \\ 0 \le \alpha \le \frac{\pi}{2} \end{cases}$

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. $2\cos\alpha = 2\sqrt{3}\sin\alpha \Leftrightarrow \alpha = \frac{\pi}{6}$

. $D_{r\alpha} = D_1 + D_2$:
$D_1 \to 2\cos\alpha \le r \le 2\sqrt{3},\ 0 \le \alpha \le \pi/6$
$D_2 \to 2\sqrt{3}\sin\alpha \le r \le 2\sqrt{3},\ \pi/6 \le \alpha \le \pi/2$

+) $I = \int_0^{\pi/6} d\alpha \int_{2\cos\alpha}^{2\sqrt{3}} r\sin\alpha\cos\alpha\,dr + \int_{\pi/6}^{\pi/2} d\alpha \int_{2\sqrt{3}\sin\alpha}^{2\sqrt{3}} r\sin\alpha\cos\alpha\,dr$

$= \int_0^{\pi/6} \sin\alpha\cos\alpha \frac{(2\sqrt{3})^2-(2\cos\alpha)^2}{2}\,d\alpha$

$+ \int_{\pi/6}^{\pi/2} \sin\alpha\cos\alpha \frac{(2\sqrt{3})^2-(2\sqrt{3}\sin\alpha)^2}{2}\,d\alpha$

$= \frac{1}{2}\int_0^{\pi/6} \sin\alpha\cos\alpha\,(12-4\cos^2\alpha)\,d\alpha + \frac{1}{2}\int_{\pi/6}^{\pi/2} \sin\alpha\cos\alpha\,(12-12\sin^2\alpha)\,d\alpha$

$= \frac{1}{2}\left(\frac{1}{8}.12 - 4.\frac{7}{64}\right) + \frac{1}{2}\left(\frac{3}{8}.12 - \frac{15}{64}.12\right)$

$= \frac{11}{8}$

71) $\iint_D (x+y)\,dxdy$; $D: \begin{cases} x \le 0 \\ x^2+y^2 \ge 1,\ x^2+y^2 \le 4 \end{cases}$

+) Put $x = r\cos\alpha$
$y = r\sin\alpha$

. $D_{r\alpha} \begin{cases} \pi/2 \le \alpha \le 3\pi/2 \\ 1 \le r \le 2 \end{cases}$

. $J = r$

+) $I = \int_{\pi/2}^{3\pi/2} d\alpha \int_1^2 r^2(\cos\alpha+\sin\alpha)\,dr = -2.\frac{7}{3} = \frac{7}{6}.(-4) = \frac{-14}{3}$

72) $\iint_D \cos(x^2+y^2)\,dxdy$ $D: \begin{cases} y \ge 0 \\ x^2+y^2 \le 9 \end{cases}$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$ $D_{r\alpha} \begin{cases} 0 \le \alpha \le \pi \\ 0 \le r \le 3 \end{cases}$

+) $I = \int_0^{\pi} d\alpha \int_0^3 r\cos(r^2)\,dr = \frac{1}{2}.\pi(\sin 9)$

$I = \iint_D \sqrt{4-x^2-y^2}\,dxdy$, $D: \begin{cases} x \geq 0 \\ x^2+y^2 \leq 4 \end{cases}$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$

. $D r\alpha \begin{cases} 0 \leq r \leq 2 \\ \frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2} \end{cases}$

+) $I = \int_{-\pi/2}^{\pi/2} d\alpha \int_0^2 r\sqrt{4-r^2}\,dr = \pi \cdot \frac{-1}{2} \int_0^2 (4-r^2)^{\frac{1}{2}}\, d\,4-r^2$

$= \frac{-\pi}{2} \cdot \frac{2}{3}\left((4-2^2)^{\frac{3}{2}} - (4-0^2)^{\frac{3}{2}}\right) = \frac{\pi}{3} \cdot 8 = \frac{8\pi}{3}$

(73) $\iint_D y e^x\,dxdy$; $D: \begin{cases} x, y \geq 0 \\ x^2+y^2 \leq 25 \end{cases}$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

. $J = r$

. $D r\alpha \begin{cases} 0 \leq \alpha \leq \pi/2 \\ 0 \leq r \leq 5 \end{cases}$

+) $I = \int_0^{\pi/2} d\alpha \int_0^5 r^2 \sin\alpha\, e^{r\cos\alpha}\,dr$

$= \int_0^5 dr \int_0^{\pi/2} r^2 \sin\alpha\, e^{r\cos\alpha}\,d\alpha$

$= \int_0^5 -r e^{r\cos\alpha} \Big|_{\alpha=0}^{\alpha=\pi/2} dr$

$= \int_0^5 re^r - r\,dr$

$= re^r - e^r - \frac{r^2}{2} \Big|_0^5 = 5e^5 - e^5 - \frac{25}{2} + 1 = 4e^5 - \frac{23}{2}$

(74) $\iint_D \arctan\frac{y}{x}\,dxdy$; $D: \begin{cases} 1 \leq x^2+y^2 \leq 4 \\ 0 \leq y \leq x \end{cases}$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

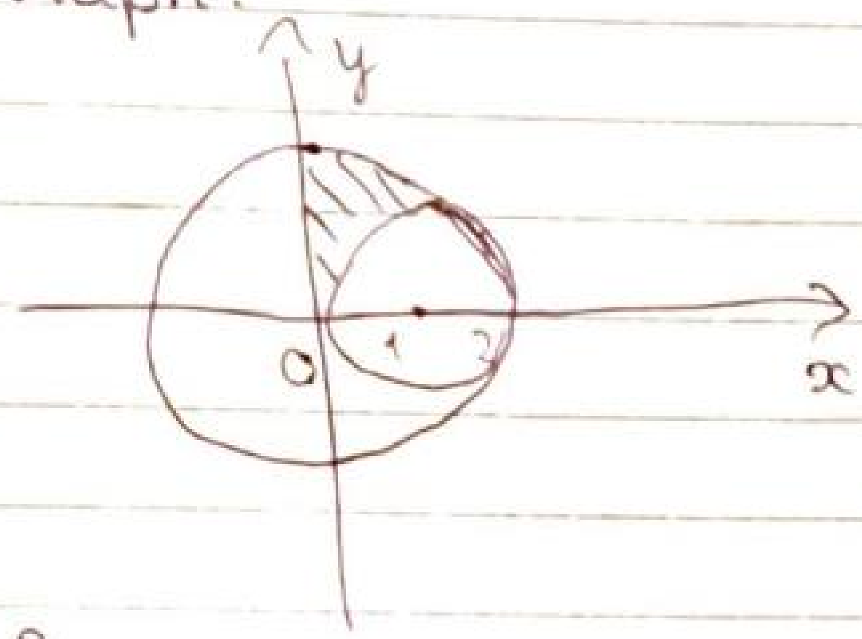
. $J = r$

. $D r\alpha \begin{cases} 1 \leq r \leq 2 \\ 0 \leq \alpha \leq \frac{\pi}{4} \end{cases}$

+) $I = \int_0^{\pi/4} d\alpha \int_1^2 \alpha \cdot r\,dr = \frac{1}{2} \frac{\pi^2}{16} \cdot \frac{1}{2}(2^2 - 1^2) = \frac{3\pi^2}{64}$

(75) $\iint_D x\,dxdy$; $D \begin{cases} x^2+y^2 \leq 4 \\ x^2+y^2 \geq 2x \\ x \geq 0, y \geq 0 \end{cases}$

+) Graph:



+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

. $D r\alpha \begin{cases} 0 \leq \alpha \leq \pi/2 \\ 2\cos\alpha \leq r \leq 2 \end{cases}$

. $J = r$

+) $I = \int_0^{\pi/2} d\alpha \int_{2\cos\alpha}^2 r^2\cos\alpha\,dr$

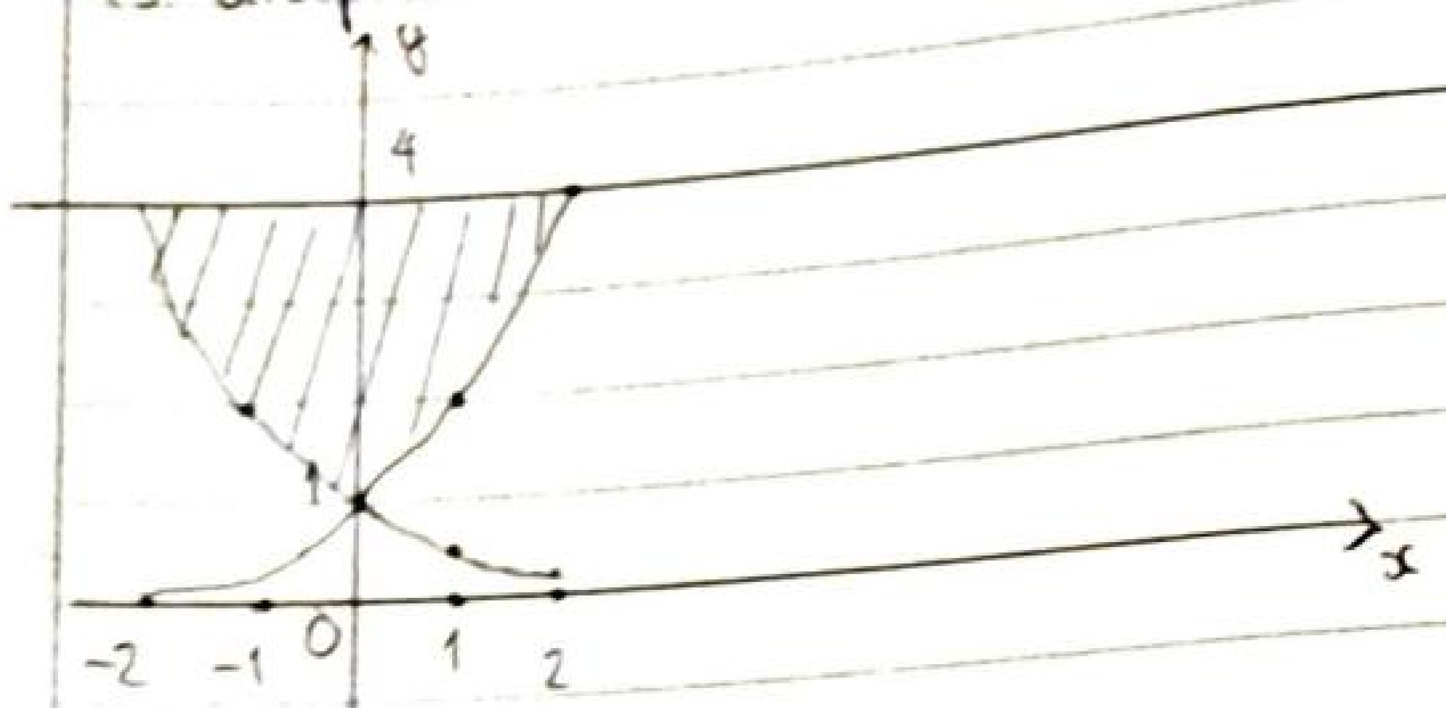
$= \int_0^{\pi/2} \cos\alpha \cdot \frac{1}{3} \cdot (8 - 8\cos^3\alpha)\,d\alpha$

$= \frac{8}{3} - \frac{\pi}{2}$

(7.6) Compute the area of the domain D bounded by

a) $\begin{cases} y=2^{x}, y=2^{-x} \\ y=4 \end{cases}$

+) Graph:



+) $A = \int_1^4 dy \int_{-\log_2 y}^{\log_2 y} dx = \int_1^4 2\log_2 y \, dy$

+) $\int 2\log_2 y \, dy = \frac{2}{\ln 2} \int \ln y \, dy = \frac{2}{\ln 2}\left(y\ln y - \int 1 dy\right)$

$= \frac{2}{\ln 2}(y\ln y - y) + C$

+) $A = \frac{2}{\ln 2}(y\ln y - y) \Big|_{y=1}^{y=4}$

$= \frac{2}{\ln 2}(4\ln 4 - 4) - \frac{2}{\ln 2}(-1)$

$= \frac{2}{\ln 2}(8\ln 2 - 4 + 1)$

$= 16 - \frac{6}{\ln 2}$

b) $\begin{cases} y^2=x, y^2=2x \\ x^2=y, x^2=2y \end{cases}$

+) Put $u=\frac{y^2}{x}$, $v=\frac{x^2}{y}$

• $D_{uv} \begin{cases} 1\le u\le 2 \\ 1\le v\le 2 \end{cases}$

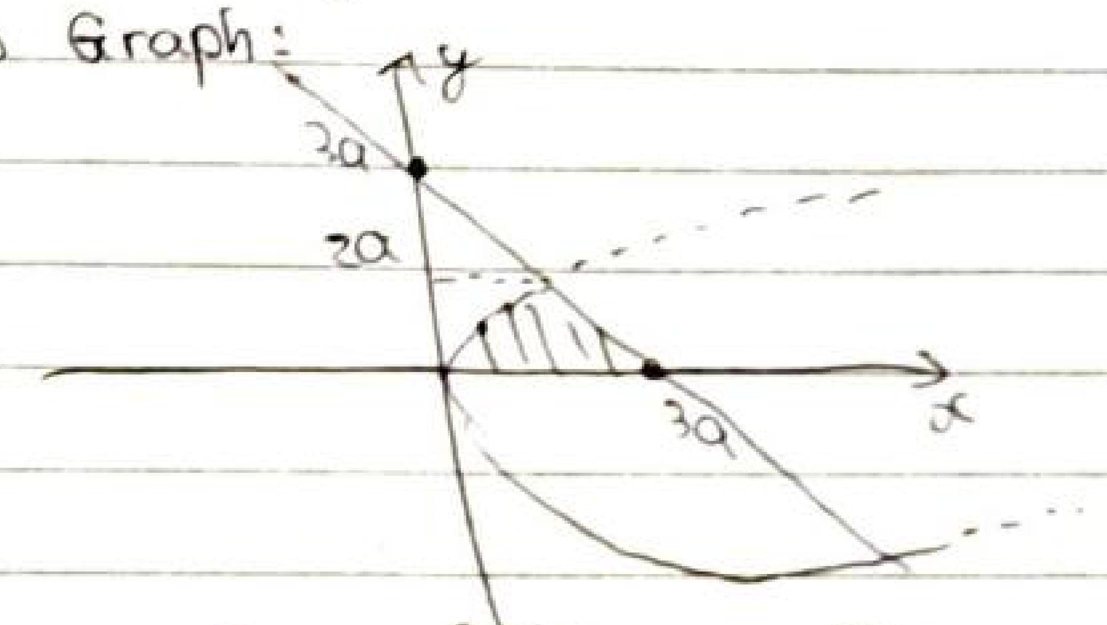
• $J^{-1} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} \frac{-y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & \frac{-x^2}{y^2} \end{vmatrix} = 1-4 = -3$

$\Rightarrow J = \frac{-1}{3}$

+) $A = \iint_{D_{xy}} dxdy = \iint_{D_{uv}} \left|\frac{-1}{3}\right| dudv = \frac{1}{3}\int_1^2 du \int_1^2 dv = \frac{1}{3}$

c) $\begin{cases} y=0, y^2=4ax \\ x+y=3a \end{cases}$

+) Graph:



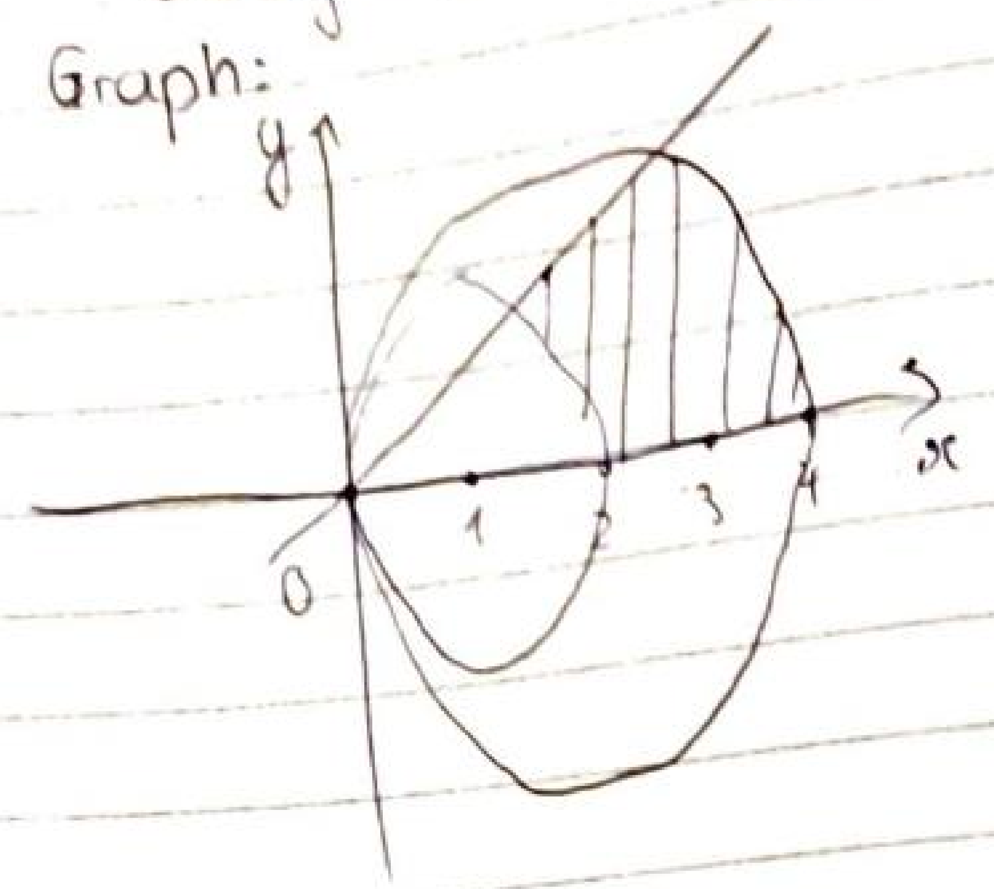
+) $A = -\int_0^{2a} dy \int_{3a-y}^{y^2/4a} dx = -\int_0^{2a} \frac{y^2}{4a} - 3a + y \, dy$

$= -\frac{y^3}{12a} + 3ay - \frac{y^2}{2} \Big|_0^{2a}$

$= -\frac{8a^3}{12a} + 3a.2a - \frac{4a^2}{2} = \frac{10}{3}a^2$

d) $\begin{cases} x^2+y^2=2x, & x^2+y^2=4x \\ x=y, & y=0 \end{cases}$

+) Graph:



+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

. $D_{r\alpha} \begin{cases} 0 \le \alpha \le \pi/4 \\ 2\cos\alpha \le r \le 4\cos\alpha \end{cases}$

. $J = r$

+) $A = \iint\limits_{D_{xy}} dxdy = \iint\limits_{D_{r\alpha}} r\,dr\,d\alpha = \int_0^{\pi/4} d\alpha \int_{2\cos\alpha}^{4\cos\alpha} r\,dr$

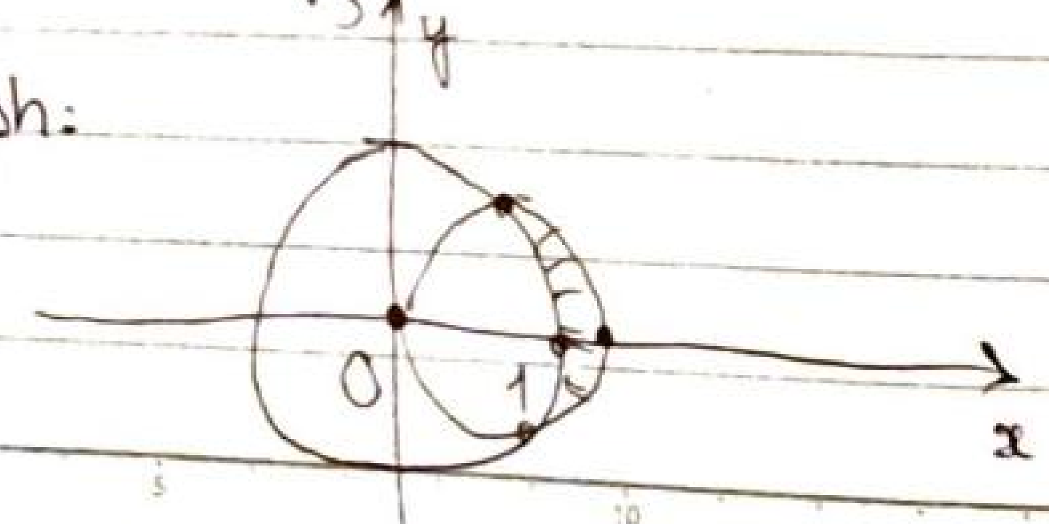
$= \int_0^{\pi/4} \frac{1}{2}(16\cos^2\alpha - 4\cos^2\alpha)\,d\alpha$

$= 6\int_0^{\pi/4} \cos^2\alpha\,d\alpha = 6\int_0^{\pi/4} \frac{\cos 2\alpha + 1}{2}\,d\alpha$

$= 6\,\frac{1}{2}\cdot\frac{1}{2} + 6\,\frac{1}{2}\cdot\frac{\pi}{4} = \frac{3}{2} + \frac{3\pi}{4}$

e) $r=1,\ r = \frac{2}{\sqrt{3}}\cos\varphi$

+) Graph:



+) Intersection of $r=1$ and $r = \frac{2}{\sqrt{3}}\cos\varphi$:

$\varphi = \frac{\pi}{6},\ \varphi = \frac{-\pi}{6}$

+) $A = \int_{-\pi/6}^{\pi/6} d\alpha \int_1^{2/\sqrt{3}\cos\alpha} r\,dr$

$= \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{4}{3}\cos^2\alpha - 1\,d\alpha = \frac{1}{2}\int_{-\pi/6}^{\pi/6} \frac{4}{3}\cdot\frac{\cos 2\alpha + 1}{2} - 1\,d\alpha$

$= \frac{1}{2}\left(\frac{4}{3}\cdot\frac{1}{2}\cdot\sin 2\alpha\cdot\frac{1}{2} + \frac{-1}{3}\alpha\right)\Big|_{-\pi/6}^{\pi/6}$

$= \frac{1}{2}\left(\frac{2\sqrt{3}}{3}\cdot\frac{1}{2} - \frac{1}{9}\pi\right) = \frac{\sqrt{3}}{6} - \frac{1}{18}\pi$

f) $(x^2+y^2)^2 = 2a^2xy$ $(a>0)$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

. $J = r$

. $r^4 = 2a^2r^2\sin\alpha\cos\alpha$

$\Rightarrow r^2 = 2a^2\sin\alpha\cos\alpha$

$\Rightarrow \begin{cases} \sin\alpha\cos\alpha \ge 0 \\ r = \sqrt{2}\,a\sqrt{\sin\alpha\cos\alpha} \end{cases}$

$\Rightarrow \begin{cases} \alpha \in [0, \pi/2] \text{ or } \alpha \in [\pi, 3\pi/2] \\ r = a\sqrt{\sin 2\alpha} \end{cases}$

+) $A = \iint\limits_{D_{xy}} dxdy = \iint\limits_{D_{r\alpha}} r\,d\alpha\,dr = \int_0^{\pi/2} d\alpha \int_0^{a\sqrt{\sin 2\alpha}} r\,dr + \int_\pi^{3\pi/2} d\alpha \int_0^{a\sqrt{\sin 2\alpha}} r\,dr$

$= \int_0^{\pi/2} \frac{a^2\sin 2\alpha}{2}\,d\alpha + \int_\pi^{3\pi/2} \frac{a^2\sin 2\alpha}{2}\,d\alpha$

$= \frac{a^2}{2} + \frac{a^2}{2} = a^2$

g) $x^3 + y^3 = axy$ $(a>0)$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

- $J = r$
- $r^3(\sin^3\alpha + \cos^3\alpha) = ar^2 \sin\alpha\cos\alpha$

$\Rightarrow r = \dfrac{a\sin\alpha\cos\alpha}{\sin^3\alpha+\cos^3\alpha}$ $a\sin\alpha\cos\alpha/(\sin^3\alpha+\cos^3\alpha)$

+) $A = \iint_{D_{xy}} dx\,dy = \int_0^{\pi/2} d\alpha \int_0 r\,dr$

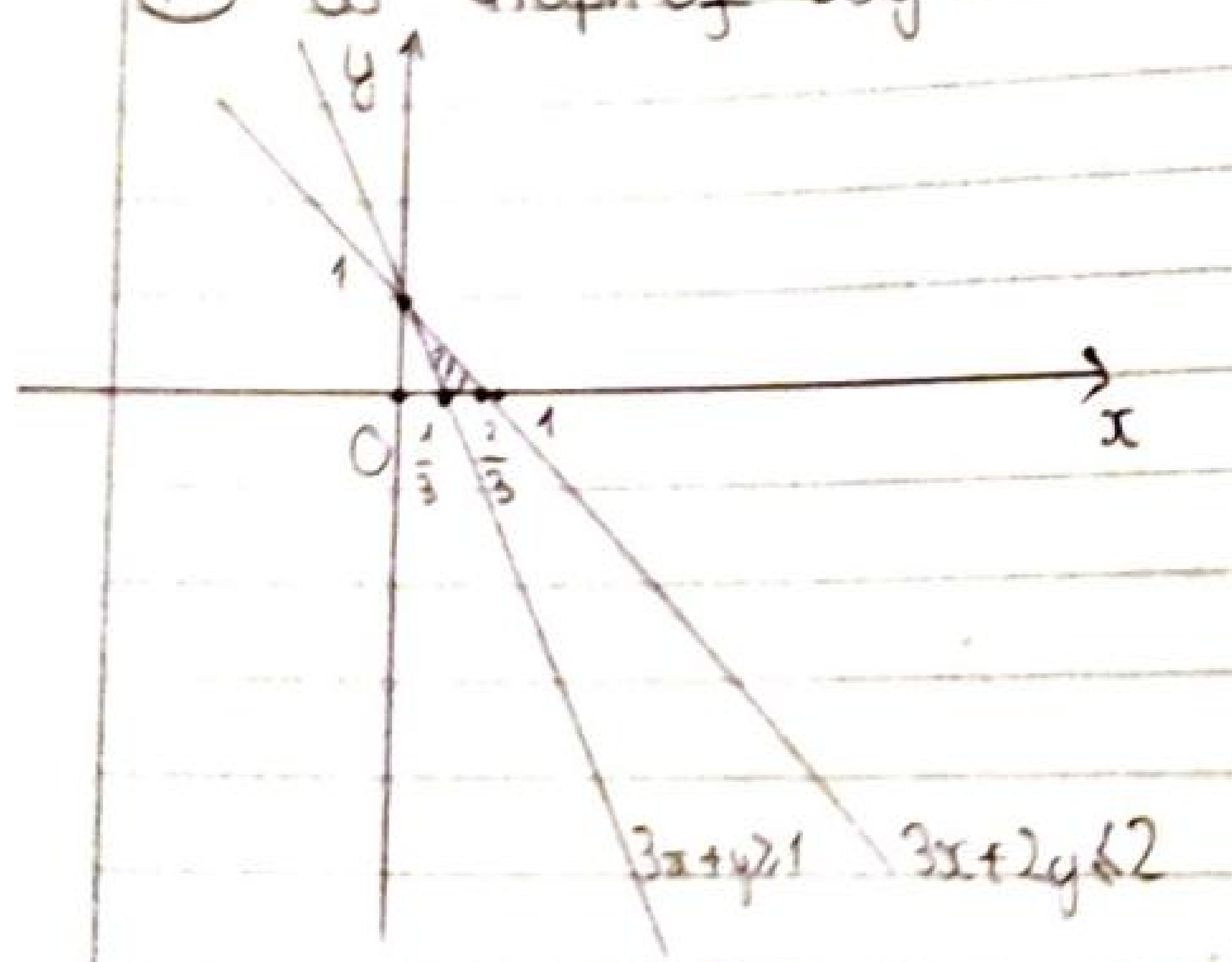
$= \dfrac{1}{6}a^2$

h) $r = a(1+\cos\alpha)$

$A = \iint_{D_{xy}} dx\,dy = \int_0^{2\pi} d\alpha \int_0^{a(1+\cos\alpha)} r\,dr = \int_0^{2\pi} \dfrac{a^2(1+\cos\alpha)^2}{2}\,d\alpha$

$= \dfrac{3\pi}{2}a^2$

(77) a) Graph of D_{xy}:



+) $V = \iint_{D_{xy}} dx\,dy \int_0^{1-x-y} dz$

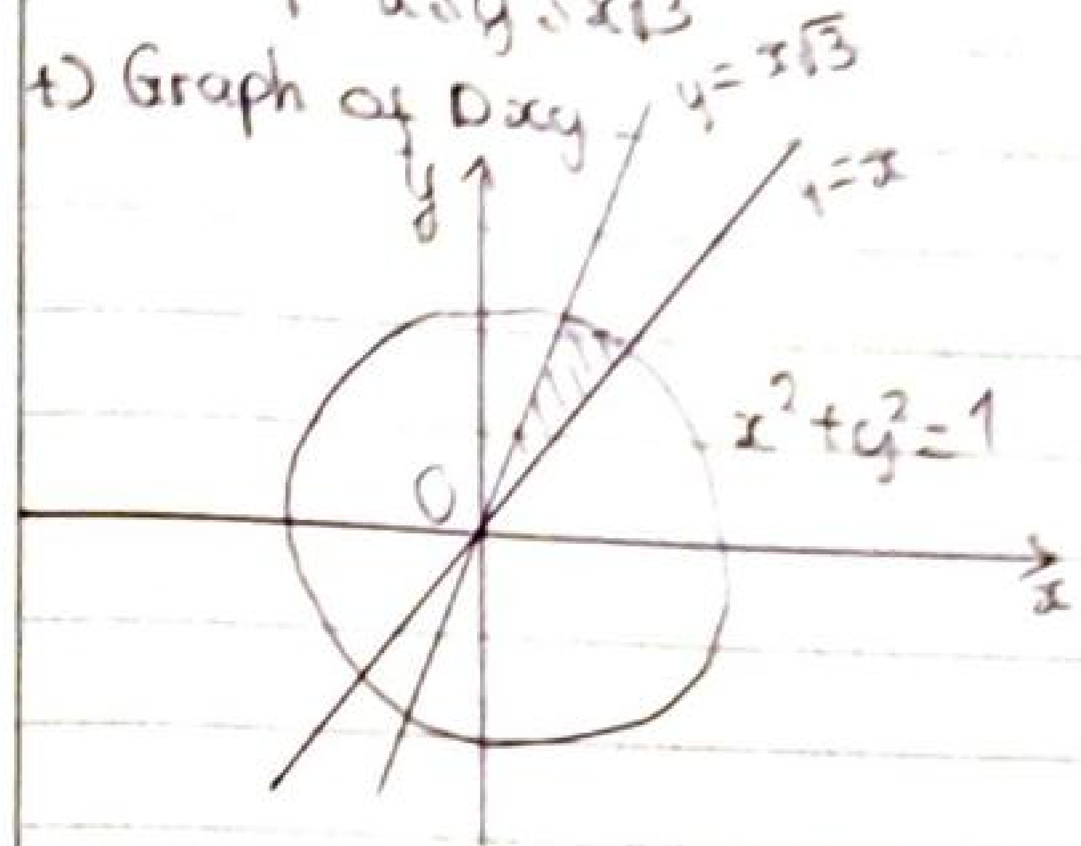
$= \int_0^1 dy \int_{\frac{1-y}{3}}^{(2-2y)/3} 1-x-y\,dx = \int_0^1 \left(\dfrac{2-2y}{3} - \dfrac{1-y}{3}\right) - \dfrac{1}{2}\left(\left(\dfrac{2-2y}{3}\right)^2 - \left(\dfrac{1-y}{3}\right)^2\right)$

$- y\left(\dfrac{2-2y}{3} - \dfrac{1-y}{3}\right) dy$

$= \dfrac{1}{18}$

b) $\begin{cases} 0 \le z \le 1 - x^2 - y^2 \\ x \le y \le x\sqrt{3} \end{cases}$

+) Graph of D_{xy}



+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

- $J = r$
- $D_{r\alpha}$ $\begin{cases} 0 \le r \le 1 \\ \dfrac{\pi}{4} \le \alpha \le \dfrac{\pi}{3} \end{cases}$

+) $V = \iint_{D_{xy}} 1-x^2-y^2\,dx\,dy = \int_{\pi/4}^{\pi/3} d\alpha \int_0^1 r(1-r^2)\,dr = \dfrac{\pi}{48}$

(78) Compute volume:

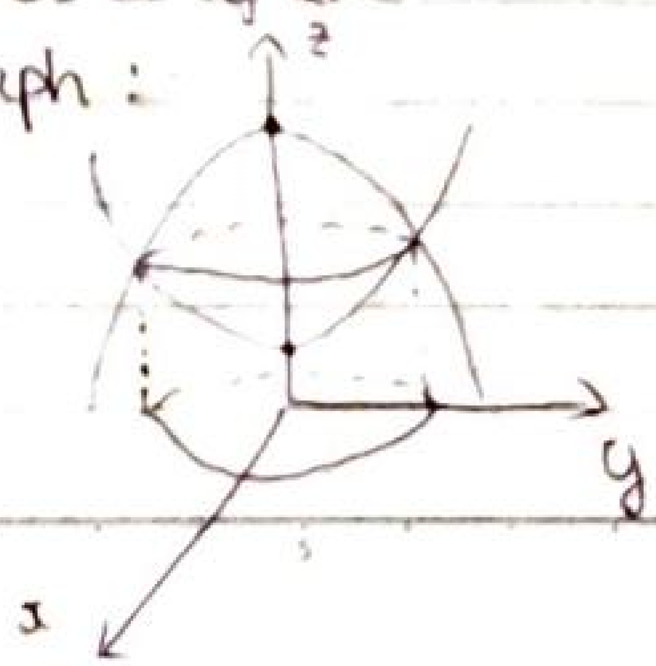
a) $\begin{cases} z = 4 - x^2 - y^2 \\ 2z = 2 + x^2 + y^2 \end{cases}$

+) $2(4-x^2-y^2) = 2 + x^2 + y^2$

$\Rightarrow 6 = 3(x^2+y^2)$

$\Rightarrow D: x^2+y^2 \le 2$

+) Graph:



+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

- $J = r$
- $D_{r\alpha}$ $\begin{cases} 0 \le r \le \sqrt{2} \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $V = \int_0^{2\pi} d\alpha \int_0^{\sqrt{2}} \left[(4-r^2) - \dfrac{(2+r^2)}{2}\right] r\,dr$

$= 2\pi \cdot \dfrac{3}{2} = 3\pi.$

b) $\begin{cases} z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}, \; z = 0 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{2x}{a} \end{cases}$

+) $V = \iint\limits_{D_{xy}: \frac{x^2}{a^2} + \frac{y^2}{b^2} \le \frac{2x}{a}} \left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) dxdy$

+) Put $x = ar\cos\alpha$

$y = br\sin\alpha$

. $J = abr$

. $D_{r\alpha}$: $\begin{cases} r^2 \le 2r\cos\alpha \\ r \ge 0 \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le 2\cos\alpha \\ -\dfrac{\pi}{2} \le \alpha \le \dfrac{\pi}{2} \end{cases}$

+) $V = \displaystyle\int_{-\pi/2}^{\pi/2} d\alpha \int_0^{2\cos\alpha} r^2 . abr\, dr$

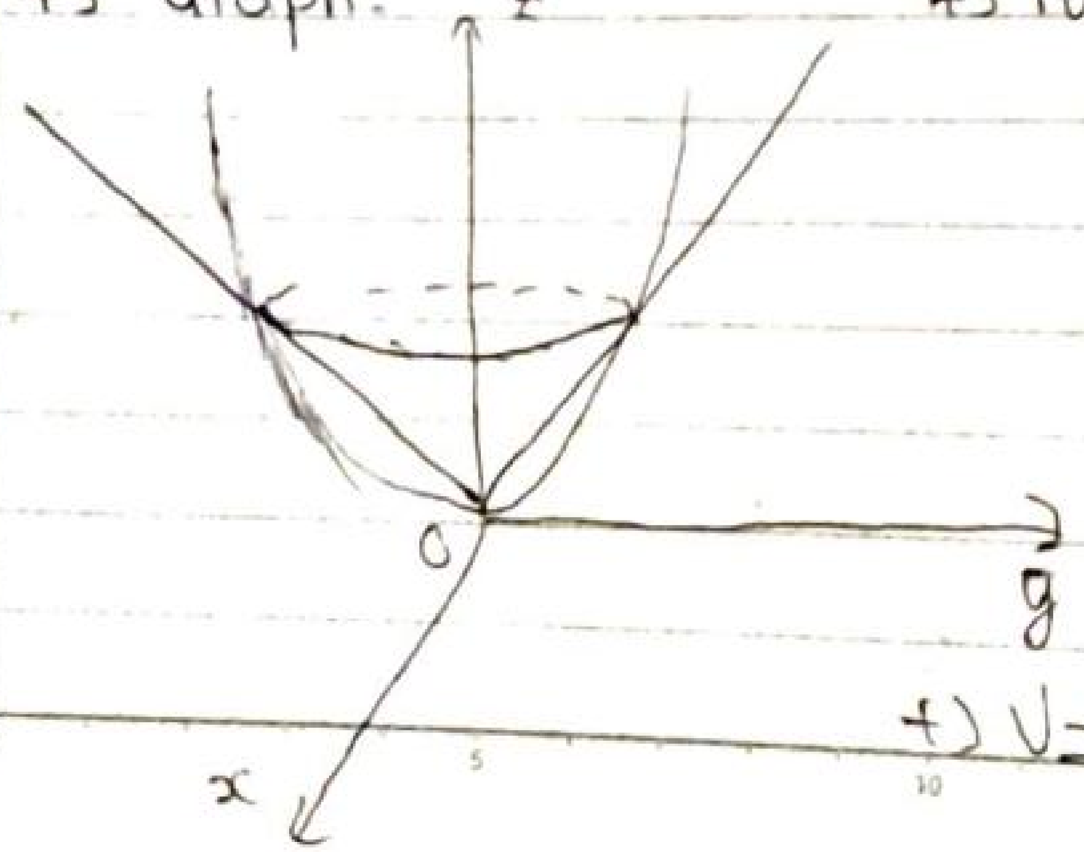
$= \dfrac{ab}{4} \displaystyle\int_{-\pi/2}^{\pi/2} 2^4 \cos^4\alpha\, d\alpha$

$= 4ab . \dfrac{3}{8}\pi = \dfrac{3}{2} ab\pi$

c) $\begin{cases} az = x^2 + y^2 \\ z = \sqrt{x^2+y^2} \end{cases}$

+) $V = \iint\limits_{D_{xy}} \sqrt{x^2+y^2} - \dfrac{x^2+y^2}{a}\, dxdy$

+) Graph:



+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

. $J = r$

. $\sqrt{r^2} \ge \dfrac{r^2}{a} \Leftrightarrow a \ge r$

. $D_{r\alpha}$ $\begin{cases} a \ge r \ge 0 \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $V = \displaystyle\int_0^{2\pi} d\alpha \int_0^a \left(r - \dfrac{r^2}{a}\right) . r\, dr$

$= 2\pi\left(\dfrac{1}{3} . a^3 - \dfrac{1}{4a} . a^4\right) = \dfrac{\pi}{6} a^3$

(79) area of: $x = y^2 + z^2$; $x \le 1$

+) $A = \iint\limits_{D_{yz}: y^2+z^2 \le 1} \sqrt{1 + x'^2_y + x'^2_z}\, dydz = \iint\limits_{D_{yz}} \sqrt{1+4y^2+4z^2}\, dydz$

+) Put $y = r\cos\alpha$

$z = r\sin\alpha$

. $J = r$

. $D_{r\alpha}$ $\begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $I = \displaystyle\int_0^{2\pi} d\alpha \int_0^1 \sqrt{1+4r^2}\, r\, dr = 2\pi . \dfrac{1}{8} \int_0^1 (1+4r^2)^{1/2} d(1+4r^2)$

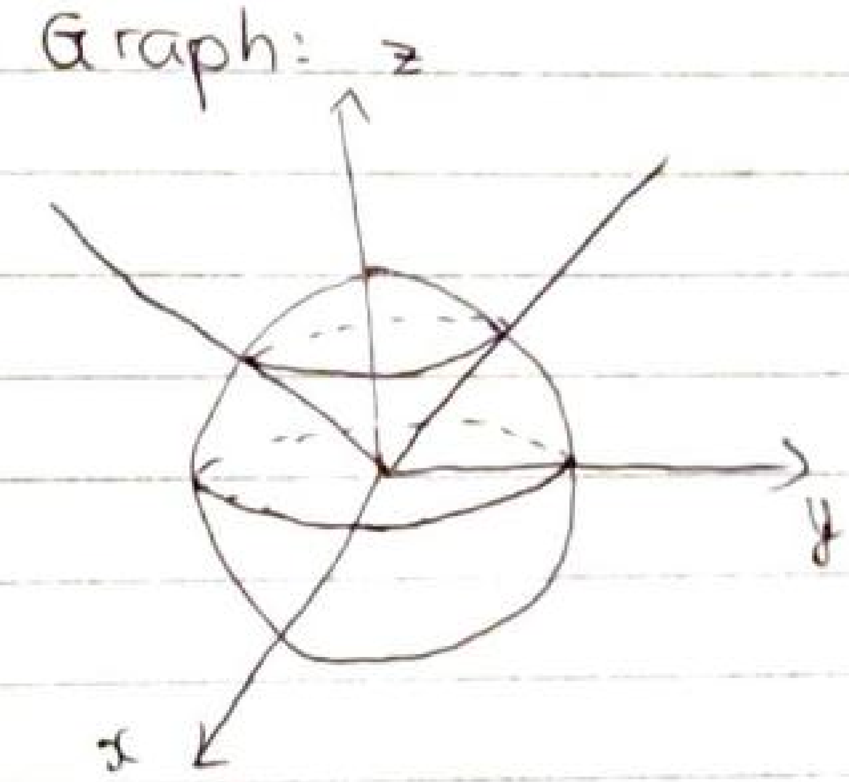
$= \dfrac{\pi}{4}\left((1+4)^{3/2} - (1+0)^{3/2}\right) . \dfrac{2}{3} = \dfrac{\pi}{6}(5\sqrt{5} - 1)$

3.2 Triple Integrals.

(80)

a) $\iiint\limits_V (x^2+y^2)\, dxdydz$; V bounded by $\begin{cases} x^2+y^2+z^2 = 1 \\ x^2+y^2 = z^2 \end{cases}$

+) Graph:



+) Symmetric property:

$I = 2\iint\limits_{D_{xy}} dxdy \displaystyle\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} (x^2+y^2)\, dz$

+) D_{xy} : $\sqrt{x^2+y^2} \le \sqrt{1-x^2-y^2}$

$\Leftrightarrow x^2+y^2 \le 1/2$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$

. $D_{r\alpha}$ $\begin{cases} 0 \le r \le 1/\sqrt{2} \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $I = 2\displaystyle\int_0^{2\pi} d\alpha \int_0^{1/\sqrt{2}} r^3\left(\sqrt{1-r^2} - r\right) dr$

$= 4\pi \displaystyle\int_0^{1/\sqrt{2}} r^3\sqrt{1-r^2} - r^4\, dr$

• $\int_0^{1/\sqrt{2}} r^3(1-r^2)^{1/2}\,dr = \frac{1}{2}\int_0^{1/\sqrt{2}} r^2\sqrt{1-r^2}\,dr^2 = \frac{1}{2}\int_0^{1/2} t\sqrt{1-t}\,dt$ (Put $1-t$

$= \frac{1}{2}\int_1^{1/\sqrt{2}} (1-a^2)\,a\,(-2a)\,da = \frac{1}{2}\int_{1/\sqrt{2}}^{1} 2a^2-2a^4\,da = \int_{1/\sqrt{2}}^{1} a^2-a^4\,da$

$= \frac{1}{3}\left(1-\frac{\sqrt{2}}{4}\right) - \frac{1}{5}\left(1-\frac{\sqrt{2}}{8}\right)$

• $\int_0^{1/\sqrt{2}} r^4 = \frac{1}{5}\cdot\frac{\sqrt{2}}{8}$

+) $I = 4\pi\left(\frac{8-5\sqrt{2}}{60}\right) = \boxed{\frac{8-5\sqrt{2}}{15}\pi.}$

$\boxed{C_2}$ +) Put $x = r\cos\alpha\sin\theta$

$y = r\sin\alpha\sin\theta$

$z = r\cos\theta$

• $J = -r^2\sin\theta$

• $D_{r\alpha\theta}$ $\begin{cases} 0 \le \alpha \le 2\pi \\ 0 \le r \le 1 \\ 0 \le \theta \le \pi/4 \end{cases}$

+) $I = 2\int_0^{2\pi} d\alpha \int_0^{\pi/4} d\theta \int_0^{1} r^2\sin^2\theta \cdot r^2\sin\theta\,dr$

$= 2\cdot 2\pi\cdot\frac{1}{5}\cdot\int_0^{\pi/4}\sin^3\theta\,d\theta$

$= \frac{4\pi}{5}\int_0^{\pi/4} -\sin^2\theta\,d\cos\theta$

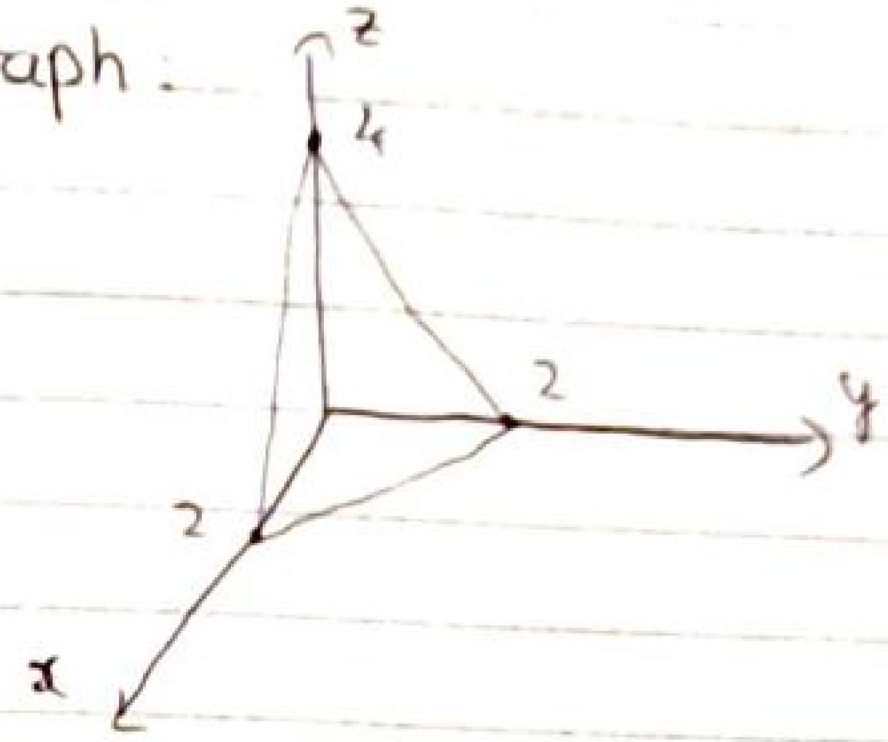
$= \frac{4\pi}{5}\int_0^{\pi/4} -1+\cos^2\theta\,d\cos\theta$

$= \frac{4\pi}{5}\left(-\cos\theta + \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi/4}$

$= \frac{4\pi}{5}\left(-\frac{\sqrt{2}}{2} + 1 + \frac{\sqrt{2}}{12} - \frac{1}{3}\right) = \boxed{\frac{8-5\sqrt{2}}{15}\pi}$

b) $\iiint_E y\,dx\,dy\,dz$; E bounded by $x=0, y=0, z=0, 2x+2y+z=4$

+) Graph:



+) $I = \int_0^2 dx \int_0^{(4-2x)/2} dy \int_0^{4-2x-2y} y\,dz$

$= \int_0^2 dx \int_0^{2-x} y(4-2x-2y)\,dy$

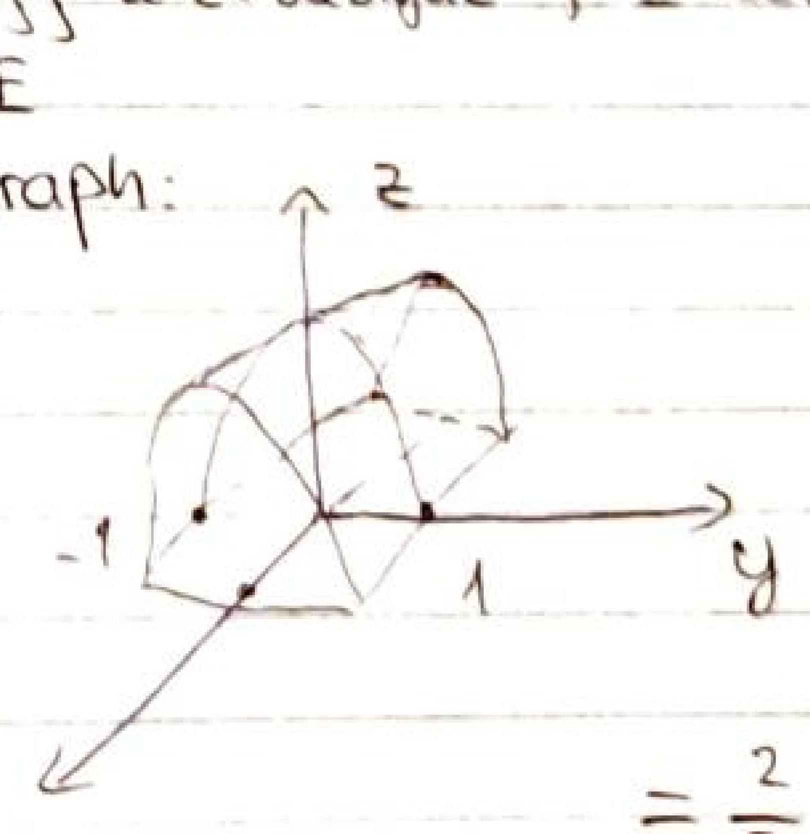
$= \int_0^2 (2-x)\left[(2-x)^2-0^2\right] - \frac{2}{3}(2-x)^3\,dx$

$= \int_0^2 \frac{1}{3}(2-x)^3\,dx = \frac{-1}{3}\int_0^2 (2-x)^3\,d(2-x) = \frac{-1}{3}\cdot\frac{1}{4}(2-x)^4\Big|_0^2$

$= \frac{4}{3}$

c) $\iiint_E x^2e^y\,dx\,dy\,dz$; E bounded by $z = 1-y^2, z=0, x=1, x=-1$

+) Graph:



+) $I = \int_{-1}^{1} dx \int_{-1}^{1} dy \int_0^{1-y^2} x^2e^y\,dz$

$= \int_{-1}^{1} dx \int_{-1}^{1} x^2e^y(1-y^2)\,dy$

$= \frac{2}{3}\int_{-1}^{1} e^y - e^y y^2\,dy$

$= \frac{2}{3}\left(-y^2e^y + 2ye^y - e^y\right)\Big|_{y=-1}^{y=1}$

$= \frac{2}{3}(-e+2e-e) - \frac{2}{3}(-e^{-1}-2e^{-1}-e^{-1})$

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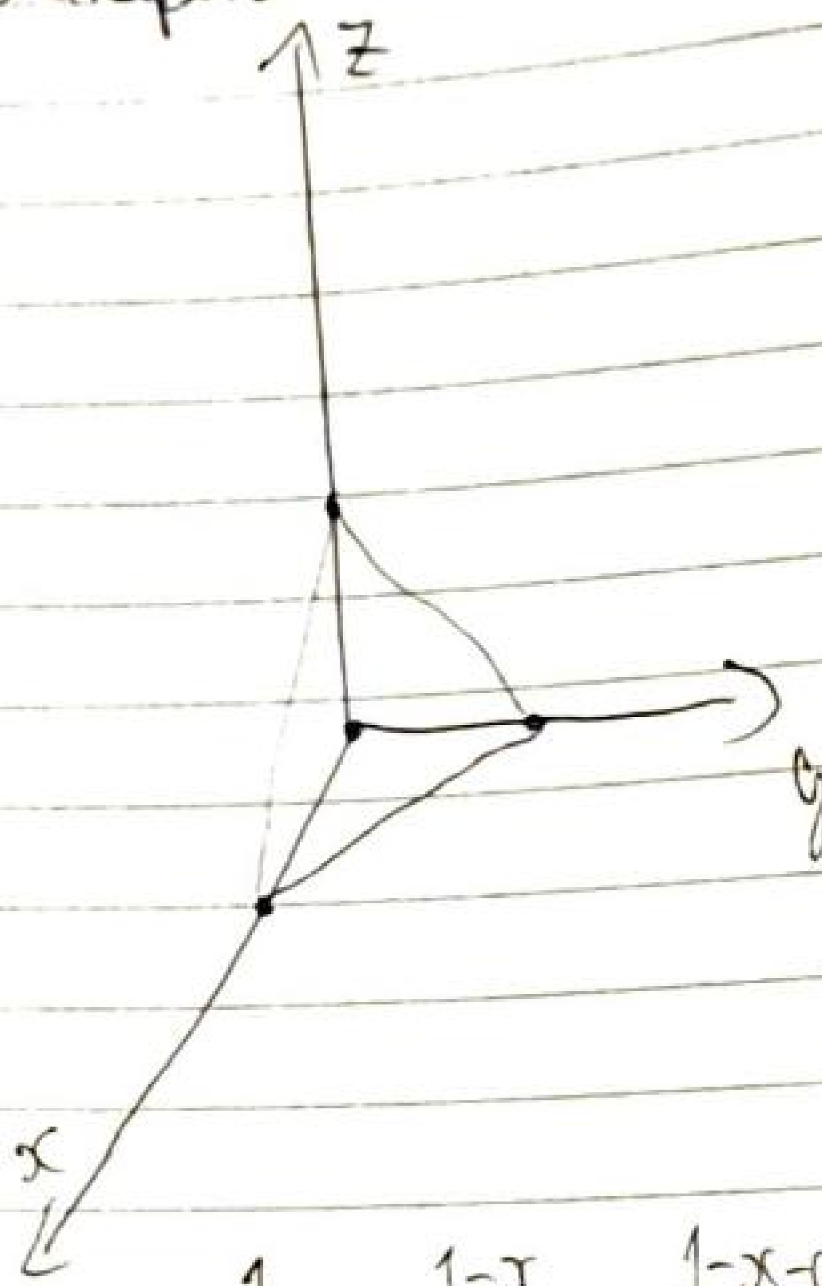
No

$= \frac{8}{35}$

[redacted] $\iiint_E xy\,dxdydz$, E: solid tetrahedron : $(0,0,0), (1,0,0), (0,1,0), (0,$

+) Graph:



+) $I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xy\,dz$

$= \int_0^1 dx \int_0^{1-x} xy(1-x-y)dy$

$= \int_0^1 dx \int_0^{1-x} y(x-x^2) - y^2x\,dy$

$= \int_0^1 \frac{1}{2}(x-x^2)(1-x)^2 - \frac{1}{3}x(1-x)^3\,dx$

$= \int_0^1 \frac{1}{6}x(1-x)^3 dx = \frac{1}{120}$

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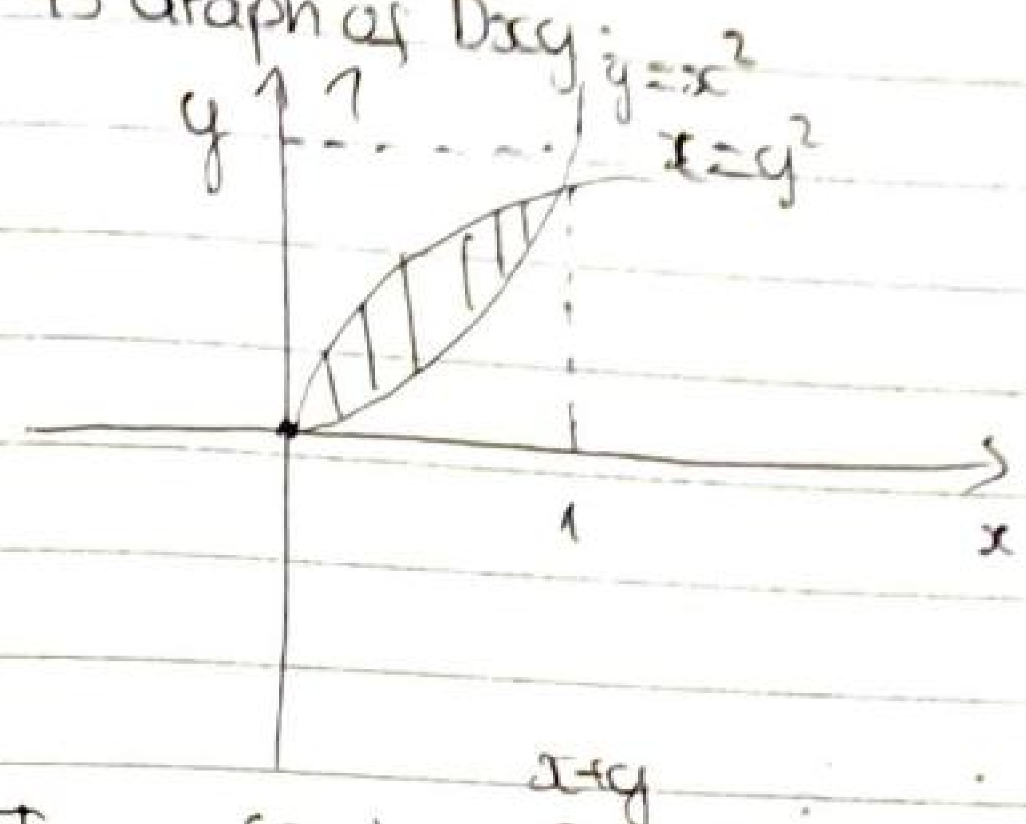
Ngày
No

d) +) Graph of Dxy



+) $I = \iint_{Dxy} dxdy \int_0^{x+y} xy\,dz$

$= \iint_{Dxy} xy(x+y)\,dxdy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} xy(x+y)dy$

$= \int_0^1 x^2\cdot\frac{1}{2}(x-x^4) + x\cdot\frac{1}{3}(x\sqrt{x}-x^6)\,dx$

$= \int_0^1 \frac{1}{2}\cdot(x^3-x^6) + \frac{1}{3}(x^{5/2}-x^7)\,dx$

$= \frac{1}{2}\cdot\left(\frac{1}{4}-\frac{1}{7}\right) + \frac{1}{3}\left(\frac{2}{7}-\frac{1}{8}\right) = \frac{3}{28}$

e) $\iiint_E xyz\,dxdydz$, E: $(0,0,0) \to (1,0,0) \to (0,1,0) \to (0,0,1)$

+) $I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} xyz\,dz = \int_0^1 dx \int_0^{1-x} xy(1-x-y)^2\cdot\frac{1}{2}\,dy$

$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} xy(x^2+y^2+1-2x-2y+2xy)\,dy$

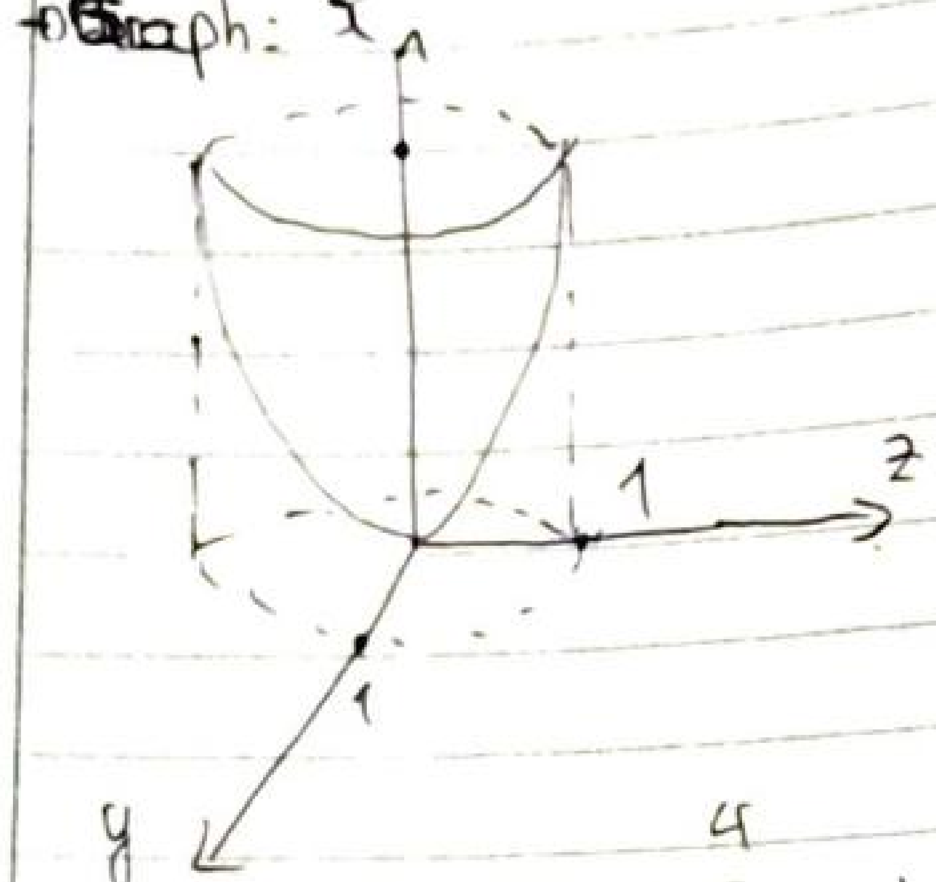
$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} y(x^3-2x^2+x) + y^2(2x^2-2x) + y^3x\,dy$

$= \frac{1}{2}\int_0^1 x(x-1)^2\cdot\frac{1}{2}(x-1)^2 + \frac{1}{3}\cdot 2x(x-1)\cdot(1-x)^3 + \frac{1}{4}x(x-1)^4\,dx$

$= \frac{1}{2}\int_0^1 (x-1)^4\left(\frac{1}{2}x - \frac{2}{3}x + \frac{1}{4}x\right)dx = \frac{1}{24}\cdot\frac{1}{30} = \frac{1}{720}$

f) $\iiint_E x\,dx\,dy\,dz$; E bounded by: $x=4y^2+4z^2$, $x=4$

+) Graph:



+) $I = \iint_{D_{yz}:\, y^2+z^2\le 1} dy\,dz \int_{4y^2+4z^2}^{4} x\,dx$

$= \frac{1}{2}\iint_{D_{yz}} (16-16(y^2+z^2)^2)\,dy\,dz$

+) Put $y=r\cos\alpha$, $z=r\sin\alpha$

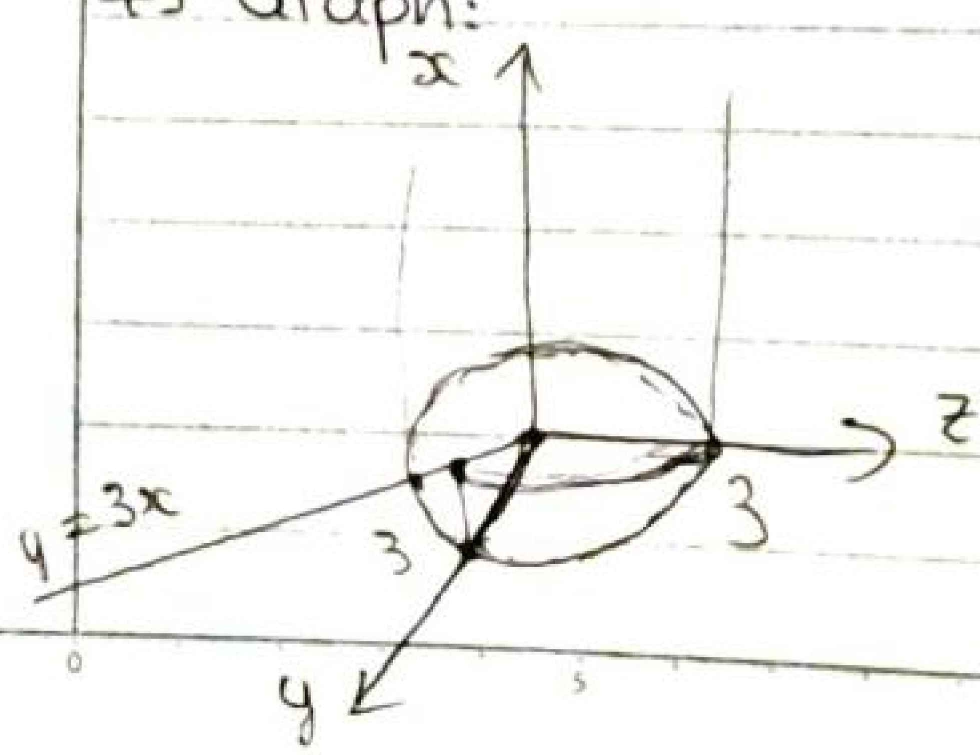
. $J=r$

. $D_{r\alpha}\begin{cases} 0\le r\le 1 \\ 0\le \alpha\le 2\pi \end{cases}$

+) $I = 8\int_0^{2\pi} d\alpha \int_0^1 (1-r^4)\,r\,dr = 8.2\pi.\frac{1}{3} = \frac{16\pi}{3}$

g) $\iiint_E z\,dx\,dy\,dz$, E: bounded by: $y^2+z^2=9$, $x=0$, $y=3x$, $z=0$ in the 1st octant

+) Graph:



+) $I = \iint_{D_{yz}} dx\,dy \int_0^{y/3} z\,dx$

$D_{yz}: y^2+z^2\le 9,\ y\ge 0,\ z\ge 0$

+) Put $y=r\cos\varphi$, $z=r\sin\varphi$

. $J=r$

. $D_{r\varphi}\begin{cases} 0\le r\le 3 \\ 0\le\varphi\le \frac{\pi}{2} \end{cases}$

$I = \int_0^{\pi/2} d\alpha \int_0^3 \frac{r^2\sin\alpha\cos\alpha \cdot r}{3}\,dr = \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{81}{4} = \frac{27}{8}$

8)

a) $\iiint_V x+y+z\,dx\,dy\,dz$, V bounded by $\begin{cases} x+y+z=\pm 3 \\ x+2y-z=\pm 1 \\ x+4y+z=\pm 2 \end{cases}$

+) Put $u=x+y+z$

$v=x+2y-z$

$w=x+4y+z$

. $J^{-1} = \frac{D(u,v,w)}{D(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{vmatrix} = 6$

$\Rightarrow J = \frac{1}{6}$. $D_{uvw}\begin{cases} -3\le u\le 3 \\ -1\le v\le 1 \\ -2\le w\le 2 \end{cases}$

+) $I = \iiint_{D_{uvw}} \frac{1}{6} u\,du\,dv\,dw$

$= \frac{1}{6}\int_{-3}^{3} u\,du \int_{-1}^{1} dv \int_{-2}^{2} dw = \frac{1}{6}.0.2.4 = 0$

b) $\iiint_V (3x^2+2y+z)\,dx\,dy\,dz$; V: $|x-y|\le 1$, $|y-z|\le 1$, $|z+x|\le 1$

+) Put $u=x-y$, $v=y-z$, $w=z+x$

. $D_{uvw}\begin{cases} |u|\le 1 \\ |v|\le 1 \\ |w|\le 1 \end{cases}$

. $x=\frac{u+w+v}{2}$, $y=\frac{-u+w+v}{2}$, $z=\frac{-u-v+w}{2}$

. $J = \begin{vmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}$

+) $I = \frac{1}{2}\iiint_{D_{uvw}} 3\frac{(u+v+w)^2}{4} + (-u+w+v) + \frac{-u-v+w}{2}\,du\,dv\,dw$

$= \frac{1}{2}\iiint_{D_{uvw}} \frac{3}{4}(u^2+v^2+w^2+2uv+2vw+2uw) + \frac{-3}{2}u + \frac{3}{2}w + \frac{v}{2}\,du\,dv\,dw$

$= \frac{1}{2}\iiint_{D_{uvw}} \frac{3}{4} \cdot 3u^2 \, du\,dv\,dw$ (Symetric property)

$= \frac{1}{2} \cdot \frac{3}{4} \cdot (1^3-(-1)^3) \cdot (1-(-1)) \cdot (1-(-1)) = 3$

c) $\iiint_V dx\,dy\,dz$, $V: |x-y| + |x+3y| + |x+y+z| \leq 1$

+) Put $u = x-y$, $v = x+3y$, $z = x+y+z$

• $J^{-1} = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 4 \Rightarrow J = \frac{1}{4}$

• $D_{uvw}: |u| + |v| + |w| \leq 1$

+) $I = \iiint_{D_{uvw}} \frac{1}{4} \, du\,dv\,dw = 8 \cdot \frac{1}{4} \iiint du\,dv\,dw = 8 \cdot \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{3}$

$D_{uvw}: u+v+w \leq 1,$
$u \geq 0, v \geq 0, w \geq 0$

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$\iiint_V x^2+y^2 \, dx\,dy\,dz$, $V: \begin{cases} x^2+y^2 \leq 1 \\ 1 \leq z \leq 2 \end{cases}$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

• $J = \begin{vmatrix} \cos\alpha & -r\sin\alpha & 0 \\ \sin\alpha & r\cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

• $D_{r\alpha z} \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \alpha \leq 2\pi \\ 1 \leq z \leq 2 \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^1 dr \int_1^2 r^3 dz = 2\pi \cdot \frac{1}{4} \cdot (2-1) = \frac{\pi}{2}$

83 $\iiint_V z\sqrt{x^2+y^2} \, dx\,dy\,dz$

a) $V: x^2+y^2 = 2x$, $z = 0, z = a$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

• $J = r$

• $D_{r\alpha z} \begin{cases} -\pi/2 \leq \alpha \leq \pi/2 \\ 0 \leq r \leq 2\cos\alpha \\ 0 \leq z \leq a \end{cases}$

+) $I = \int_0^a dz \int_{-\pi/2}^{\pi/2} d\alpha \int_0^{2\cos\alpha} zr^2 dr$

$= \frac{a^2}{2} \cdot \int_{-\pi/2}^{\pi/2} \frac{1}{3} \cdot 8\cos^3\alpha \, d\alpha$

$= \frac{a^2}{2} \cdot \frac{1}{3} \cdot 8 \cdot \frac{4}{3} = \frac{16}{9} a^2$

b) $V: x^2+y^2+z^2 \leq a^2$, $z \geq 0$ $(a > 0)$

+) Put $x = r\cos\alpha\sin\theta$

$y = r\sin\alpha\sin\theta$

$z = r\cos\theta$

• $J = -r^2\sin\theta$

• $D_{r\alpha\theta} \begin{cases} 0 \leq \alpha \leq 2\pi \\ 0 \leq \theta \leq \pi/2 \\ 0 \leq r \leq a \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^a r\cos\theta \sqrt{r^2\sin^2\theta} \cdot r^2\sin\theta \, dr$

$= \int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^a r^4 \sin^2\theta\cos\theta \, dr$

$= \frac{1}{5} a^5 \cdot \frac{1}{3} \cdot 2\pi = \frac{2}{15}\pi a^5$

84 $\iiint_V \sqrt{x^2+y^2} \, dx\,dy\,dz$; V bounded by $\begin{cases} x^2+y^2 = z^2 \\ z = 1 \end{cases}$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

• $J = r$

• $D_{r\alpha z} \begin{cases} 0 \leq z \leq 1 \\ 0 \leq r \leq z \\ 0 \leq \alpha \leq 2\pi \end{cases}$

+) $I = \int_0^1 dz \int_0^{2\pi} d\alpha \int_0^z r^2 dr = 2\pi \int_0^1 \frac{z^3}{3} dz = 2\pi \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{\pi}{6}$

85 $\displaystyle\iiint_V \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-2)^2}}$, $V: \begin{cases} x^2+y^2 \le 1 \\ |z| \le 1 \end{cases}$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

- $J = r$
- $D_{r\alpha z}$ $\begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le 2\pi \\ -1 \le z \le 1 \end{cases}$

+) $I = \displaystyle\int_0^{2\pi} d\alpha \int_{-1}^{1} dz \int_0^1 \frac{r}{\sqrt{r^2+(z-2)^2}}\,dr$

$= 2\pi \displaystyle\int_{-1}^{1} \sqrt{r^2+(z-2)^2}\,\Big|_{r=0}^{r=1} dz$

$= 2\pi \displaystyle\int_{-1}^{1} \sqrt{1+(z-2)^2} - \sqrt{(z-2)^2}\,dz$

$= -2\pi \displaystyle\int_3^1 \sqrt{1+t^2} - \sqrt{t^2}\,dt$ (Put $t = 2-z$)

$= 2\pi \displaystyle\int_1^3 \sqrt{1+t^2} - t\,dt$ ($t \ge 0\ \forall\ |z| \le 1$)

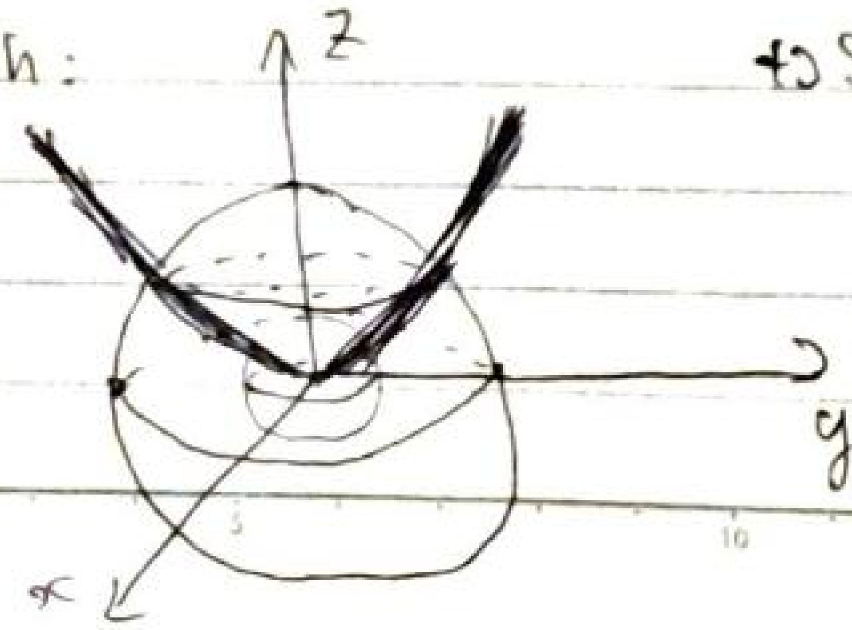
$= 2\pi \left(\frac{1}{2} t\sqrt{1+t^2} + \frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right| - \frac{t^2}{2} \right) \Big|_1^3$

$= 2\pi \left(\frac{3}{2}\sqrt{10} + \frac{1}{2}\ln(3+\sqrt{10}) - \frac{9}{2} - \frac{\sqrt{2}}{2} - \frac{1}{2}\ln(1+\sqrt{2}) + \frac{1}{2} \right)$

$= \pi\left(3\sqrt{10} + \ln\frac{3+\sqrt{10}}{1+\sqrt{2}} - \sqrt{2} - 8\right)$

86 $\displaystyle\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz$, $V: \begin{cases} 1 \le x^2+y^2+z^2 \le 4 \\ x^2+y^2 \le z^2 \end{cases}$

+) Graph:



+) Symmetric property:

$I = 2 \displaystyle\iiint_{V_1} (x^2+y^2+z^2)\,dx\,dy\,dz$

V_1 $\begin{cases} 1 \le x^2+y^2+z^2 \le 4 \\ x^2+y^2 \le z^2 \end{cases}$

+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$

- $J = -r^2\sin\theta$
- $D_{r\alpha\theta}$ $\begin{cases} 0 \le \alpha \le 2\pi \\ 0 \le \theta \le \pi/4 \\ 1 \le r \le 2 \end{cases}$

+) $I = \displaystyle\int_0^{2\pi} d\alpha \int_0^{\pi/4} d\theta \int_1^2 r^4 \sin\theta\,dr$

$= 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{1}{5} \cdot 31 = \frac{31}{5}(1-\sqrt{2})\pi$

87 $\displaystyle\iiint_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $V: x^2+y^2+z^2 \le z$

+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$

- $J = -r^2\sin\theta$
- $D_{r\alpha\theta}$: $\begin{cases} r \ge 0 \\ r^2 \le r\cos\theta \\ 0 \le \alpha \le 2\pi \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le \cos\theta \\ 0 \le \theta \le \pi/2 \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $I = \displaystyle\int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^{\cos\theta} r^3 \sin\theta\,dr$

$= 2\pi \displaystyle\int_0^{\pi/2} \frac{1}{4}\cos^4\theta \cdot \sin\theta\,d\theta = 2\pi\,\frac{1}{4}\cdot\frac{1}{5} = \frac{\pi}{10}$

88 $\displaystyle\iiint_V z\sqrt{x^2+y^2}\,dx\,dy\,dz$, $V: \frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \le 1,\ z \ge 0$ $(a, b > 0)$

+) Put $x = ar\cos\alpha\sin\theta$

$y = ar\sin\alpha\sin\theta$

$z = br\cos\theta$

- $J = -a^2 b\,r^2\sin\theta$
- $D_{r\alpha\theta}$ $\begin{cases} r^2 \le 1,\ r \ge 0 \\ 0 \le \alpha \le 2\pi \\ br\cos\theta \ge 0 \end{cases} \Leftrightarrow \begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le 2\pi \\ 0 \le \theta \le \pi/2 \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^1 br\cos\theta . a . r\sin\theta . a^2 b r^2 \sin\theta \, dr$

$= 2\pi . \frac{1}{3} . \frac{1}{5} . a^3 b^2 = \frac{2}{15} \pi a^3 b^2$

(89) $\iiint_V \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dxdydz$; $V: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

$(a, b, c > 0)$

+) Put $x = ar\cos\alpha\sin\theta$

$y = br\sin\alpha\sin\theta$

$z = cr\cos\theta$

• $J = -abcr^2\sin\theta$

• $D_{r\alpha\theta}: \begin{cases} r^2 \leq 1, r \geq 0 \rightarrow 0 \leq r \leq 1 \\ 0 \leq \alpha \leq 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^1 r^2 . abcr^2 \sin\theta$

$= 2\pi . 2 . \frac{1}{5} . abc = \frac{4\pi}{5} abc$

(90) $\iiint_V \sqrt{z - x^2 - y^2 - z^2} \, dxdydz$, $V: x^2 + y^2 + z^2 \leq z$.

+) Put $x = r\cos\alpha\sin\theta$

$y = r\sin\alpha\sin\theta$

$z = r\cos\theta + \frac{1}{2}$

• $J = -r^2\sin\theta$

• $D_{r\alpha\theta}: \begin{cases} 0 \leq \alpha \leq 2\pi \\ 0 \leq r \leq \frac{1}{2} \\ 0 \leq \theta \leq \frac{2}{\pi} \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^{1/2} r^2 \sin\theta \sqrt{\frac{1}{4} - r^2} \, dr$

$= 4\pi \int_0^{1/2} r^2 \sqrt{\frac{1}{4} - r^2} \, dr = 4\pi \int_0^{\pi/2} \frac{1}{4} \sin^2 t \sqrt{\frac{1}{4} - \frac{1}{4}\sin^2 t} . \frac{1}{2} \cos t \, dt$ $\langle$ Put $r = \frac{1}{2}\sin t \rangle$

$= 4\pi . \frac{1}{16} \int_0^{\pi/2} \sin^2 t \cos^2 t \, dt = \frac{1}{64} \pi^2$

(91) $\iiint_V (4z - x^2 - y^2 - z^2) \, dxdydz$, $V: x^2 + y^2 + z^2 \leq 4z$

+) $V \rightarrow x^2 + y^2 + (z-2)^2 \leq 4$

+) Put $x = r\cos\alpha\sin\theta$

$y = r\sin\alpha\sin\theta$

$z = 2 + r\cos\theta$

• $J = -r^2\sin\theta$

• $D_{r\alpha\theta} \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \alpha \leq 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^{2} (4 - r^2) r^2 \sin\theta \, dr$

$= 2\pi . 2 . \frac{64}{15} = \frac{256}{15}\pi$

(92) $\iiint_V xz \, dxdydz$, $V: x^2 + y^2 + z^2 - 4x - 4y - 4z \leq -2$

+) $V \rightarrow (x-2)^2 + (y-2)^2 + (z-2)^2 \leq 10$

+) Put $x = 2 + r\cos\alpha\sin\theta$

$y = 2 + r\sin\alpha\sin\theta$

$z = 2 + r\cos\theta$

$\rightarrow 1 + r\cos\alpha\sin\theta$

$1 + r\sin\theta\sin\theta$

$1 + r\cos\theta$

$(x-1)^2 + (y-1)^2 + (z-1)^2 \leq 1$

chứ @@

• $J = -r^2\sin\theta$

• $D_{r\alpha\theta} \begin{cases} 0 \leq r \leq \sqrt{10} \\ 0 \leq \alpha \leq 2\pi \\ 0 \leq \theta \leq \pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^{\sqrt{10}} (2 + r\cos\alpha\sin\theta)(2 + r\cos\theta) r^2 \sin\theta \, dr$

$(4 + 2r\cos\alpha\sin\theta + 2r\cos\theta + r^2\cos\alpha\sin\theta\cos\theta) r^2 \sin\theta$

$4r^2\sin\theta + 2r^3\sin^2\theta\cos\alpha + 2r^3\sin\theta\cos\theta + r^4\cos\alpha\sin\theta\cos\theta$

$= 4 . \frac{10\sqrt{10}}{3} . 2\pi . 2 + 2 . \frac{100}{4} . 0 . \frac{\pi}{2} + 2 . \frac{100}{4} . 2\pi . 0 + 0 = \frac{160\sqrt{10}}{3}\pi$

$$\iiint_V xz\,dxdydz \quad , V: x^2+y^2+z^2-2x-2y-2z \leq -2$$

+) $V \to (x-1)^2+(y-1)^2+(z-1)^2 \leq 1$

+) Put $x = 1 + r\cos\alpha\sin\theta$

$y = 1 + r\sin\alpha\sin\theta$

$z = 1 + r\cos\theta$

- $J = -r^2\sin\theta$
- $D_{r\alpha\theta}$ $\begin{cases} 0 \leq \alpha \leq 2\pi \\ 0 \leq r \leq 1 \\ 0 \leq \theta \leq \pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^1 (1+r\cos\alpha\sin\theta)(1+r\cos\theta).\,r^2\sin\theta\,dr.$

$= \frac{1}{3}.2\pi.2 = \frac{4\pi}{3}$

(93) $I = \iiint_V \frac{dxdydz}{(1+x+y+z)^3}$, V bounded by $\begin{cases} x=0 \\ y=0 \\ z=0 \\ x+y+z=1 \end{cases}$

+) $I = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz$

$= \int_0^1 dx \int_0^{1-x} \frac{-1}{2} \cdot \left(\frac{1}{2^2} - \frac{1}{(1+x+y)^2} \right) dy$

$= \int_0^1 dx \int_0^{1-x} \frac{-1}{8} + \frac{1}{2(1+x+y)^2} dy$

$= \int_0^1 \frac{-1}{8}(1-x) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{1+x} \right) dx$

$= \int_0^1 \frac{-3}{8} + \frac{x}{8} + \frac{1}{2} \cdot \frac{1}{x+1} dx$

$= \frac{-5}{16} + \frac{1}{2}\ln 2$

(94) $\iiint_V z\,dxdydz \quad , V: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, z \geq 0$

+) Put $x = ar\cos\alpha\sin\theta, y = br\sin\alpha\sin\theta, z = cr\cos\theta$

- $J = -abc\,r^2\sin\theta$
- $D_{r\alpha\theta}$: $\begin{cases} r^2 \leq 1, r \geq 0 \to 0 \leq r \leq 1 \\ 0 \leq \alpha \leq 2\pi \\ cr\cos\theta \geq 0, 0 \leq \theta \leq \pi \to 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^1 r\cos\theta\, r^2\sin\theta\, dr \; abc^2$

$= 2\pi.\frac{1}{2}.\frac{1}{4} = \frac{\pi}{4}abc^2$

(95) a) $I_1 = \iiint_B \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dxdydz,\ B: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$

+) Put $x = ar\cos\alpha\sin\theta, y = br\sin\alpha\sin\theta, z = cr\cos\theta$

- $J = -abc\,r^2\sin\theta$
- $D_{r\alpha\theta}$: $\begin{cases} 0 \leq r \leq 1 \\ 0 \leq \alpha \leq 2\pi \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$

+) $I_1 = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^1 r^4\sin\theta.\,abc\,dr$

$= 2\pi.\,2.\,\frac{1}{5}.abc = \frac{4}{5}\pi abc$

b) $I_2 = \iiint_C z\,dxdydz$, C bounded by $z^2 = \frac{h^2}{R^2}(x^2+y^2)$

and $z = h$

+) Put $x = r\cos\alpha, y = r\sin\alpha, z = z$

- $J = r$
- $D_{r\alpha z}$ $\begin{cases} \frac{h}{R}r \leq z \leq h \\ 0 \leq r \leq R \\ 0 \leq \alpha \leq 2\pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^R dr \int_{\frac{h}{R}r}^{h} r z\, dz$

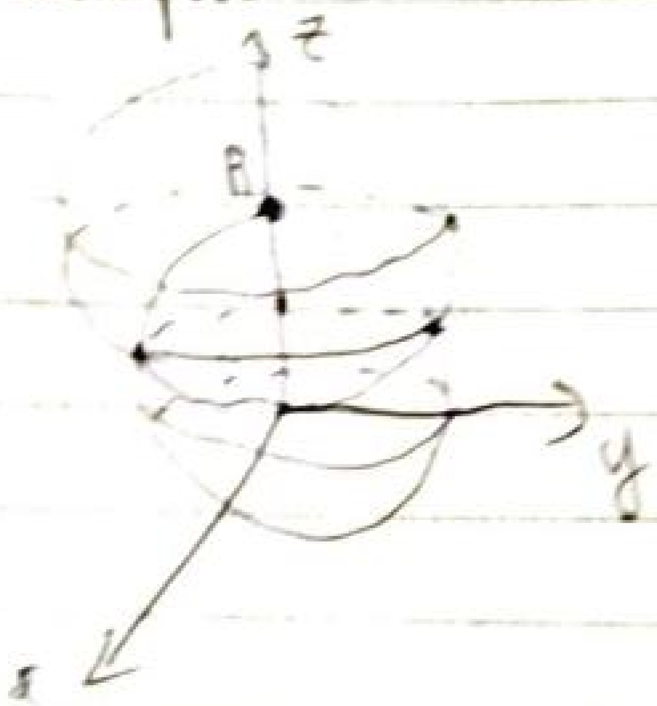
$= 2\pi \int_0^R \frac{r}{2}\left(h^2 - \frac{h^2}{R^2} r^2\right) dr$

$= 2\pi \left(\frac{h^2}{2}\cdot\frac{R^2}{2} - \frac{h^2}{2R^2}\cdot\frac{R^4}{4}\right)$

$= \frac{\pi}{4} h^2 R^2$

c) $I_3 = \iiint_D z^2\, dx\,dy\,dz$, D bounded by: $\begin{cases} x^2+y^2+z^2 \le R^2 \\ x^2+y^2+z^2 \le 2Rz \end{cases}$

+) Graph:



+) Intersection:

$R^2 = 2Rz \Leftrightarrow z = \frac{R}{2}$

$\Rightarrow$ Projection of D on Oxy

$x^2+y^2 \le \frac{3R^2}{4}$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

$z = z$

• $J = r$

• $D_{r\alpha z}$ $\begin{cases} 0 \le r \le \frac{\sqrt{3}}{2} R \\ 0 \le \alpha \le 2\pi \\ R - \sqrt{R^2-r^2} \le z \le \sqrt{R^2-r^2} \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^{\frac{\sqrt{3}}{2}R} dr \int_{R-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} z^2 r\, dz$

$= \frac{2\pi}{3} \int_0^{\frac{\sqrt{3}}{2}R} r(R^2-r^2)^{3/2} - r(R-\sqrt{R^2-r^2})^3\, dr$

$= \frac{\pi}{3} \int_0^{\frac{3R^2}{4}} (R^2-t)^{3/2} - (R-\sqrt{R^2-t})^3\, dt$

$3R^2/4$

(1) $= \frac{\pi}{3} \int_0 \left((R^2-t)^{3/2} - (R^3 - 3R^2\sqrt{R^2-t} + 3R(R^2-t) - (R^2-t)^{3/2}) \right) dt$

$= \frac{\pi}{3} \int_0^{3R^2/4} 2(R^2-t)^{3/2} - R^3 + 3R^2\sqrt{R^2-t} - 3R(R^2-t)\, dt$

$= \frac{\pi}{3}\left(2\cdot(R^2-t)^{5/2}\cdot\frac{2}{5}(-1) - 4R^3t + 3R^2(R^2-t)^{3/2}\cdot\frac{2}{3}\cdot(-1) + \frac{3}{2}Rt^2 \right)\Big|_0^{\frac{3R^2}{4}}$

$= \frac{\pi}{3}\left(-\frac{4}{5}\left(\frac{1}{32}R^5 - R^5\right) - 4R^3\cdot\frac{3}{4}R^2 + -2R^2\left(\frac{1}{8}R^3 - R^3\right) + \frac{3}{2}R\cdot\left(\frac{3}{4}\right)^2\cdot R^4 \right)$

$= \frac{\pi}{1}\cdot\frac{59}{480}R^5$

(2) Put $R^2 - r^2 = t$

$I = \frac{2\pi}{3}\cdot\frac{-1}{2} \int_0^{\frac{\sqrt{3}}{2}R} (R^2-r^2)^{3/2} - (R-\sqrt{R^2-r^2})^3\, d(R^2-r^2)$

$= \frac{\pi}{3} \int_{\frac{R^2}{4}}^{R^2} t^{3/2} - (R-\sqrt{t})^3\, dt$

$= \frac{\pi}{3} \int_{R^2/4}^{R^2} t^{3/2} - (R^3 - 3R^2\sqrt{t} + 3Rt - t\sqrt{t})\, dt$

$= \frac{\pi}{3} \int_{R^2/4}^{R^2} 2t^{3/2} - R^3 + 3R^2\sqrt{t} - 3Rt\, dt$

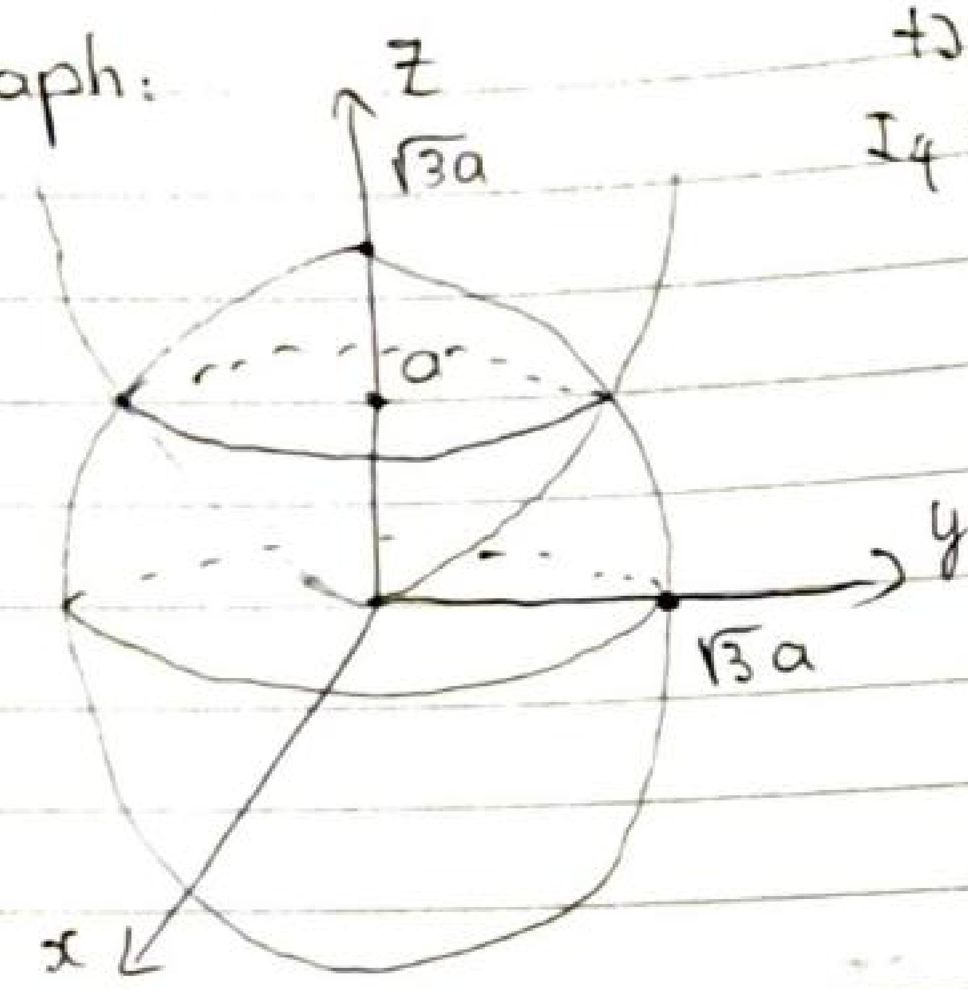
$= \frac{\pi}{3}\left(2\cdot t^{5/2}\cdot\frac{2}{5} - R^3\cdot t + 3R^2 t^{3/2}\cdot\frac{2}{3} - 3Rt^2\cdot\frac{1}{2} \right)\Big|_{R^2/4}^{R^2}$

$= \frac{\pi}{3} R^5 \left[\left(2\cdot\frac{2}{5} - 1 + 3\cdot\frac{2}{3} - 3\cdot\frac{1}{2}\right) - \left(2\cdot\left(\frac{1}{4}\right)^{5/2}\cdot\frac{2}{5} - \frac{1}{4} + 3\left(\frac{1}{4}\right)^{3/2}\cdot\frac{2}{3} - 3\cdot\left(\frac{1}{4}\right)^2\cdot\frac{1}{2}\right) \right]$

$= \pi\cdot\frac{59}{480}R^5$

d) $I_4 = \iiint\limits_V (x+y+z)^2 \, dxdydz$, V bounded by $\begin{cases} x^2+y^2 \leq 2az \\ x^2+y^2+z^2 \leq 3a^2 \end{cases}$

+) Graph:



+) Symetric property:

$I_4 = \iiint\limits_V x^2+y^2+z^2 \, dxdydz$

+) Intersection: $\begin{cases} x^2+y^2 = 2az \\ x^2+y^2+z^2 = 3a^2 \end{cases} \to z^2+2az = 3a^2$

$\begin{cases} z = a \ (z \geq 0) \\ x^2+y^2 = 2a^2 \end{cases}$

-> Projection of V on Oxy: $x^2+y^2 \leq 2a^2$

+) Put $x = r\cos\alpha$

$y = r\sin\alpha$

$z = z$

- $J = r$
- $D_{r\alpha z}$ $\begin{cases} 0 \leq \alpha \leq 2\pi \\ 0 \leq r \leq \sqrt{2}a \\ \frac{r^2}{2a} \leq z \leq \sqrt{3a^2-r^2} \end{cases}$

+) $I_4 = \int_0^{2\pi} d\alpha \int_0^{\sqrt{2}a} dr \int_{\frac{r^2}{2a}}^{\sqrt{3a^2-r^2}} r(r^2\cos^2\alpha + r^2\sin^2\alpha + z^2) \, dz$

$= 2\pi \int_0^{\sqrt{2}a} dr \int_{r^2/2a}^{\sqrt{3a^2-r^2}} r^3 + rz^2 \, dz = 2\pi \int_0^{\sqrt{2}a} r^3\left(\sqrt{3a^2-r^2} - \frac{r^2}{2a}\right) + \frac{r}{3}\left(\begin{matrix}(3a^2-r^2)\sqrt{3a^2-r^2} \\ -\frac{r^6}{8a^3}\end{matrix}\right)$

$$= 2\pi \int_0^{\sqrt{2}a} \frac{2}{3} r^3\sqrt{3a^2-r^2} - \frac{r^5}{2a} + a^2 r\sqrt{3a^2-r^2} - \frac{r^7}{24a^3} \, dr$$

+) $I_A = \int_0^{\sqrt{2}a} \frac{r^7}{24a^3} \, dr = \frac{r^8}{192a^3}\Big|_0^{\sqrt{2}a} = \frac{1}{12}a^5$

+) $I_B = \int_0^{\sqrt{2}a} \frac{2}{3} r^3\sqrt{3a^2-r^2} + a^2 r\sqrt{3a^2-r^2} \, dr$

$= \int_0^{\sqrt{2}a} \frac{1}{3} r^2\sqrt{3a^2-r^2} + \frac{a^2}{2}\sqrt{3a^2-r^2} \, dr^2$

$= -\int_{3a^2}^{a^2} \frac{1}{3}(3a^2-t)\sqrt{t} + \frac{a^2}{2}\sqrt{t} \, dt$ (Put $3a^2-r^2 = t$)

$= \int_{a^2}^{3a^2} \frac{3}{2}a^2\sqrt{t} - \frac{1}{3}t^{3/2} \, dt$

$= \frac{3}{2}a^2 . t^{3/2} . \frac{2}{3} - \frac{1}{3} . t^{5/2} . \frac{2}{5} \Big|_{a^2}^{3a^2}$

$= a^2(3a^2)^{3/2} - a^2(a^2)^{3/2} - \frac{2}{15}(3a^2)^{5/2} + \frac{2}{15}(a^2)^{5/2}$

$= a^5\left(3\sqrt{3} - 1 - \frac{2}{15} 9\sqrt{3} + \frac{2}{15} . 1\right)$

$= a^5\left(\frac{9\sqrt{3}}{5} - \frac{13}{15}\right)$

+) $I_C = \int_0^{\sqrt{2}a} \frac{r^5}{2a} \, dr = \frac{r^6}{12a}\Big|_0^{\sqrt{2}a} = \frac{2}{3}a^5$

=> $I = 2\pi(I_B - I_A - I_C) = 2\pi a^5\left(\frac{9\sqrt{3}}{5} - \frac{97}{60}\right)$

d) +) Graph: ; +) Symetric property: ;

+) Intersection: $\begin{cases} z = a \\ x^2+y^2 = 2a^2 \end{cases}$

+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$

- $J = -r^2\sin\theta$
- $D_{r\alpha\theta}$: $\begin{cases} \theta : 0 \to \cos^{-1}(\sqrt{3}/3), \ r: 0 \to \sqrt{3}a, \ \alpha: 0 \to 2\pi \\ \theta : \cos^{-1}(\sqrt{3}/3) \to \pi/2, \ r: 0 \to \frac{2a\cos\theta}{\sin^2\theta}, \ \alpha: 0 \to 2\pi \end{cases}$

+) $I = I_1 + I_2$

⊕ • $I_1 = \int_0^{2\pi} d\alpha \int_0^{\cos^{-1}(\sqrt{3}/3)} d\theta \int_0^{\sqrt{3}a} r^4 \sin\theta \, dr$

$= 2\pi \left(1 - \frac{\sqrt{3}}{3}\right) \cdot \frac{9\sqrt{3}}{5} a^5$

+) • $I_2 = \int_0^{2\pi} d\alpha \int_{\cos^{-1}(\sqrt{3}/3)}^{\pi/2} d\theta \int_0^{\frac{2a\cos\theta}{\sin^2\theta}} r^4 \sin\theta \, dr$

$= 2\pi \cdot \frac{1}{5} \int_{\cos^{-1}(\sqrt{3}/3)}^{\pi/2} \left(\frac{2a\cos\theta}{\sin^2\theta}\right)^5 \sin\theta \, d\theta$

$= 2\pi \cdot \frac{32}{5} \int_{\pi/2}^{\cos^{-1}(\sqrt{3}/3)} \left(\frac{\cos\theta}{1-\cos^2\theta}\right)^5 d\cos\theta$

$= 2\pi \, \frac{32}{5} a^5 \int_0^{\sqrt{3}/3} \left(\frac{x}{1-x^2}\right)^5 dx$

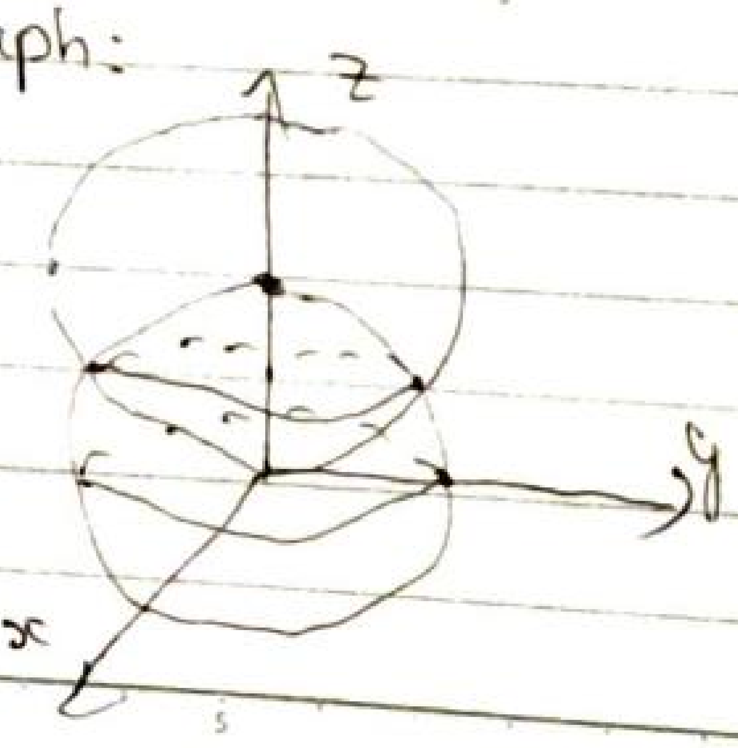
$= 2\pi \cdot \frac{32}{5} \cdot \frac{11}{384} a^5$

$\Rightarrow I = 2\pi \cdot \left(\frac{9\sqrt{3}}{5} - \frac{9}{5} + \frac{11}{60}\right) a^5$

$= 2\pi \left(\frac{9\sqrt{3}}{5} - \frac{97}{60}\right) a^5$

c) $I_3 = \iiint_D z^2 \, dx\,dy\,dz$, D bounded by $\begin{cases} x^2+y^2+z^2 \le R^2 \\ x^2+y^2+z^2 \le 2Rz \end{cases}$

+) Graph:



+) Put $x = r\cos\alpha \sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$ $\begin{cases} 0 \le \alpha \le 2\pi \\ 0 \le \theta \le \pi \\ 0 \le r \end{cases}$

• $J = -r^2 \sin\theta$

• $D \begin{cases} r^2 \le R^2 \\ r^2 \le 2Rr\cos\theta \end{cases}$ → Intersection: $\begin{cases} r = R \\ r^2 = 2Rr\cos\theta \end{cases} \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow D_{r\alpha\theta}$:

$\begin{cases} 0 \le \alpha \le 2\pi \\ \left[\begin{array}{l} 0 \le \theta \le \pi/3, \ 0 \le r \le R \\ \frac{\pi}{3} \le \theta \le \frac{\pi}{2}, \ 0 \le r \le 2R\cos\theta \end{array}\right. \end{cases}$

+) $I_1 = \int_0^{2\pi} d\alpha \int_0^{\pi/3} d\theta \int_0^R r^2\cos^2\theta \, r^2 \sin\theta \, dr$

$= 2\pi \, \frac{7}{24} \cdot \frac{R^5}{5}$

+) $I_2 = \int_0^{2\pi} d\alpha \int_{\pi/3}^{\pi/2} d\theta \int_0^{2R\cos\theta} r^2\cos^2\theta \, r^2 \sin\theta \, dr$

$= 2\pi \int_{\pi/3}^{\pi/2} \frac{(2R\cos\theta)^5}{5} \cos^2\theta \sin\theta \, d\theta$

$= 2\pi \cdot \frac{32R^5}{5} \cdot \frac{1}{2048}$

$\Rightarrow I = \pi \cdot \frac{59}{480} R^5$

(96) Volume of object bounded by Oxy, $x = 0$, $x = a$, $y = 0$, $y = b$, $z = \frac{x^2}{2p} + \frac{y^2}{2q}$ $(p > 0, q > 0)$

$V = \int_0^a dx \int_0^b dy \int_0^{\frac{x^2}{2p}+\frac{y^2}{2q}} dz = \int_0^a dx \int_0^b \frac{x^2}{2p} + \frac{y^2}{2q} \, dy$

$= \int_0^a \frac{x^2}{2p} \cdot b + \frac{1}{6q} \cdot b^3 \, dx = \frac{b}{2p} \cdot \frac{1}{3} \cdot a^3 + \frac{1}{6q} \cdot b^3 \cdot a$

$= \frac{a^3 b}{6p} + \frac{a b^3}{6q}$

(97) $\iiint \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$

$V: x^2+y^2+z^2 \le z$

+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$ $\quad \begin{cases} 0\le\alpha\le 2\pi \\ 0\le\theta\le\pi/2 \\ 0\le r\le\cos\theta \end{cases}$

• $J = -r^2\sin\theta$

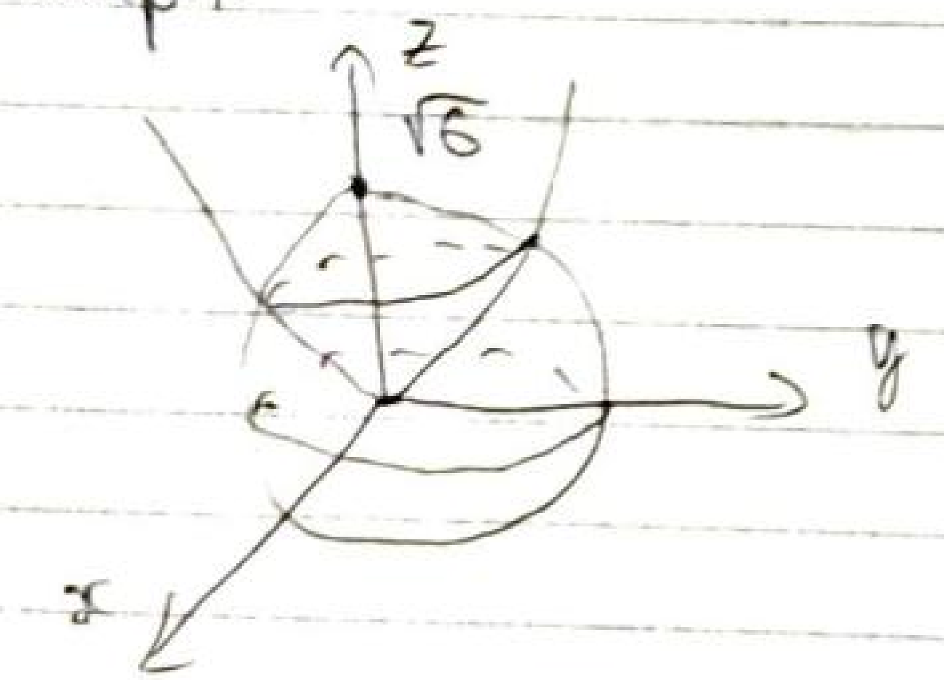
+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi/2} d\theta \int_0^{\cos\theta} r^3\sin\theta\,dr$

$= 2\pi \int_0^{\pi/2} \frac{1}{4}\cos^4\theta\sin\theta\,d\theta$

$= 2\pi\cdot\frac{1}{4}\cdot\frac{1}{5} = \frac{\pi}{10}$

(98) $I = \iiint_V z\,dx\,dy\,dz$ $\quad V \begin{cases} (x^2+y^2+z^2)^2 = 6^2 \\ z = x^2+y^2 \end{cases}$

+) Graph:



+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$ $\quad \begin{cases} 0\le r \\ 0\le\alpha\le 2\pi \\ 0\le\theta\le\pi \end{cases}$

• $J = -r^2\sin\theta$

• $D \begin{cases} r^2 \le 6 \\ r\cos\theta \ge r^2\sin^2\theta \end{cases} \rightarrow \begin{cases} r\le\sqrt{6} \\ r\le\frac{\cos\theta}{\sin^2\theta} \end{cases}$

Intersection: $\theta = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right)$

+) $I = I_1 + I_2$

• $I_1 = \int_0^{2\pi} d\alpha \int_0^{\cos^{-1}\left(\frac{\sqrt6}{3}\right)} d\theta \int_0^{\sqrt6} r\cos\theta\, r^2\sin\theta\,dr$

$= 2\pi\cdot\frac{1}{6}\cdot 9 = 3\pi$

• $I_2 = \int_0^{2\pi} d\alpha \int_{\cos^{-1}\left(\frac{\sqrt6}{3}\right)}^{\pi/2} d\theta \int_0^{\cos\theta/\sin^2\theta} r\cos\theta\, r^2\sin\theta\,dr$

$= 2\pi \int_{\cos^{-1}\left(\frac{\sqrt6}{3}\right)}^{\pi/2} \sin\theta\cdot\cos\theta\cdot\frac{1}{4}\frac{\cos^4\theta}{\sin^8\theta}\,d\theta$

$= 2\pi\cdot\frac{1}{4}\cdot\frac{4}{3} = \frac{2}{3}\pi$

$\Rightarrow I = \boxed{\frac{11}{3}\pi}$

(99) $I = \iiint_V \frac{xyz}{x^2+y^2}\,dx\,dy\,dz$

$V: \begin{cases} (x^2+y^2+z^2)^2 \le a^2xy \\ z\ge 0 \end{cases}$

+) Put $x = r\cos\alpha\sin\theta$, $y = r\sin\alpha\sin\theta$, $z = r\cos\theta$ $\quad \begin{cases} 0\le\alpha\le 2\pi \\ 0\le\theta\le\pi \\ 0\le r \end{cases}$

• $J = -r^2\sin\theta$

• $V \begin{cases} r^4 \le a^2\cos\alpha\sin^2\theta\sin\alpha \\ r\cos\theta\ge 0 \end{cases}$

$\Leftrightarrow \begin{cases} r^2 \le a^2\sin^2\theta\sin\alpha\cos\alpha \rightarrow \begin{cases} r\le\sqrt{a^2\sin\alpha\cos\alpha\sin\theta} \\ \left[\begin{array}{l} 0\le\alpha\le\frac{\pi}{2} \\ \frac{3\pi}{2}\ge\alpha\ge\pi \end{array}\right. \end{cases} \\ r\cos\theta\ge 0 \rightarrow 0\le\theta\le\frac{\pi}{2} \end{cases}$

+) $I = \iiint_{D_{r\alpha\theta}} \frac{r^3\cos\alpha\sin\alpha\sin^2\theta\cos\theta}{r^2\sin^2\theta}\cdot r^2\sin\theta\,dr\,d\alpha\,d\theta$

$= \iiint_{D_{r\alpha\theta}} r^3\sin\alpha\cos\alpha\sin\theta\cos\theta\,dr\,d\alpha\,d\theta$

+) $I_1 = \int_0^{\pi/2} dx \int_0^{\pi/2} d\theta \int_0^{\sqrt{a^2\sin x\cos x}\,.\,\sin\theta} r^3 \sin x\cos x \sin\theta\cos\theta\, dr$

$= \int_0^{\pi/2} dx \int_0^{\pi/2} \frac{1}{4} (a^2\sin x\cos x)^2 . \sin^4\theta .\, \sin x\cos x \sin\theta\cos\theta\, d\theta$

$= \int_0^{\pi/2} dx \int_0^{\pi/2} \frac{1}{4} a^4 \sin^3 x\cos^3 x \sin^5\theta\cos\theta\, d\theta$

$= \frac{1}{4} a^4 . \frac{1}{12} . \frac{1}{6} = \frac{1}{288} a^4$

+) Similarly, $I_2 = \frac{1}{288} a^4$

+) Conclusion: $I = I_1 + I_2 = \frac{1}{144} a^4$

Chapter 4: Integrals depending on a parameter.

⊛ Remind:

1. Definite integrals depending on a parameter:

a) Definition: Suppose that $f(x,y)$ is a continuous function defined on $[a,b]\times[c,d]$, then

$$I(y) = \int_a^b f(x,y)\,dx$$

is a function defined on $[c,d]$ and is called an integral depending on a parameter of function $f(x,y)$.

b) Continuity and taking limits under the integral sign

If function $f(x,y)$ is defined and continuous on the rectangle $[a,b]\times[c,d]$ then the integral $I(y)$ is continuous on $[c,d]$

$\lim_{y\to y_0} I(y) = \lim_{y\to y_0} \int_a^b f(x,y)\,dx = \int_a^b \lim_{y\to y_0} f(x,y)\,dx = \int_a^b f(x,y_0)\,dx = I(y_0)$

Example: $\lim_{y\to 0} \int_0^2 x^2\cos xy\, dx$

+) $f(x,y) = x^2\cos xy$ is continuous on R^2

=> $I(y)$ is continuous on R.

+) $\lim_{y\to 0} I(y) = I(0) = \int_0^2 x^2\,dx = \frac{8}{3}$

c) Leibniz's Theorem.

Suppose that:

i) $f(x,y)$ is continuous on $[a,b]\times[c,d]$

ii) $f'_y(x,y)$ is continuous on $[a,b]\times[c,d]$

Then $I(y)$ is differentiable on (c,d) and

$$I'(y) = \left(\int_a^b f(x,y)\,dx\right)'_y = \int_a^b f'_y(x,y)\,dx$$

Example:

a) $I(y) = \int_0^1 \arctan\frac{x}{y}\,dx$

+) $f(x,y) = \arctan\frac{x}{y}$ is continuous on $\left[\begin{array}{l} R\times(-\infty,0) \\ R\times(0,+\infty)\end{array}\right.$

+) $f'_y(x,y) = \dfrac{-x/y^2}{1+\frac{x^2}{y^2}} = \dfrac{-x}{y^2}\cdot\dfrac{y^2}{x^2+y^2} = \dfrac{-x}{x^2+y^2}$ is continuous on $\left[\begin{array}{l} R\times(-\infty,0) \\ R\times(0,+\infty)\end{array}\right.$

Leibniz => $I'(y) = \int_0^1 \frac{-x}{x^2+y^2}\,dx = \left.\frac{-\ln(x^2+y^2)}{2}\right|_0^1 = \frac{1}{2}\ln\frac{y^2}{y^2+1}$

=> $I(y) = \int \frac{1}{2}\ln\frac{y^2}{y^2+1}\,dy + C$

$= \frac{1}{2} y \ln\frac{y^2}{y^2+1} - \arctan(y) + C$

+) $I(0^+) = \frac{\pi}{2} = 0 - 0 + C \Rightarrow C = \frac{\pi}{2}$

=> $I(y) = \frac{1}{2} y\ln\frac{y^2}{y^2+1} - \arctan y + \frac{\pi}{2}$

b) $J(y)=\int_0^1 \ln(x^2+y^2)\,dx$

+) $f(x,y)=\ln(x^2+y^2)$ is continuous on $R\times R \,/\, (0;0)$

+) $f'_y(x,y)=\dfrac{2y}{x^2+y^2}$ is continuous on $R\times R \,/\, (0;0)$

$\overset{\text{Leibniz}}{\Rightarrow} J'(y)=\int_0^1 \dfrac{2y}{x^2+y^2}\,dx = 2y.\dfrac{1}{y}\arctan\dfrac{x}{y}\Big|_0^1$

$=2\arctan\dfrac{1}{y}$

$\Rightarrow J(y)=\int 2\arctan\dfrac{1}{y}\,dy+C$

$=2y\arctan\dfrac{1}{y}-2\int y.\dfrac{-1/y^2}{1+\frac{1}{y^2}}\,dy+C$

$=2y\arctan\dfrac{1}{y}+\int\dfrac{2y}{y^2+1}\,dy+C$

$=2y\arctan\dfrac{1}{y}+\ln(y^2+1)+C$

+) $I(0)=-2 \Rightarrow C=-2$

$\Rightarrow J(y)=2y\arctan\dfrac{1}{y}+\ln(y^2+1)-2$

d) If $f(x,y)$ is defined and continuous on $[a,b]\times[c,d]$

then:

$$\int_c^d I(y)\,dy=\int_c^d dy\int_a^b f(x,y)\,dx=\int_a^b dx\int_c^d f(x,y)\,dy.$$

Example: $I=\int_0^1 \dfrac{x^b-x^a}{\ln x}\,dx \;(0<a<b)$

+) $f(x,y)=x^y$ is continuous on R^2

+) $I=\int_0^1 dx\int_a^b x^y\,dy=\int_a^b dy\int_0^1 x^y\,dx$

$=\int_a^b \dfrac{1}{y+1}\,dy=\ln\dfrac{b+1}{a+1}$

2 Improper integrals depending on a parameter.

a) Definition:

- Consider $I(y)=\int_a^{+\infty} f(x,y)\,dx$, $y\in[c,d]$

- We say that the integral $I(y)$ is

i) convergent at $y_0\in[c,d]$ if $\int_a^{\infty} f(x,y_0)\,dx$ is convergent

ie, $\forall\varepsilon>0, \exists b=b(\varepsilon,y_0)>a$ (depending on ε and y_0):

$\left|I(y_0)-\int_a^A f(x,y_0)\,dx\right|=\left|\int_A^{\infty} f(x,y_0)\,dx\right|<\varepsilon$ for all $A>b$.

ii) convergent on $[c,d]$ if $I(y)$ is convergent at any $y\in[c,d]$

iii) uniformly convergent on $[c,d]$ if $\forall\varepsilon>0, \exists b=b(\varepsilon)>a$:

$\left|\int_A^{\infty} f(x,y)\,dx\right|<\varepsilon$ for all $A>b$ and $y\in[c,d]$

Example: a) $I(y)=\int_0^{+\infty} y e^{-yx}\,dx=-e^{-yx}\Big|_{x=0}^{x=\infty}=1 \;\forall y>0$

b) Prove $I(y)$ converges to 1 uniformly on $[y_0,+\infty)$ for all $y_0>0$

$\Leftrightarrow \forall\varepsilon>0, \exists b_\varepsilon>0: \int_A^{\infty} y e^{-yx}\,dx<\varepsilon$ for all $A>b_\varepsilon$, $y\in[y_0,\infty)$

$\Leftrightarrow e^{-yA}<\varepsilon$

$\Leftrightarrow e^{-y_0 b_\varepsilon}\le\varepsilon$

$\Leftrightarrow -y_0 b_\varepsilon\le\ln\varepsilon$

$\Leftrightarrow y_0 b_\varepsilon\ge\ln\dfrac{1}{\varepsilon}$ (pick $b_\varepsilon=\dfrac{1}{y_0}\ln\dfrac{1}{\varepsilon}$

$\Leftrightarrow b_\varepsilon\ge\dfrac{1}{y_0}\ln\dfrac{1}{\varepsilon}$ (Satisfied)

$b_\varepsilon=b(\varepsilon,y_0)$

c) Explain why $I(y)$ is not uniformly convergent on $(0,+\infty)$

$\dfrac{1}{y}\ln\dfrac{1}{\varepsilon}\to+\infty$ when $y\to 0^+ \Rightarrow \nexists\, b_\varepsilon$

b) Sufficient conditions for uniform convergence

Theorem (Weierstrass Criterion)

If

i) $|f(x,y)| \le g(x) \quad \forall (x,y) \in [a,+\infty) \times [c,d]$

ii) The improper integral $\int_a^{+\infty} g(x)\,dx$ is convergent

then $I(y) = \int_a^{+\infty} f(x,y)\,dx$ is uniformly convergent on $[c,d]$

c) Continuity and taking limits under the integral sign.

Theorem.

If

i) $f(x,y)$ is continuous on $[a,+\infty) \times [c,d]$

ii) $I(y) = \int_a^{+\infty} f(x,y)\,dx$ is uniformly convergent on $[c,d]$

then $I(y)$ is continuous on $[c,d]$, ie

$$\lim_{y\to y_0} I(y) = \lim_{y\to y_0} \int_a^{+\infty} f(x,y)\,dx = \int_a^{+\infty} \lim_{y\to y_0} f(x,y)\,dx = \int_a^{+\infty} f(x,y_0)\,dx.$$

d) Differentiation under the integral sign.

Theorem

If

i) $f(x,y)$ and $f'_y(x,y)$ are continuous on $[a,+\infty) \times [c,d]$

ii) $I(y) = \int_a^{+\infty} f(x,y)\,dx$ is convergent on $[c,d]$

iii) $\int_a^{+\infty} f'_y(x,y)\,dx$ is uniformly convergent on $[c,d]$

then $I(y)$ is differentiable on $[c,d]$ and $I'(y) = \int_a^{+\infty} f'_y(x,y)\,dx$

e) Integration under the Integral sign.

Theorem

If

i) $f(x,y)$ is continuous on $[a,+\infty) \times [c,d]$

ii) $I(y) = \int_a^{+\infty} f(x,y)\,dx$ is uniformly convergent on $[c,d]$

then, $I(y)$ is integrable on $[c,d]$ and

$$\int_c^d I(y)\,dy = \int_c^d \left(\int_a^{+\infty} f(x,y)\,dx \right) dy = \int_a^{+\infty} \left(\int_c^d f(x,y)\,dy \right) dx.$$

f) Integration techniques

⊛ Differentiation under the integral sign.

S_1: $I'(y) = \int_a^{+\infty} f'_y(x,y)\,dx$

S_2: $I(y) = \int I'(y)\,dy + C$

S_3: $I(y_0) = ? \to C = ?$

Remark: check the conditions.

⊛ Integration under the integral sign.

S_1: $f(x,y) = \int_c^d F(x,y)\,dy$

S_2: Change the order of integration:

$$\int_a^{+\infty} f(x,y)\,dx = \int_a^{+\infty} \left(\int_c^d F(x,y)\,dy \right) dx = \int_c^d \left(\int_a^{+\infty} F(x,y)\,dx \right) dy$$

Remark: check the conditions.

3. Euler integral

a) The Gamma function

$\Gamma(p) = \int_0^{+\infty} x^{p-1} e^{-x}\,dx\,dx$ defined on $(0,+\infty)$

Properties:

i) $\Gamma(p+1) = p\Gamma(p)$

If $\alpha \in (n, n+1]$ then $\Gamma(\alpha) = (\alpha-1)(\alpha-2)\dots(\alpha-n)\Gamma(\alpha-n)$

Specially $\begin{cases} \Gamma(1) = 1 \\ \Gamma(\frac{1}{2}) = \sqrt{\pi} \end{cases} \to \begin{cases} \Gamma(n) = (n-1)! \\ \Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} \end{cases}$

ii) Derivative of the Gamma function:

$$\Gamma^{(k)}(p) = \int_0^{+\infty} x^{p-1}(\ln x)^k e^{-x}\,dx$$

iii) $\Gamma(p).\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ $\forall\, 0<p<1$

b) The Beta function:

Form 1: $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

Form 2: $B(p,q) = \int_0^{+\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\,dx$ $\left(x = \frac{t}{t+1}\right)$

Properties:

i) $B(p,q) = B(q,p)$

ii) $B(p,q) = \frac{p-1}{p+q-1}B(p-1,q)$, if $p>1$

Specially: $B(1,1) = 1 \to B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$ $\forall m,n \in \mathbb{N}$

Trigonometric form: $B(p,q) = 2\int_0^{\pi/2} \sin^{2p-1}t\cos^{2q-1}t\,dt$

Relation between $B - \Gamma$: $B(p,q) = \frac{\Gamma(p).\Gamma(q)}{\Gamma(p+q)}$

$\boxed{\text{IMPORTANT INTEGRAL}}$

Dirichlet: $\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Gauss: $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Fresnel: $\int_0^{\infty} \sin(x^2)\,dx = \int_0^{\infty} \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

(100)

a) $\lim_{y\to 0} \int_y^{1+y} \frac{dx}{1+x^2+y^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$

b) $\lim_{y\to 0} \int_0^2 x^2\cos xy\,dx = \int_0^2 x^2\,dx = \frac{8}{3}$

(101)

a) $I(y) = \int_0^1 \arctan\frac{x}{y}\,dx$

+) $I'(y) = \int_0^1 \frac{-x}{y^2}\cdot\frac{1}{1+(\frac{x}{y})^2}\,dx = \int_0^1 \frac{-x}{x^2+y^2}\,dx$

$= -\frac{\ln(x^2+y^2)}{2}\Big|_0^1 = \ln(y) - \frac{1}{2}\ln(y^2+1)$

+) $I(y) = \int \ln(y) - \frac{1}{2}\ln(y^2+1)\,dy + C$

$= y\ln(y) - \int y\cdot\frac{1}{y}\,dy - \frac{1}{2}y\ln(y^2+1) + \frac{1}{2}\int y\cdot\frac{2y}{y^2+1}\,dy + C$

$= y\ln(y) - y - \frac{1}{2}y\ln(y^2+1) + y - \arctan y + C$

$= \frac{1}{2}y\ln\frac{y^2}{y^2+1} - \arctan(y) + C$

+) $I(0^+) = \frac{\pi}{2} = 0 - 0 + C = C$

$\Rightarrow I(y) = \boxed{\frac{1}{2}y\ln\frac{y^2}{y^2+1} - \arctan(y) + \frac{\pi}{2}}$

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b) $J(y) = \int_0^1 \ln(x^2+y^2)\,dx$

+) $J'(y) = \int_0^1 \frac{2y}{\ln(x^2+y^2)}\,dx = 2y \cdot \frac{1}{y} \cdot \arctan\left(\frac{x}{y}\right)\Big|_0^1$

$= 2\arctan(\frac{1}{y})$

+) $J(y) = \int 2\arctan\left(\frac{1}{y}\right)dy + C = 2y\arctan\left(\frac{1}{y}\right) - 2\int y \cdot \frac{-1}{y^2} \cdot \frac{1}{1+\frac{1}{y^2}} dy + C$

$= 2y\arctan\frac{1}{y} + 2\int \frac{y}{1+y^2}\,dy + C$

$= 2y\arctan\frac{1}{y} + \ln(y^2+1) + C$

+) $J(0) = -2 = 0 + 0 + C = C$

$\Rightarrow J(y) = \boxed{2y\arctan\frac{1}{y} + \ln(y^2+1) - 2}$

c) $K = \int_0^1 \frac{x^b - x^a}{\ln x}\,dx \quad (0 < a < b)$

$= \int_0^1 dx \int_a^b x^y\,dy = \int_a^b dy \int_0^1 x^y\,dx = \int_0^b \frac{1}{y+1} \cdot dy$

$= \ln(y+1) \Big|_a^b = \ln\frac{(b+1)}{a+1}$

102 → removed

103 → removed

104 Prove

a) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$ (Gauss)

b) $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ (Dirichlet)

c) $\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ (Fresnel

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d) $\int_0^\infty e^{-yx}\frac{\sin x}{x}\,dx = \frac{\pi}{2} - \arctan y.$

+) $I'(y) = \int_0^\infty -e^{-yx}\sin x\,dx$

$= \int_0^\infty e^{-yx}\,d\cos x$

$= \cos x \cdot e^{-yx}\Big|_{x=0}^{x=\infty} + \int_0^\infty \cos x \cdot y \cdot e^{-yx}\,dx$

$= -1 + y\int_0^\infty e^{-yx}\,d\sin x$

$= -1 + y\left(\sin x\, e^{-yx}\Big|_{x=0}^{x=\infty} + \int_0^\infty y\sin x\, e^{-yx}\,dx\right)$

$= -1 + y^2\int_0^\infty \sin x\, e^{-yx}\,dx$

$= -1 - y^2 I'(y)$

$\Rightarrow I'(y) = \frac{-1}{1+y^2}$

$\Rightarrow I(y) = -\arctan y + C$

+) $I(0) = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \Rightarrow 0 + C = \frac{\pi}{2} \Rightarrow C = \frac{\pi}{2}$

$\Rightarrow I(y) = \boxed{\frac{\pi}{2} - \arctan y}$

e) $\int_0^\infty \frac{\sin yx}{x(1+x^2)}\,dx = \frac{\pi}{2}(1 - e^{-y})$

+) $I(y) = \int_0^\infty \frac{\sin yx}{x(x^2+1)}\,dx$

+) $I'(y) = \int_0^\infty \frac{\cos yx}{x^2+1}\,dx$

+) $I''(y) = \int_0^\infty \frac{-\sin(yx) \cdot x}{x^2+1}\,dx = \int_0^\infty \sin yx \cdot \frac{1-(x^2+1)}{x(x^2+1)}\,dx$

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$= I(y) - \frac{\pi}{2}$

$\Rightarrow I'(y) = I'''(y)$

+) Put $I'(y) = C_1 e^{C_2 y}$

- $I'(y) = I'''(y) \Rightarrow C_1 e^{C_2 y} = C_1 \cdot C_2^2 e^{C_2 y} \Rightarrow C_2 = \pm 1$

- $I'(y) = C_3 \cdot e^{y} + C_4 \cdot e^{-y}$
  $I'(0^+) = \frac{\pi}{2} \Rightarrow C_3 + C_4 = \frac{\pi}{2}$

- $I''(y) = C_3 \cdot e^{y} - C_4 \cdot e^{-y}$
  $I''(0^+) = \frac{-\pi}{2} \Rightarrow C_3 - C_4 = \frac{-\pi}{2}$

Thus, $C_3 = 0$, $C_4 = \frac{\pi}{2}$

+) $I'(y) = \frac{\pi}{2} \cdot e^{-y}$

$\Rightarrow I(y) = \frac{-\pi}{2} e^{-y} + C$

$I(0^+) = \frac{-\pi}{2} + C = 0 \Rightarrow C = \frac{\pi}{2}$

$\Rightarrow I(y) = \frac{-\pi}{2} \cdot e^{-y} + \frac{\pi}{2} = \boxed{\frac{\pi}{2}(1 - e^{-y})}$

f) $\int_0^\infty \frac{1-\cos yx}{x^2} dx$

$= \int_0^\infty \cos(yx) - 1 \, d\frac{1}{x} = \frac{\cos(yx) - 1}{x}\Big|_0^\infty + \int_0^\infty \frac{y \sin yx}{x} dx$

$= y\frac{\pi}{2} \cdot \mathrm{sign}(y) = |y|\frac{\pi}{2}$

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k) $\lim_{y \to 0^+} \left(\int_0^{+\infty} y \cdot e^{-yx} dx \right) \neq \int_0^\infty \left(\lim_{y \to 0^+} y \cdot e^{-yx} \right) dx$

Why?

g) $\int_0^\infty \frac{x \sin yx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ay}$; $a, y \geq 0$

+) $I(y) = \int_0^\infty \frac{x \sin yx}{a^2 + x^2} dx = \frac{\pi}{2} - a^2 \int_0^\infty \frac{\sin yx}{x(a^2 + x^2)} dx$

$\Rightarrow I'(y) = -a^2 \int_0^\infty \frac{\cos yx}{x^2 + a^2} dx$

$\Rightarrow I''(y) = a^2 \int_0^\infty \frac{x \sin yx}{a^2 + x^2} dx = a^2 I(y)$

+) Put $I(y) = C_1 \cdot e^{C_2 y}$

- $I''(y) = a^2 I(y) \Rightarrow C_2^2 = a^2 \Rightarrow C_2 = \pm a$
- $I(y) = C_3 \cdot e^{ay} + C_4 \cdot e^{-ay}$
  $I(0^+) = \frac{\pi}{2} \Rightarrow C_3 + C_4 = \frac{\pi}{2}$

- $I'(0^+) = -a^2 \int_0^\infty \frac{1}{x^2 + a^2} dx = -a^2 \cdot \frac{1}{a} \arctan\left(\frac{x}{a}\right)\Big|_{x=0}^{x=\infty}$
  $= -a \cdot \frac{\pi}{2} = aC_3 - aC_4 \Rightarrow C_3 - C_4 = -\frac{\pi}{2}$

Thus $C_3 = 0$, $C_4 = \pm\frac{\pi}{2}$

+) $I(y) = \frac{\pi}{2} \cdot e^{-ay}$

h) $\int_0^\infty e^{-yx^2} dx = \frac{\sqrt{\pi}}{2\sqrt{y}}$; $y > 0$

$I(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-yx^2} d\sqrt{y}x = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{y}}$ (Gauss integral)

i) $\int_0^\infty (e^{-\frac{a}{x^2}} - e^{-\frac{b}{x^2}})dx = \sqrt{\pi b} - \sqrt{\pi a}$; $a,b>0$

$I'(a) = \int_0^\infty e^{-a/x^2}\cdot\frac{-1}{x^2}dx$

$= \int_0^\infty e^{-a/x^2}\, d(1/x)$

$= \frac{1}{\sqrt a}\int_0^\infty e^{-a/x^2}\, d\,1/x = \frac{-\sqrt\pi}{2\sqrt a}$

$\Rightarrow I(a) = -\sqrt{\pi a} + C$

$I(a=b) = 0 \Rightarrow C = \sqrt{\pi b}$

$\Rightarrow \int_0^\infty e^{-a/x^2} - e^{-b/x^2}dx = \boxed{\sqrt{\pi b} - \sqrt{\pi a}}$

j) $\int_0^{+\infty}\frac{\arctan\frac{x}{a} - \arctan\frac{x}{b}}{x}dx = \frac{\pi}{2}\ln\frac{b}{a}$; $a,b>0$

+) $I'(a) = \int_0^{+\infty}\frac{-x}{a^2}\cdot\frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{x}dx$

$= \int_0^{+\infty}\frac{-1}{x^2+a^2}dx$

$= -\frac{1}{a}\arctan\frac{x}{a}\Big|_{x=0}^{x=\infty}$

$= -\frac{1}{a}\cdot\frac{\pi}{2}$

$\Rightarrow I(a) = \frac{-\pi}{2}\ln(a) + C$

+) $I(b) = 0 \Rightarrow C = \frac{\pi}{2}\ln(b)$

$\Rightarrow \int_0^\infty\frac{\arctan\frac{x}{a} - \arctan\frac{x}{b}}{x}dx = \boxed{\frac{\pi}{2}\ln\frac{b}{a}}$

(1C5) a) $\int_0^{+\infty}\frac{e^{-\alpha x} - e^{-\beta x}}{x}dx$

+) $I'(\alpha) = \int_0^\infty -e^{-\alpha x}dx = \frac{1}{\alpha}e^{-\alpha x}\Big|_{x=0}^{x=\infty} = \frac{-1}{\alpha}$

$\Rightarrow I(\alpha) = -\ln(\alpha) + C$

+) $I(\beta) = 0 \Rightarrow C = \ln(\beta)$

$\Rightarrow I = \boxed{\frac{\ln\beta}{\alpha}}$

b) $\int_0^{+\infty}\frac{e^{-\alpha x^2} - e^{-\beta x^2}}{x^2}dx$

+) $I'(\alpha) = \int_0^\infty -e^{-\alpha x^2}dx = -\frac{1}{\sqrt\alpha}\int_0^\infty e^{-\alpha x^2}d\sqrt\alpha x = \frac{-\sqrt\pi}{2\sqrt\alpha}$

$\Rightarrow I(\alpha) = -\sqrt{\pi\alpha} + C$

+) $I(\beta) = 0 \Rightarrow C = \sqrt{\pi\beta}$

$\Rightarrow I = \boxed{\sqrt{\pi\beta} - \sqrt{\pi\alpha}}$

c) $\int_0^{+\infty}\frac{dx}{(x^2+y)^{n+1}}$

+) Put $I(y) = \frac{1}{x^2+y}$

. $I^{(1)}(y) = \frac{(-1)}{(x^2+y)^2}$

. $I^{(2)}(y) = \frac{(-1)(-2)}{(x^2+y)^3}$

. $I^{(n)}(y) = \frac{n!(-1)^n}{(x^2+y)^{n+1}}$

$\Rightarrow \int_0^{+\infty}\frac{1}{(x^2+y)^{n+1}}dx = \frac{(-1)^n}{n!}\cdot\int_0^{+\infty}I^{(n)}(y)dx = \frac{(-1)^n}{n!}\left(\int_0^{+\infty}I(y)dx\right)^{(n)}$

$= \frac{(-1)^n}{n!}\cdot\left(\frac{\pi}{2\sqrt y}\right)^{(n)} = \frac{(-1)^n}{n!}\cdot\frac{\pi}{2}\cdot\frac{(-1)^n\cdot(2n-1)!!}{2^n y^{\frac{2n+1}{2}}} = \boxed{\frac{\pi(2n-1)!!}{2^{n+1}n!\,y^{\frac{2n+1}{2}}}}$

d) $\int_0^{\infty} e^{-ax}\frac{\sin bx - \sin cx}{x}\,dx$

+) $I'(b) = \int_0^{\infty} e^{-ax}\cos bx\,dx$

$= \frac{1}{b}\int_0^{\infty} e^{-ax}\,d\sin bx = \frac{1}{b}e^{-ax}\sin bx\Big|_{x=0}^{x=\infty} - \frac{1}{b}\int_0^{\infty}\sin bx\,de^{-ax}$

$= -\frac{1}{b}\cdot(-a)\int_0^{\infty} e^{-ax}\sin bx\,dx$

$= \frac{a}{b}\cdot\frac{-1}{b}\int_0^{\infty} e^{-ax}\,d\cos bx$

$= \frac{-a}{b^2}e^{-ax}\cos bx\Big|_{x=0}^{x=\infty} + \frac{a}{b^2}\int_0^{\infty}\cos bx\,de^{-ax}$

$= \frac{a}{b^2} - \frac{a^2}{b^2}\int_0^{\infty} e^{-ax}\cos bx\,dx$

$= \frac{a}{b^2} - \frac{a^2}{b^2}I'(b)$

$\Rightarrow I'(b) = \frac{a}{b^2}\cdot\frac{b^2}{a^2+b^2} = \frac{a}{a^2+b^2}$

$\Rightarrow I(b) = a\cdot\frac{1}{a}\arctan\frac{b}{a} + C = \arctan\frac{b}{a} + C$

+) $I(c) = 0 \Rightarrow C = -\arctan\frac{c}{a}$

$\Rightarrow I = \boxed{\arctan\frac{b}{a} - \arctan\frac{c}{a}}$

e) $\int_0^{+\infty} e^{-ax}\frac{\cos bx - \cos cx}{x}\,dx$

+) $I'(b) = -\int_0^{+\infty} e^{-ax}\sin bx\,dx$

$= \frac{1}{b}\int_0^{+\infty} e^{-ax}\,d\cos bx = \frac{1}{b}e^{-ax}\cos bx\Big|_{x=0}^{x=\infty} - \frac{1}{b}\int_0^{\infty}\cos bx\,de^{-ax}$

$= \frac{-1}{b} - \frac{1}{b}\cdot(-a)\int_0^{\infty} e^{-ax}\cos bx\,dx$

$= \frac{-1}{b} + \frac{a}{b}\cdot\frac{1}{b}\int_0^{\infty} e^{-ax}\,d\sin bx$

$= \frac{-1}{b} + \frac{a}{b^2}e^{-ax}\sin bx\Big|_{x=0}^{x=\infty} - \frac{a}{b^2}\int_0^{\infty}\sin bx\,de^{-ax}$

$= \frac{-1}{b} + \frac{a^2}{b^2}\int_0^{\infty} e^{-ax}\sin bx\,dx$

$= \frac{-1}{b} + \frac{a^2}{b^2}I'(b)$

$\Rightarrow I'(b) = \frac{-1}{b}\cdot\frac{b^2}{b^2-a^2} = \frac{-b}{b^2-a^2}$

$\Rightarrow I(b) = -\frac{\ln(b^2-a^2)}{2} + C$

+) $I(c) = 0 \Rightarrow C = \frac{+\ln(c^2-a^2)}{2}$

$\Rightarrow I = \frac{1}{2}\left|\ln\left(\frac{b^2-a^2}{c^2-a^2}\right)\right| = \boxed{\frac{1}{2}\ln\frac{c^2-a^2}{b^2-a^2}}$

f) $\int_0^{+\infty} e^{-ax}\cos\gamma x\,dx$

$= \frac{1}{\gamma}\int_0^{+\infty} e^{-ax}\,d\sin\gamma x$

$= \frac{1}{\gamma}e^{-ax}\sin\gamma x\Big|_{x=0}^{x=\infty} - \frac{1}{\gamma}\int_0^{+\infty}\sin\gamma x\,de^{-ax}$

$= 0 - \frac{1}{\gamma}\cdot(-a)\int_0^{+\infty} e^{-ax}\sin\gamma x\,dx$

$= \frac{a}{\gamma}\cdot\frac{-1}{\gamma}\int_0^{\infty} e^{-ax}\,d\cos\gamma x$

$= \frac{-a}{\gamma^2}\cdot e^{-ax}\cos\gamma x\Big|_{x=0}^{x=\infty} + \frac{a}{\gamma^2}\int_0^{\infty}\cos\gamma x\,de^{-ax}$

$= \frac{a}{\gamma^2} - \frac{a^2}{\gamma^2}I \Rightarrow I = \frac{a}{\gamma^2}\cdot\frac{\gamma^2}{\gamma^2+a^2} = \boxed{\frac{a}{\gamma^2+a^2}}$

g) $\int_0^{+\infty} e^{-x^2}\cos(yx)\,dx$

+) $I'(y) = \int_0^{+\infty} e^{-x^2}(-x)\sin yx\,dx = \frac{1}{2}\int_0^{+\infty} \sin yx\, d e^{-x^2}$

$= \frac{1}{2} e^{-x^2}\sin yx \Big|_{x=0}^{x=\infty} - \frac{1}{2} y \int_0^{+\infty} e^{-x^2}\cos yx\,dx$

$= 0 - \frac{1}{2} y\, I(y)$

$= -\frac{1}{2} y\, I(y)$

+) $I(y) = C\, e^{-y^2/4}$

+) $I(0) = C = \frac{\sqrt{\pi}}{2}$

$\Rightarrow I(y) = \boxed{\frac{\sqrt{\pi}}{2} e^{-y^2/4}}$

h) $\int_{-\infty}^{+\infty} \frac{\arctan(x+y)}{1+x^2}\,dx$

+) $I'(y) = \int_{-\infty}^{+\infty} \frac{1}{(1+x^2)(1+(x+y)^2)}\,dx$

• $\frac{1}{(x^2+1)((x+y)^2+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x+y)^2+1}$

$\rightarrow 1 = (Ax+B)(x^2+2yx+y^2+1) + (Cx+D)(x^2+1)$

$$\left.\begin{array}{l} x^3:\ A+C=0 \\ x^2:\ 2yA+B+D=0 \\ x:\ (y^2+1)A+2yB+C=0 \\ 1:\ (y^2+1)B+D=1 \end{array}\right\} \Rightarrow \begin{array}{l} A = \frac{-2}{y^3+4y};\ B = \frac{1}{y^2+4} \\ C = \frac{2}{y^3+4y};\ D = \frac{3}{y^2+4} \end{array}$$

$\rightarrow I'(y) = \int_{-\infty}^{+\infty} \frac{Ax+B}{x^2+1} + \frac{C(x+y)+D-yC}{(x+y)^2+1}\,dx$

$= \frac{A}{2}\ln(x^2+1) + B\arctan(x) + \frac{C}{2}\ln((x+y)^2+1) + (D-yC)\arctan(x+y) \Big|_{x=-\infty}^{x=+\infty}$

$= B\pi + (D-yC)\pi$

$= \pi(B+D-yC)$

$= \pi\left(\frac{1}{y^2+4} + \frac{3}{y^2+4} - \frac{2}{y^2+4}\right)$

$= \pi\cdot\frac{2}{y^2+4}$

$\rightarrow I(y) = \pi\, 2\cdot\frac{1}{2}\arctan\left(\frac{y}{2}\right) + C = \pi\arctan\left(\frac{y}{2}\right) + C$

+) $I(0) = 0 = 0 + C$

$\rightarrow C = 0$

$\rightarrow I(y) = \boxed{\pi\arctan\left(\frac{y}{2}\right)}$

i) $\int_0^{+\infty} \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx \quad (a,b>0)$

+) $I'(a) = \int_0^{+\infty} -x e^{-ax^2}\,dx = \frac{1}{2a}\int_0^{+\infty} d e^{-ax^2} = \frac{1}{2a} e^{-ax^2}\Big|_0^{+\infty} = \frac{-1}{2a}$

$\rightarrow I(a) = -\frac{1}{2}\ln(a) + C$

+) $I(b) = 0 = C - \frac{1}{2}\ln(b)$

$\rightarrow C = \frac{1}{2}\ln(b)$

$\rightarrow I = \boxed{\frac{1}{2}\ln\left(\frac{b}{a}\right)}$

i) $\int_0^{+\infty} \frac{e^{-ax^3}-e^{-bx^3}}{x}\,dx \quad (a,b>0)$

+) $I'(a)=\int_0^{+\infty} -x^2e^{-ax^3}\,dx=\frac{1}{3a}\cdot e^{-ax^3}\Big|_{x=0}^{x=+\infty}=\frac{-1}{3a}$

$\to I(a)=\frac{-1}{3}\ln(a)+C$

+) $I(b)=0 \to C=\frac{1}{3}\ln(b)$

$\to I=\boxed{\frac{1}{3}\ln\frac{b}{a}}$

j) $\int_0^{\infty} \frac{e^{-ax^2}-\cos bx}{x^2}\,dx \quad (a>0)$

+) $dI(a,b)=\left(\int_0^{\infty} -e^{-ax^2}dx\right)da+\left(\int_0^{\infty}\frac{\sin bx}{x}dx\right)db$

$=\frac{-1}{\sqrt{a}}\cdot\frac{\sqrt{\pi}}{2}\,da+\frac{\pi}{2}\,db$

$\to I(a,b)=-\sqrt{\pi a}+\frac{\pi}{2}b+C$

+) $I(0,0)=0 \to C=0 \to I(a,b)=\boxed{-\sqrt{\pi a}+\frac{\pi}{2}b}$

k) $\int_0^{\pi} \ln(1+y\cos x)\,dx$

+) $I'(y)=\int_0^{\pi}\frac{\cos x}{1+y\cos x}\,dx$

• Put $z=\tan\frac{x}{2}$: $dx=\frac{2dz}{1+z^2}$; $\cos x=\frac{1-z^2}{1+z^2}$

$\to I'(y)=\int_0^{\infty}\frac{1-z^2}{1+z^2}\cdot\frac{1}{1+y\cdot\frac{1-z^2}{1+z^2}}\cdot\frac{2}{1+z^2}\,dz$

$=\int_0^{\infty}\frac{2(1-z^2)}{1+z^2}\cdot\frac{1}{1+z^2+y(1-z^2)}\,dz$

$=\int_0^{\infty}\frac{2(z^2-1)}{(z^2+1)(z^2(y-1)-y-1)}\,dz$

$=\int_0^{\infty}\frac{2}{y(z^2+1)}+\frac{2}{(y^2-y)z^2-(y^2+y)}\,dz$

$=\frac{2}{y}\cdot\int_0^{\infty}\frac{1}{z^2+1}\,dz+\frac{2}{(y^2-y)}\int_0^{\infty}\frac{1}{z^2+\frac{y^2+y}{y-y^2}}\,dz$

$=\frac{2}{y}\cdot\arctan z\Big|_{z=0}^{z=\infty}+\frac{2}{y^2-y}\cdot\frac{1}{\sqrt{\frac{y^2+y}{y-y^2}}}\cdot\arctan\frac{z}{\sqrt{\frac{y^2+y}{y-y^2}}}\Big|_{z=0}^{z=\infty}$

$=\frac{\pi}{2}\left(\frac{2}{y}+\frac{-2}{\sqrt{y^2+y}\cdot\sqrt{y-y^2}}\right)$

$=\pi\left(\frac{1}{y}-\frac{1}{y\sqrt{1-y^2}}\right)$

$\to I(y)=\pi\left(\ln(y)+\ln\left(\frac{1}{y}+\frac{\sqrt{1-y^2}}{y}\right)\right)+C$

$=\pi\ln(1+\sqrt{1-y^2})+C$

+) $I(0)=\pi\ln(2)+C=0 \to C=-\pi\ln(2)$

$\to I(y)=\boxed{\pi\ln\frac{1+\sqrt{1-y^2}}{2}}$

l) $\int_0^{\infty} e^{-x^2}\sin ax\,dx$

+) $I'(a)=\int_0^{\infty} x e^{-x^2}\cos ax\,dx=\frac{-1}{2}\int_0^{\infty}\cos ax\,d e^{-x^2}$

$=\frac{-1}{2}e^{-x^2}\cos ax\Big|_{x=0}^{x=\infty}+\frac{1}{2}\int_0^{\infty}e^{-x^2}d\cos ax$

$=\frac{1}{2}-\frac{a}{2}\int_0^{\infty}e^{-x^2}\sin ax\,dx=\frac{1}{2}-\frac{a}{2}I(a)$

$\to I''(a)=\frac{-a}{2}I'(a) \to I'(a)=C.e^{-\frac{a^2}{4}}$

+) $I'(0)=\frac{1}{2}\to C=\frac{1}{2}\to I'(a)=\frac{1}{2}e^{-\frac{a^2}{4}} \Rightarrow$ $\boxed{\text{prup}}$

m) $\int_0^\infty \frac{\sin xy}{x}\,dx \quad (y \geq 0)$

+) $y = 0 \to I = \int_0^\infty \frac{\sin 0x}{x}\,dx = \boxed{0}$

+) $y > 0 \to I = \int_0^\infty \frac{\sin xy}{xy}\,dxy = \int_0^\infty \frac{\sin t}{t}\,dt = \boxed{\frac{\pi}{2}}$

n) $\int_0^\infty e^{-ax^2}\cos bx\,dx \quad (a>0)$

+) $I'(b) = \int_0^\infty e^{-ax^2}(-x).\sin bx\,dx$

$= \frac{1}{2a}\int_0^\infty \sin bx\,d\,e^{-ax^2}$

$= \frac{1}{2a} e^{-ax^2}\sin bx \Big|_{x=0}^{x=\infty} - \frac{1}{2a}\int_0^\infty e^{-ax^2}\,d\sin bx$

$= -\frac{1}{2a}.b\int_0^\infty e^{-ax^2}\cos bx\,dx = \frac{-b}{2a} I(b)$

$\to I(b) = c.e^{\frac{-b^2}{4a}}$

+) $I(0) = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{\sqrt{a}}\int_0^\infty e^{-(\sqrt{a}x)^2}\,d\sqrt{a}x = \frac{\sqrt{\pi}}{2\sqrt{a}} = c$

$\Rightarrow I = \boxed{\frac{\sqrt{\pi}}{2\sqrt{a}} e^{-b^2/4a}}$

p) $\int_0^\infty \frac{\sin ax\cos bx}{x}\,dx = \frac{1}{2}\int_0^\infty \frac{\sin[(a+b)x] + \sin[(a-b)x]}{x}\,dx$

+) $\forall a>b: I = \frac{1}{2}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \boxed{\frac{\pi}{2}}$

+) $\forall a=b: I = \frac{1}{2}\left(\frac{\pi}{2}+0\right) = \boxed{\frac{\pi}{4}}$

+) $\forall a<b: I = \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = \boxed{0}$

Thứ Ngày No

o) $\int_0^\infty x^{2n} e^{-x^2}\cos bx\,dx = 0 \quad (n \in N)$

+) $I(b) = \int_0^\infty e^{-x^2}\cos bx\,dx$

$\to I^{(1)}(b) = \int_0^\infty e^{-x^2}(-x)\sin bx\,dx$

$\to I^{(2)}(b) = \int_0^\infty e^{-x^2}(-x).x\cos bx\,dx = -\int_0^\infty x^2 e^{-x^2}\cos bx\,dx$

$\to I^{(3)}(b) = \int_0^\infty e^{-x^2}(-x)x(-x)\sin bx\,dx$

$\to I^{(4)}(b) = \int_0^\infty e^{-x^2}(-x)x(-x)x\cos bx\,dx = \int_0^\infty x^4 e^{-x^2}\cos bx\,dx$

$\to \left[\begin{array}{l} n\%2==1: I^{(2n)}(b) = -\int_0^\infty e^{-x^2}.x^{2n}\cos bx\,dx \\ n\%2==0: I^{(2n)}(b) = \int_0^\infty e^{-x^2}.x^{2n}\cos bx\,dx \end{array}\right.$

$\to I^{(2n)} = (-1)^n\int_0^\infty e^{-x^2}.x^{2n}\cos bx\,dx$

+) $I(b) = \frac{\sqrt{\pi}}{2}.e^{-b^2/4}$

$\Rightarrow \frac{\sqrt{\pi}}{2}\left(e^{-b^2/4}\right)^{(2n)} = (-1)^n\int_0^\infty e^{-x^2}.x^{2n}\cos bx\,dx$

$\Rightarrow 0 = \boxed{(-1)^n \frac{\sqrt{\pi}}{2}\left(e^{-b^2/4}\right)^{(2n)}}$

q) $\int_0^\infty \frac{\sin ax\sin bx}{x}\,bx = \int_0^\infty \frac{1}{2}\left[\frac{\cos[(a-b)x]}{x} - \frac{\cos[(a+b)x]}{x}\right]$

+) From ⓔ: $\int_0^{+\infty} e^{-a'x}.\frac{\cos b'x - \cos c'x}{x}\,dx = \frac{1}{2}\ln\frac{c'^2-a'^2}{b'^2-a'^2}$

. $a'=0,\ b'=a-b,\ c'=a+b \to \int_0^{+\infty}\frac{\cos(a-b)x - \cos(a+b)x}{x}\,dx$

$= \frac{1}{2}\ln\frac{(a+b)^2}{(a-b)^2}$

$\Rightarrow I = \boxed{\frac{1}{2}\ln\left|\frac{a+b}{a-b}\right|}$

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a) $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx = I$

+) $\beta(p,q) = 2\int_0^{\pi/2} \sin^{2p-1} x \cos^{2q-1} x \, dx$

$\rightarrow I = \frac{\beta(7/2, 5/2)}{2} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}}{5!} \cdot \frac{1}{2} \cdot \frac{\pi}{\sin(\frac{\pi}{2})}$

$= \boxed{\frac{3\pi}{512}}$

e) $\int_0^{+\infty} \frac{1}{1+x^3} dx = I$

+) Put $t = x^3 \rightarrow dt = 3x^2 dx = 3t^{2/3} dx$

$\rightarrow I = \int_0^{+\infty} \frac{1}{3t^{2/3}} \cdot \frac{1}{1+t} dt$

$= \frac{1}{3}\int_0^{+\infty} \frac{t^{-2/3}}{t+1} dt$

+) $I = \frac{1}{3}\beta(p,q) = \frac{1}{3}\int_0^{+\infty} \frac{t^{p-1}}{(t+1)^{p+q}} dt$

$\cdot\ p = \frac{1}{3},\ q = \frac{2}{3}$

$\rightarrow I = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{1}{3}\frac{\pi}{\sin(\frac{\pi}{3})} = \boxed{\frac{2\sqrt{3}}{9}\pi}$

b) $\int_0^a x^{2n}\sqrt{a^2-x^2}\, dx \ (a>0)$

+) Put $t = \frac{x}{a} \rightarrow I = \int_0^1 (at)^{2n}\sqrt{a^2-a^2t^2} \cdot a\,dt$

$= \int_0^1 a^{2n+2} \cdot t^{2n}\sqrt{1-t^2}\, dt$

+) Put $t = \sin y$

$I = \int_0^{\pi/2} a^{2n+2} \cdot \sin^{2n} y \cdot \cos^2 y \, dy$

$= \frac{a^{2n+2}}{2} \cdot \beta\left(\frac{2n+1}{2}, \frac{3}{2}\right) = \frac{\Gamma(\frac{2n+1}{2}) \cdot \Gamma(\frac{3}{2})}{\Gamma(n+2)}$

$= \frac{a^{2n+2}}{2} \cdot \frac{(2n-1)!!}{2^n}\sqrt{\pi} \cdot \frac{1!!}{2^1}\sqrt{\pi} \cdot \frac{1}{(n+1)!}$

$= \frac{a^{2n+2}\,\pi \cdot (2n-1)!!}{2^{n+2}(n+1)!}$

c) $\int_0^{+\infty} x^{10} e^{-x^2} dx = I$

+) $I = \frac{1}{2}\int_0^{+\infty} x^9 e^{-x^2} dx^2 = \frac{1}{2}\int_0^{+\infty} t^{9/2} e^{-t} dt$

$= \frac{1}{2}\Gamma(11/2) = \frac{1}{2}\Gamma\left(5+\frac{1}{2}\right) = \frac{1}{2}\frac{(2\cdot5-1)!!\sqrt{\pi}}{2^5} = \boxed{\frac{9!!\sqrt{\pi}}{2^6}}$

d) $\int_0^{+\infty} \frac{\sqrt{x}}{(1+x^2)^2} dx$

+) Put $x = \sqrt{t}$

$\cdot\ dx = \frac{1}{2\sqrt{t}} dt$

$\cdot\ I = \int_0^{+\infty} \frac{t^{1/4}}{(1+t)^2} \cdot \frac{1}{2\sqrt{t}} dt$

$= \frac{1}{2}\int_0^{+\infty} \frac{t^{-1/4}}{(1+t)^2} dt = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{5}{4}\right)$

$= \frac{1}{2}\frac{\Gamma(3/4)\,\Gamma(5/4)}{\Gamma(2)}$

$= \frac{1}{2}\frac{1}{4}\Gamma(3/4)\Gamma(1/4) \cdot \frac{1}{\Gamma(2)}$

$= \frac{1}{2}\cdot\frac{1}{4}\frac{\pi}{\sin(\frac{\pi}{4})} \cdot \frac{1}{1!} = \boxed{\frac{\sqrt{2}}{8}\pi}$

KOKUYO

f) $\int_0^{+\infty} \frac{x^{n+1}}{(1+x^n)^2} dx$

+) Put $t = x^n \to x = t^{1/n} \to dx = \frac{1}{n} \cdot t^{\frac{1-n}{n}} dt$

• $I = \int_0^\infty \frac{t^{\frac{n+1}{n}}}{(1+t)^2} \cdot \frac{1}{n} \cdot t^{\frac{1-n}{n}} dt$

$= \int_0^\infty \frac{1}{n} \cdot \frac{t^{\frac{2}{n}}}{(1+t)^2} dt$

$= \frac{1}{n} \beta\left(\frac{n+2}{n}, \frac{n-2}{n}\right)$

$= \frac{1}{n} \cdot \frac{\frac{n+2}{n}-1}{\frac{n+2}{n}+\frac{n-2}{n}-1} \cdot \beta\left(\frac{2}{n}, \frac{n-2}{n}\right)$

$= \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{1}{1} \cdot \frac{\pi}{\sin\frac{2\pi}{n}} = \boxed{\frac{2\pi}{n^2 \sin\left(\frac{2\pi}{n}\right)}}$

g) $\int_0^1 \frac{1}{\sqrt[n]{1-x^n}} dx$

+) Put $x = t^{1/n}$

• $dx = \frac{1}{n} t^{\frac{1-n}{n}} dt$

• $I = \int_0^1 \frac{1}{n} \cdot \frac{t^{\frac{1-n}{n}}}{(1-t)^{1/n}} dt$

$= \int_0^1 \frac{1}{n} t^{\frac{1}{n}-1} \cdot (1-t)^{-1/n} dt$

$= \frac{1}{n} \beta\left(\frac{1}{n}, 1+\frac{-1}{n}\right) = \boxed{\frac{1}{n} \cdot \frac{\pi}{\sin\frac{\pi}{n}}}$

h) $\int_0^{+\infty} \frac{x^4}{(1+x^3)^2} dx$

+) Put $x = t^{1/3}$

• $dx = \frac{1}{3} t^{-2/3} dt$

• $I = \int_0^\infty \frac{t^{4/3}}{(1+t)^2} \cdot \frac{1}{3} t^{-2/3} dt$

$= \frac{1}{3} \int_0^\infty \frac{t^{2/3}}{(1+t)^2} dt$

$= \frac{1}{3} \beta\left(\frac{5}{3}, \frac{1}{3}\right)$

$= \frac{1}{3} \cdot \frac{\frac{5}{3}-1}{\frac{5}{3}+\frac{1}{3}-1} \cdot \beta\left(\frac{2}{3}, \frac{1}{3}\right)$

$= \frac{2}{9} \cdot \frac{\pi}{\sin\frac{2\pi}{3}} = \boxed{\frac{4\sqrt{3}}{27}\pi}$

Chapter 5: Line integrals.

Remind:

1. Line integrals of scalar fields

a) Definition

$\int_C f(x,y)\, ds = \int_a^b f(r(t))\, |r'(t)|\, dt$

① $C: x = x(t), y = y(t)$: $\int_a^b f(x(t), y(t)) \sqrt{x'^2(t) + y'^2(t)}\, dt$

② $C: y = y(x), a \le x \le b$: $\int_a^b f(x, y(x)) \sqrt{1 + y'^2(x)}\, dx$

③ $C: x = x(y), c \le y \le d$: $\int_c^d f(x(y), y) \sqrt{1 + x'^2(y)}\, dy$

b) Properties

• Line integrals of scalar vector fields do not depend on the direction of C.

• Physical interpretation: the mass of C is $\int_C p(x,y)\, ds$ where $p(x,y)$ is the density function.

• The length of C is $I = \int_C ds$

• Linearity and additivity.

④ $\int_C f(x,y)\,dS = \int_{\alpha_1}^{\alpha_2} f(r(\alpha)\cos\alpha, r(\alpha)\sin\alpha)\sqrt{r^2(\alpha)+r'^2(\alpha)}\,d\alpha$

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2. Line integrals of f along C with respect to x and y.

① $\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

② $\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$

③ $\int_C f(x,y)\,dx = -\int_{-C} f(x,y)\,dx$; $\int_C f(x,y)\,dy = -\int_{-C} f(x,y)\,dy$

3. Line integrals of vector fields.

a) Definition

Let F be a continuous vector field defined on a smooth curve C given by a vector function r(t), $a \le t \le b$. Then the integral of F along C is:

$$\int_C F.T\,ds = \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]\,dt$$

$$= \int_C P(x,y)\,dx + Q(x,y)\,dy$$

b) Properties

• Line integrals of vector fields depend on the direction of the curve

• Linearity and additivity.

c) Green's theorem:

⊙ Closed curve orientation:

We use the convention that the positive orientation of a simple closed curve C refers to a single counter-clockwise traversal of C.

⊙ Theorem:

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

• If ∂D is negatively oriented:

$$\int_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

• If C is not closed, we "closed off" the curve, apply Green's theorem, subtract the integral over the piece with glued on.

⊙ Area of Domain:

If $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ then $S(D) = \iint_D 1\,dx\,dy = \int_{\partial D} P\,dx + Q\,dy$

d) Independence of Path:

Assume that D is a simple domain. P, Q and their partial derivatives are continuous on $\bar{D}$. Then the following assertions are equivalent:

1. $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ for all $(x,y) \in D$

2. $\int_L P\,dx + Q\,dy = 0$ for all closed curve L contained in D

3. $\int_C P\,dx + Q\,dy$ is independent of path.

4. $F = P(x,y)\vec{i} + Q(x,y)\vec{j}$ is conservative; i.e. $\exists u(x,y)$ such that $\nabla u = F$. The function u is computed by:

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)\,dy$$

$$= \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy.$$

① Check the condition $P'_y = Q'_x$

② Choose the path that the integration is simplest

③ If we can find u: $du = Pdx + Qdy$, then

$$I = u(B) - u(A)$$

* In 3D space:

1. $\text{curl}\,\vec{F} = \vec{0} \iff \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}$

2. $\oint_C Pdx + Qdy + Rdz = 0$ for any closed curve C

3. $\int_C Pdx + Qdy + Rdz$ is independent of path.

4. $\vec{F}$ is conservative $\iff \vec{F} = \nabla u$ where

$$u = \int_{x_0}^{x} P(x, y_0, z_0)\,dx + \int_{y_0}^{y} Q(x, y, z_0)\,dy + \int_{z_0}^{z} R(x, y, z)\,dz$$

(10) Evaluate

a) $\int_C (x-y)\,ds$; $C: x^2 + y^2 = 2x$

① +) Put $x = 1 + \cos\alpha$, $D\alpha: 0 \le \alpha \le 2\pi$

$y = \sin\alpha$

+) $I = \int_0^{2\pi} (1 + \cos\alpha - \sin\alpha)\sqrt{(-\sin\alpha)^2 + \cos^2\alpha}\,d\alpha$

$= \int_0^{2\pi} (1 + \cos\alpha - \sin\alpha)\,d\alpha$

$= 2\pi.$

② +) Put $x = r\cos\alpha$

$y = r\sin\alpha$

$\bullet\ C: r^2 = 2r\cos\alpha \to \begin{cases} r = 2\cos x \\ -\frac{\pi}{2} \le x \le \frac{\pi}{2} \end{cases}$

+) $I = \int_{-\frac{\pi}{2}}^{\pi/2} (2\cos^2 x - 2\cos x \sin x)\sqrt{4\cos^2 x + (-2\sin x)^2}\,dx$

$= 4\int_{-\pi/2}^{\pi/2} (\cos^2 x - \cos x \sin x)\,dx$

$= 4\left(\frac{\pi}{2} - 0\right) = 2\pi.$

b) $\int_C y^2\,ds$, $C: \begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$, $0 \le t \le 2\pi$, $a > 0$

+) $I = \int_0^{2\pi} a^2(1 - 2\cos t + \cos^2 t)\sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt$

$= \int_0^{2\pi} a^3(1-\cos t)^2\sqrt{2 - 2\cos t}\,dt$

$= \int_0^{2\pi} a^3\sqrt{2}\,(1-\cos t)^{5/2}\,dt$

$= \int_0^{2\pi} a^3\sqrt{2}\cdot\left(2\sin^2\frac{t}{2}\right)^{5/2} dt$

$= a^3\cdot\sqrt{2}\cdot 2^{5/2}\cdot\frac{32}{15} = \frac{256}{15}a^3$

c) $\int_C \sqrt{x^2+y^2}\,ds$, $C: \begin{cases} x = (\cos t + t\sin t) \\ y = (\sin t - t\cos t) \end{cases}$, $0 \le t \le 2\pi$

$\bullet\ x^2 + y^2 = (\cos t + t\sin t)^2 + (\sin t - t\cos t)^2$

$= \cos^2 t + t^2\sin^2 t + 2t\sin t\cos t + \sin^2 t + t^2\cos^2 t - 2t\sin t\cos t$

$= t^2 + 1$

$\bullet\ x'(t) = -\sin t + \sin t + t\cos t = t\cos t$

$\bullet\ y'(t) = \cos t - \cos t + t\sin t = t\sin t$

$\bullet\ ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = t\,dt$

$\to I = \int_0^{2\pi} \sqrt{t^2+1}\;t\,dt = \frac{1}{2}\int_0^{2\pi} \sqrt{t^2+1}\,d(t^2+1) = \frac{1}{2}\cdot(t^2+1)^{3/2}\cdot\frac{2}{3}\Big|_0^{2\pi}$

$= \frac{1}{3}\left[(4\pi^2+1)^{3/2} - 1\right]$

d) $\int_C (x+y)dS$, $C: x^2+y^2=2y$.

+) Put $x = \cos\alpha$

$y = 1+\sin\alpha$

. $C: \cos^2\alpha + 1+2\sin\alpha+\sin^2\alpha = 2+2\sin\alpha$ (True)

. $D\alpha: 0 \leq \alpha \leq 2\pi$

+) $I = \int_0^{2\pi} (\cos\alpha+1+\sin\alpha)\sqrt{(-\sin\alpha)^2+(\cos\alpha)^2}\, d\alpha$

$= \int_0^{2\pi} \sin\alpha+\cos\alpha+1 = 2\pi$

e) $\int_L xy\,ds$, $L: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ x\geq 0, y\geq 0$

+) Put $x = a\cos\alpha$

$y = b\sin\alpha$

. $L: \cos^2\alpha+\sin^2\alpha = 1$ (True)

$\left.\begin{matrix} a\cos\alpha \geq 0 \\ b\sin\alpha \geq 0 \end{matrix}\right\} \Rightarrow 0 \leq \alpha \leq \frac{\pi}{2}$

+) $I = \int_0^{\pi/2} ab\cos\alpha\sin\alpha\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}\, d\alpha$

$= \int_0^1 ab\,t\sqrt{a^2t^2+b^2(1-t^2)}\,dt$ (Put $t=\sin\alpha$)

$= ab\int_0^1 t\sqrt{b^2+(a^2-b^2)t^2}\,dt$

$= \frac{ab}{2}\left[\int_0^1 \sqrt{b^2+(a^2-b^2)t^2}\, d\left(t^2(a^2-b^2)+b^2\right)\right] \cdot \frac{1}{a^2-b^2}$

$= \frac{ab}{2(a^2-b^2)} \cdot \left(b^2+(a^2-b^2)t^2\right)^{\frac{3}{2}} \cdot \frac{2}{3}\Big|_0^1$

$= \frac{ab}{3(a^2-b^2)} \cdot (a^3-b^3) = \boxed{\frac{ab(a^2+ab+b^2)}{3(a+b)}}$

f) $I = \int_L |y|\,ds$; $L: r = a(1+\cos\alpha)$ $(a>0)$

+) $I = \int_0^{2\pi} |a(1+\cos\alpha)\sin\alpha|\sqrt{a^2(1+\cos\alpha)^2+a^2(-\sin\alpha)^2}\, d\alpha$

$= \int_0^{2\pi} \sqrt{2}a^2(1+\cos\alpha)^{3/2} \cdot |\sin\alpha|\, d\alpha$

. $0 \to \frac{\pi}{2}: I_1 = \sqrt{2}\int_0^{\pi/2} a^2(1+\cos\alpha)^{3/2} \cdot \sin\alpha\, d\alpha$

$= \sqrt{2}\int_0^1 a^2(1+t)^{3/2}\, d(t+1)$ (Put $t=\cos\alpha$)

$= \sqrt{2}a^2 \cdot (1+t)^{5/2}\frac{2}{5}\Big|_0^1 = \frac{2\sqrt{2}}{5} \cdot a^2(4\sqrt{2}-1)$

. $\frac{\pi}{2} \to \pi: I_2 = \sqrt{2}\int_{\pi/2}^{\pi} a^2(1+\cos\alpha)^{3/2}\sin\alpha\, d\alpha$

$= \sqrt{2}\int_{-1}^0 a^2(1+t)^{3/2}\, d(t+1)$

$= \sqrt{2}a^2(1+t)^{5/2}\frac{2}{5}\Big|_{-1}^0 = \frac{2\sqrt{2}}{5}a^2(1-0) = \frac{2\sqrt{2}}{5}a^2$

$0 \to \pi$:

$\sqrt{2}a^2(1+t)^{5/2}\frac{2}{5}$

$= \frac{2\sqrt{2}}{5}a^2(4\sqrt{2}$

$= \frac{8.2}{5}a^2$

. Symetric property:

$I = 2(I_1+I_2) = \boxed{\frac{32}{5}a^2}$

g) $I = \int_L |y|\,ds$, $L: (x^2+y^2)^2 = a^2(x^2-y^2)$

+) Put $x = r\cos\alpha$ | $r \geq 0$

$y = r\sin\alpha$

. $L: r^4 = a^2r^2(\cos^2\alpha-\sin^2\alpha)$

$\to r^2 = a^2(\cos^2\alpha-\sin^2\alpha) = a^2\cos 2\alpha$

$\to \begin{cases} r = a\sqrt{\cos 2\alpha} \\ \left[\begin{matrix} -\frac{\pi}{4} \leq \alpha \leq \frac{\pi}{4} \\ \frac{3\pi}{4} \leq \alpha \leq \frac{5\pi}{4} \end{matrix}\right. \end{cases}$

+) $I_1 = \int_0^{\pi/4} a\sqrt{\cos 2\alpha}\,\sin\alpha\,\sqrt{a^2\cos 2\alpha + a^2\frac{\sin^2 2\alpha}{\cos 2\alpha}}\,d\alpha$

$= \int_0^{\pi/4} a^2 \sin\alpha\, d\alpha = a^2\left(1-\frac{\sqrt{2}}{2}\right)$

+) Symetric property: $I = 4I_1 = \boxed{a^2(4-2\sqrt{2})}$

(108) $\int_{ABCA} 2(x^2+y^2)dx + x(4y+3)dy$, ABCA: A(0,0), B(1,1), C(0,2)

+) AB: $t: 0\to 1$
$x=t$
$y=t$

• $I_1 = \int_0^1 2(t^2+t^2).1\,dt + t(4t+3).1\,dt$

$= \int_0^1 8t^2+3t\,dt = \frac{25}{6}$

+) BC: $t: 0\to 1$
$x=1-t$
$y=1+t$

• $I_2 = \int_0^1 2[(1-t)^2+(1+t^2)].(-1)\,dt + (1-t)(4+4t+3).1\,dt$

$= \int_0^1 -8t^2-3t+3\,dt = \frac{-7}{6}$

+) CA: $t: 0\to 1$
$x=0$
$y=2-2t$

• $I_3 = \int_0^1 0\,dt = 0$

+) $I = I_1+I_2+I_3 = \boxed{3}$

(109) $\int_{ABCDA} \frac{dx+dy}{|x|+|y|}$; A(1,0). B(0,1). C(-1,0). D(0,-1)

+) AB: $x+y=1 \Rightarrow dx+dy=0$
BC: $x-y=-1 \Rightarrow dx=dy$
CD: $x+y=-1 \Rightarrow dx+dy=0$
DA: $x-y=1 \Rightarrow dx=dy$

+) $I = \int_{BC}\frac{2dx}{y-x} + \int_{DA}\frac{2dx}{+x-y}$

$= \int_0^{-1}\frac{2}{1}\,dx + \int_0^{1}\frac{2}{1}\,dx$

$= -2+2 = \boxed{0}$

(110) $I=\int_C (xy+x+y)dx + (xy+x-y)dy$; C: $x^2+y^2=R^2$, oriented positive.

a) Directly:

+) Put $x=R\cos\alpha$, $y=R\sin\alpha$
• C: $R^2\cos^2\alpha + R^2\sin^2\alpha = R^2$ (true)
• $D\alpha$: $0\le\alpha\le 2\pi$.

+) $I = \int_0^{2\pi}[(R^2\sin\alpha\cos\alpha + R\sin\alpha + R\cos\alpha).(-R\sin\alpha) + (R^2\sin\alpha\cos\alpha + R\cos\alpha - R\sin\alpha)(R\cos\alpha)]\,d\alpha$

$= R^3.0 + R^2(-\pi) + R^2.0 + R^3.0 + R^2.\pi + R^2.0$

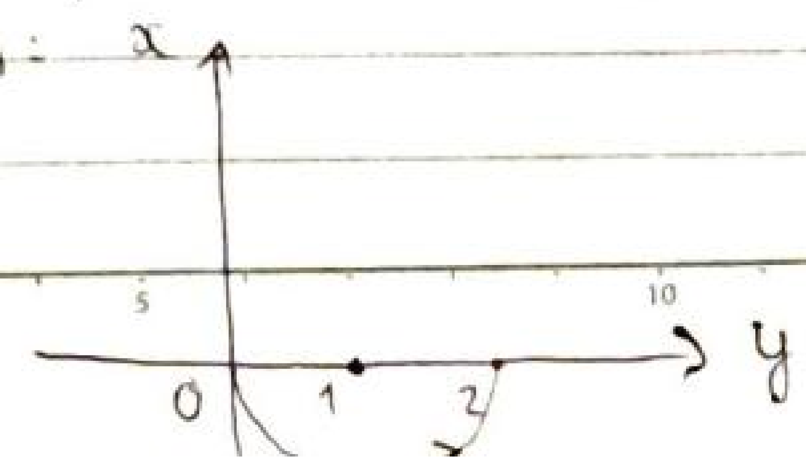
$= \boxed{0}$

b) Green's theorem:

+) Apply Green's theorem:
• $I = \iint_D (y+1)-(x+1)\,dxdy = \iint_D y-x\,dxdy = \boxed{0}$
• D: $x^2+y^2 \le R^2$

(111) C: $x^2+y^2=2x$, traced from (0,0) to (2,0)

+) Graph:



a) $\int_C (xy+x+y)dx + (xy+x-y)dy$

① + Put $x = 1+\cos\alpha$

$y = \sin\alpha$

. $D\alpha: \pi \le \alpha \le 2\pi$

+ $I = \int_\pi^{2\pi} [[(1+\cos\alpha)\sin\alpha + 1+\cos\alpha+\sin\alpha](-\sin\alpha) + [(1+\cos\alpha).\sin\alpha + 1+\cos\alpha-\sin\alpha].\cos\alpha]\, d\alpha$

$= \int_\pi^{2\pi} (-(\sin^2\alpha + \cos\alpha\sin^2\alpha + \sin\alpha + \sin\alpha\cos\alpha + \sin^2\alpha) + \cos\alpha.\sin\alpha + \cos^2\alpha\sin\alpha + \cos\alpha + \cos^2\alpha - \sin\alpha\cos\alpha)\, d\alpha$

$= -(\frac{\pi}{2} + 0 - 2 + 0 + \frac{\pi}{2}) + (0 - \frac{2}{3} + 0 + \frac{\pi}{2} + 0)$

$= -\pi + 2 - \frac{2}{3} + \frac{\pi}{2} = \frac{4}{3} - \frac{\pi}{2}$

② + Apply Green's theorem:

. $I = \iint_D (y+1)-(x+1)\,dxdy - \int_{C_1} (xy+x+y)dx + (xy+x-y)dy$

. $D: x^2+y^2 \le 2x,\ y \le 0$

. $C_1: (2,0) \to (0,0)$

+ Put $x = 1 + r\cos\alpha$

$y = r\sin\alpha$

. $D \to Dr\alpha: \begin{cases} 0 \le r \le 1 \\ \pi \le \alpha \le 2\pi \end{cases}$

. $J = r$

. $I_1 = \int_\pi^{2\pi} d\alpha \int_0^1 r(r\sin\alpha - r\cos\alpha)\,dr$

$= \frac{1}{3}(-2-0) - \frac{\pi}{2} = \frac{-2}{3} - \frac{\pi}{2}$

+ $I_2 = \int_2^0 x\,dx = -2$

$-\frac{\pi}{2}$

+ $I = I_1 - I_2 = \frac{-2}{3} + 2 = \frac{4}{3} - \frac{\pi}{2}$

b) $\int_C [x^2(y+\frac{x}{4})]dy - [y^2(x+\frac{y}{4})]dx$

① + Put $x = 1+\cos\alpha$

$y = \sin\alpha$

. $D\alpha: \pi \le \alpha \le 2\pi$

+ $I = \int_\pi^{2\pi} [(1+\cos\alpha)^2(\sin\alpha + \frac{1+\cos\alpha}{4})\cos\alpha - \sin^2\alpha(1+\cos\alpha+\frac{\sin\alpha}{4})(-\sin\alpha)]\,d\alpha$

$= \frac{1}{4}\int_\pi^{2\pi} (1+2\cos\alpha+\cos^2\alpha)(4\sin\alpha+\cos\alpha+1)\cos\alpha + \sin^2\alpha(4+4\cos\alpha+\sin\alpha)\sin\alpha\, d\alpha$

$= \frac{1}{4}\int_\pi^{2\pi} (4\cos\alpha\sin\alpha + 8\cos^2\alpha\sin\alpha + 4\cos^3\alpha\sin\alpha + \cos^2\alpha + 2\cos^3\alpha + \cos^4\alpha + \cos\alpha + 2\cos^2\alpha + \cos^3\alpha + 4\sin^3\alpha + 4\sin^3\alpha\cos\alpha + \sin^4\alpha)\,d\alpha$

$= \frac{1}{4}(0 - \frac{16}{3} + 0 + \frac{\pi}{2} + 0 + \frac{3}{8}\pi + 0 + \pi + 0 - \frac{16}{3} + 0 + \frac{3\pi}{8})$

$= -\frac{8}{3} + \frac{9}{16}\pi$

② + Apply Green's theorem:

. $I = \iint_D (2xy + \frac{3x^2}{4}) + (2xy + \frac{3y^2}{4})\,dxdy$

$- \int_{C_1} x^2(y+\frac{x}{4})dy - y^2(x+\frac{y}{4})dx$

. $D: x^2+y^2 \le 2x,\ y \le 0$

. $C_1: (2,0) \to (0,0)$

+ Put $x = r\cos\alpha,\ y = r\sin\alpha$

. $Dr\alpha \begin{cases} 3\pi/2 \le \alpha \le 2\pi \\ 0 \le r \le 2\cos\alpha \end{cases}$

. $J = r$

. $I_1 = \int_{3\pi/2}^{2\pi} d\alpha \int_0^{2\cos\alpha} r.(\frac{3}{4}r^2 + 4r^2\sin\alpha\cos\alpha)\,dr$

$$= \int_{3\pi/2}^{2\pi} \frac{3}{4}\cdot\frac{1}{4}\cdot 16\cos^4\alpha + 4\cdot\frac{1}{4}\cdot 16\cos^4\alpha \sin\alpha\cos\alpha \, d\alpha$$

$$= \frac{9}{16}\pi - \frac{8}{3}$$

+) $I_2 = \int_2^0 0\,dx = 0$

+) $I = I_1 - I_2 = \boxed{\frac{9}{16}\pi - \frac{8}{3}}$

c) $\int_C (xy + e^x\sin x + x + y)\,dx - (xy - e^{-y} + x - \sin y)\,dy$

+) Apply Green's theorem:

• $I = \iint_D [-(y+1) - (x+1)]\,dxdy - \int_{C_1} (xy+e^x\sin x + x + y)\,dx - (xy - e^{-y} + x - \sin y)\,dy$

• D: $x^2+y^2 \le 2x,\ y \le 0$

• C_1: $(2,0) \to (0,0)$

+) Put $x = 1 + r\cos\alpha$

$y = r\sin\alpha$

• $J = r$

• $D_{r\alpha}$: $\begin{cases} 0\le r\le 1 \\ \pi \le\alpha\le 2\pi \end{cases}$

• $I_1 = \int_\pi^{2\pi} d\alpha \int_0^1 (-1 - r\cos\alpha - r\sin\alpha - 2)\, r\,dr$

$= \int_\pi^{2\pi} d\alpha \int_0^1 r(-r\cos\alpha - r\sin\alpha - 3)\,dr$

$= -\frac{1}{3}\cdot(-2) - \frac{3}{2}\pi = \boxed{\frac{2}{3} - \frac{3\pi}{2}}$

+) $I_2 = \int_2^0 e^x\sin x + x\,dx$

$= \int_0^2 e^x(-\sin x) - x\,dx$

$= \frac{e^x\cos x - e^x\sin x - x^2}{2}\Big|_0^2$

$= \frac{e^2(\cos 2 - \sin 2) - 4}{2} - \frac{1}{2}$

$= \boxed{\frac{e^2(\cos 2 - \sin 2)}{2} - \frac{5}{2}}$

+) $I = I_1 - I_2 = \boxed{\frac{-3}{2}\pi + \frac{19}{6} + \frac{e^2(\sin 2 - \cos 2)}{2}}$

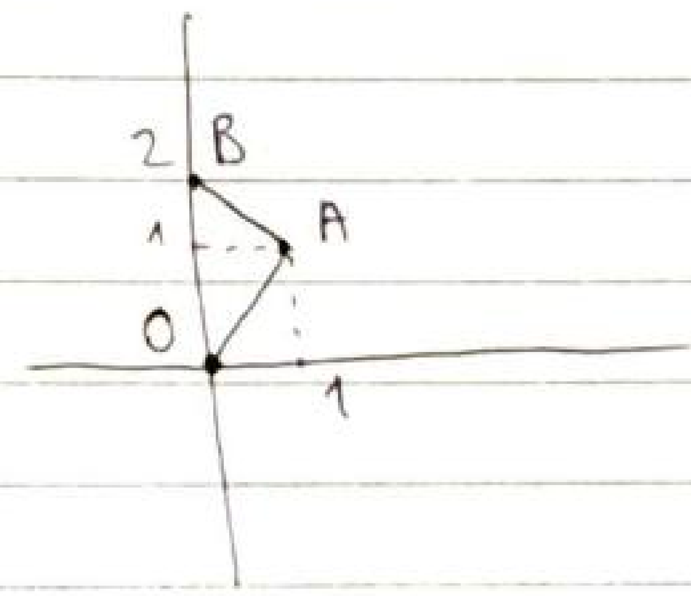
112) $\int_{OABO} e^x[(1-\cos y)\,dx - (y - \sin y)\,dy]$

OABO : O(0,0), A(1,1), B(0,2)

+) $P = e^x(1-\cos y)$

$Q = -e^x(y-\sin y)$

+) Graph:



+) Apply Green theorem:

• $I = \iint_D -e^x(y-\sin y) - e^x\sin y\,dxdy = \iint_D -e^x y\,dxdy$

• D: $\begin{cases} x: 0 \to 1 \\ y: x \to 2-x \end{cases}$

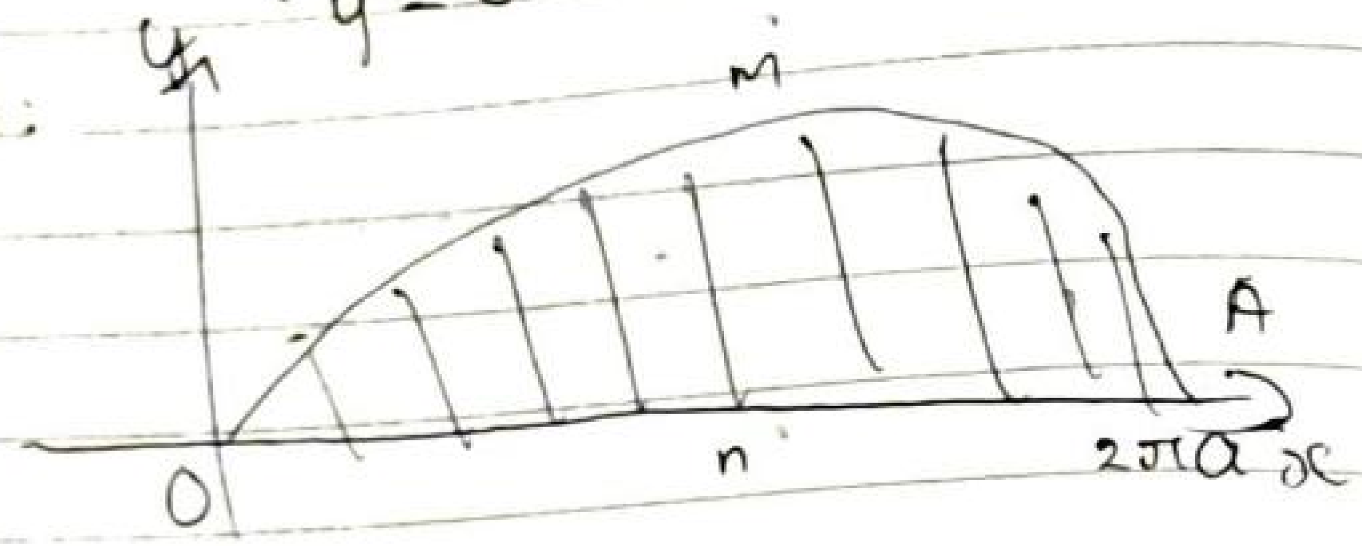
→ $I = \int_0^1 dx\int_x^{2-x} -e^x y\,dy = \int_0^1 -e^x\cdot\frac{(2-x)^2 - x^2}{2}\,dx$

$= \int_0^1 e^x(2x-2)\,dx = 2(xe^x - 2e^x)\Big|_0^1 = 2(e-2e) - 2(-2)$

$= \boxed{-2e+4}$

(113) area of $\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \\ y = 0 \end{cases}$ $(a>0)$

+) Graph:



+) $S(D) = \oint_{\partial D} x\,dy = \int_{AmO} x\,dy + \int_{OnA} x\,dy$

$= \int_{2\pi}^{0} a(\theta - \sin\theta)\,a\sin\theta\,d\theta$

$= a^2 \int_0^{2\pi} \sin^2\theta - \theta\sin\theta\,d\theta$

$= a^2(\pi - (-2\pi)) = 3\pi a^2$

(114) $I = \int_{(-2;1)}^{(3;0)} (x^4+4xy^3)dx + (6x^2y^2-5y^4)dy$

+) $P = x^4+4xy^3$

$Q = 6x^2y^2 - 5y^4$

+) $P'_y = Q'_x = 12xy^2$

$\Rightarrow \exists u(x,y): I = u(3,0) - u(-2,1)$

+) $u = \int_0^x x^4\,dx + \int_0^y 6x^2y^2 - 5y^4\,dy$

$= \frac{x^5}{5} + 2x^2y^3 - y^5$

+) $I = \frac{243}{5} - \frac{3}{5} = 48$

(115) $\int_{(1,\pi)}^{(2,2\pi)} \left(1 - \frac{y^2}{x^2}\cos\frac{y}{x}\right)dx + \left(\sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}\right)dy$

+) $P = 1 - \frac{y^2}{x^2}\cos\frac{y}{x}$

$Q = \sin\frac{y}{x} + \frac{y}{x}\cos\frac{y}{x}$

+) $P'_y = \frac{-2y}{x^2}\cos\frac{y}{x} + \frac{y^2}{x^2}\cdot\sin\frac{y}{x}\cdot\frac{-1}{x}$

$Q'_x = \frac{-y}{x^2}\cos\frac{y}{x} - \frac{y}{x^2}\cos\frac{y}{x} - \frac{y}{x}\cdot\frac{-y}{x^2}\cdot\left(-\sin\frac{y}{x}\right)$

$\Rightarrow P'_y = Q'_x$

+) $I = \int_1^2 (1 - \pi^2\cos\pi)\,dt + (\sin\pi + \pi\cos\pi)\pi\,dt$

$= (1+\pi^2) + (0-\pi)\pi = 1$

Chapter 6: Surface integrals.

1. Surface integrals of scalar fields

a) Formulations

$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v), y(u,v), z(u,v))\,|r_u \times r_v|\,du\,dv$

$= \iint_D f(x,y,z(x,y))\sqrt{1+z'^2_x+z'^2_y}\,dx\,dy$

2. Surface integrals of vector fields

a) Oriented surfaces: <two-sided surfaces>

Let S be a surface: At every point, there are 2 unit normal vectors $\vec{n}$ and $-\vec{n}$:

. If it is possible to choose a unit normal vector at every such point so that it varies continuously over S, then S is called an oriented surface. There are two possible orientations for any orientable surface.

• Conversely, S is called a nonorientable surface.

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S \vec{F}(x,y,z)\cdot\vec{n}(x,y,z)\,dS$$

b) Definition:

If $\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is a continuous vector field defined on an oriented surface S with a normal vector $\vec{n}$ then the surface integral of $\vec{F}$ over S is

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_S \vec{F}\cdot\vec{n}\,dS.$$

This integral is also called the flux of $\vec{F}$ across S.

c) Formulation

⊛ Let S be given by $r(u,v)$, then a normal vector is $\vec{N} = r_u\times r_v = (A,B,C)$

• If $N \uparrow\uparrow n$: $n = \left(\frac{A}{|r_u\times r_v|}, \frac{B}{|r_u\times r_v|}, \frac{C}{|r_u\times r_v|}\right)$

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iint_D (AP+BQ+CR)\,du\,dv.$$

• If $N \uparrow\downarrow n$:

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = -\iint_D (AP+BQ+Cr)\,du\,dv.$$

⊛ If $\widehat{(\vec{n}, Oz)} < \frac{\pi}{2}$: $\iint_S R\,dx\,dy = \iint_D R(x,y,z(x,y))\,dx\,dy$

⊛ If $\widehat{(\vec{n}, Oz)} > \frac{\pi}{2}$: $\iint_S R\,dx\,dy = -\iint_D R(x,y,z(x,y))\,dx\,dy.$

d) Ostrogradsky – Gauss theorem (The divergence theorem)

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

• If S is inword oriented, then

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = -\iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dx\,dy\,dz$$

• If S is not closed, we use "closeoff" technique.

e) Stoke's formula:

$$\int_C P\,dx + Q\,dy + R\,dz = \iint_S \overrightarrow{\text{curl}F}\cdot\vec{n}\,dS$$

where the orientation of S induces the positive orientation of the boundary curve C shown in the figure.

f) Relation between Surface Integrals of scalar and vector fields

$$\iint_S [P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\cos\gamma]\,dS$$

$$= \iint_S P(x,y,z)\,dy\,dz + Q(x,y,z)\,dz\,dx + R(x,y,z)\,dx\,dy.$$

where $n = (\cos\alpha, \cos\beta, \cos\gamma)$ is the unit normal vector of S.

116. $\iint_S \left(z + 2x + \frac{4y}{3}\right)dS$; $S: \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$; $x,y,z \geq 0$

+) $D_{xy}: \frac{x}{2} + \frac{y}{3} \leq 1$; $x,y \geq 0 \Rightarrow \begin{cases} 0 \leq x \leq 2 \\ 0 \leq y \leq 3 - \frac{3x}{2} \end{cases}$

+) $z = 4 - 2x - \frac{4y}{3}$

$\left.\begin{array}{l} \cdot\ z'_x = -2 \\ \cdot\ z'_y = -\frac{4}{3} \end{array}\right\} \Rightarrow dS = \sqrt{1^2 + (-2)^2 + \left(\frac{-4}{3}\right)^2}\,dx\,dy = \frac{\sqrt{61}}{3}\,dx\,dy$

+) $I = \iint_{D_{xy}} 4 - 2x - \frac{4y}{3} + 2x + \frac{4y}{3} = \frac{4\sqrt{61}}{3}\int_0^2 dx \int_0^{3-\frac{3x}{2}} dy = \boxed{4\sqrt{61}}$

(117) $\iint_S (x^2+y^2)\,dS$; $S: \begin{cases} z=\sqrt{x^2+y^2} \\ z\leq 1 \end{cases}$

+) $D_{xy}: \sqrt{x^2+y^2}\leq 1 \Rightarrow x^2+y^2\leq 1$

+) $z'_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$, $z'_y = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$ $\Rightarrow dS = \sqrt{1+z_x'^2+z_y'^2}\,dxdy = \sqrt{2}\,dxdy$

+) $I = \iint_{D_{xy}} (x^2+y^2)\sqrt{2}\,dxdy$

. Put $x = r\cos\alpha$, $y = r\sin\alpha$

$\to D_{r\alpha} \begin{cases} 0\leq r\leq 1 \\ 0\leq\alpha\leq 2\pi \end{cases}$; $J=r$

$\Rightarrow I = \int_0^{2\pi} d\alpha \int_0^1 r^3\sqrt{2}\,dr = \boxed{\frac{\pi}{\sqrt{2}}}$

(117) $\iint_S (x^2+y^2)\,dS$; $S \begin{cases} z = x^2+y^2 \\ 0\leq z\leq 1 \end{cases}$

+) $D_{xy}: x^2+y^2\leq 1$

+) $z'_x = 2x$, $z'_y = 2y$ $\Rightarrow dS = \sqrt{1+4x^2+4y^2}\,dxdy$.

+) $I = \iint_{D_{xy}} (x^2+y^2)\sqrt{1+4x^2+4y^2}\,dxdy$.

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$

. $D_{r\alpha} \begin{cases} 0\leq r\leq 1 \\ 0\leq\alpha\leq 2\pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^1 r^3\sqrt{1+4r^2}\,dr$

$= 2\pi\cdot\frac{1}{2}\int_0 r^2\sqrt{1+4r^2}\,dr^2$

$= \pi\int_1^5 \frac{a-1}{4}\sqrt{a}\,d\frac{a-1}{4}$ (Put $a = 1+4r^2$)

$= \frac{\pi}{16}\int_1^5 a^{3/2} - a^{1/2}\,da$

$= \frac{\pi}{16}\left(\frac{2}{5}a^{5/2} - \frac{2}{3}a^{3/2}\right)\Big|_{a=1}^{a=5}$

$= \boxed{\pi\left(\frac{5\sqrt{5}}{12}+\frac{1}{60}\right)}$

(118) $\iint_S x^2y^2z\,dS$; $S \begin{cases} z=\sqrt{x^2+y^2} \\ z\leq 1 \end{cases}$

+) $D_{xy}: x^2+y^2\leq 1$

+) $z'_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$, $z'_y = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}}$ $\Rightarrow dS = \sqrt{1+\frac{x^2}{(\sqrt{x^2+y^2})^2}+\frac{y^2}{(\sqrt{x^2+y^2})^2}}\,dxdy = \sqrt{2}\,dxdy$

+) $I = \iint_{D_{xy}} x^2y^2\sqrt{x^2+y^2}\sqrt{2}\,dxdy$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$

. $D_{\alpha r}: \begin{cases} 0\leq\alpha\leq 2\pi \\ 0\leq r\leq 1 \end{cases}$

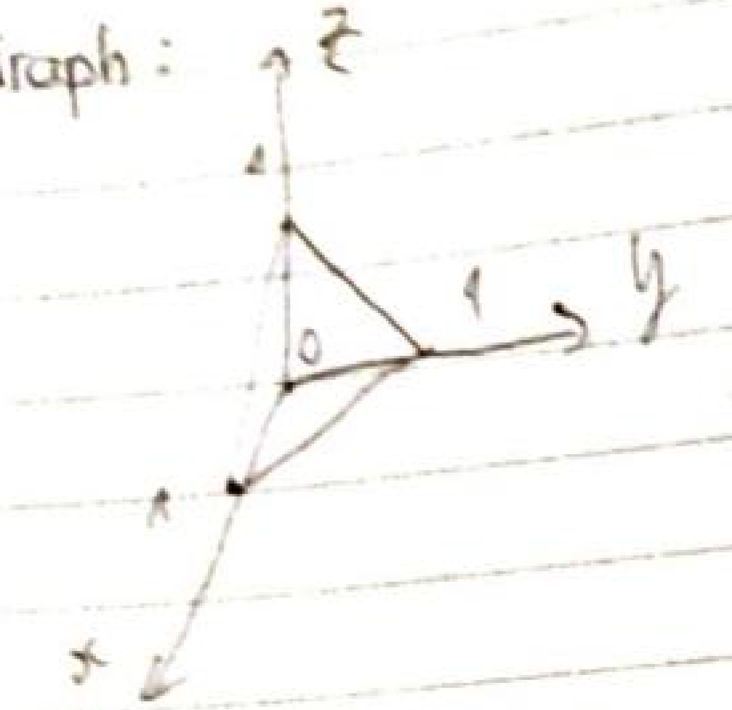
+) $I = \int_0^{2\pi} d\alpha \int_0^1 r^2\cos^2\alpha\, r^2\sin^2\alpha\sqrt{r^2}\,\sqrt{2}\,r\,dr$

$= \sqrt{2}\int_0^{2\pi}\cos^2\alpha\sin^2\alpha\,d\alpha \int_0^1 r^6$

$= \boxed{\frac{\pi\sqrt{2}}{28}}$

(119) $\iint_S \frac{dS}{(2+x+y+z)^2}$; S boundary of $\begin{cases} x+y+z\leq 1 \\ x\geq 0\ y\geq 0\ z\geq 0 \end{cases}$

+) Graph:



+) Symmetric property:

$$I = \iint_{S_1} f(x,y,z)\,dS + 3\iint_{S_2} f(x,y,z)\,dS_2$$

- $S_1: x+y+z=1;\ x,y,z \geq 0$
- $S_2: z=0;\ x+y \leq 1;\ x,y \geq 0$

+) Projection of S_1 on Oxy is S_2

$$I_1 = \iint_{S_2} \frac{1}{3^2}\sqrt{1+1+1}\,dxdy = \frac{\sqrt{3}}{9}\iint_{S_2} dxdy = \frac{\sqrt{3}}{18}$$

+) $$I_2 = \iint_{S_2} \frac{1}{(2+x+y)^2}\,dxdy = \int_0^1 dx \int_0^{1-x} \frac{1}{(2+x+y)^2}\,dy$$

$$= \int_0^1 \frac{-1}{2+x+y}\Big|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{-1}{3} + \frac{1}{2+x}\,dx$$

$$= \ln(1+x) - \frac{1}{3}x \Big|_{x=0}^{x=1}$$

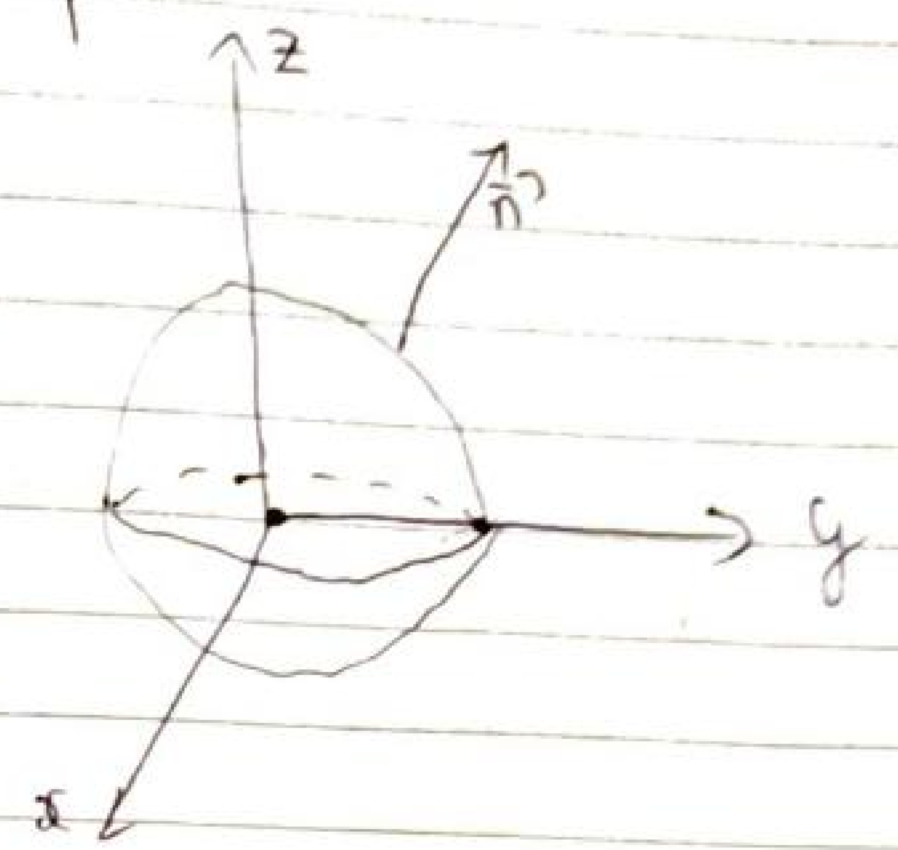
$$= \ln\frac{3}{2} - \frac{1}{3}$$

+) $I = I_1 + 3I_2 = \frac{\sqrt{3}}{18} + 3\ln\frac{3}{2} - 1$

(120) $\iint_S z(x^2+y^2)\,dxdy$, $S: x^2+y^2+z^2=1,\ z \geq 0$

oriented outward.

+) Graph:



+) $(\vec{n}, Oz) \leq \frac{\pi}{2} \Rightarrow I = \iint_{D_{xy}} \sqrt{1-x^2-y^2}\,(x^2+y^2)\,dxdy$

$D_{xy}: x^2+y^2 \leq 1$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

- $J = r$
- $D_{r\alpha} \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \alpha \leq 2\pi \end{cases}$

+) $$I = \int_0^{2\pi} d\alpha \int_0^1 \sqrt{1-r^2}\,r^2 r\,dr = 2\pi\frac{1}{2}\int_0^1 \sqrt{1-r^2}\,r^2\,dr^2$$

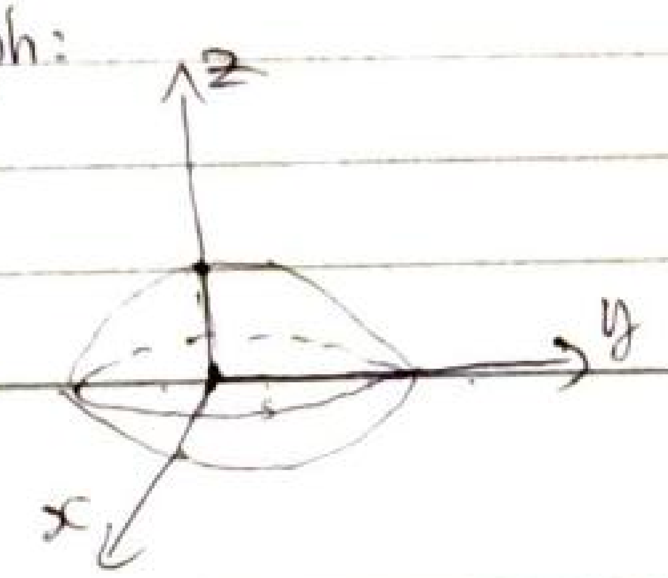
+) Put $a = 1-r^2 \to \begin{cases} da = -dr^2 \\ r^2 = 1-a \end{cases}$

- $$I = \frac{2\pi}{2}\int_0^1 \sqrt{a}\,(1-a)\,da = \pi\left(-a^{5/2}\cdot\frac{2}{5} + a^{3/2}\cdot\frac{2}{3}\right)\Big|_{a=0}^{a=1}$$

$$= \frac{4\pi}{15}$$

(121) $\iint_S y\,dxdz + z^2\,dxdy$, $S: \begin{cases} x^2+\frac{y^2}{4}+z^2=1,\ x\geq 0, y\geq 0, z\geq 0 \\ \text{oriented downward} \end{cases}$

+) Graph:



+) Put $x = \cos\alpha \sin\theta$
$y = 2\sin\alpha \sin\theta$
$z = \cos\theta$.

. $D_{\alpha\theta} \begin{cases} 0 \le \alpha \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$

+) Chose $\vec{N} = r_\theta \times r_\alpha = (\cos\alpha\cos\theta, 2\sin\alpha\cos\theta, -\sin\theta)$
$\times (-\sin\alpha\sin\theta, 2\cos\alpha\sin\theta, 0)$
$= (2\cos\alpha\sin^2\theta, \sin\alpha\sin^2\theta, 2\sin\theta\cos\theta)$

+) $\vec{N}.\vec{k} = 2\sin\theta\cos\theta \ge 0 \ \forall\, 0 \le \theta \le \frac{\pi}{2} \Rightarrow (\vec{N}, \vec{k}) \le \frac{\pi}{2}$
$(\vec{n}, \vec{k}) \ge \frac{\pi}{2}$ (S oriented downward)

$\Rightarrow I = -\iint_{D_{\alpha\theta}} 0 + \sin\alpha\sin^2\theta . 2\sin\alpha\sin\theta + 2\sin\theta\cos\theta . \cos^2\theta \, d\alpha d\theta$

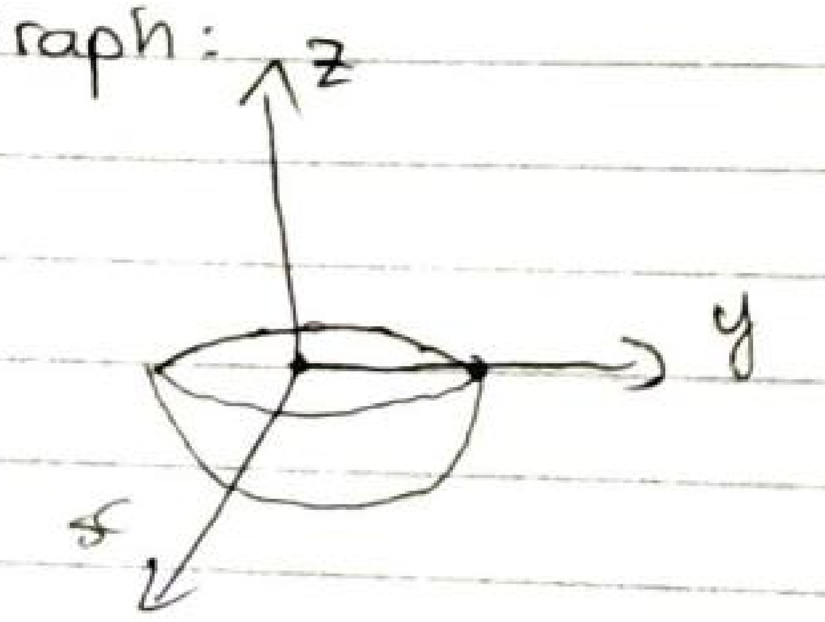
$= -\int_0^{\pi/2} d\alpha \int_0^{\pi/2} 2\sin^2\alpha\sin^3\theta + 2\sin\theta\cos^3\theta \, d\theta$

$= (-1)2\left[\frac{\pi}{4}\cdot\frac{2}{3} + \frac{\pi}{2}\cdot\frac{1}{4}\right]$

$= \frac{-7\pi}{12}$

(122) $\iint_S x^2y^2z \, dxdy$. $S: \begin{cases} x^2+y^2+z^2 = R^2, \ z \le 0 \\ \text{oriented upward.} \end{cases}$

+) Graph:



+) $(\vec{n}, \vec{k}) < \frac{\pi}{2}$ (S oriented upward)

$\Rightarrow I = \iint_{D_{xy}} x^2y^2(-\sqrt{R^2-x^2-y^2})\,dxdy$
$D_{xy}: x^2+y^2 \le R^2$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$
. $D_{r\alpha} \begin{cases} 0 \le r \le R \\ 0 \le \alpha \le 2\pi \end{cases}$
. $J = r$

+) $I = \int_0^{2\pi} d\alpha \int_0^R r^2\cos^2\alpha \, r^2\sin^2\alpha(-\sqrt{R^2-r^2})\, r\,dr$

$= \frac{-\pi}{4}\cdot\frac{1}{2}\int_0^R r^4\sqrt{R^2-r^2}\,dr^2$

+) Put $a = R^2 - r^2 \Rightarrow da = -dr^2$
$I = \frac{-\pi}{8}\int_0^{R^2} (R^2-a)^2\sqrt{a}\,da$

$= \frac{-\pi}{8}\int_0^{R^2} (R^4 - 2R^2a + a^2)\sqrt{a}\,da$

$= \frac{-\pi}{8}\left(R^4 a^{3/2}\frac{2}{3} - 2R^2 . a^{5/2} . \frac{2}{5} + a^{7/2} . \frac{2}{7}\right)\Big|_0^{R^2}$

$= \frac{-2\pi R^7}{105}$

(123) $S: x^2+y^2+z^2 = a^2$, oriented outward

a) $\iint_S x\,dydz + y\,dzdx + z\,dxdy$.

+) Apply O-G theorem:
. $I = \iiint_V 3\,dxdydz$
. $V: x^2+y^2+z^2 \le a^2$
$\Rightarrow I = 3.\frac{4\pi a^3}{3} = 4\pi a^3$

b) $\iint_S x^3dydz + y^3dzdx + z^3dxdy$.

+) Apply O-G theorem:
. $I = \iiint_V 3(x^2+y^2+z^2)\,dxdydz$
. $V: x^2+y^2+z^2 \le a^2$

Thứ

Ngày

+) Put $x = r\cos\alpha \sin\theta$

$y = r\sin\alpha \sin\theta$

$z = r\cos\theta$

• $D_{r\alpha\theta}: \begin{cases} 0 \le r \le a \\ 0 \le \alpha \le 2\pi \\ 0 \le \theta \le \pi \end{cases}$

• $J = -r^2 \sin\theta$

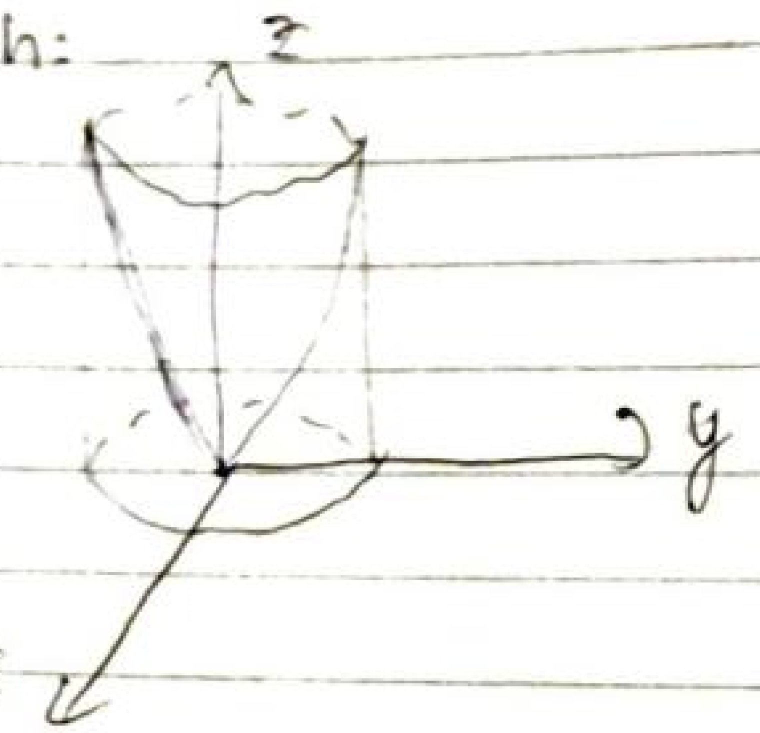
+) $I = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta \int_0^{a} 3r^4 \sin\theta \, dr$

$= 2\pi . 2 . 3 . \frac{a^5}{5} = \frac{12\pi a^5}{5}$

(124) $\iint_S y^2 z \, dx dy + xz \, dy dz + x^2 y \, dx dz$,

S is the boundary of $x \ge 0, y \ge 0, x^2+y^2 \le 1, 0 \le z \le x^2+y^2$, out word oriented.

+) Graph:



+) Apply O-G theorem:

• $I = \iiint_V (y^2 + z + x^2) \, dx dy dz$

• $V: x \ge 0, y \ge 0, x^2+y^2 \le 1, 0 \le z \le x^2+y^2$.

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

• $J = r$

• $D_{r\alpha z} \begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le \frac{\pi}{2} \\ 0 \le z \le r^2 \end{cases}$

Thứ

Ngày

No

+) $I = \int_0^{\pi/2} d\alpha \int_0^1 dr \int_0^{r^2} (r^2+z) r \, dz$

$= \frac{\pi}{2} \int_0^1 r^5 + \frac{r^5}{2} \, dr = \frac{\pi}{8}$

(125) $\iint_S x \, dydz + y \, dzdx + z \, dxdy$; S is boundary of $(z-1)^2 \ge x^2+y^2$

$a \le z \le 1$, $a > 0$, oriented outward.

+) Apply O-G theorem:

• $I = \iiint_V 3 \, dxdydz$

• $V: (z-1)^2 \ge x^2+y^2$, $a \le z \le 1$, $a > 0$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$, $z = z$

• $J = r$

• $D_{r\alpha z} \begin{cases} 0 \le \alpha \le 2\pi \\ (1-z)^2 \ge r^2 \to 1 - z \ge r \ge 0 \\ 1 \ge z \ge a \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_a^1 dz \int_0^{1-z} 3r \, dr$

$= 2\pi \int_a^1 \frac{3}{2} (1-z)^2 \quad dz$

$= 3\pi . \frac{(z-1)^3}{3} \Big|_{z=a}^{z=1}$

$= -\pi (a-1)^3 = \pi (1-a)^3$

(126) $\int_C F \, dr = ?$

C : oriented counter clock wise as view from above

a) $F(x,y,z) = (x+y^2) i + (y+z^2) j + (z+x^2) k$

C: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

+) Apply Stokes theorem:

+) $I = \iint_S \text{curl} \vec{F} . \vec{n} \, dS = \iint_S (-2z) i + (-2x) j + (-2y) k$

+) S: $x+y+z = 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$, oriented upward (right hand rule)

+) $\vec{n} = (1,1,1)/\sqrt{3}$

$\Rightarrow I = \iint_S \frac{-2}{\sqrt{3}}(x+y+z)\,dS = \frac{-2}{\sqrt{3}}\iint_S dS = \frac{-2}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = -1$

b) $F(x,y,z) = i + (x+yz^2)j + (xy-\sqrt{z})k$, C is boundary of $3x+2y+z=1$ in the first octant

+) Apply Stokes theorem:

. $I = \iint_S (x-y)\,dydz + (-y)\,dzdx + dxdy$

. S: $3x+2y+z=1$, oriented upward (right hand rule)
$x,y,z \geq 0$

+) $\vec{n} = (3,2,1)/\sqrt{14}$

$\Rightarrow I = \frac{1}{\sqrt{14}}\iint_S 3x-3y-2y+1\,dS$

$= \frac{1}{\sqrt{14}}\iint_S 3x-5y+1\,dS$

$= \iint_{D_{xy}} 3x-5y+1\,dxdy$

$= \int_0^{1/3} dx \int_0^{(1-3x)/2} 3x-5y+1\,dy$

$= \int_0^{1/3} 3x.\frac{1-3x}{2} - 5.\frac{(1-3x)^2}{2.2^2} + \frac{1-3x}{2}\,dx$

$= \frac{1}{24}$

c) $F(x,y,z) = yzi + 2xzj + e^{xy}k$, C: $x^2+y^2=16, z=5$

+) Apply Stokes theorem:

. $I = \iint_S (x.e^{xy} - 2x)\,dydz + (y)\,dzdx + (2z-z)\,dxdy$

. S: $x^2+y^2 \leq 16, z=5$, oriented upward (right hand rule)

+) $I = 5\iint_S dxdy = 5.4.4.\pi = 80\pi$

d) $F(x,y,z) = xyi + 2zj + 3yk$, C is the curve intersection of $x+z=5$ and the cylinder $x^2+y^2=9$.

+) Apply Stoke theorem:

. $I = \iint_S 1\,dydz - x\,dxdy$

. S: $\begin{cases} x+z=5 \\ x^2+y^2 \leq 9 \end{cases}$, oriented upward (right hand rule)

+) $\vec{n} = (1,0,1)/\sqrt{2}$

$\Rightarrow I = \frac{1}{\sqrt{2}}\iint_S 1-x\,dS = \iint_{D_{xy}} 1-x\,dxdy$

+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $J = r$

. $D_{r\alpha}\begin{cases} 0 \leq r \leq 3 \\ 0 \leq \alpha \leq 2\pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^3 r(1-r\cos\alpha)\,dr$

$= 2\pi.\frac{3^2}{2} = 9\pi$

Chapter 7: Field

1. Scalar fields.

a) directional derivative.

$\frac{\partial u}{\partial \vec{e}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$

$= \lim_{t\to 0} \frac{M_0 + t\vec{e}}{t}$

($\vec{e}$: unit vector)

b) Gradient

$\overrightarrow{grad}\,u(M) = \nabla u(M) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$

Theorem. If $u(x,y,z)$ is differentiable at M:

$$\frac{\partial u}{\partial \vec{\ell}}(M) = \frac{\overrightarrow{\text{grad}}\, u(M).\vec{\ell}}{\|\ell\|}$$

$\frac{\partial u}{\partial \vec{\ell}}$ rate of change of $u(x,y,z)$ at M in the direction $\vec{\ell}$?

$\boxed{\text{JAME STEWART}}$ Find the directional derivative

① $f(x,y) = e^x \sin y$ $\quad (0, \pi/3)$ $\quad v = (-6; 8)$

+) $\overrightarrow{\text{grad}}\, u(x,y) = (e^x \sin y, e^x \cos y)$

+) $\overrightarrow{\text{grad}}\, u(0, \pi/3) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

+) $\frac{\partial u}{\partial \vec{v}} = \frac{\frac{\sqrt{3}}{2}.(-6) + \frac{1}{2}.8}{\sqrt{6^2+8^2}} = \frac{-3\sqrt{3}}{10} + \frac{2}{5}$

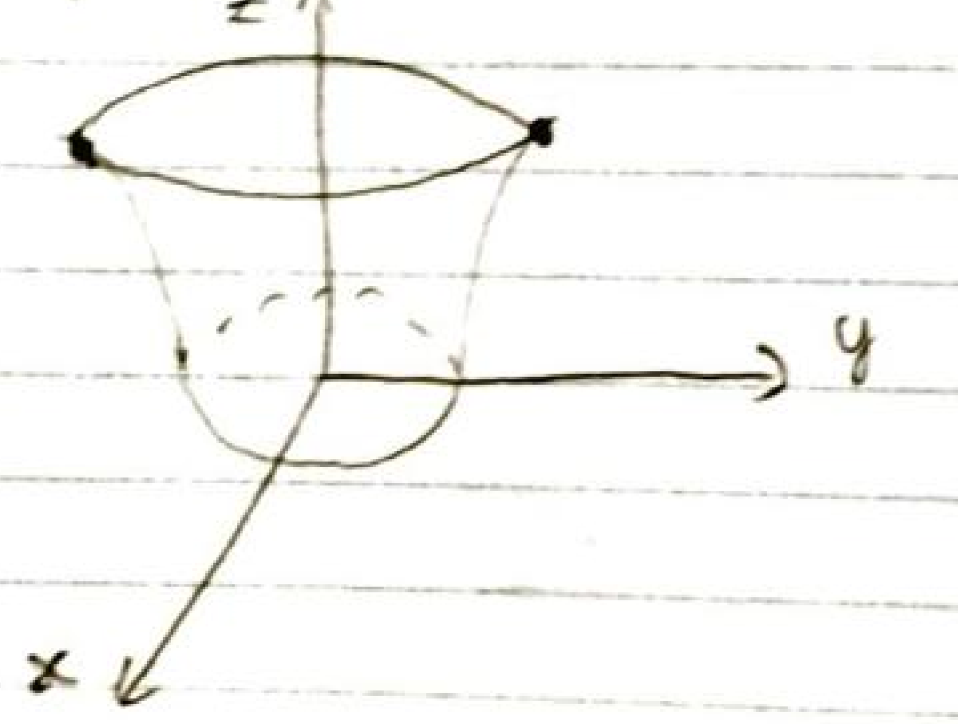
...

File 2: Calculus2_Baitap

100 $\iint_S \vec{F}.d\vec{S}$ (The flux of F across S)

a) $F(x,y,z) = x^3 y i - x^2 y^2 j - x^2 y z k$, S is the surface of the solid bounded by the hyperboloid $x^2 + y^2 - z^2 = 1$ and the planes $z = -2$, $z = 2$.

+) Graph:



+) Apply O-G theorem:

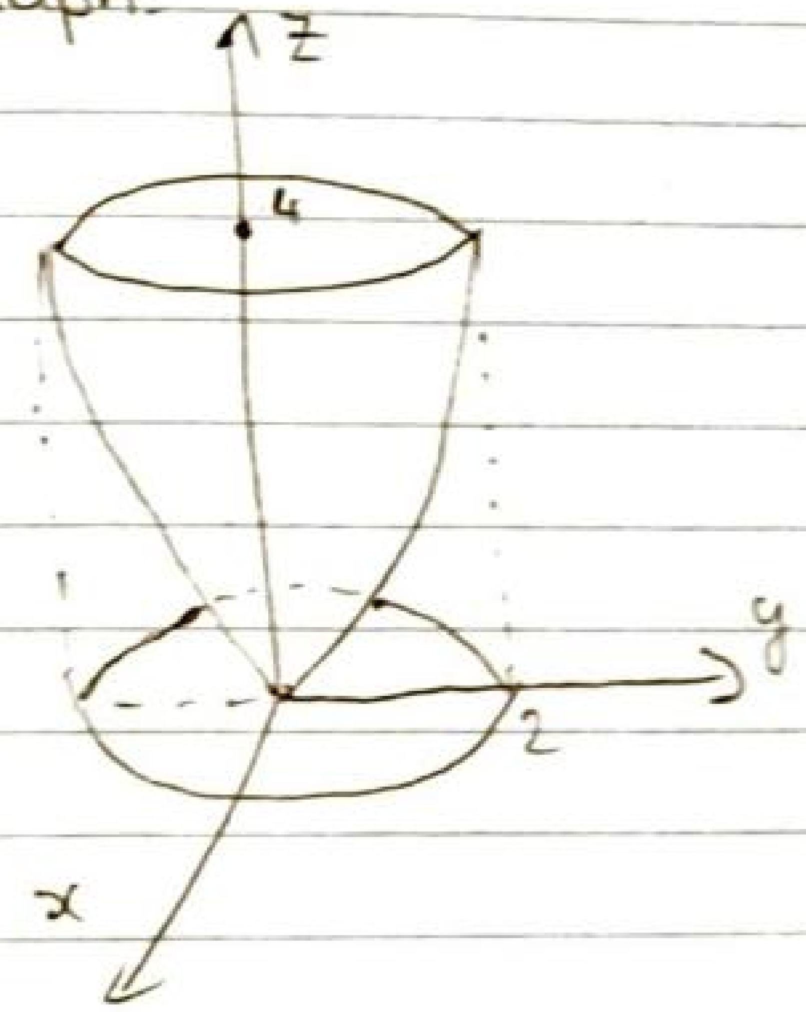
. $I = \iiint_V 3x^2 y - 2x^2 y - x^2 y \, dx dy dz = \boxed{0}$

. $V: \begin{cases} x^2 + y^2 \le 1 + z^2 \\ -2 \le z \le 2 \end{cases}$

b) $F(x,y,z) = (\cos z + x y^2) i + x e^{-z} j + (\sin y + x^2 z) k$.

S is boundary of $\begin{cases} z = x^2 + y^2 \\ z = 4 \end{cases}$

+) Graph:



+) Apply O-G theorem:

. $I = \iiint_V y^2 + x^2 \, dx dy dz$

. $V: \begin{cases} 0 \le z \le 4 \\ x^2 + y^2 \le z \end{cases}$

+) Put $x = r\cos\alpha$, $y = r \sin\alpha$, $z = z$

. $J = r$

. $D_{r\alpha} \begin{cases} 0 \le \alpha \le 2\pi \\ 0 \le r \le \sqrt{z} \end{cases}$

+) $I = \int_0^4 dz \int_0^{2\pi} d\alpha \int_0^{\sqrt{z}} r^3 dr = 2\pi \cdot \frac{1}{4} \int_0^4 z^2 dz = \boxed{\frac{32}{3}\pi}$

c) $F(x,y,z) = 4x^3z\,i + 4y^3z\,j + 3z^4k$.

$S: x^2+y^2+z^2 = R^2$

+) Apply O-G theorem:

• $I = \iiint\limits_V 12x^2z + 12y^2z + 12z^3\,dxdydz$

$= \iiint\limits_V 12z(x^2+y^2+z^2)\,dxdydz$

• $V: x^2+y^2+z^2 = R^2$

+) $f(x,y,z) = -f(x,y,-z)$

V is symetric respect to Oxy

$\Rightarrow I = \boxed{0}$

(99) $\iint\limits_S \text{curl} F.dS$

a) $F(x,y,z) = 2y\cos z\,i + e^x\sin z\,j + xe^y\,k$

$S: x^2+y^2+z^2 = 9,\ z \geq 0$, oriented upward.

+) Apply Stokes theorem:

• $I = \int\limits_C 2y\cos z\,dx + e^x\sin z\,dy + xe^y\,dz$

• $C: x^2+y^2 = 9,\ z = 0$, oriented positive (RHR)

$\Rightarrow I = \int\limits_C 2y\,dx$

+) Put $x = 3\cos\alpha,\ y = 3\sin\alpha$

• $D\alpha: 0 \leq \alpha \leq 2\pi$

• $I = \int_0^{2\pi} 2.3\sin\alpha.(-3\sin\alpha)\,d\alpha$

$= -18\pi$

b) $F(x,y,z) = x^2z^2\,i + y^2z^2\,j + xyz\,k$

$S: \begin{cases} z = x^2+y^2 \\ x^2+y^2 \leq 4 \end{cases}$, oriented upward

+) Apply Stokes theorem:

• $I = \int\limits_C x^2z^2\,dx + y^2z^2\,dy + xyz\,dz$

• $C: x^2+y^2 = 4,\ z = 4$ (oriented positive - RHR)

$\Rightarrow I = \int\limits_C 16x^2\,dx + 16y^2\,dy$

+) Put $x = 2\cos\alpha,\ y = 2\sin\alpha$

• $D\alpha: 0 \leq \alpha \leq 2\pi$

• $I = \int_0^{2\pi} 16.2^2.\cos^2\alpha.2(-\sin\alpha) + 16.2^2.\sin^2\alpha.(2\cos\alpha)\,d\alpha$

$= 128\int_0^{2\pi} \cos^2\alpha.(-\sin\alpha) + \sin^2\alpha\cos\alpha\,d\alpha$

$= 0$

(98) a) Find the center of mass of $x^2+y^2+z^2 = a^2,\ z \geq 0$, if it has constant density.

+) $m = \iint\limits_S \rho\,dS = \rho\iint\limits_S dS = \rho.2\pi a^2$

+) Symetric property: $x_G = y_G = 0$

+) $z_G = \frac{1}{m}\iint\limits_S z.\rho\,dS = \frac{1}{2\pi a^2}\iint\limits_S z\,dS$

• Put $x = a\cos\alpha\sin\theta$

$y = a\sin\alpha\sin\theta$

$z = a\cos\theta$

$\rightarrow \begin{cases} D\alpha\theta.\ 0 \leq \alpha \leq 2\pi;\ 0 \leq \theta \leq \pi/2 \\ |r_\alpha \times r_\theta| = a^2\sin\theta \end{cases}$

• $z_G = \frac{1}{2\pi a^2}.\int_0^{2\pi} d\alpha \int_0^{\pi/2} a\cos\theta.a^2\sin\theta\,d\theta$

$= \frac{1}{2\pi a^2}.2\pi.\frac{1}{2}.a^3 = \frac{a}{2}$

+) $G = \boxed{\left(0, 0, \frac{a}{2}\right)}$

b) Find the mass of a thin funnel in the shape of cone $z=\sqrt{x^2+y^2}$, $1\le z\le 4$, $\rho(x,y,z)=10-z$

$m=\iint_S (10-z)\,dS$

$=\iint_{D_{xy}} (10-\sqrt{x^2+y^2})\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy$

$=\iint_{D_{xy}} (10-\sqrt{x^2+y^2})\sqrt{2}\,dxdy$

+) Put $x=r\cos\alpha$, $y=r\sin\alpha$

• $D_{r\alpha}\begin{cases}1\le r\le 4\\ 0\le\alpha\le 2\pi\end{cases}$

• $J=r$

+) $I=\int_0^{2\pi} d\alpha\int_1^4 (10-r)\sqrt{2}.r\,dr$

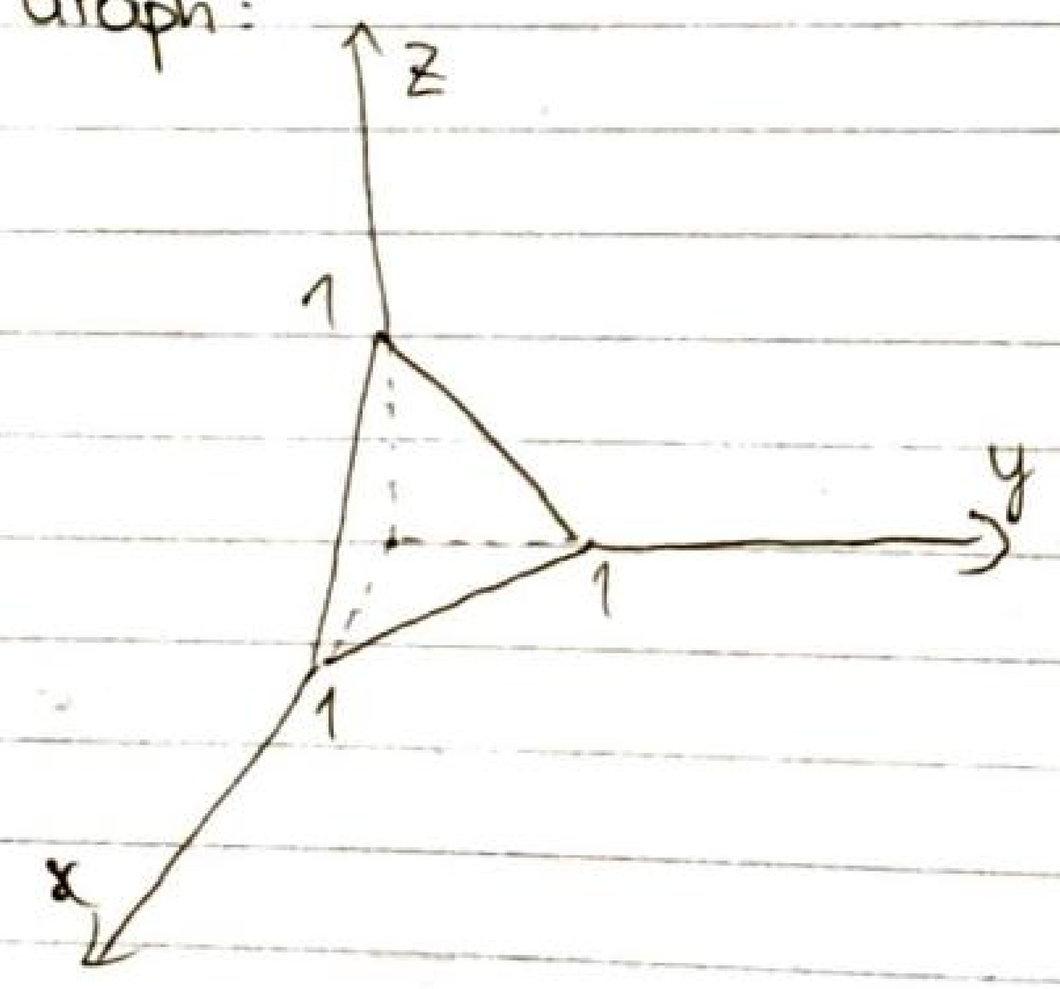
$=2\pi.\sqrt{2}.54=\boxed{108\sqrt{2}\pi}$

97) $\iint_S F.dS$: the flux of F across S (closed, outward oriented)

a) $F(x,y,z)=xze^y i - xze^y j + zk$

S: $x+y+z=1$; $x,y,z\ge 0$; downward orientation.

+) Graph:



+) $\vec{n}=-(1,1,1)/\sqrt{3}$

$\rightarrow I=\frac{-1}{\sqrt{3}}\iint_S (xze^y - xze^y + z)\,dS$

$=\frac{-1}{\sqrt{3}}\iint_S z\,dS$

$=\frac{-1}{\sqrt{3}}\iint (1-x-y)\sqrt{1^2+1^2+1^2}\,dxdy$

D_{xy}: $x+y\le 1$; $x,y\ge 0$

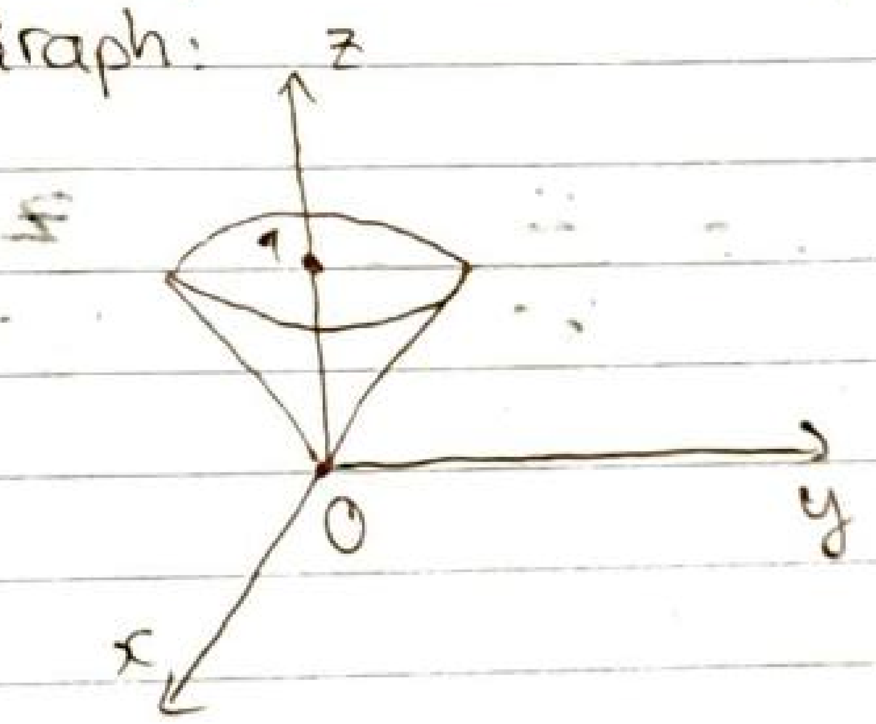
$=-\int_0^1 dx\int_0^{1-x}(1-x-y)\,dy$

$=\int_0^1 -(1-x)^2+\frac{(1-x)^2}{2}\,dx$

$=\boxed{\frac{-1}{6}}$

b) $F(x,y,z)=xi+yj+z^4k$, S is the part of the cone $z=\sqrt{x^2+y^2}$ beneath the plane $z=1$, downward orientation

+) Graph:



+) $\vec{N}=(-z'_x,-z'_y,1)=\left(\frac{-x}{\sqrt{x^2+y^2}},\frac{-y}{\sqrt{x^2+y^2}},1\right)$

$\vec{N}.\hat{k}=1>0\Rightarrow(\vec{N},\hat{k})<\frac{\pi}{2}$

+) $(\vec{n},\hat{k})>\frac{\pi}{2}\Rightarrow\vec{N}\uparrow\downarrow\vec{n}$

+) $I=-\iint_S (x,y,z^4)\frac{\vec{N}}{||N||}\,dS$

$$= -\iint_{D_{xy}} \frac{-(x^2+y^2)}{\sqrt{x^2+y^2}} + (x^2+y^2)^2\,dx\,dy \quad (D_{xy}: x^2+y^2 \le 1)$$

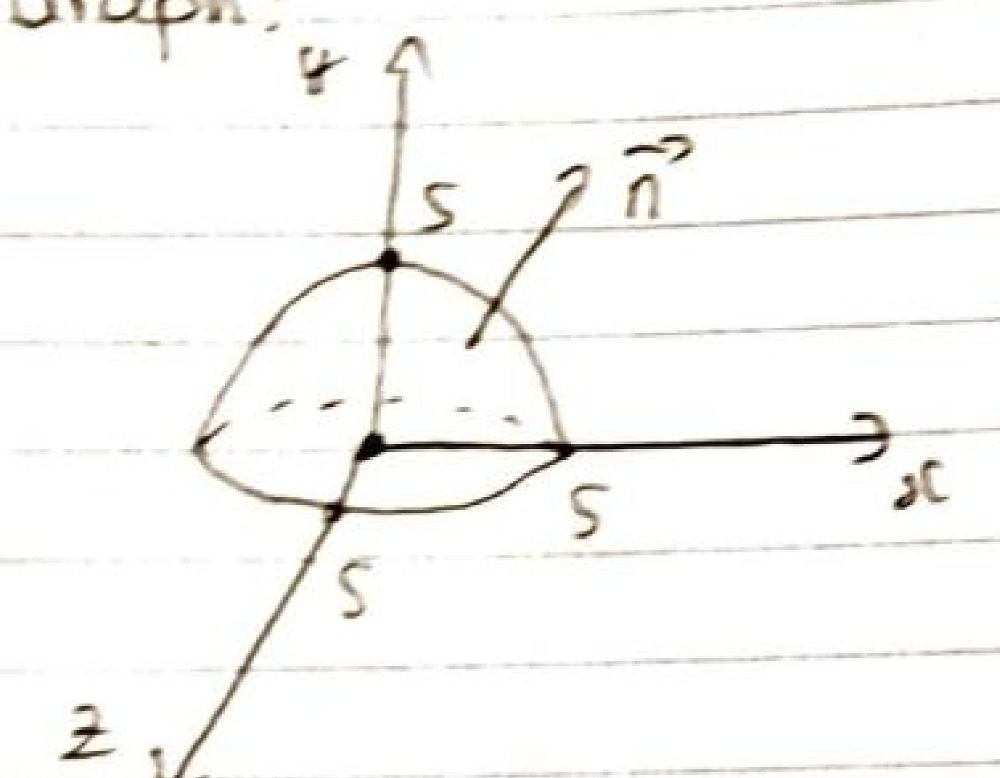
+) Put $x = r\cos\alpha$, $y = r\sin\alpha$

. $D_{r\alpha}\begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le 2\pi \end{cases}$

. $I = -\int_0^{2\pi} d\alpha \int_0^1 (-r + r^4)\,r\,dr = -2\pi \cdot \frac{-1}{6} = \boxed{\frac{\pi}{3}}$

c) $F(x,y,z) = xzi + xj + yk$, $S: x^2+y^2+z^2 = 25$, $y \ge 0$, oriented in the direction of positive y-axis.

+) Graph:



+) $\vec{N} = (-y'_x, 1, -y'_z) = \left(\frac{x}{\sqrt{25-x^2-z^2}}, 1, \frac{z}{\sqrt{25-x^2-z^2}}\right)$

$\vec{N}.\hat{j} = 1 > 0 \Rightarrow (\vec{N}, \hat{j}) < \frac{\pi}{2}$

+) $(\vec{n}, \hat{j}) < \frac{\pi}{2} \Rightarrow \vec{N} \uparrow\uparrow \vec{n}$

+) $I = \iint_S F.\frac{\vec{N}}{\|N\|}\,dS = \iint_{D_{xz}} \frac{x^2 z}{\sqrt{25-x^2-z^2}} + x + z\,dx\,dz$

$(D_{xz}: x^2+y^2 \le 25)$

+) Put $x = r\cos\alpha$, $z = r\sin\alpha$

. $D_{r\alpha}\begin{cases} 0 \le r \le 5 \\ 0 \le \alpha \le 2\pi \end{cases}$

. $I = I$

+) $I = \int_0^{2\pi} d\alpha \int_0^5 \left(\frac{r^2\cos^2\alpha\, r\sin\alpha}{\sqrt{25-r^2}} + r\cos\alpha + r\sin\alpha\right) r\,dr$

$= 0$

d) $F(x,y,z) = xyi + 4x^2j + yzk$.

$S: z = xe^y$; $0 \le x \le 1$; $0 \le y \le 1$, oriented upward.

+) $\vec{N} = (-z'_x, -z'_y, 1) = (-e^y, -xe^y, 1)$; $(\vec{N}, \hat{k}) < \frac{\pi}{2}$

+) $(\vec{n}, \hat{k}) < \frac{\pi}{2} \Rightarrow \vec{N} \uparrow\uparrow \vec{n}$

+) $I = \iint_S \vec{F}.\frac{\vec{N}}{\|N\|}\,dS = \iint_{D_{xy}} -e^y.xy - xe^y.4x^2 + y.xe^y\,dx\,dy$

$= \int_0^1 dx \int_0^1 -4x^3 e^y\,dy$

$= (-1)(e-1) = 1-e$

e) $F(x,y,z) = x^2i + y^2j + z^2k$

S is the boundary of $0 \le z \le \sqrt{1-y^2}$, $0 \le x \le 2$

+) Apply O-G theorem.

. $I = \iiint_V 2(x+y+z)\,dx\,dy\,dz$

. $V: 0 \le z \le \sqrt{1-y^2}$, $-1 \le y \le 1$, $0 \le x \le 2$

+) $I = 2\int_0^2 dx \int_{-1}^1 dy \int_0^{\sqrt{1-y^2}} x+y+z\,dz$

$= 2\int_0^2 dx \int_{-1}^1 (x+y)\sqrt{1-y^2} + \frac{1-y^2}{2}\,dy$

$= 2\int_0^2 dx \int_{-1}^1 \left(x\sqrt{1-y^2} + \frac{1-y^2}{2}\right) dy$

$= 2\left(2.\frac{\pi}{2} + 2.\frac{2}{3}\right) = 2\pi + \frac{8}{3}$

Thứ
Ngày

a) $\iint_S xy\,dS$, $S: (1,0,0) \to (0,2,0) \to (0,0,2)$

+) $S: 2x+y+z=2$; $x,y,z \geq 0$

+) $I = \iint_{D_{xy}} xy \sqrt{1^2+1^2+2^2}\, dxdy$
$D_{xy}: 2x+y \leq 2, x \geq 0, y \geq 0$

$= \sqrt{6} \int_0^1 dx \int_0^{2-2x} xy\, dy$

$= \sqrt{6} \int_0^1 x \frac{(2-2x)^2}{2} dx$

$= \frac{\sqrt{6}}{6}$

b) $\iint_S yz\,dS$; $S: x+y+z=1$; $x,y,z \geq 0$

+) $I = \iint_{D_{yz}} yz\sqrt{1^2+1^2+1^2}\, dydz$
$D_{yz}: y+z \leq 1, y \geq 0, z \geq 0$

$= \sqrt{3} \int_0^1 dy \int_0^{1-y} yz\, dz$

$= \sqrt{3} \int_0^1 y \frac{(1-y)^2}{2} dy$

$= \frac{\sqrt{3}}{24}$

c) $\iint_S yz\,dS$, $S: x = u^2, y = u\sin\theta, z = u\cos\theta$
$0 \leq u \leq 1$, $0 \leq \theta \leq \pi/2$

+) $r_u \times r_\theta = (2u, \sin\theta, \cos\theta) \times (0, u\cos\theta, -u\sin\theta) = (-u, 2u^2\sin\theta, 2u^2\cos\theta)$

+) $I = \int_0^1 du \int_0^{\pi/2} u\sin\theta\, u\cos\theta \sqrt{u^2+4u^4}\, d\theta$

$= \frac{1}{2} \int_0^1 u^3 \sqrt{1+4u^2}\, du$

$= \frac{1}{4 \cdot 2} \int_0^1 u^2 \sqrt{1+4u^2}\, du^2+1$

Thứ
Ngày No.

+) Put $t = 1+4u^2 \to u^2 = \frac{t-1}{4}$, $d(4u^2+1) = dt$

$I = \frac{1}{16} \int_1^5 \frac{t-1}{4} \sqrt{t}\, dt$

$= \frac{1}{16} \left(\frac{1}{4} \cdot t^{5/2} \cdot \frac{2}{5} - \frac{1}{4} \cdot t^{3/2} \cdot \frac{2}{3} \right) \Big|_1^5$

$= \frac{1}{16} \left(\frac{5\sqrt{5}}{3} + \frac{1}{15} \right)$

$= \frac{5\sqrt{5}}{48} + \frac{1}{240}$

d) $\iint_S z\,dS$, $S: x = y+2z^2$, $0 \leq y \leq 1$, $0 \leq z \leq 1$

+) $I = \iint_{D_{yz}} z\sqrt{1^2+(-1)^2+(-4z)^2}\, dydz$

$= \int_0^1 dy \int_0^1 z\sqrt{2+16z^2}\, dz$

$= \frac{1}{32} \int_0^1 (2+16z^2)^{1/2}\, d(2+16z^2)$

$= \frac{1}{32} (2+16z^2)^{3/2} \cdot \frac{2}{3} \Big|_0^1$

$= \frac{1}{48} \cdot \left(54\sqrt{2} - 2\sqrt{2} \right)$

$= \frac{13\sqrt{2}}{12}$

e) $\iint_S y^2\,dS$, $S: x^2+y^2+z^2=4$
$x^2+y^2 \leq 1$, $z \geq 0$

+) $I = \iint_{D_{xy}} y^2 \sqrt{1+\left(\frac{x}{\sqrt{4-x^2-y^2}}\right)^2+\left(\frac{y}{\sqrt{4-x^2-y^2}}\right)^2}\, dxdy$
$D_{xy}: x^2+y^2 \leq 1$

$= \iint_{D_{xy}} y^2 \sqrt{\frac{4}{4-x^2-y^2}}\, dxdy$

KOKUYO

Thứ
Ngày . . No.

+) Đặt $x = r\cos\alpha$, $y = r\sin\alpha$

- $J = r$
- $D_{r\alpha} \begin{cases} 0 \le r \le 1 \\ 0 \le \alpha \le 2\pi \end{cases}$

+) $I = \int_0^{2\pi} d\alpha \int_0^1 r^2 \sin^2\alpha \cdot \dfrac{2r}{\sqrt{4-r^2}}\, dr$

$= 2\pi \int_0^1 \dfrac{r^3}{\sqrt{4-r^2}}\, dr$

$= -\pi \int_0^1 r^2 (4-r^2)^{-1/2}\, d(4-r^2)$

$= -\pi \int_4^3 (4-a)\, a^{-1/2}\, da$

$= \pi \int_3^4 4a^{-1/2} - a^{1/2}\, da$

$= \pi \left(4 \cdot a^{1/2} \cdot 2 - a^{3/2} \cdot \dfrac{2}{3}\right) \Big|_3^4$

$= \pi \left(\dfrac{32}{3} - 6\sqrt{3}\right)$

Thứ
Ngày . . No.

KOKUYO