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CHAPTER: MULTIPLE INTEGRALS

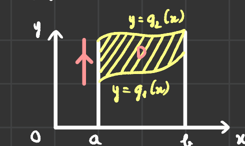
Double Integrals

Fubini's Theorem: If f is continuous on the rectangle $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$

Then:

$$\boxed{\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy}$$

Double Integrals over General Regions $\displaystyle\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$



Type I (x) $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

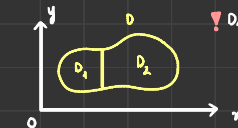
$$\boxed{\text{Then: } I = \iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

Type II (y) $D = \{ (x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y) \}$

$$\boxed{\text{Then: } I = \iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy}$$

Properties

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$



! D_1, D_2 not overlap

$$\iint_D |f(x,y)|\, dA = \iint_{D^+} f + \iint_{D^-} -f$$

Symmetry

1. If D is symmetric about Ox (Oy), f is an odd function w/r to x (y)
   Then: $\iint_D f(x,y)\, dx\, dy = 0$

2. If D is symmetric about Ox (Oy), f is an even function w/r to x (y)
   Then: $\iint_D f(x,y)\, dx\, dy = 2 \iint_{D^+} f(x,y)\, dx\, dy$

Changing variables in double integrals

Jacobian coefficient:

$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} \neq 0,\ \forall (u,v) \in D_{uv}$$

$$J^{-1} = \frac{D(u,v)}{D(x,y)} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} \neq 0$$

General:

$$\boxed{I = \iint_D f(x,y)\,dx\,dy = \iint_{D_{uv}} f\big(x(u,v),\, y(u,v)\big)\,|J|\,du\,dv}$$

Polar coordinate: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \rightarrow D_{r\theta} = \{(r,\theta),\ a \le r \le b,\ \alpha \le \theta \le \beta\}$

Then:

$$\boxed{I = \iint_{D_{r\theta}} f(x,y)\,dA = \int_\alpha^\beta\int_a^b f\big(r\cos\theta,\, r\sin\theta\big)\,r\,dr\,d\theta}$$

Extended:

1. If $D: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Then $\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$, $J = abr$

2. If $D: (x-a)^2 + (y-b)^2 = R^2$. Then $\begin{cases} x = a + r\cos\theta \\ y = b + r\sin\theta \end{cases}$, $J = r$

Applications of double integrals

Volume

If $f(x,y) \geq 0$, the volume of the domain lying above the rectangle $R = [a,b] \times [c,d]$ and below the surface $z = f(x,y)$ is:

$$V = \iint_R f(x,y)\,dA$$

Area

If $f(x,y) = 1$ over D. Then the region of D is: $A(D) = \iint_D dA$

Total mass of the lamina

$$m = \iint_D \underbrace{\rho(x,y)}_{\text{density}}\, dA$$

Moments

Moment of the lamina about the x-axis: $M_x = \iint_D y\,\rho(x,y)\,dA$

Moment of the lamina about the y-axis: $M_y = \iint_D x\,\rho(x,y)\,dA$

Centre of mass $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

Moments of inertia

$$I_x = \iint_D y^2\,\rho(x,y)\,dA$$

$$I_y = \iint_D x^2\,\rho(x,y)\,dA$$

$$I_0 = \iint_D (x^2+y^2)\,\rho(x,y)\,dA$$

(M.o.i about the origin or polar m.o.i)

Triple Integrals

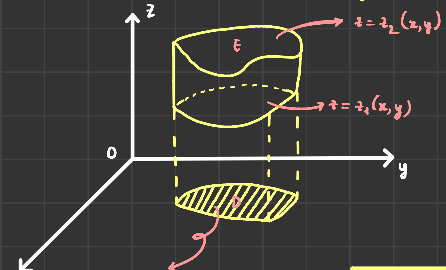
Fubini's Theorem

If f is continuous on the rec. box $B = [a,b] \times [c,d] \times [r,s]$. Then:

$$\boxed{\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f \; dz\,dy\,dx}$$

Triple Integrals over General Regions

$$E = \{ (x,y,z) \mid (x,y) \in D,\ z_1(x,y) \le z \le z_2(x,y) \}$$



FIND THE PROTECTION

$$\boxed{I = \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz \right] dA}$$

Double integral

D : type I (x) $\quad I = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dy\,dx$

D : type II (y) $\quad I = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz\,dx\,dy$

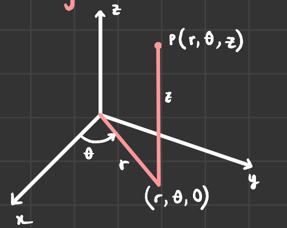
Triple Integral → Double Integral → Iterated Integrals

Changing variables in triple integrals

$$\boxed{I = \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V_{uvw}} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw}$$

For system: $\begin{cases} a_1x + b_1y + c_1z = \lambda_1 \\ a_2x + b_2y + c_2z = \lambda_2 \\ a_3x + b_3y + c_3z = \lambda_3 \end{cases} \Rightarrow J^{-1} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Cylindrical Coordinate



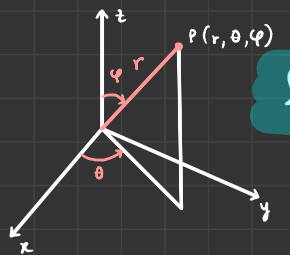
$0 \le \theta \le 2\pi$

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$, $J = r$

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

→ Best for problems involving symmetry about an axis (z in this formula)

Spherical Coordinate



$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \pi \end{cases}$

$x = r\cos\theta\sin\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\varphi$, $J = -r^2\sin\varphi$

→ Best for problems involving symmetry about a point (origin in this formula)

$$E = \{(r,\theta,\varphi) \mid a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le \varphi \le d\}$$

Extended Spherical and Cylindrical Coordinates

1. V: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ $\qquad x = ar\cos\theta\sin\varphi \qquad y = br\sin\theta\sin\varphi \qquad z = cr\cos\varphi$, $J = -abcr^2\sin\varphi$

2. V: $(x-a)^2 + (y-b)^2 + (c-z)^2 \le R^2$ $\qquad x = a + r\cos\theta\sin\varphi \qquad y = b + r\sin\theta\sin\varphi \qquad z = c + r\cos\varphi$, $J = -r^2\sin\varphi$

3. V: $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} \le 1$

$\xrightarrow{\text{Cylindrical}}$ $x = ar\cos\theta \quad y = ar\sin\theta \quad z = bz'$, $J = a^2 br$

$\xrightarrow{\text{Spherical}}$ $x = ar\cos\theta\sin\varphi \quad y = ar\sin\theta\sin\varphi \quad z = br\cos\varphi$, $J = a^2br^2\sin\varphi$

CHAPTER : LINE INTEGRALS

Definition

If f defined on a smooth curve C given by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $\langle a \le t \le b \rangle$

Then the line integral of f along C is : $\boxed{I = \int_C f(x,y)\,ds = \int_a^b f(\underbrace{x(t),(y(t)}_{\vec{r}(t)})\underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}_{|\vec{r}'(t)|}\,dt}$

Line Integrals over scalar fields

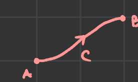
1. $y = y(x)$, $a \le x \le b$: $I = \int_C f(x,y)\,ds = \int_a^b f(x, y(x))\overbrace{\sqrt{1+y'(x)^2}}^{\text{arclength}}\,dx$

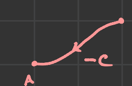
2. $x = x(y)$, $c \le y \le d$: $I = \int_C f(x,y)\,ds = \int_c^d f(x(y), y)\sqrt{1+x'(y)^2}\,dy$

3. $r = r(\varphi)$, $\varphi_1 \le \varphi \le \varphi_2$: $I = \int_C f(x,y)\,ds = \int_{\varphi_1}^{\varphi_2} f(r(\varphi)\cos\varphi, r(\varphi)\sin\varphi)\sqrt{r^2(\varphi)+r'^2(\varphi)}\,d\varphi$

$ds = \sqrt{r^2(\varphi)+r'^2(\varphi)}\,d\varphi$

→ Do not depend on the path of the curve : $\int_{-C} f(x,y)\,ds = \int_{C} f(x,y)\,ds$





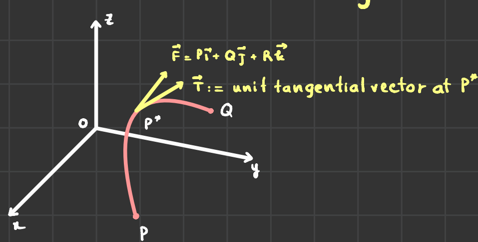
Line Integral in space

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$, $a \le t \le b$

$$I = \int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}\,dt$$

Line Integrals over vector fields

WHY ? Find Work done by a constant force in moving an object from P to Q (along the curve C)



Then: $W = \int_C \vec{F}(x,y,z) \cdot \vec{T}(x,y,z)\,ds$

Definition let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is:

$$\boxed{I = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_C \vec{F} \cdot \vec{T}\,ds}$$

We have that:

$$\left.\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t), y(t))\,x'(t)\,dt = \int_C P(x,y)\,dx \\ \int_C f(x,y)\,dx &= \int_a^b f(x(t), y(t))\,y'(t)\,dt = \int_C Q(x,y)\,dy \end{aligned}\right\} \quad I = \int_C P\,dx + Q\,dy$$

→ 3D : $I = \int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$:= circulation of F accross C if C is a closed curve

Formulations

1. $x = x(t)$, $y = y(t)$, $a \le t \le b$

$$\boxed{I = \int_C P\,dx + Q\,dy = \int_C P\big(x(t), y(t)\big)\,x'(t) + Q\big(x(t), y(t)\big)\,y'(t)\;dt}$$

2. $y = y(x)$, $a \le x \le b$

$$\boxed{I = \int_C P\,dx + Q\,dy = \int_C \big[P\,(x, y(x)) + Q\,(x, y(x))\,y'(x)\big]\,dt}$$

or use x as parameter

$x = x$, $y = y(x)$, $x \in [a,b]$

$\rightarrow I = \int_C P(x, y(x))\,dx + Q(x, y(x))\,\underset{y'(x)\,dx}{\underset{\parallel}{dy}}$

3. $x = x(y)$, $c \le y \le d$

$$\boxed{I = \int_C P\,dx + Q\,dy = \int_C \big[P\,(x(y), y)\,x'(y) + Q\,(x(y), y)\big]\,dt}$$

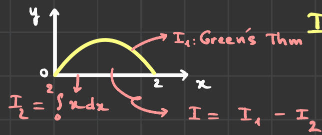
GREEN'S THEOREM

Let C be a positively oriented ($\circlearrowleft$), piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then:

$$\boxed{\int_C P\,dx + Q\,dy = +\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}$$

$\oint \equiv \int_{\partial D} \equiv$

∂D → positively oriented



If C is not closed, we "close off" the curve, apply the Theorem and subtract.

FUNDAMENTAL THEOREM FOR LINE INTEGRALS

C: Smooth curve given by $\vec{r}(t)$, $a \le t \le b$.

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}\,, \quad \nabla f(x,y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Applications of line integrals

Area of domains

Choose P and Q such that: $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$

Then:

$$\boxed{\begin{aligned} A &= \oint_C x\,dy = -\oint_C y\,dx \\ &= \frac{1}{2}\oint_C x\,dy - y\,dx \end{aligned}}$$

Independence of path

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Thm 1: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D iff $\int_C \vec{F} \cdot d\vec{r} = 0$, $\forall$ closed path C in D.

Thm 2: Suppose $\vec{F}$ is a vector field, cont. on D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D, i.e., $\exists f : \nabla f = \vec{F}$.

where $f(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)\,dy$

$= \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy$

HOW TO SOLVE ? $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

1. Check for conservativity of the vector field, i.e., $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$

2. By the Theorem, $\exists f : \nabla f = \vec{F}$. Now we have:

$f_x(x,y,z) = P(x,y,z)$ $\qquad$ $f_y(x,y,z) = Q(x,y,z)$ $\qquad$ $f_z(x,y,z) = R(x,y,z)$

$$f_x(x,y,z) = P(x,y,z) \xrightarrow{\int dx} f(x,y,z) = \alpha + g(y,z)$$

$$f(x,y,z) = \alpha + g(y,z) \xrightarrow[\text{w/r to } y]{\text{Differentiate}} \underbrace{A + g'(y,z)}_{f_y} = Q(x,y,z) \Rightarrow g(y,z) = B + h(z)$$

$$g(y,z) = B + h(z) \xrightarrow[\text{w/r to } z]{\text{Differentiate}} \underbrace{D + h'(z)}_{f_z} = R(x,y,z) \Rightarrow h(z) = ? \ (\text{const } K)$$

back sub $\qquad$ back sub

Then: $f(x,y,z) = \alpha + g(y,z) + K$

$\Rightarrow I = f(B) - f(A)$

COMPUTING INTEGRALS THAT ARE INDEPENDENT OF PATH

1. Check for conservativity of vector field
2. Choose the path such that the integration is simplest. (Might be lines, parallel zigzags...)

CHAPTER: SURFACE INTEGRALS

Surface Integrals of parametric surfaces

1. Surface area

If a smooth parametric surface S is given by:

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k} \qquad (u,v) \in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$\boxed{A(S) = \iint_D \left| \vec{r}_u \times \vec{r}_v \right| dA}$$

where

$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}$$

$$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}$$

2. Surface area of the graph of a function $z = z(x,y)$

$$\boxed{A(S) = \iint_D \underbrace{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}_{|\vec{r}_x \times \vec{r}_y|}\, dA}$$

3. Surface Integrals of parametric surfaces

Surface integral of f over the surface S is:

$$\boxed{I = \iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA}$$

Graph $z = z(x,y)$:

$$I = \iint_S f(x,y,z)\, dS = \iint_D f(x, y, z(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Surface Integrals of vector fields

WHY ? Given a fluid with density $\rho(x,y,z)$ $[kg\,m^{-3}]$ and velocity field $\vec{v}(x,y,z)$ floating through S. Then the mass of fluid per unit time accrossing S is:

$$\iint_S \underbrace{\rho \vec{v}}_{\vec{F}} \cdot \vec{n}\, dS$$

DEFINITION If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$. Then the surface integral of F over S is:

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA}$$

$:=$ Flux of $\vec{F}$ across S

$\vec{n}$: positive $\equiv$ outward

HOW TO SOLVE ?

GENERAL: Let $\vec{S}$ be given by $\vec{r}(u,v)$. Then: $\vec{N} = \vec{r}_u \times \vec{r}_v = (A, B, C)$

If $\vec{N} \uparrow\uparrow \vec{n}$:

$$\boxed{I = \iint_D (A.P + B.Q + C.R)\, du dv}$$

If $\vec{N} \uparrow\downarrow \vec{n}$:

$$\boxed{I = -\iint_D (A.P + B.Q + C.R)\, du dv}$$

SPECIAL CASES: + If S is given by $z = z(x,y)$ Then $\vec{N} = (-z'_x, -z'_y, 1)$

$$\boxed{I = \pm\iint_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA}$$

Similarly $\begin{cases} x = x(y,z) \\ y = y(x,z) \end{cases}$ Then $\vec{N} = (1, -x'_y, -x'_z)$; Then $\vec{N} = (y'_x, -1, y'_z)$

+ If $P = Q = 0 \Rightarrow I = \iint_D R(x, y, z(x,y))\, dxdy$

<Similarly for the others.>

STOKES' THEOREM

curl

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a vector field on $\mathbb{R}^3$, and the partial derivatives of P, Q, R all exist. Then the curl of $\vec{F}$ is:

$$\boxed{\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \nabla \times \vec{F}$$

nabla (del) operator

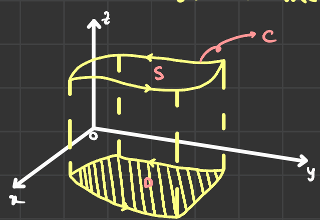
$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

divergence

$$\boxed{\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

$$= \nabla \cdot \vec{F}$$

Stokes' Theorem



Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field, then:

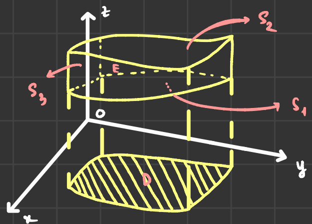
boundary

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}}$$

surface

or: $\displaystyle\int_C P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

THE DIVERGENCE THEOREM (Ostrogradsky)



! E is a whole fucking solid

Let E be a simple solid region and let S be the boundary surface of E, given positive (outward) orientation. Let $\vec{F}$ be a vector field, then:

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dv}$$

boundary surface ... solid region

or: $\iint_S P\,dy\,dz + Q\,dx\,dz + R\,dx\,dy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz$

Vector calculus

Gradient vector

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

Rate of change

$$\nabla f \cdot \vec{u} = \frac{\partial f}{\partial u}(M_0)$$

, where $\frac{\partial f}{\partial u}(M_0)$ is the directional derivative at $M_0(x_0, y_0, z_0)$ in the direction $\vec{u}$. Thus it represents the rate of change of f at M_0 in the direction $\vec{u}$.

R.o.t = max $= |\nabla f|$ if $\vec{u} \parallel \nabla f$.

- increase fastest : $\vec{u} \uparrow\uparrow \nabla f$
- decrease fastest: $\vec{u} \uparrow\downarrow \nabla f$

CHAPTER: INTEGRALS DEPENDING ON PARAMETERS

1. INTEGRALS DEPENDING ON PARAMETERS

Suppose that f is a continuous function defined on $[a,b]\times[c,d]$. Then: $I(y)=\int_a^b f(x,y)\,dx$ is a func defined on $[c,d]$ and is called "integral depending on a parameter of the func $f(x,y)$.

CONTINUITY $f(x)$ cont. on $[a,b]\times[c,d]$

$$\lim_{y\to y_0} I(y) = I(y_0)$$

$$\lim_{y\to y_0} \int_a^b f(x,y)\,dx = \int_a^b f(x,y_0)\,dx$$

DIFFERENTIATION
- $f(x,y)$ cont. on $[a,b]\times[c,d]$
- $f'_y(x,y)$ cont. on $[a,b]\times[c,d]$

$$I'(y) = \int_a^b f'_y(x,y)\,dx$$

INTEGRATION $f(x,y)$ cont. on $[a,b]\times[c,d]$

$$\int_c^d I(y)\,dy := \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

HOW TO SOLVE ?

$I(y) = \int_a^b f(x,y)\,dx$

DIFFERENTIATION

Step 1: Check for conditions (continuity of cont. of f and f'_y)

Step 2: Find $I'(y) = \int_a^b f'_y(x,y)\,dx$

Step 3: Use Newton-Leibniz to find $I(y) = \int I'(y)\,dy + C$

Step 4: Solve for const C.

INTEGRATION

Step 1: Check for conditions

Step 2: Express $f(x,y) = \int_c^d F(x,y)\,dy$

Step 3: Change the order of integration

2. IMPROPER INTEGRALS DEPENDING ON PARAMETERS

CONVEGENCE

Consider $I(y) = \int_a^{+\infty} f(x,y)\,dx$, $y \in [c,d]$. We say that $I(y)$ is:

1. Convergent at $y_0 \in [c,d]$ if: $\int_a^{\infty} f(x,y_0)$ conv., i.e., $\forall \varepsilon > 0$, $\exists b(\varepsilon, y_0) > a$ s.t:

$$\left| I(y_0) - \int_a^b f(x,y_0)\,dx \right| = \left| \int_b^{\infty} f(x,y_0)\,dx \right| < \varepsilon \qquad \forall b > b(\varepsilon, y_0)$$

2. Conv. on $[c,d]$ if $I(y)$ conv. at $\forall y_0 \in [c,d]$.
3. Uniformly conv. on $[c,d]$ if $\forall \varepsilon > 0$, $\exists b_\varepsilon > a$:

$$\left| I(y) - \int_a^b f(x,y)\,dx \right| = \left| \int_b^{\infty} f(x,y)\,dx \right| < \varepsilon \qquad \forall b > b_\varepsilon, \forall y \in [c,d]$$

Weierstrass Criterion
- $|f(x,y)| \leq g(x)$, $\forall (x,y) \in [a,+\infty) \times [c,d]$
- $\int_a^{+\infty} g(x)\, dx$ conv.

Then: $I(y)$ is uni. conv. on $[c,d]$

CONTINUITY
- $f(x,y)$ cont. on $[a,\infty) \times [c,d]$
- $I(y)$ uni. conv $\forall y \in [c,d]$

Then $\boxed{\lim_{y \to y_0} I(y) = I(y_0)}$

DIFFERENTIATION

f, f'_y cont. on $[a,\infty) \times [c,d]$

$I(y)$ uni. conv $\forall y \in [c,d]$

$I'(y)$ uni. conv $\forall y \in [c,d]$

Then $\boxed{I'(y) = \int_a^{\infty} f'_y(x,y)\, dx}$

INTEGRATION

f cont. on $[a,\infty) \times [c,d]$

$I(y)$ uni. conv $\forall y \in [c,d]$

$$\boxed{I(y) = \int_c^d \left(\int_a^b \right) = \int_a^b \left(\int_c^d \right)}$$

3. IMPORTANT INTEGRALS

DIRICHLET

$$\boxed{\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}}$$

GAUSS

$$\boxed{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$$

FRESNEL

$$\boxed{\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}}$$

4. EULER INTEGRALS

GAMMA FUNCTION

$$\boxed{\Gamma(p) = \int_0^{+\infty} x^{p-1} e^{-x}\, dx}$$ defined on $(0, +\infty)$

$$\Gamma(p+1) = p \cdot \Gamma(p)$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)!!}{2n}\sqrt{\pi}$$

$$\Gamma^{(k)}(p) = \int_0^{+\infty} x^{p-1} (\ln^k x)\, e^{-x}\, dx$$

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad \forall\, 0 < p < 1$$

BETA FUNCTION

$$B(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1}\, dx$$

$$B(p,q) = \int_0^{+\infty} \frac{x^{p-1}}{(1+x)^{p+q}}\, dx$$

$x = \frac{t}{t+1}$

$$B(p,q) = B(q=p)$$

$$B(p,q) = \frac{p-1}{p+q-1} B(p-1, q), \quad p > 1$$

$$B(p,q) = \frac{q-1}{p+q-1} B(p, q-1), \quad q > 1$$

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

$$B(p,q) = 2 \cdot \int_0^{\frac{\pi}{2}} \sin^{2p-1} t \cdot \cos^{2q-1} t\, dt \quad \text{(Trig-form)}$$

RELATION BETWEEN GAMMA & BETA FUNCTIONS

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}$$