Coordinate Rings & Uniformizing Parameters

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Here are some theorems and definitions we used to help us through our project:

Definition. *The **localization a ring** R over a prime ideal $\mathfrak{p}$ is denoted $R_{\mathfrak{p}} = S^{-1}R$, where $S = R - \mathfrak{p}$ is a multiplicative saturated set, we also say "R localized at $\mathfrak{p}$."*

Proposition. *If R is a ring with $\mathfrak{p}$ a prime ideal of R, then $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$.*

Definition. *A ring R is **Noetherian** if it satisfies the ascending chain conditions on ideals. That is, if every chain $I_1 \subset I_2 \subset I_3 \subset \cdots$ of ideals of R is eventually stationary, that is there exsits k such that $I_k = I_{k+1} = I_{k+2} = \cdots$.*

Theorem. (Hilbert's Basis Theorem) *If R is a Noetherian ring, then R[X] is a Noetherian ring.*

Theorem. *If R is a Noetherian ring and I is a proper ideal of R then R/I is also Noetherian.*

Theorem. *Every localization of a Noetherian ring is still Noetherian.*

Let's start out with what is a discrete valuation is.

Definition. *A **discrete valuation** of a field k is a surjective function $v : k^\times \to \mathbb{Z}$ such that for every $x, y \in k^\times$*

(1) $v(x \cdot y) = v(x) + v(y)$,

(2) $v(x + y) \geq min\{v(x), v(y)\}$ *if* $x \neq -y$ *. The function v is a group homomorphism where we define $v(1) = 0$ and $v(x^{-1}) = -v(x)$.*

*Let $A = \{x \in k \,|\, v(x) \geq 0\} \subseteq k$ and $\mathfrak{m} = \{x \in k \,|\, v(x) > 0\} \subseteq A$. This set A is a local domain, not a field, and $\mathfrak{m}$ is its maximal ideal. Further, $\mathfrak{m}$ is nonzero and since v is surjective, then we call A the **discrete valuation ring** or acronym-ed **DVR** of the function v.*

Definition. *An element $x \in \mathfrak{m}$ with $v(x) = 1$ is called a **uniformizing parameter**. x is irreducible because if $x = ab$ where $v(a), v(b) \geq 0$ then $v(x) = v(ab) = v(a) + v(b) = 1$ so $v(a) = 0$ or $v(b) = 0$.*

Now look at the plane curve C, defined by $f(X, Y) = 0$ and the coordinate ring for this curve is $k[X,Y]/(f)$. Next, choose a point **P** on the curve and do a change of coordinates so that the f has no constant term, that is the point **P** is now the origin.

If f has a linear term, let's do another change of coordinates so that the only linear term is Y, that is $f = Y + p(X, Y)$, where $p \in (X, Y)^2$.

Look at the localization of the coordinate ring $k[X,Y]/(f)$. Let $R = k[X,Y]/(f)$. Let $x = X+(f)$ and $y = Y+(f)$. Then the ring localized at the origin, which corresponds to the maximal ideal (X, Y), is $R_{(x,y)} = \{\bar{g}/\bar{h} \,|\, \bar{g}, \bar{h} \in R,\ h(0, 0) \neq 0\}$ where $\overline{g} = g + (f)$ and $\overline{h} = h + (f)$ and the maximal ideal of this local ring is $(x, y)R_{(x,y)} = \{\bar{g}/\bar{h} \,|\, g(0, 0) = 0\}$, by the proposition above.

By Hilbert's Basis Theorem, we know for k, a field, it is Noetherian and thus $k[X]$ is Noetherian and finally therefore $k[X][Y] = k[X, Y]$ is Noetherian. (You can also apply the Corollary of Hilbert's Basis Theorem to get this result.)

Also by the theorem above states if I is an ideal of R then R/I is also Noetherian and hence, since $k[X, Y]$ is Noetherian with the prime ideal (f) so $k[X,Y]/(f)$ is again Noetherian.

Now all the facts are here to help prove the next proposition.

Proposition. *Suppose* $(R, \mathfrak{m})$ *is a Noetherian local domain and* $\mathfrak{m} = (t)$ *is principal. Then* $\cap_{i=1}^{n} \mathfrak{m}^i = (0)$. *In fact, R is a discrete valuation ring.*

Proof. Suppose by way of contradiction that $\cap_{i=1}^{n} \mathfrak{m}^i \neq (0)$. Then $x \in \cap_{i=1}^{n} \mathfrak{m}^i$ where $x \neq 0$ and so $x = x_i t^i$ for some $x_i \in R$. Since $x \in \cap_{i=1}^{n} \mathfrak{m}^i$, then $x = x_i t^i = x_{i+1} t^{i+1}$. Now cancel like terms and get

$$x_i = x_{i+1} t \in (x_{i+1}).$$

Thus $(x_i) \subseteq (x_{i+1})$. So now there is a chain of ideals and since R is Noetherian, there exists some $n \in \mathbb{Z}^+$, where this chain of ideals become stationary:

$$(x_1) \subseteq (x_2) \subseteq \cdots \subseteq (x_k) = (x_{k+1}) = \cdots$$

Since $x \in \cap_{i=1}^{n} \mathfrak{m}^i$, then we can also write $x = x_k t^k = x_{k+1} t^{k+1}$.

Also, since $(x_k) = (x_{k+1})$ and that can only happen if one of them differs by a unit, so say $x_{k+1} = u x_k$ for some unit $u \in R$. Then,

$$x_k t^k = x_{k+1} t^{k+1} = u x_k \cdot t^{k+1},$$

next cancel like terms,

$$1 = ut$$

since R is a integral domain, we can *cancel* out terms, but this just shows that t is a unit! A contradiction to the fact that x is nonzero and hence $\cap_{i=1}^{n} \mathfrak{m}^i = (0)$

Now if this doesn't seem familiar, this is something we did on the take-home exam! ☺

Next, define a discrete valuation ν.

Let $y \in K = Frac(R)$. Then $y = \frac{a}{b}$ for some a, $b \in R$. From the take-home exam it was determined that every non-zero element of R can be written uniquely as $a = s_1 t^n$ and $b = s_2 t^m$ for some units s_1, $s_2 \in R$ and n, $m \in \mathbb{Z}^+$.

And so $y = \frac{a}{b} = \frac{s_1 t^n}{s_2 t^m} = \frac{s_1}{s_2} \cdot t^{n-m}$.

Now define the discrete valuation by

$$\nu : K \to \mathbb{Z} \;\; via \; \nu(y) = \nu(\frac{s_1}{s_2} \cdot t^{n-m}) = n - m.$$

ν is surjective because for every $n \in \mathbb{Z}^+$ there is always a t^n that is correspond to it.

Now to verify (1) of the definition of DVR, let z, $w \in Frac(R)$ then $z = \frac{c}{d}$ and $w = \frac{g}{h}$ for some $c, d, g, h \in R$. So $c = s_1 t^{n_1}$, $d = s_2 t^{n_2}$, $g = r_1 t^{m_1}$, $h = r_2 t^{m_2}$, for some units s_i, $r_i \in R$ and n_i, $m_i \in \mathbb{Z}$.

Notice $\nu(z) = n_1 - n_2$ and $\nu(w) = m_1 - m_2$.

$$\nu(z \cdot w) = \nu(\frac{c}{d} \cdot \frac{g}{h}) = \nu(\frac{cg}{dh}) = \nu(\frac{s_1 t^{n_1} \cdot r_1 t^{m_1}}{s_2 t^{n_2} \cdot r_2 t^{m_2}})$$

$$= \nu(\frac{s_1 r_1 \cdot t^{n_1+m_1}}{s_2 r_2 \cdot t^{n_2+m_2}}) = \nu(\frac{s_1 r_1}{s_2 r_2} t^{n_1+m_1-(n_2+m_2)})$$

$$= \nu(\frac{s_1 r_1}{s_2 r_2} t^{n_1+m_1-n_2-m_2})$$

$$= \nu(\frac{s_1 r_1}{s_2 r_2} t^{(n_1-n_2)+(m_1-m_2)}) = (n_1 - n_2) + (m_1 - m_2)$$
$$= v(z) + \nu(w).$$

Now to verify (2), use the same z and w. WLOG, suppose $v(z) \geq \nu(w)$.

$$z + w = \frac{c}{d} + \frac{g}{h} = \frac{s_1 t^{n_1}}{s_2 t^{n_2}} + \frac{r_1 t^{m_1}}{r_2 t^{m_2}} = \frac{s_1}{s_2} t^{n_1-n_2} + \frac{r_1}{r_2} t^{m_1-m_2}$$

$$= t^{n_1-n_2}(\frac{s_1}{s_2} + \frac{r_1}{r_2} t^{(m_1-m_2)-(n_1-n_2)})$$

And since $\frac{s_1}{s_2} + \frac{r_1}{r_2} t^{(m_1-m_2)-(n_1-n_2)}$ is a unit, we get
$$\nu(z+w) = n_1 - n_2 \geq \nu(w) = min\{\nu(w), \nu(z)\}.$$

Thus R is a discrete valuation ring (DVR) of ν. □

Notice a previous Lemma that one of the other groups did also would have helped us complete our proof:

Lemma. *Let A be a local domain, $\mathfrak{m}$ its maximal ideal. Assume that $\mathfrak{m}$ is nonzero and principal and that $\cap_{n\geq 0}\mathfrak{m}^n = (0)$. Then A is a DVR.*

Essentially the a local domain and $\mathfrak{m}$ is nonzero and principal and it was shown that $\cap_{n\geq 0}\mathfrak{m}^n = (0)$. And thus by the Lemma above $(R, \mathfrak{m})$ is a DVR.

Let's look back at our coordinate ring.
Let $R = {}^{k[X,Y]}/_{(f)}$ where $f = Y + P(X,Y)$ and $P(X,Y) \in (X,Y)^2$. Let $x = X + (f)$ and $y = Y + (f)$. Then our local ring is $R_{(x,y)} = \{\bar{g}/\bar{h} \mid \bar{g}, \bar{h} \in R,\ h(0,0) \neq 0\}$ where $\bar{g} = g + (f)$ and $\bar{h} = h + (f)$ and the maximal ideal is $(x,y)R_{(x,y)} = \{\bar{g}/\bar{h} \mid g(0,0) = 0\}$.

We are going to look at two cases:

Case 1. If f has a linear term, then $R_{(x,y)}$ is a DVR.

Case 2. If f does not have a linear term, then $R_{(x,y)}$ is not a DVR.

CASE 1:

If f has a linear term then the local ring $R_{(x,y)}$ is a discrete valuation ring. That is, we are going to find a uniformizing parameter.

Proof. First look at f, so f has the form:
$$f(X,Y) = Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 + a_{12}XY^2 + a_{30}X^3 + \cdots$$

Now, reduce it by *modulo* f and factor out a Y from each term that has a Y:
$$0 = y(1 + a_{11}x + a_{02}y + a_{12}xy + \cdots + a_{0n}y^{n-1}) + a_{20}x^2 + a_{30}x^3 + \cdots + a_{n0}x^n$$

So, moving the Y terms to the other side and get:
$$y(1 + a_{11}x + a_{02}y + a_{12}xy + \cdots + a_{0n}y^{n-1}) = -(a_{20}x^2 + a_{30}x^3 + \cdots + a_{n0}x^n)$$

Call $a_{20}x^2 + a_{30}x^3 + \cdots + a_{n0}x^n = p(x)$ and since $1 + a_{11}x + a_{02}y + \cdots + a_{0n}y^{n-1}$ is a unit, we can divide by that unit to isolate y

$$y = \frac{-p(x)}{1 + a_{11}x + a_{02}y + a_{12}xy + \cdots + a_{0n}y^{n-1}} \in (x)^2 \subseteq (x)$$

Now we claim $(x, y)R_{(x,y)} = (x)$.

$\supseteq$: Since for any $h \in (X)$, $h(0, 0) = 0$, so $\overline{h} \in (x, y)R_{(x,y)}$. So $(x) \subseteq (x, y)R_{(x,y)}$.

$\subseteq$: Since $y \in (x)^2 \subseteq (x)$ so $(x, y)R_{(x,y)} = (x)R_{(x,y)} \subseteq (x)$.

Therefore $(x) = (x, y)R_{(x,y)}$.

$R_{(x,y)}$ is a Noetherian local ring, since the ring in the beginning was Noetherian and localization of a Noetherian ring makes the localized ring Noetherian by one the theorems, with a principal maximal ideal (x) and so by the **theorem** above $\cap_{i=1}^{n}(x)^i = (0)$ (and by that fact that no element can lie in all powers of x) and so by the **lemma** we get that $R_{(x,y)}$ is a DVR. □

CASE 2:

What if f did not have a linear term? That is, what if $f \in (X, Y)^2$? Would the local ring be a DVR?

But first, let's prove a lemma.

Lemma. *If $(R, \mathfrak{m})$, a local ring with maximal ideal $\mathfrak{m}$ is a DVR then $dim_{R/\mathfrak{m}}(\mathfrak{m}^k/\mathfrak{m}^{k+1}) = 1$.*

Proof. Suppose the local ring $(R, \mathfrak{m})$ is a DVR. Then the local ring has a uniformizing parameter and so the maximal ideal $\mathfrak{m}$ is principal. Call the uniformizing parameter t, so $\mathfrak{m} = (t)$. Let $\alpha \in R/\mathfrak{m}$, $x \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$.

Then $\alpha = r + \mathfrak{m}$ and $x = ut^k + \mathfrak{m}^{k+1}$, for some r, $u \in R$, now define the multiplication between α and x by

$$\alpha \cdot x = r \cdot ut^k + \mathfrak{m}^{k+1}.$$

We claim the multiplication above is well-defined.

First, let α, $\beta \in R/\mathfrak{m}$ and $x \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$. Then $\alpha = r_1 + \mathfrak{m}$,$\beta = r_2 + \mathfrak{m}$, and $x = u't^k + \mathfrak{m}^{k+1}$, for some r_1, r_2, $u' \in R$.

Suppose $\alpha = \beta$. Then $r_1 - r_2 \in \mathfrak{m} = (t)$ so $r_1 - r_2 = u \cdot t$ for some $u \in R$.

Now look at

$$(r_1 - r_2) \cdot u't^k = (ut) \cdot u't^k = uu't^{k+1} \in \mathfrak{m}^{k+1},$$

since $uu' \in R$.

So, $(r_1 - r_2) \cdot u't^k = r_1u't^k - r_2u't^k \in \mathfrak{m}^{k+1}$, that is

$$\alpha x = r_1u't^k + \mathfrak{m}^{k+1} = r_2u't^k + \mathfrak{m}^{k+1} = \beta x.$$

Next, let $\alpha \in R/\mathfrak{m}$ and $x, y \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$. Then $\alpha = r + \mathfrak{m}$, $x = u't^k + \mathfrak{m}^{k+1}$, and $y = ut^k + \mathfrak{m}^{k+1}$ for some r, u', $u \in R$.

Suppose $x = y$. Then $u't^k - ut^k \in \mathfrak{m}^{k+1} = (t)^{k+1}$, so $u't^k - ut^k = (u' - u) \cdot t^{k+1}$.

Then,

$$r \cdot (u't^k - ut^k) = r \cdot \overline{u}t^{k+1} \in \mathfrak{m}^{k+1}$$

since ideals absorb products.

Then $r \cdot (u't^k - ut^k) = r \cdot u't^k - r \cdot ut^k \in \mathfrak{m}^{k+1}$, that is

$$\alpha x = ru't^k + \mathfrak{m}^{k+1} = rut^k + \mathfrak{m}^{k+1} = \alpha y$$

Thus this multiplication is well-defined.

So the vector space is generated by one element since everything in $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ looks like $\alpha \cdot t^k + \mathfrak{m}^{k+1}$ and thus $dim(\mathfrak{m}^k/\mathfrak{m}^{k+1}) = 1$, for any $k \in \mathbb{Z}^+$. □

To answer your question whether the local ring is a DVR when $f \in (X,Y)^2$, no, it is not! ☹

Proof. Assume by way of contradiction that $R_{(x,y)}$ is a DVR.
Then by the **lemma** above we have $dim({}^{(x,y)}/_{(x,y)^2}) = 1$.
Let $a \in {}^{(x,y)}/_{(x,y)^2}$.
Then $a = bX + cY + (x,y)^2$, for some $b, c \in R$.
Since the dimension is 1, then either X or Y is a multiple of the other. Suppose $y = \alpha x$ for some $\alpha \in R_{(x,y)}$, that is, $Y+(f) = \alpha X+(f)$ and so $Y-\alpha X \in (f)$, a contradiction! Since $(f) \subseteq (X,Y)^2$, but $Y-\alpha X \notin (X,Y)^2$ since both X and Y are both linearly independent! And so $dim({}^{(x,y)}/_{(x,y)^2}) = 2 \neq 1$.
Therefore if f does not have a linear term, then the local ring $R_{(x,y)}$is not a DVR. ☐

Notice the tangent line of the (irreducible) polynomial has not been the uniformizing parameter of the local ring $R = {}^{k[X,Y]}/_{(f)}$. Next, we will prove the tangent line of the curve cannot be the uniformizing parameter.

Remember in order for an element to be a uniformizing parameter of the DVR, its valuation must equal 1.

Proof. Suppose $g(X,Y) = b_{10}X + b_{01}Y + b_{20}X^2 + b_{11}XY + b_{02}Y^2 + \cdots + b_{n0}X^n \in k[X,Y]$ for some $a_{ij} \in k$. To find the tangent line, implicitly differentiate by X,

$$\frac{d}{dX}(g(X,Y)) = b_{10} + b_{01}\frac{dY}{dX} + b_{20}\cdot 2X + b_{11}Y + b_{11}X\cdot\frac{dY}{dX} + \cdots + b_{n0}\cdot nX^{n-1}$$

$$0 = \frac{dg}{dX}(0,0) = b_{10} + b_{01}\frac{dY}{dX}$$

$$\frac{dY}{dX} = -\frac{b_{10}}{b_{01}}$$

Which is the slope of the tangent line. Now plugging in to point-slope form of the equation where the point is the orgin and slope $-\frac{b_{10}}{b_{01}}$ to get

$$Y = -\frac{b_{10}}{b_{01}}X$$

Rearranging,

$$0 = b_{10}X + b_{01}Y.$$

Notice the tangent line is the linear terms of g. Now look at g in the ring ${}^{k[X,Y]}/_{(g)}$. Since $g \in (g)$, then $g = 0 \in {}^{k[X,Y]}/_{(g)}$.
Let $x = X + (g)$ and $y = Y + (g)$, then

$$g(x,y) = 0 = b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + \cdots + b_{n0}x^n$$

and move the tangent line terms over and get

$$b_{10}x + b_{01}y = -(b_{20}x^2 + b_{11}xy + b_{02}y^2 + \cdots + b_{n0}) \in (x,y)^2.$$

That is, the tangent line is in the second power of the ideal, that is the valuation of $b_{10}x + b_{01}y$ is 2, and hence can not be a uniformizing parameter. ☐

Back to Case 1:

Let's look back at **case 1**, where $R = {}^{k[X,Y]}/_{(f)}$ and now write out each element of *f*,

$$f(X,Y) = Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 + \cdots + a_{n0}X^n.$$

Now implicitly differentiate by *X*,

$$\frac{d}{dX}(f(X,Y)) = \frac{dY}{dX} + a_{20} \cdot 2X + a_{11}Y + a_{11}X \cdot \frac{dY}{dX} + \cdots + a_{n0} \cdot nX^{n-1}$$
$$0 = \frac{dg}{dX}(0,0) = \frac{dY}{dX}$$
$$\frac{dY}{dX} = 0$$

So the tangent line of f at the origin is $Y = 0$. And Y cannot be the uniformizing parameter because notice before that $Y \in (X)^2$ and so the valuation of Y is equal to 2.

So we found one uniformizing parameter in case 1, x, but there can be more than one!
Now look at $x + y$, which is a uniformizing parameter in the ring $R = {}^{k[X,Y]}/_{(f)}$.

Proof. Assume by way of contradiction that $x + y$ is not a uniformizing parameter, then $x + y \in (x)^2$ and previously we found that $y \in (x)^2$ so suppose $y = t \cdot x^2$ for some $t \in k[X, Y]$.
Substituting in, we get
$$x + y = x + tx^2 = x(1 + tx) \in (x)^2.$$
So that implies $x \in (x)^2$ since $(1 + tx)$ is a unit. A contradiction! Since $x \notin (x)^2$.
Thus, $x + y$ is a uniformizing parameter of $R_{(x,y)}$. $\square$

In conclusion, uniformizing parameters are unique up to units.