Quandles

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1 Introduction

Quandles can be described as an algebraic structure that can be associated with a knot. These types of algebraic structures appeared as early as 1943, but today's terminology "quandle" was due to David Joyce's doctoral dissertation in the 1980s [6]. Quandle theory might seem a bit odd at first, but it is like learning about group theory in abstract algebra for the first time. Usually the operation symbols are $\triangleright$ instead of the use of $*$ in this paper, but they stand for the same thing. Let's start off with some definitions.

Definition. A *rack* is a set X with binary operations $*$ and $*^{-1}$ such that

1. $(x * y) *^{-1} y = (x *^{-1} y) * y = x$ for all $x,\ y \in X$ *(right-invertible)*
2. $(x * y) * z = (x * z) * (y * z)$ for all $x,\ y,\ z \in X$ *(self-distributivity).*

We can view racks as a set where each element acts on the left and right as autormorphisms, with left the action is the inverse of the right one [8]. A quandle is a special kind of rack.

Definition. A *quandle* Q is a rack with one more axiom:

0. $x * x = x$ for all $x \in Q$, that is all elements in Q are *idempotent.*

Example 1. (*Conjugation Quandle*) Notice there are two types of conjugations in any group, here it will be denoted $x * y = y^{-1}xy$ and the other $x *^{-1} y = yxy^{-1}$. In fact, writing the conjugation of two elements in these ways do form a quandle!

Proof. Let G be a group. We just need to show that $x * y = y^{-1}xy$ satisfies all the quandle axioms.
First, lets show that for all $x \in G$, x is idempotent. So, then $x * x = x^{-1}xx = x$.
Next, for the second axiom let $x, y \in G$. Then

$$(x * y) *^{-1} y = (y^{-1}xy) *^{-1} y = y(y^{-1}xy)y^{-1} = (yy^{-1})x(yy^{-1}) = x$$

and

$$(x *^{-1} y) * y = (yxy^{-1}) * y = y^{-1}(yxy^{-1})y = (y^{-1}y)x(y^{-1}y) = x,$$

so $(x * y) *^{-1} y = (x *^{-1} y) * y$.
Finally for the third axiom, let $x, y, z \in G$. This time I will start from the right hand side, so

$$(x * z) * (y * z) = (z^{-1}xz) * (z^{-1}yz) = (z^{-1}yz)^{-1}(z^{-1}xz)(z^{-1}yz)$$

since G is a group, then

$$= (z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz) = z^{-1}y^{-1}(zz^{-1})x(zz^{-1})yz) = (z^{-1}y^{-1})x(yz)$$

$$= z^{-1}(y^{-1}xy)z = (y^{-1}xy) * z = (x * y) * z.$$

Therefore $x * y = y^{-1}xy$ gives us a quandle, it is exactly the set $\{x^{-1}gx \mid x, g \in G\}$ that is it is all the conjugations of all the elements of a group G. $\square$

Notice every group give a quandle with this operation. Every conjugacy class of a group G is a *subquandle* of the conjugation quandle.

Definition. A nonempty subset V of a quandle Q is a *subquandle* of Q if V forms a quandle under the same binary operation of Q.

Definition. An *involutory quandle* or *kei* X is a quandle with binary operation $* = *^{-1}$ that is, it satisfies the following axioms:

1. $x * x = x$ for all $x \in X$,
2. $(x * y) * y = x$ for all $x,\ y \in X$, and
3. $(x * y) * z = (x * z) * (y * z)$ for all $x,\ y,\ z \in X$.

We can define known knot invariants as quandles, such as the Alexander polynomial.

Example 2. (*Alexander Quandle*) Let A be a $\mathbb{Z}[t, t^{-1}]$-module. A is a quandle defined by $x * y = tx + (1-t)y$, for every $x, y \in A$.

Example 3. (*Dihedral Quandle*) Let $a, b \in \mathbb{Z}_n$ for some integer n. Define $a * b = 2b - a \mod n$. Then $\mathbb{Z}_n$ is a quandle, defined by $*$. (Notice this is an Alexander quandle where $t = -1$.)
Let's look at more concrete examples:

(a) Let $n = 3$ and so $\mathbb{Z}_3 = \{0, 1, 2\}$. Let's find how the quandle $\mathbb{Z}_3$ would look like. Looking at Table 1, we can see that $\mathbb{Z}_3 =< 0, 1, 2 \,|\, 0 * 1 = 2,\ 0 * 2 = 1,\ 1 * 2 = 0 >$.

(b) What about the quandle for $\mathbb{Z}_4$ where $\mathbb{Z}_4 = \{0, 1, 2, 3\}$?
Looking at Table 2, the quandle would be $\mathbb{Z}_4 =< 0, 1, 2, 3 \,|\, 0 * 1 = 2,\ 1 * 2 = 3,\ 2 * 3 = 0,\ 3 * 0 = 1 >$. Notice this is **NOT** commutative, unlike the quandle for $\mathbb{Z}_3$, because $0 * 1 = 2$, but $1 * 0 = 3$.

Sometimes a dihedral quandle called a *cyclic quandle* [7], but it is usually it is called a dihedral quandle since this set of reflections of a regular n-gon.

2 Knot Quandles

A question may rise is that why do we care about quandles? One of the main reasons quandles was brought up was because of the knots that can be associated with some of these quandles. In fact, quandles are knot invariants!

Definition. A *knot quandle* is defined on an oriented knot where $*$ and $*^{-1}$ are defined as where $a * b$ is defined to be *"the arc where arc a under arc b from right to left"* and $a *^{-1} b$ is *"the arc where arc a is under arc b from left to right."* Using Figure 1, as pictorial guidance.

Note. One can view the operation $*$ is defined to be the positive crossing of an oriented knot diagram and $*^{-1}$ is the negative crossings of the same diagram.

Definition. An *involutory knot quandle* is a quandle where the orientation of a knot does not matter and only record the fact that $a * b = c$ whenever *"a goes under arc b gives us c"*. Pictorially defined in Figure 2, where $a * b = c$ and $c * b = a$, satisfying Axiom 2 of the definition of involutory quandle.

Example 4. Looking at Figure 8 (below, the figure-8 knot), there is an orientation on the knot diagram, so we can determine the knot quandle as follows:

$$< a_1, a_2, a_3, a_4 \,|\, a_2 *^{-1} a_3 = a_1, a_3 *^{-1} a_4 = a_2, a_4 *^{-1} a_1 = a_3, a_1 * a_2 = a_4 >$$

The involutory quandle presentation is similar,

$$< a_1, a_2, a_3, a_4 \,|\, a_2 * a_3 = a_1, a_3 * a_4 = a_2, a_4 * a_1 = a_3, a_1 * a_2 = a_4 > .$$

Finding involutory quandles are easier to calculate only because we do not need to worry about the orientation of a knot, however we do lose some information about the knot since the regular knot quandle can contain more information (such as the Wirtinger presentation of the knot group).

Sometimes the quandle presentation of a knot is the same as the involutory quandle presentation of the same knot. For example, if we looked at the trefoil's knot quandle and its involutory knot quandle, both have the same presentation!

The next question is, when we look at knots and their quandles, can we determine anything about them? The answer is, we can.

Fact. *A knot quandle and an involutory knot quandle are knot invariants.*

Proof. I will prove a involutory knot quandle is invariant under the Reidermeister moves.

Clearly it is invariant under *R0*, planar isotopy.

For *RI*, there are two cases, in both cases, staring with one strand labeled x and apply the *RI* move. Pick one arc that does have a "loop" in it, we know that must be the same arc labeled x. Then applying the involutory crossing rule in Figure 2, to get $x * x$, and so by definition it is still x. Going backwards, (removing the loop) we can see it is still invariant, since $x * x = x$, by definition.

Now for *RII*, we start with two strands, label one x and the other y. Applying the *RII* move and get one arc crossing over the other arc. Without loss of generality say we put strand x over the strand y, if it was the other way, we can relabel the strands so that x is still put over y. Then start by labeling the bottom ends of the strands, we know it must be x and y respectively and so the new arc formed by applying the *RII* move is $y * x$. Next, moving up the strands, the strand that was labeled x will still be x since there are no new "breaks" in the over strand. To determine the last strand, we have the crossing with x as the over strand and $y * x$ as one of the known under strands, so then applying the involutory crossing rule in Figure 2, we get $(y * x) * x$ and so, by definition $(x * y) * y = x$. Conversely, notice that $(y * x) * x = y$, by Axiom 2, getting back to the uncrossed strands.

Lastly, for *RIII* there are many crossings. Without loss of generality label the main (X) crossing with y as the over arc and x as the under arc and z at one end of the strand behind the crossing on the left side. (This is all without loss of generality because with any changes we can relabel the arcs so that the labelings work in a similar way) So the top strand of the main crossing (X) is $x * y$. Then the middle arc between x and y gives us $z * x$. The last strand will be labeled using the involutory crossing rule to get $(z * x) * y$. Now that we know what the "names" of these arcs are, applying the *RIII* moves only the middle arc and then we must determine the other arcs are still named the same. Using the same labeling, label x, y, and z the same as how we started, so the other strand not labeled in the main crossing (X) in the top right corner will be the same $x * y$, by the rule. Now the middle arc that was once labeled $z * x$, will now be labeled $z * y$, since it was moved to being between y and $x * y$. Finally, the last strand that is not labeled will be at the crossing where $x * y$ is over $z * y$, so it will be labeled $(z * y) * (x * y)$ and by definition $(z * x) * y = (z * y) * (x * y)$. Conversely, this move satisfies Axiom 3.

Therefore an involutory knot quandle is a knot invariant.

For regular knot quandle, it is proved similarly, with some added cases because knot quandles are defined on oriented knots. (Figures 3 and 4 shows how the involutory quandle operations work on the Reidermeister moves.) $\square$

Pictorially it is quite easy to see how quandles are invariant under the Reidermeister moves, but to describe it in words is quite wordy.

On the other hand racks are not invariant under Reidermeister moves because of the first Reidermeister move. So, there is something called framed isotopy.

Definition. *Framed isotopy* is essentially the Reidermeister moves except the first move is replaced by a writhe-preserving double "loop" move, described by Figure 5 [6].

Definition. Let K be a tame knot. The knot diagram associated with K will give the *fundamental knot quandle*, $Q(K)$, such that it is the set of arcs in the knot diagram with the binary operations $*$ and $*^{-1}$ recording the relationships between arcs.

Note. Not all elements in $Q(K)$ is an arc in the diagram associated with the knot K. [1]

Example 5. Let K be the figure-8 knot defined by Figure 6. The involutory quandle presentation of this knot is:

$$Q(K) =< a_1, a_2, a_3, a_4 \,|\, a_3 * a_1 = a_4,\ a_4 * a_2 = a_1,\ a_3 * a_4 = a_2,\ a_2 * a_4 = a_3 >$$

to make calculations easier let $a_1 = d$, $a_2 = b$, $a_3 = a$, and $a_4 = c$, so

$$Q(K) =< a, b, c, d \,|\, a * d = c,\ c * b = d,\ a * c = b,\ b * c = a > .$$

Now, notice how do we define $c * a$ or $a * b$ or $b * a$ or $d * c$? In all of those cases, in the form $x * y$, y is never an over arc of x in those cases. So by definition, we just add those elements into the quandle, even if we try to define them to be an arc, then $Q(K)$ is not a quandle. For example let's show that $c * a$ cannot be an arc.

Proof. Assume by way of contradiction that $c * a$ is an arc. Then there are four cases:
Case 1. If $c * a = d$ then $(c * a) * a = c$ by definition, but $(c * a) * a = d * a = b \neq c$.
Case 2. If $c * a = c$ then look at $b = d * a = (c * b) * a \neq (c * a) * (b * a) = c * d = a$.
Case 3. If $c * a = b$ then $(c * a) * a = c$ by definition, but $(c * a) * a = b * a = d \neq c$.
Case 4. If $c * a = a$ then look at $b = d * a = (c * b) * a \neq (c * a) * (b * a) = a * d = c$.
In all cases it will give us a contradiction. Therefore $c * a$ cannot be an arc. $\square$

So, for the other combination of arcs $a * b$, $b * a$, $d * c$, by a similar reasoning, cannot be an arc either. What could they be? Since these quandles are just structures of an algebraic object, they don't necessarily have to be defined on a knot.

Definition. A *quandle homomorphism* is a function $f : Q_1 \to Q_2$ such that $f(x * y) = f(x) * f(y)$ for two quandles Q_1 and Q_2.

Definition. A *counting invariant* is counting the number of homomorphisms possible from one knot quandle to another quandle, denoted $|Hom(Q(K), Q)|$. Sometimes the knot is called *Q-colorable* if we are counting the ways f is a homomorphism from the knot quandle *Q(K)* to a quandle Q.

Example 6. Calculate the involutory knot quandle for the trefoil and find $|Hom(Q(trefoil), \mathbb{Z}_3)|$. Using Figure 6 we can use a multiplication table to record the crossings, to give us a quandle presentation for the trefoil:

$$< a, b, c \,|\, a * b = c,\ b * c = a,\ c * a = b > .$$

In Table 3, the multiplication table for the trefoil is very similar to the dihedral quandle for $\mathbb{Z}_3$ (Table 1). So, how many ways can we send a to? b?

$$\begin{array}{lll} a \mapsto 0 & b \mapsto 0 & \\ a \mapsto 1 & b \mapsto 1 & c \mapsto ? \\ a \mapsto 2 & b \mapsto 2 & \end{array}$$

Once two arcs have their chosen maps, the last arc, in this case c, has only one choice to be. So $|Hom(Q(trefoil), \mathbb{Z}_3)| = 9$. Notice, these are just the 3-colorings of a trefoil, pictured in Figure 6.

Fact. *Every tame knot in $\mathbb{R}^3$ has a fundamental quandle that is related to their knot group.*

The fundamental knot quandle can be presented the same way a knot group is presented, essentially the knot group defined by the Wirtinger presentation is a conjugation quandle on the generators of the knot group. Since each strand in the knot diagram has one generator and the relations are just conjugations, so this presentation is exactly the conjugation quandle with the same generators! Let's look at some examples.

Example 7. The Wirtinger presentation of the figure-8 knot, K (Figure 8), is given by following:

$$\pi_1(\mathbb{R}^3 - K) =< x_1, x_2, x_3, x_4 \,|\, x_3^{-1} x_1 x_3 = x_2,\ x_4 x_2 x_4^{-1} = x_3,\ x_1^{-1} x_3 x_1 = x_4,\ x_2 x_4 x_2^{-1} = x_1 >$$

So, now define the operation $*$ and $*^{-1}$ to be the conjugation of the elements, like how it was defined in Example 1:

$$Q(K) =< x_1, x_2, x_3, x_4 \,|\, x_1 * x_3 = x_2,\; x_2 *^{-1} x_4 = x_3,\; x_3 * x_1 = x_4,\; x_4 *^{-1} x_2 = x_1 > .$$

Notice the quandle for this knot can also be generated by $a_1 * a_3$ where a_1 is under a_3 which gives us the other arc a_2, where a_i are the arcs in a regular knot projection of the figure-8 knot. So essentially the knot group of the figure-8 knot is the same as the knot quandle.

Example 8. The Wirtinger presentation of the trefoil is as follows:

$$< x_1, x_2, x_3 \,|\, x_1^{-1} x_3 x_1 = x_2,\; x_2^{-1} x_1 x_2 = x_3,\; x_3^{-1} x_2 x_3 = x_1 >$$

and so again the quandle defined for the trefoil is also similar to how the figure-8 knot's quandle is presented:

$$< x_1, x_2, x_3 \,|\, x_3 * x_1 = x_2,\; x_1 * x_2 = x_3,\; x_2 * x_3 = x_1 > .$$

A quandle is an algebra that be associated with a knot, these axioms are derived from the Reidermeister moves on the knot diagrams. So, knots are completely classified by their fundamental quandles, since their fundamental quandles can be defined by their knot group.

3 The Fox Colorings and Quandles

Just like how the trefoil's tricolorability was defined by the counting invariant, we can generalize the colorability of a knot using the knot quandle.

Definition. An *n-coloring* of a knot diagram D associated with a knot K is such that each arc is assigned an element (color) of $\mathbb{Z}_n$ such that each crossing the sum of the values satisfy $a + c - 2b \equiv 0 \mod n$ and at at least two elements (colors) of $\mathbb{Z}_n$ are used.

Now relating the definition of n-coloring with a quandle, is quite simple (Figure 9).

Define $a * b = c$ via $c = 2b - a \mod n$, so $a * b = 2b - a \mod n$. So essentially, the dihedral quandle is the n-coloring of a knot! $|Hom(Q(K), \mathbb{Z}_n)|$ corresponds to the dihedral quandle since we know in a knot quandle $a * b = c$ from Figure 9.

So this gives us the $n - coloring$ of the knot. Looking at the multiplication table for $\mathbb{Z}_5$ in Table 4 and looking at the involutory knot quandle for the figure-8 knot, we know that the figure-8 knot is 5-colorable, so what is $|Hom(Q(figure - 8), \mathbb{Z}_5)|$?

How many nontrivial colorings do we have? What about trivial colorings? So, $|Hom(Q(fig-8), \mathbb{Z}_5)| = 125$. Thus there are 125 5-colorings of the figure-8 knot. So the counting invariant is not a strong invariant, but it is a nice way to put colorability without the use of linear algebra.

So a knot quandle can define many knot invariants that we already know. The question is, can quandles generate stronger knot invariants?

4 Why Study Quandles?

A quandle is an elementary algebraic structure that can be associated with a knot [1]. As we learn more about quandles, we hope that it may lead us into more powerful knot invariants. In fact there is a cohomology theory for quandles that are about knotted surfaces in 4-space [2], so from that theory are quandle cocycle invariants that are defined in state-sum form. According to Nelson, quandles and racks are found everywhere and current research is being done on the *Dehn quandle* of a surface as well as on homology on quandles and racks and even quandle Galois theory [5]. Interestingly Joyce related geodesics and the involutory quandle and even proved that every involutory quandle can be represented as an involutory quandle with geodesics [4]. For every quandle out there, there is not necessarily a knot or link associated with it, but as we continue to learn more about quandles there may be a strong knot invariant lurking beneath it all.

5 Figures and Tables

		b		
	$*$	0	1	2
	0	0	2	1
a	1	2	1	0
	2	1	0	2

Table 1: The multiplication table for the quandle $\mathbb{Z}_3$.

		b			
	$*$	0	1	2	3
	0	0	2	0	2
a	1	3	1	3	1
	2	2	0	2	0
	3	1	3	1	3

Table 2: The multiplication table for the quandle $\mathbb{Z}_4$.

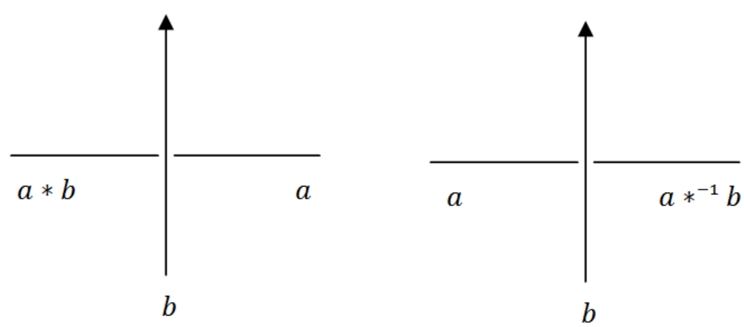

Figure 1: The definition of $*$ and $*^{-1}$ on a knot, with orientation.

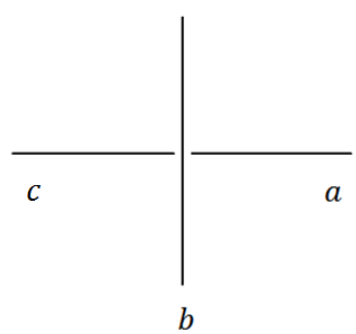

Figure 2: The crossing of a knot without orientation.

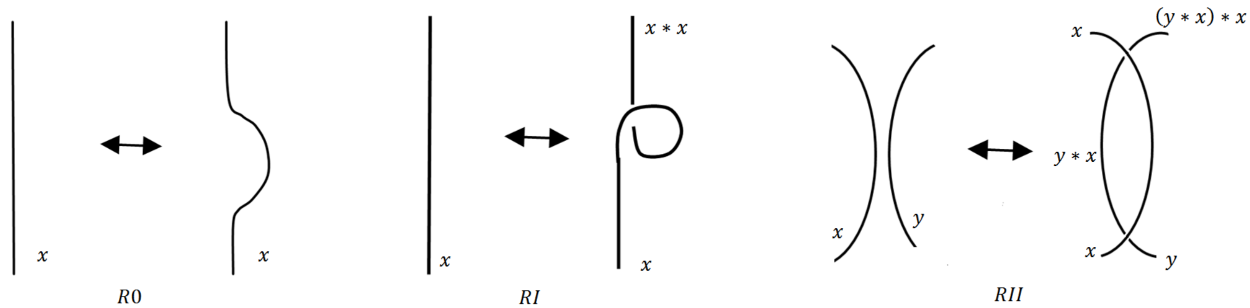

Figure 3: The involutory knot quandle operations on R0, *RI*, *RII*.

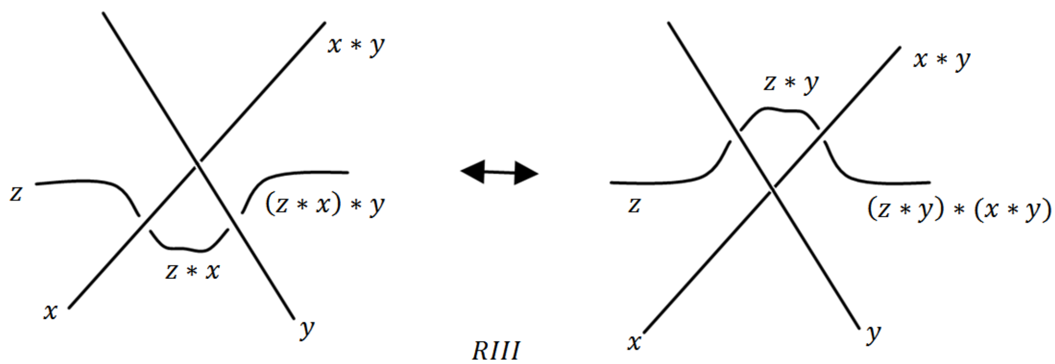

Figure 4: Involutory knot quandle operations on the third Reidermeister moves.

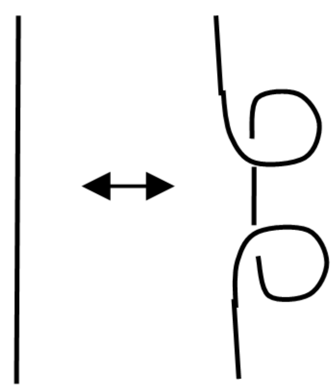

Figure 5: Framed isotopy of the *RI* move.

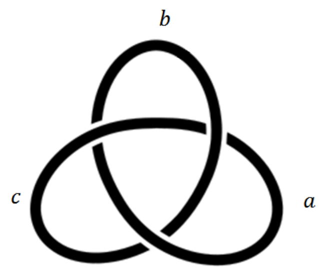

Figure 6: Trefoil knot

	$*$	a	b	c
x	a	a	c	b
	b	c	b	a
	c	b	a	c

Table 3: Multiplication table of $Q(trefoil)$.

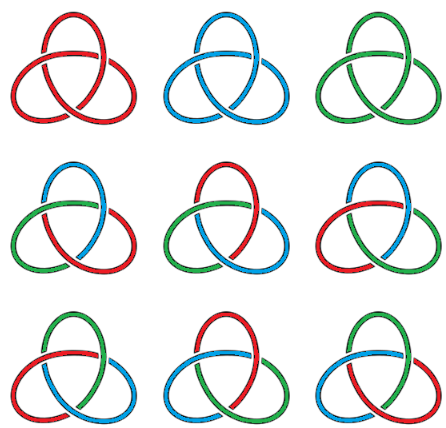

Figure 7: Pictorial representation of $|Hom(Q(trefoil), \mathbb{Z}_3)| = 9$.

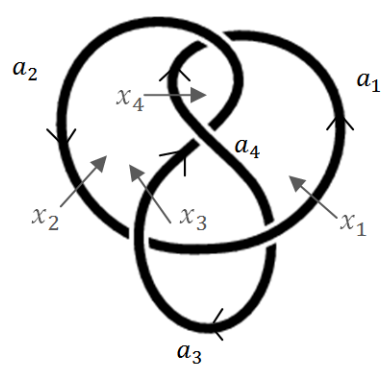

Figure 8 (what a coincidence): The figure-8 knot

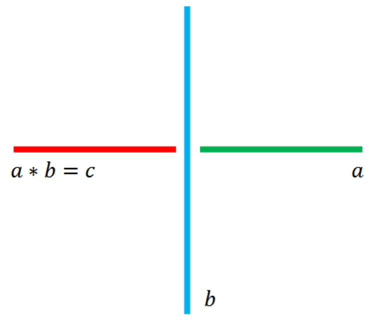

Figure 9: The coloring of a crossing defined by the dihedral quandle.

		b				
	$*$	0	1	2	3	4
	0	0	2	4	1	3
	1	4	1	3	0	2
a	2	3	0	2	4	1
	3	2	4	1	3	0
	4	1	3	0	2	4

Table 4: Multiplication table for $\mathbb{Z}_5$.

6 References

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