Derivation of MGF of Binomial Distribution

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10/5/2021

STAT 620 Quiz II Assignment

Derive mgf of binomial distribution and use it to find mean and variance.

Binomial Distribution

If a total number of **n** Bernoulli trials (success/ failures) are conducted and

- The trials must be independent
- Each trial must have the same success of probability **p**
- X is the number of success in the **n** trials

then X has the Binomial distribution with the parameters **n** and **p**

Binomial Distribution

We know for the binomial distribution

$$X \sim Bin(n, p)$$

$$S_X = \{0, 1, ...n\}$$

where the pmf is

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

MGF

By definition the MGF is

$$M_X(t) = E(e^{tX})$$

but it can easily be written in terms of summation like

$$M_X(t) = \sum_{x=0}^{n} e^{t \cdot x} \binom{n}{x} p^x (1-p)^{n-x}$$

Binomial theorem states that

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} = (x+y)^{n}$$

$$M_X(t) = \sum_{x=0}^{n} \binom{n}{x} (1-p)^{n-x} (e^t p)^n$$

$$M_X(t) = (1 - p + pe^t)^n$$

Mean or E(X) is the first derivative of the MGF

$$E(X) = M'_X(t) = \frac{d}{dt}(1-p+pe^t)^n = n(1-p+pe^t)^{n-1} \cdot (0-0+pe^t)$$

The zeroes cancel out, switch the terms around and you are left with

$$M'_X(t) = np(e^t)(1-p+pe^t)^{n-1}$$

For mean, when t=0

$$E(X) = np(e^0)(1 - p + pe^0)^{n-1}$$

E raised to the zero power is 1, the p's in the second half of the equation cancel out, one raised to any power is one and you are left with

$$E(X) = np \cdot 1(1 - p + p)^{n-1} = np \cdot 1 \cdot 1 = np$$

To find the variance you take the second derivative of the mgf

$$M_X''(t) = \frac{d}{dt} np(e^t)(1-p+pe^t)^{n-1}$$

$$M_X''(t) = np(e^t)(1-p+pe^t)^{n-1} + npe^t[(n-1)(1-p+pe^t)^{n-2} \cdot pe^t]$$

Rearrange the terms and we have

$$M''_X(t) = n \cdot p(e^t)(1-p+pe^t)^{n-1} + n(n-1)p^2(e^{t^2})(1-p+pe^t)^{n-2}$$

$$M''_X(t) = n \cdot p(1) + n(n-1)p^2(1)(1)$$

Such that

$$E(X^2) = np(1) + n(n-1)p^2 = np + (n^2 - n)p^2 = np + n^2p^2 - np^2$$

Inputing what we have derived in to the formula for variance

$$Var(X) = E(X^2) - [E(X)]^2$$

We have

$$Var(X) = np + n^2p^2 - np^2 - n^2p^2$$

where

$$[E(X)]^2 = (np)^2 = n^2p^2$$

The second and fourth term cancel each other out and we are left with

$$Var(X) = np - np^2 = np(1 - p)$$