RNG of Weibulls

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Consider the Weibull distribution:

$$f_\Theta(\theta) = \frac{\alpha}{\beta^\alpha}\theta^{\alpha-1}\exp\big(-(\theta/\beta)^\alpha\big)$$

How can we generate realizations from this distribution "by hand"?

Let $\alpha = 2$ and $\beta = 0.5$. One method is to sample from Weibull using the *inverse CDF method*.

$$F_\Theta(\theta) = \int_0^\theta f_\Theta(\theta)d\theta = \frac{\alpha}{\beta^\alpha}\int_0^\theta \theta^{\alpha-1}\exp\big(-(\theta/\beta)^\alpha\big)$$

Formally, we would use u-substitution but it's fairly obvious what the integral is:

$$F_\Theta(\theta) = \Bigg[-\exp\big(-(\theta/\beta)^\alpha\big)\Bigg]_0^\theta = 1 - \exp\big(-(\theta/\beta)^\alpha\big)$$

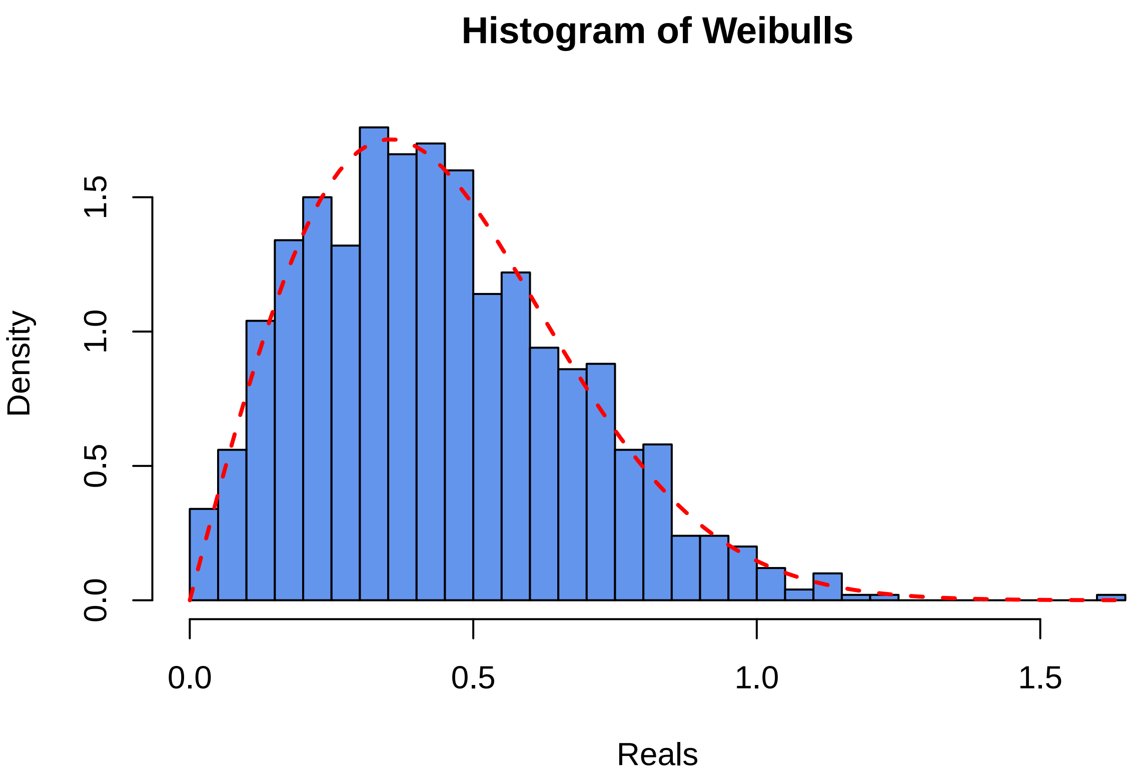
Now we invert the CDF:

$$p = 1 - \exp\big(-(\theta/\beta)^\alpha\big)$$
$$\log(1-p) = -(\theta/\beta)^\alpha$$
$$F_\Theta^{-1}(p) = \beta\big[-\log(1-p)\big]^{1/\alpha}$$

Now we apply the Inverse CDF method to sample from the Weibull with $\alpha = 2$ and $\beta = 0.5$. For b samples, repeat this b times:

1. $U^* \sim Unif[0,1]$
2. $X^* = F_\Theta^{-1}(U^*)$

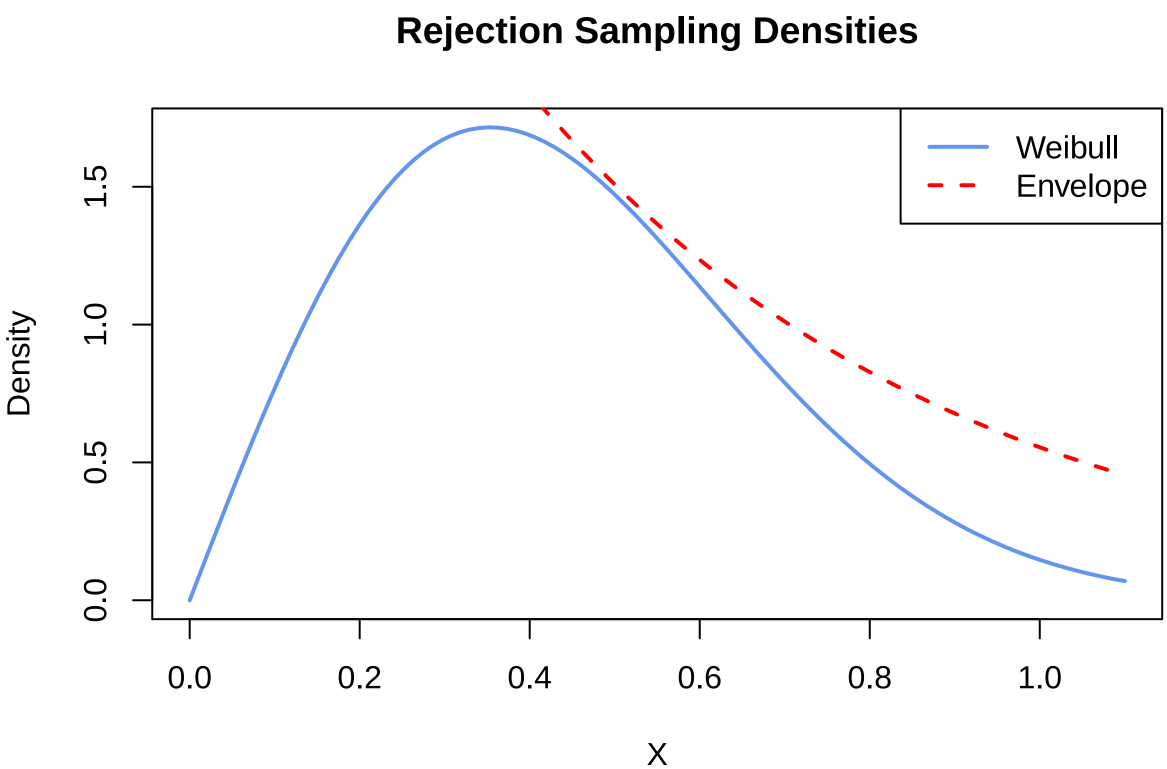
```r
# set Weibull parameters
alpha = 2
beta = 0.5

invWeibull = function(p) {
    beta * (-log(1 - p))^(1/alpha)
}
invWeibull = Vectorize(invWeibull)

Weibull_samples = invWeibull(runif(1000))

hist(Weibull_samples, col = "cornflowerblue", xlab = "Reals", main = "Histogram of Weibulls",
    freq = FALSE, breaks = 30)
curve(dweibull(x, 2, 0.5), add = TRUE, col = "red", lwd = 2, lty = 2)
```



Now let's use *rejection sampling* to sample from the Weibull distribution with $\alpha = 2$ and $\beta = 0.5$. Specifically, use the Exponential distribution with scale β and multiplier $k = \alpha$ as the envelope.

First, let's show graphically that this envelope adequately covers the Weibull distribution for all θ.

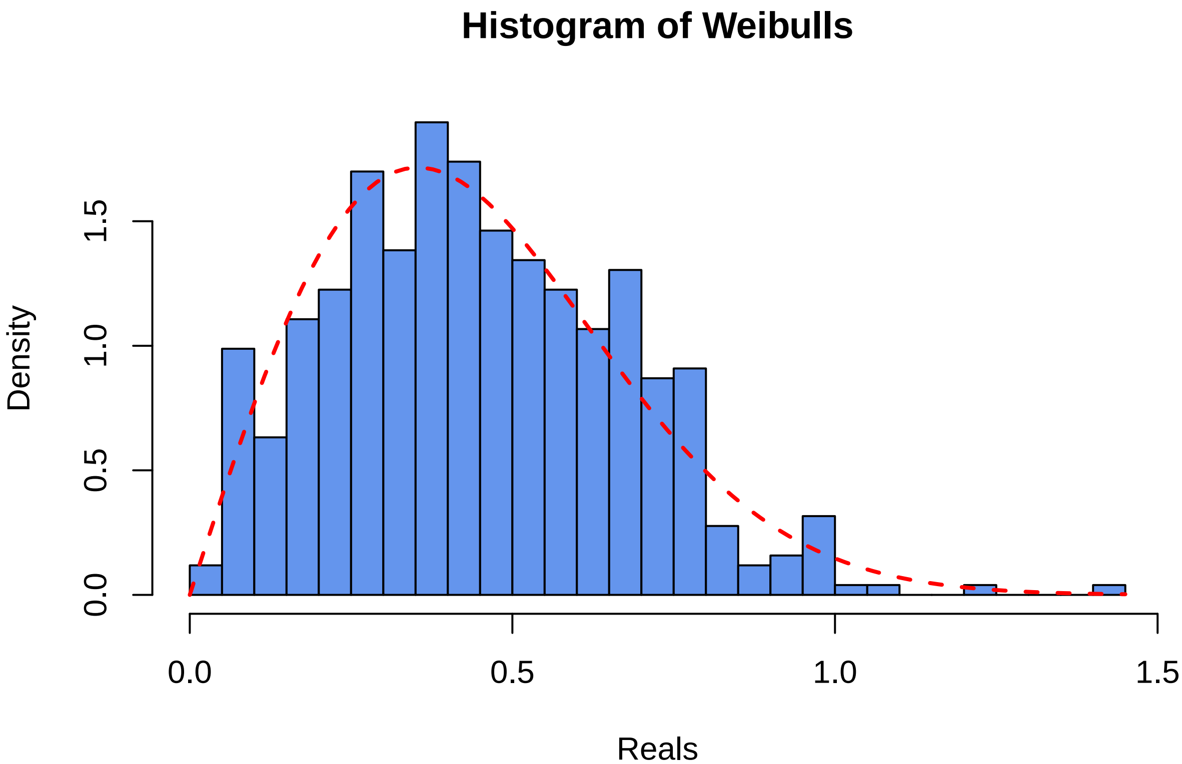
```r
curve(dweibull(x, 2, 0.5), col = "cornflowerblue", from = 0, to = 1.1, lwd = 2,
    xlab = "X", ylab = "Density", main = "Rejection Sampling Densities")
curve(2.05 * dexp(x, 2), add = TRUE, lwd = 2, col = "red", lty = 2)
legend("topright", legend = c("Weibull", "Envelope"), lwd = 2, lty = c(1, 2),
    col = c("cornflowerblue", "red"))
```



Great! Now we can proceed with the Rejection Sampling algorithm:

1. Sample $\theta^* \sim Exp(\beta)$
2. Sample $u^* \sim U[0, 1]$
3. If $u^* < \frac{f(\theta^*)}{G(\theta^*)}$ then keep θ^*. Otherwise reject it.

```
alpha = 2
beta = 0.5
B = 1000

weibull_samples_RS = numeric()

for (b in 1:B) {
    theta = rexp(1, 2)
    u = runif(1)
    if (u < dweibull(theta, 2, 0.5)/(2.01 * dexp(theta, 2))) {
        weibull_samples_RS = c(weibull_samples_RS, theta)
    }
}

hist(weibull_samples_RS, col = "cornflowerblue", xlab = "Reals", main = "Histogram of Weibulls",
    freq = FALSE, breaks = 30)
curve(dweibull(x, 2, 0.5), add = TRUE, col = "red", lwd = 2, lty = 2)
```


Histogram of Weibulls
Density
0.0
0.5
1.0
1.5
0.0
0.5
1.0
1.5
Reals